01b–An Ancient Linear System

with a modern perspective

We quote a linear algebra word problem that dates back more than two thousand years:

> *There are three types of corn. One bundle of the first type, two of the second, and three of the third total to 26 measures. Two bundles of the first type, three of the second, and one of the third total 34 measures. Lastly, three bundles of corn of the first type, two bundles of the second type, and one bundle of the third type make a total of 39 measures. How many measures make up a single bundle of each type?*

Using familiar notation we denote the quantities of each of the three types of corn by x, y, z. Then we have the following linear system of equations:

$$\left.\begin{aligned} 1x + 2y + 3z &= 26 \\ 2x + 3y + 1z &= 34 \\ 3x + 2y + z &= 39 \end{aligned}\right\} . \tag{1}$$

We'll briefly discuss how to represent a linear system of equations such as (1) using a 3×3 *coefficient matrix*, a 3×1 *column vector* (i.e., a thin matrix) representing the *unknowns* (or *variables*) x, y, z, and another 3×1 column vector representing the *right hand sides* (or RHS for short). Hence can rewrite (1) as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 26 \\ 34 \\ 39 \end{bmatrix} . \tag{2}$$

We'll need to refer to individual equations in systems such as (1) and the corresponding *rows* (horizontals) in matrix representations such as (2). For this purpose, we'll use Roman numerals (*I, II, III, IV, V, . . .)* Thus I stands for the first, top equation of (1), or the first (top) row of (2), II stands for the second equation or row, etc.

It is commonly believed that Roman numerals are inefficient for doing mathematics, so why do we use them? Well, in our discussions, we will rarely need to go past IV in referring to equations, reverting to the abstract n after that. Also, we will not use Roman numerals within *computations*, but only for *nomenclature*, and there's lots of precendence for that in modern society—just look at fancy buildings and clocks. Also, although our linear algebra example is ancient, Roman numerals may actually predate it(!)

We now proceed to solve (1). We'll using the same strategy that is said to have been used by the ancients. Followup steps may deviate, but they'll retain conceptual unity. The first step is to "triple" equation (or row) I. We write

$$I \to 3I$$

and read this to mean *take equation (or row)* I*, multiply it by* 3*, and make that the new row* I*.*

This will transform our system to a new one. The reader should reflect on why the new system has precisely the same solution set as the old one, what is sometimes called *solution equivalence.* Note that, in particular, this means that if the new system has no solutions then the original system does not either, and conversely. (The reader should review, with great care, the meaning of *conversely.*)

Here is our new system:

$$\begin{bmatrix} 3 & 6 & 9 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 78 \\ 34 \\ 39 \end{bmatrix} .$$

We now perform the *row operation* $I \to I - III$, which yields a system with a zero in the top left entry:

$$\begin{bmatrix} 0 & 4 & 8 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 39 \\ 34 \\ 39 \end{bmatrix} .$$

Although, technically, the variable x still appears in the first equation (or row), it has a zero multiplier, hence it has essentially been *eliminated* from equation I. The general idea is to eliminate as many variables as we can, in a reasonably systematic fashion, to yield a linear system that is very simple and easy to understand.

We now deviate from ancient history, taking row operation steps not necessarily following the founders, nor necessarily the most efficient or elegant:

$$II \to 3II\,; \quad III \to 2III.$$

Note that we are prescribing *two* row operations in *one fell swoop.* As we'll later recommend a one-step-at-a-time approach, why are we multi-tasking now? Well, the two row operations are independent of one another, involving different rows. Sill caution is advised. In any case, we arrive at the following system:

$$\begin{bmatrix} 0 & 4 & 8 \\ 6 & 9 & 3 \\ 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 39 \\ 102 \\ 78 \end{bmatrix} ,$$

and the situation is ripe for zeroing out the x entry in the second equation with $II \to II - III$:

$$\begin{bmatrix} 0 & 4 & 8 \\ 0 & 5 & 1 \\ 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 39 \\ 24 \\ 78 \end{bmatrix}.$$

We better not zero out any more x's (why?) Instead, we prepare to zero out the z entry in I, using $I \to I - 8II$:

$$\begin{bmatrix} 0 & -36 & 0 \\ 0 & 5 & 1 \\ 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -153 \\ 24 \\ 78 \end{bmatrix}. \tag{3}$$

Note that in this last row operation we took one row and subtracted a multiple of another row from it. This type of "compound" operation is most commonly used, and may be called a *workhorse* operation.

Our latest system, (3) has all variables except y eliminated from the first equation, which reads $(-36)y = -153$, hence $y = 153/36 = 17/4$. At this point, we could try to simplify rows II and III farther, but since row II involves only y and z, and we already know y, we yield to temptation and solve for z.

Equation II says

$$5y + z = 24,$$

so

$$(85)/4 + z = (96)/4,$$

hence

$$z = 11/4.$$

Now equation III enables us to solve for x. It says:

$$6x+4y+2z = 78, \text{ or } 6x+(68/4)+(22/4) = 78 = 312/4,$$

so

$$6x = 222/4 \text{ and } x = 37/4.$$

Notice that the solutions are not *integers* (whole numbers), but rather quarters of integers. In general, we will study linear algebra over the *real numbers*, $\mathbb{R}$, and later over the *complex numbers* $\mathbb{C}$. While many examples considered may turn out to involve only integers, this is only for aesthetic reasons, or for ease computation.

Reflecting back on what we've done, the approach seems anything but systematic. With less luck, we might have even gone around in circles, simplifying one portion of our system while making another portion more complex. And what guarantee do we have that such an approach, in general, will eventually lead to a solution, or conclusive evidence to lack thereof? Not to worry. We will take care of all these considerations. Stay tuned.

Those who wish to see the full extent of the specific steps taken by the ancients to solve the system we considered are directed to [2]. See also [1].

Figure 1: Ancient linear algebra

Bildarchiv Preussischer Kulturbesitz/Art Resource/NY. Photo by Olaf M. Teßmer.

It began in cuneiform tablets like VAT 8389. Vorderasiatisches Museum in Berlin, 12.1 by 7.9 cm.

References

[1] Grcar, J.F. (2011). Mathematicians of Gaussian Elimination., *Notices of the Amer. Math. Soc.*, 58:6, 782–787.

[2] Hart, R. (2011). *The Chinese Roots of Linear Algebra.* Johns Hopkins University Press, Baltimore.