Another look at the relation between higher order arithmetic and set theory

(joint work with Emanuele Frittaion)

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Interpretations between arithmetic and set theory

- $\mathbf{Q}$ is mutually interpretable with set theory with adjunction (not biinterpretable).
- $\mathbf{PA}$ is synonymous with $\mathbf{ZFC}_{fin} + TC$.
- $\mathbf{ATR}_0$ is biinterpretable with $\mathbf{ATR}_0^{set}$, this interpretation in particular yields many correspondences between systems of second order arithmetic and theories extending $\mathbf{ATR}_0^{set}$ (it however is not synonymous).
- Simpson noticed that the theory $\mathbf{ATR}_0^{set}$, is sufficient to construct L and proves that $L_{\omega_1^{CK}} \models \mathbf{KP}_r$

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What other correspondences are there between theories of higher order arithmetic and set theory?

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Tree interpretation

Definition
Given a tree $T \subseteq X^{<\omega}$ a bisimulation on T is a relation $B \subseteq T \times T$ that is symmetric and:

$$\sigma B \tau \leftrightarrow \forall x \in X \, (\sigma^\frown(x) \in T \rightarrow \exists y \in X \; \sigma^\frown(x) B \tau^\frown(y))$$

Any system with sufficient transfinite recursion will prove that well founded trees have a unique bisimulation.

Trees modulo bisimulations is a very natural way to interpret set theory.

1. With well founded trees one can interpret set theory with foundation.
2. One can use arbitrary trees to interpret the Aczel antifoundation axiom.
3. With trees with labels one can also interpret set theories with urelements.

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$*$-interpretation

We may define a translation $*$ which has as domain the well founded trees of sequences, equality is translated to the formula $S =^* T$ if and only if the unique bisimulation on the tree $\{\emptyset\} \cup (s)^\frown S \cup (t)^\frown T$ contains the pair (s,t). Membership is translated to $T \in^* S$ to mean that there exists some element x such that $\{\sigma : (x)^\frown \sigma \in T\} = T_{(x)} =^* S$.

Using the $*$ interpretation paired with the construction of L one can show that $\mathrm{Z}_2 \rhd \mathbf{ZFC}^-$. For higher order arithmetic one gets similar results, namely that $\mathrm{Z}_{n+2} \rhd \mathbf{ZFC}^- + \mathcal{P}^n(\omega)$ exists.

We would like to generalize this in two ways. First is to consider some system which has the γ-th iteration of the powerset of ω for a larger class of ordinals and we would like to also consider systems with restricted comprehension.

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The systems

Definition
$\mathbf{A}^-$ is the system consisting of

1. Extensionality.
2. Union.
3. Pair.
4. Infinity, (there is a least inductive set)
5. Δ_0-Comprehension.

$\mathbf{B} \equiv \mathbf{A}^-$+ Finite Powerset + Transitive closure + Regularity.
$\mathbf{C} \equiv \mathbf{B}$+ Mostowski Collapse, also referred to as axiom β.

For a given fragment $\mathbf{T} \subseteq \mathbf{ZF}$ and some suitably defined ordinal γ by $\mathbf{T}^\gamma$ we mean the system

$$\mathbf{T} + \mathcal{P}^\gamma(\omega) \text{ exists}$$

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Fixing the system A

It turns out that the theory $\mathbf{A}^-$ as is is deficient.

1. It does not ensure that ordinal addition on ω is total, nor any other reasonable notion of addition.
2. In particular, for any discrete linear order $(L, <_L)$ with minimum and no maximum there exists a model of $M \vDash \mathbf{A}^-$ such that $\omega^M \cong L$.
3. Such model can be built by taking $((L, <_L), Def(L))$, define membership by $\in_M$ is given by $i \in_M j$ if $i, j \in L$ and $i <_L j$ and $i \in_M s$ if $i \in s \in Def(L)$ and close it under union and pair.
4. The main observation is that discrete linear orders are o-minimal and admit quantifier elimination in the language $(0, S, <)$. So the standard cut will not be a set in this model.
5. Doing the construction in $\mathrm{I}\Delta_0^0 + exp$ we get $\mathbf{A}^-$ in $\mathrm{I}\Delta_0^0 + exp$ by simply formalizing the proof.

To fix this let $\mathbf{A} \equiv \mathbf{A}^- + \omega^{<\omega}$ exists. ($\mathbf{A}^- +$ axiom $\beta +$ Cartesian product ensures the existence the graphs of addition and multiplication on ω).

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An alternative interpretation

To carry out the $*$ interpretation we need to be able to code sequences.

1. $\mathbf{A}^\gamma$ proves that there is Δ_0 definable coding for sequence of $\mathcal{P}^\gamma(\omega)$ with elements of $\mathcal{P}^\gamma(\omega)$
2. However, $\mathbf{A}^\gamma$ does not prove the existence of bisimulation for all well founded trees of coded sequences.
3. We can however restrict ourselves to a smaller class of well founded trees to which $\mathbf{A}^\gamma$ can prove the existence of bisimulations.
4. This gives rise to the $\mathbf{b}*$ translation, which has the advantage that it preserves formula complexity.

We have $k \geq 0$

$$\mathbf{A}^\gamma + \Sigma_k \text{ Separation} \rhd \mathbf{B}^\gamma + \Sigma_k \text{ Separation}$$

The definition of the trees in the $\mathbf{b}*$ translation is technical but it captures the fact that the trees corresponding to the transitive closure and finite powerset have lots of repetitions.

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Interpreting B

Let $n, k \in \omega \cup \{\infty\}$

$$\mathsf{ACA}_0 + \Sigma^1_n\text{-}\mathsf{CA}_0 + \Sigma^1_k\text{-}\mathsf{IND} \rhd \mathbf{B} + \Sigma_n \text{ Separation} + \Sigma_k \text{ Foundation}$$

Using the same translation one can get more correspondences between extensions of $\mathbf{ACA}_0$ and $\mathbf{B}$, for example

$$\mathbf{ATR}_0 \rhd \mathbf{B} + \Delta_1 \text{ Separation} + \Delta_0 \text{ Fixed Point}$$

even interpretations for theories of classes, for example

$$\mathbf{NBG} \rhd \mathbf{B}^\gamma + \gamma \text{ is inaccessible}$$

In such interpretations $\mathcal{P}^\gamma(\omega)$ is preserved, so the theories on the right hand side are $\mathcal{L}_2$ conservative extensions.

The theories $\mathbf{Z}^\gamma \equiv \mathbf{A}^\gamma +$ Separation and $\mathbf{B}^{\gamma+1}$ are conservative for all sentences with quantifiers bounded by $\mathcal{P}^\gamma(\omega)$.

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Fixed points and bisimulations

Bisimulations can be naturally described using fixed points. In particular, if we want the existence of bisimulations the appropriate axiom to assume is that of Δ_0 Fixed point. We get the following

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$$\mathbf{A}^\gamma + \Sigma_{k+1} \text{ Separation} \rhd \mathbf{C}^\gamma + \Sigma_k \text{ Separation} + \Delta_0 \text{ Least Fixed Point.}$$

For the case where $\gamma = 0$ it was shown by Simpson that in a sense it is optimal.

It is possible to iterate the Hartogs construction in $\mathbf{C}^\gamma$ to prove that $\aleph_\gamma$ exists.

Improvements in the non $cf(\gamma) = \omega$ case

We observe that if γ is either a successor or $cf(\gamma) > \omega$ we may code members of $[\mathcal{P}^\gamma(\omega)]^\omega$ with elements of $\mathcal{P}^\gamma(\omega)$.

If we are working in $\mathbf{A}^\gamma + DC$ a tree $T \subseteq (\mathcal{P}^\gamma(\omega))^{<\omega}$ being well founded is equivalent to a Δ_0 formula with $\mathcal{P}^\gamma(\omega)$ as parameter.

Of course we could avoid DC if $\mathcal{P}^\gamma(\omega)$ came equipped with a well ordering. The next best thing would to be working with trees on a well ordering of order type $\aleph_\gamma$. In the case where γ is limit such a well order exists.

In the case where $\gamma = \beta + 1$ such well order is not ensured. We may construct a quasi order $(W, \leq_W)$ whose equivalence classes will be of order type $\aleph_\gamma$. In particular, W will be the equivalence classes of well orderings on some $(U, <_U)$ where U will have order type $\aleph_\beta$.

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More

Let the domain of the interpretation be well founded trees $T \subseteq Seq(W)$ such that there exists an $n \in \omega$ such that for any $\sigma \in T$ of length $> n$

$$\forall x, y \in W\, (\sigma^\frown(x) \in T \wedge \sigma^\frown(y) \in T \to (y = x \vee x <_W y \vee y <_W x))$$

Such an translation will yield the following interpretations For all $k \geq 0$ and γ which is a successor or has uncountable cofinality:

$\mathbf{A}^\gamma + \Sigma_{k+1}$ Separation $\rhd \mathbf{C} + \Sigma_{k+1}$ Separation $+ \aleph_\gamma$ exists $+ V = H^+(\aleph_\gamma)$

However, it turns out that for any γ we have that

$$\mathbf{C}^\gamma \vdash Con(\mathbf{A}^\gamma + \text{ Foundation})$$

and therefore $\mathbf{A}^\gamma \not\rhd \mathbf{C}^\gamma$

The constructible universe

With some work it can be shown that $\mathbf{C} + \Delta_0$ Fixed Point proves that for any ordinal α L_α exists and that L is absolute and has a definable well ordering.

Pairing this with the $*$ translation we get the following interpretations

1. When $\gamma = 0$ or $cf(\gamma) = \omega$

$$\mathbf{B}^\gamma + \Sigma_{k+2} \text{ Separation} \rhd \mathbf{C}^\gamma + \Sigma_{k+1} \text{ Strong Collection} + V = L$$

$$\mathbf{B}^\gamma + \Delta_{k+2} \text{ Separation} \rhd \mathbf{C}^\gamma + \Sigma_{k+1} \text{ Collection} + V = L$$

2. When $\gamma = \beta + 1$ or $cf(\gamma) > \omega$

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However, $\mathbf{C}^\gamma + \Sigma_{k+1}$ Separation $+ \Sigma_{k+1}$ cannot prove Σ_{k+1} reflection holds in L, it turns out that Σ_{k+2} induction on ω is needed (McKenzie).

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The $cf(\gamma) = \omega$ case

We cannot do better in the case where $\gamma = 0$ or $cf(\gamma) = \omega$. The case $cf(\gamma) = \omega$ behaves similarly as in the countable case. In particular

$$\bigcup : [\mathcal{P}^{\gamma}(\omega)]^{\omega} \to \mathcal{P}^{\gamma+1}(\omega)$$

is surjective. If we have choice, for example if we are working in $\mathbf{C}^{\gamma} + V = L$ then we will have Skolem functions for $\mathcal{P}^{\gamma}(\omega)$, this allows us to write a formula which is $\Pi^{1,set}_{1}$ over $\mathcal{P}^{\gamma}(\omega)$.

In terms of translation this means that any formula which is $\Sigma^{1,set}_{n+1}(\mathcal{P}^{\gamma}(\omega))$ can be expressed by a Σ_n formula. $\mathbf{C}^{\gamma} + V = L + \Sigma_k$ Separation will prove that $\mathcal{P}^{\gamma+1}(\omega) \models \Sigma_{k+1}$ Separation. Using the $\mathbf{b}*$ translation in $\mathcal{P}^{\gamma+1}(\omega)$ will yield a model of $\mathbf{B}^{\gamma} + \Sigma_{k+1}$ Separation. So in particular

$$\mathbf{C}^{\gamma} + \Sigma_{k+1} \text{ Separation} \rhd \mathbf{B}^{\gamma} + \Sigma_{k+2} \text{ Separation}$$

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Limits

$$\mathbf{A}^\gamma + \Sigma_{k+1} \text{ Separation} \vdash Con(\mathbf{A}^\gamma + \Sigma_k \text{ Separation })$$

so in particular

$$\mathbf{A}^\gamma + \Sigma_k \text{ Separation} \quad \not\rhd \mathbf{A}^\gamma + \Sigma_{k+1} \text{ Separation}$$

For γ successor and uncountable cofinality we have

$$\mathbf{C}^\gamma + \Sigma_k \text{ Separation } + \Sigma_k \text{ Collection } \not\rhd \mathbf{A}^\gamma + \Sigma_{k+1} \text{ Separation}$$

If $\beta < \gamma$ then

$$\mathbf{A}^\gamma + \Delta_0 \text{ Fixed Point} \vdash Con(\mathbf{ZF}^- + \mathcal{P}^\beta(\omega) \text{ exists})$$

Δ_{n+1} Separation

The systems $\mathbf{A}^\gamma + \Delta_{n+1}$ Separation without any additional induction turns out to be conservative over $\mathbf{A}^\gamma + \Sigma_n$ separation.

Theorem (McKenzie)
$\mathbf{KP} + \Sigma_{k+2}$ Collection $+ V = L$ is Π_{k+3} conservative over $\mathbf{KP} + \Sigma_{k+1}$ Separation $+ \Sigma_{k+1}$ Collection $+ V = L$.

The following proof makes use of cuts which can be arbitrarily high in a model of $\mathbf{KP} + V = L$ so analogous results for $\mathbf{C}^\gamma$ hold. Pairing this with Shoenfield Absolutness it follows that $\mathbf{A} + \Delta_{n+1}$ Separation is Σ^1_2 conservative over $\mathbf{A} + \Sigma_n$ Separation. With more induction on ω we get a different picture.

Theorem (McKenzie)
$\mathbf{KP} + \Sigma_{k+2}$ Collection $+ \Sigma_{k+2}$ Induction proves the consistency of $\mathbf{KP} + \Sigma_{k+1}$ Separation $+ \Sigma_{k+1}$ Collection $+$ Foundation.

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Solidity and Tightness

Definition
A theory $\mathbf{T}$ is said to be tight if for all $\mathbf{U}, \mathbf{V} \supseteq \mathbf{T}$ then $\mathbf{U}$ and $\mathbf{V}$ are bi-interpretable if and only if $\mathbf{U} = \mathbf{V}$

A theory $\mathbf{T}$ is said to be solid if for any $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \models \mathbf{T}$ such

$$\mathcal{M}_0 \rhd \mathcal{M}_1 \rhd \mathcal{M}_2$$

and $\mathcal{M}_0$ is definably isomorphic to $\mathcal{M}_2$ we have that $\mathcal{M}_1$ is definably isomorphic to $\mathcal{M}_2$

Examples are PA (Visser), Z_2, Z_n, and $\mathbf{ZF}$(Enayat) and any extension. However, $\mathbf{ZFC}^-$ is not tight (Hamkins Freire).

Non tightness of ZF^{γ}

The idea of Hamkins and Freire is to take a model $M \models \mathbf{ZFC} + 2^{\aleph_0} = \aleph_2 + \mathbf{MA}+$ there exists a definable almost disjoint family of reals of size $\aleph_1$ and then interpret $H(\aleph_2)^M$ in $H(\aleph_1)^M$ via almost disjoint coding. However, in general, natural generalization of forcing axioms to notions of forcing satisfying $\aleph_{\gamma+1}$.c.c. fail to be consistent.

Fortunately, the posets needed to force an almost disjoint coding of subsets of $\aleph_{\gamma+1}$ via a subset of $\aleph_\gamma$ are nice enough to give an iteration which is $\aleph_{\gamma+1}$c.c. and $< \aleph_\gamma$ closed. This allows us to construct a model of $\mathbf{ZFC}$ in which $H(\aleph_{\gamma+1})$ and $H(\aleph_{\gamma+2})$ are bi-interpretable and $\mathcal{P}^{\gamma}(\omega) \subseteq H(\aleph_{\gamma+1})$.

With some work you can formalize the theories of $H(\aleph_{\gamma+1})$ and $H(\aleph_{\gamma+2})$ gives us two distinct bi-interpretable extensions of $\mathbf{ZFC}^{-} + \mathcal{P}^{\gamma}(\omega)$ exists, therefore showing non tightness.

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Coding using bisimulations

The case where $\gamma = 0$ we have that $\mathsf{ATR}_0 \vdash Con(\mathsf{ACA}_0)$ so in particular you have that $\mathsf{ACA}_0 \not\trianglerighteq \mathsf{ATR}_0$ and therefore $\mathbf{B} \not\trianglerighteq \mathbf{C}$. Some work is needed to prove that $\mathbf{C}^\gamma \vdash Con(\mathbf{B}^\gamma)$, to do so one needs to show that $\mathbf{C}$ is able to prove that for any transitive set X the set $Th(X)$ satisfying the Tarski conditions exists.

To do so, we construct for each formula φ and sequences of elements $\sigma \in X^{<\omega}$ trees $T_{\varphi(\sigma)}$ $T^{\top}_{\varphi}$ and $T^{\perp}_{\varphi}$ where the last two do not depend on σ such that $\varphi(\sigma) \leftrightarrow T_{\varphi(\sigma)} \in^* T^{\top}_{\varphi} \leftrightarrow T_{\varphi(\sigma)} \notin^* T^{\perp}_{\varphi}$. Relative to a sufficiently large well founded tree T and the collapsing function π of T we will have that $Th(X, \in)$ will be Δ_0 definable.

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To do so, we construct for each formula φ and sequences of elements $\sigma \in X^{<\omega}$ trees $T_{\varphi(\sigma)}$ $T_\varphi^\top$ and $T_\varphi^\bot$ where the last two do not depend on σ such that $\varphi(\sigma) \leftrightarrow T_{\varphi(\sigma)} \in^* T_\varphi^\top \leftrightarrow T_{\varphi(\sigma)} \notin^* T_\varphi^\bot$. Relative to a sufficiently large well founded tree T and the collapsing function π of T we will have that $Th(X, \in)$ will be Δ_0 definable.

Thank you for your attention!