ORBITAL RESONANCE

Orbit-orbit and spin-orbit resonances explained with Newtonian mechanics

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Schrödinger's cats

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1 Abstract

The orbital structure of the Solar system has many unique features, each one of them entailing constraints that inform us about the principal mechanisms responsible for its formation and evolution. As with all science, it is in our interest to create a theoretical framework that makes sense of the current observed state of our planetary system in order to understand the world we live in and to be able to correctly predict future outcomes.

One such particular attribute, inherent not only to our Solar system but to most planetary distributions across the Cosmos is the tendency for orbital resonances, that is, many orbital systems possess pairs of celestial objects which have characteristic frequencies or periods that can be correlated by means of simple numerical expressions. It is yet unclear whether this is a common trait that derives naturally from the evolution of a system or if this by itself plays an important role in the emergence of stable celestial distributions.

In the following paper, our aim is to deduce an easily understandable theoretical basis that describes the mechanisms underlying two different instances of orbital resonance (spin-orbit coupling and orbit-orbit coupling), using Newtonian mechanics as groundwork. While there are many published resources available that handle this topic, most involve advanced mathematics and Hamiltonian mechanics, both unknown to the majority of high school students. Therefore, our work uses Newton's method for treating celestial mechanics and Kepler's laws instead of delving into advanced concepts. Also, we have not used the principles of General Relativity in our model. This does not come at a great cost, as it has an irrelevant impact on the results obtained regarding the particular situations that are presented. Furthermore, we have assured to only make justified approximations, which make the necessary formulae integrable and simplify the needed calculations without failing to meet the expected accuracy.

The following framework is also paired with experimental data for the sake of validation.

1.1 Introduction

When we refer to orbital resonance, we have two main different types of circumstances in mind (there are more, but these are the most prevalent). The first effect is spin-orbit coupling, where a rotating body with asymmetric density distribution may produce a gravitational field which varies periodically and couples with its orbital motion. The second is orbit-orbit coupling, where bodies orbiting in a gravitational field exert periodic influence upon each other, with either stabilising or perturbing outcomes. In the case of two bodies with stable orbits around a third one, this leads to the fact that the relation between their orbital periods may be expressed as:

$$\frac{T_1}{T_2} = \frac{p}{q}$$

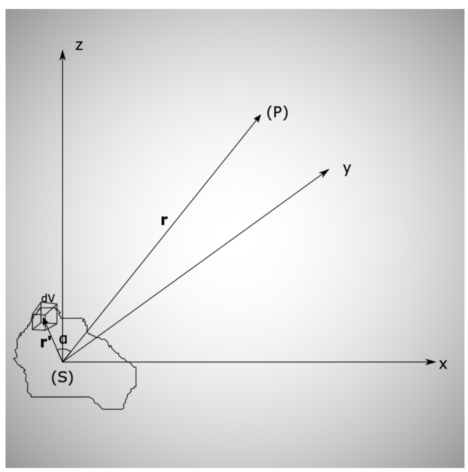
where p and q are small integers.

2 Spin-orbit resonance

This type of orbital resonance involves a celestial body for which the period of rotation around its own axis and the period of revolution around another body are in a ratio of small integers. Examples of such resonance are: the Moon around the Earth (1:1), Mercury around the Sun (3:2), Deimos and Phobos around Mars (1:1), dwarf planet Pluto and its satellite (already locked in synchronous rotation). In what follows, we will explain why certain ratios (1:1, 3:2, 1:2) are particularly stable and present the mechanism responsible for those resonant states: tidal locking. To test the model, we will apply it to the case of the Earth-Moon system and predict its evolution towards synchronous rotation (1:1 resonance).

2.1 Explanation for the stability

Let's consider the motion of a satellite around a planet. Assume it orbits in an ellipse with eccentricity e, has moments of inertia about the principal axes I_x, I_y, I_z and spins about the z axis. We will use multipole expansion to approximate its potential at the site of the planet (in the figures S is the satelite of mass m and P the planet of mass M):



$$\phi = -G \int \frac{1}{\mid \mathbf{r} - \mathbf{r'} \mid} \rho dV.$$

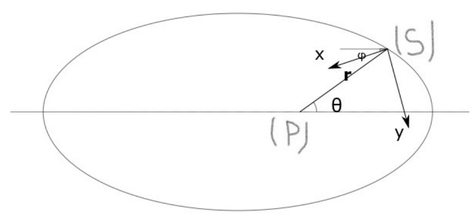
The function $\frac{1}{|\mathbf{r}-\mathbf{r'}|}$ is the generating function for Legendre polynomials, so:

$$\phi = -G \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\alpha) \rho dV.$$

Dipole term vanishes, so we keep only monopole and quadrupole terms. After some manipulations, we arrive at:

$$\phi = -\frac{Gm}{r} - \frac{3}{2}\frac{G(I_x x^2 + I_y y^2 + I_z z^2)}{r^5} + \frac{1}{2}\frac{G(I_x + I_y + I_z)}{r^3}.$$

The force acting on the planet:



$$F_x = -M\frac{\partial \phi}{\partial x}.$$

$$F_y = -M\frac{\partial \phi}{\partial y}.$$

$$F_z = -M\frac{\partial \phi}{\partial z}.$$

The z component of the torque:

$$\tau_z = F_y x - F_y x.$$

After long calculations, we get:

$$\tau_z = \frac{3GMxy(I_x - I_y)}{r^5}.$$

$$x = r\cos(\theta - \phi).$$

$$y = r\sin(\theta - \phi).$$

The equation of motion:

$$\frac{d^2\phi}{dt^2} + \frac{2GM(I_x - I_y)\sin(2\theta - 2\phi)}{2I_z r^3} = 0.$$

It is well known that:

$$\frac{GM}{a^3} = (\frac{d\theta}{dt})^2.$$

Also:

$$r = \frac{a}{1 + e\cos\theta}.$$

Approximately:

$$\frac{d^2\phi}{dt^2} + \frac{3}{2}\frac{(I_x - I_y)}{I_z}(1 + 3e\cos\theta)\sin(2\phi - 2\theta) = 0.$$

This becomes:

$$\frac{d^2\phi}{dt^2} + \frac{3}{2}\frac{(I_x - I_y)}{I_z}(\sin(2\phi - 2\theta) + \frac{3e}{2}\sin(2\phi - \theta) + \frac{3e}{2}\sin(2\phi - 3\theta)) = 0.(1)$$

The first term in the paranthesis coresponds to 1:1 resonance, the second to 1:2, the third to 3:2. To see why this ratios are stable let's assume we're close to one of the resonances (say 3:2). Then ($\frac{d\phi}{dt} = \omega_z$):

$$2\frac{d\phi}{dt} = 3\frac{d\theta}{dt} + \delta\omega_z.$$

$$2\phi = 3\theta + \delta\omega_z t.$$

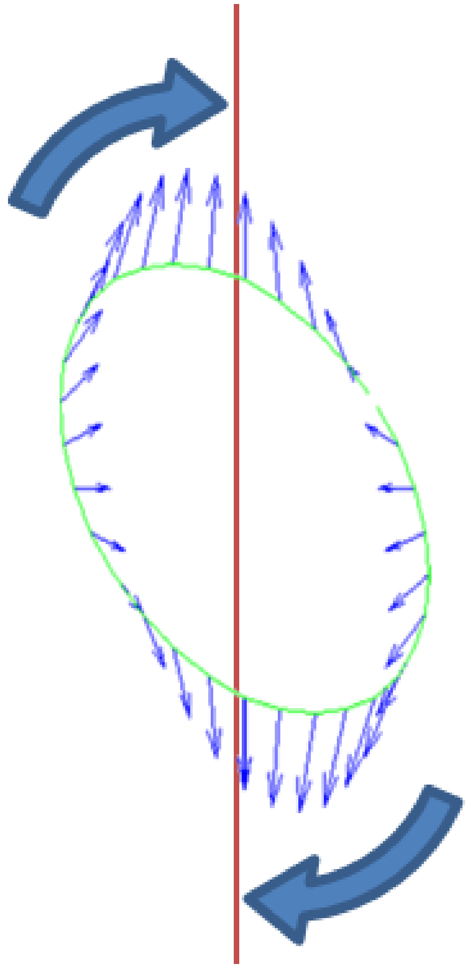
Averaging (1) over many periods, the terms corresponding to other resonances average to zero and we're left with:

$$< \frac{d^2\phi}{dt^2} > + \frac{3e}{2}\frac{(I_x - I_y)}{I_z}\delta\omega_z t = 0.$$

If $\delta\omega_z > 0$, $\frac{d^2\phi}{dt^2} < 0$, which means the tendency will be to come back to resonance, which means stability. The degree of stability varies with eccentricity e and asphericity ($\frac{I_x - I_y}{I_z}$). For example, the Moon has e=0.0549 so the more stable resonance is 1:1. Mercury has e=0.205 so all the resonances have comparable stabilities. The actual state of its 3:2 resonance between its rotation and the orbital mortion around the Sun depends on the history of the planet.

2.2 How they got there - qualitatively

We proved the stability of the observed resonance, but we also have to present a possible mechanism responsible for spin-orbit resonance: tidal locking. Let's consider the Moon. The nonuniformity of Earth's gravitational field created tidal bulges (see figure below) on the Moon. Because of friction, these bulges lead the Earth-Moon line if the Moon rotates faster than it orbits, a torque will act on the Moon, which will tend two equalize the rotation and revolution periods. That is why we only see one side of the Moon. A similar phenomenon is determing the gradual equalization of Earth's rotation (day) and revolution around Earth-Moon center of mass periods. Next, we will try to build a quantitative model and explain the observed lenghtening of an Earth day (2.3 ms/century) and increase in Earth-Moon distance (3.8cm/century). However, before they reach synchronous rotation, Sun will become a red giant. We will use the observed angle between tidal bulges on Earth and Earth-Moon line (10 degrees). A similar calculation was performed at IPhO 2009 T1, but we believe that model to be oversimplifying.



2.3 Quantitative predictions

First, let's calculate the shape of the ocean surface on Earth due to tides from the Moon (mass M). This will have the shape of an equipotential in the system rotating about CM at

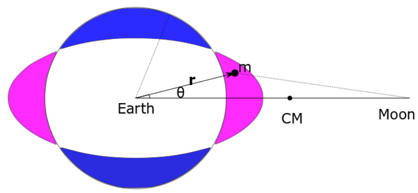
$$\omega^2 = \frac{G(M + M_E)}{L^3}.$$

Potential from Moon attraction:

$$V_M = -\frac{GMm}{\sqrt{L^2 + r^2 - 2rL\cos\theta}}.$$

Potential from Earth:

$$V_E = \frac{GM_E m}{r}.$$



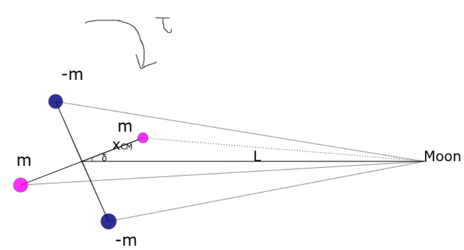
'Centrifugal potential':

$$V_{cf} = -\frac{mG(M+M_E)}{2L^3}(r^2 + L^2\frac{M^2}{(M+M_E)^2} - 2rL\frac{M}{M+M_E}).$$

After imposing the condition of constant potential and some calculations we get:

$$r = R + \frac{R^4 M}{2L^3 M_E}(3\cos^2\theta - 1)$$
$$h = r - R.$$

We will calculate the torque on the Moon from this mass distribution -τ and the torque on Earth from the Moon will be τ. If we perform a multipole expansion, we notice that the monopole term gives no torque, the dipole moment vanishes and we are left only with quadrupole term. We will substitute this by an ideal quadrupole: the positive masses at the center of mass of the pink coloured bulges and the negative masses at the blue coloured missing water (see figures)



$$m = \frac{4\pi}{3\sqrt{3}}\frac{R^6 M}{M_E L^3}.$$

$$m = 3.68 * 10^{16} kg.$$

An approximate calculation gives:

$$x_{CM} = \frac{\sqrt{3}}{4}R.$$

Using the figure above we arrive at the following expression for the torque on the Earth tending to allign the bulges:

$$\tau = \frac{9GMmR^2 \sin 2\delta}{4L^3}.$$

We know that $I_E = 8 * 10^{37} kgm^2$. Writing the second law for rotational motion:

$$-\tau = I_E \frac{d\Omega}{dt}$$

and substituting numerical values:

$$\frac{d\Omega}{dt} = -6.34 * 10^{-22} s^{-2}$$

$$\frac{dT}{dt} = 2.3 ms/century$$

which is in very good agreement with experimental value. Using the conservation of angular momentum and keeping only the significant terms

$$J = I_E \Omega + ML^2 \omega$$

and $\omega^2 = \frac{G(M+M_E)}{L^3}$ we arrive at

$$\frac{dL}{dt} = -\frac{2I_E}{ML\omega} \frac{d\Omega}{dt}$$

and numerically

$$\frac{dL}{dt} = 4.2 m/century$$

in acceptable agreement with the observed value of $3.8 m/century$.

We attached the full intermediate calculations that led to the above formulas in the *Appendix*. (or see pages 19-21)

3 Orbit-orbit resonance

3.1 Orbit-orbit resonance:qualitative model

Observations made on certain moons or planets in our solar system led astronomers to the conclusion that the periods of their orbits are linked in a very peculiar way. Celestial bodies' whose orbital periods are related by a ratio of small integers exhibit a periodical gravitational influence that gives them stability.

If their orbital periods do not respect the rule of small integer ratio, their evolution is either a chaotic one (no unusual interaction-Earth and Mars), either they get expelled from the system. The latter is best exemplified in gaps asteroid or planetary rings.

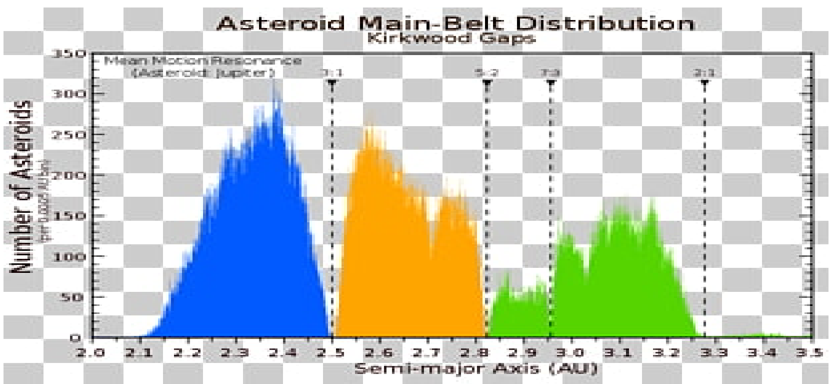
0.5 

Figure 1: Kirkwood belt gaps due to resonance with Jupiter

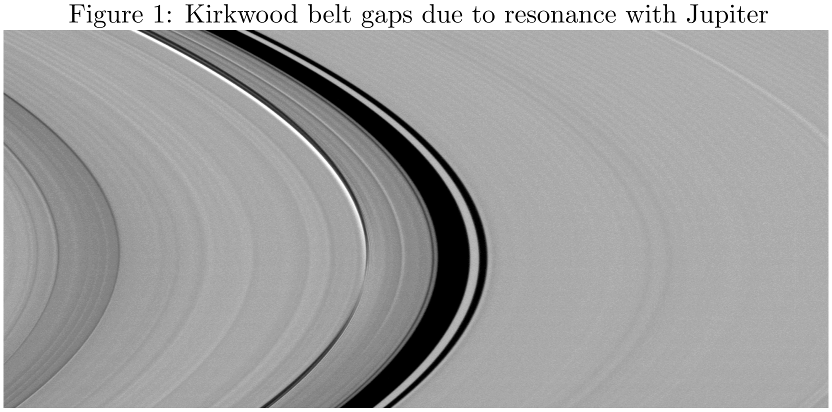
0.5 

Figure 2: The eccentric Titan Ringlet in the Columbo Gap of Saturn's C Ring

Figure 3: Observed examples of orbital resonance

The observations lead to the conclusions that their orbital periods have the following properties at resonance:

1) system with high stability →gap of only 1 between the two integers:
$\frac{T_1}{T_2} = \frac{m+1}{m}$ where $m \in N^*$
2) systems with low stability→gap of more than 1 between the two integers e.g.: $\frac{8}{5}$ $\frac{7}{4}$

3.2 Orbit-orbit resonance: rigorous model

The rigorous model consists of the analysis of a three body system: the main planet/star M, a small mass m_1 orbiting M and a much larger perturbation object m_2. ($m_2 >> m_1$)

The system in which we will be working is a rotational one attached to the center of mass of M and m_2.

First we will calculate angular frequency for the 2-body system consisting of point masses M and m2:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m_2)}$$

$$\Omega^2 = \frac{G(m_2+M)}{(r_2+\rho)^3}$$

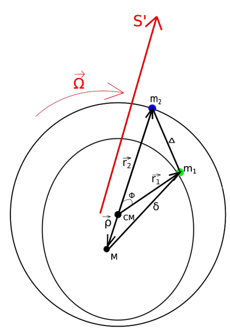


Figure 4: Rotational system

We will now study the motion of infinitesimally mass m1 in the rotational reference frame S'.

Using notations from the sketch above:

$$\vec{F_1'} = \frac{GMm_1}{\delta^3}(\vec{\rho} - \vec{r_1}) + \frac{Gm_1m_2}{\Delta^3}(\vec{r_2} - \vec{r_1}) - 2m_1\vec{\Omega} \times \dot{\vec{r_1}} - m_1\vec{\Omega} \times (\vec{\Omega} \times \vec{r_1})$$

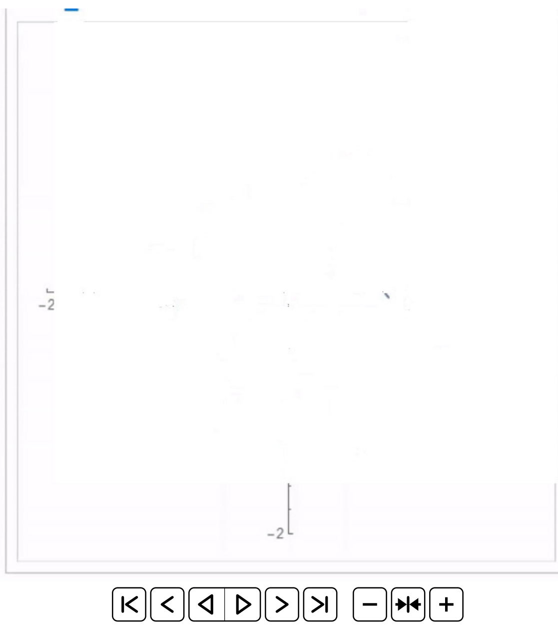
Now using Newton's second law:

$$m_1\ddot{\vec{r_1}} = \vec{F_1'}$$

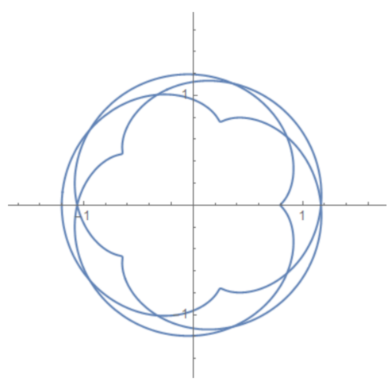
Using the expression for the force from above and double cross product rule yields:

$$\ddot{\vec{r_1}} + 2\Omega \times \dot{\vec{r_1}} = \frac{GM}{\delta^3}(\vec{\rho} - \vec{r_1}) + \frac{Gm_2}{\Delta^3}(\vec{r_2} - \vec{r_1}) + \Omega^2 \vec{r_1}$$

A simulation of the above equation made using Wolfram Mathematica (viewable with Adobe Reader):



And the final frame:



Next step is projecting vectors on axes. So:

$$\vec{r_1} = (x, y, z) \qquad \vec{\rho} = (-\rho, 0, 0)$$

$$\vec{r_2} = (r_2, 0, 0) \qquad \vec{\Omega} = (0, 0, \Omega)$$

Defining an potiential function so that:

$$\ddot{x} - 2\Omega\dot{y} = \frac{\partial U}{\partial x}$$

$$\ddot{y} + 2\Omega\dot{x} = \frac{\partial U}{\partial y}$$

$$\ddot{z} = \frac{\partial U}{\partial z}$$

$$\frac{\partial}{\partial y}(\frac{\partial U}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial U}{\partial y}) = 0$$

Yields

$$U = \frac{GM}{\delta} + \frac{Gm_2}{\Delta} + \frac{\Omega^2 r_1^2}{2}$$

Now using center of mass proprieties and geometric relations:

$$M\vec{\rho} + m_2\vec{r_2} = \vec{0}$$

$$\rho^2 = r_1^2 + \delta^2 + 2r_1\delta cos\Phi$$

$$\Delta^2 = r_1^2 + r_2^2 - 2r_1r_2 cos\Phi$$

Assuming that orbit excentricities are relative small, we can neglect centrifugal term variance. Making this assumption, the potential variance from equilibrium motion is:

$$\delta U = \frac{Gm_2}{\Delta} - \frac{Gm_2 r_1 cos\Phi}{r_2^2}$$

3.3 Numerical analysis

Previously, we showed that the potential variance takes the form:

$$\delta U = \frac{Gm_2}{\Delta} - \frac{Gm_2 r_1 cos(\Phi)}{r_2^2}$$

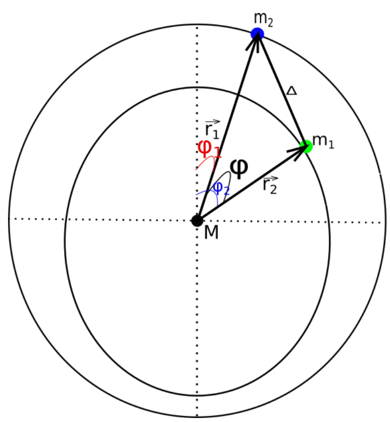


Figure 5: The schematics of our rigorous model

In which we use the formulas for coordinates on an ellipse and cosine rule for Δ:

$$r_1 = \frac{a_1(1-e_1^2)}{1+e_1 cos(\phi_1)} \qquad r_2 = \frac{a_2(1-e_2^2)}{1+e_2 cos(\phi_2)} \qquad \Delta = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 cos(\phi)}$$

In order to achieve resonance, the the forces of interaction must be at a maximum when the two bodies are the closest: both ϕ_1 and ϕ_2 are equal to 0. Thus, $\phi = \phi_2 - \phi_1$ is also zero. This translates into the potential variance canceling out.

We remind that the potential variance was relative to equilibrium thus it represents the derivative of the potential. For a function to have maximum value, its derivative is zero. Because the variation of interaction is characterised by the potential variance the latter has to be zero.

$$\delta U = 0$$

First we work on our equation by forcing Gm_2:

$$Gm_2(\frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 cos(\phi)}} - \frac{r_1 cos(\Phi)}{r_2^2}) = 0$$

By plugging in r_1 and r_2 in the previous parenthesis and squaring it we have:

$$r_2^4 = r_1^2(r1^2 + r_2^2 - 2r_1 r_2 cos(\phi))cos(\phi)^2$$

$$a_2^4(\frac{1-e_2^4}{1+e_2 cos\phi_2})^4 = [a_1^4(\frac{1-e_1^4}{1+e_1 cos\phi_1})^4 + a_1^2 a_2^2(\frac{(1-e_2^2)(1-e_1^2)}{(1+e_2 cos(\phi_2)(1+e_1 cos(\phi_1)})^2 -$$

$$-2a_1^3 a_2(\frac{1-e_1^3}{1+e_1 cos(\phi_1)})^3 \frac{1-e_2}{1+e_2 cos(\phi_2)}]cos(\phi)^2 \qquad (*)$$

As mentioned before we take $\phi_1 = \phi_2 = \phi = 0$, therefore $cos(\phi_1) = cos(\phi_2) = cos(\phi) = 1$.

In our model we used the general equations which involved any arbitrary e_1 and e_2.

After a thorough analysis of measured data on planets' and asteroids' orbital eccentricity (data taken from NASA official website), we came to the conclusion that the moons of Jupiter which are known to be in resonance (Io, Europa and Ganymede), have negligible eccentricities: 0.0041, 0.0094, respectively 0.0011. Thus, we will use a second order approximation for e_1 and e_2 with the formula:

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)x^2}{2}, x << 1$$

In our approximation we keep only the terms which have powers less than or equal to 4 e.g. e_1^4, e_1^4, $e_1 e_2^3$ or $e_2 e_1^3$. After performing the approximations on the terms in equation (*) we end up with the following expression:

$$a_2^4(1-4e_2-10e_2^2+16e_2^3+30e_2^4) = a_1^4(1-4e_1-10e_1^2+16e_1^3+30e_1^4)+$$

$$+a_1^2a_2^2(1-2e_1-2e_2+4e_1^3+4e_2^3+3e_1^4+3e_2^4+4e_1e_2+5e_1^2e_2^2+6e_1^2e_2+6e_1e_2^2-8e_1e_2^3-8e_2e_1^3)-$$

$$-2a_1^3a_2(1-3e_1-6e_1^2-2e_2+6e_1e_2+12e_1^2e_2-6e_1e_2^2-12e_1^2e_2^2+2e_2^2-e_2^3+3e_1e_2^3+9e_1^3+12e_1^4-$$

$$-18e_1^3e_2) \qquad (**)$$

We will now solve (**) for $\frac{a_1}{a_2}$ and $\frac{T_1}{T_2} = (\frac{a_1}{a_2})^{3/2}$ by forming new equations with the symmetrical terms containing e_1 and e_2.

Note that these values for $\frac{T_1}{T_2}$ correspond to two adjacent objects in orbit, thus the solutions account for more than two objects in orbit which respect the condition that two neighbouring orbits have the set $\frac{T_1}{T_2}$ e.g 1:2:4:8 or 1:3:6 etc.

Case ID	Power and eccentricity	$\frac{a_1}{a_2}$	$\frac{T_1}{T_2}$
a)	e_2^0 or e_1^0	1.62	2.062
b)	e_2^1	0.917	0.878
c)	e_1^1	1 and 0.5	1 and 0.353
d)	e_2^2	1.357	1.58
e)	e_1^2	1.20	1.31
f)	e_2^3	1.716	2.48
g)	e_1^3	0.82 and 0.3	0.742 and 0.164
h)	e_2^4	$\sqrt{10}$	5.62
i)	e_1^4	0.645 and 0.155	0.52 and 0.061
j)	$e_1^2 e_2$	0.25	0.125
k)	$e_1^3 e_2$	0.222	0.104
l)	$e_1 e_2$	0.333	0.192

N.B. other arrangements of e_1 and e_2 did not lead to real solutions for $\frac{a_1}{a_2}$ and $\frac{T_1}{T_2}$ respectively.

Let us take a look at the set of values given by our simplified model:

i) From a) we obtain the approximate value of $\frac{T_1}{T_2} = 2.062$ which implies a 1:2.062:4.124 etc. resonance. By comparison with public data, these values correspond to the three moons of Jupiter known to be in resonance: Io, Europa and Ganymede.

Their orbital periods are as follows (in Earth days):

$T_i = 1.7691, \quad T_e = 3.5512, \quad T_g = 7.1546.$

Relative to the inner one, Io, these lead to a 1:2.0073:4.0044 resonance which is very close to the one predicted by our model. Therefore, we have a prime numerical confirmation that our simplified model is a correct one.

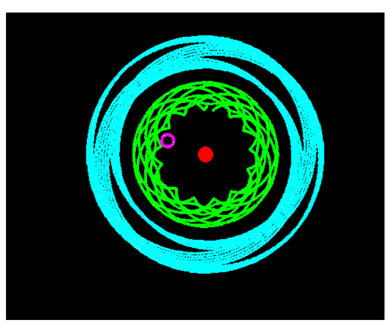


Figure 6: Io-Europa-Ganymede 1:2:4 resonance as seen from a rotating frame attached to Io. Pink→ Io, Green→Europa, Blue→Ganymede,In the center→ Jupiter

ii) From c) we obtain the approximate value of $\frac{T_1}{T_2} = 1$ which implies a 1:1 resonance. This special case of resonance lies at the bottom, of both Lagrange points theory and the clearance of space between objects with similar orbits due to their gravitational interaction.

iii) From d) we obtain the approximate value of $\frac{T_1}{T_2} = 1.58 \approx 1.6$ which implies a 5:8 resonance. As has been discovered recently (Danielsson and Ip, 1972), the Earth and Toro form an 8:5 resonance system. In the simulation you can see the 5 characteristic areas. Therefore, we have a second numerical confirmation that our simple model is a correct one.

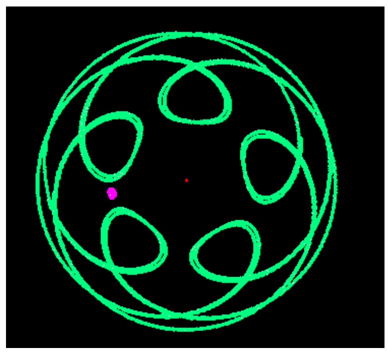


Figure 7: Earth-Toro 8:5 resonance as seen from a rotating frame attached to the Earth. Pink→ Earth, Green→1685 Toro,In the center→ the Sun

iv) From e) we obtain the approximate value of $\frac{T_1}{T_2} = 1.31 \approx \frac{4}{3}$ which implies a 3:4 resonance. In the simulation you can see the 3 characteristic areas. This result of our model corresponds to the 3:4 resonance of Tian and Hyperion, moons of Saturn.

Their orbital periods are as follows (in Earth days):
$T_t = 15.94542, \quad T_h = 21.27661.$

Relative to Titan $\frac{T_h}{T_t} \approx 1.334$, thus we can safely say that our simplified model holds true with known observations.

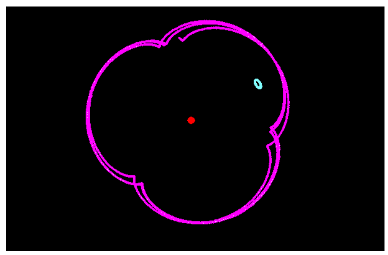


Figure 8: Titan-Hyperion 4:3 resonance as seen from a rotating frame attached to the Titan. Blue→ Titan, Pink→Hyperon, In the center→ Saturn

v) From f) we obtain the approximate value of $\frac{T_1}{T_2} = 2.48 \approx 2.50$ which implies a 2:5 resonance. This result of our model corresponds to the previously mentioned Kirkwood asteroid belt gaps.

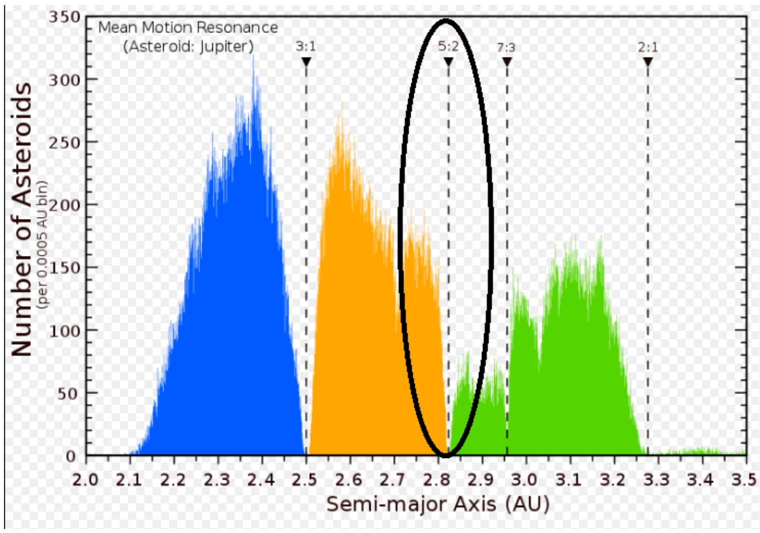


Figure 9: Highlight of our result being found in data taken from nature

We attached the full intermediate calculations that led to the above formulas in the *Appendix*. (or see pages 22-23)

3.4 Conclusions

In conclusion, the simplified model for orbit-orbit resonance we proposed accounts for a number of real-life examples of the phenomena. Although the approximations of the potential variance could have been done more rigorously using higher mathematics, for simplicity we preferred to stick to a second order approximation.

4 Appendix

Here we attached all the calculations we wrote while developing the mathematical model.

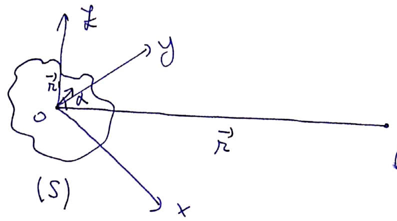
These were the intermediary formulas that lead to the main equations in the previous sections.

The formulas are, in order of appearance:

From the spin-orbit section: first 11 photos

The orbit-orbit section: the next 6 photos

Hero Why stable?



$$V(\vec{r}') = -G\int \frac{1}{|\vec{r} - \vec{r}'|}\rho\, d\vartheta$$

the function $\frac{1}{|\vec{r}' - \vec{r}'|}$ is ~~called~~ the generating function for Legendre polynomials so:

$$V(\vec{r}') = -G\sum_{M=0}^{\infty}\frac{1}{r^{M+1}}\int (r')^M P_M(\cos\alpha)\rho(\vec{r}')d\vartheta'$$

dipole term vanishes

we keep only monopole and quadrupole terms:

$$V(\vec{r}') \approx -G\left[\frac{1}{r}\int\rho(\vec{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\vec{r}')\,\cancel{d\tau}\,d\vartheta' + \frac{1}{r^3}\int r'^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\vec{r}')d\vartheta'\right]$$

$$= -\frac{Gm}{r} - \frac{G}{r^3}\int\left[\frac{3}{2}\left(\frac{\vec{r}'\vec{r}'}{r}\right)^2 - \frac{x'^2}{2} - \frac{y'^2}{2} - \frac{z'^2}{2}\right]\rho(\vec{r}')d\vartheta'$$

$$\cancel{= -\frac{GM}{r} - \frac{G}{r^3}\int\left(\frac{3}{2}\; x'x + y'y + z'z\right.}$$

$$= -\frac{GM}{r} - \frac{3}{2}\frac{G}{r^5}\int(x'x + y'y + z'z)^2\rho(\vec{r}')d\vartheta' +$$

$$+ \frac{G}{2r^3}(I_x + I_y + I_z) = -\frac{GM}{r} - \frac{3}{2}\frac{G}{r^5}\int(x'^2x^2 +$$

$$+ y'^2y^2 + z'^2z^2 + 2x'xy'y + 2x'z'xz + 2y'z'yz)\,\rho(\vec{r}')d\vartheta'$$

from the definition of principal ~~axis~~ axes we know:

$$\int\rho(\vec{r}')\,x'y'\,d\vartheta' = 0$$ (the off-diagonal terms in the inertia tensor)

$$V(\vec{r}') = -\frac{GM}{r} - \frac{3}{2}\frac{G}{r^5}(I_x x^2 + I_y y^2 + I_z z^2) + \frac{1}{2}\frac{G}{r^3}(I_x + I_y + I_z)$$

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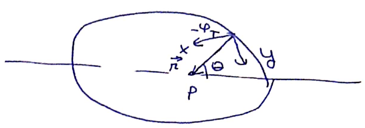

$$F_x = -M\frac{\partial U}{\partial x}$$

$$\vec{M} = \vec{r}\times\vec{F} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ x & y & z\\ F_x & F_y & F_z\end{vmatrix}$$

$$M_z = xF_y - yF_x$$

$$M_z = I_z\frac{d\omega_z}{dt} = I_z\ddot{\varphi}$$

$$x = r\cos(\theta - \varphi)$$

$$y = r\sin(\theta - \varphi)$$

$$F_x = +\frac{GMm}{r^2}\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} - \frac{1}{2}\frac{G I_x M}{r^4}\cdot 3\frac{x}{r} - \frac{3}{2}\frac{G I_y M}{r^5}x - \frac{3}{2}\frac{G M I_z}{r^5}x + \frac{15}{2}\frac{GM}{r^7}x(I_x x^2 + I_y y^2 + I_z z^2) - \frac{3}{2}\,2\,\frac{GM}{r^5}I_x x = \frac{GMmx}{r^3} -$$

$$-\frac{3}{2}\frac{GM}{r^5}x(I_x + I_y + I_z) + \frac{15}{2}\frac{GM}{r^7}x(I_x x^2 + I_y y^2 + I_z z^2) - 3\frac{GM}{r^5}I_x x$$

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$$\cancel{\tau_z = \frac{GMmxy}{r^3} - \frac{3}{2}\frac{GMy}{r^5}x(I_x + I_y + I_z) + \frac{15}{2}\frac{GM}{r^7}xy(I_x x^2 + I_y y^2 + I_z z^2)}$$

$$\tau_z = -x\cdot\frac{3GM}{r^5}I_y y + \frac{3GM}{r^5}xy I_x =$$

$$= \frac{3GM}{r^5}xy(I_x - I_y)$$

$$I_z\ddot{\varphi} = -\frac{3GM}{r^{\cancel{5}3}}(I_x - I_y)\,\cancel{r^2}\,\frac{\sin 2(\theta - \varphi)}{2}$$

$$\cancel{\ddot{\varphi} =}$$

$$\ddot{\varphi} + \frac{3}{2}\frac{GM(I_x + I_y)}{I_z\; r^3}\sin(2(\theta + \varphi)) = 0$$

$$\cancel{\frac{GM}{r^3} = \dot{\theta}^2} \qquad \frac{GM}{a^3} = \dot{\theta}^2$$

$$\ddot{\varphi} + \frac{3}{2}\left(\frac{a}{r}\right)^3\frac{I_y - I_x}{I_z}\sin(2(\varphi - \theta)) = 0$$

$$r = \frac{a}{1 + e\cos\theta}$$

$$\left(\frac{a}{r}\right)^3 = (1 + e\cos\theta)^3 \approx 1 + 3e\cos\theta$$

$$\ddot{\varphi} + \frac{3}{2}\frac{I_y - I_x}{I_z}(1 + 3e\cos\theta)\sin(2\varphi - 2\theta) \cong 0$$

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$$\cos\theta \sin(2\varphi - 2\theta) = \frac{1}{2}\left(\sin(\theta + 2\varphi - 2\theta) + \sin(\cancel{\theta - 2\varphi}\ 2\varphi - 3\theta)\right)$$

$$\ddot{\varphi} + \frac{3}{2}\frac{I_y - I_x}{I_z}\Big[\underbrace{\sin(2\varphi - 2\theta)}_{1:1\ \text{res}} + \frac{3e}{2}\underbrace{\sin(2\varphi-\theta)}_{1:2\ \text{res}} + \frac{3e}{2}\underbrace{\sin(2\varphi - 3\theta)}_{3:2\ \text{resonance}}\Big) \simeq 0 \quad (1)$$

To see why those ratios are stable let's assume we're close to one of the resonances (3:2) then ($\dot{\varphi} = \omega_z$)

$$2\dot{\varphi} = 3\dot{\theta} + \delta\omega_z$$

$$2\varphi \cong 3\theta + \delta\omega_z t$$

averaging (1) we get rid of all the terms except:

$$\langle \ddot{\varphi} \rangle + \frac{3}{2}\frac{I_y - I_x}{I_z}\frac{3e}{2}\delta\omega_z t \approx 0$$

if $\delta\omega_z > 0 \Rightarrow \langle\ddot{\varphi}\rangle < 0 \Rightarrow \langle\dot{\omega}_z\rangle < 0$

$\Rightarrow$ it is stable

with the degree of ~~stabl~~ stability proportional to the ~~e~~ eccentricity and asphericity $\left(\frac{I_x - I_y}{I_z}\right)$ 15

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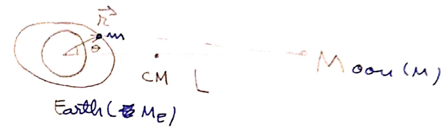

$$L\frac{M}{M_E + M} \qquad r = R + h$$

$$\omega^2 = \frac{G(M + M_E)}{L^3}$$

$$V_M = -\frac{GMm}{\sqrt{L^2 + r^2 - 2rL\cos\theta}} \approx -\frac{GMm}{L}\left[1 + \frac{r}{L}\cos\theta + \frac{r^2}{L^2}\frac{3\cos^2\theta - 1}{2} + \dots\right]$$

$$V_E = -\frac{GM_E m}{r}$$

$$V_{cf} = -m\omega^2(r^2 + L'^2 - 2rL'\cos\theta)$$

$$V_{tot} = \text{const}$$

$$V_{cf} = -\frac{mG(M+M_E)}{2L^3}\left(r^2 + L^2\frac{M^2}{(M+M_E)^2} - 2r\frac{ML}{M+M_E}\cos\theta\right)$$

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$$V_{cf} = -\frac{mG(M+M_E)}{2L^3}r^2 - \frac{mGM^2\cancel{L^2}}{2L^{\cancel{3}}(M+M_E)} + \frac{\cancel{2}GmM\cos\theta\, r}{\cancel{2}L^2}$$

~~$V = V_0 +$~~

$$V - V_0 = -\frac{GMm}{2L^3}r^2(3\cos^2\theta - 1) - \frac{Gm(M+M_E)}{2L^3}r^2 - \frac{GM_E m}{r}$$

$$r^2 \approx R^2 + 2Rh$$

$$\frac{1}{r} \approx \frac{1}{R} - \frac{h}{R^2}$$

$$V - V_0 \approx -\frac{GMm}{2L^3}r^2(3\cos^2\theta - 1) - \frac{Gm(M+M_E)}{2L^3}2Rh + \frac{GM_E m}{R^2}h$$

$$\frac{GM_E m}{R^2}h = \frac{GMm}{2L^3}R^2(3\cos^2\theta - 1)$$

$$\boxed{h = \frac{R^4}{2L^3}\frac{M}{M_E}(3\cos^2\theta - 1)}$$

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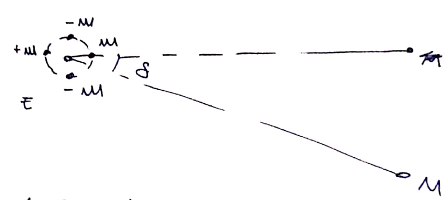
Model



$\cos\theta_0 = \frac{1}{\sqrt{3}}$ ~~$\Rightarrow \theta_0 = \frac{\pi}{6}$~~

Calculation of mass m:

$$m = 2\rho_w\int_0^{2\pi}\int_0^{\theta_0}\int_R^{R+h} r\sin\theta\, d\theta\, dr\, r\, d\varphi =$$

$$= 2\rho_w 2\pi\int_0^{\theta_0}\sin\theta\, d\theta\,\frac{(R+h)^3 - R^3}{3} \approx$$

$$\approx 4\pi\rho_w\int_0^{\theta_0}\sin\theta\, d\theta\,\frac{R^3 + 3R^2h - R^3}{3}$$

$$= 2\pi\rho_w R^2\frac{R^4 M}{L^3 M_E}\int_0^{\theta_0}\sin\theta\, d\theta\,(3\cos^2\theta - 1)$$

$$= -2\pi\rho_w\frac{R^6 M}{L^3 M_E}\int_1^{\frac{1}{\sqrt{3}}}(3\cos^2\theta - 1)\, d(\cos\theta)$$

$$= 2\pi\rho_w\frac{R^6 M}{L^3 M_E}\int_{\frac{1}{\sqrt{3}}}^1 (3x^2 - 1)\, dx =$$

$$= 2\pi\rho_w\frac{R^6 M}{M_E L^3}\left(3\frac{1 - \left(\frac{1}{\sqrt{3}}\right)^3}{3} - 1 + \frac{1}{\sqrt{3}}\right) = \cancel{\frac{4\pi\rho_w}{3\sqrt{3}}}$$

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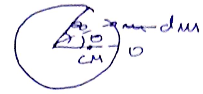
$$m = \frac{4\pi \rho \omega}{3\sqrt{3}} \frac{R^6 M}{M_E L^3} = 3{,}68 \cdot 10^{16}\,kg$$

Calculation of Center of Mass



$$x_{CM} = \int_0^{\theta_0} \frac{R\cos\theta \, dm}{m} = \int_0^{\theta_0} \frac{R\cos\theta}{m} 2\pi\rho\omega$$

$$\ldots R^2 \sin\theta d\theta = \frac{2\pi\rho\omega R^3}{m} \frac{R^4 M}{2L^3 M_E} \int_{\frac{1}{\sqrt{3}}}^{1} \cos\theta(3\cos^2\theta - 1)\, d(\cos\theta)$$

$$= R \frac{\cancel{2\pi}\rho\omega R^{\cancel{2}}}{\frac{4\pi\rho\omega}{3\sqrt{3}} \frac{R^6 M}{M_E L^3}} \frac{R^4 M}{2L^3 M_E} \int_{\frac{1}{\sqrt{3}}}^{1} (3x^3 - x)\, dx$$

$$= R \frac{3\sqrt{3}}{4} \left(3 \frac{1 - \frac{1}{9}}{4} - \frac{1 - \frac{1}{3}}{2} \right)$$

$$= R \frac{\sqrt{3}}{4} = 0{,}43 R$$

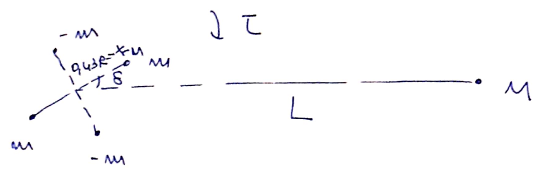
~~$2 \; 0{,}43 R \, m = 2R m_0$~~

~~$m_0 = 0{,}43 m$~~

ПБ

Calculation of torque (τ)

$\delta = 10^0$



$$\tau = \frac{GMm\,x_{CM}L\sin\delta}{(L^2 + x_{CM}^2 - 2Lx_{CM}\cos\delta)^{\frac{3}{2}}} - \frac{GMm\,x_{CM}L\sin\delta}{(L^2 + x_{CM}^2 + 2Lx_{CM}\cos\delta)^{\frac{3}{2}}} +$$

$$+ \frac{GMm\,x_{CM}L\cos\delta}{(L^2 + x_{CM}^2 - 2Lx_{CM}\sin\delta)^{\frac{3}{2}}} - \frac{GMm\,x_{CM}L\cos\delta}{(L^2 + x_{CM}^2 + 2Lx_{CM}\sin\delta)^{\frac{3}{2}}}$$

$$\tau \approx \frac{\cancel{12}^{6}\, Gm \cdot 0{,}43^2 R^2 \sin 20^0}{\cancel{2} L^3} = 5{,}08 \cdot 10^{16}\,Nm$$

$$I_E \frac{d\Omega}{dt} = -\tau$$

$$\frac{d\Omega}{dt} = 6{,}34 \cdot 10^{-22}\,s^{-2}$$

$$\frac{dT}{dt} = -\frac{2\pi}{\Omega^2} \frac{d\Omega}{dt} = 7{,}48 \cdot 10^{-13} \frac{s}{s}$$

$$\approx 2{,}3 \frac{ms}{100\,years}$$

$$\mathcal{L} \cong I_E \Omega + ML^2\omega$$

$$d\Omega = -\frac{ML\omega}{2I_E} dL$$

$$\Rightarrow \frac{dL}{dt} = 4{,}2 \frac{m}{100\,years}$$

$$\frac{Gm_2}{a_2}\cdot\frac{1+e_2\cos\varphi_2}{1-e_2^2}\left(\frac{1}{\sqrt{\left(\frac{a_1}{a_2}\right)^2\cdot\left(\frac{1-e_1^2}{1-e_2^2}\cdot\frac{1+e_2\cos\varphi_2}{1+e_1\cos\varphi_1}\right)^2+1+}}\right.$$

$$-2\,\frac{a_1}{a_2}\,\frac{1-e_1^2}{1-e_2^2}\cdot\frac{1+e_2\cos\varphi_2}{1+e_1\cos\varphi_1}\cos\phi\Big) -$$

$$-\cos\phi\cdot\frac{a_1}{a_2}\cdot\frac{1-e_1^2}{1-e_2^2}\cdot\frac{1+e_2\cos\varphi_2}{1+e_1\cos\varphi_1}\cos\phi\Big)$$

$$\frac{Gm_2}{a_2}\,(1+e_2\cos\varphi_2)\left(\frac{1}{\sqrt{\left(\frac{a_1}{a_2}\right)^2\left(\frac{1+e_2\cos\varphi_2}{1+e_1\cos\varphi_1}\right)^2+1-2\frac{a_1}{a_2}\frac{1+e_2\cos\varphi_2}{1+e_1\cos\varphi_1}\cos\phi}}\right.$$

$$-\cos\phi\,\frac{a_1}{a_2}\,\frac{1+e_2\cos\varphi_2}{1+e_1\cos\varphi_1}\cancel{\cos\phi}\Big)$$

$$\frac{1+e_2\cos\varphi_2}{\sqrt{\left(\frac{a_1}{a_2}\right)^2(1+2e_2\cos\varphi_2-2e_1\cos\varphi_1)+1-2\frac{a_1}{a_2}(1+e_2\cos\varphi_2-e_1\cos\varphi_1)\cos\varphi}} -$$

$$-\frac{a_1}{a_2}\cos\phi\cdot\frac{1+2e_2\cos\varphi_2}{1+e_1\cos\varphi_1}$$

$$a_2^4(1-4e_2^2+6e_2^4)(1-4e_2-6e_2^2)=a_1^4(1-4e_1^2+6e_1^4)\cdot(1-4e_1-6e_1^2)+$$

$$+a_1^2a_2^2(1-2e_2^2+e_2^4)(1-2e_1^2+e_1^4)(1-2e_2-e_2^2)(1-2e_1-e_1^2)$$

$$-2a_1^3a_2(1-3e_1^2+3e_1^4)(1-e_2)(1-e_2+e_2^2)(1-3e_1-3e_1^2).$$

$$a_2^4(1-4e_2-6e_2^2-4e_2^2+16e_2^3+24e_2^4+6e_2^4-\cancel{24e_2^5}-$$
$$-36e_2^6)=a_1^4(1-4e_1-6e_1^2-4e_1^2+16e_1^3+24e_1^4+6e_1^4)+$$

$$+a_1^2a_2^2(1-2e_1^2+e_1^4-2e_2^2+4e_2^2e_1^2-\cancel{2e_2^2e_1^4}+e_2^4)(1-2e_1-e_1^2-2e_2+4e_1e_2+2e_2e_1^2-e_2^2+2e_1e_2^2+e_1^2e_2^2)$$

$$-2a_1^3a_2(1-e_2-3e_1^2+3e_1^2e_2+3e_1^4)(1-3e_1-3e_1^2-e_2+3e_1e_2+3e_1^2e_2+e_2^2-3e_1e_2^2-3e_1^2e_2^2).$$

$$+a_1^2a_2^2(1-2e_1^2+e_1^4+e_2^4-2e_2^2+4e_2^2e_1^2)(1-2e_1-2e_2-e_1^2-e_2^2+$$
$$+4e_1e_2+e_1^2e_2^2+2e_2e_1^2+2e_1e_2^2)-$$

$$-2a_1^3a_2(1-e_2-3e_1^2+3e_1^2e_2+3e_1^4)(1-3e_1-3e_1^2-e_2+$$
$$+3e_1e_2+3e_1^2e_2+e_2^2-3e_1e_2^2-3e_1^2e_2^2)$$

$$a_2^4(\ldots)=a_1^4(\ldots)+a_1^2a_2^2(1-2e_1-2e_2-e_1^2-e_2^2+4e_1e_2+e_1^2e_2^2+$$
$$+2e_2e_1^2+2e_1e_2^2-2e_1^2+4e_1^3+4e_1^2e_2+2e_1^4+2e_1^2e_2^2$$
$$-8e_1^3e_2+e_1^4+e_1^4-2e_2^2+4e_1e_2^2+4e_2^3+$$
$$+2e_2^2e_1^2+2e_2^4-8e_1e_2^3+4e_2^2e_1^2)$$

$$+a_1^2a_2^2(1-2e_1-2e_2+4e_1^2+4e_1^3+3e_1^4+3e_2^4+4e_1e_2+5e_1^2e_2^2+$$
$$+6e_1^2e_2+6e_1e_2^2-8e_1e_2^3-8e_2e_1^3).$$

$$-2a_1^3a_2(1-3e_1-3e_1^2-e_2+3e_1e_2+3e_1^2e_2+e_2^2-3e_1e_2^2-3e_1^2e_2^2$$
$$-e_2+3e_1e_2+3e_1^2e_2+e_2^2-3e_1e_2^2-3e_1^2e_2^2-e_2^3+3e_1e_2^3$$
$$-3e_1^2+9e_1^3+9e_1^4+3e_1^2e_2-9e_1^3e_2-3e_1^2e_2^2$$
$$+3e_1^2e_2-9e_1^3e_2-3e_1^2e_2^2+3e_1^4).$$

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$$-2a_1^3a_2(1-3e_1-3e_1^2-2e_2+6e_1e_2+12e_1^2e_2+e_2^2-6e_1e_2^2-12e_1^2e_2^2$$
$$+e_2^2-e_2^3+3e_1e_2^3-3e_1^2+9e_1^3+9e_1^4+3e_1^4-18e_1^3e_2)$$

$$-2a_1^3a_2(1-3e_1-6e_1^2-2e_2+6e_1e_2+12e_1^2e_2+e_2^2-6e_1e_2^2-$$
$$-12e_1^2e_2^2+e_2^2-e_2^3+3e_1e_2^3-\cancel{3e_1^3}+9e_1^3+9e_1^4+$$
$$+3e_1^4-18e_1^3e_2)$$

$\bullet$ $a_2^4\cdot 30=3a_1^2a_2^2.$

$a_2^2\cdot 10=a_1^2 \Rightarrow$ $\frac{a_1}{a_2}=\sqrt{10}.$ $\frac{m_1}{m_2}=\cancel{0{,}44}\ 5{,}62.$

$\bullet$ $0=a_1^4\cdot 30+3a_1^2a_2^2-2a_1^3a_2\cdot 12.$

$0=30a_1^2+3a_2^2-24a_1a_2$

$30\left(\frac{a_1}{a_2}\right)^2-24\,\frac{a_1}{a_2}+3=0$; $\frac{a_1}{a_2}=0{,}645$ $\left(\frac{m_1}{m_2}\right)^{-2/3}$ $0{,}52$

$=0{,}155.$ $0{,}061$

$\bullet$ $-14a_2^4=+2a_1^2a_2^2+3a_1^3a_2$

$4a_2^4=2a_1^2a_2^2+3a_1^3a_2$

$4=2\left(\frac{a_1}{a_2}\right)^2+3\left(\frac{a_1}{a_2}\right)^3$ $\frac{a_1}{a_2}=0{,}914 \Rightarrow \frac{m_1}{m_2}=$

$\frac{m_1}{m_2}=0{,}878.$

• $0=16a_1^4+4a_1^2a_2^2-18a_1^3a_2$. (V)

$$16\left(\frac{a_1}{a_2}\right)^4-18\left(\frac{a_1}{a_2}\right)^3+4\left(\frac{a_1}{a_2}\right)^2=0$$

$$16\left(\frac{a_1}{a_2}\right)^2-18\frac{a_1}{a_2}+4=0$$

~~1:3 Hyperion. Titan~~

$\boxed{\frac{a_1}{a_2}=0{,}82\ \frac{m_1}{m_2}\ \cancel{1{,}347}}$ (V)

$=0{,}30$ $\cancel{0{,}085}$.

$\frac{m_1}{m_2}=0{,}742$

$\frac{m_1}{m_2}=0{,}64$

• $0=5a_1^2a_2^2+24a_1^3a_2$

• $0=6a_1^2a_2^2-24a_1^3a_2$

$6a_1^2a_2^2=24a_1^3a_2$ (V).

$6a_2=24a_1 \qquad \frac{a_1}{a_2}=\frac{1}{4}\Rightarrow\frac{m_1}{m_2}=\frac{1}{8}$?

• ~~$0=6a_1^2a_2^2+12a_1^3a_2$~~

• ~~$0=-8a_1^2a_2^2-6a_1^3a_2$~~

• $0=-8a_1^2a_2^2+36a_1^3a_2$. (V)

$8a_1^2a_2^2=36a_1^3a_2$

$8a_2=36a_1\Rightarrow\frac{a_1}{a_2}=\frac{2}{9}\Rightarrow\frac{m_1}{m_2}=$ ~~0,254~~ $0{,}104$.

• $0=-4a_1^4-2a_1^2a_2^2+6a_1^3a_2$ (V)

$4a_1^4+2a_1^2a_2^2-6a_1^3a_2=0$

$4a_1^2+2a_2^2-6a_1a_2=0$

$4\left(\frac{a_1}{a_2}\right)^2-6\frac{a_1}{a_2}+2=0$

$\frac{a_1}{a_2}=1$ $\quad\boxed{\frac{m_1}{m_2}=1}$

$=0{,}5$ $\quad\boxed{=\cancel{2{,}83}}$ $0{,}353$

• $-10a_2^3=-4a_1^3a_2$ (V)

$10a_2^3=4a_1^3\Rightarrow\frac{a_1}{a_2}=1{,}357$ $\Rightarrow$

$\frac{m_1}{m_2}=1{,}58$

• ~~$10a_1$~~ $0=-10a_1^4+12a_1^3a_2$

(V) $10a_1=12a_2\Rightarrow\frac{a_1}{a_2}=1{,}2\Rightarrow$

$\frac{m_1}{m_2}=1{,}31$

Hyp. Titan.

5 Bibliography

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