04b.
Gauche RREF:
What Does It Say About Solutions?

> **B-movie** | ˈbēˌmo͞ovē |
>
> noun
>
> a low-budget movie, especially (formerly) one made for use as a companion to the main attraction in a double feature: *[as modifier] : a B-movie actress.*

This installment in our *Really Short Readings* continues a new paradigm. We label this reading $04b$, with "b" meant to suggest something like a *B movie*. In contrast with the regular Really Short Readings, which aim to introduce and reflect upon new topics and ideas, b readings are meant to reinforce basic points, remove confusion, or simply call attention to it.

In reading the passages below the reader may wish to review or have on hand earlier readings with concrete examples of row reducing a matrix using Gauss-Jordan and, separately, reaching RREF by a Gauche approach.

We have established two ways to take a matrix M and transform it into its RREF: the Gauss-Jordan algorithm, a systematic sequence of row operations, and the Gauche approach, a sequence of vector selection and sequencing powered by linear combinations and the concept of *Span*. The former approach allows us to work on solutions of linear systems as we work towards RREF: to solve the linear system $M\vec{x} = \vec{b}$, form the augmented matrix $(M|\vec{b})$. Proceed to row reduce this matrix. Notice that in this process, the coefficient matrix M *drives* and the vector $\vec{b}$ comes along for the ride. That is, $\vec{b}$ does not *influence* the selection of row reduction actions, but does get transformed by them. (Reader: are we anthropomorphizing here? Can vectors exert influence?) On the other hand, forming $RREF(M)$ by the Gauche approach deals entirely with the coefficient matrix and does not leave us with a sequence of row operations to use to solve a linear system. So the question comes back to mind:

> How does the Gauche path to RREF help us solve linear systems?

The Gauche approach to RREF is perhaps best viewed as a means to increase and improve understanding of the RREF concept and to interpret the RREF form of a matrix more clearly. Still, it does also provide some dividends which can be applied to solving linear systems.

Let's begin with homogeneous systems. Notice that if $\vec{b} = \vec{0}$ then in row reducing the augmented matrix $(M|\vec{b})$ the augmentation vector (to the right of $|$) remains $\vec{0}$ throughout the process. Thus, knowing $RREF(M)$ by Gauche means is just as good here as knowing it by Gauss-Jordan means. Recall:

> The pivot columns of $RREF(M)$ correspond to *dependent* variables for the system $M\vec{x} = \vec{0}$. The non-pivot columns correspond to *free* variables for $M\vec{x} = \vec{b}$.

This tells us lots about solutions for the homogeneous system $M\vec{x} = \vec{0}$ associated with M For example, for the 3×3 walking matrix, $W \equiv \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ the RREF is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Thus we see that W has two dependent variables (x_1, x_2) and one free variable (x_3). to solve $W\vec{x} = \vec{0}$, choose any values for x_3, say t and then the values for x_1 and x_2 are determined: $x_2 = -2t$ and $x_3 = t$. This tells us that the system $W\vec{x} = \vec{0}$ has infinitely many solutions. Since W is a square matrix, we may declare that W is a *singular* matrix.

What about a possibly inhomogeneous variant, say $W\vec{x} = \vec{b}$? This last system could be inconsistent. For instance,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 1 \end{array}\right)$$

is inconsistent. Can $RREF(W)$ help us see this (without performing row operations directly on the augmented vector)? It can!

Solving this last system amounts to finding unknowns x_1, x_2, x_3 so that

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

This simply says that $\vec{b}$ is a linear combination of ther column vectors of W. But we know (from Gauche RREF) that the third column of W is a linear combo of the first two columns. So we don't "need" the third column. (Reader: elaborate.) So we need to find scalars x_1, x_2 to solve

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

and if we cannot find such scalars then the system $W\vec{x} = \vec{b}$ is inconsistent. We have reduced our problem to a system of three equations in two unknowns. The reader will have no trouble showing that this system has no solution, so the original system is inconsistent.

Here is a useful generalization.

Theorem. Let $M\vec{x} = \vec{b}$ be a linear system. Then this system is consistent precisely when the column vector $\vec{b}$ is a linear combination of the column vectors of $M, \vec{m}_1, \ldots \vec{m}_k$. In this case, any solution vector $\vec{x}$ manifests a linear combination of the columns of M yielding $\vec{b}$:

$$x_1\vec{m}_1 + x_2\vec{m}_2 + \ldots + x_k\vec{m}_k = \vec{b}. \tag{2}$$

Moreover, when the system is consistent, we may manifest (2) with all free variables x_i set to zero.

This last assertion follows because, by the Gauche approach, non-pivot columns are linear combinations of pivot columns.

The Gauche approach can also help us with consistent non-homogeneous linear systems. The upshot is that, to solve a non-homogeneous linear system, we merely have to find one particular solution, and then add that solution to the totality of solutions of the accompanying homogeneous system. For instance, in a previous reading we presented solutions of the system

$$\left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & 3 \\ 2 & -4 & 1 & 1 & 2 \\ 1 & -2 & -2 & 3 & 1 \end{array}\right).$$

in the following form:

$$\left\{ \begin{pmatrix} 3-t \\ t \\ -2 \\ -2 \end{pmatrix} \middle| t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 3 \\ 0 \\ -2 \\ -2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

The solutions are presented as one particular solution added to arbitrary solutions of the accompanying homogeneous system. This principle could use some elaboration. However, this is a “b” reading, and we don’t want it to run too long. The reader is invited to provide elaboration, and we will also add some, upon request.

The overall upshot of this discussion is that, while the Gauche approach does not aim directly to solve a particular linear system, it can help us with that endeavor, and more.