Chapter 1-Linear Systems

A **system of linear equation** or **linear system** is a collection of linear equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n = & b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n = & b_3 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n = & b_m \end{cases} \tag{1}$$

1. it has m equations, n variables.$m > n$ or $m \le n$
2. **A solution of the linear system:** n-tuple $(x_1, \cdots, x_n)$ satisfies all Eqs.
3. **Solution sets:** all solutions
4. **Equivalent linear systems:** if the solution sets of two linear systems are the same, these two linear systems are equivalent. 3 cases.

Case 1. Interchange the position of two equations.

$$\begin{cases} 3x_1 - 5x_2 + 8x_3 = -4 \\ 1x_1 + 2x_2 - 4x_3 = 5 \\ -2x_1 + 6x_2 + 1x_3 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 1x_1 + 2x_2 - 4x_3 = 5 \\ 3x_1 - 5x_2 + 8x_3 = -4 \\ -2x_1 + 6x_2 + 1x_3 = 3 \end{cases}$$

Case 2. Multiply an equation by a nonzero constant

$$\begin{cases} 3x_1 - 5x_2 + 8x_3 = -4 \\ 1x_1 + 2x_2 - 4x_3 = 5 \\ -2x_1 + 6x_2 + 1x_3 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 6x_1 - 10x_2 + 16x_3 = -8 \\ 1x_1 + 2x_2 - 4x_3 = 5 \\ -2x_1 + 6x_2 + 1x_3 = 3 \end{cases}$$

Case 3. Add a multiple of one equation to another

$$\begin{cases} 3x_1 - 5x_2 + 8x_3 = -4 \\ 1x_1 + 2x_2 - 4x_3 = 5 \\ -2x_1 + 6x_2 + 1x_3 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 3x_1 & -5x_2 + 8x_3 = -4 \\ 1x_1 & +2x_2 - 4x_3 = 5 \\ & 10x_2 - 7x_3 = 13 \end{cases}$$

Write linear systems in the matrix form $A\mathbf{x} = \mathbf{b}$

Linear system (1) can be written as :

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \quad \textbf{OR} \quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} (A\mathbf{x} = \mathbf{b})$$

The above left matrix is called **augmented matrix**. **Coefficient matrix** $= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

Elementary row operations for matrices:

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Add a multiple of one row to another row

Equivalent matrices:

if one matrix can be obtained from the other by applying a sequence of elementary row operations.

$$\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases} \xRightarrow[\text{matrix}]{\text{augmented}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -5 & -1 & 2 \\ -4 & 13 & -12 & 11 \end{array}\right]$$

$$\xRightarrow[\text{with}]{\text{equivalent}} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xRightarrow[\text{linear system}]{\text{corresponding}} \begin{cases} x_1 - 3x_2 + 2x_3 & = -1 \\ 0x_1 + x_2 - 5x_3 & = 4 \\ 0x_1 + 0x_2 + x_3 & = 3 \end{cases} \xRightarrow{\text{solution}} \begin{cases} x_1 & = 50 \\ x_2 & = 19 \\ x_3 & = 3 \end{cases}$$

Question: what is the solution for the original linear system?

Every type of elementary row operation to the augmented matrix corresponds to one type of equivalent linear systems:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -5 & -1 & 2 \\ -4 & 13 & -12 & 11 \end{array}\right] \xRightarrow[\text{linear system}]{\text{corresponding}} \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases}$$

$$\xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -5 & 4 \\ -4 & 13 & -12 & 11 \end{array}\right] \xrightarrow[\text{to Eq.2}]{\text{(-2)} \times \text{Eq.1 is added}} \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 0x_1 + 1x_2 - 5x_3 = 4 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases}$$

$$\xrightarrow{4R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -5 & 4 \\ 0 & 1 & -4 & 7 \end{array}\right] \xrightarrow[\text{to Eq.3}]{\text{(4)} \times \text{Eq.1 is added}} \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 0x_1 + 1x_2 - 5x_3 = 4 \\ 0x_1 + 1x_2 - 4x_3 = 7 \end{cases}$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[\text{to Eq.3}]{\text{(-1)} \times \text{Eq.2 is added}} \begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 0x_1 + 1x_2 - 5x_3 = 4 \\ 0x_1 + 0x_2 + 1x_3 = 3 \end{cases}$$

Rewrite:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xRightarrow[\text{linear system}]{\text{corresponding}} \begin{array}{cccc} x_1 & -3x_2 & +2x_3 & =-1 \\ & x_2 & -5x_3 & =4 \\ & & x_3 & =3 \end{array}$$

Then use "**Back-Substitution** " to solve for the solution sets:

$x_3 = 3 \qquad \Rightarrow x_2 = 4 + 5x_3 = 19 \qquad \Rightarrow x_1 = -1 - 2x_3 + 3x_2 = 50$

Therefore, the solution is $\begin{cases} x_1 & = 50 \\ x_2 & = 19 \\ x_3 & = 3 \end{cases}$

Solve Linear System by Gaussian Elimination

Some concepts about a matrix:

1. Leading Term: the leading term of a row is the leftmost nonzero term in that row.
2. **(Row) Echelon Form (REF):** the matrix is in echelon form if
   - Every leading term is in a column to the left of the leading term of the row below it.
   - Any zero rows are at the bottom of the matrix

   Note: the above two conditions imply that in echelon form will have zeros filling out the column below each of the leading terms

Example (1)

Are the following matrices are in echelon form? if not, tell why

1. $\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$ (Yes)

2. $\begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (No. Leading term of R_2 is to the right (not left) of the leading term of R_3)

3. $\begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (Yes)

4. $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (No, Zero row is not at the bottom)

5. $\begin{bmatrix} 1 & 5 & 2 & 9 \\ 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (Yes)

6. $\begin{bmatrix} 2 & -3 \\ 0 & 7 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (Yes)

Gaussian Elimination: Write the linear system in augmented matrix form, reduce this matrix into row echelon form by elementary operations, then solve it in backwards way

Example (2)

Find the set of solutions for the given linear system

$$\begin{cases} 2x_1 + 6x_2 - 9x_3 - 4x_4 = 0 \\ -3x_1 - 11x_2 + 9x_3 - x_4 = 0 \\ x_1 + 4x_2 - 2x_3 + x_4 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 6 & -9 & -4 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 1 & 4 & -2 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 1 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 2 & 6 & -9 & -4 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + 3R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & -2 & -5 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + 2R_2 \to R_3} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right].$$ *4 variables, but just 3 "useful" equations, there is* **ONE free variable**.

BACKWARD WAY *to select a free variable:* $x_4 = s$.

BACKWARD WAY *to solve for the solution set:*
$x_4 = s, \quad x_3 = 2x_4 = 2s, \quad x_2 = -2x_4 - 3x_2 = -8s, \quad x_1 = -x_4 + 2x_3 - 4x_2 = 35s.$

Therefore, $\boxed{x_1 = 35s, x_2 = -8s, x_3 = 2s, x_4 = s, s \in \mathbf{R}}$. *Infinitely many solutions.*

Different value for s **leads to different solution for the system**. *Eg.* $s = 0 \Rightarrow$ *trivial solution.*

Number of solutions for linear systems

For any linear system, there must be a unique (the system is independent), or infinity many (the system is dependent) or no solution(s). The system is consistent if it has solution (unique or ∞), inconsistent if it doesn't have any solution.

$$\begin{cases} 6x_1 - 10x_2 = 0 \\ -3x_1 + 5x_2 = 8 \end{cases}$$

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 6 & -10 & 0 \\ -3 & 5 & 8 \end{array}\right]$$

$\Downarrow$ echelon form

$$\left[\begin{array}{cc|c} 6 & -10 & 0 \\ 0 & 0 & 8 \end{array}\right]$$

$$\Downarrow$$

$0 = 8$ (*false*)

No solution!

$$\begin{cases} 4x_1 + 10x_2 = 14 \\ -6x_1 - 15x_2 = -21 \end{cases}$$

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 4 & -10 & 14 \\ -6 & -15 & -21 \end{array}\right]$$

$\Downarrow$ echelon form

$$\left[\begin{array}{cc|c} 4 & -10 & 14 \\ 0 & 0 & 0 \end{array}\right]$$

We have a free variable, **BACKWARD WAY** to select the free variable: $x_2 = s$, then solve $x_1 = \frac{7+5s}{2}$, $s \in \mathbb{R}$.

infinitely many solution!

$$\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ 2x_1 - 5x_2 - x_3 = 2 \\ -4x_1 + 13x_2 - 12x_3 = 11 \end{cases}$$

$\Downarrow$ echelon form

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

BACKWARD WAY to solve it: $x_3 = 3$, then solve $x_2 = 19$, then solve $x_1 = 50$

unique solution!

Homogeneous linear systems:

In linear system (1), if all right hand sides are 0s, i.e $b_1 = b_2 = \cdots = b_m = 0$, this linear system is called homogeneous linear system. Any homogeneous system has at least one trivial solution

$$x_1 = 0, x_2 = 0, \cdots, x_n = 0$$

Solve Linear System by Gaussian-Jordan Elimination

Suppose that the matrix is already in echelon form,

1. Pivot positions : are those that contain a leading term
2. Pivot: is a nonzero number in a pivot position
3. Pivot columns: are the columns that contain pivot positions

Reduced (Row) Echelon Form(RREF): a matrix is in Reduced (Row) Echelon Form if

- It is in echelon form.
- All pivots are 1.
- The only nonzero term in a pivot column is in the pivot position

eg. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Example (3)

For the following matrices, identify all pivots, pivot columns, then determine whether they are in RREF.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
(Yes)

$$\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
(Yes)

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \text{(No)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{(No)}$$

Gaussian-Jordan Elimination: Write the linear system in augmented matrix form, reduce this matrix into reduced row echelon form by elementary operations, then solve it.

Example (2-Continue)

Use Gaussian-Jordan to find the set of solutions for the given linear system

$$\begin{cases} 2x_1 + 6x_2 - 9x_3 - 4x_4 = 0 \\ -3x_1 - 11x_2 + 9x_3 - x_4 = 0 \\ x_1 + 4x_2 - 2x_3 + x_4 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 6 & -9 & -4 & 0 \\ -3 & -11 & 9 & -1 & 0 \\ 1 & 4 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_1+2R_3\to R_1]{R_2-3R_3\to R_2} \left[\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_1-4R_2\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -35 & 0 \\ 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \text{ (RREF)}$$

$x_4 = s, \quad x_3 = 2x_4 = 2s, \quad x_2 = -8x_4 = -8s, \quad x_1 = 35x_4 = 35s$

Therefore, $\boxed{x_1 = 35s, x_2 = -8s, x_3 = 2s, x_4 = s, s \in \mathbf{R}}$. *Infinitely many solutions.*

Different value for s leads to different solution for the system. *Eg.* $s = 0 \Rightarrow$ *trivial solution.*

Application-Curve fitting

Need to know: through n points, we can find a **unique** polynomial of degree $n-1$ that fits all points. eg. two points determines one line $f(x) = ax + b$, which is a polynomial of degree 1.

Example (4)

Find the quadratic polynomial whose graph goes through the points $(-2, 10)$, $(0, 6)$, and $(2, 18)$

Solution

Any quadratic polynomial can be written as: $y(x) = ax^2 + bx + c$, *we need to find* a, b, c

$$\begin{cases} 4a - 2b \quad +c = 10 & (-2, 10) \\ \qquad\qquad c = 6 & (0, 6) \\ 4a + 2b \quad +c = 18 & (2, 18) \end{cases} \iff \begin{bmatrix} 4 & -2 & 1 \\ 0 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 18 \end{bmatrix} \iff \left[\begin{array}{ccc|c} 4 & -2 & 1 & 10 \\ 0 & 0 & 1 & 6 \\ 4 & 2 & 1 & 18 \end{array}\right]$$

$$\xrightarrow{R_3-R_1\to R_3} \left[\begin{array}{ccc|c} 4 & -2 & 1 & 10 \\ 0 & 0 & 1 & 6 \\ 0 & 4 & 0 & 8 \end{array}\right] \xrightarrow[\frac{1}{4}R_2\to R_2]{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|c} 4 & -2 & 1 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{array}\right] \Longrightarrow \begin{cases} a = \dfrac{10 - c + 2b}{4} = 2 \\ b = 2 \\ c = 6 \end{cases}$$

Therefore, $\boxed{f(x) = 2x^2 + 2x + 6}$