Equations of motion for scalar in Schwarzschild-AdS

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We begin with the $d = 4$ Schwarzschild-AdS metric in ingoing Eddington-Finkelstein coordinates,

$$ds^2 = -A(r)dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where

$$A(r) = 1 - \frac{2M}{r} + \frac{r^2}{L^2}. \tag{2}$$

Here M is the black hole mass parameter, and L the AdS radius.

The wave equation in spherical symmetry takes the form

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) &(3)\\ &= \frac{2}{r}A\phi' + A\phi'' + A'\phi' + 2\dot{\phi}' + \frac{2}{r}\dot{\phi}, &(4)\end{aligned}$$

where the dot and prime denote partial derivatives with respect to v and r, respectively.

To integrate the equations, it is useful to define the derivative

$$d_+\phi = \dot{\phi} + \frac{1}{2}A\phi'. \tag{5}$$

Then the equation (3) takes the form

$$0 = \frac{1}{r}A\phi' + 2\left(d_+\phi\right)' + \frac{2}{r}d_+\phi. \tag{6}$$

We would like to treat ϕ and $d_+\phi$ as our dynamical variables. Given $\phi(v = v_0, r)$ on some time slice, (6) can be integrated in space to obtain $d_+\phi(v = v_0, r)$. Next, $\dot{\phi}(v = v_0, r)$ is obtained from (5), which upon time integration gives $\phi(v = v_0 + \Delta v, r)$. The procedure then repeats to generate the full solution.

The difficulty with integrating equations (5)–(6) directly is that the spatial coordinate $r \in [r_0, \infty)$. So, we also compactify the spatial coordinate by defining a new coordinate $\rho = 1/r$. This gives rise to a compact domain $\rho \in [0, 1/r_0]$. We define new functions,

$$\begin{aligned} \phi(v, r) &= \varphi(v, 1/r) = \varphi(v, \rho), &(7)\\ d_+\phi(v, r) &= \Pi(v, 1/r) = \Pi(v, \rho). &(8)\end{aligned}$$

After this transformation, (5)–(6) take the form

$$\begin{aligned} \partial_v\varphi &= \Pi + \frac{\rho^2}{2}A\partial_\rho\varphi, &(9)\\ \partial_\rho\Pi &= \frac{1}{\rho}\Pi - \frac{\rho}{2}A\partial_\rho\varphi, &(10)\end{aligned}$$

respectively.

The general solution for φ can be studied asymptotically in a power series about the boundary $\rho = 0$. We obtain,

$$\varphi(v, \rho) = \varphi_0(v) + L^2\dot{\varphi}_0(v)\rho + \varphi_3(v)\rho^3 + \left(\frac{ML^4}{2}\dot{\varphi}_0(v) - L^2\dot{\varphi}_3(v)\right)\rho^4 + O(\rho^5), \tag{11}$$

where $\varphi_0(v)$ and $\varphi_3(v)$ are two free functions of time, which may be regarded as the Dirichlet and Newmann boundary data. To impose our reflecting boundary condition, we impose $\varphi_0(v) = 0$; $\varphi_3(v)$ will be determined by the equations of motion. Thus, we require that, asymptotically,

$$\varphi(v, \rho) = O(\rho^3), \tag{12}$$

$$\Pi(v, \rho) = O(\rho^2). \tag{13}$$

Finally, we rescale the dynamical variables by powers of ρ such that they both go to zero linearly at the boundary. That is, we define,

$$\hat{\varphi} = \frac{\varphi}{\rho^2}, \tag{14}$$

$$\hat{\Pi} = \frac{\Pi}{\rho}. \tag{15}$$

In terms of these variables, we have

$$\partial_v \hat{\varphi} = \frac{1}{\rho}\hat{\Pi} + A\rho\hat{\varphi} + \frac{1}{2}A\rho^2\partial_\rho\hat{\varphi}, \tag{16}$$

$$\partial_\rho \hat{\Pi} = -A\rho\hat{\varphi} - \frac{1}{2}A\rho^2\partial_\rho\hat{\varphi}. \tag{17}$$

We would like to solve these equations, imposing the boundary condition $\hat{\Pi}(v, 0) = 0$ when integrating (17). (For the boundary condition $\hat{\varphi}(v, 0) = 0$, it should be sufficient to impose this initially.) Both of the above equations have singular points at $\rho = 0$ [recall, $A \to 1/(\rho^2 L^2)$ as $\rho \to 0$], so we must treat this end point with special care at the level of the equations as well. Using L'Hôpital's rule, we find that in the limit,

$$\lim_{\rho\to 0} \partial_v \hat{\varphi} = 0, \tag{18}$$

$$\lim_{\rho\to 0} \partial_\rho \hat{\Pi} = -\left.\frac{3}{2L^2}\partial_\rho\hat{\varphi}\right|_{\rho=0}. \tag{19}$$

In the code, we solve (16)–(17), careful to treat the $\rho = 0$ boundary as above.