10. Great Expectations for the Determinant

- $M \longrightarrow \det(M)$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow ad - bc$
- $\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \longrightarrow$ $dgjm - chjm - dfkm + bhkm + cflm - bglm - dgin + chin + dekn - ahkn - celn + agln + dfio - bhio - dejo + ahjo + belo - aflo - cfip + bgip + cejp - agjp - bekp + afkp$

10.1 Asking for alot

We would like a systematic way to assign a single real number t to a (square) matrix M so that important properties of M (whatever those may be) can be read off from the single number t. There are many ways to do this, and no one way does everything we may desire. The *determinant* is one approach, and it provides a great deal of insight. For a competitor, check out the concept of *trace*, coming up a little later. And by the way, why restrict to *square* matrices? So we will define a function called $\det(\cdot)$, from *square* matrices to scalars:

$$M \longrightarrow \det(M).$$

10.2 Great Expectations

We impose a bunch of expectations on the det of a square matrix (M or L), and following the *name it to tame it* principle, we will assign these names; we will also illustrate and elaborate on their meanings.

- *Normalized:* $\det(Id_{n\times n}) = 1$.
- *Alternating:*
  If L is obtained from M by switching two rows, then $\det(L) = -\det(M)$.
- *Multi-linear:*
  The function $\det(M)$ is linear in a given row
  if all other rows are "frozen".

What do we mean by these properties? Let's first illustrate in the context of small cases. Then we can elaborate and generalize.

The *normalization* condition is simple enough. In the 2×2 case it simply says:

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

This is like setting a standard unit of length, or mass, or temperature. [The reader is invited to challenge this point later on, when we have exhibited many more properties of $\det(\cdot)$.]

The *alternating* condition is also simple to visualize in the 2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \qquad (1)$$

i.e., "switching rows changes the sign of the determinant", though this statement must be parsed carefully. Notice a consequence of (1):

If a (square) matrix has two equal rows then its determinant is zero.

We'll say more about this later on.

Multilinearity, the third condition, says that, viewing $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a function of a, b, with c, d treated as fixed constants, this is a *linear* function of a, b:

$$\det \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{pmatrix} = \det \begin{pmatrix} a_1 & b_1 \\ c & d \end{pmatrix} + \det \begin{pmatrix} a_2 & b_2 \\ c & d \end{pmatrix},$$

i.e., $\det(\cdot)$ "respects" row addition. As part of linearity, we also expect $\det(\cdot)$ to respect scaling. For any scalar λ, we expect

$$\det \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix} = \lambda \cdot \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (2)$$

Note that in (2) we are scaling the row, i.e., scaling each row entry by *the same* scalar. Observe that throughout this *mutilinearlity* discussion the second row has been fixed: $\begin{pmatrix} c & d \end{pmatrix}$. This is what we mean by the second row being *frozen*.

We say that $\det(\cdot)$ is *linear* in the first row if the second row is frozen. So why the adjective *multilinear*? Because we also expect $\det(\cdot)$ to be linear in the second row if the first row is frozen.

10.3 Supersized determinants

At this point we could say "and so forth for larger sized square matrices". But we have to be careful. For $M_{n\times n}$ we expect $\det(M)$ to be *linear* in a given row of M if **all other rows** of M are frozen. For instance,

$$\det\begin{pmatrix} a_1+\lambda a_2 & b_1+\lambda b_2 & c_1+\lambda c_2 \\ d & e & f \\ g & h & i \end{pmatrix}$$
$$= \det\begin{pmatrix} a_1 & b_1 & c_1 \\ d & e & f \\ g & h & i \end{pmatrix} + \lambda \det\begin{pmatrix} a_2 & b_2 & c_2 \\ d & e & f \\ g & h & i \end{pmatrix},$$

where we have sneakily combined "respecting" row addition and row scaling in one fell swoop. We remind the reader that, while we have illustrated linearity in the first row while the others are frozen, we expect *multilinearity*: linearity in each row, when all other rows are frozen.

10.4 Nagging questions

All these expectations are fine and good. But is there a function that meets such fussy requirement? At the risk of spoiling the plot, we answer in the affirmative. In the 2×2 case we have the definition

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv ad - bc,$$

and we invite the reader to verify that all expectations are met. In fact, this definition is *forced upon us* by the fundamental properties expected of the determinant, and it's not difficult to show that.

What about determinants of larger sized square matrices? There are multiple approaches to these, including formulas, recursions, and algorithms. We'll elaborate on these elsewhere, emphasizing that we'll always think back to the three fundamental properties itemized above. We make salient

> The Gauss-Jordan elimination process
> that yields $RREF(M_{n\times n})$
> also yields a computation of $\det(M)$.

We will provide illustrations of this fact elsewhere, but here is some motivation. As we row reduce M, the workhorse row operations will preserve the determinants of the corresponding matrices. (Reader: elaborate.) If we switch two rows, we change the determinant by a sign. If we scale a row, we scale the determinant by the same scalar. So we can "track" or "journal" the effect of our row reduction steps on the determinant of the matrices involved. When we reach $RREF(M)$, if it turns out to be $Id_{n\times n}$ then we know that the value of the final determinant is 1, and we just have to account for the "journaled" steps we took to get there.

What if $RREF(M)$ is not the identity matrix? Then $RREF(M)$ must have a zero row and, by multilinearity $\det(RREF(M)) = 0$ hence $\det(M) = 0$. Since the invertible matrices are precisely the ones whose $RREF$ is the identity, we have (sort of) proven a familiar sounding fact:

Theorem. *Let M be a square matrix. Then M is invertible if and only if* $\det(M) \neq 0$.

Note that this assertion is quite useful as a *fact*, but does not serve well as a *definition* of the concept of matrix invertibility; that concept is best defined by the "undoing" of the action of a matrix viewed as a function.

We conclude by asserting that the determinant has the same fundamental properties for columns as it does for rows:

> The determinant function is
> multilinear and alternating
> in columns, as well as in rows.