13. Eigenvalues and Eigenvectors Using Row Reduction

13.1 *Looking for λ in all the row places*

Consider the matrix M:

$$\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}.$$

We are looking for *eigenvalues* of M. That is, we wish to find scalars λ for which there is a non-zero vector $\vec{x}$ with

$$\boxed{M\vec{x} = \lambda\vec{x}}.$$

i.e., the effect of multiplying the matrix M by $\vec{x}$ is equivalent to scaling $\vec{x}$ by λ. One can also restate this as follows:

The image of the vector $\vec{x}$ when multiplied by the matrix M is along the **same direction** as $\vec{x}$.

[The input vector $\vec{x}$ must be nonzero and hence have a direction. The output vector $M\vec{x}$ is forgiven if it is zero ($\vec{0}$) and then said to have the same direction as $\vec{x}$ (just in this one context–don't make a habit of it...)]

We can rewrite our goal as

$$\boxed{(M - \lambda Id_{2\times 2})\,\vec{x} = \vec{0}}. \tag{1}$$

Here $Id_{2\times 2}$ is the 2×2 identity matrix. `[Reader: verify that the two goal forms are equivalent.]` Since we wish to find a vector $\vec{x} \neq \vec{0}$ satisfying (1), we can rephrase this as the goal of

finding a non-trivial *solution* *to the homogeneous system* (1)

and this is possible precisely when the matrix $(M - \lambda Id_{2\times 2})$ is *singular, i.e., non-injective*, i.e. has an RREF with a zero row (in our square matrix context). Filling out this matrix, we are looking for scalars λ for which the following matrix is non-injective:

$$\begin{pmatrix} -1-\lambda & 2 \\ -6 & 6-\lambda \end{pmatrix}.$$

We perform row operations $II \to II/(-6)$ followed by $I \to I + (1+\lambda)II$ to obtain:

$$\begin{pmatrix} 0 & 2 + \frac{(1+\lambda)(\lambda-6)}{6} \\ 1 & \frac{\lambda-6}{6} \end{pmatrix}. \tag{2}$$

Recall that row operations preserve injectivity and non-injectivity properties of square matrices and also solutions of associated homogeneous systems (and *surjectivity/non-surjectivity* and *singularity* properties too). From our experience with row operations, or consultation with Sherlock Holmes, we note that (2) is non-injective when the top row is zero. `[Reader: explain in full why the matrix in (2) is singular, or non-injective, precisely when the top row is purely zero.]` We write

$$2 + \frac{(1+\lambda)(\lambda-6)}{6} = 0.$$

This occurs when $2 \cdot 6 + (1+\lambda)(\lambda-6) = 0$, i.e., when $\lambda^2 - 5\lambda + 6 = 0$, and this holds precisely when $\lambda = 2$ or $\lambda = 3$.

`We pause to reflect and observe that our (presumably) purely linear algebraic considerations have led us to a non-linear equation: a` *`quadratic equation`* `(for λ). We shall have occasion to re-reflect on this point.`

When $\lambda = 2$ the matrix (2) is:

$$\begin{pmatrix} 0 & 0 \\ 1 & \frac{-4}{6} \end{pmatrix}.$$

which is row equivalent to

$$\begin{pmatrix} 0 & 0 \\ 3 & -2 \end{pmatrix}. \tag{3}$$

Although (3) is not quite in RREF, we can already see that the solutions of the associated homogeneous system to (3) are:

$$\operatorname{span}\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}. \tag{4}$$

Thus (4) gives the 2-eigenspace of the matrix M.

Recall that the 2-eigenspace of M is the set of all vectors $\vec{x}$ ($\vec{0}$ included) for which

$$M\vec{x} = 2 \cdot \vec{x} \quad .$$

Similarly, the 3-eigenspace of M is given by:

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}. \tag{5}$$

We encourage the reader to do the row reduction work leading to (5). Thus M has a basis of eigenvectors, an *eigenbasis*:

$$\left\{ \vec{b}, \vec{c} \right\}, \text{ where } \vec{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Using this basis we have *diagonalized* M into the following form:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

What do we mean by this? Well, every 2×1 vector $\vec{v}$ may be written as $\alpha\vec{b} + \gamma\vec{c}$, where the scalars α, γ are uniquely determined by $\vec{v}$. (This is the *basis* property of $\{\vec{b}, \vec{c}\}$.) Now the action of the matrix M on the vector $\vec{v}$ is

$$M\vec{v} = M(\alpha\vec{b} + \gamma\vec{c}) = 2 \cdot \alpha\vec{b} + 3 \cdot \gamma\vec{c}.$$

Thus, in the basis $\{\vec{b}, \vec{c}\}$, M acts like the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, scaling each basis vector by the respective eigenvalue.

13.2 *Eigenvectors by RREF–in general*

Let T be an $n \times n$ matrix. To find eigenvectors for T using RREF, follow the procedure below.

- Form the matrix $S \equiv S(\lambda) \equiv T - \lambda Id_{n \times n}$.
- Find all values of λ for which $S(\lambda)$ is non-injective.
- For each value of λ with $S(\lambda)$ non-injective, find solutions of the homogeneous system $S(\lambda)\vec{x} = \vec{0}$. This is the λ–eigenspace of T; the nontrivial solutions are the λ-eigenvectors of T.

Note that the middle step leads to the need to solve an n^{th} degree polynomial equation in λ.

13.3 *An Existential Question*

Does every square matrix have eigenvalues? The answer is no in the context of real numbers. Let

$$J \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Take a would-be eigenvector for J, $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then

$$J\vec{v} = \lambda\vec{v}\,; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} b \\ -a \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

The rightmost equation says $b = \lambda a, , -a = \lambda b$. Thus $b = \lambda(-\lambda b)$ or $b = -\lambda^2 b$. If b is nonzero, we can divide by it and obtain $1 = -\lambda^2$, which is impossible over the real numbers. Hence $b = 0$ and since $-a = \lambda b$, a is zero as well. So our would-be eigenvector for J, $\vec{v}$ must be identically zero, and hence it does not qualify for eigenvector status. So J has no eigenvectors and hence no eigenvalues.

We can reach this conclusion by row operation methods as well. Note that $J - \lambda Id$ is the matrix

$$J \equiv \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix},$$

which is row equivalent to the matrix below via the row operation $I \longrightarrow I - \lambda II$:

$$J \equiv \begin{pmatrix} 0 & 1 + \lambda^2 \\ -1 & -\lambda \end{pmatrix}. \tag{6}$$

When is (6) non-injective? Never, since the top right entry of (6) is never zero (for λ a real number, that is). This suggests extending the analysis to the complex domain. Sticking to the reals for the moment, we note that J is a *skew-symmetric* materix: $J^t = -J$. *Symmetric* matrices, on the other hand, (M is symmetric if $M^t = M$) are *guaranteed* to have eignevectors, with eigenspaces as ample as one could hope for. And there is a marvelous, calculus-based proof this by Herb Wilfe [1].

References

[1] Herb Wilfe, Amer. Math. Monthly, *An algorithm-inspired proof of the spectral theorem in E^n*, 1981.