04. A Gauche Perspective on RREF

1 What is a linear combo?

Take two vectors in the same *sandbox*, that is, two vectors of the same shape (later to be phrased as two vectors in a given *vector space*), for instance,

$$\vec{u} \equiv \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad \vec{w} \equiv \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Then an expression of the form

$$5 \cdot \vec{u} + 7 \cdot \vec{w}$$

is said to be a *linear combination* of the set of vectors $\{\vec{u}, \vec{w}\}$. More generally, if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are vectors in a given sandbox, i.e., all of the same shape, and $a_1, a_2, \ldots, a_k$ are scalars (numbers), then an expression of the form

$$a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + \ldots + a_k \cdot \vec{v}_k = \sum_{i=1}^{k} a_i \cdot \vec{v}_i,$$

is said to be a *linear combination of the vectors* $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. We will usually dispense with the multiplicative dot (as in $5 \cdot \vec{u}$), with tacit understanding that there is scaling there ($5\vec{u}$). Also, with apologies to the language academy, we will often resort to the shorter expression *linear combo.*

2 What is the sound of one hand clapping?

As is often done in the mathematical literature, we stretch our jargon to the point of degeneracy. (By the way, what's *jargon*?) So we will also speak of a linear combination of a set consisting of just single vector, e.g., $5\vec{u}$. This simply means a scaling of that vector. And finally, we can even speak of *a linear combo of a set consisting of no vectors at all* (the empty set $\emptyset$). This simply stands for *the zero vector of the respective sandbox* $\vec{0}$. In the example above, this is $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$, and our future jargon will have it as *the zero vector of the respective vector space.* This usage is consistent with more basic notational conventions, as we articulate below. Recall the familiar Σ notation for sums, e.g.,

$$\sum_{i=1}^{3}(2i+1) \equiv (2\cdot 1+1) + (2\cdot 2+1) + (2\cdot 3+1).$$

A somewhat degenerate instance of this notation is $\sum_{i=1}^{1}(2\cdot 1+1) \equiv (2\cdot 1+1)$, a sum with just one summand. Even more degenerately, we can consider an *empty sum*, e.g.,

$$\sum_{i=1}^{-5}(2i+1).$$

By convention, we start with the lower limit for the index and increment by 1 until we reach the upper limit. In this last display, no index qualifies and the sum is an empty sum, which is set to equal zero, by convention: $\sum_{i=1}^{-5}(2i+1) \equiv 0$. Similarly, an empty product stands for 1, by convention:

$$\prod_{i=1}^{3}(2i+1) \equiv (2\cdot 1+1)\cdot(2\cdot 2+1)\cdot(2\cdot 3+1)\ ,$$

and $\Pi_{i=1}^{-5}(2i+1) \equiv (2\cdot 2+1) \equiv 1$. The reader may now wish to go back and review the degenerate cases of the linear combinations above. (But please do not go into an infinite reading loop.)

3 A different, more natural path to RREF.

We already know that, given a matrix A, there is one and only one matrix in *RREF* that is row equivalent to A: "RREF is unique". We have motivated a proof for this, and the literature abounds with other proofs, but still leaves us wanting. These proofs seem *unnatural.* If $RREF(A)$ is unique, then every aspect of $RREF(A)$ should reflect some property of A. Of course, the *RREF* concept has some conventions built in, e.g., *left-to-right*, so those could also be reflected.

4 The Gauche procedure: an example.

We outline a procedure below which leads us to *RREF* and makes the above naturality reflections salient. Starting with a $p \times k$ matrix A, we will flag certain columns as *keepers* and designate others as *subordinate*, maintaining a journal for our experience.

While the procedure can be applied to a general matrix A, it may help to keep a specific matrix in mind, we'll keep each of the following two matrices in mind:

$$\begin{pmatrix} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

We examine the columns of a matrix, from left to right, and select certain columns as *keepers.* For a given column, we ask:

> *Can we present this column as a linear combination of the columns to the left which are already selected?*

We will call this the *Left-Leaning Question*, or *LLQ*.

For the first column, the *LLQ* asks: "Is this column nonzero?". For the second column, the *LLQ* asks: "Is this column a multiple of the first column?" And so on.

We first apply the procedure tersely to a specific matrix, and then articulate the procedure in full generality and detail. Take the matrix

$$T \equiv \begin{pmatrix} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{pmatrix}.$$

We'll review and act on the columns of T from left to right. The first column is not identically zero, so we declare it a "keeper", and journal it with the vector $\vec{J}_1 \equiv \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$.
Next, examine the second column of T. It is not a scalar multiple of the first column of T, so we declare this a keeper, and journal it with $\vec{J}_2 \equiv \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$. The third column of T is, by inspection, presentable as a linear combo of the first and second columns (with scalings $3, 1$ respectively:

$$\begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix} = 3 \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \qquad (*)$$

How did we obtain this relation? That's a side-issue. We want to focus on the *Gauche* procedure here. But just to allay worries, note that one can arrive at $(*)$ via a little trial and error, or by setting up a small linear system (3 equations in 2 unknowns; see the "fifth column" discussion below).

Because of $(*)$, we won't keep the third column vector, calling it *subordinate* and we'll journal our action with the vector $\vec{J}_3 \equiv \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, which encodes the recipe for producing this vector as a linear combination of keepers. Similarly, the fourth column of T is subordinate, and journaled with $\vec{J}_4 \equiv \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$. The fifth and final column vector of T is not presentable as a linear combo of previous keepers. **`[Reader: justify this, or rotate the page and read the box below.]`**

> (Take a times the first column of T and add it to b times the second column, and look at the top and bottom entries. To produce the fifth column of T, we need $2a+b=2$ and also $a+b=2$. This implies that $a=0$, and then we run into trouble with the middle entries of our vectors.)

So we declare the fifth column of T a keeper and journal it with $\vec{J}_5 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Now form a 3×5 matrix using the vectors we journaled, in the order we journaled them $(\vec{J}_1 \quad \vec{J}_2 \quad \cdots \quad \vec{J}_5)$:

$$\begin{pmatrix} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This turns out to be the *RREF* of T (!)

5 The Gauche procedure in full generality

We now outline our procedure for a general matrix A, with column vectors $\vec{A}_1, \ldots, \vec{A}_k$. For the very first column $\vec{A}_1$, since we have no columns selected yet, the *LLQ* question above is tantamount to asking

Is this column identically zero?

This is consistent with the concept of *linear combination of the empty set of vectors* If this column is identically zero, we journal it by writing down the zero $p \times 1$ vector $\vec{J}_1 \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. If this column is not identically zero we declare this vector a *keeper* and journal it by writing down a $p \times 1$ column vector with a 1 in the first slot and zeros elsewhere: $\vec{J}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Next, we take the second column vector of A, $\vec{A}_2$, and ask the Left-Leaning Question (LLQ) which, in this case, is tantamount to asking if this vector $\vec{A}_2$ may be presented as a scalar multiple of the first column of A, consistent with our linear combination of a single vector convention. (Readers who are very careful, or very fussy, or both, may object, noting that step one has two possible outcomes and that these outcomes should perhaps be dealt with separately. However, a moment's reflection shows that the end results will be the same.) If $\vec{A}_2$ is presentable as $\lambda \vec{A}_1$, we add to our journal the vector $\begin{pmatrix} \lambda \\ 0 \\ \cdots \\ 0 \end{pmatrix}$. In general, if we focus on the ℓ^{th} column of A, $\vec{A}_\ell$, we ask the LLQ: is $\vec{A}_\ell$ is presentable as a linear combination of the selected columns, say $\vec{A}_\ell = \sum_{i=1}^m a_i \vec{A}_{s_i}$, where the thus far selected columns are $\vec{A}_{s_1}, \ldots \vec{A}_{s_m}$, m being the number of keepers accumulated prior to the focus on column $\vec{A}_\ell$? If so, we call $\vec{A}_\ell$ *subordinate*, i.e., not a keeper, and we journal this action with the vector $\vec{J}_\ell \equiv \sum_{i=1}^k a_k \vec{e}_i$, where, for each q, $\vec{e}_q$ stands for the $p \times 1$ vector with a 1 in entry q and zeros elsewhere, i.e., the q^{th} column of the $p \times p$ identity matrix. If $\vec{A}_\ell$ is not presentable as a linear combination of "keepers" (previously selected column vectors), we include $\vec{A}_\ell$ in our keeper list, and journal it with $\vec{e}_\ell$. Forming a $p \times k$ matrix from the journaled columns, in journaled order, we have a matrix which turns out to be $RREF(A)$.

The reader is invited to apply the Gauche Procedure, for practice, to the follow matrix:

$$\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix}.$$

In future writing we will take another look at the uniqueness property of RREF, using a Gauche perspective.

See the next page for a *SageMath* script with one, conventional, step-by-step process, row reducing the matrix T to its RREF. But first a couple of quotes:

> *Only nothing can come from nothing*
> (Latin: *Ex nihilo nihil fit*)
>
> Parmenides
> *Ancient Greek philosopher*

> *Only the zero vector can come from nothing*
> (The span of the empty set is the (appropriate) zero vector)
>
> Linear Algebra convention

(Continue onto next page.)

Row Reduction of T using `SageMath`

The script below provides SageMath commands that row reduce the matrix T "by hand". That is, the arithmetic, organization and update of matrix entries is done by the computer. But the choice of row reduction steps was made by a human. To run SageMath one can use the following web page:

`https://sagecell.sagemath.org/`

For a quick reference sheet for doing linear algebra with SageMath, look here (one long line, split-printed here):

`wiki.sagemath.org/quickref?action=AttachFile&do=`
`view&target=quickref-linalg.pdf`

or simply do a web search for the keyword string "sage quickref linear algebra". The book provides an introduction to SageMath. One can probably guess the meaning of each step, with a little help from the quick reference pages. The "QQ" string in the first line tells SageMath that the Matrix M has rational numbers for entries. This allows for fractions; SageMath defaults to integers (whole numbers).

```
M=matrix(QQ,[[2,1,7,-7,2],[-3,4,-5,-6,3],[1,1,4,-5,2]]) ; M
M.add_multiple_of_row(0,2,-2) ; M
M.add_multiple_of_row(1,2,3) ; M
M.add_multiple_of_row(2,0,1) ; M
M.add_multiple_of_row(1,0,7) ; M
M.rescale_row(0,-1) ; M
M.rescale_row(1,(-1)/5) ; M
M.swap_rows(0,2) ; M
M.swap_rows(1,2) ; M
M.add_multiple_of_row(1,2,-2) ; M
```