AST 231: Problem Set 2

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Problem 1. Calculate the distance to a star with the following observed characteristics. Be sure to state where you obtained any relevant information that you used in the calculation. V = 12.25, B-V = +0.32, Spectral Type = A0V. What is the parallax angle for this star?

Answer 1. Our approach is to first find distance using the given information. Then, use that distance to compute parallax. We will calculate the distance using the distance modulus:

$$\mathrm{V} - \mathrm{M_V} = 5\log_{10}(d) + \mathrm{A_V} \tag{1}$$

Using Rochester's color database[1] we get that $\mathrm{M_V} = 1.11$ and that $(\text{B-V})_{\text{intrinsic}} = 0$ for an A0V star. We can use the intrinsic color and the observed color to the the color excess, E(B-V), which we can then use to get our extinction factor ($\mathrm{A_V}$). We do these steps using the following equations:

$$\text{E(B-V)} = \text{B-V}_{\text{obs}} - \text{B-V}_{\text{intrinsic}} \tag{2}$$

and

$$\mathrm{A_V} = 3.1 \cdot \text{E(B-V)} \tag{3}$$

Thus, plugging our observed color and the intrinsic color into equation 2, we get:

$$\text{E(B-V)} = 0.32 - 0 = 0.32$$

And then plugging our excess color into equation 3:

$$\mathrm{A_V} = 3.1 \cdot 0.32 = 0.992$$

Now, we have all the components needed to use the distance modulus. Therefore, plugging in all our values:

$$12.25 - 1.11 = 5\log_{10}(d) - 0.992$$

$$\boxed{d = 1070.5 \text{ pc}}$$

[1] https://www.pas.rochester.edu/~emamajek/EEM_dwarf_UBVIJHK_colors_Teff.txt

To find parallax, we use the following equation:

$$\mathrm{d[pc]} = \frac{1}{\mathrm{p[arcsec]}} \tag{4}$$

Thus, we our calculated distance into equation 4 to get our parallax:

$$1070.5 = \frac{1}{\mathrm{p}}$$

$$\boxed{p = 0.0009''}$$

So, the parallax angle for our star is 0.0009 arcseconds.

Problem 2. Calculate the temperature at which the number density of hydrogen atoms in the first excited state is $\frac{1}{20}$ of the number density of hydrogen atoms in the ground (fundamental) state. [Note: this is just a small variation of Question 1.7 from your textbook and the answer to that question is provided in the book. So, an intelligent strategy would be – do that problem first and be sure you get the right answer.]

Answer 2. The ratio of number densities between two energy levels is given by the `Boltzmann Equation`:

$$\frac{n_i}{n_j} = \frac{\mathrm{g}_i}{\mathrm{g}_j} \exp\left(\frac{-(\mathrm{E}_i - \mathrm{E}_j)}{kT}\right) \tag{5}$$

Since we are considering the first excited state of hydrogen to the ground state, let i = 2 and j = 1. Thus, the Boltzmann equation becomes

$$\frac{n_2}{n_1} = \frac{\mathrm{g}_2}{\mathrm{g}_1} \exp\left(\frac{-(\mathrm{E}_1 - \mathrm{E}_2)}{kT}\right)$$

where we are given that $\frac{n_2}{n_1} = \frac{1}{20}$. We then solve for the rest of our variables.

We know that g_n, the statistical weight (or degeneracy) is given by:

$$g_n = 2n^2 \tag{6}$$

Therefore, $g_1 = 2(1)^2 = 2$ and $g_2 = 2(2)^2 = 8$.
Next, the energy levels (in eV) is given by the equation:

$$\mathrm{E}_n = 13.6\left[1 - \frac{1}{n^2}\right] \tag{7}$$

Hence, $E_1 = 13.6(1 - 1) = 0$ and $E_2 = 13.6(1 - \frac{1}{4}) = 10.2$ eV.
Finally, we know that the Boltzmann constant is $k = 8.617 \times 10^{-5}$ eV K^{-1}. This gives us all the necessary information to solve for temperature. Thus, plugging all into the Boltzmann equation, we get:

$$\frac{1}{20} = \frac{8}{2}\exp\left(\frac{-10.2}{8.617\times 10^{-5}\cdot T}\right)$$

Rearranging for temperature:

$$T = \frac{-10.2}{8.617\times 10^{-5}\cdot \ln\left(\frac{1}{20}\cdot\frac{2}{8}\right)} = \boxed{27,012\text{ K}}$$

If this was the hydrogen population in a star, a temperature of 27,000 K would give it a spectral type of about B1.

Problem 3. What is the ionization fraction of HI at a depth where T = 8000 K and P = 140 dyne/cm^2 in a star composed of pure hydrogen. You may approximate the partition function of neutral hydrogen, U_I in the book's notation, to be equal to the statistical weight of the ground state, namely g = 2. [Note that this is a small variation of Question 1.9 from your textbook, so try that one first and make sure you get the answer given in the back of the book.]

Answer 3. The ionization fraction of HI (f_1) can be found using the following equation:

$$f_1 = \frac{n_1}{n_1 + n_2} \tag{8}$$

where n_1 and n_2 are the number densities of the first and second energy state, respectively. To compute these number densities, we will start with the `Saha equation`:

$$\frac{n_2}{n_1} = \frac{1}{n_e}\left(\frac{2\pi\cdot \mathrm{m}_e\cdot kT}{h^2}\right)^{\frac{3}{2}}\frac{2U_2}{U_1}\exp\left(\frac{-\mathrm{E}_{\text{ion}}}{kT}\right) \tag{9}$$

Assuming the star is composed of pure hydrogen, we set $n_e = n_2$ because every ionized hydrogen (n_2) will release exactly one free electron (n_e). Thus, there will be a one-to-one correspondence. Using this fact, we can substitute n_e for n_2 in the Saha equation and rearrange it so that it looks as follows:

$$\frac{n_2^2}{n_1} = \left(\frac{2\pi\cdot \mathrm{m}_e\cdot kT}{h^2}\right)^{\frac{3}{2}}\frac{2U_2}{U_1}\exp\left(\frac{-\mathrm{E}_{\text{ion}}}{k\cdot T}\right)$$

Notice that the right hand side of our modified Saha equation are all known constants[2] :

$$\begin{aligned}
m_e &= 9.109\times 10^{-31}\text{ kg}\\
k &= 1.381\times 10^{-23}\text{ J}\cdot\text{K}^{-1}\\
T &= 8000\text{ K}\\
h &= 6.626\times 10^{-34}\text{ J}\cdot\text{s}\\
\mathrm{E}_{\text{ion}} &= 2.18\times 10^{-18}\text{ J}\\
U_2 &= 1\\
U_1 &= 2
\end{aligned}$$

[2] We assume $U_2 = 1$ based off example 1.7 in *An Introduction to Stellar Astrophysics.*

Plugging in all our values, we get:

$$\frac{n_2^2}{n_1} = \left(\frac{2\pi \cdot 9.109 \times 10^{-31} \cdot 1.381 \times 10^{-23} \cdot 8000}{(6.626 \times 10^{-34})^2}\right)^{\frac{3}{2}} \frac{2 \cdot 1}{2} \exp\left(\frac{-2.18 \times 10^{-18}}{1.381 \times 10^{-34} \cdot 8000}\right)$$

$$\frac{n_2^2}{n_1} = 4.59 \times 10^{18} \text{ m}^{-3}$$

This generates an equation with two unknowns. Thus, we continue by making use of the fact that we have all the components required (pressure and temperature) to solve to n_{tot} in the `ideal gas law` equation:

$$P = n_{\text{tot}} kT \tag{10}$$

We plug in T = 8000 K, the K constant, and P = 14 N $\cdot$ m^{-2}. Therefore,

$$n_{\text{tot}} = \frac{P}{kT} = \frac{14}{1.38 \times 10^{-23} \cdot 8000} = 1.268 \times 10^{20} \text{ m}^{-3}$$

We want to relate n_{tot} to n_1 and n_2, which we can do based off, once again, the assumption that the star is composed of pure hydrogen. Because of this, we know that only three particles/atoms in the star: neutral hydrogen (n_1), ionized hydrogen (n_2), and free electrons (n_e). Therefore, we can write the following:

$$n_{\text{tot}} = n_1 + n_2 + n_e = n_1 + 2n_2 = 1.268 \times 10^{20} \text{ m}^{-3}$$

We now have two equations with two unknowns:

$$\begin{array}{ccc} n_1 + 2n_2 = 1.268 \times 10^{20} \text{ m}^{-3} & \text{and} & \frac{n_1^2}{n_1} = 4.59 \times 10^{18} \text{ m}^{-3} \\ \downarrow & & \downarrow \\ n_2 = \frac{1}{2}\left(1.26 \times 10^{20} - n_1\right) & \text{and} & n_2^2 = 4.59 \times 10^{18} \cdot n_1 \end{array}$$

Plugging the left equation into the right equation, we get one quadratic function in terms of one variable:

$$\left(\frac{1}{2}(1.26 \times 10^{20} - n_1)\right)^2 = 4.59 \times 10^{18} \cdot n_1$$

We use the quadratic formula to solve for n_1. This produces two solutions: $n_1 = \{8.686 \times 10^{19}, 1.85 \times 10^{20}\}$. Physically, n_1 can not be greater than n_{tot}, which is what the second solution suggests. Therefore, we continue with the first solution and say that

$$n_1 = 8.686 \times 10^{19} \text{ m}^{-3}$$

We can now plug n_1 into either of the equations used for the systems of equations. We choose the left one. Therefore,

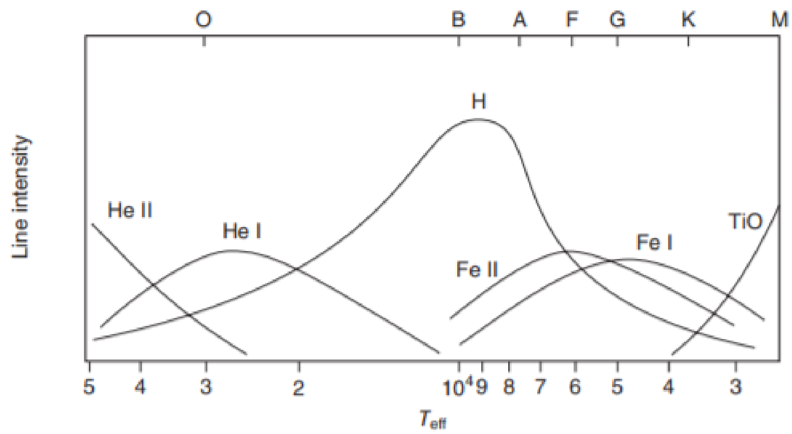
$$n_2 = \left(\frac{1}{2}(1.26 \times 10^{20} - n_1)\right)^2 = \left(\frac{1}{2}(1.26 \times 10^{20} - 8.686 \times 10^{19})\right)^2 = 1.997 \times 10^{19} \text{ m}^{-3}$$

We now have all the components necessary to compute f_1. Plugging in n_1 and n_2 into equation 8:

$$f_1 = \frac{n_1}{n_1 + n_2} = \frac{8.686 \times 10{-}^{19}}{8.686 \times 10^{19} + 1.997 \times 10^{19}} = \boxed{0.81 = 81\%}$$

So the ionization fraction of HI at a depth of T = 8000 K and P = 140 dyne / cm^2 is 81%.

We verify that our answer is in the correct range using figure 1.9 from *An Introduction to Stellar Astrophysics*:



We see that at T = 8000 K, HI is a very prominent feature. Thus, we would expect the composition of the star to be primarily HI. This fact is supported from our calculation, evidence that our answer matches what is expected.

Problem 4. Calculate the P(r) inside a sphere of radius $R_\star$ with a constant density ρ. (This is Question 2.2 from the book.)

Answer 4. To calculate the pressure as a function of radius, we will start with the `hydrostatic equilibrium equation`:

$$\frac{\mathrm{dP}}{\mathrm{dr}} = -\rho(r)g(r) \tag{11}$$

In this equation, is given that $\rho(r) = \rho$, a constant. The gravitational acceleration, $g(r)$, is given by:

$$g(r) = -\frac{\mathrm{GM}(r)}{r^2} \tag{12}$$

Thus, the hydrostatic equilibrium equation for this system is as follows:

$$\frac{\mathrm{dP}}{\mathrm{dr}} = -\rho\frac{\mathrm{GM}(r)}{r^2}$$

To solve the above differential, we will need to solve for M(r), which we can do with the another fundamental equation of stellar structure:

$$\frac{\mathrm{dM(r)}}{\mathrm{dr}} = 4\pi r^2 \rho(r) \tag{13}$$

$$\mathrm{dM(r)} = 4\pi r^2 \rho(r)\ \mathrm{dr}$$

We proceed by integrating both sides

$$\int_0^{\mathrm{M(r)}} \mathrm{dM(r)} = \int_0^r 4\pi r^2 \rho(r) \mathrm{dr}$$

$$\mathrm{M(r)} = 4\pi\rho \int_0^r r^2 \mathrm{dr}$$

$$\mathrm{M(r)} = \frac{4\pi\rho r^3}{3}$$

Having the functional form of M(r), we can plug all our knows into the hydrostatic equilibrium equation:

$$\mathrm{dP} = -\rho \frac{G}{r^2} \frac{4\pi\rho r^3}{3}\ \mathrm{dr} = -\frac{4}{3} G\rho^2 \pi r\ \mathrm{dr}$$

We integrate both sides. The left side from P(r) to 0 and the right side from r to R_*. Thus,

$$\int_{\mathrm{P(r)}}^0 \mathrm{dP} = -\int_r^{\mathrm{R}_*} \frac{4}{3} G\rho^2 \pi r \mathrm{dr}$$

$$\int_{\mathrm{P(r)}}^0 \mathrm{dP} = -\frac{4}{3} G\rho^2 \pi \int_r^{\mathrm{R}_*} r\ \mathrm{dr}$$

$$-\mathrm{P(r)} = -\frac{4}{3} G\rho^2 \pi \left(\frac{R_*^2}{2} - \frac{r^2}{2}\right)$$

$$\boxed{\mathrm{P(r)} = \frac{2}{3} G\rho^2 \pi \left(R_*^2 - r^2\right)}$$

The boxed equation above gives the pressure of the star as a function of radius.

Problem 5. Calculate the gravitational potential energy of a fictitious star which has $\mathrm{M}_\star$ and radius $\mathrm{R}_\star$, that has a density profile $\rho(r) = \rho_c(1 - \frac{r}{R_\star})$. Give your answer in terms of the central density, ρ_c, and in terms of $\mathrm{M}_\star$ and $\mathrm{R}_\star$. (This is only slightly different from Question 2.3 in the book.)

Answer 5. The gravitational potential potential energy of a body is given by

$$\Omega = -\int_0^{M_*} \frac{\mathrm{GM(r)}}{r} dM \tag{14}$$

We need to solve for the M(r) component which is given by a fundamental equation of stellar structure:

$$\frac{\mathrm{dM(r)}}{dr} = 4\pi r^2 \rho(r)$$

Plugging in for the $\rho(r)$ given:

$$\frac{\mathrm{dM(r)}}{dr} = 4\pi r^2 \rho_c \big(1 - \frac{r}{R_*}\big)$$

We now solve for M(r):

$$\int_0^{\mathrm{M(r)}} \mathrm{dM(r)} = \int_0^r 4\pi r^2 \rho_c \big(1 - \frac{r}{R_*}\big)\mathrm{dr}$$

$$\mathrm{M(r)} = 4\pi\rho_c \int_0^r \big(r^2 - \frac{r^3}{R_*}\big)\mathrm{dr}$$

$$\mathrm{M(r)} = 4\pi\rho_c \big(\int_0^r r^2\mathrm{dr} - \frac{1}{R_*}\int_0^r r^3\mathrm{dr}\big)$$

$$\mathrm{M(r)} = 4\pi\rho_c \big(\frac{r^3}{3} - \frac{r^4}{4R_*}\big)$$

Now that we have the functional form of M(r), we can plug it into the equation for gravitational potential energy where $\mathrm{dM(r)} = 4\pi r^2 \rho_c \big(1 - \frac{r}{R_*}\big)\mathrm{dr}$. So,

$$\Omega = -\int_0^{R_*} \frac{G}{r}\big(4\pi\rho_c\big)\big(\frac{r^3}{3} - \frac{r^4}{4R_*}\big)4\pi r^2 \rho_c \big(1 - \frac{r}{R_*}\big)\mathrm{dr}$$

$$= -16\pi^2 G\rho_c^2 \int_0^{R_*} r^2\big(\frac{r^3}{3} - \frac{r^4}{4R_*}\big)\big(1 - \frac{r}{R_*}\big)\mathrm{dr}$$

$$= -16\pi^2 G\rho_c^2 \int_0^{R_*} \frac{r^4}{3} - \frac{r^5}{3R_*} - \frac{r^5}{4R_*} + \frac{r^6}{4R_*^2}\ \mathrm{dr}$$

$$= -16\pi^2 G\rho_c^2 \int_0^{R_*} \frac{4r^4}{12R_*^2} - \frac{4r^5 R_*}{12R_*^2} - \frac{3r^5 R_*}{12R_*^2} + \frac{3r^6}{12R_*^2}\ \mathrm{dr}$$

$$= -16\pi^2 G\rho_c^2 \int_0^{R_*} \frac{1}{12R_*^2}\big(4r^4 R_*^2 - 4r^5 R_* - 3r^5 R_* + 3r^6\big)\ \mathrm{dr}$$

$$= \frac{-16\pi^2 G\rho_c^2}{12R_*^2}\int_0^{R_*}\left(4r^4R_*^2 - 4r^5R_* - 3r^5R_* + 3r^6\right)\,\mathrm{dr}$$

$$= \frac{-16\pi^2 G\rho_c^2}{12R_*^2}\left(4R_*^2\frac{R_*^5}{5} - 4R_*\frac{R_*^6}{6} - 3R_*\frac{R_*^6}{6} + 3\frac{R_*^7}{7}\right)$$

$$= \frac{-16\pi^2 G\rho_c^2}{12}\left(4\frac{R_*^5}{5} - 4\frac{R_*^5}{6} - 3\frac{R_*^5}{6} + 3\frac{R_*^5}{7}\right)$$

$$= \frac{-16\pi^2 G\rho_c^2 R_*^5}{12}\left(\frac{4}{5} - \frac{4}{6} - \frac{3}{6} + \frac{3}{7}\right)$$

$$= \frac{-16\pi^2 G\rho_c^2 R_*^5}{12}\left(\frac{23}{210}\right)$$

$$\boxed{\Omega = -\frac{26}{315}\pi^2 G\rho_c^2 R_*^5}$$

We have an answer in terms of R_* and ρ_c. We now want to have it in terms of M_*. To do this, we use the M(r) equation we found at the beginning of this problem:

$$\mathrm{M(r)} = 4\pi\rho_c\left(\frac{r^3}{3} - \frac{r^4}{4R_*}\right)$$

$$\mathrm{M}(R_*) = 4\pi\rho_c\left(\frac{R_*^3}{3} - \frac{R_*^4}{4R_*}\right) = 4\pi\rho_c\left(\frac{R_*^3}{3} - \frac{R_*^3}{4}\right)$$

$$= 4\pi\rho_c\left(\frac{4R_*^3 - 3R_*^3}{12}\right) = \frac{\pi}{3}\rho_c R_*^3$$

$$\mathrm{M}_* = \frac{\pi}{3}\rho_c R_*^3$$

We want to solve for ρ_c and plug it into the final form of Ω that we calculated:

$$\mathrm{M}_* = \frac{\pi}{3}\rho_c R_*^3 \rightarrow \rho_c = \frac{3\mathrm{M}_*}{\pi R_*^3}$$

Plugging into Ω:

$$\Omega = -\frac{26}{315}\pi^2 G\left(\frac{3\mathrm{M}_*}{\pi R_*^3}\right)^2 R_*^5$$

$$\Omega = -\frac{26}{315}\pi^2 G\left(\frac{9\mathrm{M}_*^2}{\pi^2 R_*^6}\right)R_*^5 = \frac{-26\cdot 9}{315}\left(\frac{G\mathrm{M}_*^2}{R_*}\right)$$

$$\boxed{\Omega = \frac{-26}{35}\left(\frac{G\mathrm{M}_*^2}{R_*}\right)}$$

The boxed equation above gives the gravitational potential energy in terms of M_* and R_*.