02. Row reduction

Consider the linear system below, expressed as an *augmented matrix*:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & \alpha \\ 2 & 1 & 1 & \beta \\ 1 & 1 & 1 & \gamma \end{array}\right). \tag{1}$$

Each row of the matrix in (1) represents a equation in our system, and the "augmentation" entry to the right of the bar represents the right hand side of the equation. So, for instance, taking x, y, z as our unknowns, the first equation of our system is: $1 \cdot x + 2 \cdot y + 1 \cdot z = \alpha$.

(We take a moment to point out that this equation has a *geometric* interpretation. It represents a *plane* in xyz space, and changing the right hand side scalar α shifts the plane to a new one, parallel to the original; more about this later.)

To solve the system we proceed by "zeroing out" the leftmost entry of the second equation and the leftmost entry of the third equation using row operations. (Note that we do not literally zero things out– that would not be useful. Rather, we perform operations on equations and matrix rows which, among other things, result in certain coefficients acquiring zero values.) To streamline notation we will use *left to right and top to bottom ordering,* with appropriate apologies to those that prefer alternative orderings. (Look up "chirality" and "down under".) So, using *Newspeak*[1], we restate our plan as follows: we will try to zero out the first entry of the second equation and the first entry of the third equation. We proceed with the following row operations:

$$II \to II - 2I\,; \qquad III \to III - I.$$

In prose, replace equation two by equation two minus 2 times equation one; then replace equation three by equation three minus equation one.

We obtain

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & \alpha \\ 0 & -3 & -1 & \beta - 2\alpha \\ 0 & -1 & 0 & \gamma - \alpha \end{array}\right). \tag{2}$$

[1] See George Orwell's book *1984*.

We now pause for some important reflection. The reader is encouraged to rewrite (1) in full as a system of linear equations with full notation (lots of " + " signs) and the variables x, y, z. Then the row operations can be reinterpreted and validated as *operations on equations*, or *"equation operations"*, justified by standard rules of algebra and arithmetic. Reinterpreting (2) as a system of linear equations, we see that if x, y, z solves (1) then x, y, z also solves (2). Conversely (lookup *converse*), if x, y, z solves (2) then we can apply similar equation operations (specifically $III \to III + I\,;\ II \to II + 2I$) to show that x, y, z solves (1) as well. Similar considerations apply to other row operations (scaling by a nonzero number, or swapping two rows). The upshot is:

$\boxed{\textit{Row operations preserve solution sets}}$

With this important perspective in mind, we continue, aiming next to zero out the second entry of the second row. We use $II \to II - 3III$, obtaining:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & \alpha \\ 0 & 0 & -1 & \beta - 3\gamma + \alpha \\ 0 & -1 & 0 & \gamma - \alpha \end{array}\right). \tag{3}$$

Notice that we did not "ruin" any of our "hard-earned" zeros. We now zero the 3^{rd} entry of the first row, sans ruination (if you know what I mean): $I \to I + II$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2\alpha + \beta - 3\gamma \\ 0 & 0 & -1 & \beta - 3\gamma + \alpha \\ 0 & -1 & 0 & \gamma - \alpha \end{array}\right). \tag{4}$$

We can now zero the 2^{nd} entry of the 1^{st} row: $I \to I + 2III$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \beta - \gamma \\ 0 & 0 & -1 & \beta - 3\gamma + \alpha \\ 0 & -1 & 0 & \gamma - \alpha \end{array}\right). \tag{5}$$

We can easily solve for x, y, z using (5), but for form's sake, let's replace the -1s by $+1$ in the second and $3rd$ rows of the coefficient matrix. (Notice that this is not *literally* what we are doing.) We invoke $II \to (-1)II$ and $III \to (-1)III$ and then, for further adornment, we transpose (or *"swap"* the 2^{nd} and 3^{rd} rows: $II \longleftrightarrow III$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \beta - \gamma \\ 0 & 1 & 0 & \alpha - \gamma \\ 0 & 0 & 1 & -\alpha - \beta + 3\gamma \end{array}\right). \tag{6}$$

This system is trivial to solve; it also enables us to solve the original system (1), no matter which right

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hand side is prescribed:

$$x = \beta - \gamma,\ y = \alpha - \gamma,\ z = \alpha - \beta + 3\gamma.$$

This is clearly the unique solution to equation (5).

But why is it the unique solution to (1)*?* Because (5) is *row equivalent* to (1) and row operations preserve solution sets, as observed above, right after (2). [Reader: define *row equivalent*.]

We now step back and reflect once more. The treatment of (1) suggests that given *any* linear system we can apply row operations to obtain a simplified system with the same solution set, and that there seems to be an optimal "most simplified" matrix type to aim for, one that cannot be further simplified in a useful manner. Contrast this with other objects, e.g., a bicycle or a piece of software. Is there a design of a bicycle that cannot be further improved? Is there an unimprovable piece of software? The situation here, in this context of linear algebra, seems to be special indeed.

To make the suggestion above rigorous we have to demonstrate that we will always be able to make progress by row reduction steps, never getting *stuck*, until we reach the unimprovable form. Moreover, we have to spell out what constitutes this ultimate form.

Indeed, there is a systematic algorithm for doing this and there are axioms describing when a linear system is in this purported most simplified form, known as *Row Reduced Echelon Form*, or *RREF*. The algorithm relieves us of the burden of having to make arbitrary choices, and directs steady progress.

To see the algorithm idea, look back at (3). From (3) we proceeded to zero out the 3^{rd} entry of the first row and then the 2^{nd} entry of the first row. Another linear system solver might have proceeded the other way around. Roughly speaking, the algorithm seeks to build the leftmost possible column with a leading 1, i.e., a *pivot*, then the second leftmost such column, etc., moving zero rows downward. The precise algorithm is spelled out elsewhere in these writings, and in most books.

How about the "unimprovable form"? What properties must it have? First, that

> *in each row, sweeping from the left, the first nonzero entry we encounter (if any) should be a* 1*.*

(Otherwise we could divide by it, normalize, and have a simpler, better form). These leading 1s will be called *pivots*; they can be used to zero out all other entries in a given column with the aforementioned leading 1. We can then arrange the pivots so that

> *as we move from upper to lower rows, the pivot entries move from left to right.*

(Otherwise we could perform row swaps to attain this). Finally, require that

> *all purely zero rows are at the bottom.*

(Otherwise we can swap rows, as needed, to manifest this condition.)

The three highlighted conditions above give rise to the RREF, which is an early and important example of a *normal form*, a concept which occurs in many parts of mathematics and which can produce numerous insights. In the coming sections we will explore and apply the RREF. Before closing, though, we summarize the aforementioned properties in a more verbally graphic way. This can safely be left for later readings.

The Requirements of RREF

A matrix E is in RREF if it satisfies the follow conditions.

- **Pivots** Sweeping each row of E from the left, the first nonzero scalar encountered, if any, is a 1. We call this entry, along with its column, a *pivot*.
- **Pivot Column Insecurity** In a pivot column, the scalar 1 encountered in the row sweep is the only non-zero entry in its column.
- **Downright Conventional** If a pivot scalar 1 is to the right of another, it is also lower down.
- **Bottom Zeros** Rows consisting entirely of zeros, if they appear, are at the bottom of the matrix.

The label *Pivot Insecurity* requires explanation. We think of the pivot scalar 1's as insecure: they don't want competition from other nonzero entries along their column. Sorry pivots–row insecurity cannot be accommodated.

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