AST 231: Problem Set 4

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Problem 1. Calculate the mean molecular weight for the following gasses:

Answer 1. The mean molecular weight (μ) is given by equation 5.112 in the textbook:

$$\frac{1}{\mu} = \frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_{\text{electron}}} \tag{1}$$

where we can calculate μ_{ion} and μ_{ion} using the following equations:

$$\frac{1}{\mu_{\text{ion}}} = \sum \frac{X_i}{A_i} \tag{2}$$

and

$$\frac{1}{\mu_{\text{electron}}} = \sum \frac{X_i(z+1)}{A_i} \tag{3}$$

respectively. Here, X_i is the mass fraction of a species i, A_i is the number of nucleons in the nucleus of the species, and z_i is the charge of the nucleus of the species.

Thus, we can plug in equations 2 and 3 into 1 to get the mean molecular weight in physical terms:

$$\frac{1}{\mu} = \sum \frac{X_i}{A_i} + \sum \frac{X_i(z+1)}{A_i} \tag{4}$$

We use equation 4 in the following questions below.

a) Pure Hydrogen, which is 100% neutral

The mas fraction of pure neutral hydrogen is $X_1 = 1$ and the number of nucleons in the nucleus is $A_1 = 1$ as well (as given by the hint below). Therefore, plugging into equation 4,

$$\frac{1}{\mu} = \sum_{1} \frac{X_i}{A_i} = \frac{X_1}{A_1} = \frac{1}{1} = 1 \rightarrow \boxed{\mu = 1}$$

b) Pure Hydrogen, which is 100% ionized

Since the hydrogen is fully ionized, we use second term in equation 4 (essentially, just equation 3). Here, $X_1 = A_1 = z_1 = 1$. Therefore,

$$\frac{1}{\mu} = \sum_{1} \frac{X_i(z_i+1)}{A_i} = \frac{X_1(z_1+1)}{A_1} = \frac{1(1+1)}{1} = \frac{2}{1} \rightarrow \boxed{\mu = \frac{1}{2}}$$

c) Pure Hydrogen, which is 50% neutral and 50% ionized

We are dealing with neutral and ionized hydrogen so we use both parts of equation 4. Because we are dealing with Hydrogen again, $X_1 = 0.5$ and $A_1 = z_1 = 1$. Plugging into the equation,

$$\frac{1}{\mu} = \sum_{1} \frac{X_i}{A_i} + \sum_{1} \frac{X_i(z_i+1)}{A_i} = \frac{X_1}{A_1} + \frac{X_1(z_1+1)}{A_1}$$

$$= \frac{0.5}{1} + \frac{0.5(1+1)}{1} = 1.5 \rightarrow \boxed{\mu = \frac{2}{3}}$$

d) Neutral gas that is 75% Hydrogen and 25% Helium

Now, we have neutral Hydrogen ($X_1 = 0.75, A_1$) and neutral Helium ($X_2 = 0.25, A_2 = 4$, as given by the hint below). This means we will be using only the first part of equation 4. Plugging into the equation:

$$\frac{1}{\mu} = \sum_{2} \frac{X_i}{A_i} = \frac{X_1}{A_1} + \frac{X_2}{A_2} = \frac{0.75}{1} + \frac{0.25}{4} = 0.8125 \rightarrow$$

$$\boxed{\mu = 1.23}$$

e) Fully ionized gas that is 75% Hydrogen and 25% Helium

Again, we have Hydrogen with $X_1 = 0.75$ and $A_1 = 1$ and Helium with $X_2 = 0.25$ and $A_2 = 4$. This time, the gas is fully ionized, so we will use the second term in equation 4 with $z_1 = 1$ and $z_2 = 2$. Plugging in the values:

$$\frac{1}{\mu} = \sum_{2} \frac{X_i(z_i+1)}{A_i} = \frac{X_1(z_1+1)}{A_1} + \frac{X_2(z_2+1)}{A_2}$$

$$= \frac{0.75(1+1)}{1} + \frac{0.25(2+1)}{4} = 1.6875 \rightarrow \boxed{\mu = 0.593}$$

f) Fully ionized gas that is 73% Hydrogen, 25% Helium and 2% metals

In this ionized gas there is Hydrogen with $X_1 = 0.73$ and $A_1 = z_1 = 1$, Helium with $X_2 = 0.25$, $z_2 = 2$, and $A_2 = 4$, and metals with $X_3 = 0.02$, $z_3 = 2$, and $A_3 = 16$, as given by the hint below. We are using only the second term in equation 4. Thus, plugging in the values:

$$\frac{1}{\mu} = \sum_{3} \frac{X_i(z_i+1)}{A_i} = \frac{X_1(z_1+1)}{A_1} + \frac{X_2(z_2+1)}{A_2} + \frac{X_3(z_3+1)}{A_3}$$

$$= \frac{0.73(1+1)}{1} + \frac{0.25(2+1)}{4} + \frac{0.02(2+1)}{16} = 1.69 \rightarrow \boxed{\mu = 0.606}$$

[Hints and Notes: Fraction refer to Òmass fractionsÓ not Ònumber fractionsÓ, i.e. 75% Hydrogen means that 75% of the MASS of the star is Hydrogen, not that 75% of the atoms are Hydrogen. Remember that Òmean molecular weightÓ (μ) is given in units of the mass of a hydrogen atom. Hence, it is unitless itself. In these calculations, you can consider the mass of a hydrogen atom to be 1, whether neutral or ionized, since the mass of the electron is negligible. Free electrons add numbers of particles but no significant mass. Hence, ionization reduces the mean molecular weight. Helium atoms can be taken to be of mass 4. Metal atoms are dominated by Carbon and Oxygen, and can be assumed to have mass 16. Fully ionized metal atoms can be assumed to have 1 electron and 1 neutron for every proton. The difference in mass between the proton and the neutron may be neglected in these calculations.]

Problem 2. The Kelvin-Helmholtz time, often referred to as just the "Kelvin time", is the length of time it would take a star to radiate away an energy equal to one-half of its current gravitational potential energy, assuming it was radiating at its current luminosity. As a star contracts, it puts one-half of the energy gained into thermal (kinetic) energy of its particles, satisfying the Virial Theorem. That leaves one-half to radiate away. As discussed in class and in your text book, stable, gravitationally bound, systems (stars, clusters of stars, clusters of galaxies, etc.) always have a time averaged gravitational potential energy that is twice the size of their time averaged kinetic energy. Before it was recognized that stars have internal energy sources (nuclear fusion), it was thought that their only means of gaining energy was by contracting under their own self-gravity and the Kelvin time was thought to be the full lifetime of the star. Now we recognize it as the pre-main sequence lifetime ... the time it takes the star to get hot enough inside to begin nuclear fusion. Calculate the Kelvin time for the Sun using its current luminosity and the fact that pre-main sequence stars are fully convective and, therefore obey the $P \propto \rho^{\gamma}$ "polytropic" relationship, with $\gamma = \frac{5}{3}$.

[Hint: You will need to find a relationship between gravitational potential energy (also known as "Binding Energy") and polytropic index. You will find this buried in the extensive supplementary material on polytropes that is linked on our course Web site.]

Answer 2. We are told that the Kelvin-Helmholtz time (t_K) is the length of time it would take a star to radiate 0.5 of its current gravitational potential energy (Ω)assuming that it radiates at its current luminosity (L). This translates to the following equation:

$$t_{\mathrm{K}} = \frac{0.5\Omega}{\mathrm{L}} \tag{5}$$

Calculating the Kelvin time for the sun translates equation 5 to

$$t_{\mathrm{K}} = \frac{0.5\Omega_{\odot}}{\mathrm{L}_{\odot}}$$

We now have to find both components to the equation. We can find the gravitational potential energy using the following equation found on pg. 6 of the supplemental material on polytropes:

$$\Omega = \left(\frac{3}{5-n}\right)\left(\frac{\mathrm{GM}^2}{\mathrm{R}}\right)$$

Using the properties of the sun and physical values, we have the following values:

$$\begin{aligned} \mathrm{G} &= 6.67 \times 10^{-11}\ \mathrm{m}^{-3}\ \mathrm{kg}^{-1}\ s^{-2} \\ \mathrm{M}_\odot &= 1.989 \times 10^{30}\ \mathrm{kg} \\ \mathrm{R}_\odot &= 6.963 \times 10^{8}\ \mathrm{m} \end{aligned}$$

To solve for n, the polytropic index, we use the following equation:

$$n = \frac{1}{\gamma - 1} \tag{6}$$

We are given that $\gamma = \frac{5}{3}$ so

$$n = \frac{1}{\frac{5}{3} - 1} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

Now we have all the components to find Ω. So plugging all of them in:

$$\Omega = \left(\frac{3}{5-n}\right)\left(\frac{\mathrm{GM}^2}{\mathrm{R}}\right) = \Big(\frac{3}{5-\frac{3}{2}}\Big)\Big(\frac{(6.67 \times 10^{11})(1.989 \times 10^{30})^2}{6.963 \times 10^8}\Big) = 3.248 \times 10^{41}\ \mathrm{m}^2\ \mathrm{kg}\ \mathrm{s}^{-2}$$

We know that $\mathrm{L}_\odot = 3.828 \times 10^{26}$ W, where the watt is also $\mathrm{W} = \mathrm{kg}\ \mathrm{m}^2\ \mathrm{s}^{-3}$. Plugging this value and our calculated Ω value into equation 1, we can solve for the Kelvin time:

$$t_K = \frac{0.5\Omega}{L} = \frac{0.5 \cdot 3.248 \times 10^{41}}{3.828 \times 10^{26}}$$

$$= \boxed{4.24 \times 10^{14} \text{secs} = 1.34 \times 10^{7}\ \text{yrs}}$$

Thus, a PMS star with the physical characteristics of the sun would reach the main sequence in 1.34×10^7 years.