AST 231: Problem Set 3

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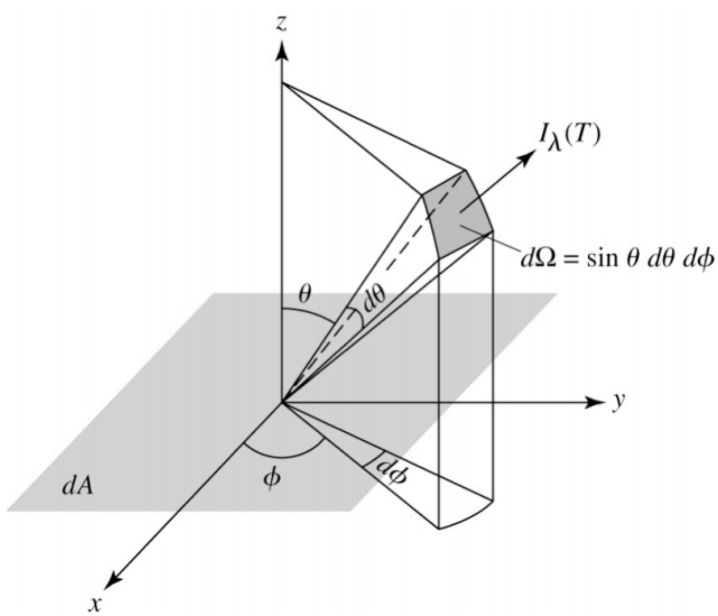
Problem 1. What is the solid angle subtended by the portion of the celestial sphere between right ascension 0 hr and 1 hr and between declination 59 degrees and 60 degrees? Give your answer in square degrees. As always, show your work.

Answer 1. The image we want to have in mind when going about this problem is the following (taken from lecture 6 slide 1):



where solid angle is Ω, right ascension (RA) is ϕ and (90° - declination) is θ. The equation we want to consider is the following:

$$\mathrm{d}\Omega = \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \tag{1}$$

Thus,

$$\int \mathrm{d}\Omega = \Omega = \int\int \sin\theta \mathrm{d}\theta\mathrm{d}\phi$$

We are given our bounds of integration, so we carry out the integral as follows:

$$\Omega = \int_0^{15} \int_{90-60}^{90-59} \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

$$\Omega = \int_0^{15} \mathrm{d}\phi \int_{30}^{31} \sin\theta \mathrm{d}\theta$$

$$\Omega = \phi\Big|_0^{15} \Big(- \cos\theta \Big)\Big|_{30}^{31}$$

$$\Omega = 15^\circ \Big(- \cos(31^\circ) + \cos(30^\circ) \Big)$$

$$\Omega = 15^\circ \Big(0.0089 \Big)$$

The output of a trigonometric function is given in radians so before multiplying, we must convert to degrees.

$$0.0089 \cdot \frac{180}{\pi} = 0.5075^\circ$$

Therefore,

$$\Omega = 15^\circ \Big(0.5075^\circ \Big)$$

$$\boxed{\Omega = 7.6 \text{ sq}^\circ}$$

The solid angle subtended by the portion of the celestial sphere between RA 0 hr and 1 hr and between 59 degrees and 60 degrees is thus 7.6 sq°.

Problem 2. Estimate the mean free path of a photon in the stellar interior. State your assumptions.

Answer 2. We know that the mean free path of a photon is given by the following equation:

$$\langle l \rangle = \frac{1}{n_e \sigma} \tag{2}$$

where n_e is the number density of electrons and σ is the cross section of an electron. We know that σ is a constant given by Thompson scattering: 6.65×10^{-29} m^2. To get the mean free path, we now only need to know the electron number density inside of a star. Because our answer will vary depending on the star, we choose to consider the interior of the sun (or a sun-like star). We know that the sun is 75% hydrogen and 25% helium. To calculate the population of each of these elements, we divide 75% the mass of the sun (2×10^{30} kg) by the mass of hydrogen (1.67×10^{-27} kg) and 25% the mass of the sun by the mass of helium (6.646×10^{-27} kg). Therefore, we get that

$$\text{number}_{\text{HI}} = \frac{2 \times 10^{30} \cdot 0.75}{1.67 \times 10^{-27}} = 8.98 \times 10^{56}$$

and

$$\text{number}_{\text{He}} = \frac{2 \times 10^{30} \cdot 0.25}{6.646 \times 10^{-27}} = 7.5 \times 10^{55}$$

To get an electron number density, we continue by assuming that the number density of protons is equal to the number density of electrons. Because we know that hydrogen has 1 proton and helium has 2 protons, we can multiply the number of hydrogen and helium by their respective proton count to get the number of protons. We can sum the two numbers to get the total number of protons in the sun. Therefore,

$$\text{number}_{\text{proton,HI}} = 8.98 \times 10^{56} \cdot (1) = 8.98 \times 10^{56}$$

and

$$\text{number}_{\text{proton,He}} = 7.5 \times 10^{55} \cdot (2) = 1.5 \times 10^{56}$$

By our assumption the number of electrons is equal to the number of protons. So,

$$\text{number}_{\text{e}} = \text{number}_{\text{proton,HI}} + \text{number}_{\text{proton,He}} = 8.98 \times 10^{56} + 1.5 \times 10^{56} = 1.04 \times 10^{57}$$

To get a number density of electrons, we now divide by the volume of the sun, which we solve for given that it has a radius of 6.957×10^{8} meters:

$$V_{\odot} = \frac{4}{3}\pi(6.957 \times 10^{8})^{3} = 1.41 \times 10^{27} \text{ m}^{3}$$

So the number density of electrons is:

$$n_e = \frac{1.04 \times 10^{57}}{1.41 \times 10^{27}} = 7.37 \times 10^{29} \text{ m}^{-3}$$

We now have all we need to solve for the mean free path of a photon inside the sun. So plugging in n_e and σ :

$$\langle l \rangle = \frac{1}{n_e \sigma} = \frac{1}{7.37 \times 10^{29} \cdot 6.65 \times 10^{-29}}$$

$$\boxed{\langle l \rangle = 0.02 \text{ m}}$$

Thus, inside the interior of a sun-like star, a photon can travel 0.02 meters before interacting with something else.

Problem 3. What is the mean free path of a photon in the Universe just before it becomes transparent? Note that the main opacity source is electron scattering, that the free electrons come from ionized H atoms and that the composition of the Universe at this time is 75% H and 25% He (by mass). For definiteness, let's take the transition point to be the time when 50% of the H atoms are ionized. [Hint: you will need to look up the current mass density of the Universe (not including Dark Matter or Dark Energy! ... i.e. the "baryonic" mass density) and the current temperature of the Universe (near 2.7 K, but you might want to be more exact). Then you will have to scale these back in time, accounting for the expansion of space. The scale factor is $(z+1)^{-1}$ where z is the redshift, defined as $\frac{\Delta\lambda}{\lambda}$. For example, in the Universe today, z = 0, so the scale factor is 1. At z = 1, all lengths are 1/2 of what they are in today's Universe. By Wien's law, then, the Universe is hotter by a factor of 2. Its mass density is also higher by a factor of $2^3 = 8$, since all that mass is packed into a volume that is smaller by that factor. So, you know how temperature and density scale with z. Therefore, first, find the value of z that leads to a 50% ionization fraction when the Saha equation is applied. Then calculate the mean free path of photons under those conditions of density and temperature for the given composition.]

Answer 3. As the question suggests, we first use the Saha equation to find the redshift at which the universe first becomes transparent. We are told that at this stage, the ionization fraction is 50%. So we have that

$$f_1 = \frac{n_1}{n_1 + n_2} = 0.5 \rightarrow n_1 = n_2$$

Thus, we are going to find the redshift that make the Saha equation equal to $\frac{n_2}{n_1} = 1$. Therefore,

$$\frac{n_2}{n_1} = 1 = \frac{1}{n_e}\left(\frac{2\pi \cdot \mathrm{m}_e \cdot kT}{h^2}\right)^{\frac{3}{2}} \frac{2U_2}{U_1} \exp\left(\frac{-\mathrm{E}_{\mathrm{ion}}}{kT}\right) \tag{3}$$

From the question, we can also relate temperature and n_e to redshift. Due to the scale factor, the temperature scales back as follows:

$$T(z) = (1+z)T_0 \tag{4}$$

where $T_0 = 2.735$ K, the temperature of the universe now.

We know that the total mass density of the universe today is 9.9×10^{-30} g$\cdot$ cm^{-3} and that baryonic mattter makes up 4.6% percent of that matter[1]. This gives us $\rho_{0,H} = 4.55 \times 10^{-31}$ g $\cdot$ cm^{-3} = 4.55×10^{-28} kg $\cdot$ m $^{-3}$. We need to also scale back this mass density, which we are told in the question can be done with a scale factor of $(z+1)^3$. Once we scale to right before the universe became transparent, we know that 75% of matter was hydrogen and that 25% of that hydrogen was ionized. All in one equation, we get that

$$\rho_{0,H} = 4.55 \times 10^{-28} \cdot (\frac{3}{4})(\frac{1}{2})(z+1)^3$$

To turn this into an electron number density, we can divide by the mass of a proton (since $n_{\mathrm{HII}} = n_e$). Therefore, with $m_p = 1.672 \times 10^{-27}$ kg,

$$n_e = 4.55 \times 10^{-28} \cdot (\frac{3}{4})(\frac{1}{2})(z+1)^3 \frac{1}{1.672 \times 10^{-27}}$$

This gives us an equation that relates electron number density to the scale factor:

$$n_e(z) = 0.1019 \text{ m}^{-3}(z+1)^3 \tag{5}$$

Recapping what we just did and introducing all the constants for the Saha equation:

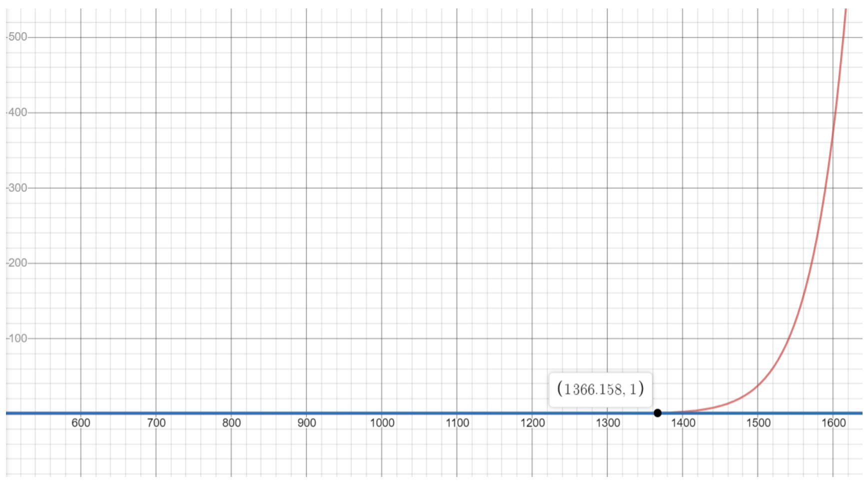
[1] https://wmap.gsfc.nasa.gov/universe/uni_matter.html

$$
\begin{aligned}
m_e =& \quad 9.109\times10^{-31}\ \mathrm{kg}\\
k =& \quad 1.381\times10^{-23}\ \mathrm{J\cdot K}\\
T =& \quad 2.735(1+z)\\
h =& \quad 6.626\times10^{-34}\ \mathrm{J\cdot s}\\
\mathrm{E_{ion}} =& \quad 2.719\times10^{-18}\ \mathrm{J}\\
U_2 =& \quad 1\\
U_1 =& \quad 2\\
n_e =& \quad 0.1019(1+z)^3
\end{aligned}
$$

Plugging all these into the equation,

$$1=\frac{1}{0.1019(1+z)^3}\left(\frac{2\pi\cdot9.109\times10^{-31}\cdot1.381\times10^{-23}\cdot(1+z)\cdot2.735}{(6.626\times10^{-34})^2}\right)^{\frac{3}{2}}\frac{2\cdot1}{2}\exp\left(\frac{-2.719\times10^{-18}}{8.617\times10^{-5}\cdot2.735\cdot(1+z)}\right)$$

To solve for z, we plot the left hand side and the right hand side as a function of z and check where they intercept. We generate the following plot:



This tells us that in order to make Saha work under our conditions, $z = 1366$, very early in the universe.

Now that we have our z value, we want to compute the mean free path:

$$\langle l\rangle=\frac{1}{n_e\sigma} \tag{6}$$

where σ is the cross section. In this case, it is the Thompson scattering constant (cross section of an electron) of $6.65\times10^{-29}\ \mathrm{m}^{-2}$. We can use our equation of n_e as a function of z to compute the electron number density at a redshift of 1366. This will give:

$$n_e(1366)=0.1019\cdot(1+1366)^3=260526717$$

Therefore,

$$\langle l\rangle=\frac{1}{n_e\sigma}=\frac{1}{260526717\cdot6.65\times10^{-29}}$$

$$\boxed{\langle l\rangle=5.77\times10^{19}\ \mathrm{m}}$$

So a photon can travel, on average, 5.77×10^{19} m before interacting with something else.

Problem 4. Use the Saha equation to compare the number density of H^- atoms to the number density of HI atoms capable of absorbing in the Paaschen continuum in the solar atmosphere (i.e. n = 3 level HI atoms). Assume a temperature of 5800 K and an electron pressure of 20 dynes cm^{-2}, typical of an optical depth of about 2/3 in the solar photosphere.

Answer 4. We want to compare the number density of H^- atoms to the number density of HI atoms with an electron in the n = 3 level. This is not possible with just one equation since we are comparing the population of an ion at a specific energy level to the population of a specific ion. Instead, we must employ the `Boltzmann Equation`

$$\frac{n_i}{n_{\text{ion}}} = \frac{\mathrm{g}_i}{\mathrm{U}_{\text{ion}}} \exp\left(\frac{-\mathrm{E}_i}{kT}\right) \tag{7}$$

to get the fraction of HI atoms at an n = 3 energy level to all HI atoms. Then, we will use the `Saha Equation`

$$\frac{n_{\text{HI}}}{n_{\text{H}^-}} = \frac{1}{n_e}\left(\frac{2\pi \cdot \mathrm{m}_e \cdot kT}{h^2}\right)^{\frac{3}{2}} \frac{2U_3}{U_{\text{ion}}} \exp\left(\frac{-\mathrm{E}_{\text{ion}}}{kT}\right) \tag{8}$$

to get the fraction of H^- over HI atoms.

We first work on computing the fraction of HI atoms at an n = 3 energy level to all HI atoms (i.e - the Boltzmann Equation). Here, i = 3. The degeneracy (or statistical weight) is found using the equation

$$g_n = 2n^2 \tag{9}$$

so $g_3 = 2(3)^2 = 18$. The partition function is found with the equation

$$\mathrm{U}_{\text{ion}} = \sum^{3} g_n \exp\left(\frac{E_{\text{ion}}}{kT}\right) \tag{10}$$

so, using $k = 8.617 \times 10^{-5}$ eV $\cdot$ K^{-1} and T = 5800 K:

$$\mathrm{U}_{\text{ion}} = 2 \cdot \exp(0) + 8 \cdot \exp(\frac{-10.2}{8.617 \times 10^{-5} \cdot 5800}) + 18 \cdot \exp(\frac{-12.08}{8.617 \times 10^{-5} \cdot 5800}) \approx 2$$

Additionally, the energies are given by

$$E_n = 13.6\Big(1 - \frac{1}{n^2}\Big) \text{ eV} \tag{11}$$

.

From the above, we get the following numbers to plug into the boltzmann equation:

$$\begin{array}{rc} g_3 = & 18 \\ \mathrm{U_{ion}} = & 2 \\ \mathrm{E_3} = & 12.08 \text{ eV} \\ k = & 8.617\times 10^{-5} \text{ eV}\cdot\text{K}^{-1} \\ T = & 5800 \text{ K} \end{array}$$

Plugging in:

$$\frac{n_3}{n_{\text{ion}}} = \frac{18}{2}\cdot\exp\left(\frac{-12.08}{8.617\times 10^{-5}\cdot 5800}\right) = 2.78\times 10^{-10}$$

Now, we can find the fraction of H^- to HI atoms using the Saha equation. In the Saha equation, we will use the following numbers:

$$\begin{array}{rc} m_e = & 9.109\times 10^{-31} \text{ kg} \\ k = & 1.381\times 10^{-23} \text{ J}\cdot\text{K} \\ T = & 5800 \text{ K} \\ h = & 6.626\times 10^{-34} \text{ J}\cdot\text{s} \\ \mathrm{E_{ion}} = & 0.754 \text{ eV} \\ U_{\mathrm{H}^-} = & 2 \\ U_{\mathrm{HI}} = & 2 \\ n_e = & 2.49\times 10^{19} \end{array}$$

where we found n_e from the ideal gas law

$$P_e = n_e kT \tag{12}$$

$$\rightarrow n_e = \frac{P_e}{kT} = \frac{2}{1.381\times 10 - 23\cdot 5800} = 2.49\times 10^{19}$$

Plugging everything into Saha,

$$\frac{n_{\mathrm{HI}}}{n_{\mathrm{H}^-}} = \frac{1}{2.49\times 10^{19}}\left(\frac{2\pi\cdot 9.109\times 10^{-31}\cdot 1.381\times 10^{-23}\cdot 5800}{(6.626\times 10^{-34})^2}\right)^{\frac{3}{2}}\frac{2\cdot 2}{2}\exp\left(\frac{-0.754}{8.617\times 10^{-5}\cdot 5800}\right)$$

$$\frac{n_{\mathrm{HI}}}{n_{\mathrm{H}^-}} = 1.89\times 10^7$$

Now, multiplying the Saha equation by the Boltzmann equation and then taking the reciprocal, we get our desired result:

$$\left(\frac{n_3}{n_{\text{ion}}}\right)\left(\frac{n_{\mathrm{HI}}}{n_{\mathrm{H}^-}}\right) = \frac{n_3}{n_{\mathrm{H}^-}} = 0.00527$$

Therefore,

$$\boxed{\frac{n_{\mathrm{H}^-}}{n_3} = 190}$$

This tells us that there are 190 n_{H^-} atoms for every HI atom with an energy level of n = 3, which are rare and unstable (get kicked to ionized or to ground quickly).