09b. A One-Sided Approach to the Two-Sided Matrix Inverse

9.1 Left is right.

Below we shall not assume that a matrix is square unless specifically so articulated.

Definition. Let M be a matrix. We say that M is *surjective* if the linear system

$$M\vec{w} = \vec{z}$$

has a solution for every right hand side vector $\vec{z}$. We say that M is *injective* when the homogeneous linear system $M\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

Theorem 1. *Let B be a surjective matrix. Assume that for some matrix A we have $AB = Id$, the identity matrix. Then $BA = Id$ as well. That is:*

If a surjective matrix has a left inverse then that left inverse is also a right inverse.

Proof. We want to understand the vector BA. Let $\vec{x}$ be a vector for which $BA\vec{x}$ is defined. Since B is surjective, there is a vector $\vec{y}$ so that $B\vec{y} = \vec{x}$. Thus $BA\vec{x} = BA(B\vec{y}) = B(AB)\vec{y} = B\vec{y} = \vec{x}$. Thus BA takes every (apppropriate) vector $\vec{x}$ to $\vec{x}$, so BA is the identity matrix. □

Note that in both statement and proof we don't specify the size and shape of the matrices and vectors involved. In particular, we don't assume that B is square. This follows from the hypotheses. The reader is invited to specify the number of rows and columns involved (say B is $m \times n$), and everything would work out. But the proof as written seems to *find flow*, in the sense of *Mihaly Csikszentmihaly* .[6, 7], we retain it.

9.2 Surjection and Inversion

Theorem 2. *Let B be a surjective matrix. Then there is a matrix C so that $BC = Id$.*

Definition. We will denote by $\vec{e}_j$ the $n \times 1$ column vector with a "1" in entry j and zeros elsewhere. The number of entries of this column, i.e., the height of $\vec{e}_j$, will be clear from the context. Thus $\vec{e}_1 \equiv \begin{pmatrix} 1 & 0 & \cdots 0 \end{pmatrix}^t$.

For example, if $B = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$ we have the linear systems

$$B \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad B \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with solution vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$. This example is continued after the proof.

Proof. For j indexing the rows of B, concatinating (juxtaposing?) these column vectors $\{\vec{c}_j\}$ we obtain the matrix $C \equiv \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_{\text{last}} \end{pmatrix}$. Then $BC = Id$ simply encodes the equations

$$\left\{ B\vec{c}_j = \vec{e}_j \qquad\qquad j = 1, 2, \ldots, \text{last} . \right.$$

□

Back to the example before the proof: we concatinate $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ to obtain the matrix

$$Q \equiv \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix},$$

and the reader can check that $B \cdot Q$ equals the 2×2 identity matrix.

9.3 Surjectivity, Injectivity, and RREF

Theorem 3. *Let M be a matrix. Then*

i) M is surjective if and only if every row of $RREF(M)$ is a pivot row.

ii) M is injective if and only if every column of $RREF(M)$ is a pivot column.

iii) If M is square then M is injective if and only if M is surjective.

Proof. i) If $RREF(M)$ has a non-pivot row, that row must be identically zero. Hence the bottom row of $RREF(M)$ is zero as in the left matrix in (1):

$$\begin{pmatrix} * & * & \cdots * \\ * & * & \cdots * \\ \vdots & \vdots & \vdots \\ * & * & \cdots * \\ 0 & 0 & \cdots 0 \end{pmatrix} \; ; \; \left(\begin{array}{ccc|c} * & * & \cdots * & 0 \\ * & * & \cdots * & 0 \\ \vdots & \vdots & \vdots & 0 \\ * & * & \cdots * & 0 \\ 0 & 0 & \cdots 0 & 1 \end{array}\right) . \tag{1}$$

We can augment $RREF(M)$ with a column vector to represent an inconsistent linear system, as in the right matrix of (1). Reversing row operations that took M to $RREF(M)$ and applying them to this augmented matrix we obtain an inconsistent linear system with coefficient matrix M, so M is not surjective.

A linear system is inconsistent only when its augmented matrix RREF has a zero row in the coefficient portion and a nonzero scalar in the augmentation entry of the same row. So if every row of $RREF(M)$ is a pivot row, M is surjective.

ii) Consider the columns of $RREF(M)$. Each pivot column corresponds to a *dependent variable* [3] for the homogeneous system $M\vec{x} = \vec{0}$. Nonpivot columns corresponds to *free variables.* If $RREF(M)$ has a nonpivot column then $M\vec{x} = \vec{0}$ has multiple solutions, one for each choice of values for the free variables. Thus if M is injective then M has no free variables: every column of $RREF(M)$ is a pivot column.

iii) If M is square then having all rows be pivot rows is equivalent to having all columns be pivot columns. Hence surjectivity implies injectivity and conversely. $\square$

9.4 Two-Sided Inverses

Theorem 4. *Let B be a surjective square matrix. Then B has a two-sided inverse.*

Proof. By Theorem 2, there is a matrix C with $BC = Id$. If $C\vec{x} = \vec{0}$ then $\vec{x} = BC\vec{x} = B\vec{0} = \vec{0}$, so that C is injective. By Theorem 3, part *iii)*, C is surjective. By Theorem 1, CB=Id. $\square$

Corollary. *If A and B are square matrices with $AB = Id$ then $BA = Id$.*

Proof. The matrix B is injective, by an argument we met above. Thus B is surjective, by Theorem 3. By Theorem 1, $BA = Id$. $\square$

We can paraphrase the Corollary as follows:

If a square matrix has a one-sided inverse then that inverse is two-sided.

Of course, two-sided inverses are unique by a familiar argument. If M has a left inverse L and a right inverse R then $LMR = L(MR) = L(Id) = L$ and $LMR = (LM)R = (Id)R = R$.

Concluding Remarks We have been cavalier in defining *injective, surjective* and allied notions, as matrices have both left and right versions of these. So if the reader finds the presentation to be one-sided, who is one to argue? On the other hand, these narrow definitions are workable in the short term and can be expanded later in a course.

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