7.0 Definition

07. Describing the Null Space – The Hound of the Basis Vectors

In the future, everyone will be world-famous for 15 minutes.

Andy Warhol, 1968

Here we will outline and illustrate a procedure for describing the *null space* of a matrix in a rather precise way.

7.1 The 4×4 Walking matrix and its null space.

Take the 4×4 *walking around matrix*, W_4, or just W for short.

$$W \equiv W_4 \equiv \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Let's first work out the RREF of W *Gauchely.* Notice that the first column vector (or just *column* for short, or $\vec{w}_1$ for shorter) of W is nonzero, hence a keeper. The second column, $\vec{w}_2$, is not expressible as a scalar multiple of the first `[Reader:show.]` Hence the second column is a keeper as well. It is not difficult to show that columns $\vec{w}_3, \vec{w}_4$ are expressible as linear combinations of $\vec{w}_1$ and $\vec{w}_2$:

$$\vec{w}_3 = (-1)\vec{w}_1 + 2\vec{w}_2 \quad ; \quad \vec{w}_4 = (-2)\vec{w}_1 + 3\vec{w}_2. \tag{1}$$

The scalars in (1) are easily found by trial and error, or by solving two 2×2 linear systems. `[Reader:provide details.]` Hence

$$RREF(W) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

We now proceed to describe the *null space* (a.k.a. the *kernel* of the matrix W). That is, we wish to describe the space of all solutions to the homogeneous system $W\vec{x} = \vec{0}$. We note that the RREF of W has precisely the same null space. `[Reader:why?]`

7.2 Independent Variables Get 15 Minutes of Fame

The RREF of W, (2), has two pivot columns: $\vec{w}_1, \vec{w}_2$ and two non-pivot columns $\vec{w}_3, \vec{w}_4$. We recall that this implies that x_1, x_2 are *dependent* variables and x_3, x_4 are *free* (or *independent*) variables. That is, we may choose freely values for x_3, x_4 and then solve uniquely for values of x_1, x_2. Thus the values of the latter are dependent on the freely chosen values of the former. To describe the space of all solutions to the homogeneous system $W\vec{x} = \vec{0}$ we will make our free choices in an explicit, systematic way.

We start with the last free variable, in this case x_4, and work backwards. We give each free variable its "fifteen minutes of fame". That is we choose the value 1 for this free variable and choose the value 0 (zero) for all other free variables. In this case, we choose

$$x_4 = 1 \quad , \quad x_3 = 0.$$

Then the second row of the homogenous system associated with (2) says $0x_1 + x_2 + 2x_3 + 3x_4 = 0$. Note that the x_1 term has a zero multiplier in the last expression; we may omit such terms in the future. Solving for x_2 we have $x_2 = -3$. Then the first row of (2) leads to $x_1 - 2x_4 = 0$, so $x_1 = 2$. The solution vector we found for $W\vec{x} = \vec{0}$ is

$$\vec{c}_4 \equiv \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Next, we give the free variable x_3 its fifteen minutes of fame. We set $x_3 = 1$ and we set all other free variables equal to zero; in this case, $x_4 = 0$. Then we solve for x_2 and x_1 using (2) to obtain

$$\vec{c}_3 \equiv \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

Note that we can express *any* solution of the system $W\vec{x} = \vec{0}$ as a linear combination of these last two "null" vectors:

$$s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

Why can "any" solution of the homogeneous system $W\vec{x} = \vec{0}$ be expressed in the form (3)? Let's perform the addition in (3) to obtain

$$\begin{pmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{pmatrix}.$$

Now, since x_3, x_4 are free variables for the homogeneous linear system $W\vec{x} = \vec{0}$, for any scalars s, t we have a solution of the form $\begin{pmatrix} * \\ s \\ t \end{pmatrix}$, where "$*$" represents the top two entries, and we can solve uniquely for the $*$ components, because of the dependent variable properties of x_3, x_4, confirming (3).

7.3 The Hound of the Basis Vectors

We note that the set (actually *multi-set*) of vectors $\{\vec{c}_3, \vec{c}_4\}$ has the following property:

> *Every vector in $Null(W_4)$ is uniquely expressible as a linear combination of the vectors $\vec{c}_3$ and $\vec{c}_4$.*

In anticipation of upcoming discussions, we phrase the above property as follows:

> *The set $\{\vec{c}_3, \vec{c}_4\}$ is a* basis *for the vector space $Null(W_4)$.*

In general, if V is a vector space (a sandbox for vector addition and for scaling), we say that $\mathcal{B} \equiv \{\vec{\sqsubseteq}_\infty, \ldots, \vec{\sqsubseteq}_\|\}$ is a *basis* for the vector space V if every vector in $\vec{u}$ in V can be expressed, in a unique way, as linear combination of the vectors in $\mathcal{B}$. That is, given $\vec{u} \in V$, there is one and only one set of scalars $a_1, \ldots a_k$ so that
$\vec{u} = a_1\vec{v}_1 + \ldots + a_k\vec{v}_k$.

One can say that a set of vectors $\mathcal{B}$ is a basis for a vector space V if it satisfies the *Goldilocks Condition*: not too small, not too big; just right. That is, $\mathcal{B}$ is large enough: linear combinations of its vectors can represent *any* vector in V. ($\mathcal{B}$ "spans" V.) Also, $\mathcal{B}$ is small enough: it has no redundancies. Each vector of V can be presented as a linear combination of vectors in $\mathcal{B}$ in just one way. ($\mathcal{B}$ is "linearly independent".)

We note that a vector space can have *many bases* (degeneracies aside). In the near-term, we will discuss bases and allied concepts extensively.

7.4 Inverse Problems–An Intriguing Converse?

There is a relatively young subject in mathematics called *Inverse Problems*, and it is widely used in imaging technology, Earthquake prediction, and more. An oversimplified, yet still useful description of it reads: *recovering the question from the answer*. We can appreciate an instance of it in the current Linear Algebra context.

If we are given as data the solution set to a linear system of equations we cannot possibly recover the system from the data. For instance, the following three linear systems all have the same solution set:

$$\begin{cases} 2x+ & 3y & = 4 \\ 3x+ & 2y & = 1 \end{cases}; \begin{cases} 3x+ & 2y & = 1 \\ 2x+ & 3y & = 4 \end{cases}; \begin{cases} 4x+ & 6y & = 8 \\ 3x+ & 2y & = 1 \end{cases}.$$

On the other hand, perhaps we can recover the original linear system *up to row operations*. This is tantamount to recovering the RREF of the original system. Now gazing at the solution space (3) and comparing it with the RREF of W, (2), we see that some scalars appear in both. Could this is coincidence? Surely not (Sherlock Holmes).

We know that if two linear systems are *row-equivalent* (one can be obtained from the other by a sequence of row operations), then the two systems have the same solution set. Could the *converse* be also true? `[Reader:what do mathematicians mean by "converse"?]` We ask:

> *If two linear systems have the same solution set, does it follow that they are row equivalent?*

The similarity between (3) and (2) suggests that, at least under certain conditions, this may be true. The reader is encouraged to explore this question and look for more information about it in upcoming forums.