11. Computing Determinants Using RREF

(and getting the matrix inverse for free)

11.1 Determinants in a Nutshell

We assume here that the reader is familiar with the abstract properties that determine the determinant, namely, that the determinant is alternating, is multilinear, and takes on the value 1 on the Identity matrix of size $n \times n$. Let M be an $n \times n$ matrix. In a nutshell, to compute $\det(M)$, perform row operations to put M in RREF. If the end result has a zero row, M is singular and $\det(M) = 0$. If the end result has no zero row, it must be the identity matrix of size $n \times n$. In this case, M is non-singular, $\det(M) \neq 0$ and, in order to compute $\det(M)$, we will need to review the row reduction steps leading to $RREF(M)$. This deserves a new paragraph.

11.2 Determined to exploit RREF work

We take a number J (for *Journal*, perhaps), and set it equal to 1. Computer scientists may think of this as an *accumulator*. More generally, we will update the value of J as we perform row operations leading to $RREF(M)$. The final value will be precisely $\det(M)$. Starting with $J = 1$, we keep J as is every time we perform a *workhorse operation*, i.e., a row operation that subtracts a multiple of one row from another. If we switch two rows, we replace the value of J by (-1) times the value of J. If we scale a row by the nonzero scalar α, we divide J by α to obtain the updated value of J. These updates of J are directly motivated by the abstract properties of the determinant function. The reader is invited to elaborate on each of the corresponding details. Below we'll illustrate this for a specific 3×3 matrix.

11.3 A 3×3 determinant computation

Let M be the following matrix:

$$M \equiv \begin{pmatrix} 3 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

We set $J = 1$ and perform the row operations $II \to II - I$, obtaining

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

The updated value of J is the same as the previous value, 1. Next we perform

$I \to I - 3III$, obtaining

$$\begin{pmatrix} 0 & -4 & -8 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Next, perform $I \to I - 4II$ and we have

$$\begin{pmatrix} 0 & 0 & -12 \\ 0 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Now write $III \to III + 2II$, obtaining

$$\begin{pmatrix} 0 & 0 & -12 \\ 0 & -1 & 1 \\ 1 & 0 & 5 \end{pmatrix}.$$

Now scale row I by $-1/(12)$ obtaining

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 5 \end{pmatrix}.$$

We have tacitly updated the value of J to be 1 in each of the last few steps. This time we need to divide J by $-1/(12)$, so the updated value of J is -12.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 5 \end{pmatrix}.$$

Now perform $III \to III - 5I$ and then $II \to II - I$, obtaining

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Notice that we have taken the bold and risky and not recommended practice of performing two row operations before writing down results. But these are simple and *uncoupled* (reader: define) operations and the risk is small. Next scale row II by the factor (-1) and then switch row I and III. We obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We need to divide the value of J (-12) by (-1) because of the last scaling operation, and then multiply it by (-1) because of the row switch. We have obtained the identity matrix and the value of J, (-12), is also the value of $\det(M)$.

It should be noted that we were a little foolhardy above. We updated the value of J at each step while not knowing if M is singular or not. Had M been singular, the J updating work would have been "wasted". So some would argue that one should perform row reduction without J updates and only go back and update if the matrix in question turns out to be non-singular. The *just in time* efficiency experts may wish to consider this when computing determinants using $RREF$.

11.4 Appendix: Sagemath

We can perform the above row operations using the free, open source, computer mathematics tool *Sage*, also known as *SageMath*. We employ the following web page to perform our computations:

`https://sagecell.sagemath.org`

The various row operations we performed are presented below. Note that we"thickened" the original matrix, adding a copy of the 3×3 identity matrix, and "carried" it along, applying the same row operations. As a result, when we arrived at the RREF of our original matrix M, the augmented portion gave us the *inverse* of M:

$$K \equiv M^{-1} = \begin{pmatrix} \frac{1}{12} & \frac{1}{3} & -\frac{1}{4} \\ \frac{7}{12} & -\frac{2}{3} & \frac{1}{4} \\ -\frac{5}{12} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}.$$

The reader should verify that both $M \cdot K$ and $K \cdot M$ equal the identity matrix $Id_{3\times 3}$.

To see the working matrix at a particular stage, erase (or comment out with #) all the steps that follow.

```
M=matrix(QQ,[[3,2,1],[3,1,2],[1,2,3]]) ; M
M.add_multiple_of_row(1,0,-1) ; M
M.add_multiple_of_row(0,2,-3) ; M
M.add_multiple_of_row(0,1,-4) ; M
M.add_multiple_of_row(2,1,2) ; M
M.rescale_row(0,(-1/12)) ; M
M.add_multiple_of_row(2,0,-5) ; M
M.add_multiple_of_row(1,0,-1) ; M
M.rescale_row(1,(-1)) ; M
M.swap_rows(0,2) ; M

[    1     0     0  1/12   1/3  -1/4]
[    0     1     0  7/12  -2/3   1/4]
[    0     0     1 -5/12   1/3   1/4]

#K=M.submatrix(0,3,3,3) ; K

T=(matrix(QQ,[[3,2,1],[3,1,2],[1,2,3]])).submatrix(0,0,3,3) ; T

T*M ; M*T


# Quick Reference link for Sage  Linear Algebra:
# wiki.sagemath.org/quickref?action=AttachFile&do=get&target=quickref-linalg.pdf
#
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