ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)

ORGANISATION OF ISLAMIC COOPERATION (OIC)

Department of Computer Science and Engineering (CSE)

MID SEMESTER EXAMINATION **SUMMER SEMESTER, 2021-2022**
DURATION: 1 HOUR 30 MINUTES **FULL MARKS: 75**

CSE 4803: Graph Theory

Programmable calculators are not allowed. Do not write anything on the question paper.
Answer **all 3 (three)** questions. Figures in the right margin indicate full marks of questions whereas corresponding CO and PO are written within parentheses.

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1. a) "If every vertex of a simple graph G has degree 2, then G is a circuit." — Do you agree with the statement? Justify your answer. 6 (CO1) (PO1)

Solution:
The statement is FALSE.
Such a graph can be a disconnected graph with each component being a circuit. If infinite graphs are allowed, then the graph can be an infinite path.
Rubric:

- 2 point for identifying the statement as False
- 4 points for providing counterexample

b) Three actors and their managers are on one side of a river, along with a boat that can hold one or two people. The goal is to take all of them to the other side of the river. In one move, one or two people can use the boat (given it is on their side) to go to the other side of the river. However, the managers are jealous. On any side, no actor can be in the presence of another manager unless their own manager is also present.

With a brief explanation, answer the following questions:

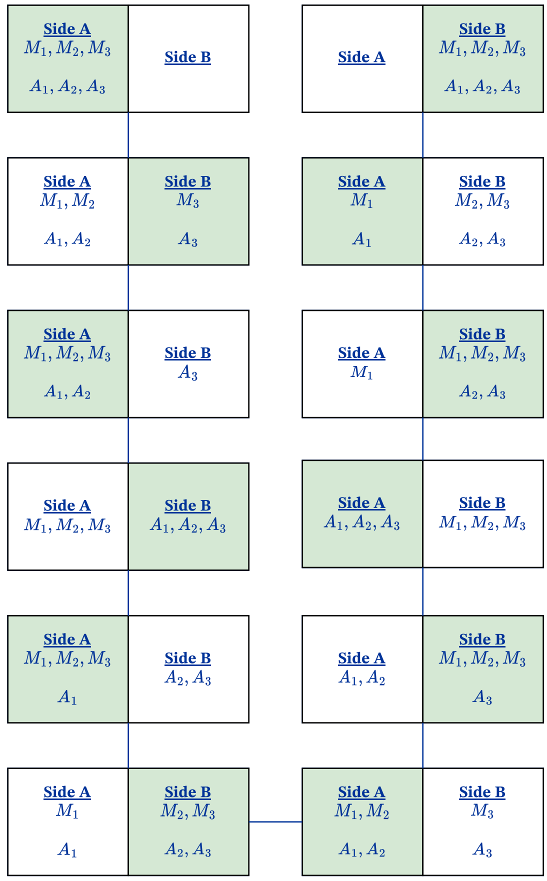
i. Formulate the problem as a graph by identifying the vertices and edges. 6 (CO2) (PO2)

Solution:
Each state of the problem will describe the position of the manager (M_1, M_2, M_3) and actors (A_1, A_2, A_3) on each side of the river. The vertices of the graph will represent each possible state.
Two vertices will be connected by a bidirectional edge if it is possible to take one or two people from one side of the river (given that side has the boat in that state) to the other side to transition from one valid state to another.
Rubric:

- 2 points for condition
- 2 points for vertices
- 2 points for edges

ii. Recommend a solution to the problem using graph theoretic techniques. 10 (CO3) (PO3)

Solution:
The graph representation may look like the following:



The solution can be derived by starting from the top-left vertex, traversing the edges sequentially, and stopping at the top-right vertex.

Rubric:

- 6 points for drawing a partial graph
- 4 points for the solution

2. a) Show that any two simple connected graphs with n vertices, all of degree two, are isomorphic. 7 (CO1) (PO1)

Solution:

Any two simple connected graphs with n vertices all of degree two are isomorphic because such graphs are essentially circuits, and any two cycles with the same number of vertices are isomorphic. To see why, consider two circuits with n vertices.

To establish an isomorphism between the two circuits, it suffices to find a bijection between their vertices that preserves the adjacency relationships between vertices. Since both circuits have the same number of vertices and all vertices have degree two, such a bijection always exists, and it can be constructed by arbitrarily pairing up the vertices in one circuit with the corresponding vertices in the other circuit.

Rubric:

- 3 points for considering both of them are circuits
- 4 points for bijection between the vertices

b) In a round-robin tournament every player plays against every other player. You are asked 12 (CO3) (PO3) to schedule a tournament of 144401 players. Each player can play at most two matches per day.

Determine the number of days required to finish the tournament in the shortest possible time.

Solution:

A round-robin tournament among n players can be represented using a complete graph with n vertices. A Hamiltonian Circuit in that graph can represent the games played in a day, where each player plays against two other players. To schedule the game each day, we need to find the number of edge-disjoint Hamiltonian Circuits in that complete graph.

According to Theorem 2.8, in a complete graph with n vertices there are $(n-1)/2$ edge-disjoint Hamiltonian Circuits, if n is an odd number ≥ 3. So, for $n = 144401$, we get:

$$\begin{aligned}\text{Number of days required} &= \frac{n-1}{2} \\ &= \frac{144401-1}{2} \\ &= 72200\end{aligned}$$

That means, at least 72200 days are required to complete the tournament.

Rubric:

- 2 points for mentioning that its a complete graph
- 2 points for identifying that we need a Hamiltonian Circuit
- 2 points for the circuits being edge-disjoint in consecutive days
- 2 points for the theorem/formula
- 4 points for the result

3. a) Determine the relationship between the following pairs: 7×3 (CO1) (PO1)

i. Minimum Vertex Cover and Independent Set

Solution:

Since vertex cover selects vertices to cover at least one endpoint of each edge, if we remove all the vertices that are in the vertex cover, the remaining vertices are going to form an independent set. This is because removing the vertex cover will result in all the edges having at least one endpoint removed from them. That means, the remaining vertices are not connected with each other.

If we remove $MVC(G)$ from G, the remaining $V - MVC = n - |MVC(G)|$ vertices form an independent set. In fact, we can imagine that removing $MVC(G)$ might have removed some of the vertices which could have been a part of the independent set. That means, the largest independent set can be even greater than $(n - |MVC(G)|)$. We get:

$$\alpha(G) \geq n - |MVC(G)|$$
$$|MVC(G)| \geq n - \alpha(G) \tag{1}$$

On the other hand, if we remove the largest independent set from G, the remaining $n - \alpha(G)$ vertices are going to be a part of the vertex cover. Because all the vertices in the independent set has one endpoint of all the edges. In fact, we can imagine that the minimum vertex cover is even smaller than this. We get:

$$|MVC(G)| \leq n - \alpha(G) \tag{2}$$

From Equation 1 and 2, we get:

$$|MVC(G)| = n - \alpha(G)$$

If we denote, $|MVC(G)|$ with $\beta(G)$, we get:

$$n = \alpha(G) + \beta(G)$$

Rubric:

- 2 points for MVC with IS
- 2 points for IS with MVC
- 3 point for combination

ii. Maximum Matching and Minimum Vertex Cover

Solution:
A maximum matching selects all the independent edges in a graph, G. So if we remove the edges (and their corresponding 2 vertices) that are in a maximum match from G, the remaining graph should not contain any edges. Because if there were any edges, they would have been added to the maximum matching.
On the other hand, independent set considers pairs of vertices having no edges between them. So the cardinality of the independent set should be at least equal to the remaining number of vertices. In fact, the cardinality of the independent set can be higher. We get:

$$\alpha(G) \geq n - 2\alpha'(G) \tag{3}$$

Recall that, $\alpha(G) = n - \beta(G)$. We get:

$$n - \beta(G) \geq n - 2\alpha'(G)$$

$$2\alpha'(G) \geq \beta(G) \tag{4}$$

When we are considering the minimum vertex cover, we select the vertices in such a way that for each edge, at least one vertex is selected. That means, from our example G, at least one vertex should be selected from each edge in the maximum matching. So the cardinality of the vertex cover should be at least as much as the cardinality of maximum matching. By combining this fact with Equation 4, we get:

$$\alpha'(G) \leq \beta(G) \leq 2\alpha'(G) \tag{5}$$

Rubric:

- 2 points for MM with MVC
- 2 points for MVC with MM
- 3 point for combination

iii. Edge Cover and Maximum Matching

Solution:
Consider that we have a connected graph, G. Here, $\alpha'(G)$ denotes the maximum matching. By selecting the edges that are in our maximum matching, we can cover their corresponding vertices.

However, there might be some unmatched vertices in the graph. Since the graph is connected (no isolated vertex), these unmatched vertices should be connected to the matched vertices using some edge. To cover these vertices, we need to select some more edges. To be specific, we need to select $n - 2\alpha'(G)$ edges. We get:

$$\beta'(G) = \alpha'(G) + n - 2\alpha'(G)$$
$$\beta'(G) = n - \alpha'(G)$$
$$\alpha'(G) + \beta'(G) = n \tag{6}$$

Rubric:

- 4 points for explanation
- 3 point for combination

b) Assume that we have N engineering students who will graduate soon. After graduation, the students want to get into one of the top M engineering universities to continue their higher studies. Each student has created a list of $k\,(1 \leq k \leq M)$ universities among the M universities based on their preference. Analyzing their choices, you found that each university appears exactly in k different lists.

i. Formulate the problem as a graph by identifying the vertices and edges. 6 (CO2) (PO2)

Solution:
The problem can be represented using a bipartite graph, where the partite set A consists of all the students as vertices and the partite set B consists of all the universities. There will be an edge between the vertices from partite set A to partite set B to denote the preference of a student.
The degree of each vertex will be k. It is to be noted that this scenario is only possible when $N = M$.
Rubric:

- 2 points for vertices
- 2 points for edges
- 2 points for condition

ii. Show that it is possible to assign all the students to one of their preferred universities, while also ensuring that no two students go to the same university. 7 (CO1) (PO1)

Solution:
As shown in the previous answer, the graph representing the students, universities, and their corresponding choices is a k-regular bipartite graph. One of the corrolaries of Hall's Theorem is if G is k-regular ($k \geq 1$) bipartite graph, then it has a perfect matching. That means, we can assign all the students to one of their preferred universities, while also ensuring that no two students go to the same university.
Rubric:

- 3 points for identifying the graph
- 2 points for the theorem
- 2 points for the conclusion