



Robotics 1

Direct kinematics

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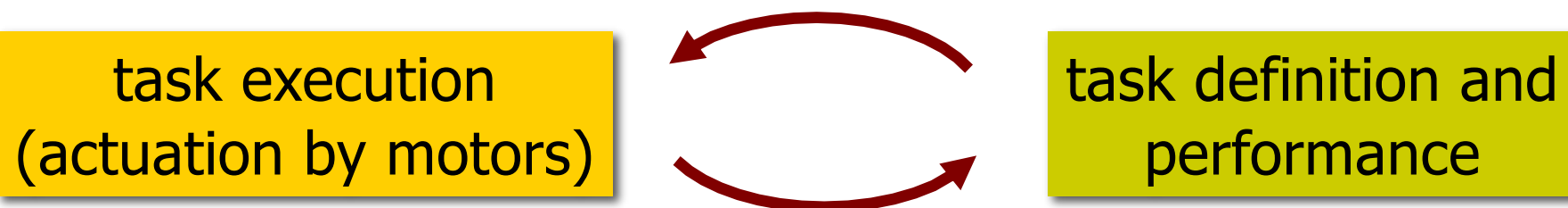
Kinematics of robot manipulators

- study of ...
geometric and timing aspects of **robot motion**,
without reference to the causes producing it
- robot seen as ...
an (open) **kinematic chain** of rigid bodies
interconnected by (revolute or prismatic) joints



Motivations

- functional aspects
 - definition of robot workspace
 - calibration
- operational aspects

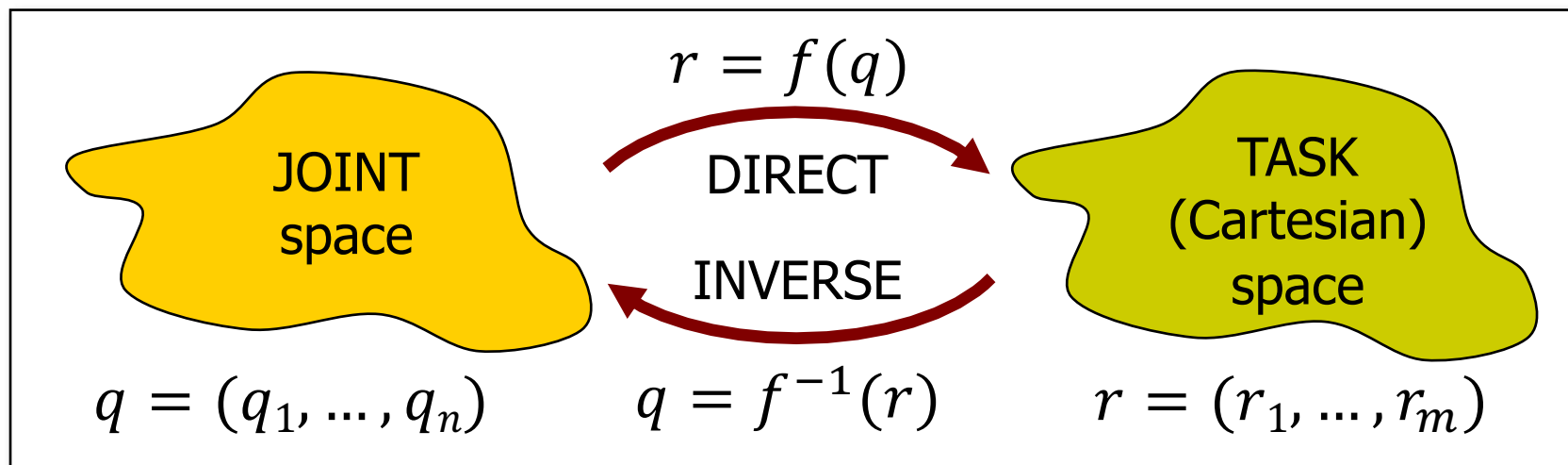


two **different** "spaces" related by kinematic (and dynamic) maps

- trajectory planning
- programming
- motion control

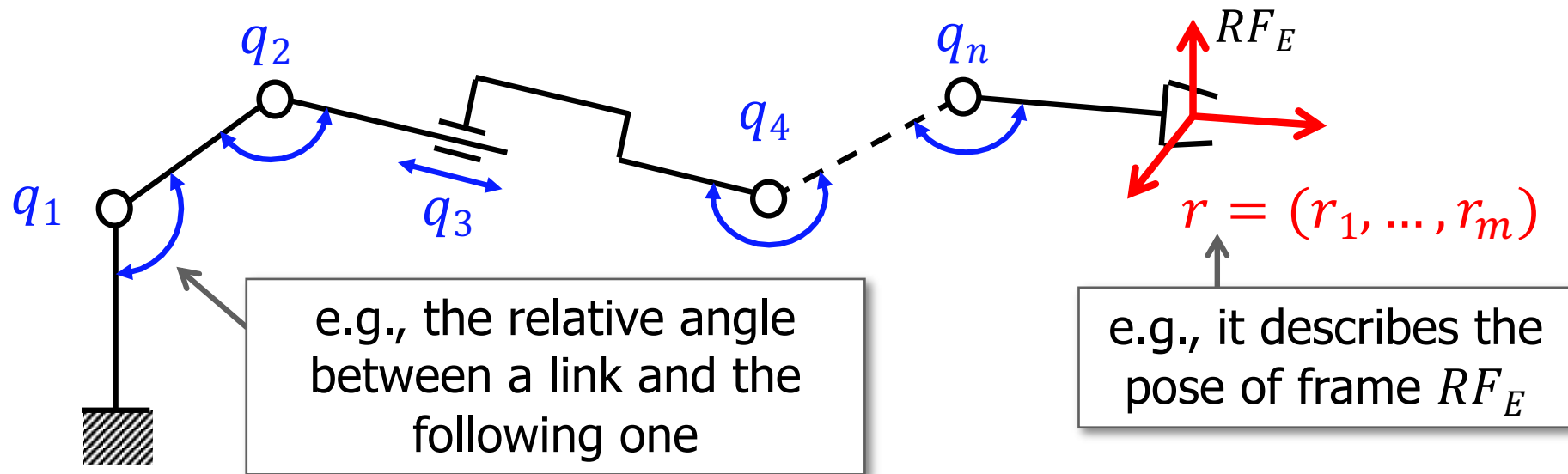
Kinematics

formulation and parameterizations



- choice of parameterization q
 - **unambiguous** and **minimal** characterization of robot configuration
 - $n = \#$ degrees of freedom (dof) = $\#$ robot joints (rotational or translational)
- choice of parameterization r
 - compact description of position and/or orientation (**pose**) variables of interest to the required task
 - usually, $m \leq n$ and $m = 6$ (but none of these is strictly necessary)

Open kinematic chains

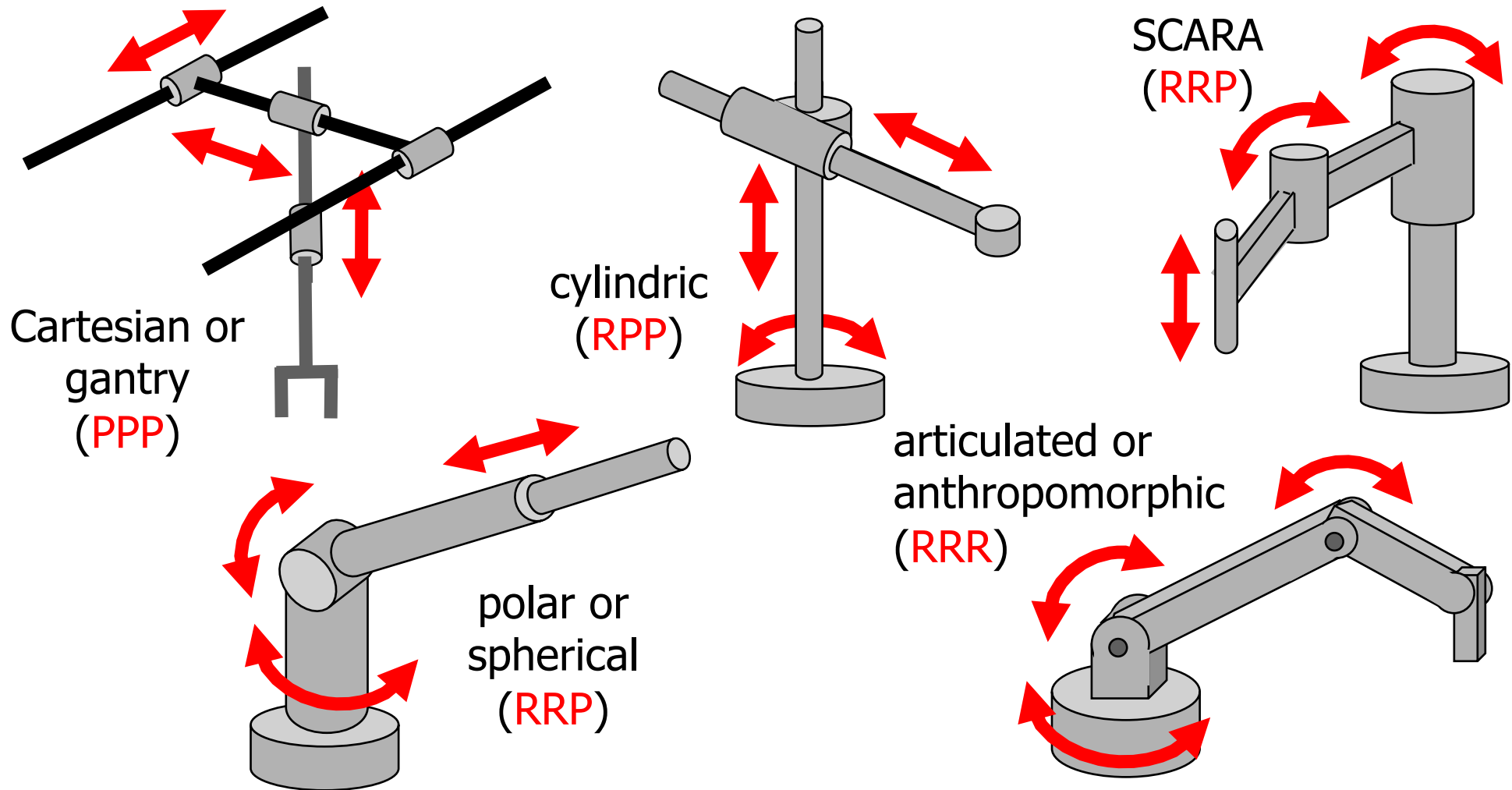


- $m = 2$
 - pointing in space
 - positioning in the plane
- $m = 3$
 - orientation in space
 - positioning and orientation in the plane
- $m = 5$
 - positioning and pointing in space (like for spot welding)
- $m = 6$
 - positioning and orientation in space
 - positioning of two points in space (e.g., end-effector and elbow)



Classification by kinematic type

first 3 dofs only



R = 1-dof rotational (revolute) joint

P = 1-dof translational (prismatic) joint



Direct kinematic map

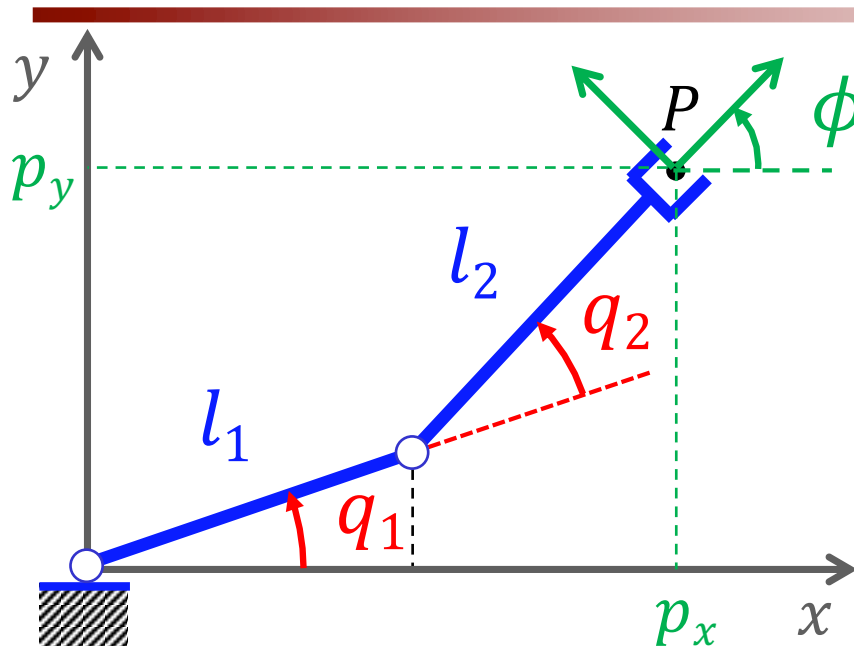
- the structure of the **direct kinematics** function depends on the chosen r

$$r = f_r(q)$$

- methods for computing $f_r(q)$
 - geometric/**by inspection**
 - **systematic**: assigning **frames attached to the robot links** and using homogeneous transformation matrices

Direct kinematics of 2R planar robot

just using inspection...



$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad n = 2$$

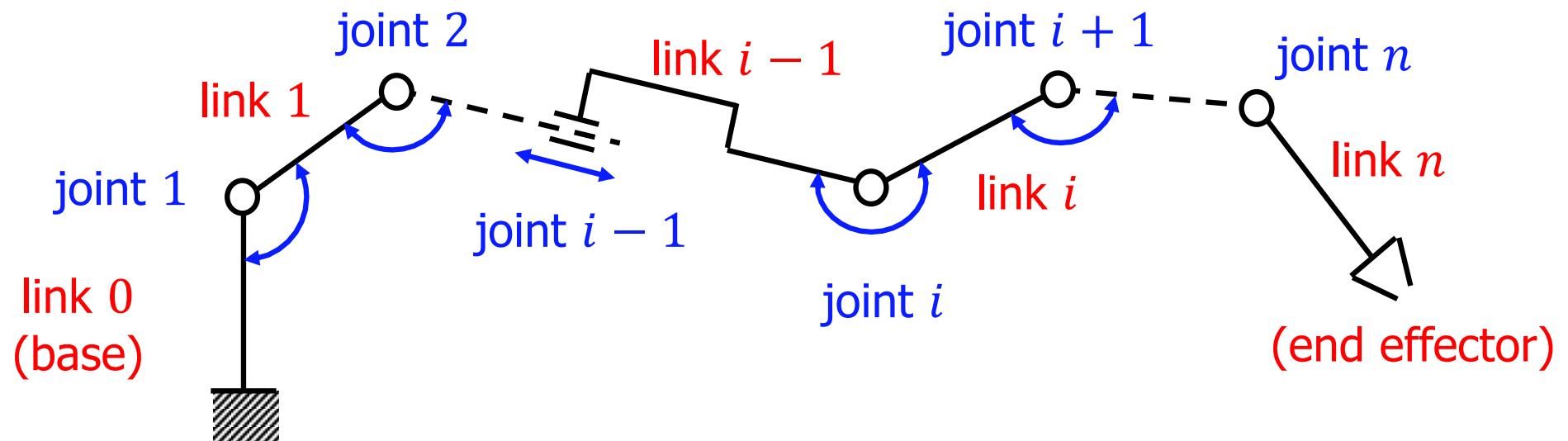
$$r = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} \quad m = 3$$

$$\begin{aligned} p_x &= l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ p_y &= l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \\ \phi &= q_1 + q_2 \end{aligned}$$

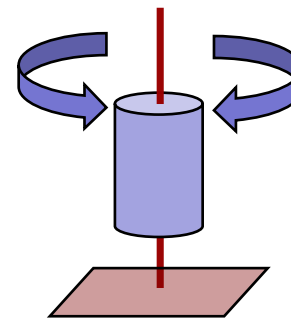
for more general cases, we need a 'method'!



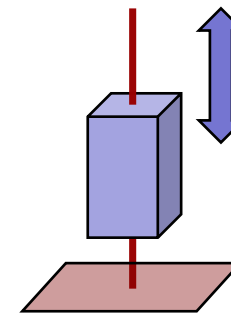
Numbering links and joints



icon representation of joint types
for the manipulator skeleton



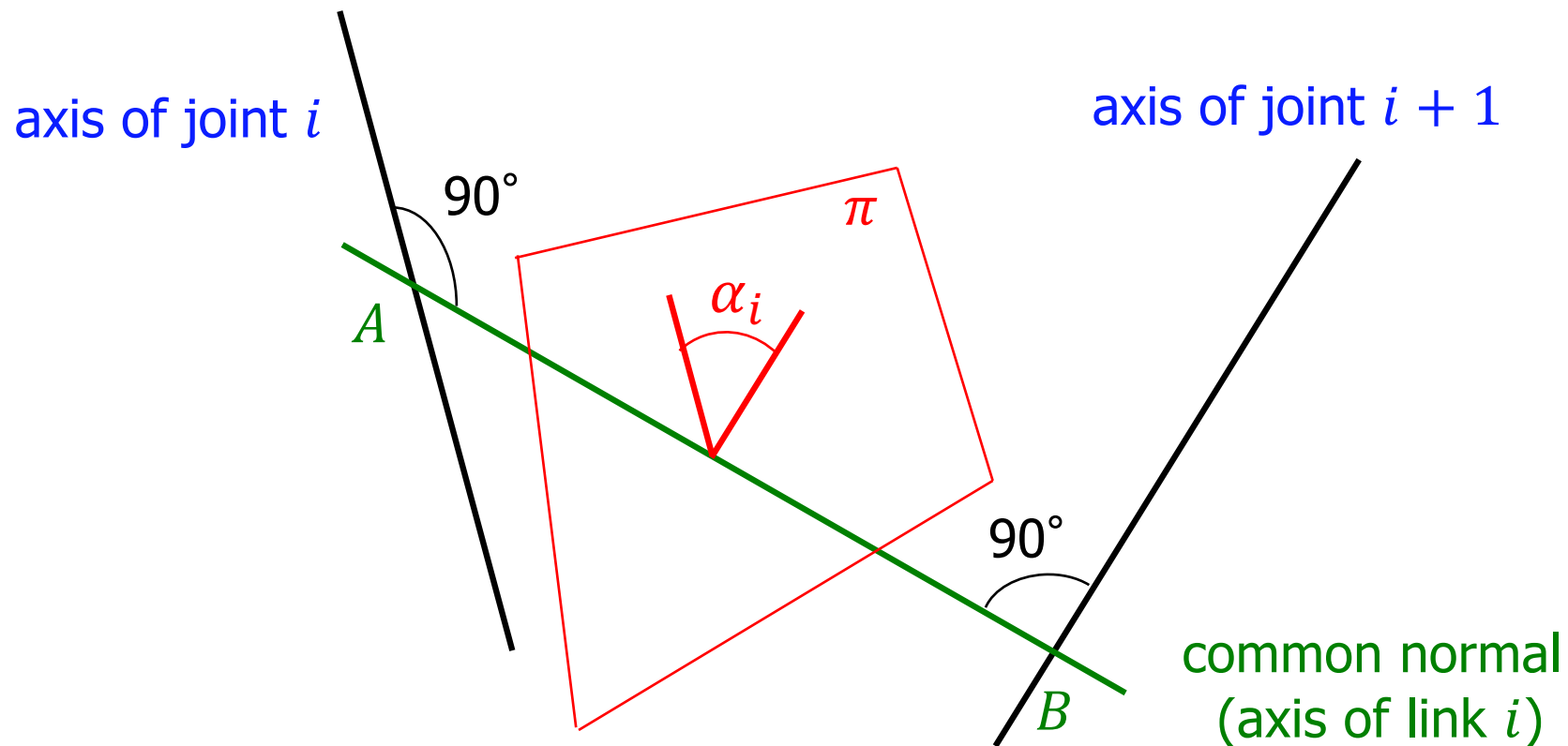
revolute



prismatic

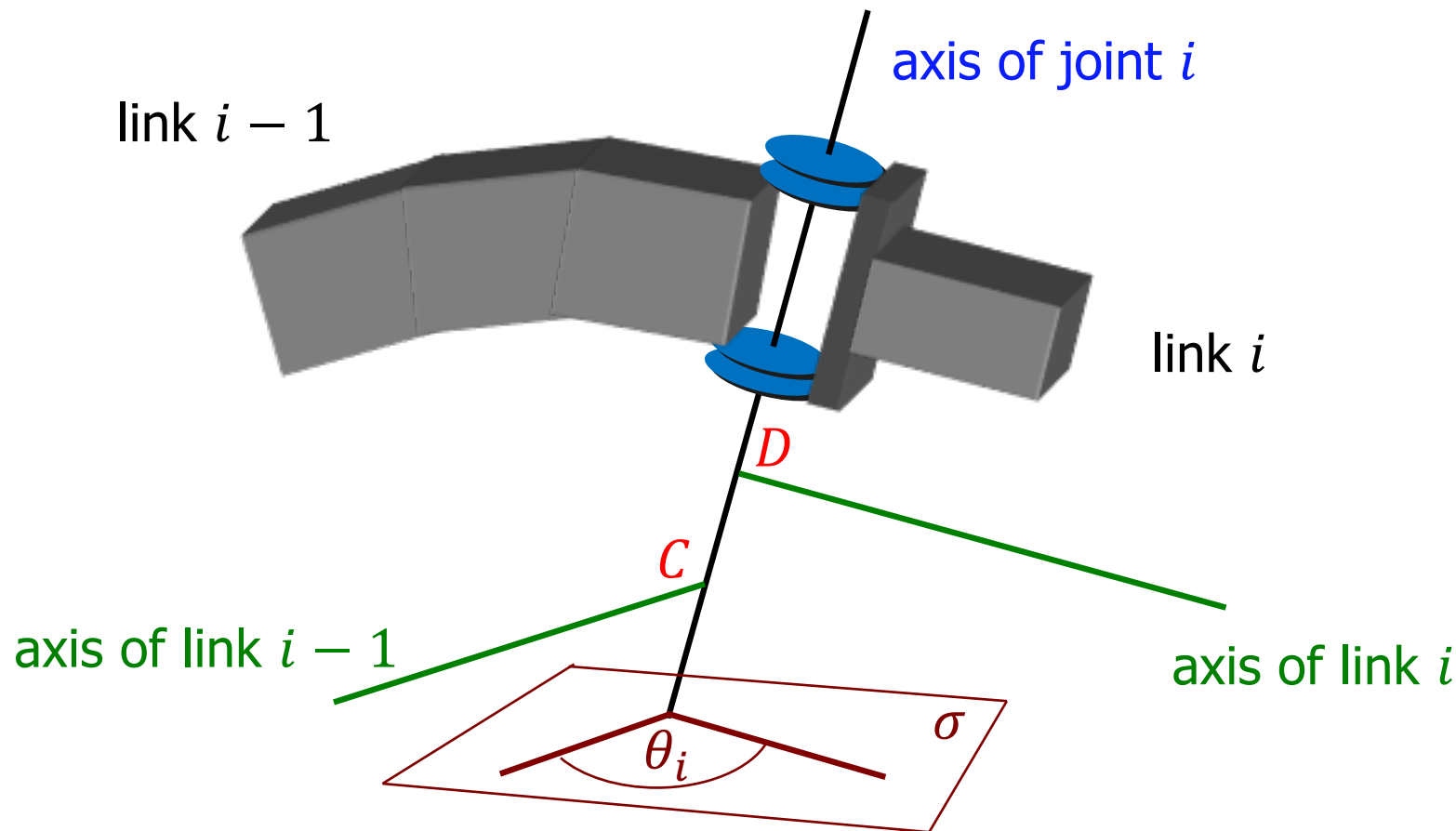


Spatial relation between joint axes



- a_i = **displacement** AB between joint axes (always well defined)
- α_i = **twist angle** between joint axes
- projected on a plane π orthogonal to the link axis
- always constant!
- with sign (pos/neg)!

Spatial relation between link axes



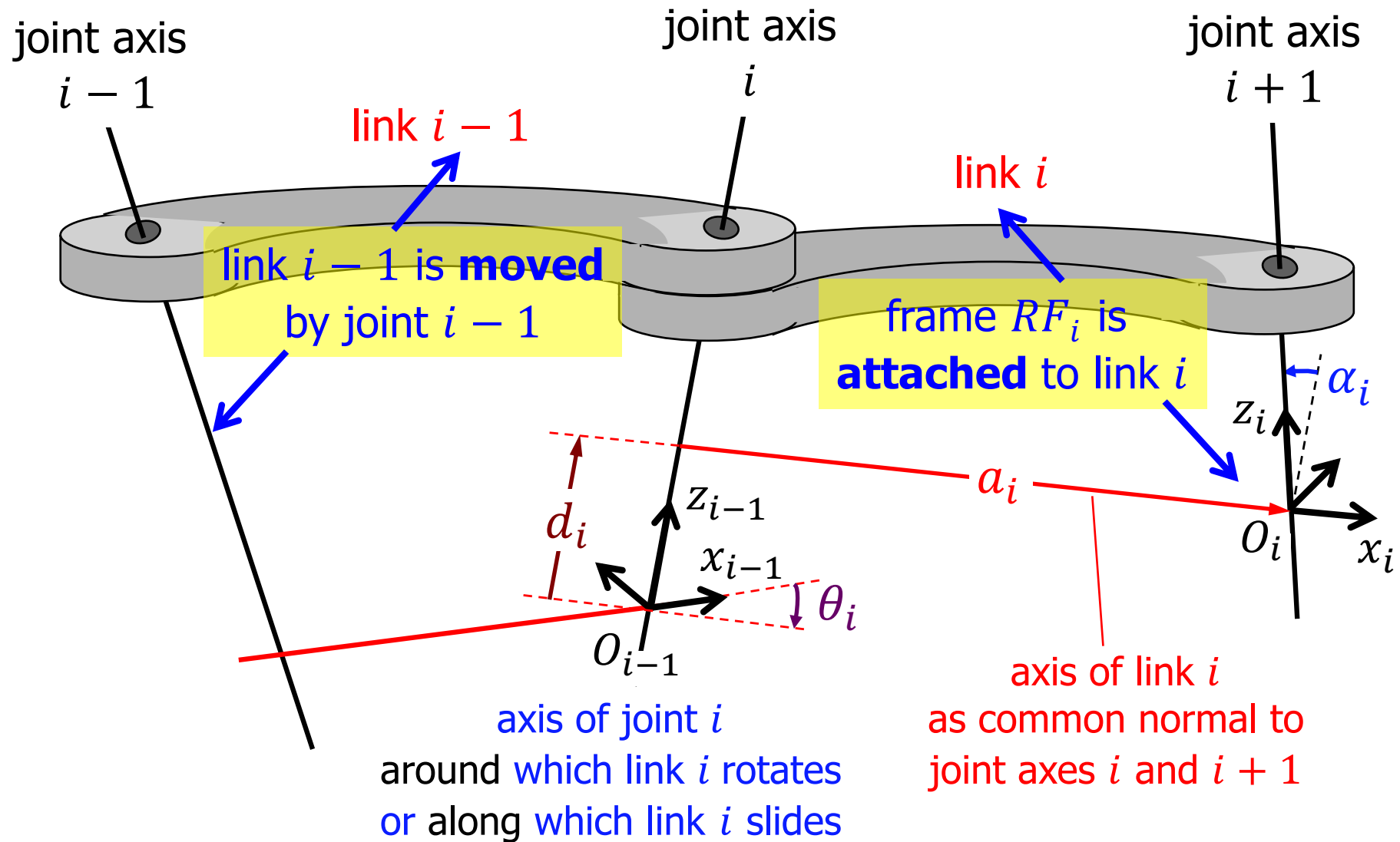
$d_i = \text{displacement } CD$ (a variable if joint i is prismatic)

$\theta_i = \text{angle between link axes}$ (a variable if joint i is revolute)
 — projected on a plane σ orthogonal to the joint axis

with sign
(pos/neg)!

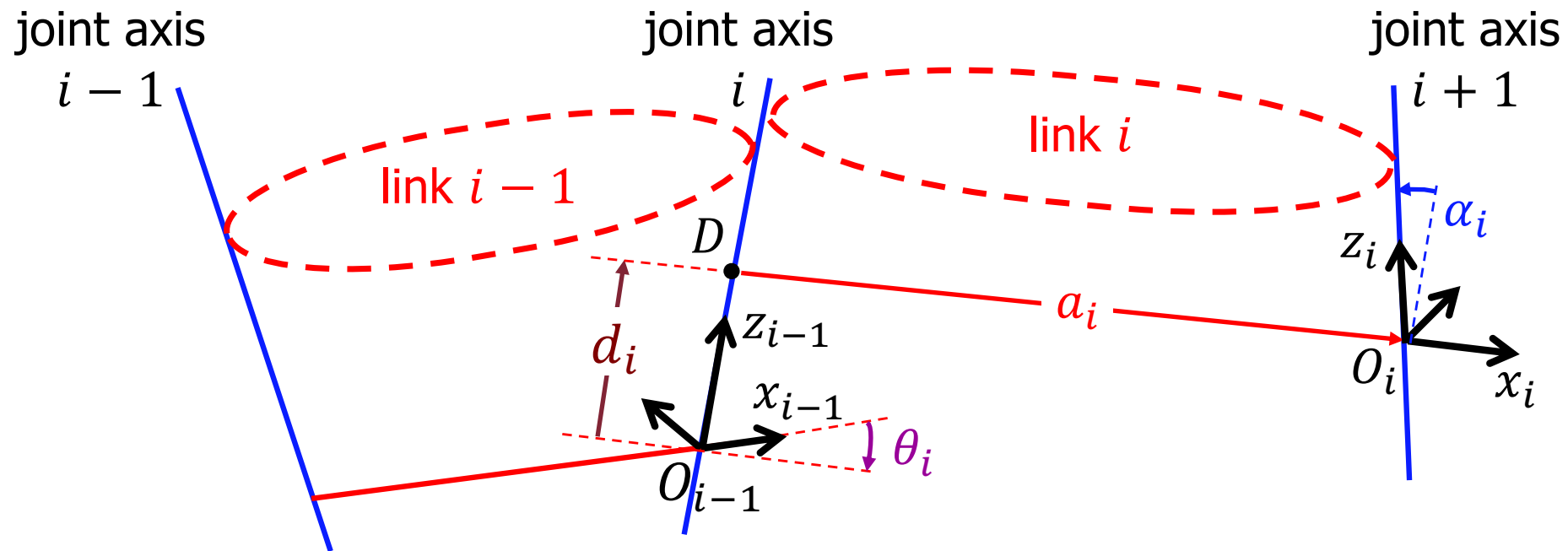


Denavit-Hartenberg (DH) frames





Definition of DH parameters



- unit vector z_i along **axis** of joint $i + 1$
- unit vector x_i along the **common normal** to joint i and $i + 1$ axes ($i \rightarrow i + 1$)
- a_i = distance DO_i , + if oriented as x_i , always constant (= 'length' of link i)
- d_i = distance $O_{i-1}D$, + if oriented as z_{i-1} , **variable** if joint i is **PRISMATIC**
- α_i = **twist** angle from z_{i-1} to z_i around x_i , + if CCW, always constant
- θ_i = angle from x_{i-1} to x_i around z_{i-1} , + if CCW, **variable** if joint i is **REVOLUTE**



DH layout made simple

a popular 3-minute illustration...



video

<https://www.youtube.com/watch?v=rA9tm0gTln8>

- **note:** the author of this video uses r in place of a , and does not add subscripts!



Homogeneous transformation

between successive DH frames (from frame $i - 1$ to frame i)

- roto-translation (screw motion) around and along z_{i-1}

$${}^{i-1}A_{i'}(q_i) = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the product of these two matrices commutes!

rotational joint $\Rightarrow q_i = \theta_i$

prismatic joint $\Rightarrow q_i = d_i$

- roto-translation (screw motion) around and along x_i

$${}^{i'}A_i = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos \alpha_i & -\sin \alpha_i & 0 \\ 0 & \sin \alpha_i & \cos \alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

← always a constant matrix



Denavit-Hartenberg matrix

J. Denavit and R.S. Hartenberg, "A kinematic notation for lower-pair mechanisms based on matrices," *Trans. ASME J. Applied Mechanics*, **23**: 215–221, 1955

$${}^{i-1}A_i(q_i) = {}^{i-1}A_{i'}(q_i) {}^{i'}A_i = \begin{bmatrix} \cos \theta_i & -\cos \alpha_i \sin \theta_i & \sin \alpha_i \sin \theta_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \alpha_i \cos \theta_i & -\sin \alpha_i \cos \theta_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}$$

compact notation: $c = \cos$, $s = \sin$

super-compact notation (if feasible): $c_i = \cos q_i$, $s_i = \sin q_i$

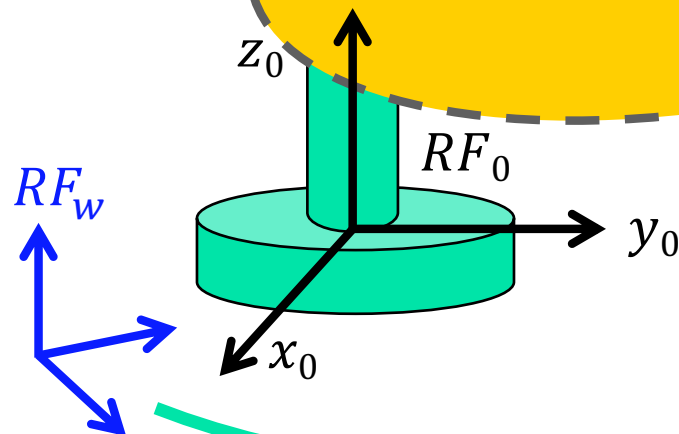
Direct kinematics of robot manipulators



description 'internal'
to the robot using

- $q = (q_1, \dots, q_n)$
- product of DH matrices

$${}^0A_1(q_1) {}^1A_2(q_2) \dots {}^{n-1}A_n(q_n)$$



description 'external'
to the robot using

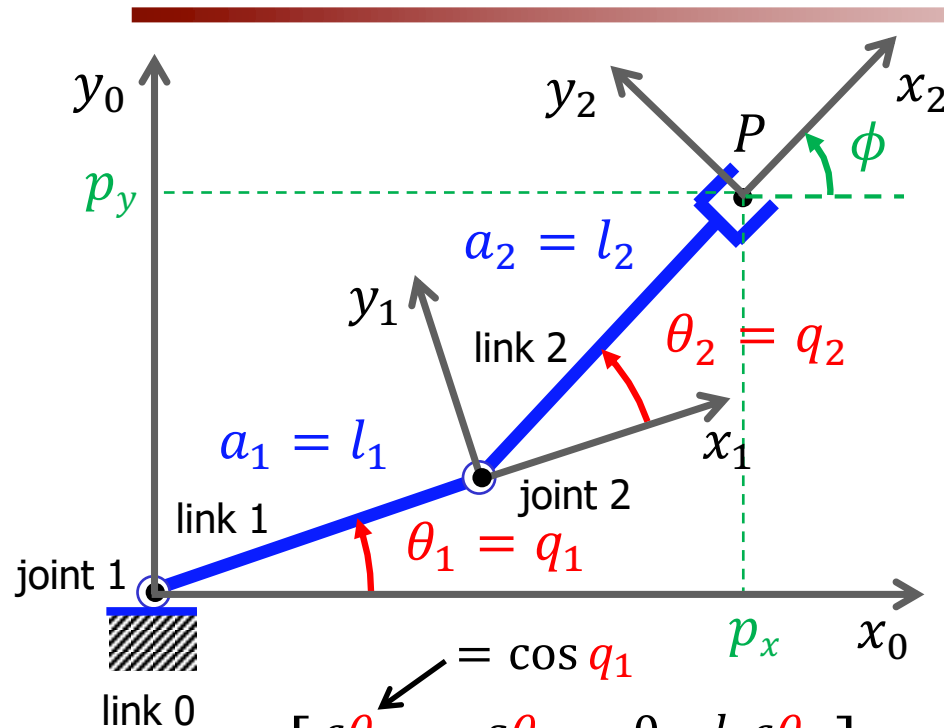
$$\begin{aligned} \bullet \quad {}^wT_E &= \begin{bmatrix} {}^wR_E & {}^wp_{wE} \\ 0^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} n & s & a & p \\ & 0^T & & 1 \end{bmatrix} \end{aligned}$$

$$\bullet \quad r = (r_1, \dots, r_m)$$

$$\begin{aligned} {}^wT_E &= {}^wT_0 {}^0A_1(q_1) {}^1A_2(q_2) \dots {}^{n-1}A_n(q_n) {}^nT_E \\ r &= f_r(q) \end{aligned}$$

alternative representations of the **direct kinematics**

Direct kinematics of 2R planar robot using DH frame assignment...



z_0, z_1, z_2 outgoing from plane

i	α_i	a_i	d_i	θ_i
1	0	l_1	0	q_1
2	0	l_2	0	q_2

$$= \cos(q_1 + q_2)$$

$${}^0A_2(q) = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0A_1(\theta_1) = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & l_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & l_1s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1A_2(\theta_2) = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & l_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

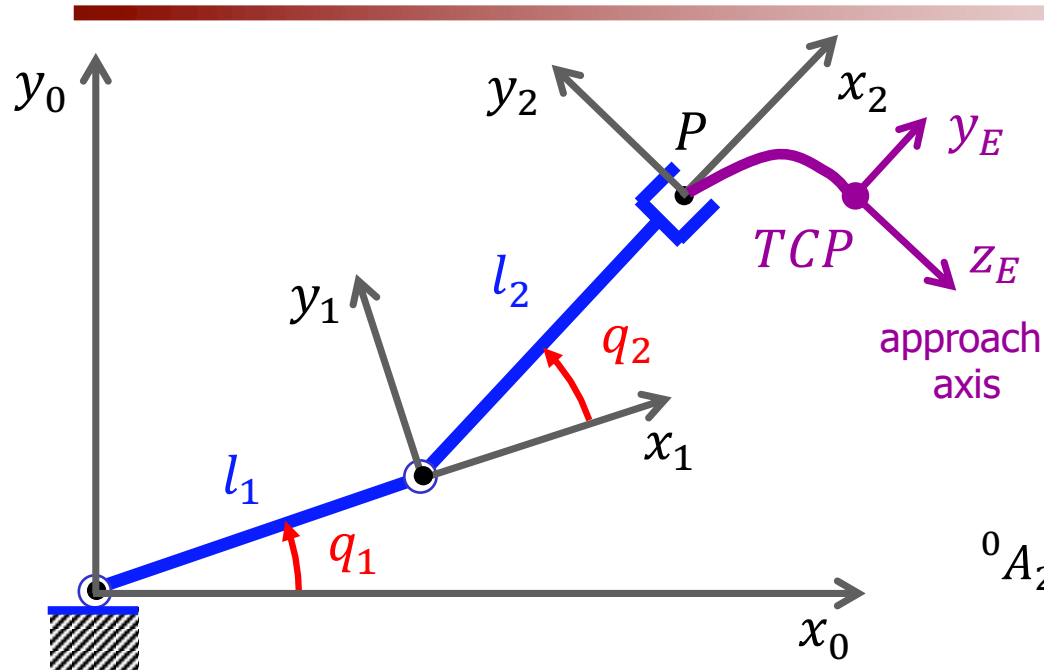
$${}^0p_{\text{hom}} \rightarrow \begin{bmatrix} p_x \\ p_y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 0 \\ 1 \end{bmatrix} = {}^0A_2(q) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \\ 0 \\ 1 \end{bmatrix}$$

$$\phi = q_1 + q_2 \quad (\text{extracted from } {}^0R_2(q))$$



Direct kinematics of 2R planar robot

TCP location on the robot end effector



i	α_i	a_i	d_i	θ_i
1	0	l_1	0	q_1
2	0	l_2	0	q_2

$${}^0A_2(q) = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Tool Center Point TCP and associated end-effector frame RF_E

$${}^2T_E = \begin{bmatrix} 0 & 1 & 0 & {}^2TCP_x \\ 0 & 0 & -1 & {}^2TCP_y \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} {}^0TCP(q) \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0TCP_x(q) \\ {}^0TCP_y(q) \\ 0 \\ 1 \end{bmatrix} = {}^0A_2(q) \begin{bmatrix} {}^2TCP_x \\ {}^2TCP_y \\ 0 \\ 1 \end{bmatrix} = {}^0T_E(q) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = {}^0A_2(q) {}^2T_E$$

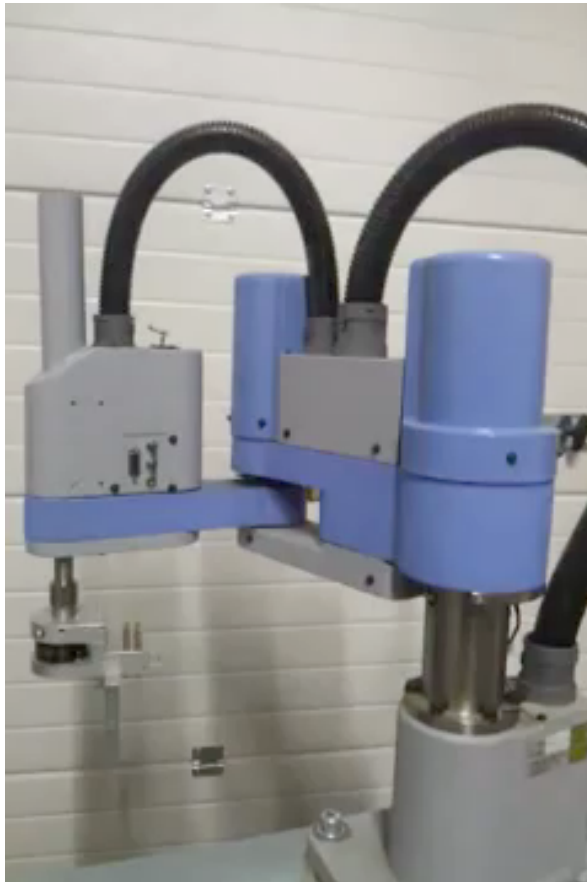


Ambiguities in defining DH frames

- frame 0: origin and x_0 axis are arbitrary
- frame n : z_n axis is not specified
 - however, x_n **must** intersect and be chosen orthogonal to z_{n-1}
- positive direction of z_{i-1} (up/down on axis of joint i) is arbitrary
 - choose one, and try to 'avoid flipping over' to the next one
- positive direction of x_i (back/forth on axis of link i) is arbitrary
 - if successive joint axes are incident, we often take $x_i = z_{i-1} \times z_i$
 - when natural, follow the direction 'from base to tip'
- if z_{i-1} and z_i are parallel (common normal not uniquely defined)
 - O_i is chosen arbitrarily along z_i , still trying to 'zero out' parameters
- if z_{i-1} and z_i are coincident, normal x_i axis can be chosen at will
 - this case occurs **only** if the two joints are of different kind (P/R or R/P)
 - again, try using 'simple values' (e.g., 0 or $\pm\pi/2$) for constant angles

DH assignment for a SCARA robot

video



Sankyo SCARA 8438



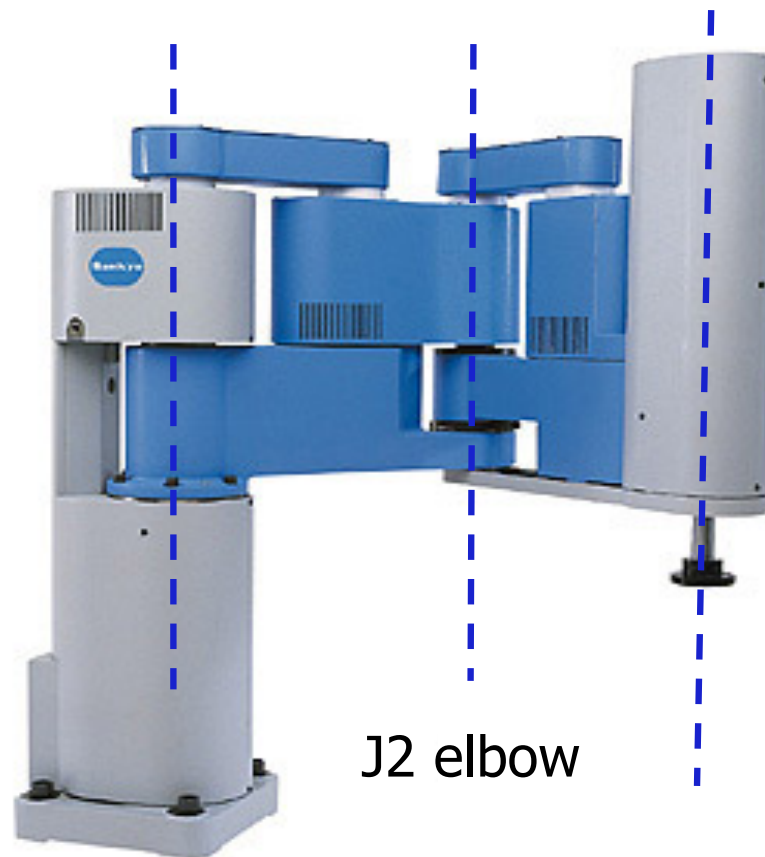
Sankyo SCARA SR 8447

Step 1: joint axes

all parallel
(or coincident)



twist angles
 $\alpha_i = 0$ or π



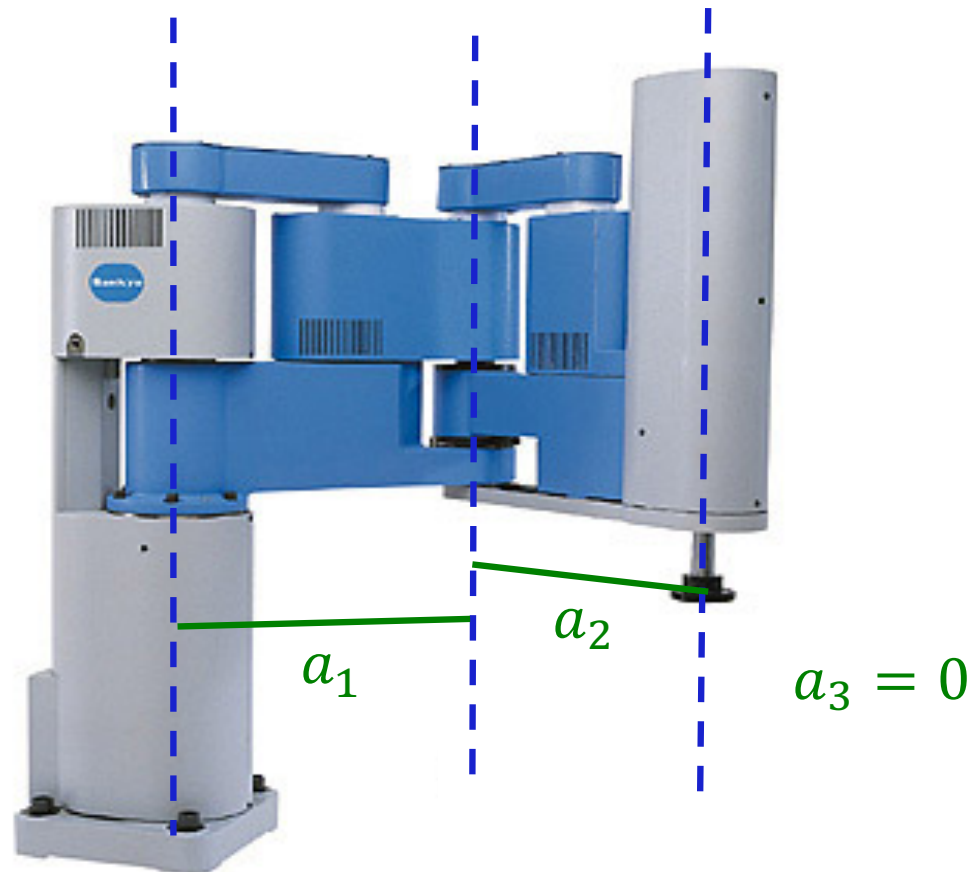
J2 elbow

J3 prismatic
 \equiv
J4 revolute

J1 shoulder

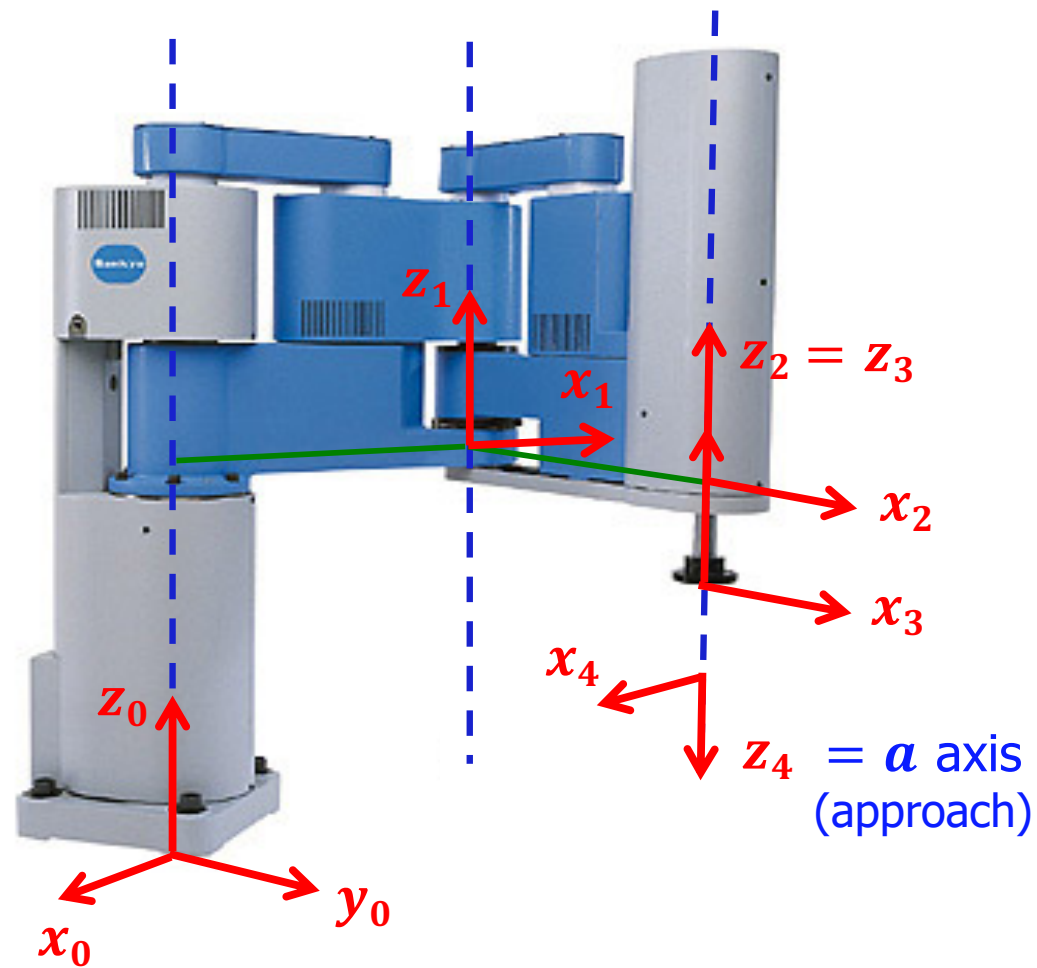
Step 2: link axes

the vertical 'heights'
of the **link axes**
are arbitrary
(for the time being)

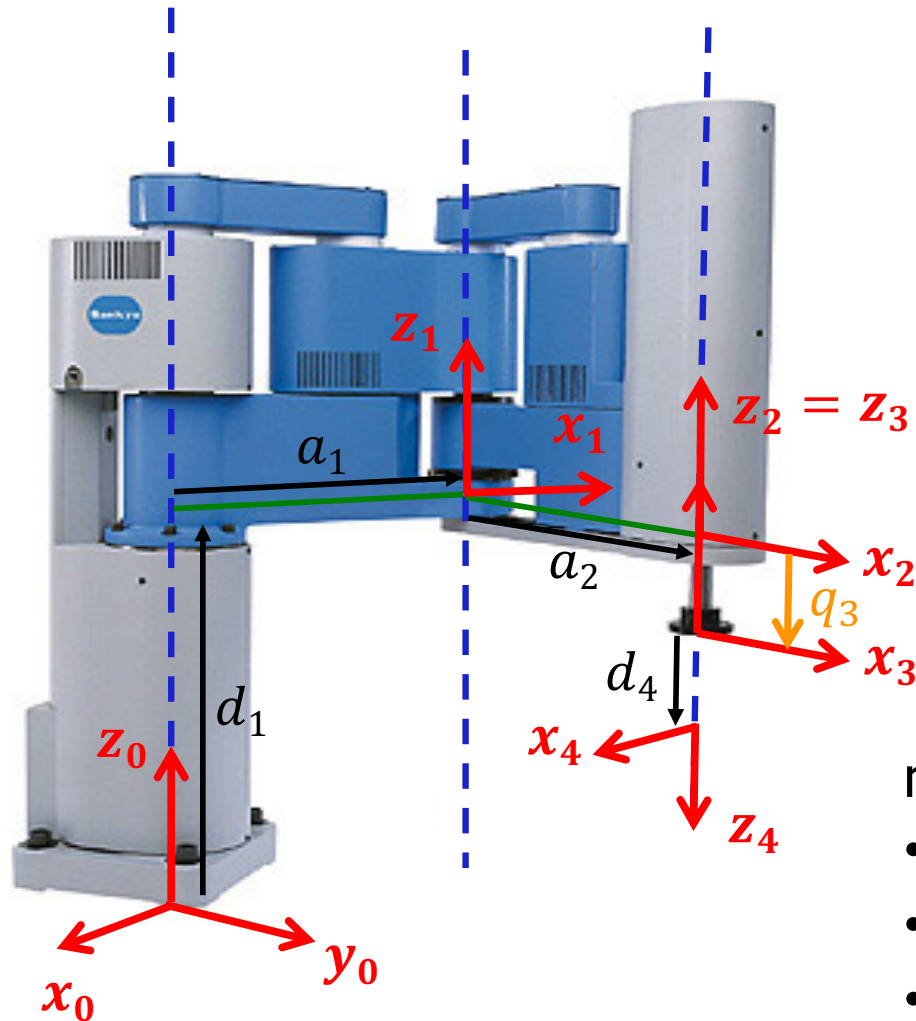


Step 3: frames

axes y_i for $i > 0$
are not shown
(nor needed; they form
right-handed frames)



Step 4: DH table of parameters



i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	q_1
2	0	a_2	0	q_2
3	0	0	q_3	0
4	π	0	d_4	q_4

note that

- d_1 and d_4 could be set = 0
- $d_4 < 0$ (opposite to z_3)
- $q_3 < 0$ in this configuration
- similarly, here $q_1 > 0$, $q_2 < 0$, $q_4 < 0$



Step 5: DH transformation matrices

$${}^0A_1(q_1) = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1s\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1A_2(q_2) = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2A_3(q_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} q &= (q_1, q_2, q_3, q_4) \\ &= (\theta_1, \theta_2, d_3, \theta_4) \end{aligned}$$

$${}^3A_4(q_4) = \begin{bmatrix} c\theta_4 & s\theta_4 & 0 & 0 \\ s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & -1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Step 6a: direct kinematics

homogeneous matrix wT_E as product of the ${}^{i-1}A_i(q_i)$'s

$${}^0A_2(q_1, q_2) = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0A_3(q_1, q_2, q_3) = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & d_1 + q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(q_1, q_2, q_4) = \begin{bmatrix} n & s & a \end{bmatrix} \quad p = p(q_1, q_2, q_3)$$

$${}^wT_E = {}^0A_4(q_1, q_2, q_3, q_4) = \begin{bmatrix} c_{124} & s_{124} & 0 & a_1c_1 + a_2c_{12} \\ s_{124} & -c_{124} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_1 + q_3 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$({}^wT_0 = {}^4T_E = I)$$



Step 6b: direct kinematics

as task vector $r \in \mathbb{R}^m$

$${}^0A_4(q_1, q_2, q_3, q_4) = \begin{bmatrix} \boxed{\begin{matrix} c_{124} & s_{124} & 0 \\ s_{124} & -c_{124} & 0 \\ 0 & 0 & -1 \end{matrix}} & \boxed{\begin{matrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ d_1 + q_3 + d_4 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

extract $\alpha_z \in \mathbb{R}$ from $R(q_1, q_2, q_4)$

take $p \in \mathbb{R}^3$ as such from $p(q_1, q_2, q_3)$

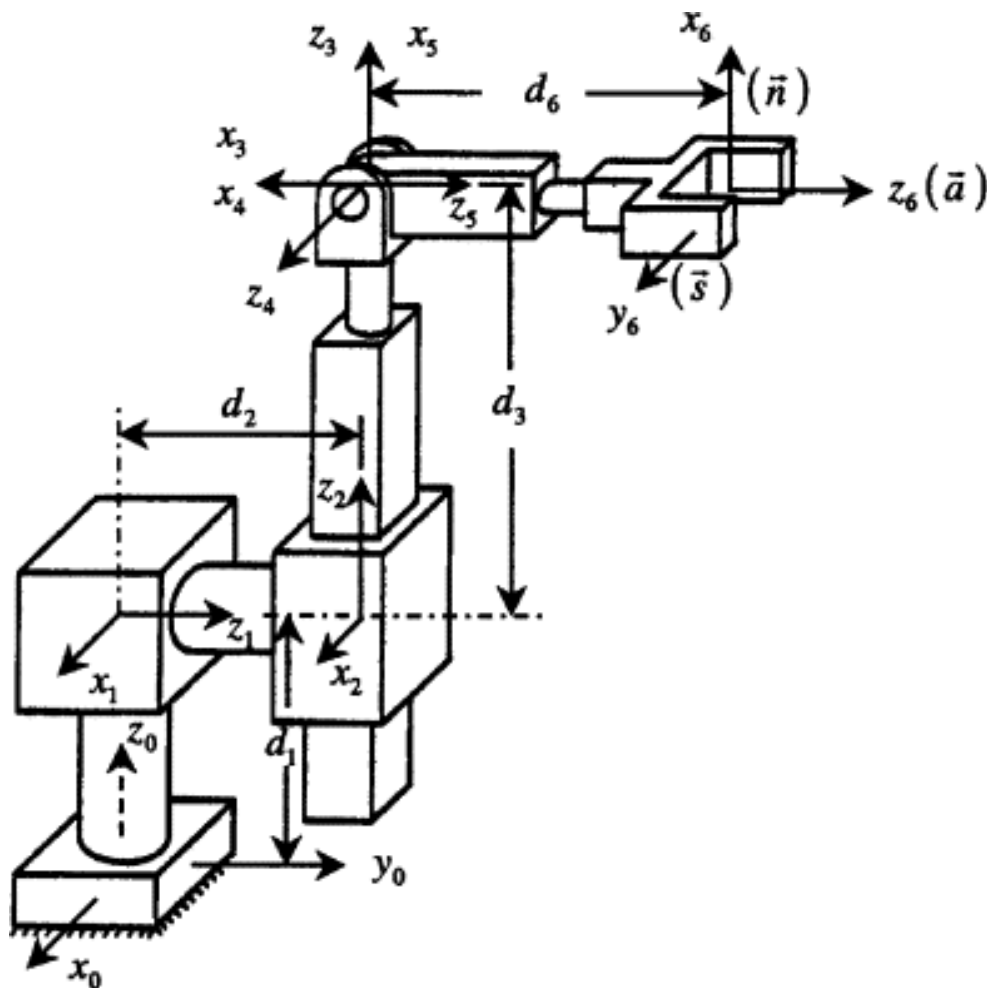
$$r = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha_z \end{bmatrix} = f_r(q) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ d_1 + q_3 + d_4 \\ q_1 + q_2 + q_4 \end{bmatrix} \in \mathbb{R}^4$$

MATLAB code available on web site: [dirkin_SCARA.m](#)



Stanford manipulator

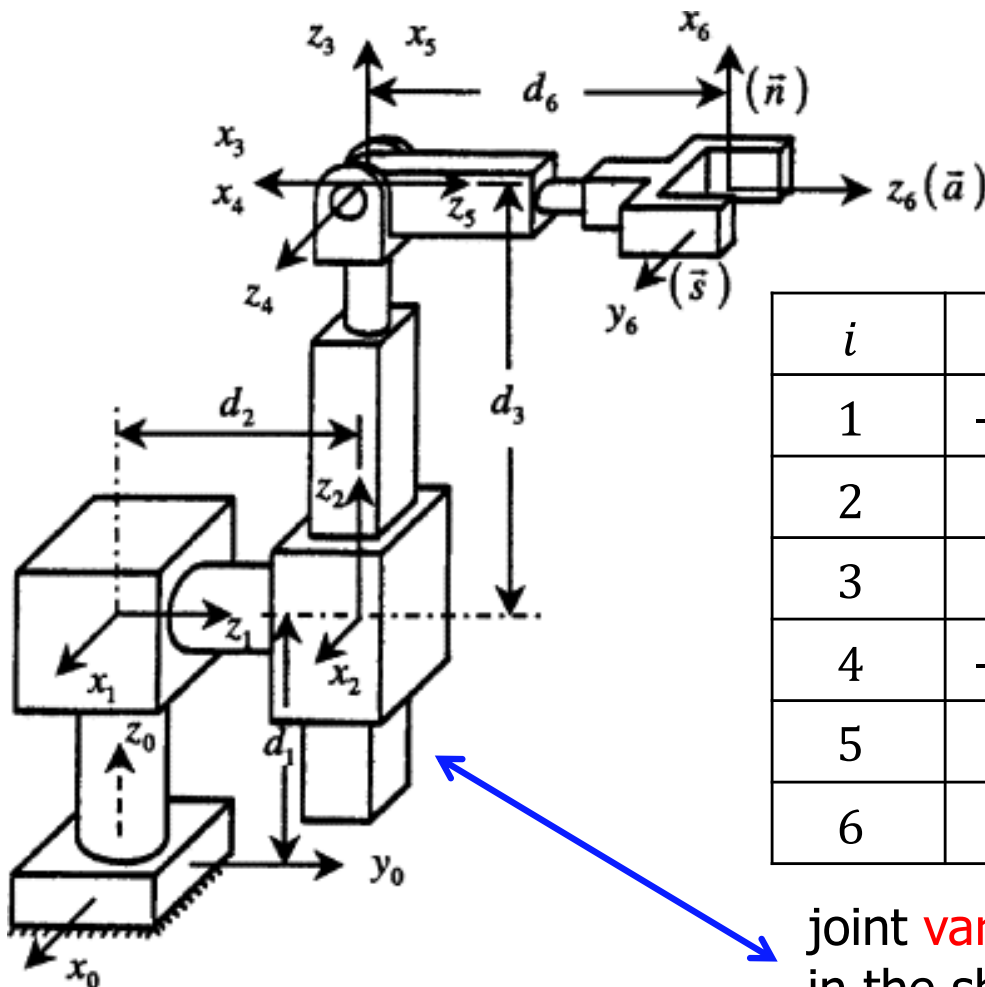
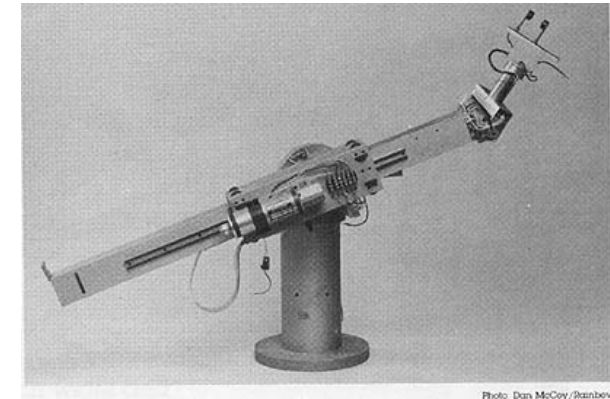
- 6-dof: 2R-1P-3R (spherical wrist)



- robot with **shoulder** offset
- 'one possible' DH assignment of frames is shown
- determine the associated
 - **table of DH parameters**
 - homogeneous transformation matrices
 - direct kinematics
- write a program for computing the direct kinematics
 - **numerically** (Matlab), given a q
 - **symbolically** (Mathematica, Maple, Symbolic Manipulation Toolbox of Matlab, ...)

DH table for Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)



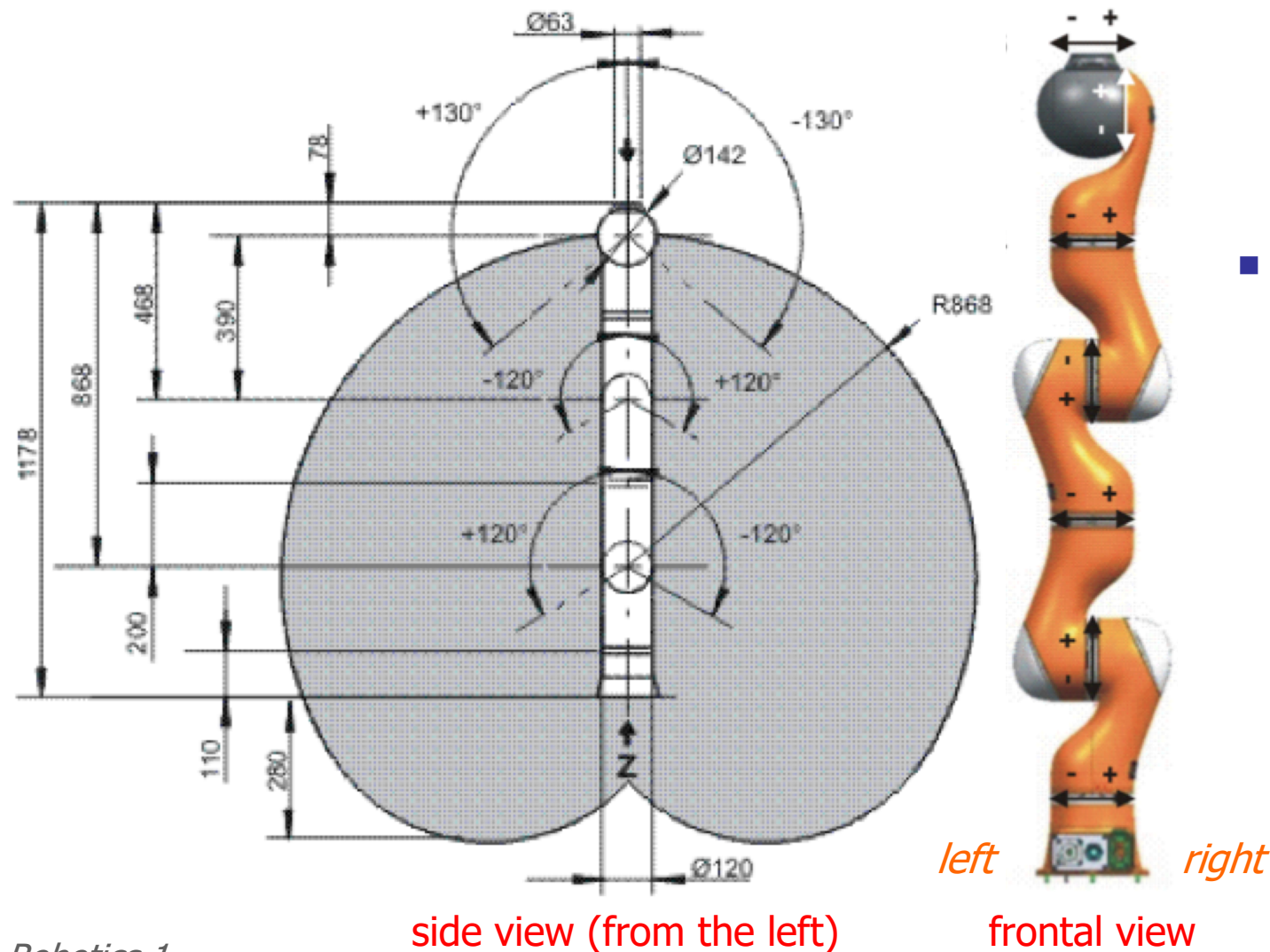
i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 > 0$	$q_1 = 0$
2	$\pi/2$	0	$d_2 > 0$	$q_2 = 0$
3	0	0	$q_3 > 0$	$-\pi/2$
4	$-\pi/2$	0	0	$q_4 = 0$
5	$\pi/2$	0	0	$q_5 = -\pi/2$
6	0	0	$d_6 > 0$	$q_6 = 0$

joint **variables** are in **red**, while their **values** in the shown robot configuration are in **blue**

KUKA LWR 4+

- 7R (no offsets, spherical shoulder and spherical wrist)

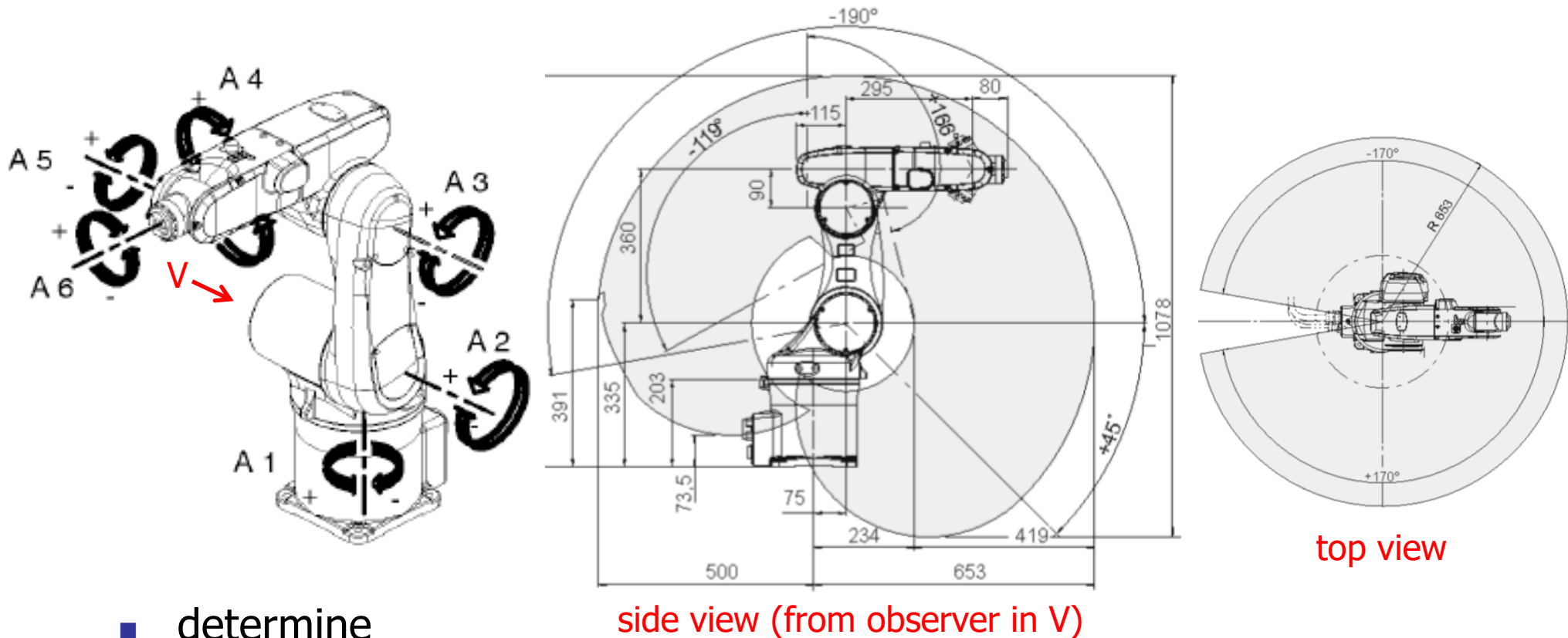
available at
DIAG Robotics Lab



- determine
 - frames and table of DH parameters
 - homogeneous transformation matrices
 - direct kinematics
 - d_1 and d_7 can be set = 0 or not (as needed)

KUKA KR5 Sixx R650

- 6R (offsets at shoulder and elbow, spherical wrist)



- determine
 - frames and table of DH parameters
 - homogeneous transformation matrices
 - direct kinematics (symbolic & numeric)

available at
DIAG Robotics Lab



Appendix:

Modified DH convention

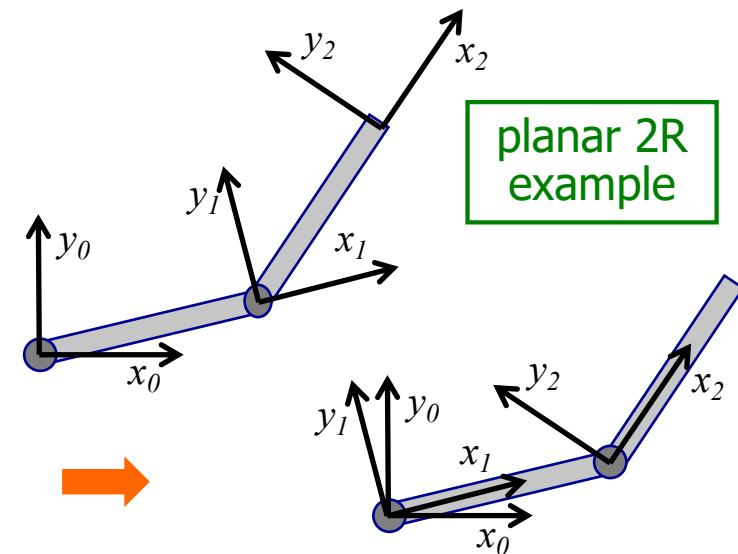
- a **modified** version introduced in J. Craig's book "Introduction to Robotics" (1986) and aligned for the indexing by Khalil and Kleinfinger (ICRA, 1986)
 - has z_i axis on joint i
 - a_i & α_i = distance & twist angle from z_{i-1} to z_i , measured along & about x_{i-1}
 - d_i & θ_i = distance & angle from x_{i-1} to x_i , measured along & about z_i
 - **source of much confusion**... if you are not aware of it (or don't mention it!)
 - convenient with link flexibility: a rigid frame at the base, another at the tip...

classical
(or distal)

$${}^{i-1}A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

modified
(or proximal)

$${}^{i-1}A_i^{\text{mod}} = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_i \\ c\alpha_i s\theta_i & c\alpha_i c\theta_i & -s\alpha_i & -d_i s\alpha_i \\ s\alpha_i s\theta_i & s\alpha_i c\theta_i & c\alpha_i & d_i c\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



modified DH tends to place frames
'at the base' of each link