# Math-4541 (Solutions)
## Mid Exam

**1(a)** Find the four fourth roots of $z = 1+i$

Sol$^n$: $r = |z| = \sqrt{2}$, $Arg\text{-}z = \theta = \pi/4$

$$\therefore z^{1/4} = (\sqrt{2})^{1/4}\left[\cos\left(\frac{\pi/4 + 2k\pi}{4}\right) + i\sin\left(\frac{\pi/4+2k\pi}{4}\right)\right]$$

$k = 0, 1, 2, 3$

$\therefore k=0, \quad z_0 =$

$k=1, \quad z_1 =$

$k=2, \quad z_2 =$

$k=3, \quad z_3 =$

**1(b)** Compute $z^9$ for $z = 1+\sqrt{3}i$

Here, $r = 2$, $\theta = \pi/3$

$$\therefore z^9 = 2^9\left(\cos 9\pi/3 + i\sin 9\pi/3\right) = -512$$

**1(c)** Solve $z^2(1-z^2) = 16$

Sol$^n$: $z^4 - z^2 + 16 = 0$

$\Rightarrow z^4 + 8z^2 + 16 - 9z^2 = 0$

$\Rightarrow (z^2+4)^2 - 9z^2 = 0$

$\Rightarrow (z^2+4+3z)(z^2+4-3z) = 0$

$\therefore z = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}i, \ \frac{3}{2} \pm \frac{\sqrt{7}}{2}i$

Q-2. (a) Compute the limit of $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$

Sol$^n$: $\lim_{z \to i} z^2 = -1$, $\lim_{z \to i} z^4 = 1$

$$\therefore \lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{3+i+1+2i}{1+i} = \frac{4+3i}{1+i}$$

$$= \frac{7}{2} - \frac{1}{2}i$$

2(b) Show that the complex function $f(z) = 2x^2 + y + i(y^2 - x)$ is not analytic at any point.

Sol$^n$:

$$f(z) = 2x^2 + y + i(y^2 - x)$$

$\therefore u = 2x^2 + y$ $\quad v = y^2 - x$.

$\therefore \frac{\partial u}{\partial x} = 4x$ $\quad \frac{\partial v}{\partial y} = 2y$

$\frac{\partial u}{\partial y} = 1$ $\quad \frac{\partial v}{\partial x} = -1$

Here $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ but $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ (Equal only at $y = 2x$ line) but There no neighbourhood or open disk about $z$ in which $f$ is diff at every pts.

$\therefore f(z)$ is not analytic

2(i) verify that the function $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane. Also, find the harmonic conjugate function of $u$.

Soln: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial^2 u}{\partial x^2} = 6x$, $\frac{\partial u}{\partial y} = -6xy - 5$, $\frac{\partial^2 u}{\partial y^2} = -6x$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$\therefore$ $u(x,y)$ satisfies Laplace's equ. showed

2nd Part:

$$\frac{\partial v}{\partial x} = 6xy + 5, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\therefore v(x,y) = 3x^2y - y^3 + h(x)$$

$$\therefore \frac{\partial v}{\partial x} = 6xy - 0 + h'(x)$$

$$\therefore h'(x) = 5$$

$$\therefore h(x) = 5x + C$$

$$\therefore v(x,y) = 3x^2y - y^3 + 5x + C \text{ (Ans)}$$

3(a) Find all solutions to the equation $\sin z = 5$

Soln:

$$\sin z = 5$$

$$\Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 5$$

$$\Rightarrow e^{2iz} - 10ie^{iz} - 1 = 0$$

$$\Rightarrow (e^{iz})^2 - 2ie^{iz}.5 - 1 = 0$$

$$\therefore e^{iz} = \frac{10i \pm \sqrt{-100 - 4.1.(-1)}}{2.1} = 5i \pm 2\sqrt{6}i = (5 \pm 2\sqrt{6})i$$

$$\therefore e^{iz} = (5 \pm 2\sqrt{6})i$$

$$\therefore iz = \ln\{(5 \pm 2\sqrt{6})i\}$$

$$\therefore z = -i[\ln(5 \pm 2\sqrt{6})i]$$

$$\therefore z = -i\ln\left[(5+2\sqrt{6})i\right] = -i\left[\ln(5+2\sqrt{6}) + i\left(\frac{\pi}{2}+2n\pi\right)\right]$$

as $\operatorname{Arg}\left[(5+2\sqrt{6})i\right] = \frac{\pi}{2} + 2n\pi$

$$\therefore z = \frac{(4n+1)\pi}{2} - i\ln(5\pm 2\sqrt{6}) \quad \text{where}$$

$n = 0, \pm 1, \pm 2, \cdots$

3(b) $\int_C \bar{z}\,dz$ where $C$ is given by $x = 3t,\ y = t^2,\ -1 \le t \le 4$ (Ans)

Soln: $z = x+iy = 3t + i\cdot t^2$

$$\therefore f(z) = \bar{z} = 3t - it^2$$

$$\therefore z'(t) = 3 + 2ti$$

$$\therefore \int_C \bar{z}\,dz = \int_{-1}^{4}(3t - it^2)(3+2it)\,dt = \int_{-1}^{4}\left[2t^3 + 9t + 3t^2 i\right]dt$$

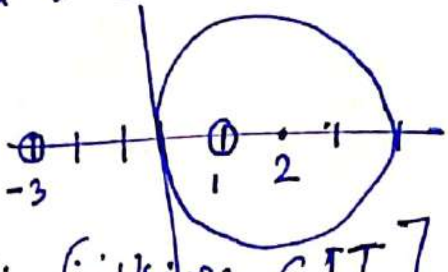
$$= \int_{-1}^{4}(2t^3+9t)\,dt + i\int_{-1}^{4}3t^2\,dt$$

$$= 195 + 65i \quad \text{(Ans)}$$

3 (c) Evaluate: $\oint_C \frac{5z+7}{z^2+2z-3}\,dz$ where C is a circle represented by $|z-2|=2$

Sol^n: $\oint_C \frac{5z+7}{z^2+2z-3}\,dz = \oint_C \left(\frac{3}{z-1} + \frac{2}{z+3}\right) dz$

∴ At $z=1$ and $z=-3$ $f(z)$ is not analytic but only $z=1$ lies within the contour C




$\therefore \oint_C \frac{5z+7}{z^2+2z-3}\,dz$

$= 3(2\pi i) + 2(0) = 6\pi i$ [∵ Using CIT]

(Ans)