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Higher-order nonlinearity of bent functions

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- We call n-variable Boolean functions or Boolean functions in dimension n the functions from the n-dimensional vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$ to $\mathbb{F}_2$.
- Their set is denoted by $\mathcal{B}_n$, where n is the number of variables of Boolean functions.
- Given a basis, the field $\mathbb{F}_{2^n}$ can be identified with the vector space $\mathbb{F}_2^n$. Thus the input of Boolean functions will also be considered in the field $\mathbb{F}_{2^n}$.

Representation of Boolean functions

- Truth Table:

x_1	x_2	x_3	$f(x)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

corresponding to 3-variable Boolean function $f(x_1, x_2, x_3) = x_1x_2x_3 + x_2x_3 + x_3$ in ANF.

- Algebraic Normal Form:

$$f(x_1, \ldots, x_n) = \bigoplus_{I \subseteq \{1,\ldots,n\}} a_I \left(\prod_{i \in I} x_i \right).$$

- Univariate Representation:

$$f(x) = \sum_{i=0}^{2^n-1} \delta_i x^i.$$

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Definition 1 (Trace Function)

Let $F = \mathbb{F}_{2^m}$, $K = \mathbb{F}_{2^n}$ where $m \mid n$. We may view F as a subfield of K. If α is an element of K, its trace relative to the subfield F is defined as follows:

$$\operatorname{tr}_F^K(\alpha) = \alpha + \alpha^{2^m} + \alpha^{2^{2m}} + \cdots + \alpha^{2^{(\frac{n}{m}-1)m}}.$$

When no confusion is likely to arise, we will simply write the trace function as $\operatorname{tr}_m^n(\alpha)$.

Remark:
Trace function tr_m^n is a F-linear tranformation from K onto F and is balanced.

Walsh transform and Hamming distance

Definition 2 (Walsh Transform)

We call the Walsh transform of a Boolean function f the Fourier transform of the function $(-1)^{f(x)}$, and we denote it by W_f:

$$W_f(u) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+u \cdot x}.$$

Definition 3 (Hamming Distance)

The Hamming distance between $f, g \in \mathcal{B}_n$ is given by

$$d_H(f,g) = |\{x \in \mathbb{F}_2^n | f(x) \neq g(x)\}|.$$

Definition 4 (Algebraic Degree)

The degree of Boolean function f is denoted by $deg(f)$ and is called the algebraic degree of the function: $deg(f) = \max\{|I| : a_I \neq 0\}$, where $|I|$ denotes the size of I.

Example 5

$f = x_1x_2x_3 + x_2x_3 + x_3$ is a 3-variable Boolean function over $\mathbb{F}_2^n$ with $deg(f) = 3$. The Hamming distance between f and $g = x_3$ is 1.

Remark:

The Hamming distance between Boolean function f and affine function $l_a = a \cdot x$ equals

$$d_H(f, l_a) = 2^{n-1} - \frac{W_f(a)}{2}.$$

The rth-order nonlinearity is an important parameter of a Boolean function f:

Definition 6 (rth-order Nonlinearity)

The rth-order nonlinearity of f is defined as the minimum Hamming distance from f to all the functions of algebraic degrees at most r:

$$nl_r(f) = \min_{g \in \mathcal{B}_n, deg(g) \leq r} d_h(f, g).$$

Remark:

The first-order nonlinearity of f is usually called the nonlinearity of f and is denoted by $nl(f)$. The nonlinearity can be computed through the Walsh transform:

$$nl(f) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_{2^n}} |W_f(a)| .$$

Assume $k \geq 3$, we are aimed to give a lower bound on the third-order nonlinearity of the simplest $\mathcal{PS}$ bent function

$$f(x,y) = \mathrm{tr}_1^k\left(\frac{\lambda x}{y}\right) \tag{1}$$

where $(x,y) \in \mathbb{F}_{2^k}^2$, $\lambda \in \mathbb{F}_{2^k}^*$, $\mathrm{tr}_1^k(x) = \sum_{i=0}^{n-1} x^{2^i}$ is the trace function from $\mathbb{F}_{2^k}$ to $\mathbb{F}_2$ and $\frac{x}{y}$ is defined to be 0 if $y = 0$.

Lemma 7

Let f be any n-variable Boolean function and r be a positive integer smaller than n. Then we have

$$nl_r(f) \geq 2^{n-1} - \frac{1}{2}\sqrt{2^{2n} - 2\sum_{a\in\mathbb{F}_{2^n}} nl_{r-1}(D_\alpha f)}.$$

where $D_\alpha f(x) = f(x) + f(x+\alpha)$ is the derivative of f at point α.

Lemma 8

Let $n \geq 3$ be an arbitrary integer. We define

$$L = \left| \left\{ c \in \mathbb{F}_{2^n} \middle| \mathrm{tr}_1^n \left(\frac{1}{c^2+c+1} \right) = \mathrm{tr}_1^n \left(\frac{c^2}{c^2+c+1} \right) = 0 \right\} \right|.$$

Then we have $L = 2^{n-2} + \frac{3}{4}(-1)^n \widehat{I_1}(1) + \frac{1}{2}\left(1-(-1)^n\right)$, *where* $\widehat{I_1}(1) = 1 - \sum_{t=0}^{\lfloor n/2 \rfloor} (-1)^{n-t} \frac{n}{n-t} \binom{n-t}{t} 2^t$.

Lemma 9

Assume $k \geq 2$, let $N_{i,j} = \left|\left\{x \in \mathbb{F}_{2^k} \middle| \mathrm{tr}_1^k(\theta_1 x + \gamma_1) = i, \mathrm{tr}_1^k(\theta_2 x + \gamma_2) = j\right\}\right|$ where $\gamma_1, \gamma_2 \in \mathbb{F}_{2^k}$ and $\theta_1, \theta_2 \in \mathbb{F}_{2^k}^*$ are distinct, then $N_{0,0} = 2^{k-2}$.

Proof.

We have $N_{0,0} + N_{0,1} + N_{1,0} + N_{1,1} = 2^k$ and $N_{0,0} + N_{0,1} = 2^{k-1}$, $N_{1,1} + N_{0,1} = 2^{k-1}$, then we get $N_{0,0} = N_{1,1}$. Besides,
$N_{0,0} + N_{1,1} = \left|\left\{x \in \mathbb{F}_{2^k} \middle| \mathrm{tr}_1^k((\theta_1 + \theta_2)x + (\gamma_1 + \gamma_2)) = 0\right\}\right| = 2^{k-1}$ since $\theta_1 \neq \theta_2$.
Therefore $N_{0,0} = 2^{k-2}$. □

Lemma we need

Lemma 10

Assume $k \geq 3$*, let*
$N_{i_1,i_2,i_3} = \left|\left\{x \in \mathbb{F}_{2^k} \middle| \mathrm{tr}_1^k\left(\theta_1 x + \gamma_1\right) = i_1, \mathrm{tr}_1^k\left(\theta_2 x + \gamma_2\right) = i_2, \mathrm{tr}_1^k\left(\theta_3 x + \gamma_3\right) = i_3\right\}\right|$,
where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{F}_{2^k}$ *and* $\theta_1, \theta_2, \theta_3 \in \mathbb{F}_{2^k}^*$ *are distinct and satisfy* $\theta_3 \neq \theta_1 + \theta_2$*. Then* $N_{0,0,0} = 2^{k-3}$.

Proof.

The equations

$$\begin{cases} N_{0,0,0} + N_{0,0,1} = 2^{k-2} \\ N_{0,0,0} + N_{0,1,0} = 2^{k-2} \\ N_{0,0,0} + N_{1,0,0} = 2^{k-2}. \end{cases} \tag{2}$$

will lead to $N_{0,0,1} = N_{0,1,0} = N_{1,0,0}$.
With the same reason we also obtain $N_{0,1,1} = N_{1,0,1} = N_{1,1,0}$.
Because $\theta_1 + \theta_2 + \theta_3 \neq 0$, we can get equations:

$$\begin{aligned} & N_{1,0,1} + N_{1,1,0} + N_{0,1,1} + N_{0,0,0} \\ = & \left| \left\{ x \in \mathbb{F}_{2^k} \middle| \mathrm{tr}_1^k \left((\theta_1 + \theta_2 + \theta_3) x + (\gamma_1 + \gamma_2 + \gamma_3) \right) = 0 \right\} \right| \\ = & 2^{k-1}. \end{aligned} \tag{3}$$

Proof of Lemma 10

Proof.

Combine $N_{0,0,1} = N_{0,1,0} = N_{1,0,0}$, $N_{0,1,1} = N_{1,0,1} = N_{1,1,0}$, equations (3) with equations $N_{0,0,0} + N_{0,0,1} + N_{0,1,0} + N_{0,1,1} = 2^{k-1}$, we obtain the result $N_{0,0,1} = N_{0,1,1}$. Therefore from equations (2) and equations (3) we have

$$\begin{cases} N_{0,0,0} + N_{0,0,1} = 2^{k-2} \\ N_{0,0,0} + 3N_{0,0,1} = 2^{k-1}. \end{cases} \tag{4}$$

and the solution is $N_{0,0,0} = N_{0,0,1} = 2^{k-3}$. $\square$

Let us consider the Walsh transform of the second-order derivative of f at points $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{F}_{2^k}^2$.
We have

$$\begin{aligned}
& W_{D_\beta D_\alpha f}(\mu, \nu) \\
= & \sum_{x \in \mathbb{F}_{2^k}} \sum_{y \in \mathbb{F}_{2^k}} (-1)^{\mathrm{tr}_1^k\left(\frac{\lambda x}{y} + \frac{\lambda(x+\alpha_1)}{y+\alpha_2} + \frac{\lambda(x+\beta_1)}{y+\beta_2} + \frac{\lambda(x+\alpha_1+\beta_1)}{y+\alpha_2+\beta_2} + \mu x + \nu y\right)} \\
= & \sum_{y \in \mathbb{F}_{2^k}} (-1)^{\mathrm{tr}_1^k\left(\frac{\lambda\alpha_1}{y+\alpha_2} + \frac{\lambda\beta_1}{y+\beta_2} + \frac{\lambda(\alpha_1+\beta_1)}{y+\alpha_2+\beta_2} + \nu y\right)} \\
& \times \sum_{x \in \mathbb{F}_{2^k}} (-1)^{\mathrm{tr}_1^k\left(\left(\frac{\lambda}{y} + \frac{\lambda}{y+\alpha_2} + \frac{\lambda}{y+\beta_2} + \frac{\lambda}{y+\alpha_2+\beta_2} + \mu\right) x\right)} \\
= & \begin{cases} 2^k \sum_{y \in S} (-1)^{\mathrm{tr}_1^k\left(\frac{\lambda\alpha_1}{y+\alpha_2} + \frac{\lambda\beta_1}{y+\beta_2} + \frac{\lambda(\alpha_1+\beta_1)}{y+\alpha_2+\beta_2} + \nu y\right)}, & \text{if equation (5) has solutions;} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

The solutions of equation (5)

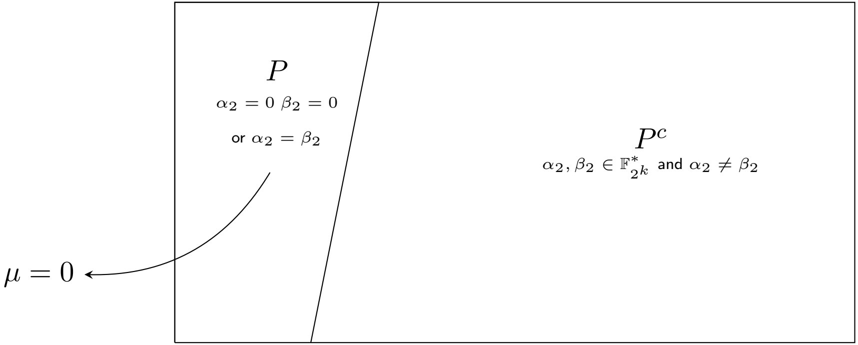
Consider the solutions of the equation:

$$\frac{\lambda}{y} + \frac{\lambda}{y+\alpha_2} + \frac{\lambda}{y+\beta_2} + \frac{\lambda}{y+\alpha_2+\beta_2} = \mu. \tag{5}$$

- If $\alpha_2 = \beta_2$ or $\alpha_2 = 0$ or $\beta_2 = 0$, then equation (5) has 0 solution when $\mu \neq 0$ and has 2^k solutions otherwise;
- If $\alpha_2, \beta_2 \in \mathbb{F}_{2^k}^*,\ \alpha_2 \neq \beta_2$, then equation (5) has $0, 4$ or 8 solutions depending on μ.

The partition of α, β-I

Figure: The partition of α, β



Consider the solutions of the equation:

$$\frac{\lambda}{y} + \frac{\lambda}{y+\alpha_2} + \frac{\lambda}{y+\beta_2} + \frac{\lambda}{y+\alpha_2+\beta_2} = \mu. \tag{5}$$

If $\alpha_2, \beta_2 \in \mathbb{F}_{2^k}^*,\ \alpha_2 \neq \beta_2$:

- If $\{0, \alpha_2, \beta_2, \alpha_2+\beta_2\}$ are solutions of (5), then we have:

$$\lambda(\alpha_2^2+\beta_2^2+\alpha_2\beta_2) + \mu(\alpha_2^2\beta_2+\alpha_2\beta_2^2) = 0. \tag{c-1}$$

- If $\{y_0, y_0+\alpha_2, y_0+\beta_2, y_0+\alpha_2+\beta_2\}$ are solutions of (5), then we have:

$$\mu \neq 0, \mathrm{tr}_1^k\left(\frac{\lambda\alpha_2}{\mu\beta_2(\alpha_2+\beta_2)}\right) = 0 \text{ and } \mathrm{tr}_1^k\left(\frac{\lambda\beta_2}{\mu\alpha_2(\alpha_2+\beta_2)}\right) = 0. \tag{c-2}$$

- Note that we can always find

$$\mu = \frac{\lambda(\alpha_2^2 + \beta_2^2 + \alpha_2\beta_2)}{\alpha_2^2\beta_2 + \alpha_2\beta_2^2}$$

satisfying condition (c-1) for α_2, β_2, leading to at least 4 solutions for equation (5).

- Thus, for all points $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{F}_{2^k}^2$ such that $\alpha_2, \beta_2 \in \mathbb{F}_{2^k}^*$ and $\alpha_2 \neq \beta_2$, there always exists (μ, ν) such that

$$W_{D_\beta D_\alpha f}(\mu, \nu) = 2^k \sum_{y \in S} (-1)^{\mathrm{tr}_1^k\left(\frac{\lambda\alpha_1}{y+\alpha_2} + \frac{\lambda\beta_1}{y+\beta_2} + \frac{\lambda(\alpha_1+\beta_1)}{y+\alpha_2+\beta_2} + \nu y\right)}$$

where $\{0, \alpha_2, \beta_2, \alpha_2 + \beta_2\} \subseteq S$.

The solutions of equation (5)

- For all points $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{F}_{2^k}$ such that $\alpha_2, \beta_2 \in \mathbb{F}_{2^k}^*$ and $\alpha_2 \neq \beta_2$, when condition (c-1) holds true, μ is determined, thus we can check whether condition (c-2) is true or false:
  - If condition (c-2) is false, then

$$S = \{0, \alpha_2, \beta_2, \alpha_2 + \beta_2\}.$$

  - If condition (c-2) is true, then

$$S = \{0, \alpha_2, \beta_2, \alpha_2 + \beta_2, y_0, y_0 + \alpha_2, y_0 + \beta_2, y_0 + \alpha_2 + \beta_2\}.$$

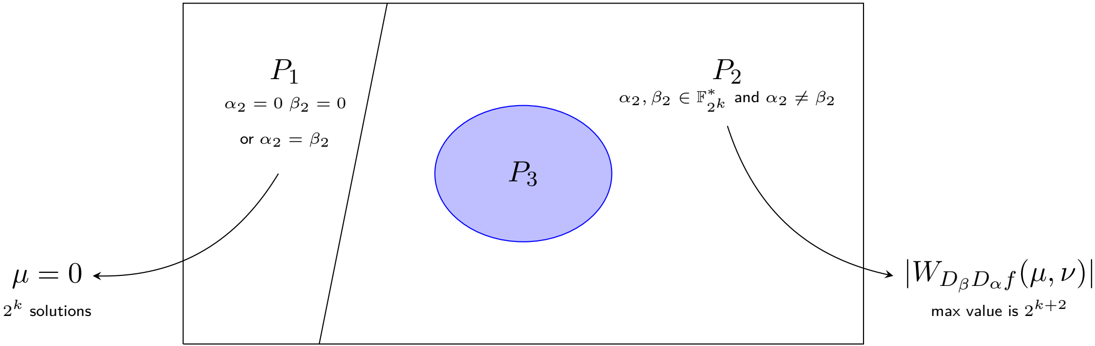
- If $S=\{0,\alpha_2,\beta_2,\alpha_2+\beta_2\}$, we have

$$W_{D_\beta D_\alpha f}(\mu,\nu) = \begin{cases} 2^{k+2}\cdot(-1)^{f^*}, \text{if } \operatorname{tr}_1^k(\mu\alpha_1+\nu\alpha_2)=0 \text{ and } \operatorname{tr}_1^k(\mu\beta_1+\nu\beta_2)=0 \\ 0, \text{ otherwise.} \end{cases}$$

where $f^*=\operatorname{tr}_1^k\left(\frac{\lambda\alpha_1}{y'+\alpha_2}+\frac{\lambda\beta_1}{y'+\beta_2}+\frac{\lambda(\alpha_1+\beta_1)}{y'+\alpha_2+\beta_2}+\nu y'\right)$ and $y'\in S$.
Note that we always have $v\in\mathbb{F}_{2^k}$ such that $\operatorname{tr}_1^k(\mu\alpha_1+\nu\alpha_2)=0$ and $\operatorname{tr}_1^k(\mu\beta_1+\nu\beta_2)=0$ by Lemma 9, so we conclude $\max_{\mu,\nu}|W_{D_\beta D_\alpha f}(\mu,\nu)|=2^{k+2}$ for all points $\alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2)\in\mathbb{F}_{2^k}^2$ such that $\alpha_2,\beta_2\in\mathbb{F}_{2^k}^*$, $\alpha_2\neq\beta_2$ and $S=\{0,\alpha_2,\beta_2,\alpha_2+\beta_2\}$.

The partition of α, β-II

Figure: The partition of α, β



- If $S = \{0, \alpha_2, \beta_2, \alpha_2+\beta_2, y_0, y_0+\alpha_2, y_0+\beta_2, y_0+\alpha_2+\beta_2\}$, we have

$$
\begin{aligned}
& W_{D_\beta D_\alpha f}(\mu,\nu) \\
= & \begin{cases} 2^{k+3}\cdot(-1)^{f_0^{**}}, \text{if } \operatorname{tr}_1^k(\mu\alpha_1+\nu\alpha_2)=0, \operatorname{tr}_1^k(\mu\beta_1+\nu\beta_2)=0 \text{ and } f_0^{**}+f_1^{**}=0 \\ 0, \text{ otherwise.} \end{cases}
\end{aligned}
$$

where $f_0^{**}+f_1^{**} = \operatorname{tr}_1^k\left(\frac{\lambda\alpha_1}{\alpha_2}+\frac{\lambda\beta_1}{\beta_2}+\frac{\lambda(\alpha_1+\beta_1)}{\alpha_2+\beta_2}+\frac{\lambda\alpha_1}{y+\alpha_2}+\frac{\lambda\beta_1}{y+\beta_2}+\frac{\lambda(\alpha_1+\beta_1)}{y+\alpha_2+\beta_2}+\nu y\right)$ and $y \in \{y_0, y_0+\alpha_2, y_0+\beta_2, y_0+\alpha_2+\beta_2\}$.

We need to prove two things:

- There exists $\mu \in \mathbb{F}_{2^k}$ such that S has 8 elements, i.e. condition (c-1) and (c-2) are both true.
- And there exists $\nu \in \mathbb{F}_{2^k}$ satisfying

$$\begin{cases} \mathrm{tr}_1^k\left(\mu\alpha_1 + \nu\alpha_2\right) = 0 \\ \mathrm{tr}_1^k\left(\mu\beta_1 + \nu\beta_2\right) = 0 \\ f_0^{**} + f_1^{**} = 0. \end{cases} \tag{6}$$

- Since we will always find $\mu = \frac{\lambda(\alpha_2^2+\beta_2^2+\alpha_2\beta_2)}{\alpha_2^2\beta_2+\alpha_2\beta_2^2} \in \mathbb{F}_{2^k}^*$ making condition (c-1) true, we take μ into condition (c-2) to get

$$\begin{cases} \mathrm{tr}_1^k\left(\frac{1}{\gamma^2+\gamma+1}\right) =0 \\ \mathrm{tr}_1^k\left(\frac{\gamma^2}{\gamma^2+\gamma+1}\right) =0. \end{cases} \qquad (*)$$

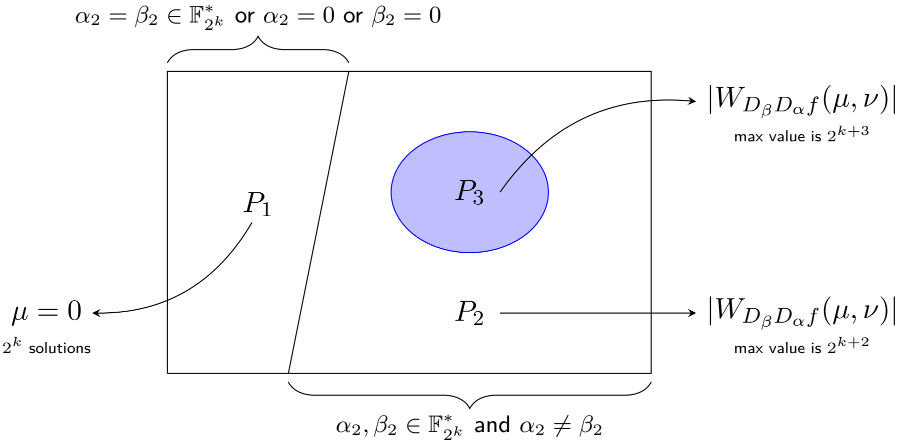
  where $\gamma = \frac{\beta_2}{\alpha_2} \in \mathbb{F}_{2^k} \setminus \mathbb{F}_4$.

- According to Lemma 8, the number of $\gamma = \frac{\beta_2}{\alpha_2}$ satisfying equations above is L.

- According to Lemma 10, there always exist $\nu \in \mathbb{F}_{2^k}$ that equations (6) hold true, which means:
  - There always exist (μ,ν) s.t. $\max_{\mu,\nu}|W_{D_\beta D_\alpha f}(\mu,\nu)| = 2^{k+3}$ for all points $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2) \in \mathbb{F}_{2^k}^2$ such that $\alpha_2, \beta_2 \in \mathbb{F}_{2^k}^*$, $\alpha_2 \neq \beta_2$ and $\gamma = \frac{\beta_2}{\alpha_2}$ satisfies equation (*).
  - Therefore, we can always find (μ,ν) s.t. $\max_{\mu,\nu}|W_{D_\beta D_\alpha f}(\mu,\nu)| = 2^{k+2}$ with $\alpha_2, \beta_2 \in \mathbb{F}_{2^k}^*, \alpha_2 \neq \beta_2$ and $\gamma = \frac{\beta_2}{\alpha_2}$ in other cases.

The partition of α, β-III

Figure: The partition of α, β



For every points $\alpha = (\alpha_1, \alpha_2) \in \mathbb{F}_{2^k} \times \mathbb{F}_{2^k}^*$, there exist L different β_2 contributing to $|W_{D_\beta D_\alpha f}(\mu, \nu)| = 2^{k+3}$, $2^k - 2 - L$ different β_2 leading to $|W_{D_\beta D_\alpha f}(\mu, \nu)| = 2^{k+2}$, while β_1 can be any element of $\mathbb{F}_{2^k}$.

Thus, for all points $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{F}_{2^k}^2$:

- When $\alpha = (\alpha_1, \alpha_2)$ is fixed, $\alpha_2 \neq 0$ and β runs over $\mathbb{F}_{2^k}^2$, we have

$$\max_{\mu,\nu} |W_{D_\beta D_\alpha f}(\mu, \nu)| = \begin{cases} 2^{k+3}, 2^k L \text{ times} \\ 2^{k+2}, 2^k(2^k - 2 - L) \text{ times} \\ *, \quad 2^{k+1} \text{ times.} \end{cases}$$

- When $\alpha = (\alpha_1, 0)$ is fixed and β runs over $\mathbb{F}_{2^k}$, we have

$$\max_{\mu,\nu} |W_{D_\beta D_\alpha f}(\mu, \nu)| = *, \quad 2^{2k} \text{ times.}$$

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Thank You

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