12. Computing Determinants Using Smaller Determinants: *The Laplace Expansion*

Pierre-Simon Laplace (1749-1827) is credited with the idea of expressing the determinant of a given matrix in terms of determinants of smaller submatrices. Here we will focus on the *idea of why* such an expression is possible, motivating it directly from the basic properties expected of the determinant function. We'll assume that the reader is familiar with the abstract properties that determine the determinant, namely, that the determinant is alternating, is multilinear (both in rows and columns), and takes on the value 1 on the Identity matrix of size $n \times n$.

We also recall that a *square* matrix T is *singular* if the system $T\vec{x} = \vec{0}$ has more than just the trivial solution $\vec{x} = \vec{0}$. (Equivalent conditions for T to be singular include that it is not *injective* or not *surjective*, or that the system $T\vec{x} = \vec{b}$ has no solution for some right hand side $\vec{b}$; indeed, any of these conditions implies the rest.) A consequence of these properties, which is useful but not mission-critical, is that the determinant of a singular matrix is always zero. (Reader: show. Hint: the square matrix T is singular $\iff$ $RREF(T)$ has a zero row.)

We first introduce a principle which will help us transition from larger to small matrices.

12.1 The All Or Nothing Principle

Recall that the *row space* of a matrix T, $RowSp(T)$, is the *span* of the row vectors of T, i.e., the set of all possible linear combinations of the rows or T. For some particular matrices T, $RowSp(T)$ consists of *all* row vectors of the same shape as the rows of T. Here is an example,, which the reader should develop:

$$T \equiv \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

In this case, we say that the row space of T is "everything".

Lemma (All or Nothing Principle)**.** *Let T be a square matrix. Then either the* row space *of T is "everything", or $RREF(T)$ has a zero row.*

Proof. If the row space of T is not "everything" then some row of $RREF(T)$ is not a pivot row. `[Reader: show this, using the given fact that` T `is square along with concepts of` *`injectivity`* `and` *`surjectivity`*`.]` In this case $RREF(T)$ must have a zero row. Conversely, if $RREF(T)$ has a zero row then $RowSp(T)$ consists of linear combinations of the $(n-1)$ top rows of $T_{n\times n}$, which cannot fill the space of all $1 \times n$ row vectors. `[Reader: elaborate.]` □

Let M be a 3×3 matrix. In a nutshell, to compute $\det(M)$, we will find an expression using 2×2 determinants that does the job. The ideas immediately carry over to the general case, showing how to compute the determinant of an $n \times n$ matrix using determinants of $(n-1) \times (n-1)$ matrices. Below is a formula that we will derive using RREF ideas. The formula is sometimes used to *define* the determinant function, and then the familiar properties we have used are proved as consequences oxf the definition. Here is the *Laplace* formula in the 3×3 context:

$$\det \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = (m_{11}) \det \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix}$$
$$- (m_{21}) \det \begin{pmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{pmatrix}$$
$$+ (m_{31}) \det \begin{pmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{pmatrix} .$$

For those who prefer numbers, here is a concrete example:

$$\det \begin{pmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{pmatrix} = (3) \det \begin{pmatrix} 1 & 6 \\ -1 & 2 \end{pmatrix}$$
$$- (4) \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
$$+ (-3) \det \begin{pmatrix} 2 & -1 \\ 1 & 6 \end{pmatrix}$$
$$= 3 \cdot 8 - 4 \cdot 3 + (-3) \cdot 13 = -27 .$$

On the right hand side above we have (determinants) of three 2×2 submatrices of the 3×3 matrix M on the left. The first submatrix is obtained from M by "masking" the 1^{st} row and 1^{st} column of M, and is denoted by $M_{1,1}$, or a similar notation. Similarly, the second and third 2×2 (sub)matrices are denoted by $M_{2,1}$ and $M_{3,1}$, respectively. The determinants of these 2×2 matrices are called *minors* of M. (Sometimes the matrices are called minors, sometimes the determinants of the matrices are called minors. Check each source with care.) There are also signs in front of the scalars m_{11}, m_{21} and m_{31}. The signs follow the alternating pattern $+ - + - \cdots$. The sign in front of each scalar (or minor) is called a *cofactor.* Check the book for terminology. Here our purpose is merely to see how such a formula follows from RREF considerations.

Before proceeding to derive the Laplace formula, we note that there are many variations on it, e.g., here we "march along the first column. There are formulas corresponding to other marching paths. For instance, we can march along any column, or any row. All such formulas may be derived using the ideas below.

We now proceed with the derivation. Take the matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} .$$

Write the leftmost column of M as a sum of three particularly simple vectors:

$$\begin{pmatrix} m_{11} \\ m_{21} \\ m_{31} \end{pmatrix} = \begin{pmatrix} m_{11} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ m_{21} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ m_{31} \end{pmatrix}.$$

Thus we can view the "first" (read: leftmost) column of M as the sum of three vectors and, using the *multilinearity* principle that the determinant of the sum is the sum of the determinants (applied to expressing the first column as a sum, and freezing all other columns), we have:

$$\begin{aligned} \det \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} &= \det \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} \\ &+ \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} \\ &+ \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \end{aligned}$$

Using multilinearity again, this time in the context of scaling the first column vector, we express the leftmost summand in this last equation as

$$\det \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} = m_{11} \det \begin{pmatrix} 1 & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix}.$$

We now wish to perform row operations to "zero-out the m_{12} and m_{13} entries of the matrix in the right hand side above. First we will row operate on the two bottom rows of this matrix. Note that row operations on $\left(\begin{smallmatrix} 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{smallmatrix}\right)$ are morally equivalent to row operations on $\left(\begin{smallmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{smallmatrix}\right)$. We simply "carry" the left zero entries along. (Reader: discuss the meaning of *moral equivalence* in mathematics.) In effect, we are row reducing a 2×2 matrix which is imbedded in a 2×3 shell.

If this 2×2 matrix is non-singular, its RREF will be $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the RREF of $\left(\begin{smallmatrix} 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{smallmatrix}\right)$ will be $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ (the identity 2×2, housed in a 2×3 shell). Then we can "use" the imbedded identity matrix to zero out the m_{12} and m_{13} entries of the 3×3 matrix above. (Reader: elaborate on this.) Therefore, we could have zeroed out the the m_{12} and m_{13} entries *without* row reducing the bottom. (Reader: justify). In this case we have

$$\det \begin{pmatrix} 1 & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix}.$$

On the other hand, if the 2×2 matrix $\left(\begin{smallmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{smallmatrix}\right)$ is singular, its RREF is of the form $\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$ (reader: elaborate) and hence

$$\begin{pmatrix} 1 & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix}$$

is row reducible to

$$\begin{pmatrix} 1 & m_{12} & m_{13} \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix},$$

which is a singular matrix, with determinant zero. Thus, in all cases we have

$$\det \begin{pmatrix} 1 & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} = \det \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix}. \tag{1}$$

The reader should gather the subplots above so as to justify the "in all cases" assertion. In particular, the equality in (1) has not been "earned" and requires a little bit of elaboration. The equality is valid because, the 3×3 matrix already has $RREF$-ready first row and first column, so computing the 3×3 determinant via row operations is entirely equivalent to computing the determinant of the lower right 2×2 matrix by row operations, with the same numerical result.

We now evaluate the next determinant in the sum expansion of $\det(M)$:

$$\begin{aligned} \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} &= (m_{21}) \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ 1 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} \\ &= (-1)(m_{21}) \det \begin{pmatrix} 1 & m_{22} & m_{23} \\ 0 & m_{12} & m_{13} \\ 0 & m_{32} & m_{33} \end{pmatrix}. \end{aligned}$$

In this last step we switched two rows and changed the sign of the determinant. Now, using reasoning above above, we express this last block-form 3×3 determinant in terms of a 2×2 determinant:

$$(-1) \det \begin{pmatrix} 1 & m_{22} & m_{23} \\ 0 & m_{12} & m_{13} \\ 0 & m_{32} & m_{33} \end{pmatrix} = (-1) \det \begin{pmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} &= (m_{21}) \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ 1 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix} \\ &= (-1)(m_{21}) \det \begin{pmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{pmatrix}. \end{aligned}$$

Finally, using similar reasoning we obtain

$$\begin{aligned} \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} &= (m_{31}) \det \begin{pmatrix} 0 & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 1 & m_{32} & m_{33} \end{pmatrix} \\ &= (-1)(m_{31}) \det \begin{pmatrix} 1 & m_{32} & m_{33} \\ 0 & m_{22} & m_{23} \\ 0 & m_{12} & m_{13} \end{pmatrix} \\ &= (-1)(-1)(m_{31}) \det \begin{pmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{pmatrix}. \end{aligned}$$

Now putting "everything" together,

$$\begin{aligned} \det \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} &= (m_{11}) \det \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix} \\ &- (m_{21}) \det \begin{pmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{pmatrix} \\ &+ (m_{31}) \det \begin{pmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{pmatrix}. \end{aligned}$$

In conclusion we remark that for large matrices the *Laplace Expansion*, also known as the *Cofactor expansion* requires much more computational work than the row reduction approach. This is dramatized by a table in the book *Linear Algebra: A Geometric Approach* by T. Shifrin and M. Adams which counts how many scalar multiplications are required to compute determinants using the Laplace expansion:

n	cofactors	row operations
2	2	2
3	6	8
4	24	20
5	120	40
6	720	70
7	5,040	112
8	40,320	168
9	362,880	240
10	3,628,800	330

Notice that for systems of size 5×5 and larger, the number of computations required for the cofactor expansion is astronomically larger than for row reduction.