Algebras for a Functor

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Motivation

Modelling inductive types.

F-Algebras

category C, endofunctor $F\colon C \to C$

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$$FA \xrightarrow{\quad\alpha\quad} A$$

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$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\ FB \arrow[r, "\beta"] & B \end{tikzcd}$$

Initial Objects

Such an object I, that for every object X,
there exist a **unique** morphism $I \to X$.

Lambek Lemma

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If $I = (A, \alpha)$ is an initial algebra, then A is isomorphic to FA via α.

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$$
\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "Fi"'] & A \arrow[d, "i"] \\
F(FA) \arrow[r, "F\alpha"] & FA
\end{tikzcd}
$$

Polynomial Functor

- project only defines them on **Sets**
- $PX = \sum_{i \in I} X^{A_i}$, for $A : I \to \mathbf{Set}$
- natural numbers from $PX = 1 + X$

Initial algebra for polynomial functors

- Tree has a constructor Node $\sum_{i \in I} \text{Tree}^{A_i}$
- initial object is the F-algebra of the Tree

$$\begin{tikzcd}
\sum_{i \in I} \text{Tree}^{A_i} \arrow[r, "\alpha"] \arrow[d, "Pf"'] & \text{Tree} \arrow[d, "f"] \\
\sum_{i \in I} B^{A_i} \arrow[r, "\beta"'] & B
\end{tikzcd}$$

Future work

- implement presentation of natural numbers with F-algebras
- implement presentation of lists with F-algebras
- generalize polynomial functors
- generalize existance of initial algebras

Problems in Implementation

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```
; _∘_  = λ f g → record {
    f = F-Algebra-Morphism.f f ∘ F-Algebra-Morphism.f g ;
    commutes = glue {!  4!} (F-Algebra-Morphism.commutes f)
    (F-Algebra-Morphism.commutes g)  }
```

Sources

Awodey, Steve (2010). *Category Theory*. 2nd. USA: Oxford University Press, Inc. ISBN: 0199237182.

nLab authors (May 2022). *initial algebra of an endofunctor*. URL: http://ncatlab.org/nlab/show/initial%20algebra%20of%20an%20endofunctor.