Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^t/(1-\gamma)\|Q_0-Q^*\|$$

Value Iteration

Reinforcement Learning

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Recap

Sequential Decision Making

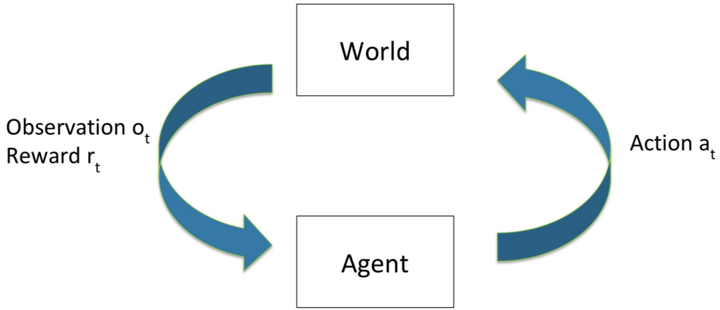
The agent interacts with the environment:

- at discrete timesteps;
- by receiving observations o_t and reward r_t from the environment;
- by taking actions a_t;

The state is a function of the history:

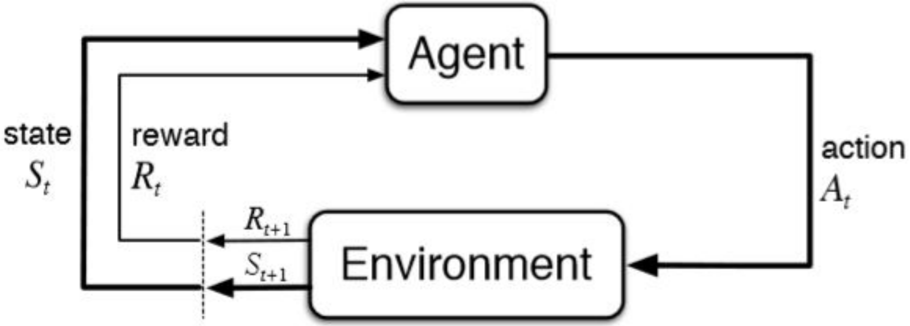
$$s_t = f(h_t)$$

and it is typically hidden or unknown



Markov Decision Process (MDP)

- Set of states S
- Set of actions A



Alternative notation

$s_{t+1} \sim p(.|s_t,a_t)$ or

$s' \sim p(.|s,a)$

Sequential Decision Making under Markov Assumption

- Markovian transition dynamics
- Full Observability
- The transition dynamics *T* is (generally) stochastic $p(s_{t+1}|s_t,a_t)$

Policy

A policy π:

- is a mapping from (all) states to actions;
- determines how agents select actions;
- can be deterministic ($a = \pi(s)$) or stochastic ($\pi(a|s)$ or $p(a|s)$ or $a \sim \pi(.|s)$)

Value Function/Q-Function

- We estimate the goodness of states and actions based on their value
- It's also a measure to compare policies

$V^{\pi}(s_t) = \mathbb{E}_{\pi}[r_t+\gamma r_{t+1}+\gamma^2 r_{t+2}+\gamma^3 r_{t+3}+...|s_t] = \mathbb{E}[\sum^{\infty}_{h=0}\gamma^h r_h|s_0=s_t,a_h=\pi(s_h),s_{h+1}\sim p(.|s_h,a_h)]$

$Q^{\pi}(s_t,a_t) = \mathbb{E}[\sum^{\infty}_{h=0}\gamma^h r_h|(s_0,a_0)=(s_t,a_t),a_{h+1}=\pi(s_h),s_{h+1}\sim p(.|s_h,a_h)]$

For infinite horizon MDPs there always exists a deterministic policy π* such that

$$V^{\pi^*}(s)\geq V^{\pi}(s)\ \forall\ s,\pi$$

meaning that π* (optimal policy) dominates all other policies π in each state

Discount Factor

Setting $\gamma = 1$ for infinite tasks is a bad idea!

Note that $\sum^{\infty}_{h=0}\gamma^{h}$ is a geometric series and for γ in [0,1] this is equivalent to $1/(1-\gamma)$

So, the value of γ approximately determines how many steps ahead we are considering

E.g., γ=0.99 -> 99 timesteps ahead

Bellman Equation

The value of a certain state is expanded in terms of the current reward and the value of the next states according to the policy

r here is function of s and π(s)

$$V^{\pi}(s_t) = \mathbb{E}_{\pi}[r_t+\gamma r_{t+1}+\gamma^2 r_{t+2}+\gamma^3 r_{t+3}+\ldots|s_t] = r_t+\gamma\mathbb{E}_{s'\sim p(.|s,\pi(s))}[V^{\pi}(s')]$$

$$Q^{\pi}(s_t,a) = r_t+\gamma\mathbb{E}_{s'\sim p(.|s,a)}[V^{\pi}(s')]$$

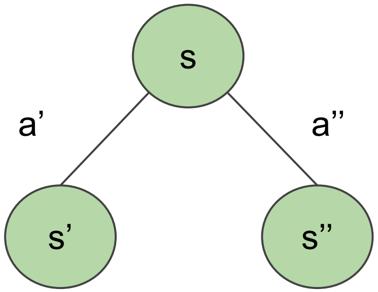
r here is function of s and a

As a result V(s) = Q(s,π(s))

Bellman Optimality Example

$$V^*(s)=\max_a[r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}V^*(s')]$$

- Try a', get r(s,a'), compute $Q^*(s,a')=r(s,a')+\gamma V^*(s')$
- Try a'', get r(s,a''), compute $Q^*(s,a'')=r(s,a'')+\gamma V^*(s'')$



Assume we know V^* at s' and s''

$$V^*(s)=\max_{a',a''}\{Q^*(s,a'),Q^*(s,a'')\}$$

Bellman Optimality (Theorem 1)

$$V^*(s)=\max_a[r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}V^*(s')]$$

given $\hat{\pi}=\mathrm{argmax}_a Q^*(s,a)$, we can show $V^{\hat{\pi}}=V^*$

This implies $\pi^*=\mathrm{argmax}_a Q^*(s,a)$ is an optimal policy

Bellman Optimality (Theorem 2)

For any V, if $V(s)=\max_a[r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}V(s')]$ for all s, then $V(s)=V^*(s)$

This means we can focus on one step at each time (leaving the remaining “problem” to V(s’), and any V that satisfies this formula is in fact V^*

End - Recap

Policy Evaluation

Question: given

- an MDP (S, A, T, R, γ)
- a policy π

how can we compute the goodness of π, i.e. V^π?

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WHY?

Policy Evaluation

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- an MDP (S, A, T, R, γ)
- a policy π

how can we compute the goodness of π, i.e. V^{π}?

WHY?

There are A^S possible policies, and we want to find the optimal one! To find it, we need to be able to evaluate it

Exact Policy Evaluation

Given (S, A, T, R, γ) and π, what is V^{π}?

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We know that **for ALL states**, Bellman equation holds

$$V^{\pi}(s) = r+\gamma\mathbb{E}_{s'\sim p(.|s,\pi(s))}[V^{\pi}(s')]$$

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How many linear constraints (equations) do we have?

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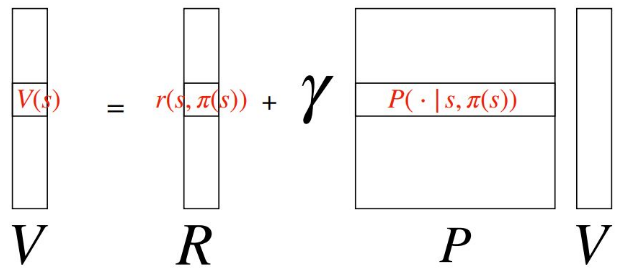
Exact Policy Evaluation

We know that **for ALL states**, Bellman equation holds

$$V^{\pi}(s) = r+\gamma\mathbb{E}_{s'\sim p(.|s,\pi(s))}[V^{\pi}(s')]$$

We can combine all the constraints together:

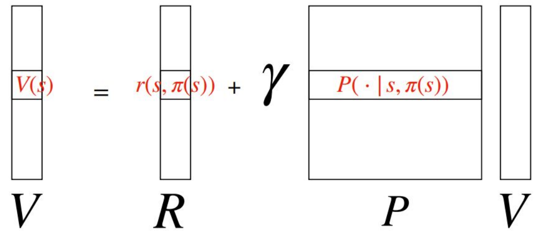


Credits: Wen Sun

Exact Policy Evaluation

Since we have this set of constraints

$$V = R + \gamma PV$$



we can solve for V as

$$V = (I - \gamma P)^{-1}R$$

Exact Policy Evaluation

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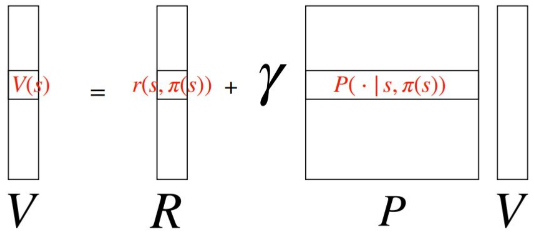

we can solve for V as

$$V = (I - \gamma P)^{-1} R$$

:(Nice but computationally expensive: inverting the matrix is $O(S^3)$

Fixed-Point Iteration

What is a fixed-point? A point where holds

$$x = f(x)$$

How can we find such points?

- Initialize x_0
- Repeat $x_{i+1} = f(x_i)$
- Stop at convergence where x is found and does not change anymore

Contractions

Convergence to a fixed-point is possible thanks to the existence of **contraction mappings**

f: M->M (M is a metric space) is a contraction mapping if:

$$|f(x) - f(x')| \leq k|x-x'| \text{ for } k \text{ in } [0, 1)$$

Contraction Operator

In the simplest case the contraction mapping can be an operator as simple as a matrix, e.g. O:

$$|OV - OV'| \leq \gamma|V - V'|$$

(we can replace k with γ as they have the same range)

Iterative Policy Evaluation

- Initialize V_0 in [0, 1/(1-γ)] (typically 0)
- Until convergence:

$$V_{i+1} = R + \gamma P V_i$$

(note: this is using matricial form because it's doing it for all states)

Iterative Policy Evaluation

- Initialize V_0 in [0, 1/(1-γ)] (typically 0)
- Until convergence:

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For each iteration it's $O(S^2)$

$$V^{t+1}(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^t(s')$$

Iterative Policy Evaluation Theorem

$$V^{\pi}(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^{\pi}(s')$$

At the end we have, for all s in S

$$||V^t(s) - V^{\pi}(s)|| \leq \gamma^t ||V^0 - V^{\pi}||$$

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$$\begin{aligned}
\forall s, & \left| V^{t+1}(s) - V^\pi(s) \right| \\
&= \left| r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,\pi(s))} V^t(s') - \left(r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,\pi(s))} V^\pi(s') \right) \right| \\
&= \gamma \left| \mathbb{E}_{s' \sim P(\cdot|s,\pi(s))} V^t(s') - \mathbb{E}_{s' \sim P(\cdot|s,\pi(s))} V^\pi(s') \right| \\
&\leq \gamma \mathbb{E}_{s' \sim P(\cdot|s,\pi(s))} \left| V^t(s') - V^\pi(s') \right| \\
&\leq \gamma \parallel V^t - V^\pi \parallel_\infty
\end{aligned}$$

Iterative Policy Evaluation Theorem

At the end we have, for all s in S

$$||V^t(s)-V^\pi(s)|| \leq \gamma^t||V^0-V^\pi||$$

$$\| V^{t+1} - V^\pi \|_\infty \leq \gamma \| V^t - V^\pi \|_\infty \leq \gamma^{t+1} \| V^0 - V^\pi \|_\infty$$

Iterative Policy Evaluation: Iterations

For iterative PE to find an ϵ accurate value function, we need a number of iterations n, with computational cost $O(S^2 \ln(1/\epsilon))$:

$$\gamma^n \|V^0 - V^\pi\| \leq \epsilon$$

$$\ln\left(\frac{\| V^0 - V^\star \|_\infty}{\epsilon}\right) \Big/ \ln(1/\gamma)$$

How to Find the Optimal Policy?

Now, what we're really interested in is finding the optimal policy π^*

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Let’s use Bellman optimality! ...and the Bellman Operator

Bellman Backup is a Contraction

- Infinity norm: $||V|| = \max_s |V(s)|$
- Set $\gamma < 1$
- Define the (non-linear) BV operator as a Bellman equation applied to V:

$$BV = \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim p(.|s,a)}[V(s')])$$

Alternative notation TV

Bellman Backup is a Contraction

$$
\begin{aligned}
\|BV_k - BV_j\| &= \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\| \\
&\leq \max_a \left\| \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') - R(s,a) - \gamma \sum_{s' \in S} P(s'|s,a) V_j(s') \right) \right\| \\
&= \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \max_a \left\| \gamma \|V_k - V_j\| \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

Bellman Backup is a Contraction

$$
\begin{aligned}
\|BV_k - BV_j\| &= \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\| \\
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&= \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \max_a \left\| \gamma \|V_k - V_j\| \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

If you apply B to two different value functions, distance between value functions shrinks after applying Bellman equation to each

Bellman Operator for Q

$$TQ(s,a) = r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}\max_{a'}[Q(s',a')])$$

Since Q: S x A -> $\mathbb{R}$, then also TQ: S x A -> $\mathbb{R}$

Value Iteration

All of this also holds for V^*

We can obtain $Q^* = TQ^*$, since Q^* is a fixed-point solution to $Q = TQ$

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- Initialize $||Q_0||$ in $[0, 1/(1-\gamma)]$ (typically 0)
- Until convergence, for all states and actions:

$$Q_{i+1} = TQ_i$$

We know the Bellman operator is a contraction!

Value Iteration

All of this also holds for V^*

We can obtain $Q^* = TQ^*$, since Q^* is a fixed-point solution to $Q = TQ$

- Initialize $||Q_0||$ in $[0, 1/(1-\gamma)]$ (typically 0)
- Until convergence, for all states and actions:

$$Q_{i+1} = TQ_i$$

$$||Q_{i+1} - Q^*|| = ||TQ_i - TQ^*|| \leq \gamma||Q_i - Q^*|| \leq \gamma^{i+1}||Q_0 - Q^*||$$

Policy from Value Iteration

We know that $\pi^*(s) = \text{argmax}_a Q^*(s,a)$, and since $Q_i(s,a) \cong Q^*(s,a)$ we could choose

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

Policy from Value Iteration

$$\pi_i(s) = \mathrm{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^t/(1-\gamma)||Q_0-Q^*||$$

Policy from Value Iteration

$\pi_i(s) = \mathrm{argmax}_a Q_i(s,a)$

What is the quality of such policy? For all states

$V^{\pi_i}(s) \geq V^\star(s) - 2\gamma^t/(1-\gamma)||Q_0-Q^\star||$

$$
\begin{aligned}
V^{\pi^t}(s) - V^\star(s) &= Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^\star(s)) \\
&= Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^t(s)) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s)) \\
&= \gamma \mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^t(s,\pi^t(s)) + Q^t(s,\pi^\star(s)) - Q^\star(s,\pi^\star(s)) \\
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Add and subtract

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\end{aligned}$$

Expand and get rid of r

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...and you're left with this

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&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) + Q^\star(s, \pi^t(s)) - Q^t(s, \pi^t(s)) + Q^t(s, \pi^\star(s)) - Q^\star(s, \pi^\star(s)) \\
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&= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^t(s)) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s, \pi^t(s)) - Q^t(s, \pi^t(s)) + Q^t(s, \pi^\star(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
\end{aligned}
$$

same as before

Policy from Value Iteration

$$\pi_i(s) = \operatorname{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^t/(1-\gamma)\|Q_0 - Q^*\|$$

$$\begin{aligned}
V^{\pi^t}(s) - V^\star(s) &= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^t(s)) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s, \pi^t(s)) - Q^t(s, \pi^t(s)) + Q^t(s, \pi^\star(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
\end{aligned}$$

just exploit this

$$\|Q_i - Q^*\| \leq \gamma^i \|Q_0 - Q^*\|$$

Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^t/(1-\gamma)||Q_0 - Q^*||$$

$$\begin{aligned}
V^{\pi^t}(s) - V^\star(s) &= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^t(s)) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}&\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s, \pi^t(s)) - Q^t(s, \pi^t(s)) + Q^t(s, \pi^\star(s)) - Q^\star(s, \pi^\star(s)) \\
\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))}&\left(V^{\pi^t}(s') - V^\star(s')\right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
\end{aligned}$$

and again

$$||Q_i - Q^*|| \leq \gamma^i ||Q_0 - Q^*||$$

Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^t/(1-\gamma)||Q_0 - Q^*||$$

$$\begin{aligned}
V^{\pi^t}(s) - V^\star(s) &= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^t(s)) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) + Q^\star(s, \pi^t(s)) - Q^t(s, \pi^t(s)) + Q^t(s, \pi^\star(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
\end{aligned}$$

repeat and get 1/(1-γ)

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Policy from Value Iteration

If we want an ϵ error on the quality of the policy, to determine the number of iterations i we can just solve for it in this equation

$$2\gamma^{i}/(1-\gamma)||Q_0-Q^*|| \leq \epsilon$$