Equations of motion for Einstein-scalar system

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1 General setting

We study the Einstein equation with negative cosmological constant, coupled to a real massless scalar field,

$$G_{ab} - \frac{3}{L^2} g_{ab} = T^{\phi}_{ab}, \tag{1}$$

where

$$T^{\phi}_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi. \tag{2}$$

Here L is the AdS radius, and we work in $d = 4$.

We also impose spherically symmetry, and we work in ingoing Eddington-Finkelstein coordinates,

$$ds^2 = -A(v,r)dv^2 + 2dvdr + \Sigma(v,r)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3}$$

If we set

$$A(v,r) = 1 - \frac{2M}{r} + \frac{r^2}{L^2}, \qquad \Sigma(v,r) = r, \tag{4}$$

this reduces to the Schwarzschild-AdS black hole. We will of course keep A and Σ as free functions, as they will be determined by the Einstein equation.

2 Equations of motion

The wave equation takes the form

$$0 = 2\frac{\Sigma'}{\Sigma} d_+\phi + 2\frac{d_+\Sigma}{\Sigma}\phi' + 2(d_+\phi)', \tag{5}$$

while the Einstein equation is equivalent to

$$0 = -\frac{1}{2} - \frac{3\Sigma^2}{2L^2} + (d_+\Sigma)\Sigma' + \Sigma(d_+\Sigma)', \tag{6}$$

$$0 = \frac{2}{\Sigma^2} - 4\frac{(d_+\Sigma)\Sigma'}{\Sigma^2} + 2(d_+\phi)\phi' + A'', \tag{7}$$

$$0 = d_+d_+\Sigma + \frac{1}{2}\Sigma(d_+\phi)^2 - \frac{1}{2}(d_+\Sigma)A', \tag{8}$$

$$0 = \frac{1}{2}\Sigma(\phi')^2 + \Sigma''. \tag{9}$$

Note that we expressed time derivatives in terms of

$$d_+ f = \dot{f} + \frac{1}{2} A f', \tag{10}$$

where prime and dot denote partial derivatives with respect to r and v, respectively.

Examining the above equations, we see that, given $\phi(v = v_0, r)$, we can integrate as follows:

1. Solve (5), (6), (7), and (9) for $d_+\phi$, $d_+\Sigma$, A and Σ, respectively, at time $v = v_0$ by integrating radially.

2. Knowledge of A and $d_+\phi$ gives rise to the time derivative $\partial_v \phi$ via (10). This may be integrated in time to obtain $\phi(v = v_0 + \Delta v, x)$.

Equation (8) is not needed.

While this basic strategy will eventually be implemented, there are several challenges that must be addressed. First, we require a finite spatial domain, so we define a new variable $\rho = 1/r$. There is also some gauge freedom in the equations, which must be fixed. We also require boundary data for the fields at $r \to \infty$, which is a regular singular point that must be treated carefully.

3 Compact radial coordinate $\rho = 1/r$

One difficulty with integrating the equations directly is that the spatial coordinate $r \in [r_0, \infty)$. Thus, we compactify the spatial coordinate by defining a new coordinate $\rho = 1/r$. This gives rise to a compact domain $\rho \in [0, 1/r_0]$. We define new functions,

$$\phi(v, r) = \varphi(v, 1/r) = \varphi(v, \rho), \tag{11}$$
$$d_+\phi(v, r) = \Pi(v, 1/r) = \Pi(v, \rho), \tag{12}$$
$$\Sigma(v, r) = \sigma(v, 1/r) = \sigma(v, \rho), \tag{13}$$
$$d_+\Sigma(v, r) = s(v, 1/r) = s(v, \rho), \tag{14}$$
$$A(v, r) = \alpha(v, 1/r) = \alpha(v, \rho). \tag{15}$$

After this transformation, (5), (6), (7), and (9) take the form

$$\partial_\rho \Pi = -\frac{\partial_\rho \sigma}{\sigma} \Pi - \frac{s}{\sigma} \partial_\rho \varphi, \tag{16}$$
$$\partial_\rho s = -\frac{1}{2\rho^2 \sigma} - \frac{3\sigma}{2L^2 \rho^2} - \frac{s \partial_\rho \sigma}{\sigma}, \tag{17}$$
$$\partial_\rho^2 \alpha = -\frac{2}{\sigma^2 \rho^4} - \frac{4 s \partial_\rho \sigma}{\rho^2 \sigma^2} + \frac{2}{\rho^2} \Pi \partial_\rho \varphi - \frac{2}{\rho} \partial_\rho \alpha, \tag{18}$$
$$\partial_\rho^2 \sigma = -\frac{1}{2} \sigma (\partial_\rho \varphi)^2 - \frac{2}{\rho} \partial_\rho \sigma, \tag{19}$$

respectively, while the time derivative of φ is

$$\partial_v \varphi = \Pi + \frac{\rho^2}{2} \alpha \partial_\rho \varphi. \tag{20}$$

4 Boundary conditions

In order to impose boundary conditions we must first study the asymptotic solution at infinity. We expand the general solution in a power series about $\rho = 0$, careful to enforce that the boundary metric be the Einstein static universe $\mathbb{R} \times S^2$ (standard for global AdS), and that the scalar field satisfy "Dirichlet" conditions. We obtain,

$$\varphi(v,\rho) = \varphi_3\rho^3 + \left(\frac{3}{2}L^2\lambda\varphi_3 + L^2\dot{\varphi}_3\right)\rho^4 + O(\rho^5), \tag{21}$$

$$\sigma(v,\rho) = \frac{1}{\rho} + \frac{1}{2}L^2\lambda - \frac{3}{20}\varphi_3^2\rho^5 + O(\rho^6), \tag{22}$$

$$\alpha(v,\rho) = \frac{1}{L^2\rho^2} + \frac{\lambda}{\rho} + \left(1 + \frac{1}{4}L^2\lambda^2 - L^2\dot{\lambda}\right) - 2M\rho + ML^2\lambda\rho^2 + O(\rho^3), \tag{23}$$

where $\varphi_3(v)$ and $\lambda(v)$ are functions of time, and M is a constant representing the ADM mass. The function $\lambda(v)$ represents a gauge freedom to redefine the radial coordinate $1/\rho \to 1/\rho + \lambda(v)$, and may be chosen arbitrarily. The ADM mass may also be chosen freely. The function $\varphi_3(v)$, on the other hand, will arise as a consequence of the equations of motion.

Asymptotic behavior for the "time derivative" fields may be easily determined using (10),

$$\Pi(v,\rho) = -\frac{3}{2L^2}\varphi_3\rho^2 + O(\rho^3), \tag{24}$$

$$s(v,\rho) = \frac{1}{2L^2\rho^2} + \frac{\lambda}{2\rho} + \left(\frac{1}{2} + \frac{L^2}{8}\lambda^2\right) - M\rho + \frac{1}{2}M^2L^2\lambda\rho + O(\rho^3). \tag{25}$$

Finally, we define new (hatted) dynamical variables that have better behavior as $\rho \to 0$. We take

$$\hat{\varphi} = \frac{1}{\rho^2}\varphi, \tag{26}$$

$$\hat{\Pi} = \frac{1}{\rho}\Pi, \tag{27}$$

$$\tag{28}$$

as we did in the case with no backreaction, so that both scalar field variables have linear falloff to 0 as $\rho \to 0$. For the metric variables, we take

$$\hat{\sigma} = \sigma - \frac{1}{\rho}, \tag{29}$$

$$\hat{s} = s - \frac{1}{2L^2}\sigma^2 - \frac{1}{2}, \tag{30}$$

$$\hat{\alpha} = \alpha - \frac{1}{L^2}\sigma^2 - 1. \tag{31}$$

These definitions differ slightly from those of Chesler and Yaffe, which worked in the Poincaré patch and thus do not have the subtraction of $\frac{1}{2}$ and 1 in (30) and (31), respectively. We also define two new auxiliary variables,

$$\hat{\tau} \equiv \partial_\rho\hat{\sigma}, \tag{32}$$

$$\hat{\beta} \equiv \partial_\rho\hat{\alpha}, \tag{33}$$

such that the final equations of motion only contain first spatial derivatives.

The $\rho \to 0$ behavior of the new fields is

$$
\begin{aligned}
\hat{\varphi} &= \varphi_3 \rho + O(\rho^2) && \to 0, && (34)\\
\hat{\Pi} &= -\frac{3}{2L^2}\varphi_3 \rho + O(\rho^2) && \to 0, && (35)\\
\hat{\sigma} &= \frac{1}{2}L^2\lambda + O(\rho^5) && \to \frac{1}{2}L^2\lambda, && (36)\\
\hat{\tau} &= O(\rho^4) && \to 0, && (37)\\
\hat{s} &= -M\rho + \frac{1}{2}M^2L^2\lambda\rho^2 + O(\rho^3) && \to 0, && (38)\\
\hat{\alpha} &= -L^2\dot{\lambda} - 2M\rho + O(\rho^2) && \to -L^2\dot{\lambda}, && (39)\\
\hat{\beta} &= -2M + 2M^2L^2\lambda\rho + O(\rho^2) && \to -2M. && (40)
\end{aligned}
$$

We will impose as boundary data the right hand side limits above (for all but $\hat{\varphi}$, which we integrate in time rather than radially). For the gauge condition, we set $\lambda = 0$, so the only nontrivial boundary condition is $\hat{\beta}(\rho = 0) = -2M$.

5 Equations in final form

The equations of motion are obtained in *Mathematica* by substituting these new variables. We obtain,

$$\begin{aligned}\partial_\rho\hat{\Pi} = & -\frac{\rho\hat{\sigma}^2\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)}-\frac{2\hat{\sigma}\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)}-\frac{\hat{\varphi}}{L^2\rho(\rho\hat{\sigma}+1)}-\frac{\rho^2\hat{\sigma}^2\partial_\rho\hat{\varphi}}{2L^2(\rho\hat{\sigma}+1)}-\frac{\rho\hat{\sigma}\partial_\rho\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)}\\ & -\frac{\partial_\rho\hat{\varphi}}{2L^2(\rho\hat{\sigma}+1)}-\frac{\hat{\Pi}\rho\hat{\tau}}{\rho\hat{\sigma}+1}-\frac{\hat{\Pi}\hat{\sigma}}{\rho\hat{\sigma}+1}-\frac{\rho\hat{\varphi}}{\rho\hat{\sigma}+1}-\frac{2\rho\hat{s}\hat{\varphi}}{\rho\hat{\sigma}+1}-\frac{\rho^2\hat{s}\partial_\rho\hat{\varphi}}{\rho\hat{\sigma}+1}-\frac{\rho^2\partial_\rho\hat{\varphi}}{2(\rho\hat{\sigma}+1)},\end{aligned} \tag{41}$$

$$\begin{aligned}\partial_\rho\hat{s} = & -\frac{3\rho\hat{\sigma}^2\hat{\tau}}{2L^2(\rho\hat{\sigma}+1)}-\frac{3\hat{\sigma}\hat{\tau}}{L^2(\rho\hat{\sigma}+1)}-\frac{3\hat{\tau}}{2L^2\rho(\rho\hat{\sigma}+1)}-\frac{\rho\hat{\tau}}{2(\rho\hat{\sigma}+1)}-\frac{\rho\hat{s}\hat{\tau}}{\rho\hat{\sigma}+1}\\ & +\frac{\hat{s}}{\rho(\rho\hat{\sigma}+1)},\end{aligned} \tag{42}$$

$$\partial_\rho\hat{\alpha} = \hat{\beta}, \tag{43}$$

$$\begin{aligned}\partial_\rho\hat{\beta} = & -\frac{2\hat{\beta}\rho\hat{\sigma}^2}{(\rho\hat{\sigma}+1)^2}-\frac{4\hat{\beta}\hat{\sigma}}{(\rho\hat{\sigma}+1)^2}-\frac{2\hat{\beta}}{\rho(\rho\hat{\sigma}+1)^2}+\frac{4\rho^4\hat{\sigma}^4\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{16\rho^3\hat{\sigma}^3\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}\\ & -\frac{2\rho^2\hat{\sigma}^2\hat{\tau}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{2\hat{\tau}}{L^2\rho^2(\rho\hat{\sigma}+1)^2}+\frac{24\rho^2\hat{\sigma}^2\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}-\frac{4\rho\hat{\sigma}\hat{\tau}^2}{L^2(\rho\hat{\sigma}+1)^2}\\ & -\frac{2\hat{\tau}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{2\hat{\sigma}^2\hat{\tau}}{L^2(\rho\hat{\sigma}+1)^2}+\frac{4\hat{\sigma}\hat{\tau}}{L^2\rho(\rho\hat{\sigma}+1)^2}+\frac{16\rho\hat{\sigma}\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{4\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}\\ & +\frac{\rho^6\hat{\sigma}^4\partial_\rho\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{4\rho^5\hat{\sigma}^3\partial_\rho\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{6\rho^4\hat{\sigma}^2\partial_\rho\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{4\rho^3\hat{\sigma}\partial_\rho\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}+\frac{\rho^2\partial_\rho\hat{\varphi}^2}{L^2(\rho\hat{\sigma}+1)^2}\\ & +\frac{4\rho^5\hat{\sigma}^4\partial_\rho\hat{\varphi}\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)^2}+\frac{16\rho^4\hat{\sigma}^3\partial_\rho\hat{\varphi}\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)^2}+\frac{24\rho^3\hat{\sigma}^2\partial_\rho\hat{\varphi}\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)^2}+\frac{16\rho^2\hat{\sigma}\partial_\rho\hat{\varphi}\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)^2}+\frac{4\rho\partial_\rho\hat{\varphi}\hat{\varphi}}{L^2(\rho\hat{\sigma}+1)^2}\\ & +\frac{4\hat{\Pi}\rho^2\hat{\sigma}^2\hat{\varphi}}{(\rho\hat{\sigma}+1)^2}+\frac{8\hat{\Pi}\rho\hat{\sigma}\hat{\varphi}}{(\rho\hat{\sigma}+1)^2}+\frac{4\hat{\Pi}\hat{\varphi}}{(\rho\hat{\sigma}+1)^2}-\frac{2\hat{\tau}}{(\rho\hat{\sigma}+1)^2}+\frac{4\hat{s}}{\rho^2(\rho\hat{\sigma}+1)^2}-\frac{4\hat{s}\hat{\tau}}{(\rho\hat{\sigma}+1)^2}\\ & +\frac{2\hat{\Pi}\rho^3\hat{\sigma}^2\partial_\rho\hat{\varphi}}{(\rho\hat{\sigma}+1)^2}+\frac{4\hat{\Pi}\rho^2\hat{\sigma}\partial_\rho\hat{\varphi}}{(\rho\hat{\sigma}+1)^2}+\frac{2\hat{\Pi}\rho\partial_\rho\hat{\varphi}}{(\rho\hat{\sigma}+1)^2},\end{aligned} \tag{44}$$

$$\partial_\rho\hat{\sigma} = \hat{\tau}, \tag{45}$$

$$\partial_\rho\hat{\tau} = -2\rho^2\hat{\sigma}\hat{\varphi}^2-\frac{2\hat{\tau}}{\rho}-2\rho\hat{\varphi}^2-\frac{1}{2}\rho^4\hat{\sigma}\partial_\rho\hat{\varphi}^2-\frac{\rho^3\partial_\rho\hat{\varphi}^2}{2}-2\rho^3\hat{\sigma}\partial_\rho\hat{\varphi}\hat{\varphi}-2\rho^2\partial_\rho\hat{\varphi}\hat{\varphi}, \tag{46}$$

We shall integrate these equations along radial slices, given the boundary data and φ on an initial time slice.

Once $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\varphi}$ and $\hat{\Pi}$ are known, we integrate $\hat{\varphi}$ forward in time, using (20), which now takes the form

$$\begin{aligned}\partial_v\hat{\varphi} = \hat{\alpha}\rho\hat{\varphi}+\frac{\rho\hat{\sigma}^2\hat{\varphi}}{L^2}+\frac{\hat{\varphi}}{L^2\rho}+\frac{2\hat{\sigma}\hat{\varphi}}{L^2}+\frac{\rho^2\hat{\sigma}^2\partial_\rho\hat{\varphi}}{2L^2}+\frac{\rho\hat{\sigma}\partial_\rho\hat{\varphi}}{L^2}\\ +\frac{\partial_\rho\hat{\varphi}}{2L^2}+\frac{\hat{\Pi}}{\rho}+\rho\hat{\varphi}+\frac{1}{2}\hat{\alpha}\rho^2\partial_\rho\hat{\varphi}+\frac{\rho^2\partial_\rho\hat{\varphi}}{2}.\end{aligned} \tag{47}$$

These equations must be treated carefully at $\rho = 0$, where some terms become ill-defined. Taking limits as $\rho \to 0$, we obtain

$$\lim_{\rho\to 0} \partial_\rho \hat{\Pi} = -\frac{3}{2L^2}\partial_\rho \hat{\varphi}, \tag{48}$$

$$\lim_{\rho\to 0} \partial_\rho \hat{s} = -M, \tag{49}$$

$$\lim_{\rho\to 0} \partial_\rho \hat{\alpha} = -2M, \tag{50}$$

$$\lim_{\rho\to 0} \partial_\rho \hat{\beta} = 2M^2 L^2 \lambda, \tag{51}$$

$$\lim_{\rho\to 0} \partial_\rho \hat{\sigma} = 0, \tag{52}$$

$$\lim_{\rho\to 0} \partial_\rho \hat{\tau} = 0. \tag{53}$$

These results are consistent with (34)–(40), as expected.