QM Logistics Assignment 1

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1 Mathematical Model

Variables and notation

- I: set of points of interest
- e_i^A: binary decision variable set equal to 1 if point i is an ambulance site, 0 otherwise
- e_i^H: binary decision variable set equal to 1 if point i is a helicopter site, 0 otherwise
- r_{ij}^k: binary parameter equal to 1 if vehicle k $\in$ {A, H} from potential site i is able to attend to an emergency in point j, 0 otherwise

We first compute r_{ij}^k from the given data using equation 1. Note that according to this formula, even for servicing an emergency in the same point of interest, a helicopter or an ambulance is still assumed to take their start-up time.

$$r_{ij}^k = \begin{cases} 1, & \text{if } s^k + \dfrac{d_{ij}^k}{v^k} \leq 20 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Model

The objective function minimises the total costs of the ambulance and helicopter sites. The constraints ensure that for every interest point j, there is at least one ambulance or helicopter that can service it within 20 minutes.

$$\begin{aligned} \min \quad & \sum_{k \in \{A,H\}} \sum_{i \in I} c^k e_i^k \\ \text{s.t.} \quad & \sum_{k \in \{A,H\}} \sum_{i \in 1} r_{ij}^k e_i^k \geq 1 && \forall\, j \in I \\ & e_i^k \in \mathbb{B} && \forall\, i \in I,\ k \in A, H \end{aligned}$$

2 Objective Value & Solution

The CPLEX algorithm works very well on such mixed-integer programs. The problem is solved in 21 iterations and we find the minimum cost for covering every point of interest to be 120 million.

- Five ambulances should be deployed at coordinates: (26,79), (14,13), (91,17), (45,84), (75,5).
- Two helicopters should be deployed at coordinates: (34,43), (70,68).

3 Visual Representation

Figure 1 shows a scatter plot of the 50 interest points, where the ambulance posts and helicopter sites are discretely circled in black. The blue diamonds represent the coverage area of the respective ambulance posts, while the yellow and orange circles represent the coverage of the two helicopter sites. The validity of our solution is henceforth confirmed, seeing as every point of interest is indeed serviced by at least one provider of emergency care.

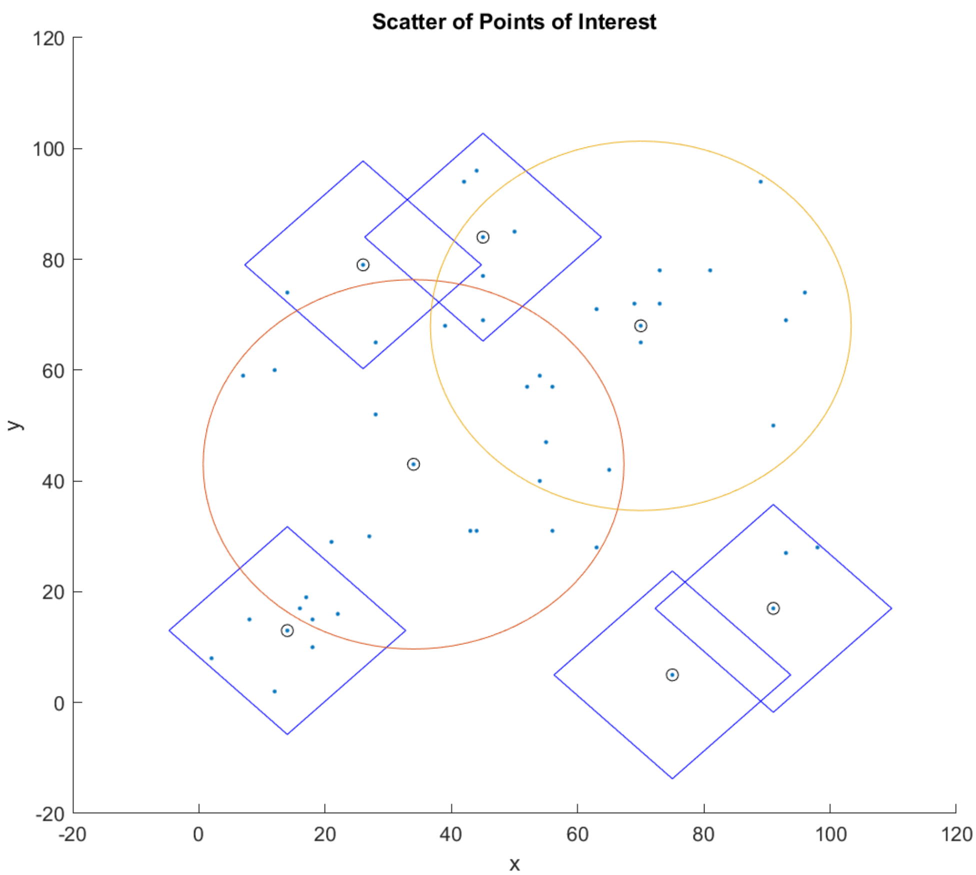

Figure 1: Points of interest, emergency care providers and their coverage.

4 Jerry's Model

On top of the notation used in section 1 we introduce the following notation:

- q_j: binary decision variable set equal to 1 if node j can be serviced within T.
- z_j: binary auxiliary variable used to model the if-then constraint involving q_j.
- p_{ij}^k: binary parameter equal to 1 if vehicle k from potential site i is able to attend to an emergency in 2T. We compute p_{ij}^k from the given data using equation 2.

$$p_{ij}^k = \begin{cases} 1, & \text{if } s^k + \dfrac{d_{ij}^k}{v^k} \leq 40 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We introduce the new constraint that every point of interest must be able to be serviced in time 2T. Furthermore we introduce the constraint that at least a fraction 1 - α of the points of interest must be serviced within time T.

$$
\begin{aligned}
\min \quad & \sum_{k\in\{A,H\}} \sum_{i\in I} c^k e_i^k \\
\text{s.t.} \quad & \sum_{k\in\{A,H\}} \sum_{i\in I} p_{ij}^k e_i^k \geq 1 && \forall\, j \in I \\
& 1 - \sum_{k\in\{A,H\}} \sum_{i\in I} r_{ij}^k e_i^k \leq z_j && \forall\, j \in I \\
& q_j \leq 1 - z_j && \forall\, j \in I \\
& \sum_{j\in I} q_j \geq (1-\alpha)n \\
& e_i^k, q_j, z_j \in \mathbb{B} && \forall\, i \in I,\ k \in A, H
\end{aligned}
$$

5 Solving Jerry's Model

For every level of α between 0 and 1, the solver was able to find an optimal solution. Figure 2 shows how the objective value changes as α increases; we note the clear descending trend until $\alpha = 0.5$, after which one helicopter site becomes enough to reach only $1 - \alpha$ points of interest within 20 minutes and costs are thus capped at 35 million. This is the case because of the very large distances helicopters can cover with 20 more minutes to spare, allowing one central helicopter site to reach every point of interest within 40 minutes. It seems that Jerry's cost cutting strategy is effective, if the sole target is cost minimization.

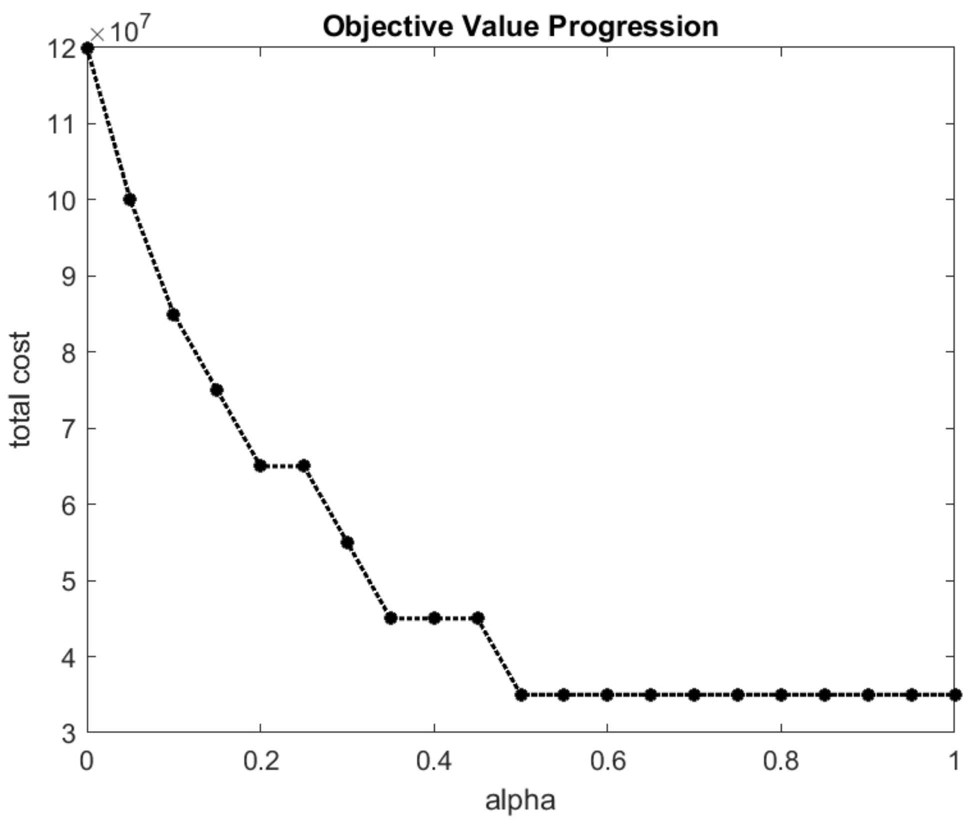

Figure 2: Total project costs plotted against the fraction of interest points served poorly.

Looking towards the solutions themselves rather than the objective values, we plot the number of helicopter sites and ambulance posts generated by each iteration of the model solver against the respective levels of α. Figure 3 predictably shows a decreasing tendency in vehicle stocks to mirror the decrease in total cost. Interesting observations include how, for every $\alpha \geq 0.1$, one helicopter site suffices, and how the required number of ambulance posts drops sharply for $\alpha > 0.25$, reaching 0 when $\alpha = 0.5$.

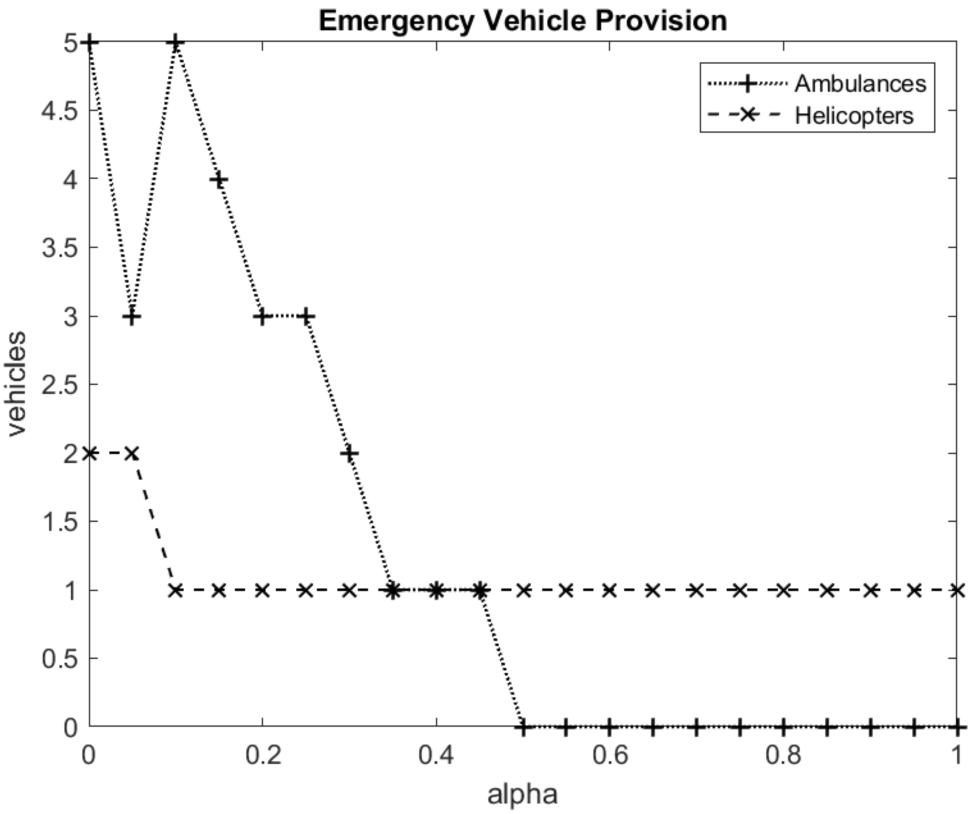

Figure 3: Number of ambulance posts and helicopter sites plotted against the fraction of interest points served poorly.

We draw confidence in these results from the fact that running this new model with $\alpha = 0$ returns the same emergency site distribution and the same objective value as the first model presented in the report.

6 Stochastic Adjusted Model

Because each point must be reached by at least one vehicle within 2T guaranteed, we redefine p^k_{ij} to reflect the worst-case scenario travel times of each vehicle, using equations 3 and 4 .

$$p^A_{ij} = \begin{cases} 1, & \text{if } s^A + \dfrac{4d^A_{ij}}{v^A} \leq 40 \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

$$p^H_{ij} = \begin{cases} 1, & \text{if } s^H + \dfrac{1.1d^H_{ij}}{v^H} \leq 40 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Furthermore we introduce P^k_{ij}, representing the probability that a vehicle of type k can reach point j from point i in time T. We obtain P^k_{ij} from the CDF of the travel time multiplier x^k.

$$\begin{aligned} P^k_{ij} &= P(s^k + x^k \frac{d^k_{ij}}{v^k}\ leqT) \\ &= P(x^k \leq (T - s^k)\frac{v^k}{d^k_{ij}}) \\ &= F_{x^k}((T - s^k)\frac{v^k}{d^k_{ij}}) \end{aligned} \tag{5}$$

The values of P^k_{ij} for different distances are shown in tables 1 and 2. The distances shown in km represent the upper threshold for the respective probability.

Table 1: P^A_{ij} for different maximum distances

d^A_{ij}	37.5	18.75	9.375	4.6875
P^A_{ij}	0.5	0.85	0.95	1

Table 2: P^H_{ij} for different maximum distances

d^H_{ij}	37.037	33.334	30.303
P^H_{ij}	0.25	0.75	1

Using these values, we construct the model as follows:

$$\begin{aligned} \min \quad & \sum_{k\in\{A,H\}} \sum_{i\in I} c^k e^k_i & \\ \text{s.t.} \quad & \sum_{k\in\{A,H\}} \sum_{i\in 1} p^k_{ij} e^k_i \geq 1 & \forall\, j \in I \\ & \prod_{i\in 1} \prod_{k\in\{A,H\}} (1 - P^k_{ij} e^k_i) \leq 1 - \beta & \forall\, j \in I \\ & e^k_i \in \mathbb{B} & \forall\, i \in I,\ k \in A, H \end{aligned}$$

7 Extreme Probability Thresholds

For $\beta = 0$, it is enough to service every point in 2T. As the plots of Jerry's model have shown, a single centrally-located helicopter is fast enough - given the relative size of the map - to reach every interest point, even when delayed slightly under the worst-case stochastic factor. Ambulances thus become redundant additional costs, leading to the optimal solution requiring no ambulance posts.

At the other end of the spectrum, when $\beta = 1$, it is essentially imposed that all points of interest *always* be served within T. As a result, even if its probability is very small, the quadruple travel time of ambulances cannot be afforded. This makes the low-variance helicopters a much more attractive option, even if significantly more helicopter sites need to be built.

For the levels of β between the two extremes, the high variance of ambulance travel times can be compensated by additional ambulance posts or helicopter sites serving a single point such that the joint probability of timely service is increased to whatever non-extreme threshold desired. Ambulances remain valuable for improving the travel time to outlying points of interest such as those in the bottom-left and bottom-right corners of the map, which is reflected in the model solver's preference for them in optimal solutions.