**09.** ***The inverse of a matrix***

## 9.1 What do we want from an inverse?

In recent discussions we viewed a matrix *as a function.* So if $A$ is a $p \times q$ matrix we think of the equation

$$A\vec{x} = \vec{y}$$

as stating that the function $A$ takes the $q \times 1$ column vector $\vec{x}$ and "sends it" to the $q \times 1$ column vector $\vec{y}$. We have also viewed matrix multiplication as a manifestation of *the composition* of functions. Thus, in the spirit of calculus, we would like *the inverse of the matrix* $A$ to be a matrix $B$ which, when viewed as a function, "undoes" the matrix (eh, function) $A$:

$$B(A(\vec{x})) = \vec{x}.$$

We may recall from Calculus that

$$ln(e^c) = c \qquad ; \qquad e^{ln(w)} = w. \tag{1}$$

Thus, we think of the logarithmic $ln(\cdot)$ function and the exponential $e^{(\cdot)}$ function as being inverses of one another. Note that some care is required in (1) above. For while the number $c$ is unrestricted, the number $w$ must be a positive real number. So the inverse behavior in both directions of (1) is not absolutely the same. (The natural log function has only positive reals in its domain.) Still, we'd like something similar for matrices.

Thus, given the matrix $A$ we'd like to find a matrix $B$ which "undoes $A$" in the additional sense that $A(B(\vec{y}))=\vec{y}$. Now the identity matrix $Id$, or just $I$ for short, is the matrix analog of the Calculus function $f(x) = x$, because $Id$ takes a column vector $\vec{y}$ to itself: $(Id)\vec{y} = \vec{y}$. Stating matrix inverse properties in terms of the identity matrix $Id$, we'd like $BA = Id_{q\times q}$ and $AB = Id_{p\times p}$. General considerations quickly show that this can only be possible if $A$ is a *square matrix*: $p = q$. (Later we'll see that for non-square matrices some partial inversion may be possible, but a total, two-sided inverse is out of the question).

## 9.2 Experimenting with the inverse concept

Take the matrix $T \equiv \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$, where $\alpha$ is a fixed constant. The reader can verify that

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \alpha x_1 + x_2 \end{pmatrix}.$$

Thus $T$ manifests the familiar row operation of adding to row $II$ the $\alpha$ multiple of row $I$. From our row operations days we know how to "undo" this: just subtract $\alpha$ times row $I$ to row $II$. This is manifested by the matrix $S \equiv \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$, and the reader will readily verify that $ST = TS = Id_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus we write $S = T^{-1}$ and $T = S^{-1}$; $S, T$ are inverses of one another. Now let's try to invert $H \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

The reader will verify that $H\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$. A moment's reflection shows that there is no hope of "undoing" $H$. For instance $H\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. If $K$ is a candidate for the matrix inverse of $H$, we'd expect $KH\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. But $H\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so we would need $K\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Yet no matrix $K$ can multiply $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to yield anything but an array full of zeros.

## 9.3 Invertibility and RREF

We have a first principle.

**Proposition.** *A square matrix $H$ is invertible* only if *$RREF(H)$ is the identity matrix.*

Let's read the proposition carefully. We are asserting that if the matrix $H$ is to have any hope of being invertible, its RREF better be the identity matrix. We are not (yet!) asserting that this condition is also sufficient.

*Proof.* Suppose that $RREF(H)$ is *not* the identity matrix of the same size as $H$. Then $RREF(H)$ has a non-pivot column. Hence the equation $H\vec{x} = \vec{0}$ has at least one free variable, hence multiple (non-trivial!) solutions. So there are non-zero vectors $\vec{x}$ with $H\vec{x} = \vec{0}$. So no candidate inverse matrix to $H$, say $K$ can have the property $KH\vec{x} = \vec{x}$ for all $x$, as in the example above. ($K\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ would have to equal $\vec{x} \neq \vec{0}$, yet $K\vec{0}$ must equal $\vec{0}$; contradiction). □

It turns out that the necessary condition in the last proposition is also sufficient. To wit,

**Proposition.** *Let $H$ be a square matrix whose RREF is the identity matrix. Then $H$ is invertible, and the Gauss-Jordan algorithm can help us compute the inverse of $H$, $K$.*

### 9.4 The Oversized Augmentation Algorithm

Before proving this proposition, let's explore an example. Take $H = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$. We will "thickly" augment $H$ with the $2 \times 2$ identity matrix:

$$\left(\begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right). \tag{2}$$

Now let's row reduce (2) using the conventional Gauss-Jordan algorithm, starting with $I \to I - 3II$, obtaining

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{array}\right).$$

Now apply $II \to II - I$ obtaining

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 4 \end{array}\right).$$

Let's write this as $(Id|K)$, where $K$ represents the resulting augmented slot, after RREF operations. As we'll see, the augmented matrix slot manifests the desired matrix inverse. First note the following:

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Thus $H$ applied to the columns of $K = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$ yields the columns of the identity matrix. In a sense, $K$ anticipates the action of $H$ and "pre-un-does it". Now verify:

$$\begin{aligned} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} &= \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Id_{2\times 2}. \end{aligned}$$

Thus, if we let $K \equiv \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$ then $KH = HK = Id_{2\times 2}$, and $K = H^{-1}; H = K^{-1}$. This is not a coincidence. For *any* square matrix $H$ with $RREF(H) = Id$, if we augment $H$ with the identity matrix of the same size as $H$ and row reduce, the augmentation at RREF time will be the inverse of $H$, every time!

**Proposition.** *Let $M$ be $q \times q$ square matrix, and augment $M$ with the identity matrix of the same size $(M|Id)$. If the RREF of $M$ is $I_{n\times n}$ then the RREF of $(M|Id)$ is of the form*

$$(Id|L),$$

*where $L$ is the inverse of $M$: $ML = LM = Id$.*

To prepare for the(straightforward) proof of the proposition, we recall the notation $\vec{e}_i$. This is a column vector of some understood-from-the-context size, with all zero entries, except for a 1 in the $i^{th}$ slot: $(0\dots010\dots0)^t$. So, for instance, the identity matrix of size $n \times n$ can be described in terms of its columns: $Id = (\vec{e}_1\vec{e}_2\dots\vec{e}_n)$. Also, the columns of a matrix have an interpretation through action on the $\vec{e}_i$.

Let $T$ be a matrix, square or not, with columns $\vec{t}_1\vec{t}_2\dots\vec{t}_r$. Then the first column of $T$ is "where $T$ sends $\vec{e}_1$": $\vec{t}_1 = T\vec{e}_1$; the second column of $T$ is "where $T$ sends $\vec{e}_2$": $\vec{t}_2 = T\vec{e}_2$, etc. Here is an example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} ; \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \text{, etc.}$$

*Proof.* Let's examine what we really mean by the broadly augmented matrix $(M|Id)$. It is really a compact form of $n$ "ordinary" augmented matrices $(M|\vec{e}_1), (M|\vec{e}_2), \dots, (M|\vec{e}_n)$, where $n$ is the square size of $M$. Thus we are trying to solve $n$ equations using RREF: $M\vec{x} = \vec{e}_1$, $M\vec{x} = \vec{e}_2$, $\dots$, $M\vec{x} = \vec{e}_n$. Since $M$'s RREF is the identity matrix we know that each of these linear systems has a unique solution, say, $\vec{\ell}_1, \vec{\ell}_2, \dots, \vec{\ell}_n$. Put these columns into a matrix:

$$L \equiv \left(\vec{\ell}_1 \dots \vec{\ell}_n\right).$$

By our earlier considerations, the matrix $L$ "sends" $\vec{e}_1$ to $\vec{\ell}_1$, $\dots$, and sends $\vec{e}_n$ to $\vec{\ell}_n$. Now the columns $\vec{\ell}_i$ were chosen precisely so that $M$ "sends" $\vec{\ell}_1$ to $\vec{e}_1$, $\dots$, $M$ "sends" $\vec{\ell}_n$ to $\vec{e}_n$. So $M$ and $L$ perform precisely reverse actions involving the basis $\vec{e}_1, \dots \vec{e}_n$ and hence are function inverses of one another (after some linear combination considerations). We can also view the matrix product $ML$ as follows:

$$ML = M\left(\vec{\ell}_1 \dots \vec{\ell}_n\right) = (\vec{e}_1 \dots \vec{e}_n) = Id_{n\times n},$$

where we have used the definition of the $\ell_i$ column vectors. It is shown independently that for square matrices $L, M$, if $ML = Id$ then $LM = Id$ follows automatically. This concludes a somewhat belabored proof, though belabored for a purpose: manifold illustration and elucidation. ∎