ROBO 5000 Fall 2025
Instructor: Dr. Leo Beuken

CP4 Report: Dynamics & Control

Due Date .... December 5, 2025
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Contents

1 Introduction

Challenge Problem 4 involved deriving the equations of motion for a inverted pendulum cart, simulating the nonlinear differential equations, and creating an animation showing the result. Additionally, a controller was implemented that is capable of keeping the pendulum within 20° of the vertical for a period of 5 seconds given a random initial angle. The results were plotted for three different initial conditions.

2 Approach

The equations of motion were derived for the system. First the motion of the cart was considered. Summing the forces on the cart resulted in the following:

$$F - F_b + R_x = M\ddot{x}$$

Where F_b is the force due to friction and and R_x is the x component of the reaction force at the pendulum pivot. Using the known equation for force due to friction results in:

$$F - b\dot{x} - R_x = M\ddot{x}$$

Next, consider the position of the point mass at the end of the rod, expressed in the inertial frame.

$$\vec{r_p} = (x - l\sin\theta)\hat{x_i} + (l\cos\theta)\hat{y_i}$$

Taking a derivative of this equation with respect to time yields the velocity of the point mass, expressed in the inertial frame.

$$\vec{v_p} = (\dot{x} - \dot{\theta}l\cos\theta)\hat{x_i} + (-\dot{\theta}l\sin\theta)\hat{y_i}$$

Taking a second derivative with respect to time yields the acceleration of the point mass expressed in the inertial frame.

$$\vec{a_p} = (\ddot{x} - \ddot{\theta}l\cos\theta + \dot{\theta}^2 l\sin\theta)\hat{x_i} + (-\ddot{\theta}l\sin\theta - \dot{\theta}^2 l\cos\theta)\hat{y_i}$$

Consider the sum of forces on the point mass in the x-direction:

$$R_x = ma_{p,x}$$

Substituting in the expression for the x component of the acceleration:

$$R_x = m(\ddot{x} + \dot{\theta}^2 l\sin\theta - \ddot{\theta}l\cos\theta)$$

Substituting this result back into the sum of forces on the cart results in the following equation:

$$(M+m)\ddot{x} + b\dot{x} - m\dot{\theta}^2 l\sin\theta + m\ddot{\theta}l\cos\theta = F$$

In order to obtain the second equation of motion, consider an orthonormal unit vector that is perpendicular to the pendulum at all times.

$$\hat{x}_B = \cos\theta\hat{x_i} + \sin\theta\hat{y_i}$$

Recall the equation of motion for the point mass:

$$\sum F = m\vec{a}$$

Where $\vec{a}$ has already been derived for the point mass and the forces acting on the point mass are just the reaction force and the force due to gravity. Next, both sides of this equation are dotted with the aforementioned orthogonal unit vector. The result for the left hand side is:

$$\hat{x}_B \cdot (R_x\hat{x}_i + R_y\hat{y}_i - mg\hat{y}_i)$$

Since the entire reaction force vector is orthogonal to the unit vector, by definition, all of that portion falls out and is equal to zero. Thus the result on the left and side is simply the magnitude of the unit vector, which is one, times the magnitude of the weight vector, times the sine of the angle. So, the result on the left hand side is:

$$-mg\sin\theta$$

Next, computing the dot product of the unit vector with the previously derived acceleration vector results in:

$$m\hat{x}_B \cdot ((\ddot{x} - \ddot{\theta}l\cos\theta + \dot{\theta}^2 l\sin\theta)\hat{x}_i + (-\ddot{\theta}l\sin\theta - \dot{\theta}^2 l\cos\theta)\hat{y}_i)$$

$$m(\cos\theta\hat{x}_i + \sin\theta\hat{y}_i) \cdot ((\ddot{x} - \ddot{\theta}l\cos\theta + \dot{\theta}^2 l\sin\theta)\hat{x}_i + (-\ddot{\theta}l\sin\theta - \dot{\theta}^2 l\cos\theta)\hat{y}_i)$$

$$m(\ddot{x}\cos\theta - \ddot{\theta}l\cos^2\theta + \dot{\theta}^2 l\sin\theta\cos\theta - \ddot{\theta}l\sin^2\theta - \dot{\theta}^2 l\sin\theta\cos\theta)$$

$$m(\ddot{x}\cos\theta - l\ddot{\theta})$$

Equating the results of the LHS and RHS yields the second equation of motion for the system.

$$l\ddot{\theta} = g\sin\theta - \ddot{x}\cos\theta$$

The next step was create a system of first order differential equations. This involved first substituting the equation for $\ddot{\theta}$ into the equation for $\ddot{x}$, and then substituting that entire result back into the equation for $\ddot{\theta}$, yielding two equations, each with only one second order derivative. Then, the two equations were split into a system of four first order differential equations, given below.

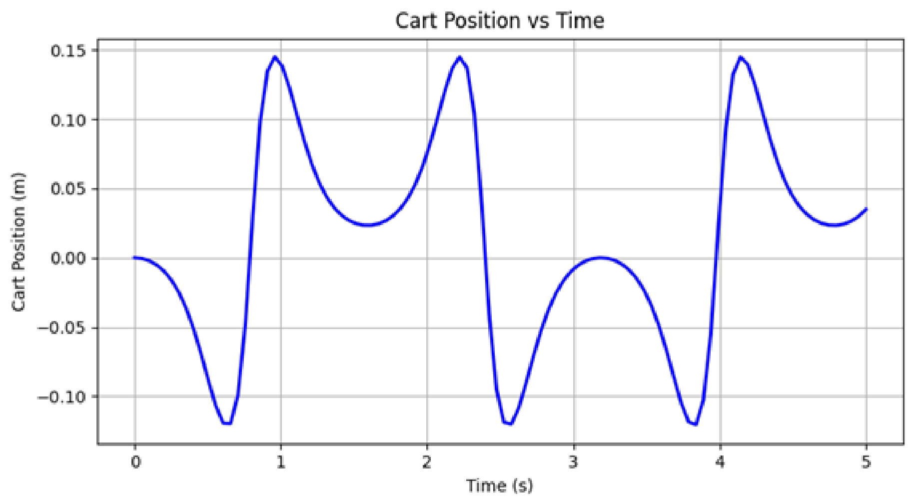
$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{F - bx_2 + mx_4^2 l\sin x_3 - mg\sin x_3\cos x_3}{(M+m) - m\cos^2 x_3}, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= \frac{g\sin x_3}{l} + \frac{\cos x_3}{l}\left(\frac{F - bx_2 + mx_4^2 l\sin x_3 - mg\sin x_3\cos x_3}{(M+m) - m\cos^2 x_3}\right).
\end{aligned}$$

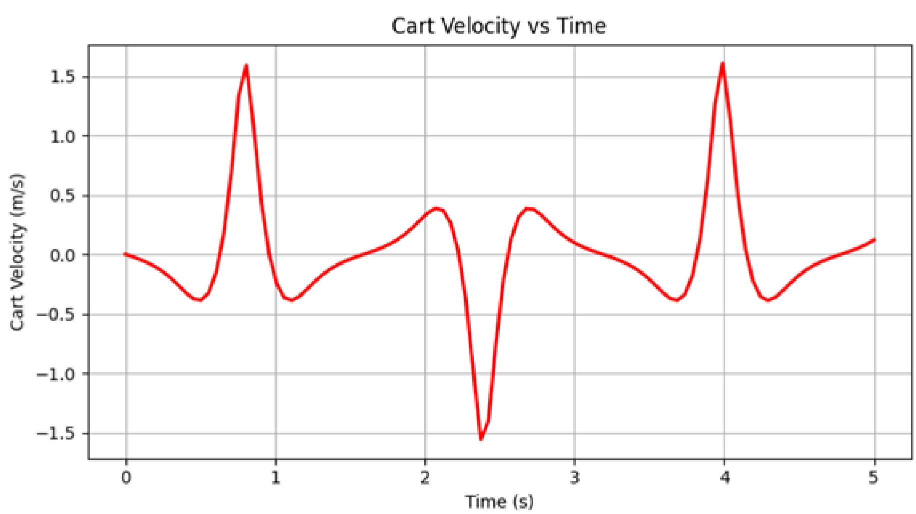
When tackling the second portion, the system had to be linearized in order to create a controller to keep the pendulum within bounds. Utilizing the Python controls toolbox, an LQR controller was created with the linearized system. In the cost, the angle was highly penalized in order to maintain the 20° bounds on the pendulum movement. The system was simulated a 0°, 2°, 5°, 8°, and 10°

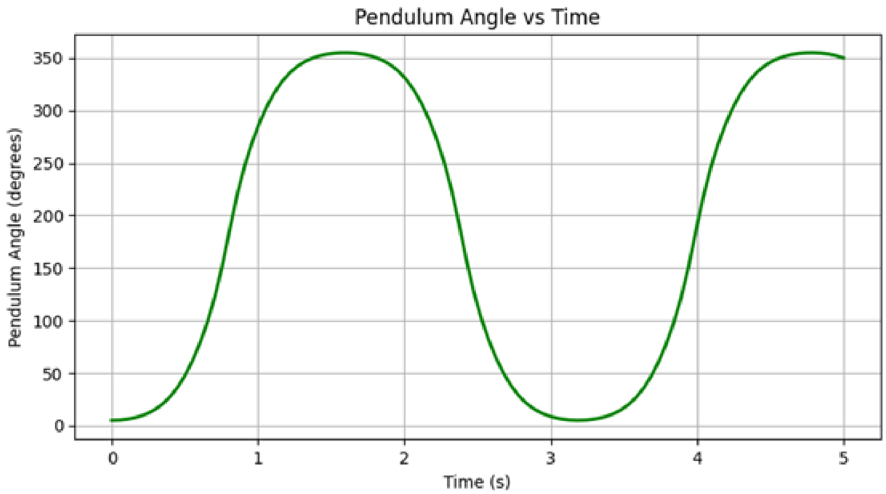
For the bonus question, the controller was adjusted to highly penalize the angle of the pendulum and have zero initial conditions for the system. The step input was added as well.

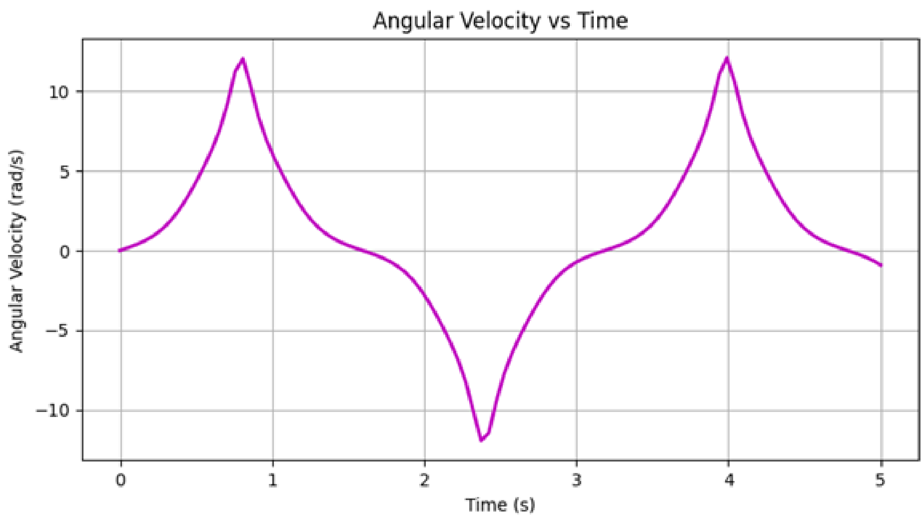
3 Results

The nonlinear differential equations were simulated in Python for a duration of 5 seconds with a 5° initial angle perturbation away from the vertical. The video showing the animation is saved in the repository, titled *simulation_5_seconds.mp4*. The plots below show the cart's position and velocity, as well as the angular position and velocity of the pendulum over time.





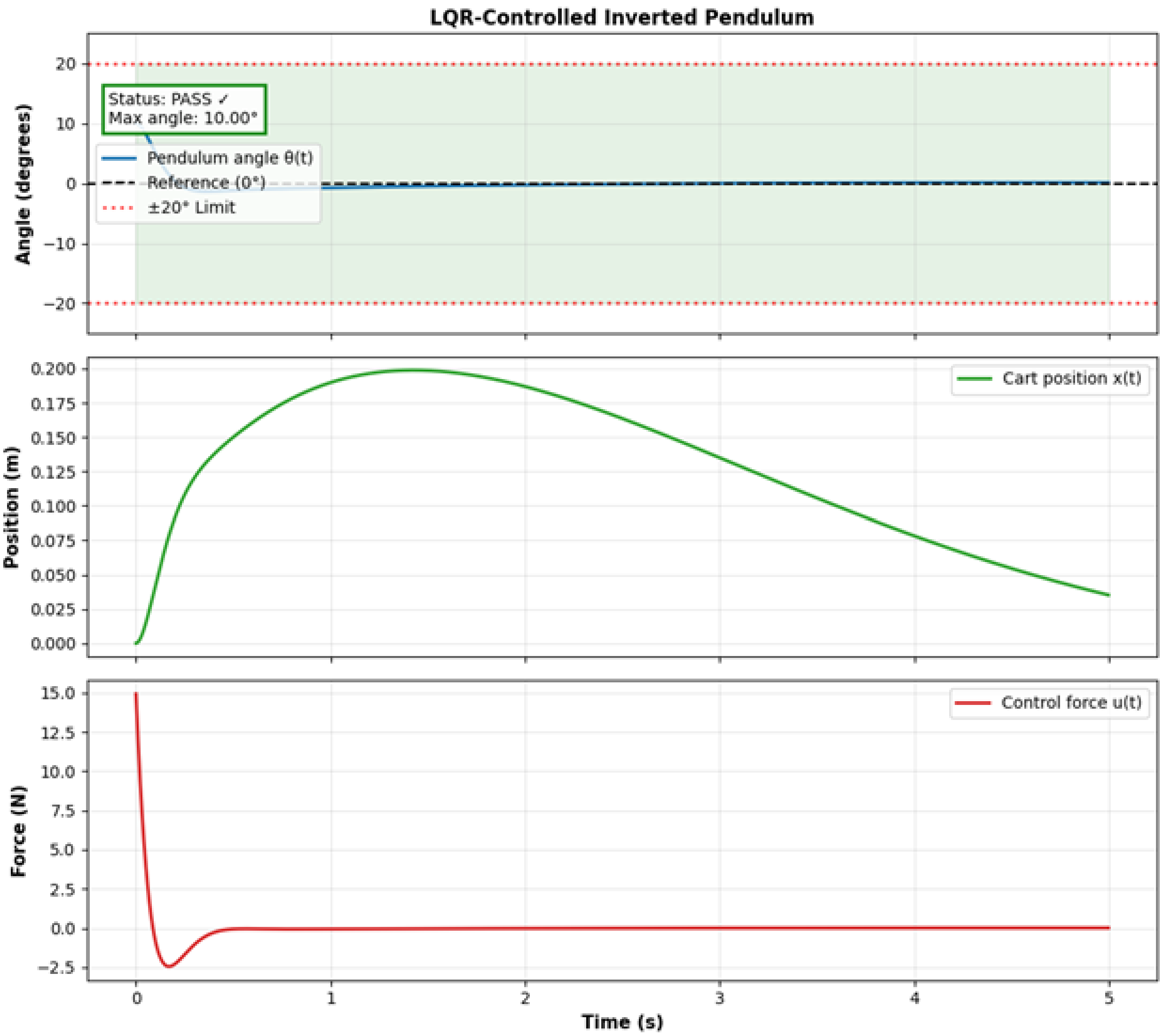




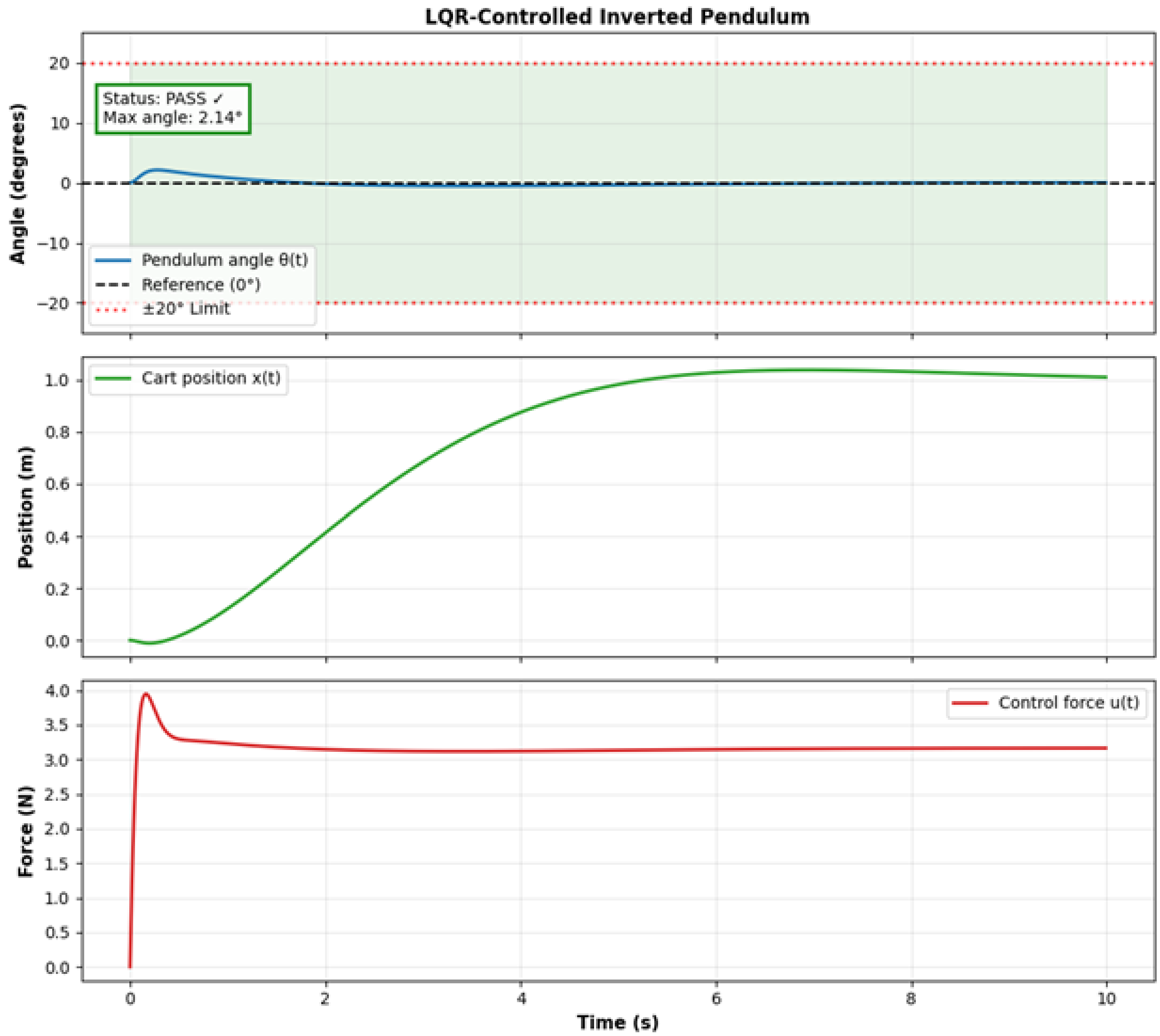
The LQR controller was highly effective in keeping the system in the desired +/- 20° pendulum bounds. Through the various tests at different initial angles, the system performed well. The "tuning knob" in Q for the angle deflection was pushed incredibly low while still showing desired performance. The only time the pendulum did not meet the 20° bounds with the controller was when the initial starting angle was greater than 20°.

For the bonus problem, the LQR controller remained highly effective. Increasing penalty for the angle in Q allowed for the pendulum to remain in the upright position with minimal change in angle.

The plots below show the angle, cart position, and force applied over a period of 5 seconds. The animation of the LQR controller in action is located in the repository.



The plots below show the result when performing a unit step to the right. The corresponding animation is located in the repository.



4 Issues Encountered

When deriving the systems of equations for the non-linear and linear systems, getting the signs correct was incredibly difficult. The orientation of the pendulum and the initial condition of the pendulum's position were quite tricky. Upon initial deployment of our code, the system performed in a way that simply violated the laws of physics. Once everything was corrected, the implementation of the system worked well.

5 Conclusions

The inverted pendulum is a fascinating problem, and requires very precise derivation for the equations of motion. One sign error can create catastrophic errors. This was a good practice in the realm of "all models are wrong, some models are useful".