Geometric series

$$\boxed{1+r+r^2+r^3+\dots = \frac{1}{1-r}} \quad \text{for } -1<r<1$$

Proof: Label $1+r+r^2+r^3+\dots = S$

$$\Rightarrow \quad 1+r\underbrace{[1+r+r^2+r^3+\dots]}_{S} = S$$

$$\Rightarrow \quad 1+rS = S$$

Solve for $S$ $\quad S = \frac{1}{1-r}$

Partial geometric sum:

$$\boxed{1+r+r^2+\dots+r^N = \frac{1-r^{N+1}}{1-r}}$$

Proof:

Recall $\quad 1+r+r^2+r^3+\dots = \frac{1}{1-r}$

$$\Rightarrow \quad \underbrace{1+r+r^2+\dots+r^N}_{\text{Call this } S_N} + r^{N+1}+\dots = \frac{1}{1-r}$$

$$\Rightarrow \quad S_N + r^{N+1} + r^{N+2} + \dots = \frac{1}{1-r}$$

$$\Rightarrow \quad S_N + r^{N+1}\underbrace{[1+r+r^2+\dots]}_{\text{Geometric series}} = \frac{1}{1-r}$$

$$\Rightarrow \quad S_N + r^{N+1}\left[\frac{1}{1-r}\right] = \frac{1}{1-r}$$

Rearranging, we have

$$S_N = \frac{1-r^{N+1}}{1-r}$$

How can we use these?

Yield to maturity on a perpetuity/consol

Writing in terms of sum of all payments:

$$P = \frac{F}{1+i} + \frac{F}{(1+i)^2} + \frac{F}{(1+i)^3} + \dots$$

$$= \frac{F}{1+i}\underbrace{\left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \dots\right]}_{\text{Geometric series with } r=\frac{1}{1+i}}$$

$$= \frac{F}{1+i}\left[\frac{1}{1-\frac{1}{1+i}}\right] = \frac{F}{1+i}\left[\frac{1}{i/1+i}\right]$$

$$= \frac{F}{\cancel{1+i}}\left[\frac{\cancel{1+i}}{i}\right]$$

$$\Rightarrow \boxed{P = F/i}$$

Similarly, for a fixed-payment loan with $N$ payments

$$P = \frac{F}{1+i} + \frac{F}{(1+i)^2} + \dots + \frac{F}{(1+i)^N}$$

$$= \frac{F}{1+i}\underbrace{\left[1 + \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^{N-1}}\right]}_{\text{Partial sum with } r=\frac{1}{1+i} \text{ for } \underline{N-1} \text{ terms}}$$

$$= \frac{F}{1+i}\left[\frac{1-\left(\frac{1}{1+i}\right)^N}{1-\left(\frac{1}{1+i}\right)}\right] \Rightarrow \boxed{P = \frac{F}{i}\left[1 - \frac{1}{(1+i)^N}\right]}$$