Exact approaches for the pattern minimization problem
Supplementary material

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1. Introduction

The supplementary material is organized as follows: a structured presentation of the proposed upper and lower bounds; additional implementation details for both our approaches and those from Martin et al. (2022) and Cui et al. (2015); additional experiments analyzing model sizes and linear relaxation quality; an assessment of applicability when overproduction is prohibited, analyzing computation times, solution quality, pattern counts, and surplus under allowed overproduction; adaptations of our approaches for alternative strategies that trade off the number of distinct cutting patterns against overproduction; and finally, formal proofs for all propositions presented in the paper.

2. A detailed explanation of the structured usage of the upper and lower bounds proposed

As follows, we present a general algorithm to find upper and lower bounds on the frequency of the cutting patterns for Formulation 1. The algorithm relies on Propositions 2 to 5, presented in Section 3.2, and on Algorithm 1: A preprocessing algorithm to determine upper bounds on the frequency of the cutting patterns, presented in Section 3.4.1.

The algorithm takes as input the number of cutting patterns used, $\mathcal{K}$, and an upper bound on the frequency of execution of the first cutting pattern, Φ^1, which can be determined using the algorithm presented in Section 3.4.1. Additionally, the algorithm also has as input the vector $\overline{\mathbf{d}} = [d_{s_1}, d_{s_2}, \ldots, d_{s_{|I|}}]$, where s_i represents the indices of the demand vector arranged in non-increasing order, such that if $i_1 \geq i_2$, then $d_{s_{i_1}} \geq d_{s_{i_2}}$, as defined in Section 3.2. As outputs, the algorithm returns the upper and lower bounds on the frequency of the cutting patterns. The following algorithm is used to determine the upper and lower bounds proposed in the paper:

The first four conditional operators in Algorithm 1 are related to Propositions 2 to 4, while the fifth conditional operator is related to the general case in which the propositions cannot be applied. Furthermore, the values of Θ^k are obtained via the closed-form expression presented in Section 3.2.

3. Remarks on the Cutting Stock Problem with Setup Cost (CSP-S)

We highlight that the contributions proposed in the previous sections are not limited to the context of the Pattern Minimization Problem (PMP). In fact, the improvements developed in this work can be reframed to suit the CSP-S with minimal changes.

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Algorithm 1 An algorithm to determine upper and lower bounds on the frequency of the cutting patterns

1: **for** k from 2 to $\mathcal{K}$ **do**
2:    **if** $d_{s1} > kd_{s2}$ **then**
3:       $\Phi^k = \min\{\lfloor (\underline{\beta} - max\{0, \mathcal{K} - k\})/k \rfloor, \lfloor d_{s_1}/k' \rfloor, \Phi^{k-1}\}$
4:    **else**
5:       **if** $k = 2$ **then**
6:          $\Phi^k = \min\{\lfloor (\underline{\beta} - max\{0, \mathcal{K} - k\})/k \rfloor, max\{d_{s_2}, \lfloor d_{s_1}/2 \rfloor\}, \Phi^{k-1}\}$
7:       **end if**
8:       **if** $k = 3$ and $2d_{s_2} \leq d_{s_1}$ and $3d_{s_2} > d_{s_1}$ **then**
9:          $\Phi^k = \min\{\lfloor (\underline{\beta} - max\{0, \mathcal{K} - k\})/k \rfloor, d_{s_2}, \Phi^{k-1}\}$
10:       **end if**
11:       **if** $k = 3$ and $2d_{s_2} > d_{s_1}$ **then**
12:          $\Phi^k = \min\{\lfloor (\underline{\beta} - max\{0, \mathcal{K} - k\})/k \rfloor, max\{d_{s_3}, \lfloor d_{s_1}/2 \rfloor\}, \Phi^{k-1}\}$
13:       **end if**
14:       **if** $k > 3$ **then**
15:          $\Phi^k = \min\{\lfloor (\underline{\beta} - max\{0, \mathcal{K} - k\})/k \rfloor, \Phi^{k-1}\}$
16:       **end if**
17:    **end if**
18: **end for**
19: **for** k from 1 to $\mathcal{K}$ **do**
20:    $\Theta^k = \left\lceil \frac{\beta - \overline{\Phi}^{k-1}}{\mathcal{K} - k + 1} \right\rceil$
21: **end for**
22: **return** Θ^k and Φ^k, for $k = 1, \ldots, \mathcal{K}$.

An intrinsic part of the PMP definition is a predefined upper bound on the number of used objects, given by $\underline{\beta}$. Such a limitation is fundamental to the development of most of the improvements proposed in Section 3. Even though the CSP-S has no pre-established limitations in the problem definition, a valid upper bound on the number of used objects, given by $\overline{\beta}$, can be determined via a simple two-stage method in the CSP-S context.

The first stage consists of solving the associated Bin Packing Problem (BPP) and obtaining a set of cutting patterns corresponding to an optimal solution to the problem. Then, in the second stage, the frequency of the cutting patterns previously determined is post-processed to be equal to the highest value of demand of the items generated by the execution of the cutting patterns. This method was proposed in Martin et al. (2022) as a means to determine an initial value for $\overline{\beta}$.

Naturally, the initial bound may be too loose to produce significant improvements during the early stages of the process. However, it is worth emphasizing that this bound can be iteratively refined using the solution for the CSP-S obtained in the previous iteration, as demonstrated in Martin et al. (2022). Within a bi-objective approach, we expect the bounds to progressively improve as the algorithm advances, ultimately enhancing its overall performance.

Therefore, inequality (9) can be applied in the context of the CSP-S by switching $\underline{\beta}$ with $\overline{\beta}$. As a consequence, the propositions and valid inequalities, as well as the preprocessing procedure, can be applied to the CSP-S by changing $T_{\underline{\beta}}$ to $T_{\overline{\beta}}$ whenever necessary.

4. Additional details regarding the implementation specifics in each paper

The approaches in Martin et al. (2022) were implemented in C++ and utilized GUROBI v.9.1.1 as the ILP solver. The experiments were conducted on a PC equipped with an Intel Xeon E5-2680 processor (2.7 GHz), limited to 4 threads, 32 GB of RAM, and running Ubuntu 16.04 LTS.

The sequential heuristic from Cui et al. (2015) was implemented in C++, utilizing the CPLEX solver (v 12.5), and executed on a computer with an Intel Core i7-3632QM CPU (2.20 GHz) and 8 GB of RAM.

All experiments reported from the main manuscript were conducted on a Dell Precision 3591 workstation equipped with an Intel Core Ultra 9 185H processor (16 cores, 22 threads) and 64 GB of RAM, running Windows 11 Pro. The computational environment used Julia v1.11.5 with Gurobi v12.0.1 as the optimization solver, utilizing Gurobi's default parameters for all computations.

To solve the instances generated by the CUTGEN1 generator, the authors Martin et al. (2022) adopted a weighted sum approach. They assigned a weight 10 times greater to minimizing the number of objects than to minimizing the number of different cutting patterns, following the same strategy as Cui et al. (2015). We also employed this strategy when implementing the model proposed in Cui et al. (2015).

It is worth noting that this strategy may produce solutions that do not minimize total trim loss, although it did not occur in our experiments. To prevent such outcomes, one could assign a higher weight to the objective of minimizing trim loss—for example, a weight equal to the total demand of all item types. While we tested higher weights, they negatively impacted the computational performance of the model proposed in Cui et al. (2015). Therefore, we opted to use the exact same weights as the original paper to better reflect the model's performance.

In Martin et al. (2022), for the Fiber instances, the authors adopted a bi-objective approach, aiming to identify all efficient solutions. This contrasts with our work, which focuses solely on solutions that achieve minimum trim loss.

4.1. Analysis of the size of the formulations

In the model proposed in Cui et al. (2015), there are a total of $2 * |P|$ variables and $|I| + |P|$ constraints, while inFormulation 2, there are $|Q| + 2 * |R|$ variables and $|I| + |R| + 1$ constraints. Since $|Q| + |R| = |P|$, we conclude that our proposed formulation has practically the same size as the model proposed in Cui et al. (2015) if $|Q| = 0$ and a smaller size if $|Q| > 0$.

As for the formulation proposed in Martin et al. (2022), we have $|F_k| = \Phi^k + 1$, thus the model has a total of $\sum_{k=1}^{\mathcal{K}} (\Phi^k + 1)(|I| + 1) - \mathcal{K}|I|$ variables. InFormulation 2, we consider a lower bound on the frequency of execution of the cutting patterns; as a consequence, we have $|F_k| = \Phi^k - \Theta^k + 1$. Moreover, the variables x_{kfi} are only defined for $f \leq \Phi_i^k$ due to Proposition 6 and the preprocessing strategy presented in Section 3.4.2 from the main manuscript. Thus, this model has a total of $\sum_{k=1}^{\mathcal{K}} (\Phi^k - \Theta^k + 1) + \sum_{k=1}^{\mathcal{K}} \sum_{i \in I} (\Phi_i^k - \Theta^k + 1)$ variables. Furthermore, the model proposed in Martin et al. (2022) has a total of $\sum_{k=1}^{\mathcal{K}} \Phi^k + |I| + 2\mathcal{K} - 1$ constraints, while Formulation 1 has $2 \sum_{k=1}^{\mathcal{K}} (\Phi^k - \Theta^k + 1) + |I| + 3\mathcal{K} - 2$ constraints. The numbers of variables

and constraints of the formulations are summarized in Table 1.

Table 1: Expressions for the number of variables and constraints considering Formulations 1 and 2 as well as the models proposed in Martin et al. (2022) and Cui et al. (2015).

Model	Number of variables	Number of constraints
Martin et al. (2022)	$\sum_{k=1}^{\mathcal{K}}(\Phi^k + 1)(|I| + 1) - \mathcal{K}|I|$	$\sum_{k=1}^{\mathcal{K}}\Phi^k + |I| + 2\mathcal{K} - 1$
Formulation 1	$\sum_{k=1}^{\mathcal{K}}(\Phi^k - \Theta^k + 1) + \sum_{k=1}^{\mathcal{K}}\sum_{i \in I}(\Phi_i^k - \Theta^k + 1)$	$2\sum_{k=1}^{\mathcal{K}}(\Phi^k - \Theta^k + 1) + |I| + 3\mathcal{K} - 2$
Cui et al. (2015)	$2 * |P|$	$|I| + |P|$
Formulation 2	$|Q| + 2 * |R|$	$|I| + |R| + 1$

Clearly, Formulation 1 is more compact than the model proposed in Martin et al. (2022) in terms of variables; as for constraints, the comparison is not clear since, while the lower bounds are helpful in reducing the size of the model, there are new valid inequalities added to the formulation.

The number of variables and constraints of the four models strongly depends on the parameters of the instances, which makes it difficult to compare the sizes of the models. For the sake of clarity, we present in this section the mean values of variables and constraints for all four models based on the dataset provided in Umetani et al. (2003). Additionally, we evaluate the effect of preprocessing strategies on the model sizes and present the results both with and without these strategies. In the computational experiments carried out in this section, we divide the dataset provided in Umetani et al. (2003) into two subsets according to the length of the objects to be cut ($L = 5180$ and $L = 9080$). Furthermore, we consider the limit of 150000 cutting patterns in the cutting pattern generation stage.

An important parameter for Formulation 1 and the model proposed in Martin et al. (2022) is the value of $\mathcal{K}$. In this sense, we use the smallest known number of distinct cutting patterns at which a solution with minimum trim loss was found in the literature, obtained in Martin et al. (2022) or Cui et al. (2015).

The results are presented in Table 2. The first seven rows of Table 2 show the results for instances with an object length of 5180, while the remaining rows present the results for instances with an object length of 9080. The first column identifies the model being used; specifically, we consider Formulation 1 without preprocessing and Formulation 1 with preprocessing strategies (Formulation 1 + Prep). The second and third columns display each formulation's average number of variables and constraints, respectively.

In terms of variables, Formulation 1 is the most compact, especially when paired with the preprocessing strategies. As for constraints, the model proposed in Martin et al. (2022) stands out. AlthoughFormulation 2 has a high number of variables and constraints, the model is more compact than the formulation proposed in Cui et al. (2015). We highlight that inFormulation 2 and the model proposed in Cui et al. (2015), the size of the model is proportional to the length of the objects, while in Formulation 1 and the model proposed in Martin et al. (2022), the number of variables and constraints is inversely proportional to the length of the objects.

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Table 2: Average value of variables and constraints considering Formulations 1 and 2 as well as the models proposed in Martin et al. (2022) and Cui et al. (2015).

Fiber instances for $L = 5180$		
Model	Average number of variables	Average number of constraints
Martin et al. (2022)	2656.10	221.80
Formulation 1	1705.35	404.15
Formulation 1+Prep	1307.60	331.65
Cui et al. (2015)	12247.10	6134.55
Formulation 2	11648.30	5536.75
Fiber instances for $L = 9080$		
Model	Average number of variables	Average number of constraints
Martin et al. (2022)	1240.25	111.00
Formulation 1	772.00	188.25
Formulation 1+Prep	651.50	157.65
Cui et al. (2015)	162786.80	81404.40
Formulation 2	156592.45	75211.05

4.2. Analysis of the quality of linear programming relaxation

To evaluate the strength of the formulations, we compare the quality of their linear programming relaxation. For this analysis, it is essential to generate all cutting patterns to ensure a fair and meaningful evaluation. Consequently, as previously mentioned, we exclude instances with an intractable number of cutting patterns, as their generation would be unfeasible. Instead, we conduct the experiments using the first five instances of each subset of the dataset provided in Umetani et al. (2003), for which the complete set of cutting patterns can be generated. For Formulation 1 and the model proposed in Martin et al. (2022), we consider the same values of $\mathcal{K}$ used in Section 4.1.

The results are presented in Table 3. As in Table 2, the first column specifies the model under evaluation, including Formulation 1 without preprocessing and Formulation 1 with preprocessing strategies (Formulation 1 + Prep). The second and third columns display the average value of the linear relaxation for each formulation for the instances with object lengths equal to 5180 and 9080, respectively. In addition to the linear relaxations of the four formulations addressed, we also presented the average value of the optimal solutions for the instances to provide a baseline for the values of the linear relaxations.

Table 3: Average value of the linear relaxation for Formulations 1 and 2, as well as the models proposed in Martin et al. (2022) and Cui et al. (2015), along with the average value of the optimal solution for the instances.

	Average value of the linear relaxation	
Model	Fiber_5180	Fiber_9080
Optimal value	55.28	31.08
Martin et al. (2022)	52.73	30.08
Formulation 1	54.93	30.92
Formulation 1+Prep	54.99	30.94
Cui et al. (2015)	54.62	30.26
Formulation 2	55.01	30.98

Among the tested models,Formulation 2 presented the best results regarding the quality of linear programming relaxation. Furthermore, the preprocessing strategies employed provide a minor improvement in the quality of the linear relaxation.

5. Remarks on overproduction

In this section, we present the necessary modifications to adapt the two proposed approaches for the case in which overproduction of items is not permitted. Additionally, we discuss the relationship between the bounds on cutting pattern frequencies proposed here and those introduced in prior works, clarifying distinctions and similarities in their application.

We also examine the criteria for selecting cutting patterns in each scenario, highlighting how these criteria vary depending on whether overproduction is allowed. Notably, the approaches proposed in this research can be easily adapted to the case in which the demand for required items has to be strictly satisfied.

In this regard, Propositions 1 to 5 remain valid across all scenarios considered. Furthermore, the upper bounds proposed in this study can be replaced by bounds proposed in the literature that specifically address cases where overproduction is not permitted. Additionally, the pattern reduction criteria used in Formulation 2 must be adjusted to the scenario where overproduction of items is not allowed. To establish a comprehensive basis, this section begins with a brief review of the literature on bounds proposed in earlier works, followed by a comparison of our proposals with these existing contributions.

5.1. Review of cutting patterns frequencies bounds in the no-overproduction scenario

For the scenario in which overproduction is not allowed, considering an arbitrary cutting pattern k and α_i^k as the number of items of type i that are produced by cutting an object according to the cutting pattern k, Vanderbeck (2000) shows that the frequency of the cutting pattern k must be equal to or lower than:

$$\min_{i \in I} \left\lfloor \frac{d_i}{\alpha_i^k} \right\rfloor. \tag{1}$$

The idea of the upper bound (1) is that if the cutting pattern k is executed with frequency higher than (1), then the demand for the item type i is exceeded, implying an unfeasible solution. The upper bound (1) is refined in Alves and de Carvalho (2009) by noting that the trim loss resulting from the execution of a cutting pattern cannot be higher than the total trim loss. In a scenario in which overproduction is not allowed, inequality (9) from the main manuscript becomes:

$$\sum_{k=1}^{K} f_k^* \bar{\ell}_k \leq L\underline{\beta} - \sum_{i \in I} \ell_i d_i = T_{\underline{\beta}}. \tag{2}$$

Consequently, the frequency of execution of an arbitrary cutting pattern k must be equal to or lower than

$$\min \left\{ \left\lfloor \frac{T_{\underline{\beta}}}{\bar{\ell}_k} \right\rfloor, \min_{i \in I} \left\lfloor \frac{d_i}{\alpha_i^k} \right\rfloor \right\}. \tag{3}$$

The probing procedure introduced in Aloisio et al. (2011) can be applied to refine the upper bound (3), albeit at the cost of increased computation time. Notably, this approach is applicable to both scenarios: with and without overproduction of items.

5.2. Comparison between bounds

The allowance of overproduction of items inherently weakens the upper bound (3). Since overproduction is permitted, we cannot rely on the strict requirement of exact demand satisfaction to further tighten this upper bound. In this context, the bound $\min_{i \in I} \left\lfloor \frac{d_i}{\alpha_i^k} \right\rfloor$ is not valid for scenarios in which a surplus of items is allowed. Instead, we can only limit the overproduction of items by observing that the trim loss resulting from overproduction cannot exceed the total trim loss, which leads to Propositions 7 and 8, as well as inequalities (18) and (19) of the main paper.

However, the upper bound $\left\lfloor \frac{T_\beta}{\overline{\ell_k}} \right\rfloor$ is valid regardless of the scenario addressed and serves as the basis for inequality (16) of the main paper. Despite being valid for both scenarios, the probing procedure proposed in Aloisio et al. (2011) requires prior knowledge of the cutting patterns. Consequently, we instead use the preprocessing strategy presented in Section 3.4 of the main paper. We highlight that the probing procedure can result in stronger bounds compared to our proposed preprocessing strategy.

To adapt Approach 1 to handle the no-overproduction scenario, it is sufficient to change the demand constraints to equality instead of greater than or equal to, and to replace $\left\lfloor \frac{T_\beta + \ell_i d_i}{\ell_i f} \right\rfloor$ with $\left\lfloor \frac{d_i}{\alpha_i^k} \right\rfloor$ in Proposition 7. Moreover, the preprocessing strategy resulting from Proposition 8 cannot produce a better bound than the one proposed in Vanderbeck (2000); consequently, it can be removed from the algorithm in the no-overproduction scenario.

Even though stronger bounds can be applied to Formulation 1, there is no guarantee that Approach 1 will perform better in the no-overproduction scenario. As stated in Martin et al. (2022), when demand must be met exactly, the number of cutting patterns necessary to find a solution that minimizes trim loss may increase, which can affect the performance of Approach 1.

As for Approach 2, a few modifications are also needed. Formulation 2 itself is valid for the no-overproduction case; the only change necessary is to modify the demand constraints to equality instead of a greater-than-or-equal-to inequality. However, the pattern reduction criterion (iv) is not valid for the no-overproduction case and must be removed. Moreover, the bound $\overline{f} = \max_{i \in I^j} \left\lceil \left\{ \frac{d_i}{\alpha_i^j} \right\} \right\rceil$ can be substituted by the tighter bound $\min_{i \in I^j} \left\lfloor \frac{d_i}{\alpha_i^j} \right\rfloor$.

6. Additional discussion on item overproduction

Ensuring exact demand fulfillment or allowing surplus in the production of items is an important aspect of the problem, which can lead to different solutions. Notably, a solution that minimizes the number of cutting patterns may produce significantly different quantities of items compared to one that uses a small, though not minimal, number of cutting patterns. Moreover, overproduction generally entails additional costs, such as storage, handling, and potential waste.

However, allowing overproduction of items can be advantageous in practical settings, as it may reduce the number of different cutting patterns required. The number of different cutting patterns is directly related to setup costs, as each new cutting pattern requires readjusting the position of the cutting tools on the machine, which consumes time and raises production costs. Reducing the number of different cutting

patterns simplifies production planning and leads to operational efficiencies, including shorter setup times and more streamlined processes.

Both scenarios—restricting overproduction and allowing it—are relevant and should be analyzed. By solving both cases, we can provide a more comprehensive understanding of the trade-offs involved and offer flexible solutions for different production environments. In this section, we present computational experiments to evaluate the effectiveness of our proposed approaches in a scenario that prohibits item overproduction. Furthermore, we compare the solutions obtained in these experiments with those reported from the main manuscript, focusing on two key aspects: the number of distinct cutting patterns used and the total amount of overproduced items.

6.1. Results for the Fiber and the CUTGEN1 instances in a no-overproduction scenario

The computational experiments presented in this supplementary material were conducted using the following setup: the approaches were implemented in the Julia programming language (version 1.9.4) with the Gurobi solver (version 11.0.3) as the ILP solver, and executed on a computer equipped with an Intel i7-8700 processor (3.20 GHz) and 16 GB of RAM.

In this section, we conduct computational experiments using Approach 1 and Approach 2 paired with Formulation 1 and Formulation 2, respectively. The modifications presented in Section 5 of the main paper (Remarks on Overproduction) are applied to adapt both approaches for the non-overproduction scenario. In Table 4, we present the results for the Fiber instances considering both approaches in a no-overproduction scenario. The table is divided vertically into instances with lower object lengths and instances with higher object lengths.

We utilize the same notation as in the first part of our computational experiments: Obj represents the number of objects used in the PMP solution ($\underline{\beta}$); P_{F_1} and T_{F_1} denote the number of different patterns used and the time required to run Algorithm 3 (an exact algorithm for the PMP with minimum trim loss), presented from the main manuscript, with Formulation 1, while P_{F_2} and T_{F_2} represent the number of different patterns used and the time required for Approach 2 paired with Formulation 2. Cases where no feasible solution was found are represented by the " $-$ " character, and the number of cutting patterns is formatted in bold or italics to indicate optimality or that the limit of 150,000 cutting patterns is reached in the second approach, respectively. As follows, we present Table 4 and a brief discussion on the results of the computational experiments:

From the results presented in Table 4, it is evident that Approach 1 can be adapted to the no-overproduction scenario while maintaining strong performance. For the Fiber instances with an object length of 5180, the approach yielded better results in the no-overproduction scenario, likely benefiting from the stronger bound discussed in Section 5 of the main paper (Remarks on Overproduction). In these instances, the approach identified 17 optimal solutions, compared to 11 optimal solutions in the scenario where item overproduction is allowed.

For instances with an object length of 9080, a total of 19 and 18 optimal solutions were identified using Approach 1 in the scenarios without and with overproduction, respectively. However, a minimum trim loss solution was achieved only for instance Fiber29_9080 in the scenario with overproduction.

The strong performance of Approach 2 for the instances with an object length of 5180 is maintained in the no-overproduction scenario. Once again, the approach successfully determined feasible solutions for all

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Table 4: Number of objects used, number of different cutting patterns employed, and computational time required to solve the Fiber instances considering the Approaches 1 and 2 together with Formulation 1 and Formulation 2 and the models proposed in Martin et al. (2022) and Cui et al. (2015).

Instance	Obj	P_{F1}	T_{F1}	P_{F2}	T_{F2}	Instance	Obj	P_{F1}	T_{F1}	P_{F2}	T_{F2}
06_5180	33	**5**	1.40	**5**	1.21	06_9080	19	**3**	0.33	**3**	4.12
07_5180	33	**4**	0.47	**4**	1.48	07_9080	19	**3**	0.08	**3**	2.58
08_5180	86	**4**	0.46	**4**	1.49	08_9080	48	**3**	0.44	**3**	0.95
09_5180	53	**6**	3.00	**6**	0.09	09_9080	29	**4**	0.83	**4**	0.73
10_5180	69	**5**	1.21	**5**	3.21	10_9080	39	**4**	1.06	**4**	5.27
11_5180	67	**5**	0.49	**5**	2.48	11_9080	38	**4**	1.03	**4**	4.06
13a_5180	56	**5**	0.80	**5**	2.79	13a_9080	32	**4**	1.15	**4**	160.47
13b_5180	28	**4**	0.25	**4**	4.50	13b_9080	16	**3**	0.54	-	-
14_5180	47	**8**	176.99	**8**	8.81	14_9080	27	**4**	0.94	*5*	tl
15_5180	57	**5**	0.98	**5**	3.20	15_9080	32	**4**	0.95	**4**	9.79
16_5180	82	**8**	385.37	**8**	218.35	16_9080	47	**5**	9.45	-	-
17_5180	83	**7**	129.22	7	1800.00	17_9080	47	**4**	3.08	*6*	tl
18_5180	96	**7**	67.61	**7**	517.51	18_9080	54	**5**	4.87	-	-
19_5180	133	**7**	85.98	7	1800.00	19_9080	73	7	3114.83	-	-
20_5180	32	**7**	17.94	8	1800.00	20_9080	19	**4**	4.05	-	-
23_5180	141	-	-	**11**	47.41	23_9080	80	**7**	396.90	-	-
26_5180	190	**8**	1515.23	9	1800.00	26_9080	107	**5**	7.38	-	-
28a_5180	83	-	-	11	1800.00	28a_9080	48	**5**	5.17	-	-
28b_5180	117	-	-	**15**	687.77	28b_9080	67	**6**	8.35	-	-
29_5180	62	**7**	5.54	11	1800.00	29_9080	35	**5**	5.76	-	-

instances. Furthermore, it identified 14 optimal solutions in the no-overproduction scenario compared to 13 in the overproduction scenario.

Conversely, Approach 2 struggled with instances where the object length was 9080. For these instances, the approach achieved minimum trim loss for only 9 instances in the no-overproduction scenario, whereas it succeeded for 15 instances in the overproduction scenario.

Table 5 presents the average number of cutting patterns and computational time required by Approach 1 and Approach 2 for instances generated by the CUTGEN1 generator. The table follows the same notation as previous ones, with the addition that the number of PMP solutions found for each class is shown in parentheses—except when a feasible solution is determined for all instances of the class.

Both approaches performed better in the scenario with overproduction of items. In the scenario with overproduction, Approach 1 identified 56 feasible solutions, including 44 optimal ones, compared to 51 feasible solutions and 39 optimal ones in the scenario without overproduction. Similarly, Approach 2 demonstrated weaker performance in the scenario without overproduction, identifying 56 feasible solutions, of which 44 were optimal. In contrast, it achieved 62 feasible solutions, including 52 optimal ones, when overproduction was allowed.

6.2. Analysis of the trade-off between number of different cutting patterns and overproduction of items

The results obtained via Formulation 1 indicate a reasonable trade-off between item overproduction and setup costs. For the 16 instances where a minimum total trim loss was determined for Fiber 5180, the average overproduction using Approach 1 paired with Formulation 1 was 0.28%, with values ranging from

Table 5: Number of objects used, number of different cutting patterns employed, and computational time required to solve the instances generated via the CUTGEN1 generator considering the Approaches 1 and 2 together with Formulation 1 and Formulation 2 and the models proposed in Martin et al. (2022) and Cui et al. (2015). The numbers in parentheses represent the number of PMP solutions found (omitted when all instances were solved).

Class	Obj	P_{F1}	T_{F1}	P_{F2}	T_{F2}	Class	Obj	P_{F1}	T_{F1}	P_{F2}	T_{F2}
1	11.4	**3.2**	1.41	*4.0(2)*	931.64	2	109.4	5.0	590.36	*6.0(2)*	tl
3	23.2	**4.8**	4.23	-	-	4	225.8	7.0(4)	2193.36	-	-
5	42.4	**7.0**	30.46	-	-	6	420.8	-	-	-	-
7	51.2	**7.4**	3.72	**7.4**	0.47	8	510.0	9.25(4)	2426.74	**8.6**	91.17
9	100.4	14.0(4)	1455.40	**13.6**	262.78	10	1000.8	-	-	16.4	770.07
11	175.4	-	-	*24(1)*	tl	12	1756.0	-	-	*34(1)*	tl
13	62.8	**8.8**	2.49	**8.8**	0.04	14	625.4	9.33(3)	986.249	**9.8**	0.10
15	125.2	15.6	1067.18	**15.6**	0.34	16	1251.2	-	-	**18.40**	34.43
17	221.4	29(1)	3364.58	**27.8**	12.06	18	2213.6	-	-	33.0	1578.60

0% to 1.64%. Similarly, for the 20 instances with object length of 9080, the average overproduction using the same approach and formulation was 1.07%, with values ranging from 0% to 5.46%.

For the 56 instances generated by CUTGEN1 where a minimum trim loss was achieved, the average overproduction using Approach 1 paired with Formulation 1 was 12.93%, with values ranging from 0.6% to 20.63%.

In terms of the number of cutting patterns, for the 11 instances where optimality was confirmed by the solver for the fiber instances with object length of 5180, there was an increase of 4.16%. Additionally, considering the 18 fiber instances with object length of 9080, there was an increase of 5.63%.

Moreover, for the CUTGEN1 classes where optimality was achieved using Formulation 1 in both scenarios (Classes 1, 3, 7, and 13), there was an increase in the number of different cutting patterns by 0%, 14.3%, 19.35%, and 12.82%, respectively, averaging 11.62%.

While Approach 1 demonstrates a reasonable balance between item overproduction and the reduction in setup costs—suggesting a fair trade-off between the two scenarios addressed—Approach 2 shows unsatisfactory results in terms of item overproduction.

As mentioned in Section 4.1 of the main paper (Stage 1: Cutting pattern generation), the reduction criterion (vi) may result in excessive levels of overproduction, which is reflected in the results obtained by Approach 2. When combined with criterion (v), these reduction criteria lead to cutting patterns that disproportionately produce the item type with the smallest length, resulting in extreme surplus levels for this item type.

For example, in Class 7, the mean overproduction value reached an average of 115.40% for items, which is clearly unacceptable. Therefore, postprocessing strategies are necessary to mitigate the overproduction caused by Approach 2.

To address this issue, we propose a postprocessing strategy that adapts the model proposed in Martin et al. (2022) to handle item overproduction. A solution for the PMP consists of a set of cutting patterns K and their corresponding execution frequencies. Thus, in a feasible solution, let f_k represent the frequency of execution of cutting pattern $k \in K$.

The core idea of the postprocessing strategy is to use the model proposed in Martin et al. (2022), considering that $\mathcal{K}$ cutting patterns are executed, each with a fixed frequency equal to f_k, for $k \in K$.

Additionally, we introduce a set of variables s_i to account for the overproduction of item type i. The objective of the model is to minimize item overproduction. The model used in the postprocessing stage is presented as follows:

$$\min \sum_{i \in I} s_i \tag{4}$$

$$\sum_{i=1}^{|I|} \ell_i x_{ki} \leq L, \qquad\qquad k \in K, \tag{5}$$

$$\sum_{k \in K} f_k x_{ki} = d_i + s_i, \qquad\qquad i \in I, \tag{6}$$

$$x_{ki} \in Z^+, \qquad\qquad k \in K, i \in I, \tag{7}$$

$$s_i \in Z^+, \qquad\qquad i \in I. \tag{8}$$

The objective function (4) corresponds to minimizing the overproduction of items. Constraints (5) ensure the feasibility of the cutting patterns, noting that every cutting pattern is executed with a frequency greater than zero. For $i \in I$, constraints (6) ensure that the total production of each item type equals the sum of its demand and overproduction, thereby satisfying the demand requirements and accurately accounting for overproduction. The remaining constraints define the domain of the variables.

It is important to note that a solution for model (4)-(8) corresponds to a PMP solution that uses the same number of objects and different cutting patterns as the original input solution. However, the new solution minimizes the overproduction of items effectively.

We implemented the postprocessing strategy with a time limit of 50 seconds to solve model (4)-(8). The time limit was reached in only one instance; nevertheless, the postprocessing strategy proved to be highly effective in reducing item overproduction. For the CUTGEN1 classes where optimality was achieved by Approach 2 in both scenarios (Classes 7, 8, 9, 13, 14, 15, 16, and 17), the initial overproduction percentages were 115.40%, 127.79%, 95.60%, 20.60%, 5.42%, 25.20%, 25.00%, and 10.95%, respectively. After applying the postprocessing strategy, these values were significantly reduced to 10.00%, 11.60%, 4.30%, 5.80%, 19.90%, 4.20%, 7.45%, and 0.80%, respectively.

Regarding the number of different cutting patterns, there was an average increase of 16.52% for the CUTGEN1 classes where optimality was achieved by Approach 2 in both scenarios. Moreover, for the Fiber instances where optimality was achieved in both scenarios, the average increase in the number of different cutting patterns was 2.67% for instances with an object length of 5180 and 11.54% for instances with an object length of 9080.

The results presented highlight the relationship between the overproduction of items and the number of different cutting patterns. Overall, there is a balanced trade-off between these two factors. Therefore, it is essential to carefully assess which scenario should be implemented in practical applications, considering the specific characteristics of the situation to align with the interests of the decision-makers.

Additionally, our results suggest that cutting pattern criteria should be used with caution, as they may lead to high levels of overproduction. In such cases, postprocessing strategies become imperative to reduce the overproduction of items and should be implemented alongside the proposed approaches.

Minimizing the overproduction of items can also be viewed as another objective of the problem. In this regard, both approaches proposed in this study could be adapted to adopt a multi-objective framework, where both the number of cutting patterns and item type surplus are minimized simultaneously. This would allow for the generation of a Pareto front for each instance, offering multiple efficient solutions that address the trade-off comprehensively. Alternatively, a simpler approach could involve penalizing overproduction in the objective functions of Formulation 1 and Formulation 2, providing a solution that balances setup costs and item overproduction.

It is important to note that while these solution approaches are relatively straightforward to implement, they significantly increase the problem's complexity, making instances even more challenging to solve.

7. Alternative approaches to balancing the number of different cutting patterns and overproduction

Both approaches proposed in this paper can be adapted to either include the overproduction of item types as an additional objective or penalize overproduction within the objective function. To achieve this, we can adopt the same strategy as the postprocessing approach presented in Section 7.2, which introduces a set of variables, s_i, to account for the overproduction of items i. In this framework, inequalities (4) and (24) are modified as follows:

$$\sum_{k \in K} \sum_{f \in F_k \setminus \{0\}} j x_{kfi} = d_i + s_i, \quad \forall i \in I,$$

$$\sum_{s \in R} \alpha_i^s r_s + \sum_{u \in Q} \alpha_i^u q_u = d_i + s_i, \quad \forall i \in I.$$

To explicitly address item overproduction as an additional objective, one can augment Formulation 1 and Formulation 2 with the objective of minimizing $\sum_{i \in I} s_i$. The resulting multi-objective problem can then be solved using conventional multi-objective optimization methods.

Alternatively, to address overproduction using a penalizing strategy, the term $\sum_{i \in I} s_i$, multiplied by a small weight, can be added to the original objective functions of Formulation 1 and Formulation 2. This modification prioritizes minimizing the number of cutting patterns while simultaneously reducing item overproduction as a secondary objective.

8. Mathematical demonstrations

As follows we provide the complete mathematical demonstrations for the propositions introduced in the main text, establishing the theoretical foundations of our work.

Proposition 1: Consider K, $\underline{\beta}$, and Φ^k previously defined. Moreover, for an arbitrary $\overline{k}$, for $\overline{k} \in K$, let $\overline{\Phi}^{\overline{k}}$ represent the cumulative sum of the maximum frequencies of the first k cutting patterns to be executed. Specifically $\overline{\Phi}^{\overline{k}}$ is given by:

$$\overline{\Phi}^{\overline{k}} = \sum_{k \in K, k \leq \overline{k}} \Phi^k.$$

Assume that the cutting patterns are sorted in non-increasing order of their frequencies. Let Θ^k represent a lower bound on the frequency for an arbitrary cutting pattern $\overline{k}$, where $\overline{k} \in K$. The following inequality must hold:

$$\Theta^{\overline{k}} \geq \left\lceil \frac{\underline{\beta} - \overline{\Phi}^{\overline{k}-1}}{\mathcal{K} - \overline{k} + 1} \right\rceil .$$

Proof. Suppose that, in a feasible solution, the $\overline{k}^{\text{th}}$ cutting pattern is performed with a frequency lower than $\left\lceil \frac{\underline{\beta} - \overline{\Phi}^{\overline{k}-1}}{\mathcal{K} - \overline{k} + 1} \right\rceil$. Since the cutting patterns are sorted in non-increasing order of their frequencies, any cutting pattern k for $k \geq \overline{k}$ will have a frequency less than or equal to $\left\lceil \frac{\underline{\beta} - \overline{\Phi}^{\overline{k}-1}}{\mathcal{K} - \overline{k} + 1} \right\rceil - 1$.

As a result, the total number of objects used in the solution must be less than or equal to

$$\overline{\Phi}^{\overline{k}-1} + (\mathcal{K} - \overline{k} + 1) \left(\left\lceil \frac{\underline{\beta} - \overline{\Phi}^{\overline{k}-1}}{\mathcal{K} - \overline{k} + 1} \right\rceil - 1 \right) .$$

By definition, the total number of objects used in the solution cannot be lower than $\underline{\beta}$. Thus, we have the inequality:

$$\underline{\beta} \leq \overline{\Phi}^{\overline{k}-1} + (\mathcal{K} - \overline{k} + 1) \left(\left\lceil \frac{\underline{\beta} - \overline{\Phi}^{\overline{k}-1}}{\mathcal{K} - \overline{k} + 1} \right\rceil - 1 \right) ,$$

which implies

$$\underline{\beta} \leq \overline{\Phi}^{\overline{k}-1} + (\mathcal{K} - \overline{k} + 1) \left(\left\lceil \frac{\underline{\beta} - \overline{\Phi}^{\overline{k}-1}}{\mathcal{K} - \overline{k} + 1} \right\rceil - 1 \right) < \overline{\Phi}^{\overline{k}-1} + \underline{\beta} - \overline{\Phi}^{\overline{k}-1} = \underline{\beta},$$

which is a contradiction. Therefore, the frequency of the $\overline{k}^{\text{th}}$ cutting pattern must be greater than or equal to

$$\left\lceil \frac{\underline{\beta} - \overline{\Phi}^{\overline{k}-1}}{\mathcal{K} - \overline{k} + 1} \right\rceil .$$

$\square$

Proposition 2: The frequency of the second cutting pattern f_2^*, is bounded by $\max\{d_{s_2}, \lfloor d_{s_1}/2 \rfloor\}$.

Proof. It is evident that if $f_1^* \leq d_{s_2}$ the proposition is satisfied. Now, notice that if $f_1^* > d_{s_2}$, then the item type s_1 must be generated by the execution of the first cutting pattern. Therefore, $d_{s_1}^2$ must be less than or equal to $d_{s_1} - f_1^*$. If $d_{s_1} - f_1^* \leq d_{s_2}$ we must have $f_2^* \leq d_{s_2}$; otherwise, since $f_2^* \leq f_1^*$ and $f_2^* \leq d_{s_1} - f_1^*$, we have, $f_2^* \leq \lfloor d_{s_1}/2 \rfloor$. Therefore, it follows that $f_2^* \leq \max\{d_{s_2}, \lfloor d_{s_1}/2 \rfloor\}$.

$\square$

Proposition 3: If $2d_{s_2} \leq d_{s_1}$ and $3d_{s_2} > d_{s_1}$, the frequency of the third cutting pattern f_3^* is bounded by d_{s_2}.

Proof. Whether the first or second cutting pattern is executed with a frequency less than or equal to d_{s_2}, the proposition is verified. If both the first and the second cutting pattern are executed with frequency greater than d_{s_2} then the item s_1 must be generated in the execution of the two cutting patterns and, consequently, the value of $d_{s_1}^3$ must be less than or equal to $d_{s_1} - f_1^* - f_2^*$. As $f_1^*, f_2^* > d_{s_2}$, so $d_{s_1}^3 \leq d_{s_1} - 2d_{s_2}$ which implies $f_3^* \leq d_{s_1} - 2d_{s_2}$. Since $3d_{s_2} > d_{s_1}$, then $d_{s_1} - 2d_{s_2} < d_{s_2}$, therefore, the frequency of f_3^* is bounded by d_{s_2}. $\square$

Proposition 4: If $2d_{s_2} > d_{s_1}$, the frequency of the third cutting pattern f_3^* is bounded by $\max\{d_{s_3}, \lfloor d_{s_1}/2 \rfloor\}$.

Proof. Suppose $f_1^*, f_2^* > \lfloor d_{s_1}/2 \rfloor$ and $f_1^*, f_2^* > d_{s_3}$. Thus, there are two possibilities: either the item type s_1 is generated by the execution of the first cutting pattern and the item type s_2 by the execution of the second cutting pattern, or the item type s_2 is generated by the execution of the first cutting pattern and the item type s_1 by the execution of the second cutting pattern. Regardless of which case happens, we must have $d_{s_1}^3 < d_{s_1} - \lfloor d_{s_1}/2 \rfloor - 1$ and $d_{s_2}^3 < d_{s_2}^3 - \lfloor d_{s_1}/2 \rfloor - 1$ and, as a consequence, $f_3^* \leq \max\{d_{s_3}, \lfloor d_{s_1}/2 \rfloor\}$. Naturally, if f_1^* or $f_2^* \leq \lfloor d_{s_1}/2 \rfloor$ or f_1^* or $f_2^* \leq d_{s_3}$ the proposition is valid. $\square$

Proposition 5: Let c be an integer number greater than 1. If $d_{s_1} > cd_{s_2}$, then the frequency of a cutting pattern k', such as $k' \leq c$, is bounded by $\lfloor d_{s_1}/k' \rfloor$.

Proof. Suppose the item type s_1 is not generated by some cutting pattern k', where $k' \leq c$. In this case, due to Proposition 2, we must have $f_{k'}^* \leq ds_2$. Furthermore, since $d_{s_1} > cd_{s_2}$, we conclude that $f_{k'}^* \leq ds_2 \leq \lfloor d_{s_1}/k' \rfloor$.

In a more general sense, as the cutting patterns are sorted in non-increasing order according to their frequencies, if any cutting pattern k_2 executed previously to the cutting pattern k' does not generate the item type s_1, then $f_{k'}^* \leq f_{k_2}^* \leq ds_2 \leq \lfloor d_{s_1}/k' \rfloor$.

On the other hand, if the item type s_1 is generated from the execution of the first k cutting patterns, for all $k < k'$, we have $d_{s_1}^{k'} \leq d_{s_1} - \sum_{k=1}^{k'-1} f_k^*$. It is evident that if $f_k^* \leq \lfloor d_{s_1}/k \rfloor$, for any $k < k'$, Proposition 3 is valid. Thus, suppose that $f_k^* > \lfloor d_{s_1}/k' \rfloor$, for all $k < k'$. In that case, we have $d_{s_1}^{k'} \leq d_{s_1} - \sum_{k=1}^{k'-1} f_k^* \leq d_{s_1} - (k'-1)\lceil d_{s_1}/k' \rceil \leq \lfloor d_{s_1}/k' \rfloor$, which completes the proof. $\square$

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