14. Eigenvalues: candidates by trace and determinant

Recall that *the trace* of a square matrix M is the sum of the diagonal entries of M. Thus if the entries of M are (m_{ij}), where i and j range independently from 1 to n, then $trace(M) = \sum_{k=1}^{n} m_{kk}$. For the purpose of illustration, we'll work with a specific matrix:

$$M \equiv \begin{pmatrix} -12 & 30 \\ -5 & 13 \end{pmatrix} .$$

We also recall the *determinant* function of a square matrix. For a 2×2 matrix, say $H \equiv \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have the familiar formula $\det(H) = ad - bc$. We now compute:

$$\text{trace }(M) = 1\,; \qquad \det(M) = -6\,.$$

Recall that finding eigenvalues and eigenvectors of a square matrix is a multi-stage and partly nonlinear problem. In order to find an eigenvalue of the square matrix T, we typically have to solve the *characteristic equation* $\det(T - \lambda Id) = 0$ for the scalar(s) λ. If T is $n \times n$, this is a polynomial equation of degree n. Once we find the possible values of λ we can then locate eigenvectors with eigenvalue λ by looking for nontrivial solutions $\vec{x}$ of the homogeneous linear system $(T - \lambda Id)\vec{x} = \vec{0}$.

We will investigate eigenvalues and eigenvectors for the matrix M using a heuristic approach that involves the concepts of *trace* and *determinant*:

Heuristic Principle

- trace(M) is a *candidate* for *sum*-of-eigenvalues of M.
- det(M) is a *candidate* for *product*-of-eigenvalues of M.

We use the term *candidates* because we need the next stage of investigation, looking for non-trivial solutions of $(M - \lambda Id)\vec{x} = \vec{0}$, to locate eigenvectors to accompany the candidate eigenvalues. (Later we'll see that the characteristic equation $\det(M - \lambda Id) = 0$ may "overpromise", in the sense that it may suggest a larger set of eigenvectors than may actually be found. But we digress.)

For our particular matrix M, the candidate (two) eigenvalues add up to 1 and multiply to -6. So let's call one candidate eigenvalue b; then the other eigenvalue is $1 - b$ and we have

$$b(1 - b) = -6\,,$$

or $b - b^2 = -6$, or $b^2 - b - 6 = 0$, or $(b-3)(b+2) =$, with solutions $b = +3$ or $b = -2$. Let's now look for eigenvectors $\vec{v}$ with eigenvalue $\lambda = +3$. We form the linear system

$$(M - \lambda Id)\vec{v} = \vec{0}\,,$$

or

$$\left[\begin{pmatrix} -12 & 30 \\ -5 & 13 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ,$$

or, substituting $\lambda = 3$ and performing some arithmetic,

$$\begin{pmatrix} -15 & 30 \\ -5 & 10 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ,$$

We may safety entrust the solution of this linear system to the reader. Using the *fifteen minutes of fame* algorithm we obtain the solution $v_2 = 1, v_1 = 2$, forming the vector $(2\,1)^t$, which stands for $\vec{v} \equiv \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$. The reader should verify that $M \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = 3 \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, thereby confirming that the vector $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ is an eigenvector of the matrix M, with eigenvalue 3.

Next, to find eigenvectors with eigenvalue -2, form the linear system $(M - \lambda Id)\vec{w} = \vec{0}$, substitute -2 for λ and solve for the entries of $\vec{w}$, namely w_1, w_2. Using the *fifteen minutes of fame* approach this yields the nontrivial solution vector $\left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right)$, which, verifiably, is an eigenvector of the matrix M with eigenvalue -2: $M \left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right) = (-2) \left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right)$.

Readers may decide for themselves if the heuristic of using trace and determinant to hunt for eigenvalues is preferable to other methods. For matrices larger than 2×2, the trace and determinant information represent a smaller portion of the totality of information about eigenvalues.

For contrast, consider the following, larger, matrix discussed in the Beezer book:

$$B = \begin{pmatrix} -2 & 1 & -2 & -4 \\ 12 & 1 & 4 & 9 \\ 6 & 5 & -2 & -4 \\ 3 & -4 & 5 & 10 \end{pmatrix} .$$

This matrix has trace 7, which is easily computed, and determinant 8, which row reduction easily reveals. Still, we expect as many as four distinct eigenvalues for B, and the heuristic principle in the box above may lead us to guess that the "spectrum" (the set of eigenvalues) of B is $\{1, 2, 2, 2\}$, but this is just guesswork.

The characteristic polynomial of B turns out to be $8 - 20\lambda + 18\lambda^2 - 7\lambda^3 + \lambda^4 = (\lambda - 1)(\lambda - 2)^3$. Thus we expect one eigen-direction with eigenvalue $\lambda = +1$ and a linearly independent set of three eigen-directions with eigenvalue $\lambda = +2$. However, computation in the book yields the $+1$ eigenvector

$$\vec{u} \equiv \begin{pmatrix} -1 \\ 3 \\ 3 \\ 0 \end{pmatrix}$$

and the $(+2)$ eigenvector

$$\vec{w} \equiv \begin{pmatrix} -1 \\ 2 \\ -1 \\ 2 \end{pmatrix} .$$

The $(+1)$ eigenspace of B is spanned by $\{\vec{u}\}$ and $(+2)$ eigenspace of B is spanned by $\{\vec{w}\}$. (The reader should verify these statements using matrix-vector multipllication.) The former seems satisfactory but the latter seems "deficient"–we expect three eigen-directions, as "promised" by the zeros of the characteristic polynomials. This deficiency is reflective of the field of read numbers and is "remedied" by going to the field of complex numbers.

Here is smaller example. Let

$$K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

The characteristic polynomial of K is $\chi_K)(\lambda) \equiv \det(K - \lambda Id) = (1 - \lambda)^2$. The zeros of the characteristic polynomial of K are $(+1)$ with multiplicity 2. So we expect two independent eigen-directions. Yet the only eigen-direction is the span of the $(+1)$ eigenvector $\binom{1}{0}$.

The reader should verify the statements above. Again, there appears to be a "deficiency". We say that the eigenvalue $(+1)$ has *algebraic multiplicty* 2 (for the multiplicity of the root 2 of the characteristic polynomial $\chi_K(\lambda)$) and *geometric multiplicity* 1, because the $(+1)$-eigenspace is of dimension 1.

Looking back at the matrix B, the eigenvalue $(+2)$ has algebraic multiplicity 3 and geometric multiplicity 1; the matrix B, the eigenvalue $(+1)$ has algebraic multiplicity 1 and geometric multiplicity 1. The reader is encouraged to review and verify these assertions.