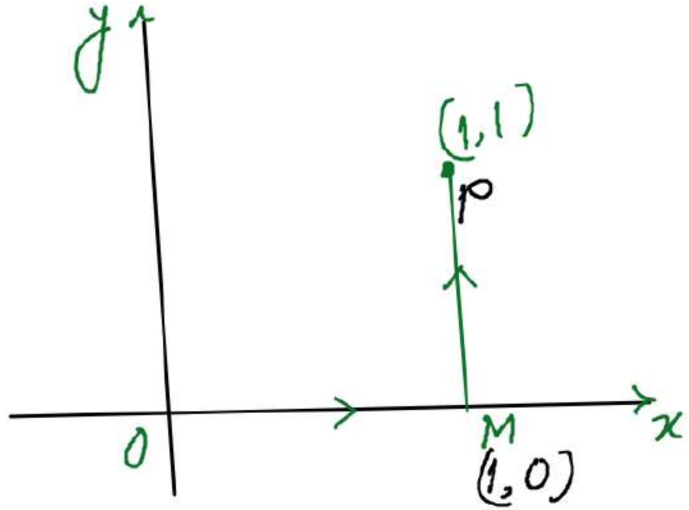
Quiz - 2

Math - 4541

Solution:

Q - (i) $\displaystyle\int_0^{1+i} (x - y + ix^2)\,dz$



$$\therefore \int_0^{1+i} (x - y + ix^2)\,dz = \int_{OM} (x - y + ix^2)\,dz + \int_{MP} (x - y + ix^2)\,dz \quad \text{——— (i)}$$

$\therefore$ Along OM, $z = x + iy = x + i \cdot 0 = x$

$\therefore dz = dx$

$$\therefore \int_0^1 (x-0+ix^2)\,dx$$

$$= \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + i/3$$

Along MP: $z = x+iy = 1+iy$

$$\therefore dz = i\,dy$$

$$\therefore \int_{MP} (x-y+ix^2)\,dz = \int_0^1 (1-y+i)\,i\,dy = \int_0^1 (i-iy-1)\,dy$$

$$= \left[iy - iy^2/2 - y\right]_0^1 = i - i/2 - 1$$

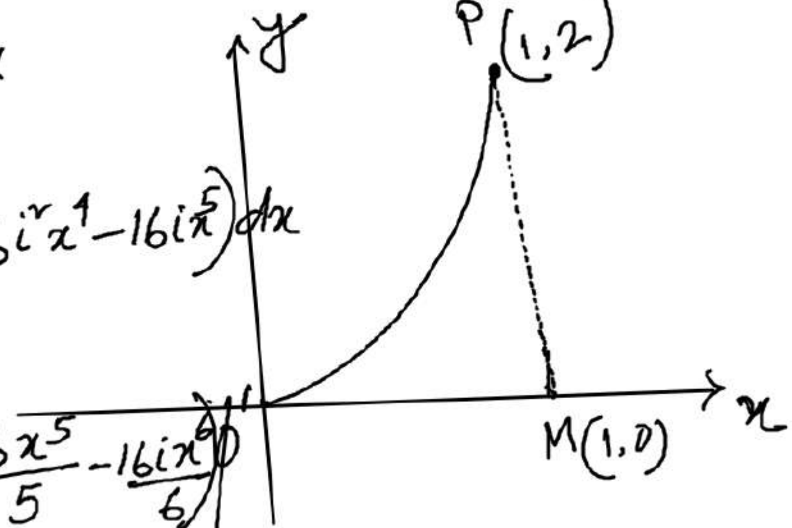
$$\therefore \int_0^{1+i} (x-y+ix^2)\,dz = \frac{1}{2} + i/3 + i - i/2 - 1 = -1/2 + \frac{5}{6}i \text{ (Ans.)}$$

Q-2. $\oint_C z^2\,dz$, C is the parabolic path $y = 2x^2$

Solution: Along the curve $y = 2x^2$, $z = x + iy = x + 2ix^2$

$\therefore dz = (1+4ix)\,dx$

$\therefore z^2 = x^2 + 4ix^3 - 4x^4$



$$\therefore \oint_C z^2\,dz = \int_0^1 (x^2+4ix^3-4x^4)(1+4ix)\,dx$$

$$= \int_0^1 (x^2+4ix^3-4x^4) + \int_0^1 (4ix^3+16i^2x^4-16ix^5)\,dx$$

$$= \left.\frac{x^3}{3} + \frac{4i\,x^4}{4} - \frac{4x^5}{5}\right|_0^1 + \left(\frac{4ix^4}{4} - \frac{16x^5}{5} - \frac{16ix^6}{6}\right)_0^1$$

$$= \frac{1}{3} + i - \frac{4}{5} + i - \frac{16}{5} - \frac{8}{3}i = -\frac{11}{3} - \frac{2i}{3} = -\frac{1}{3}(11+2i) \text{ (Ans)}$$

$\frac{5-12-48}{15}$

$-\frac{\cancel{55}\,11}{\cancel{15}\,3} = -\frac{11}{3}$

Q-3/ Evaluate $\lim_{z\to i} \dfrac{z^3+(1-3i)z^2+(i-3)z+2+i}{z-i}$ Usin LH Rule

Solution:

$$\lim_{z\to i} \frac{3z^2+2(1-3i)z+(i-3)+0+0}{1-0}$$

$$= -3+2i+6+i-3$$

$$= 3i \quad \text{(Ans)}$$

Q-4/ Conjugate harmonic function of $\ln(x^2+y^2)$ is

Solution: Let, $u(x,y) = \ln(x^2+y^2)$

$\therefore u_x = \dfrac{1}{x^2+y^2} \cdot 2x, \quad u_y = \dfrac{2y}{x^2+y^2}$

According to CR equation

$$v_x = -u_y = -\frac{2y}{x^2+y^2} \quad \text{—— (i)}$$

And $u_x = v_y = \dfrac{2x}{x^2+y^2}$

$$\therefore v = \int \frac{2x}{x^2+y^2}\,dy = 2\tan^{-1}\left(\frac{y}{x}\right) + g(x)$$

$$\therefore v_x = 2 \cdot \frac{1}{1+y^2/x^2} \cdot \left(-\frac{y}{x^2}\right) + g'(x)$$

$$v_x = \frac{-2y}{x^2+y^2} + g'(x) \quad \text{—— (ii)}$$

From (i) & (ii) $\qquad g'(x) = 0$

$\therefore g(x) = c \quad \therefore$ Harmonic Conjugate is $2\tan^{-1}\left(\frac{y}{x}\right) + c$

Q-5 //

$$\int_{[z_1, z_2]} (x^2+y^2)\, dz \quad \text{where } z_1 = 2+i,\ z_2 = -1-i$$

Solution: $z\bar{z} = x^2+y^2$

$\therefore$ Let, $f(z) = x^2+y^2$

Parametric equation of a line joining z_1 and z_2 is

$$z(t) = z_1(1-t) + z_2 t$$

$$= (2+i)(1-t) + (-1-i)t$$

$$= (-3-2i)t + 2+i$$

$$\therefore \bar{z} = (-3+2i)t + 2-i$$

$$\therefore dz = (-3-2i)\,dt$$

$$\therefore \int_{[z_1, z_2]} (x^2+y^2)\,dz = \int_0^1 \{(-3-2i)t + 2+i\}\{(-3+2i)t + 2-i\}(-3-2i)\,dt$$

$$= (-3-2i)\int_0^1 \{13t^2 + (-3-2i)(2-i)t + (2+i)(-3+2i)t + 5\}\,dt$$

$$= -4 - \tfrac{8}{3}i$$