08.Calming the Column Space
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Relaxing the Row Space

Here we will illustrate and get to know the concepts of *column space* and *row space* for a matrix. We begin with telegraphic mention of needed concepts. The approach is informal and readers should consult the book and other short readings for more detailed exposition, discussion and review of these important items. We will typeset these telegraphic preliminaries in smaller type, not to bring business to optometrists, but rather to encourage the reader to consult deeper sources.

8.1 What's a Vector Space?

A *vector space* V (book section: Vector Space Properties) is a set, a sandbox of objects we call *vectors*, with defined operations of *vector addition* and *scaling a vector by a number*, and with *"expectations"* on these operations, e.g., vector addition is commutative ($\vec{u}+\vec{v}=\vec{v}+\vec{u}$) and associative ($[\vec{a}+\vec{b}]+\vec{c}=\vec{a}+[\vec{b}+\vec{c}]$), etc. In this reading we will work with the vector space of all 4×1 column vectors and the vectors space of all 1×4 row vectors. The operations in each of these vectors spaces is defined entry by entry. Using the abstract concept of a *vector space* enables us to treat these two examples, and many others, in the same vein, furthering understanding and efficiency.

8.2 Linear Combinations and Span

The noun *linear combination* refers to a collection of vectors in a vector space. It consists of scaling each vector by a number (a *scalar*) and adding the results. Thus, if $\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_{641}$ are vectors in a vector space, a linear combination of these vectors is an expression of the form

$$t_1\vec{w}_1 + t_2\vec{w}_2 + \ldots + t_{641}\vec{w}_{641},$$

where $t_1, t_2, \cdots, t_{641}$ are ("any") numbers. We now assert:

A linear combo of linear combos is a linear combo.

The reader is urged to expand and fully deploy this thought. The *span* of a set of vectors in a vector space is the set of all possible linear combinations of these vectors. Thus, the span of $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_{641}\}$ is the set of all vectors of the form $t_1\vec{w}_1+t_2\vec{w}_2+\ldots+t_{641}\vec{w}_{641}$, where the scalars t_i range over all possible 641-tuples of choices of real numbers.

8.3 Column Space and Row Space

Definition. The *column space* of the matrix M is the span of the column vectors of M. The *row space* of the matrix M is the span of the row vectors of M.

Take the 4×4 *walking around matrix*, W_4, or just W for short.

$$W \equiv W_4 \equiv \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Consider a linear system of the form $W\vec{x}=\vec{b}$, where $\vec{x}$ is a column vector of unknowns x_1, x_2, x_3, x_4. Expanding the notation we have the system

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \tag{1}$$

Expanding the LHS of (1) we have

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = b_1 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = b_2 \\ 9x_1 + 10x_2 + 11x_3 + 12x_4 = b_3 \\ 13x_1 + 14x_2 + 15x_3 + 16x_4 = b_4 \end{cases}.$$

We can also write this as a linear combination of the columns of W:

$$x_1 \begin{pmatrix} 1 \\ 5 \\ 9 \\ 13 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 6 \\ 10 \\ 14 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 7 \\ 11 \\ 15 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 8 \\ 12 \\ 16 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}. \tag{2}$$

Thus the system $W\vec{x}=\vec{b}$ may be re-interpreted to state:

Find scalars to express the vector $\vec{b}$ as a linear combination of the column vectors of the matrix W.

The unknowns to be found, x_1, x_2, x_3, x_4, respectively, are precisely these scalars. Hence, the *span of the column vectors of the matrix W, ColSp(W)*, is the set of all linear combinations of the columns of W and, at the same time, the set of all right hand sides $\vec{b}$ for which the linear system $W\vec{x}=\vec{b}$ is *consistent.* The same logic applies to any matrix, not just W. Let's boast about this.

> Let M be a matrix. Then the *column space of* M, ColSp(W), is, by definition, the set of all linear combinations of the columns of M. At the same time, ColSp(M) is the set of all right hand side vectors $\vec{b}$ for which the linear system $M\vec{x}=\vec{b}$ is consistent.

8.4 Bases for Column and Row Spaces

We seek a concrete and precise way to describe the column space of a matrix. First observe that

Row operations do not preserve the column space of a matrix.

(We read this mathematically: a particular row operation may preserve the row space of a particular matrix, but in general we cannot expect the column space to be preserved. For example, if $T = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, apply the row operation $II \to II - I$ to obtain $Q = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The column space of T is the set of all column vectors of the form $\left(\begin{smallmatrix} t \\ t \end{smallmatrix}\right)$, where t is a real number. The column space of Q is the set of all column vectors of the form $\left(\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right)$, where t is a real number. Clearly, the above row operation on T does not preserve the column space. On the other hand, the row operation $I \to 5I$ applied to T does preserve row operations. On the other hand, we assert

Row operations preserve the row space of a matrix.

This is because row operations replace rows of M by linear combinations of rows of M in a reversible way. (Reader: elaborate.) So, does $RREF(W)$, a result of a series of row operations applied to W, tell us nothing about the column space of W? On the contrary! It tells us lots.

The reader can work out the following:

$$RREF(W) = \begin{pmatrix} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (3)$$

where we have adorned the pivot 1s with boxes to fit. Now leveraging the *Gauche perspective* on RREF (see the "Gauche" Really Short Reading, or the arXiv posting `https://arxiv.org/abs/2005.06275`), we declare:

The columns of W corresponding to the pivot columns of $RREF(W)$ form a basis $\mathcal{B}$ for the column space of W.

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The non-pivot columns of $RREF(W)$ give recipes for presenting the corresponding columns of W as linear combinations of said basis. $\mathcal{B}$.

Indeed, this assertion is at the heart of the Gauche way to produce $RREF(W)$. Inspecting (3), we'll see that the first two columns of W for a *basis*(a really good conept; see below) for the column space of W, that the third column of W is presentable as (-1) times the first column plus (2) times the second column of W, and that the fourth column of W is presentable as (-2) times the first column of W plus (3) times the second column of W. (Reader: verify directly and explicitly, not relying on Gauche wisdom.) Hence, looking at linear combinations of the column vectors of W, any linear combo involving all four columns can be rewritten as a linear combo of just the first two. (Reader: verify.) Hence the span of the columns of W equals the span of the set of just the first two columns of W. Moreover, this last spanning set is "lean", a *basis*: it spans each vector in our column space, and each vector so spanned can only be spanned in one way; there is no room for duplication here.

Exercise Inspect the columns of W again. Notice that in each column, the second entry is the average of the first and third entries, and the third entry is the average of the second and third. Prove that these two conditions (with nothing else added) describe precisely the column space of W. Use the basis $\mathcal{B}$ above to manifest the proof.

What about the row space $RowSp(W)$? This is the same as $RowSp(RREF(W))$, hence, from (3), we see that the first two rows of $RREF(W)$ are pivot rows and they form a basis of $RowSp(W)$, which is the set of all linear combinations of these two rows, each resulting row vector being presentable in one and only one way as such a linear combination.