REVERSE MATHEMATICS OF REGULAR $CSCS$ AND A FEW TOPOLOGICAL CHARACTERIZATIONS OF ATR_0

GIORGIO G. GENOVESI

Abstract. We look at the reverse mathematics of characterization theorems of regular countable second countable spaces (or $CSCS$ for short). We prove that arithmetic comprehension is equivalent over $\mathbf{RCA}_0$ to every T_3 $CSCS$ being metrizable and characterize the T_3 spaces which are metrizable over $\mathbf{RCA}_0$. We show that the effectively T_2 effectively compact $CSCS$ over $\mathbf{RCA}_0$ are precisely the spaces which are effectively homeomorphic to the upper limit topology of a well orders with a maximum element. We show that arithmetic comprehension is equivalent to T_2 compact $CSCS$ are well orderable. We also fomalize in $\mathbf{ACA}_0$ Lynn's theorem that every zero dimensional separable space is homeomorphic to the order topology of a linear order. We also show that every T_2 locally compact $CSCS$ is well orderable and every T_3 scattered $CSCS$ is completely metrizable are both equivalent to arithmetic transfinite recursion over $\mathbf{RCA}_0$. We also find a few equivalent statements to Π^1_1 comprehension.

Contents

0. Introduction

Reverse mathematics, in the broadest definition, is the study of the necessary axioms needed to prove a given theorem. Most of the time, the given theorem is one that can be expressed in the seemingly poor language of second order arithmetic. Over a suitable base theory, one can use a series of codes to talk about pairs, n-tuples, finite sets and sequences, functions, k-ary relations, countable rings and groups, complete separable metric spaces and Banach spaces, and many other objects that are treated in ordinary mathematics. This allows us to express, in the language of second order arithmetic, many classical theorems of ordinary mathematics and find what axioms are needed to prove them.

The language of second order arithmetic is limited in terms of cardinality. So, one can only hope to formalize a modest part of topology. Two approaches have been made. The first is the study of MF spaces done by Mummert [16], in which the points of a space will be identified as second order objects. This approach is similar to how Polish spaces are formalized in second order arithmetic. It turns out that over a strong fragment of second order arithmetic, MF spaces are precisely the Polish spaces [16]. Another approach is to consider Countable Second Countable Space (or $CSCS$ for short). Frieman and Hirst had studied $CSCS$ spaces in the form of countable subsets of complete metric spaces [5][6][10]. Montálban and Greenberg had looked at compact $CSCS$, which they reffered to as very countable spaces. The study of $CSCS$ in general over $\mathbf{RCA}_0$ started with Dorais [3], who shows that arithmetic comprehension is needed for many basic topological facts. For example, in the absence of arithmetic comprehension, there can be infinite discrete spaces that are compact or products of compact spaces that are not compact.

In general topology, regular Hausdorff $CSCS$ are rather simple. By Urysohn's metrization theorem, they must be metrizable. Any metric space that has cardinality less than the continuum must have a basis of open balls that are equal to their closure. In particular, any regular Hausdorff $CSCS$ will be zero dimensional and can therefore be embedded into the rationals using a variation of Sierpinski's theorem [18]. Furthermore, by a theorem of Lynn [13], we have that zero-dimensional separable spaces are linearly orderable. It is natural to ask what set existence axioms are needed to carry out these characterizations. In the first part of this work, we show that $\mathbf{ACA}_0$ is sufficient to prove all of these characterizations for all regular Hausdorff $CSCS$. We also have that $\mathbf{RCA}_0$ is sufficient to show most of these characterizations for regular Hausdorff $CSCS$ that have additional structure that codes the regularity of the space.

In the second part of the paper, we will consider scattered and locally compact Hausdorff $CSCS$. We have that scattered Hausdorff regular $CSCS$ are precisely the countable completely metrizable spaces, and locally compact Hausdorff $CSCS$ are the well orderable spaces. Both of these characterizations turn out to be equivalent to arithmetic transfinite recursion. In proving these characterizations, we get a series of other interesting topological principles equivalent to arithmetic transfinite recursion. We will also provide a few statements equivalent to Π^1_1 comprehension.

Part 1. Reverse Mathematics of Regular *CSCS*

1. Notation

For a poset $(P, \leq_P)$ and $S \subseteq P$ we write:

$$\uparrow S = \{p \in P : \exists s \in S\,(s \leq_P p)\}$$

$$\downarrow S = \{p \in P : \exists s \in S\,(p \leq_P s)\}$$

We call $\uparrow S$ the upwards closure of S in P and $\downarrow S$ the downwards closure of S in P.

We use the French notation for intervals. That is, for a linear order $(L, <_L)$ and $a, b \in L$ we write:

$$[a,b]_{<_L} = \{l \in L : a \leq_L l \leq_L b\}$$

$$]a,b]_{<_L} = \{l \in L : a <_L l \leq_L b\}$$

$$[a,b[_{<_L} = \{l \in L : a \leq_L l <_L b\}$$

$$]a,b[_{<_L} = \{l \in L : a <_L l <_L b\}$$

If L has a minimal element, we will denote it usually by 0. We consider $-\infty$ to be a new element strictly smaller than any element of L and $+\infty$ or ∞ to be an element that is strictly greater than any element of L. By $L+1$, we mean the the linear order with field $L \cup \{\infty\}$ and the order is extended so that ∞ is greater than any other element of L.

By $\mathbb{N}^{<\mathbb{N}}$ we mean the set of all finite sequences of natural numbers and by $2^{<\mathbb{N}}$ we mean the set of all finite 0-1 valued sequences. Given a sequence $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we write:

$$[[\sigma]] = \{f : \mathbb{N} \to \mathbb{N} : \forall i < |\sigma|\, f(i) = \sigma(i)\}$$

A tree is a subset T of $\mathbb{N}^{<\mathbb{N}}$ such that $T = \downarrow T$. We say that $f : \mathbb{N} \to \mathbb{N}$ is a branch in T if for all $n \in \mathbb{N}$ we have $f|_{<n} \in T$. By $[T]$, we mean the set of branches of T. We say that a tree is well founded if $[T] \neq \emptyset$.

Given $A \subseteq \mathbb{N}$ and $e, x \in \mathbb{N}$ we write $\Phi_e^A(x)$ to mean the Turing machine with index e, oracle A, and input x. We write $\Phi_e^A(x)\downarrow_{\leq t}$ to mean that the Turing machine with index e and oracle A halts at input x in less than t steps and we write $\Phi_e^A(x)\downarrow$ to mean that there exists a t such that $\Phi_e^A(x)\downarrow_{\leq t}$ and $\Phi_e^A(x)\uparrow$ if such t doesn't exist. We write $B \leq_T A$ if there exists an $e \in \mathbb{N}$ such that Φ_e^A is the characteristic function of B.

2. Reverse Mathematics

We briefly introduce the Big Five systems of reverse mathematics and some classic theorems. More details can be found in [20],[9], and [4].

$\mathbf{RCA}_0$ is the system consisting of the axioms of a discretely ordered commutative semiring, induction for Σ^0_1 formulas, and comprehension for Δ^0_1 predicates.

$\mathbf{WKL}_0$ is the system $\mathbf{RCA}_0$ plus the statement that any infinite subtree of the binary tree $2^{<\mathbb{N}}$ has an infinite branch.

$\mathbf{ACA}_0$ is the system $\mathbf{RCA}_0$ plus comprehension for all arithmetical formulas.

$\mathbf{ATR}_0$ is the system $\mathbf{RCA}_0$ plus arithmetic transfinite recursion, which is the statement that for any well order $(L, <_L)$ with least element 0, any set X, and any arithmetical formula φ there exists a sequence of sets $(X_j)_{j\in L}$ such that $X_0 = X$ and for all $j \in L \setminus \{0\}$:

$$X_j = \{n \in \mathbb{N} : \varphi(n, (X_i)_{i<j})\}$$

Informally, we can view X_j as being the set obtained by applying some arithmetic procedure given by φ to X j many times.

Π^1_1**-CA** is the system $\mathbf{RCA}_0$ plus comprehension for Π^1_1 formulas.

Proposition 2.1 $\mathbf{RCA}_0$ proves the following:

(1) For any strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, $\mathrm{rng}(f)$ exists.
(2) $\mathbf{B}\Sigma^0_1$: for every Σ^0_1 formula φ we have:

$$\forall a\ (\forall x < a\ \exists y\ \varphi(x, y)) \to (\exists b\ \forall x < a\ \exists y < b\ \varphi(x, y))$$

(See [20, Exercise II.3.14])

Proposition 2.2 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic comprehension.
(2) For every injective function $f : \mathbb{N} \to \mathbb{N}$, the range of f exists (See [20, Lemma III.1.3]).
(3) For every set X, the Turing jump $X' = \{e \in \mathbb{N} : \Phi^X_e(e)\downarrow\}$ exists (See [4, Corollary 5.6.3]).
(4) König's lemma: every infinite, finitely branching tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has a branch (See [20, Theorem III.7.2]).

Theorem 2.3 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic transfinite recursion.
(2) For every sequence of trees $(T_i)_{i\in\mathbb{N}}$ such that $\forall i \in \mathbb{N}; |[T_i]| \leq 1$ then the set $\{i \in \mathbb{N} : [T_i] \neq \emptyset\}$ exists (See [20, Theorem V.5.2]).
(3) Weak comparability of well orderings: for any two well orders L and W, there either exists an increasing function from L to W or an increasing function from W to L (See [4, Section 12.1]).
(4) Strong comparability of well orderings: for any two well orders L and W, either L is isomorphic to an initial segment of W or W is isomorphic to an initial segment of L (See [20, Section V.6]).

Theorem 2.4 (See [20, Theorem V.8.3], [4, Corollary 12.1.14]) $\mathbf{ATR}_0$ proves Σ^1_1 choice, or rather for every Σ^1_1 formula φ we have:

$$\forall n\ \exists X\ \varphi(n, X) \to \exists (X_n)_{n\in\mathbb{N}}\ \forall n\ \varphi(n, X_n)$$

Theorem 2.5 (See [20, Section V.1]) For any Σ^1_1 formula $\varphi(X)$ there exists a Δ^0_0 formula $\theta(\sigma, \tau)$ such that $\mathbf{ACA}_0$ proves that:

$$\forall X\ \varphi(X) \leftrightarrow \exists f\ \forall m\ \theta(X|_{\leq m}, f_{\leq m})$$

the formula to the righthand side of the biconditional is called the Kleene normal form of φ.

Observation 2.6 The formula θ in the previous theorem can be seen as defining a tree with respect to X. So over $\mathbf{ACA}_0$ being a ill founded tree is a universal Σ^1_1 formula and being a well founded tree is a universal Π^1_1 formula.

Definition 2.7 Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. The Kleene Brouwer order on T is given by:

$$\sigma <_{\mathrm{KB}} \tau \leftrightarrow \tau \sqsubseteq \sigma \vee \exists j\ (\forall i < j\ \sigma(i) = \tau(i) \wedge \sigma(j) < \tau(j))$$

$\mathbf{ACA}_0$ proves that every T is well founded if and only if it is well ordered with respect to the Kleene Brouwer order (See [20, Section V.1]).

Corollary 2.8 (See [20, Lemma VI.1.1]) over $\mathbf{RCA}_0$ the following are equivalent:

(1) Π^1_1 comprehension.
(2) For any sequence of trees $(T_i)_{i\in\mathbb{N}}$ the set $\{i \in \mathbb{N} : T_i \text{ is well founded}\}$ exists.

3. Countable Second Countable Spaces

Definition 3.1 A topological space is said to be second countable if it has a countable basis of open sets. We say that a topological space is first countable if the neighborhood filter of every point has a countable cofinal sequence. It is easy to check that for a countable space, being first countable is equivalent to being second countable.

Observation 3.2 One may wonder if all countable spaces are second countable. This is not the case. A simple procedure to construct countable but not second countable spaces is to take any downwards directed family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N}_{>0})$ that does not have a countable cofinal sequence, and define on $\mathbb{N}$ the topology in which every $n \in \mathbb{N}_{>0}$ is isolated and the basic open neighborhoods of 0 are the ones of the form $\{0\} \cup S$ where $S \in \mathcal{F}$.

To find such directed subsets of $\mathcal{P}(\mathbb{N}_{>0})$ one can use the fact that there exists a family $\mathcal{B}$ of size $2^{\aleph_0}$ of infinite subsets of $\mathbb{N}_{>0}$ such that the intersection of any two members is finite. Let $\mathcal{F}$ be the family of all finite intersections of the complements of elements in $\mathcal{B}$. We have that $\mathcal{F}$ is downwards directed, but it does not have a cofinal sequence.

Definition 3.3 (Dorais [3, Definition 2.2]) In the context of second order arithmetic, we have that a Countable Second Countable Space (or *CSCS* for short) is a tuple: $(X, (U_i)_{i\in\mathbb{N}}, k)$ where X is a subset of $\mathbb{N}$, for each $i \in \mathbb{N}$ $U_i \subseteq X$ and $k : X \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is such that:

$$x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$$

A set $A \subseteq X$ is open if for all $x \in A$ $\exists i \in \mathbb{N}\,(x \in U_i \subseteq A)$, and a set $C \subseteq X$ is closed if it is the complement of an open set.

Definition 3.4 (Dorais [3, Definition 2.4]) Given a *CSCS* $(X, (U_i)_{i\in\mathbb{N}}, k)$, an open code is a partial function $f : \mathbb{N} \to \mathbb{N}$. The open collection coded by f is the collection $A = \{x \in X : \exists n \in \mathbb{N}\, x \in U_{f(n)}\}$. Similarly, a closed code is a partial function $g : \mathbb{N} \to \mathbb{N}$. The closed collection coded by g is the collection $C = \{x \in X : \forall n \in \mathbb{N}\, x \notin U_{g(n)}\}$.

In general, $\mathbf{RCA}_0$ may not be sufficient to show that a coded open or closed collection exists since it is respectively Σ^0_1 and Π^0_1 definable. Similarly, we have that $\mathbf{RCA}_0$ is not sufficient to prove that every open or closed set has a code. A coded open collection that exists is called an effectively open set, and a coded closed collection which exists is called an effectively closed set.

Observation 3.5 The coded open collections will be closed under finite intersections and countable unions. Given A_0 and A_1 open collections with codes f_0 and f_1, the partial function $h : \mathbb{N}\times\mathbb{N}\times X \to \mathbb{N}$ given by $h(x, i, j) = k(x, f_0(i), f_1(j))$ will code $A_0 \cap A_1$. Given a sequence of coded open collections $(A_i)_{i\in\mathbb{N}}$ with codes $(f_i)_{i\in\mathbb{N}}$ then the partial function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $F(i, n) = f_i(n)$ is the code for the open collection $\bigcup_{i\in\mathbb{N}} A_i$. Similarly, coded closed collections are closed under finite unions and countable intersections.

Definition 3.6 Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a *CSCS*, we say that a sequence of open sets $(A_n)_{n\in\mathbb{N}}$ is a sequence of uniformly effectively open sets if there exists a sequence $(f_n)_{n\in\mathbb{N}}$, such that for all $n \in \mathbb{N}$ f_n is an open code for A_n. Similarly, $(C_n)_{n\in\mathbb{N}}$ is a sequence of uniformly effectively closed sets if there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ g_n is an open code for C_n.

Definition 3.7 (Dorais [3, Definition 2.5]) Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ and $(Y, (V_i)_{i\in\mathbb{N}}, k')$ be *CSCS*, we say that a function $f : X \to Y$ is effectively continuous if there exists a function $v : X \times \mathbb{N} \to \mathbb{N}$ such that for every $x \in X$ and $j \in \mathbb{N}$ such that $f(x) \in V_j$ then $x \in U_{v(x,j)}$ and $f(U_{v(x,j)}) \subseteq f(V_j)$. We say that the function v verifies that f is a homeomorphism. We say that $f : X \to Y$ is effectively open if there exists a function $v : X \times \mathbb{N} \to \mathbb{N}$ such that for every $x \in X$ and $j \in \mathbb{N}$ such that $x \in U_j$ then $f(x) \in V_{v(x,j)}$ and $V_{v(x,j)} \subseteq f(U_j)$. If f is effectively continuous and has an effectively continuous inverse, or equivalently, f is an effectively continuous and effectively open bijection, then we say f is an effective homeomorphism. We will write $X \cong Y$ to say that X is effectively homeomorphic to Y. We say that $f : X \to Y$ is an effective embedding if f is effectively continuous and there is a function $v : X \times \mathbb{N} \to \mathbb{N}$ such that for all $x \in X$ and $i \in \mathbb{N}$ such that $x \in U_i$ then $x \in f^{-1}(V_{v(x,i)}) \subseteq U_i$. We will say that the function v verifies that f is effectively open with its range.

Observation 3.8 Since over $\mathbf{ACA}_0$ every $CSCS$ has a function k we will often omit it from the definition of $CSCS$. Similarly, we won't distinguish homeomorphism from effective homeomorphism over $\mathbf{ACA}_0$.

Notation: We consider all real numbers to be represented as sequences $(a_i)_{i\in\mathbb{N}}$ where $a_0 \in \mathbb{Z}$ and $a_i \leq 1$ for $i \neq 0$. The number such a sequence represents is:

$$a_0 + \sum_{i>0} \frac{a_i}{2^i}$$

We point out that the representation for a real number is not unique since the sequence $(0, 1, 1, \dots)$ will represent the same number as the sequence $(1, 0, 0, \dots)$.

We would like to be able to give a $CSCS$ structure to countable metric spaces. In general, for a metric space (X, d), the open balls $B(x, r)$ are Σ^0_1 definable but may not be Π^0_1 definable relative X. So Δ^0_1 comprehension does not suffice to show that the open balls exist. One solution is to simply consider $CSCS$ with a weak basis as is done in [4, Definition 10.8.2] in which the elements of the basis are the images of partial functions rather than being sets. We present another approach to giving countable metric spaces a $CSCS$ structure.

We note that if the ball is clopen, that is, $B(x, r) = \overline{B(x, r)}$, then:

$$B(x, r) = \{y \in x : d(x, y) <_{\mathbb{R}} r\} = \{y \in x : d(x, y) \leq_{\mathbb{R}} r\}$$

Since strict inequality between reals is Σ^0_1 and weak inequality is Π^0_1, we have that $B(x, r)$ is Δ^0_1 definable. Working over $\mathbf{RCA}_0$, any clopen ball exists by Δ^0_1 comprehension. So, the problem of finding a $CSCS$ structure for a metric space is reduced to finding a basis of clopen balls.

Proposition 3.9 $\mathbf{RCA}_0$ proves that for any countable metric space (X, d) there exists a real $a \in \mathbb{R}$ such that:

$$\forall q \in \mathbb{Q}_{>0}\, \forall x, y \in X\, (d(x, y) \neq q \cdot a)$$

and that there is a function k such that $(X, (B(x, q \cdot a))_{q\in\mathbb{Q}_{>0}, x\in X}, k)$ is a $CSCS$.

Proof: Let $(t_n)_{n\in\mathbb{N}}$ be an enumeration of all triples $(x, y, q) \in X \times X \times \mathbb{Q}_{>0}$. We define a real $a = (a_i)_{i\in\mathbb{N}}$ recursively as follows. At step $i \in \mathbb{N}$ let $t_i = (x, y, q)$ and let $q \cdot d(x, y) = b$ which we write out as $b = (b_j)_{j\in\mathbb{N}}$ define:

$$(a_{2i+1}, a_{2i+2}) = \begin{cases} (0, 1) & \text{if} \quad (b_{2i+1}, b_{2i+2}) \neq (0, 1) \\ (1, 0) & \text{if} \quad (b_{2i+1}, b_{2i+2}) = (0, 1) \end{cases}$$

Given $q \in \mathbb{Q}_{>0}$ and $x, y \in X$ then we have that $d(x, y) \neq \frac{1}{q} \cdot a$ and so a has the desired property.

The collection $(B(x, q \cdot a))_{q\in\mathbb{Q}_{>0}, x\in X}$ exists by Δ^0_1 comprehension. Let $(q_i)_{i\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$ and let $k : X \times (\mathbb{Q}_{>0} \times X)^2 \to \mathbb{Q}_{>0} \times X$ be the function given by:

$$k(x, (y, p), (z, r)) = (x, q_j)$$

where:

$$j = \min\{i \in \mathbb{N} : d(x, y) <_{\mathbb{R}} (p - q_i) \cdot a \wedge d(x, z) <_{\mathbb{R}} (r - q_i) \cdot a\}$$

or rather j is the least number such that $q_j \cdot a <_{\mathbb{R}} p \cdot a - d(x, y)$ and $q_j \cdot a <_{\mathbb{R}} r \cdot a - d(x, z)$. By construction of a we also have that:

$$j = \min\{i \in \mathbb{N} : d(x, y) \leq_{\mathbb{R}} (p - q_i) \cdot a \wedge d(x, z) \leq_{\mathbb{R}} (r - q_i) \cdot a\}$$

So j can be searched effectively. By the triangle inequality, we have:

$$B(x, q_j \cdot a) \subseteq B(y, p \cdot a) \cap B(z, r \cdot a)$$

and by construction of a, and the fact that d separates points we have that k is recursive and so it exists by Δ^0_1 comprehension. □

The previous proposition tells us that any countable metric space has a basis of clopen sets. We will see later that any *CSCS* that distinguishes points (is T_0) and has a basis of clopen sets is homeomorphic to a metric space.

Definition 3.10 Let (X, d) be a countable metric space, and $a \in \mathbb{R}_{>0}$ such that:

(1) $$\forall q \in \mathbb{Q}_{>0}\, \forall x, y \in X\, (d(x, y) \neq q \cdot a)$$

The *CSCS* structure of (X, d) is $(X, (B(x, q \cdot a))_{q\in\mathbb{Q}_{>0}, x\in X}, k)$. It is straightforward to show that for any $a, b \in \mathbb{R}_{>0}$ satisfying (1) the spaces $(X, (B(x, q\cdot a))_{q\in\mathbb{Q}_{>0}, x\in X}, k)$ and $(X, (B(x, q\cdot b))_{q\in\mathbb{Q}_{>0}, x\in X}, k)$ will be effectively homeomorphic. In particular, the *CSCS* structure on (X, d) is unique up to effective homeomorphism. We say that a *CSCS* is metrizable if it is effectively homeomorphic to the *CSCS* structure of a metric space.

Definition 3.11 Let $((X^j, (U^j_i)_{i\in\mathbb{N}}, k^j))_{j\in\mathbb{N}}$ be a sequence of *CSCS* spaces we define the topological disjoint union of the sequence of spaces as being the space:

$$\left(\coprod_{j\in\mathbb{N}} X^j\ ,\ (U^j_i \times \{j\})_{i,j\in\mathbb{N}}\ ,\ k'\right)$$

Where $\coprod_{j\in\mathbb{N}} X^j = \{(x, j) : x \in X^j\}$ and $k'((x, j), (a, j), (b, j)) = k^j(x, a, b)$. We have that for every $j \in \mathbb{N}$ $X^j \times \{j\}$ is a clopen set and the map from X^j to $\coprod_{j\in\mathbb{N}} X^j$ given by $x \mapsto (x, j)$ is an effective embedding. Furthermore, we observe that the disjoint union of the spaces is computable relative to the sequence of *CSCS*.

Definition 3.12 Given a linear order $(L, <_L)$ the order topology on L is the space $(L, (\,]a, b[\,)_{a,b\in L\cup\{+\infty,-\infty\}}, k)$ where the function k is given by

$$k(x,]a, b[,]x, y[\,) = \ ]\max\{a, x\}, \min\{b, y\}[$$

We use the convention that $\forall a \in L\, (a < +\infty \wedge -\infty < a)$. We will assume that every linear order has the order topology unless specified. We define the upper limit topology on $(L, <_L)$ to be the order topology where sets of the form $]a, b]$, with $a, b \in L$, are added to the basis.

Observation 3.13 If L is well ordered, its order topology coincides with its upper limit topology since $]a, b] =]a, b + 1[$ where $b + 1$ is either the successor of b or $+\infty$ if b is maximal. However, we will see later that over $\mathbf{RCA}_0$, the two topologies are not, in general, effectively homeomorphic.

Observation 3.14 Let $(L, <_L)$ be a linear order $S \subseteq L$ be a set that is downwards closed with respect to $<_L$ then S is open in the upper limit topology since $S = \bigcup_{s\in S}\,]-\infty, s]$.

Proposition 3.15 Over $\mathbf{RCA}_0$ arithmetic comprehension is equivalent to the existence, for all sets X and sequence of sets $(A_i)_{i\in\mathbb{N}}$, of the set $\{i \in \mathbb{N} : X \subseteq A_i\}$.

Proof: Let X be any set and let $(B_e)_{e\in\mathbb{N}}$ be the sequence of sets given by:

$$A_e = \{t \in \mathbb{N} : \Phi_e^X \uparrow_{\leq t}\}$$

then $\mathbb{N} \setminus X' = \{e \in \mathbb{N} : \mathbb{N} \subseteq B_e\}$. □

Proposition 3.16 (Chan [1]) Over $\mathbf{RCA}_0$ the following are equivalent:

1. Arithmetic comprehension.
2. For every $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ and every subset $A \subseteq X$ the closure $\overline{A}$ exists.
3. For every countable metric space (X, d) and every subset $A \subseteq X$ the closure $\overline{A}$ exists.

Proof: It is clear that 1 implies 2 and 3 since the definition of closure is arithmetic.

($3 \to 1$) Let A be a set we define a subspace X of $\mathbb{Q} \times \mathbb{N}$ containing $\{0\} \times \mathbb{N}$ and for all natural numbers $n \geq 0$ we have $(\frac{1}{n}, e) \in X$ if and only if $\Phi_e^A \downarrow_{\leq n}$ otherwise $(n, e) \in X$. The metric space X is computable and therefore it exists by Δ_1^0 comprehension. We have therefore for all $n \in \mathbb{N}$ that $e \in A'$ if and only if $(0, e)$ is in the closure of $\{(q, e) \in X : q > 0\}$. Thus, if the closure exists, so does the Turing jump of A. □

4. Separation Axioms

Definition 4.1 A topological space X is said to be Kolmogorov or T_0 if for every $x, y \in X$ there exists an open set U such that $x \in U \leftrightarrow y \notin U$.

Definition 4.2 The Kolmogorov quotient of a $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ is the quotient space $X/\sim_K$ where the equivalence relation $\sim_K$ is given by:

$$x \sim_K y \leftrightarrow (\forall i \in \mathbb{N}\ x \in U_i \leftrightarrow y \in U_i)$$

We may identify the Kolmogorov quotient with the subspace of X given by the least elements of the $\sim_K$-equivalence classes. The Kolmogorov quotient is always a T_0 space. In general, the existence of the Kolmogorov quotient requires arithmetic comprehension. To see this, let $A \subseteq \mathbb{N}$ be a set and consider the space $(\mathbb{N}, (U_i)_{i\in\mathbb{N}}, k)$ where:

$$U_{(n,t)} = \begin{cases} \{2n, 2n+1\} \text{ if } \neg\Phi_e^A(e) \downarrow_{\leq t} \\ \{2n\} \text{ if } \Phi_e^A(e) \downarrow_{\leq t} \end{cases}$$

the Kolmogorov quotient of this $CSCS$ will compute the jump of A.

Definition 4.3 A $CSCS$ is said to be T_1 if every singleton is closed.

Definition 4.4 A $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ is said to be Hausdorff or T_2 if for every distinct $x, y \in X$ there exists $i, j \in \mathbb{N}$ such that:

$$x \in U_i \qquad y \in U_j \qquad U_i \cap U_j = \emptyset$$

Definition 4.5 (Dorais [3, Definition 6.1]) We say that a space is effectively T_2 if there exists a pair of functions $H_0, H_1 : [X]^2 \to \mathbb{N} \times \mathbb{N}$ such that:

$$(x \in U_{H_0(x,y)}) \wedge (y \in U_{H_1(x,y)}) \wedge (U_{H_0(x,y)} \cap U_{H_1(x,y)} = \emptyset)$$

We observe that the functions H_0 and H_1 are arithmetically definable. So over $\mathbf{ACA}_0$ every T_2 $CSCS$ is effectively T_2.

Proposition 4.6 If $(X, (U_i)_{i\in\mathbb{N}}, k)$ has an effective continuous injection to an effectively T_2 *CSCS* then X is effectively T_2.

Proof: Let $f : (X, (U_i)_{i\in\mathbb{N}}, k) \to (Y, (V_i)_{i\in\mathbb{N}}, k')$ be an effective continuous injection and let H_0^Y and H_1^Y witness that $(Y, (V_i)_{i\in\mathbb{N}}, k')$ is effectively T_2. Let $v : X \times \mathbb{N} \to \mathbb{N}$ verify that f is continuous, meaning that for all $x \in X$ and $i \in \mathbb{N}$ such that $f(x) \in V_i$ we have that:

$$x \in U_{v(x,i)} \subseteq f^{-1}(V_i)$$

For all $x, y \in X$ we define:

$$H_0^X(x, y) = v(x, (H_0^Y(f(x), f(y)))$$
$$H_1^X(x, y) = v(y, (H_1^Y(f(x), f(y)))$$

then H_0^X and H_X^1 witness that X is effectively T_2. □

Theorem 4.7 (Dorais [3, Example 7.4]) If there exists a function $f : \mathbb{N} \to \mathbb{N}$ whose image does not exist then there is a T_2 *CSCS* which is not effectively T_2. Equivalently, over $\mathbf{RCA}_0$, arithmetic comprehension is equivalent to every T_2 *CSCS* being effectively T_2.

Proof: Assume there is an injective function $f : \mathbb{N} \to \mathbb{N}$ whose image does not exist. Let $<_f$ be the order on $\mathbb{N} \cup \{\infty\}$ given by:

- $\forall n \in \mathbb{N}\ n <_f \infty$.
- $\forall n, m \in \mathbb{N}\ n <_f m \leftrightarrow f(n) < f(m)$.

Seeking a contradiction, assume that $(\mathbb{N} \cup \{\infty\}, <_f)$ with the order topology is effectively T_2 and let H_0 and H_1 be the functions that witness it. We have that n is the $<_f$ successor of m if and only if $m <_f n$ and $H_0(m, n) = (y, n)$ for some $y <_f n$ and $H_1(m, n) = (m, x)$ for some $x >_f m$. Since the image of f is unbounded, every $n \in \mathbb{N}$ has an $<_f$ successor. We can therefore define recursively g where $g(0)$ the $<_f$-minimal element and $g(n + 1)$ is the $<_f$-successor of $g(n)$. So $f \circ g$ is strictly increasing and has the same image as f, which implies that $\mathrm{rng}(f)$ exists, contradicting our initial assumption. □

Observation 4.8 The space constructed in the proof above is a well order with the order topology. While the upper limit topology and the order topology coincide on a well order, they may not be effectively homeomorphic. This is because the upper limit topology is always effectively T_2, and arithmetic comprehension is equivalent to the order topology of any well order is effectively T_2. However, it is straightforward to show that over $\mathbf{RCA}_0$, the identity is an effective homeomorphism between the upper limit topology and the order topology if and only if the successor function for the order exists.

Proposition 4.9 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) arithmetic comprehension.
(2) every well order with the upper limit topology is homeomorphic to a well order with the order topology.

Proof: $(1 \to 2)$ follows from the fact that the successor function on a well order is arithmetically definable.

$(2 \to 1)$ Let $f : \mathbb{N} \to \mathbb{N}$ be an injection, $(X, <_f)$ be as in the proof of Theorem 4.7 and let ∞_X denote the maximal element of X. We show that $\mathrm{rng}(f)$ exists, which by 2.2 implies arithmetic comprehension. By assumption, X with the upper limit topology is effectively homeomorphic to some well order $(W, <_W)$ with the order topology. Since X is effectively T_2, we also have that W, with the order topology, is effectively T_2. As in the proof of Theorem 4.7 we have that the successor function for W will exist and so the upper limit topology on W is effectively homeomorphic to the order topology. Since X has one limit point and doesn't contain an infinite closed discrete subspace, we have that W will be order isomorphic to $\mathbb{N}+n$ for some $n \in \mathbb{N}$. Up to permuting $n-1$ many isolated points of W we may assume that $(W, <_W) = (\mathbb{N}+1, <_{\mathbb{N}})$. Let $g : X \to \mathbb{N}+1$ be an effective homeomorphism and let v the function which witnesses that g is effectively open. Let $\infty_{\mathbb{N}}$ denote the maximal element of $\mathbb{N}+1$, have that:

$$\forall n \in \mathbb{N}\ (]v(\infty_X, n), \infty_{\mathbb{N}}]_{<_{\mathbb{N}}} \subseteq g(\,]n, \infty_X]_{<_f})$$

Which implies that $\forall n\ \forall m\ (m \leq_f n \to g^{-1}(m) \leq_{\mathbb{N}} v(\infty_X, n))$. So in particular we have that n is the $<_f$ successor of m if and only if

$$\forall y \leq_{\mathbb{N}} v(\infty_X, n)\ (g^{-1}(y) \leq_{\mathbb{N}} m \vee g^{-1}(y) \geq_{\mathbb{N}} n)$$

The $<_f$ successor function is Δ^0_1 definable, so it exists by Δ^0_1 comprehension. Using the successor function as in the proof of Theorem 4.7, we get that the range of f exists. □

Definition 4.10 A $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ is said to be regular if for every point $x \in X$ and any $i \in \mathbb{N}$ such that $x \in U_i$ there are open sets V and W such that:

$$x \in V \subseteq X \setminus W \subseteq U_i$$

Definition 4.11 A $CSCS$ that is T_0 and regular is said to be T_3. So over $\mathbf{ACA}_0$, the study of regular spaces is reduced to the study of T_3 spaces since we may always restrict ourselves to the Kolmogorov quotient.

Observation 4.12 All T_3 spaces are T_2.

Definition 4.13 [3, Definition 6.4] A $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ is said to be effectively regular if for every effectively closed set C of X and any $x \notin C$ there exist open collections U_0 and U_1 for open sets such that $x \in U_0 \subseteq X \setminus U_1 \subseteq X \setminus C$.

Definition 4.14 We say that a $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ is uniformly regular or uniformly effectively regular if there is a pair of functions R_0 and R_1 such that for all $x, y \in X$ and $i \in \mathbb{N}$ such that $x \in U_i$ and $y \notin U_i$ the following holds:

$$x \in U_{R_0(x,i)} \subseteq U_i$$

$$y \in U_{R_1(x,i,y)} \subseteq X \setminus U_{R_0(x,i)}$$

We will say that a $CSCS$ is uniformly T_3 if it is T_0 and uniformly regular. Intuitively, $R_0(x,i)$ is the index of a neighborhood of x contained in U_i and the sequence $(R_1(x,i,y))_{y\notin U_i}$ codes an open collection containing $X \setminus U_i$ and that is disjoint from $U_{R_0(x,i)}$. That is, we have:

$$x \in U_{R_0(x,j)} \subseteq X \setminus \left(\bigcup_{y \notin U_j} U_{R_1(x,j,y)} \right) \subseteq X \setminus U_i$$

Observation 4.15 Subspaces of regular $CSCS$ are regular and subspaces of uniformly effectively regular $CSCS$ are uniformly effectively regular.

Proposition 4.16 $\mathbf{RCA}_0$ proves that for any pair of $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ and $(Y, (V_i)_{i\in\mathbb{N}}, k')$ such that X effectively embeds into Y, if Y is uniformly regular then X is uniformly regular .

Proof: We first observe that if $Z \subseteq Y$ is a subset and R^Y_0, R^Y_1 are the functions witnessing that Y is effectively regular, then their restrictions to Z witness that Z is effectively regular. So it suffices to prove that if X is effectively homeomorphic to Y then X is also effectively regular.

Let $f : X \to Y$ be an effective homeomorphism and v_0 verify that f is continuous and v_1 verify that f^{-1} is continuous (See Definition 3.7). Let R_0^Y, R_1^Y be the functions given by uniform effective regularity of Y. We define:

$$R_0^X(x,i) = v_0(x, R_0^Y(f(x), v_1(f(x),i)))$$

We have that:

$$x \in U_{R_0^X(x,i)} \subseteq f^{-1}(V_{R_0^Y(f(x),v_1(f(x),i))}) \subseteq U_i$$

We define for $y \notin U_i$:

$$R_1^X(x,i,y) = v_0(y, R_1^Y(f(x), v_1(f(x),i), f(y)))$$

Since $y \notin U_i$ then $f(y) \notin V_{v_1(f(x),i)}$ which means that $R_1^Y(f(x), v(f(x),i), f(y))$ is well defined. By definition of R_1^Y we have that:

$$f(y) \in V_{R_1^Y(f(x),v_1(f(x),i),f(y))} \subseteq Y \setminus V_{R_0^Y(f(x),v_1(f(x),i))} \subseteq Y \setminus f(U_{R_0^X(x,i)})$$

In particular, we get that:

$$y \in U_{R_1^X(x,i,y)} \subseteq f^{-1}(V_{R_1^Y(f(x),v_1(f(x),i),f(y))}) \subseteq f^{-1}(Y \setminus f(U_{R_0^X(x,i)})) = X \setminus U_{R_0^X(x,i)}$$

This means that the function R_0^X and R_1^X have the desired properties, and so X is uniformly effectively regular. □

Proposition 4.17 $\mathbf{RCA}_0$ proves that every uniformly T_3 $CSCS$ is effectively T_2.

Proof: Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a uniformly T_3 $CSCS$ and let $x, y \in X$ be distinct points. Since X is T_0, let i be the least number such that $x \in U_i \leftrightarrow y \notin U_i$. If $x \in U_i$ then we let $H_0(x,y) = R_0(x,i)$ and $H_1(x,y) = R_1(x,i,y)$ otherwise if $y \in U_i$ we set $H_0(x,y) = R_1(x,i,y)$ and $H_1(x,y) = R_0(x,i)$. □

Proposition 4.18 $\mathbf{RCA}_0$ proves that every linear order $(L, <_L)$ that has an effectively T_2 order topology, is also uniformly T_3.

Proof: Let H_0 and H_1 be the functions that witness that the order topology on L is effectively T_2. Given $y <_L x <_L z$ we define:

$$R_0(x,(y,z)) = k(H_0(x,y), H_0(x,z))$$

and:

$$R_1(w,x,(y,z)) = \begin{cases} H_1(x,y) \cup]-\infty, y] \text{ if } w \leq_L y \\ H_1(x,z) \cup [z, +\infty[\text{ if } z \leq_L w \end{cases}$$

The functions R_0 and R_1 witness that the order topology on L is uniformly T_3. □

Corollary 4.19 Over $\mathbf{RCA}_0$ arithmetic comprehension is equivalent to every regular $CSCS$ is uniformly regular.

Proposition 4.20 Over $\mathbf{RCA}_0$ arithmetic comprehension is equivalent to all regular effectively Hausdorff $CSCS$ being uniformly T_3.

Proof: We show arithmetic comprehension proves all regular $CSCS$ are uniformly regular. Given a regular $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ then for each $(x,j) \in X \times \mathbb{N}$ such that $x \in U_j$ we define:

$$R_0(x,j) = \min\{s \in \mathbb{N} : x \in U_s \subseteq \overline{U_s} \subseteq U_j\}$$

and for all $y \in X \setminus U_j$ we define:

$$R_1(x,j,y) = \min\{s \in \mathbb{N} : y \in U_s \wedge U_s \cap U_{R_0(x,j)} = \emptyset\}$$

The R_0 and R_1 have the desired properties and are arithmetically definable. So $\mathbf{ACA}_0$ proves every regular $CSCS$ is uniformly regular.

We show the converse. Let A be a set. We show that the Turing jump of A exists, which will imply arithmetic comprehension. We define:

$$X^e = \{-\infty\} \cup ((\mathbb{Z} \cup \{+\infty\}) \times \{t \in \mathbb{N} : \Phi_e^A(e) \downarrow_{\leq t}\})$$

We define on X^e the following topology:

$$U_0 = \{-\infty\} \cup (\mathbb{Z} \times \{t \in \mathbb{N} : \Phi_e^A(e) \downarrow_{\leq t}\})$$

For all $(m, n) \in (\mathbb{Z} \times \{t \in \mathbb{N} : \Phi_e^A(e) \downarrow_{\leq t}\})$ we define

$$U_{4(m,n)+1} = \{(m, n)\}$$

$$U_{4(m,n)+2} = \{(+\infty, n)\} \cup \{(l, n) : l \geq m\}$$

$$U_{4(m,n)+3} = X^e \setminus \{(l, s) : (s = +\infty) \vee (l \geq m \wedge s \leq n)\}$$

$$U_{4(m,n)+4} = \begin{cases} X^e & \text{if} \quad \neg\Phi_e^A(e) \downarrow_{\leq n} \\ \{-\infty\} \cup \{(l, s) : l \leq m \wedge \Phi_e^A(e) \downarrow_{\leq s}\} & \text{if} \quad \Phi_e^A(e) \downarrow_{\leq n} \end{cases}$$

We define H_0 and H_1 on X^e by:

- $H_0(-\infty, (m, n)) = 4(m, n) + 3$ and $H_1(-\infty, (m, n)) = 4(m, n) + 1$
- $H_0(-\infty, (+\infty, n)) = 4(m, n) + 3$ and $H_1(-\infty, (+\infty, n)) = 4(m, n) + 2$
- $H_0((+\infty, l), (+\infty, n)) = 4(0, l) + 2$ and $H_1((+\infty, l), (+\infty, n)) = 4(0, n) + 2$
- $H_0((+\infty, l), (m, n)) = 4(m + 1, l) + 2$ and $H_1((+\infty, l), (m, n)) = 4(m, n) + 1$
- $H_0((p, l), (m, n)) = 4(p, l) + 1$ and $H_1((p, l), (m, n)) = 4(m, n) + 1$

To define k let $a = 4(i, j) + r$ and $b = 4(p, l) + s$ and let $c \geq \max\{|i|, j, |p|, l\}$ then we define:

- $k((m, n), a, b) = 4(m, n) + 1$
- $k((\infty, n), a, b) = 4(c, n) + 2$
- $k(-\infty, a, b) = \begin{cases} 4(-c, c) + 4 \text{ if } \Phi_e^A(e) \downarrow_{\leq c} \\ 4(-c, c) + 3 \text{ otherwise} \end{cases}$

It is straightforward to show that the functions H_0, H_1, and k have the desired properties and that X^e is an effectively T_2 $CSCS$. To show that X^e is regular, we first observe that every basic open set, except perhaps U_0, is clopen and $-\infty$ is the only non isolated point of U_0. If $\Phi_e^A(e) \uparrow$ then $X^e = \{-\infty\}$ otherwise let t be the least number such that $\Phi_e^A(e) \downarrow_{\leq t}$ then:

$$-\infty \in U_{4(0,t)+4} \subseteq X^e \setminus \bigcup_{s \geq t} U_{4(1,s)+2} \subseteq U_0$$

Which proves that X^e is regular.

Let X be the disjoint union of the spaces X^e. X will be regular and effectively Hausdorff, and so by assumption, we have that X will be uniformly T_3. Let R_0 and R_1 be the functions given by the uniform regularity of X. Given $e \in \mathbb{N}$ we have $\Phi_e^A(e) \downarrow_{\leq n}$ if and only if for all $m \in \mathbb{N}$ $U_{4(m,n)+4} \subseteq U_0$ in the space X^e. So if $\Phi_e^A(e) \downarrow$ then there is some $m \in \mathbb{N}$ such that $R_0((-\infty, e), (0, e)) = (4(m, n)+4, e)$, where by $(-\infty, e)$ we mean $-\infty$ of the space X^e, and $(0, e)$ is the index for U_0 in X^e. So by construction of X^e we have $\Phi_e^A(e) \downarrow_{\leq n}$. In particular, $A' \leq_T R_0$, which means the Turing jump of A exists. $\square$

Observation 4.21 We observe that the space in the previous proof is scattered. So, arithmetic comprehension is equivalent to all T_3 scattered spaces being uniformly T_3.

The following technical result will be helpful in the next part.

Theorem 4.22 Over $\mathbf{RCA}_0$, every effectively T_2 T_3 scattered $CSCS$ effectively embeds into a linear order implies arithmetic comprehension.

Proof: $(2 \to 1)$ Fix a set $A \subseteq \mathbb{N}$. Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be as in the proof of Proposition 4.20. Let $(L, <_L)$ be a linear order, $f : X \to L$ an effective embedding, and v_0 and v_1 be functions that witness respectively that f is effectively continous and effectively open. For each e, we define m_e to be the unique number such that:

$$v_0((-\infty, e), v_1((-\infty, e), 0)) = 4(m_e, n) + r = s_e$$

where $r \in \{3, 4\}$ and $n \in \mathbb{N}$, or $s_e = 0$. We have that the map $e \mapsto m_e$ is recursive relative to v_0 and v_1, so it exists by Δ^0_1 comprehension. Assume that $\Phi^A_e(e)\downarrow$, we have that $v_1((-\infty, e), 0)$ will be the index of some interval $]a, b[_{<_L}$. By construction that $U_{s_e} \subseteq f^{-1}(]a, b[_{<_L}) \subseteq U_0$. We also have that $f^{-1}([a, b]_{<_L})$ is a closed set contained in U_0 plus at most two other points. If $r = 3$ or $s_e = 0$, then the closure of U_{s_e} contains infinitely many points which are not in U_0. But the closure of U_{s_e} is contained in U_0 plus two other points. So $r = 4$. By construction of X we have that $U_{4(m_e,n)+4} \subseteq U_0$ if and only if $\Phi^A_e(e)\downarrow_{\leq m_e}$. So we have that $\Phi^A_e(e)\downarrow \leftrightarrow \Phi^A_e(e)\downarrow_{\leq m_e}$, which means that A' exists since it is computable from the map $e \mapsto m_e$. Since every set has a Turing jump by Theorem 2.2, we have arithmetic comprehension. □

5. Characterizations of Dimension 0

Definition 5.1 In general topology, a space is said to be zero-dimensional or of dimension 0 if it has a basis of clopen sets.

Definition 5.2 Given a $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ we say that

- X has a basis of clopen sets if for all $i \in \mathbb{N}$ the set U_i is also closed.
- X has an algebra of clopen sets if there exists $Int : \mathbb{N}^2 \to \mathbb{N}$ such that $U_{Int(i,j)} = U_i \cap U_j$ and $Comp : \mathbb{N} \to \mathbb{N}$ such that $U_{Comp(i)} = X \setminus U_i$.

Observation 5.3 For any sequence of sets $(A_i)_{i\in\mathbb{N}}$ the algebra generated by them is Δ^0_1 definable and so are functions Int and $Comp$ that compute respectively intersections and complements. So for any zero dimensional $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ there is an $CSCS$ with algebra of clopen sets $(X, (V_i)_{i\in\mathbb{N}}, Int, Comp)$ such that $(U_i)_{i\in\mathbb{N}}$ is a subsequence of $(V_i)_{i\in\mathbb{N}}$ and every V_i is open in $(X, (U_i)_{i\in\mathbb{N}}, k)$. This means that the identity $X \to X$ is a homeomorphism. It will not, in general, be an effective homeomorphism. What is needed to ensure the identity is an effective homeomorphism is a function that encodes complements in some way.

Definition 5.4 Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a $CSCS$ we say that X is effectively zero dimensional if there is a function $G : X \times \mathbb{N} \to \mathbb{N}$ such that if $x \in X \setminus U_i$ then $x \in U_{G(x,i)} \subseteq X \setminus U_i$.

Observation 5.5 $\mathbf{RCA}_0$ proves that any T_0 effectively zero dimensional $CSCS$ is effectively T_2.

Observation 5.6 Let $(L, <_L)$ be a linear order, the partial function:

$$G(z, (x, y)) = \begin{cases} (-\infty, x) & \text{if} \quad z \leq x \\ (y, +\infty) & \text{if} \quad z > y \end{cases}$$

witnesses that the upper limit topology is effectively zero dimensional.

Proposition 5.7 $\mathbf{RCA}_0$ proves that every effectively zero dimensional $CSCS$ is uniformly regular.

Proof: Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be an effectively zero dimension $CSCS$ and G a function that witnesses that X is effectively zero dimensional. For all $x \in X$ and $i \in \mathbb{N}$ such that $x \in U_i$ we define $R_0(x, i) = i$ and for all $y \notin U_i$ we define $R_1(x, i, y) = G(y, i)$. It is easy to verify that R_0 and R_1 witness that X is uniformly regular. □

Definition 5.8 Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a $CSCS$, $x \in X$, and a set $I = (i_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have $x \in U_{i_n}$ then we define K_I as:

(1) $K_I(x, 0) = i_0$
(2) $K_I(x, n+1) = k(x, K(x, n, I), i_{n+1})$

$\mathbf{RCA}_0$ proves the existence of K_I for all $CSCS$ and all I since it proves functions are closed under primitive recursion. If F is a finite set such that $\forall i \in F\ x \in U_i$ then we define $K(x, F) = K_F(x, |F|)$. We observe that:

$$x \in U_{K(x,F)} \subseteq \bigcap_{i\in F} U_i$$

This will allow us to find effectively sufficiently small neighborhoods of x.

We show that countable regular spaces are zero dimensional. To do so, we first show that a strong form of normality holds for uniformly regular $CSCS$.

Theorem 5.9 $\mathbf{RCA}_0$ proves that for any uniformly regular $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ and any two sequences of coded closed collections $(C_{0,e})_{e\in\mathbb{N}}$ and $(C_{1,e})_{e\in\mathbb{N}}$ such that for all $e \in \mathbb{N}$ we have that $C_{0,e} \cap C_{1,e} = \emptyset$ then there exists a sequence of clopen sets $(D_e)_{e\in\mathbb{N}}$ such that for all $e \in \mathbb{N}$ $C_{0,e} \subseteq D_e \subseteq X \setminus C_{1,e}$.

Proof: We prove the case for one pair (C_0, C_1) of disjoint closed collections coded by f_0 and f_1 respectively. The general case will follow from the fact the construction can be carried out uniformly. To construct D we will define recursively two increasing sequences of closed sets $(C_0^n)_{n\in\mathbb{N}}$ and $(C_1^n)_{n\in\mathbb{N}}$ such that $C_0^0 = C_0$ and $C_1^0 = C_1$ and for all n the sets C_0^n and C_1^n are disjoint. We also define two increasing sequences of open sets $(A_0^n)_{n\in\mathbb{N}}$ and $(A_1^n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ and $j \leq 1$ we have $A_j^n \subseteq C_j^n$. We would like that $\bigcup_{n\in\mathbb{N}} A_0^n \cup A_1^n = X$ and in such case we define $D = \bigcup_{n\in\mathbb{N}} A_0^n$ which by construction will be clopen and $C_0 \subseteq D \subseteq X \setminus C_1$. At every step, we will pick an $x \notin A_0^n \cup A_1^n$ and find a small enough neighborhood U_l such that for some $j \leq 1$ we have $x \in U_l \subseteq X \setminus C_j^n$. We then set $A_{1-j}^{n+1} = A_{1-j}^n \cup U_{R_0(x,l)}$ and $C_{1-j}^{n+1} = C_{1-j}^n \cup \bigcap_{y\notin U_l}(X \setminus U_{R_1(x,l,y)})$.

To be able to construct the sequences $(C_0^n)_{n\in\mathbb{N}}, (C_1^n)_{n\in\mathbb{N}}, (A_0^n)_{n\in\mathbb{N}}$, and $(A_1^n)_{n\in\mathbb{N}}$ we will want them to be coded as finite objects. We will code A_j^n with a finite set F_J^n of indices such that $A_j^n = \bigcup_{i\in F_j^n} U_i$. We will code C_j^n as a finite set of pairs E_j^n such that:

$$C_j^n = \bigcap_{i\in\mathbb{N}}(X \setminus U_{f_j(i)}) \cup \bigcup_{(x,l)\in E_j^n} \left(\bigcap_{y\notin U_l} (X \setminus U_{R_1(x,l,y)}) \right)$$

In general, C_j^n may not exist since it is Π_1^0 definable using parameters.

Let $X = (x_n)_{n\in\mathbb{N}}$ and let $F_0^0 = F_1^0 = E_0^0 = E_1^0 = \emptyset$. Given C_0^n, C_1^n and A_0^n, A_1^n, if $A_0^n \sqcup A_1^n = X$ then we are done. Otherwise, let m be the least number such that $x_m \in X \setminus (A_0^n \cup A_1^n)$. Let (i, j) be the least pair such that $j \leq 1$ and:

$$(\exists s \leq i\ x_m \in U_{f_j(i)}) \wedge \forall (z, l) \in E_j^n\ \exists y \leq i\ x_m \in U_{R_1(z,l,y)}$$

The formula above states that in i steps, we verify x_m is not in C_j^n. Let s be the least number such that $x_m \in U_{f_j(s)}$ and define:

$$I = \{R_1(z, l, y) : (z, l) \in E_j^n \wedge y = \mu w \leq i\ x_m \in U_{R_1(z,l,w)}\} \cup \{f_j(s)\}$$

which will be a finite set by $\mathbf{B}\Sigma_1^0$ and will be coded. We then define

$$F_{1-j}^{n+1} = F_{1-j}^n \cup \{R_0(x_m, K(x_m, I))\} \quad \text{and} \quad F_j^{n+1} = F_j^n$$

and

$$E_{1-j}^{n+1} = E_{1-j}^{n} \cup \{(x_m, K(x_m, I))\} \quad \text{and} \quad E_j^{n+1} = E_j^n$$

We now verify that the sets A_j^{n+1}, C_j^{n+1}, A_{1-j}^{n+1}, and C_{1-j}^{n+1} have the wanted properties. By construction $A_j^{n+1} = A_j^n \subseteq C_j^n = C_j^{n+1}$. We have that:

$$A_{1-j}^{n+1} = A_{1-j}^{n} \cup U_{R_0(x_m,K(x_m,I))} \subseteq C_{1-j}^{n+1} \cup \bigcap_{y \notin U_{K(x_m,I)}} X \setminus U_{R_1(x_m,K(x_m,I),y)} = C_{1-j}^{n+1}$$

since $A_{1-j}^n \subseteq C_{1-j}^n$ and by definition of R_0 and R_1 we have:

$$U_{R_0(x_m,K(x_m,I))} \subseteq \bigcap_{y \notin U_{K(x_m,I)}} X \setminus U_{R_1(x_m,K(x_m,I),y)}$$

We prove that $C_j^{n+1} \cap C_{1-j}^{n+1} = \emptyset$. We have by construction that:

$$C_j^{n+1} = C_j^n \quad \text{and} \quad C_{1-j}^{n+1} = C_{1-j}^n \cup \bigcap_{y \notin U_{K(x_m,I)}} X \setminus U_{R_1(x_m,K(x_m,I),y)}$$

By definition of the function R_1 and K we have:

$$\bigcap_{y \notin U_{K(x_m,I)}} X \setminus U_{R_1(x_m,K(x_m,I),y)} \subseteq U_{K(x_m,I)} \subseteq \bigcap_{i \in I} U_i$$

It, therefore, suffices to show that $C_j^n \cap \bigcap_{i \in I} U_i = \emptyset$. Let $r \in C_j^n$ using the definition of C_j^n we have the following cases:

Case 1: $r \in \bigcap_{i \in \mathbb{N}} X \setminus U_{f_j(i)}$ then $r \notin U_{f_j(s)}$ and so $r \notin \bigcap_{i \in I} U_i$ since $f_j(s) \in I$.

Case 2: There is $(z, l) \in E_j^n$ such that $r \in \bigcap_{y \notin U_l} X \setminus U_{R_1(z,l,y)}$. By definition of I there is some y such that $R_1(z, l, y) \in I$ and so $r \notin U_{R_1(z,l,y)}$.

Therefore, the sequences $(C_0^n)_{n\in\mathbb{N}}, (C_1^n)_{n\in\mathbb{N}}, (A_0^n)_{n\in\mathbb{N}}$, and $(A_1^n)_{n\in\mathbb{N}}$ have the wanted properties. Since the sequences $(F_0^n)_{n\in\mathbb{N}}$, $(F_1^n)_{n\in\mathbb{N}}$, $(E_0^n)_{n\in\mathbb{N}}$, and $(E_1^n)_{n\in\mathbb{N}}$ are defined recursively we have that they exist by Δ_1^0 comprehension. As observed before we have that the set

$$D = \bigcup_{n \in \mathbb{N}} A_0^n = X \setminus \bigcup_{n \in \mathbb{N}} A_1^n$$

will be the wanted clopen set. D exists Δ_1^0 comprehension since it and its complement are both Σ_1^0 definable. □

Observation 5.10 Applying the previous result to the *CSCS* $(\mathbb{N}, (\{i\})_{i\in\mathbb{N}}, k)$, where $k(i, i, i) = i$, gives us the classic recursion theory result that any pair of disjoint Π_1^0 sets can be separated by a recursive set.

Corollary 5.11 $\mathbf{RCA}_0$ proves that any uniformly regular *CSCS* is effectively homeomorphic to a *CSCS* with a basis of clopen sets and a function G.

Proof: Let R_0 and R_1 be the functions that witness that X is uniformly T_3. For each pair (x, i), where $x \in U_i$, let $C_0^{(i,x)}$ be the closure of the point x; which by regularity is contained in $U_{R_0(x,i)}$, and $C_1^{(x,i)} = X \setminus \bigcup_{y \notin U_i} U_{R_1(x,i,y)}$. We point out that the closure of x may not be a set, but it will be a coded closed collection. By Theorem 5.9 we have that there exists a sequence of clopen sets $(D_{(x,i)})_{x\in U_i}$ such that for all $x \in U_i$ we have $x \in D_{(x,i)} \subseteq X \setminus \bigcup_{y\notin U_i} U_{R_1(x,i,y)} \subseteq U_i$. We define k'. Given $x \in D_{(y,j)} \cap D_{(z,l)}$, by the construction carried out in Theorem 5.9 we have that:

$$D_{(y,j)} = \bigcup_{n\in\mathbb{N}} A^n_{0,(y,j)} = \bigcup_{n\in\mathbb{N}} \bigcup_{s\in F^n_{0,(y,j)}} U_s \qquad \text{and } D_{(z,j)} = \bigcup_{n\in\mathbb{N}} A^n_{0,(z,l)} = \bigcup_{n\in\mathbb{N}} \bigcup_{t\in F^n_{0,(z,l)}} U_t$$

Let $m = \min\{n \in \mathbb{N} : x \in A^n_{0,(y,j)} \cap A^n_{0,(z,l)}\}$ and define:

$$s = \min\{s' \in F^n_{0,(y,j)} : x \in U_{s'}\} \qquad \text{and} \qquad t = \min\{t' \in F^n_{0,(z,l)} : x \in U_{t'}\}$$

a let $k'(x,(y,i),(z,j)) = (x,k(x,s,t))$. It is straightforward, using R_0, to show that $(X,(D_{(x,i)})_{x\in U_i},k')$ is a $CSCS$ with a basis of clopen sets and it is effectively homeomorphic to $(X,(U_i)_{i\in\mathbb{N}},k)$. We define a function G witnessing that X is effectively zero dimensional. Let $x \in X$ and $i \in \mathbb{N}$ be such that $x \in U_i$. Observing the construction carried out in Theorem 5.9 we have that $X \setminus D_{(x,i)} = \bigcup_{n\in\mathbb{N}} A^n_1 = \bigcup_{n\in\mathbb{N}} \bigcup_{j\in F^n_1} U_j$. Given $y \notin D_{(x,i)}$ let $m = \min\{n \in \mathbb{N} : y \in A^n_1\}$ and define $G(y,(x,i)) = (y, \min\{j \in F^n_1 : y \in U_j\})$. We have that G witnesses that $(X,(D_{(x,i)})_{x\in U_i},k')$ is effectively zero dimensional and is recursively defined, so it exists by Δ^0_1 comprehension. □

Proposition 5.12 A $CSCS$ $(X,(U_i)_{i\in\mathbb{N}},k)$ with a basis of clopen sets and a function G is effectively homeomorphic to a $CSCS$ with an algebra of clopen sets.

Proof: Let $(V_i)_{i\in\mathbb{N}}$ be the sequence of sets given by $V_{2j} = U_i$ and $V_{2i+1} = X \setminus U_i$. We then define $A_m = \bigcap_{i\in m} \bigcup_{j\in i} V_i$. We define $Int(k,m)$ to be the number that codes the union of m and n. For $m \in \mathbb{N}$ coding the set:

$$\{\{a_{0,0},\dots,a_{0,n_0}\},\dots,\{a_{r,0},\dots,a_{r,n_r}\}\}$$

then $Comp(m)$ will code the set of all sets of the form:

$$\{a^c_{0,f(0)}, a^c_{1,f(1)},\dots,a^c_{r,f(r)}\}$$

where $f : r \to \max\{n_j : j \le r\}$ is a coded function such that for all $j \le r$ we have $f(j) \le n_j$ and where $a^c = a-1$ if a is odd and $a^c = a+1$ if a is even. One observes that $A_{Comp(m)} = X \setminus A_m$.

Let $Id : (X,(U_i)_{i\in\mathbb{N}},k) \to (X,(A_m)_{m\in\mathbb{N}},Int,Comp)$ be the identity function from X to X, we show it's an effective homeomorphism. The function $(x,i) \mapsto \{\{2i\}\}$ witnesses that Id is effectively continuous. We need to show that there is a function v such that for any (x,m) such that $x \in A_m$ then $x \in U_{v(x,m)} \subseteq A_m$. We define v on the complexity of the set m codes. If $m = \{\{2i\}\}$ then we set $v(x,m) = i$. If $m = \{\{2i+1\}\}$ then $v(x,m) = G(x,i)$. For $m = \{\{a_0,\dots,a_n\}\}$ where $a_0 < a_1 < \dots < a_n$ then we define $v(x,m) = v(x,\{\{a_0\}\})$. Finally for any $m = \{m_0,\dots,m_r\}$ then let $I = \{v(x,\{m_i\}) : i \le r\}$ we define:

$$v(x,m) = K(x,I)$$

The function v is defined primitively recursively, and so it exists by Δ^0_1 comprehension, and it verifies that the identity is effectively open between $(X,(U_i)_{i\in\mathbb{N}},k)$ and $(X,(A_m)_{m\in\mathbb{N}},Int,Comp)$. □

Theorem 5.13 $\mathbf{RCA}_0$ proves that for all $CSCS$ $(X,(U_i)_{i\in\mathbb{N}},k)$ the following are equivalent:

(1) X is uniformly T_3.
(2) X is T_0 and effectively homeomorphic to a $CSCS$ with an algebra of clopen sets.
(3) X is effectively homeomorphic to a subspace of $\mathbb{Q}$.
(4) X is metrizable.

Proof: The case in which X is finite is trivial, so we assume X is infinite and let $(x_i)_{i\in\mathbb{N}}$ be the elements of X enumerated increasingly. $(3 \to 4)$ is trivial.

($1 \to 2$) follows from Corollary 5.11 and Proposition 5.12. ($2 \to 1$) follows from the fact that any $CSCS$ with an algebra of clopen sets is effectively zero dimensional, which by Proposition 5.7 is uniformly T_3.

($4 \to 2$) We already saw in Proposition 3.9 that all countable metric spaces have a $CSCS$ structure that consists of clopen balls. In particular given a metric space (X, d) there exists an $a \in \mathbb{R}_{>0}$ such that $(X, (B(x, q \cdot a))_{q \in \mathbb{Q}_{>0}, x \in X}, k)$ is a $CSCS$ with a basis of clopen sets. We define $G : X \times (X \times \mathbb{Q}_{>0}) \to (X \times \mathbb{Q}_{>0})$ which will send $(x, (y, q))$, where $x \notin B(y, q \cdot a)$, to $(x, \frac{1}{n})$ where n is the least n such that $\frac{1}{2^n} \cdot a < d(x, y) - q \cdot a$. Since the $CSCS$ structure of X is effectively zero dimensional, by Proposition 5.12, it is effectively homeomorphic to a space with an algebra of clopen sets.

($2 \to 3$) Assume that X has an algebra of clopen sets. We will show there is an embedding f from X to the rational points of the Cantor space, which we will denote as $2^{\mathbb{N}} \cap \mathbb{Q}$. We observe that $2^{\mathbb{N}} \cap \mathbb{Q}$ is a $CSCS$ with an algebra of clopen sets of the form $[[\tau]] = \{g \in 2^{\mathbb{N}} \cap \mathbb{Q} : \tau \sqsubseteq g\}$, and it effectively embeds into $[0, 1] \cap \mathbb{Q}$.

We construct recursively the following things:

- A total surjection $\phi : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all $i < j < 2n$ we have:
$$\{m < 2n : x_i \in U_{\phi(m)}\} \neq \{m < 2n : x_j \in U_{\phi(m)}\}$$
The function ϕ will define a sequence of basic open sets $(U_{\phi(n)})_{n \in \mathbb{N}}$. For $\tau \in 2^{<\mathbb{N}}$ we write:
$$N_\tau = \{x \in X : \forall m < |\tau| \ (x \in U_{\phi(m)} \leftrightarrow \tau(m) = 1)\}$$
Since we are assuming that X has an algebra of sets, we have that for every sequence τ that:
$$N_\tau = \left(\bigcap_{i < |\tau| \wedge \tau(i) = 1} U_{\phi(i)} \right) \cap \left(\bigcap_{i < |\tau| \wedge \tau(i) = 0} U_{Comp(\phi(i))} \right)$$
So there is some j such that $N_\tau = U_j$. Furthermore, there is a uniformly effective procedure to find an index j from τ. In general, N_τ might be empty, but this will not affect the construction.
- $f(x_0), \ldots, f(x_n) \in 2^{<\mathbb{N}} \cap \mathbb{Q}$.
- A function $r_n : 2^{\leq 2n} \to 2^{\leq 2n}$ which is monotone, level preserving and for all $\sigma \in \text{rng}(r_n)$ there is an $i \leq n$ such that $f(x_i) \in N_\sigma$. We also require that for all $i \leq n$ we have $f(x_i) \in r_n(seq_{2n}(x_i))$ where $seq_{2n}(x_i)$ is a sequence of length $2n$ such that:
$$seq_{2n}(x_i)(j) = \begin{cases} 1 & \text{if} \quad x \in U_{\phi(j)} \\ 0 & \text{if} \quad x \notin U_{\phi(j)} \end{cases}$$
This ensures us that for all τ of length n $f(N_\tau) = f(X) \cap [[r_n(\tau)]]$ and that f will be injective. Finally, we will require for all $m < n$ that $r_m \subseteq r_n$.

Assume we have $f(x_0), \ldots, f(x_n), r_n, \phi|_{<2n}$ and that for all $\sigma \in 2^{=2n}$ $[[r_n(\sigma)]]$ contains a unique $f(x_i)$.

Step 1: We set $\phi(2n) = \min \mathbb{N} \setminus \text{rng}(\phi|_{<2n})$, this ensures us that ϕ will be surjective. Given $\tau \in 2^{<\mathbb{N}}$ of length $2n - 1$ there exists exactly one $j \leq n$ such that $f(x_j) \in [[r_n(\tau)]]$. Let $i \leq 1$ be such that $f(x_j) \in [[r_n(\tau)^\frown(i)]]$, we define $r'_n(\tau^\frown(0)) = r'_n(\tau^\frown(1)) = r_n(\tau)^\frown(i)$.

Step 2: There is, by construction of r'_n, exactly one $j \leq n$ such that $r'_n(seq_{2n+1}(x_{n+1})) = r'_n(seq_{2n+1}(x_j))$. We define:

$$\phi(2n+1) = \min\{l \in \mathbb{N} \setminus \mathrm{rng}(\phi|_{<2n+1}) : x_{n+1} \in U_l \leftrightarrow x_j \notin U_l\}$$

That is, the first index for a basic open set not in the range of $\phi|_{<2n+1}$ that separates x_{n+1} and x_j. The search for such l will eventually terminate since we are assuming X is T_0. Assume that $x_j \in U_{\phi(2n+1)}$ and $x_{n+1} \notin U_{\phi(2n+1)}$, the proof for the other case is analogous. Let ρ be the unique sequence of length $2n+2$ such that $f(x_j) \in [[\rho]]$ and let $i = \rho(2n+1)$. For all $\sigma \in 2^{=2n+1}$ such that $r'_n(\sigma) = r'_n(seq_{2n+1}(x_{n+1})) = r'_n(seq_{2n+1}(x_j))$ we define:

$$r_{n+1}(\sigma^\frown(0)) = r'_n(\sigma)^\frown(1-i)$$

$$r_{n+1}(\sigma^\frown(1)) = r'_n(\sigma)^\frown(i) = \rho$$

For all other sequences of length $2n+2$ we define r_{n+1} as in step 1 .

In either case, we are ensured that for all $j \leq n$, we have:

$$f(x_j) \in [[r_{n+1}(seq_{2n+2}(x_j))]]$$

Step 3: We define $f(x_{n+1})$ to be the least Cantor rational in the open set $[[r_{n+1}(seq_{2n+2}(x_{n+1}))]]$. This ends the construction.

Given the functions f, ϕ and $r = \bigcup_{n\in\mathbb{N}} r_n$ we show that f is an effective homeomorphism with its range.

We first show that the range of f exists. By definition $y \in \mathrm{rng}(f) \leftrightarrow \exists x \ f(y) = x$, and so the range is Σ^0_1 definable relative to f. We also have that $y \in \mathrm{rng}(f)$ if and only if for all $n \in \mathbb{N}$ there is a $\sigma \in 2^{=n}$ such that $y \in [[r_n(\sigma)]]$, which is a Π^0_1 definition. So we have that $\mathrm{rng}(f)$ is Δ^0_1 definable and therefore exists by Δ^0_1 comprehension. We note that Π^0_1 definition of $\mathrm{rng}(f)$ tells us it is a closed subspace of $2^{\mathbb{N}} \cap \mathbb{Q}$.

Let $x_j \in X$ be a point and $U_{\phi(i)}$ an open set containing it. We wish to effectively find an open neighborhood of $f(x)$ that is contained in $f(U_{\phi(i)})$. Let $n > \max\{i, j\}$, we have that $seq_{2n}(x_j)(i) = 1$ and so:

$$x_j \in N_{seq_{2n}(x_j)} \subseteq U_{\phi(i)}$$

Which by construction implies:

$$f(x_j) \in f(N_{seq_{2n}(x_j)}) = f(X) \cap [[r_n(seq_{2n}(x_j))]] \subseteq f(U_{\phi(i)})$$

This implies that $f : X \to f(X)$ is effectively open.

Let $[[\tau]]$ be a basic open set of the Cantor space and $f(x_j) \in [[\tau]]$. By construction we have that $\tau \in \mathrm{rng}(r)$, since r is level preserving we can effectively find $\rho = r^{-1}(\tau)$. We have that $x \in N_\rho$ and $f(N_\rho) \subseteq [[\tau]]$ by construction. This shows that f is effectively continuous. Therefore f is an effective embedding of X into $2^{<\mathbb{N}} \cap \mathbb{Q}$. □

The implication $(1 \to 3)$ is essentially due to Sierpinski [18], who uses a back and forth construction to show that any pair of countable sets of $\mathbb{R}^n$ without isolated points are homeomorphic. That the back and forth method, used by Sierpinski can be carried out in $\mathbf{RCA}_0$ for effectively zero dimensional T_0 spaces was first observed by Soldà unpublished.

Theorem 5.14 (Soldà unpublished) $\mathbf{RCA}_0$ proves that any two effectively zero dimensional $CSCS$ without isoltated points are effectively homeomorphic.

Corollary 5.15 Any non empty uniformly T_3 $CSCS$ without isolated points is effectively homeomorphic to $\mathbb{Q}$ with the order topology.

Proof: By Corollary 5.11 any uniformly T_3 $CSCS$ is effectively zero dimensional, so by Theorem 5.14, any uniformly T_3 space without isolated points will be homeomorphic to $\mathbb{Q}$. ☐

Observation 5.16 Since over $\mathbf{ACA}_0$ all T_3 $CSCS$ are uniformly T_3 we have that $\mathbf{ACA}_0$ proves every T_3 space is homeomorphic to a subset of $\mathbb{Q}$ and is homeomorphic to a space with an algebra of clopen sets.

Theorem 5.17 The following are equivalent over $\mathbf{RCA}_0$:

(1) Arithmetic comprehension.
(2) Every T_3 effectively T_2 $CSCS$ is effectively homeomorphic to a metric space.
(3) Every T_3 effectively T_2 $CSCS$ is effectively homeomorphic to a $CSCS$ with an algebra of clopen sets.
(4) Every T_3 effectively T_2 $CSCS$ is effectively homeomorphic to a subset of the rationals.

Proof: By Theorem 5.13, 2,3, and 4 are equivalent over $\mathbf{RCA}_0$. By Proposition 5.7, all effectively zero dimensional $CSCS$, and therefore all $CSCS$ with an algebra of clopen sets, are uniformly regular. By Proposition 4.16, 4 is equivalent to every regular effectively T_2 $CSCS$ is uniformly regular, which is equivalent arithmetic comprehension over $\mathbf{RCA}_0$ by Theorem 4.20. ☐

It is clear that a $CSCS$ that is not T_0 cannot be homeomorphic to a subspace of the rationals. However, for non T_0 uniformly regular spaces, we have the following result.

Proposition 5.18 $\mathbf{RCA}_0$ proves that every uniformly regular $CSCS$ is effectively homeomorphic to a $CSCS$ with a pseudometric.

Proof: Recall that a pseudometric on a set X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ that is symmetric and the triangle inequality holds. Let $(X, (U_i)_{i\in\mathbb{N}})$ be a uniformly regular $CSCS$ by Corollary 5.11 and Proposition 5.12 we can assume without loss of generarity that X has an algebra of clopen sets. Let $U[x] = \{i \in \mathbb{N} : x \in U_i\}$ we define d by:

$$d(x, y) = \sum_{k\in U[x]\triangle U[y]} \frac{1}{2^k}$$

The function d is symmetric, and the triangle inequality follows from the following property of the symmetric difference:

$$\forall A, B, C\ (A\triangle C) \subseteq ((A\triangle B) \cup (B\triangle C))$$

By definition of d we have:

$$B(x, \frac{1}{2^k}) = N_{U^k[x]} = \{y \in X : \forall i < n; (x \in U_i \leftrightarrow y \in U_i)\}$$

So we have that $B(x, \frac{1}{2^k})$ is the boolean combination of clopen sets and is, therefore, clopen. We show that the identity $Id : (X, (U_i)_{i\in\mathbb{N}}, Int, Comp) \to (X, (B(x, \frac{1}{2^k}))_{k\in\mathbb{N}}, k)$ is an effective homeomorphism. For $x \in X$ and $i \in \mathbb{N}$ such that $x \in U_i$ then we have that $x \in B(x, \frac{1}{2^i}) \subseteq U_i$ and so the map $(x, i) \mapsto i$ verifies that the identity is effectively open. The map that sends (x, k) to the code for the open set:

$$\left(\bigcap_{i<k\wedge x\in U_i} U_i\right) \cap \left(\bigcap_{i<k\wedge x\notin U_i} X \setminus U_i\right)$$

is computable relative to the functions $Comp$ and Int and it verifies that the identity is effectively continuous. ☐

Just as in the T_0 case, we get the following result:

Theorem 5.19 $\mathbf{RCA}_0$ proves that for any uniformly T_3 $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ the following are equivalent:

1. (1) X is uniformly regular.
2. (2) X is effectively homeomorphic to a $CSCS$ with an algebra of clopen sets.
3. (3) X is effectively zero dimensional.
4. (4) X is pseudometrizable.

6. Compactness

Definition 6.1 A $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ is said to be compact if for every $I \subseteq \mathbb{N}$ such that $X = \bigcup_{i\in I} U_i$ there exists a finite $a \subseteq I$ such that $X = \bigcup_{i\in a} U_i$. The covering relation for the space X is the set $C = \{a \in \mathbb{N} : X = \bigcup_{i\in a} U_i\}$. We say that X is effectively compact if it is compact and has a covering relation. We say that X is sequentially compact if every sequence contains a converging subsequence. A subset of a $CSCS$ is said to be compact or effectively compact if its subspace topology is respectively compact or effectively compact.

Lemma 6.2 (Dorais [3, Proposition 7.5]) $\mathbf{RCA}_0$ proves that every linear order with the order topology has a covering relation.

Observation 6.3 (Dorais [3]) Over $\mathbf{ACA}_0$ a $CSCS$ is compact if and only if it is sequentially compact.

Observation 6.4 $\mathbf{RCA}_0$ proves that the union of two effectively compact sets is effectively compact. In particular, Given C_0, C_1 the covering relations of K_0, K_1 respectively, then $C_0 \cap C_1$ is the covering relation for $K_0 \cup K_1$.

Theorem 6.5 Over $\mathbf{RCA}_0$ the following are equivalent:

1. (1) Arithmetic comprehension.
2. (2) Every compact $CSCS$ is effectively compact (Dorais [3, Example 3.5]).
3. (3) Every compact uniformly T_3 $CSCS$ is effectively compact.

Proof: $(1 \to 2)$ since the covering relation for a $CSCS$ is arithmetically definable. $(2 \to 3)$ is trivial. For $(3 \to 1)$ consider the $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ where $X = \mathbb{N} \cup \{\infty\}$:

$$U_{2i} = \{i\}$$

$$U_{2(n,s)+1} = \{\infty\} \cup \{j \in \mathbb{N} : j \geq n \wedge \forall e \in s\ \Phi^A_e(e)\downarrow_{\leq j} \to \exists r < j \Phi^A_e(e)\downarrow_{\leq r}\}$$

that is all the $j \geq n$ such that for all $e \in s$ the Turing machine of index e does not halt for the first time at step j. We define k that sends $(n, a, b) \mapsto 2n$ and $(\infty, 2(i,s)+1, 2(j,t)+1)$ to $2(\max\{i,j\}, r)+1$ where r is the codes the union of s and t. X is uniformly T_3 follows from the fact that the basis consists of clopen sets and all points except ∞ are isolated. By assumption, X is effectively compact and, therefore, has a covering relation C. We have that $\{2(0,\{e\})+1\} \in C \leftrightarrow \Phi^A_e(e)\uparrow$ and so $A' \leq_T C$. □

Proposition 6.6 (Dorais [3, Proposition 3.2]) $\mathbf{RCA}_0$ proves that being effectively compact is preserved under effective homeomorphism.

Proposition 6.7 (Dorais [3, Proposition 3.6]) $\mathbf{RCA}_0$ proves that for any compact $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ any effectively closed $C \subseteq X$ subset is a compact subspace.

Proposition 6.8 (Dorais [3, Proposition 6.3]) $\mathbf{RCA}_0$ proves that for any effectively T_2 $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ and $K \subseteq X$ an effectively compact set, then K is effectively closed.

Modifying Dorais' proof of the previous proposition, we get the following results:

Proposition 6.9 $\mathbf{RCA}_0$ proves that for any effectively T_2 *CSCS* $(X,(U_i)_{i\in\mathbb{N}},k)$ and $(K_n,C_n)_{n\in\mathbb{N}}$ sequence where K_n is an effectively compact subset of X and C_n is the covering relation for K_n, then $(K_n)_{n\in\mathbb{N}}$ is uniformly effectively closed.

Proof: Let H_0,H_1 the functions that witness that X is effectively T_2. Define $p:\mathbb{N}\times X\to\mathbb{N}$ such that for any $n\in\mathbb{N}$ and $x\notin K_n$ we have:

$$p(n,x)=\min\{F \text{ code for a finite set } : \{H_0(x,y):y\in F\}\in C_n\}$$

We observe that p is computable relative to $(K_n,C_n)_{n\in\mathbb{N}}$ and so it exists by Δ^0_1 comprehension. We define $s:\mathbb{N}\times X\to\mathbb{N}$ where $s(n,x)=K(\{H_1(x,y):y\in p(n,x)\},x)$. We have that by construction:

$$x\in U_{s(n,x)}\subseteq\bigcap_{y\in p(n,x)}U_{H_1(x,y)}\subseteq X\setminus\bigcup_{y\in p(n,x)}U_{H_0(x,y)}\subseteq X\setminus K_n$$

We have that s exists by Δ^0_1 comprehension and, for each n, we have that $s(n,\cdot)$ is a code for the closed set K_n, so $(K_n)_{n\in\mathbb{N}}$ is uniformly effectively closed. □

Theorem 6.10 $\mathbf{RCA}_0$ proves every effectively compact effectively T_2 *CSCS* is uniformly T_3.

Proof: Let $(X,(U_i)_{i\in\mathbb{N}},k)$ be an effectively compact effectively T_2 *CSCS*. Let C be the covering relation of X and H_0 and H_1 be functions that witness that X is effectively T_2. We define R_0 and R_1 on X. For any $x\in U_i$ we have that $\{U_i\}\cup\{U_{H_1(x,y)}:y\notin U_i\}$ is a covering of X. Let $F\subseteq X\setminus U_i$ be the least finite set such that $\{i\}\cup\{H_1(x,y):y\in F\}\in C$. Define:

$$R_0(x,i)=K(x,\{H_0(x,y):y\in F\})$$

and for each $z\notin U_i$ we define:

$$R_1(x,i,z)=\min\{y\in F:z\in U_{H_1(x,y)}\}$$

By definition, we have:

$$x\in U_{R_0(x,i)}\subseteq\bigcap_{y\in F}U_{H_0(x,y)}\subseteq X\setminus\bigcup_{y\in F}U_{H_1(x,y)}=X\setminus\bigcup_{z\notin U_i}U_{R_1(x,i,z)}\subseteq U_i$$

So the functions R_0 and R_1 have the desired properties and are recursive relative to the sets and functions $C,(X,(U_i)_{i\in\mathbb{N}},k)$, H_0, and H_1 so they exist by Δ^0_1 comprehension. □

Corollary 6.11 $\mathbf{ACA}_0$ proves that every T_2 compact *CSCS* is T_3.

Proposition 6.12 (See [8, Proposition 6.18]) $\mathbf{RCA}_0$ proves that every well order with a maximal element is compact with respect to its order topology.

Proof: Let L be a well order with its order topology and let $(A_i)_{i\in\mathbb{N}}$ be an open covering of L. Assume that $(A_i)_{i\in\mathbb{N}}$ does not have a finite subcover, we show that L has an infinite descending sequence $(x_i)_{i\in\mathbb{N}}$. We define x_0 to be the maximal element of L and i_0 the least i such that $x_0\in A_i$. Assume we have defined $x_0,\dots,x_n$ and $i_0,\dots,i_n$ such that $\bigcup_{k\le n}A_{i_k}$ is upwards closed. Since $\bigcup_{k\le n}A_{i_k}$ is open its $<_L$-least element cannot be a limit point of L so we set $x_{n+1}=\max L\setminus\bigcup_{k\le n}A_{i_k}$ and $i_{n+1}=\min\{i:x_{n+1}\in A_i\}$. Since all the elements of the basis are intervals, we have that $\bigcup_{k\le n+1}A_{i_k}$ will also be upwards closed, and we can effectively find x_{n+1} since A_n is of the form $]x_{n+1},z[$. We observe that $(x_i)_{i\in\mathbb{N}}$ is computable, so it exists by Δ^0_1 comprehension and is strictly decreasing in L, which contradicts the assumption that L is well ordered. □

Observation 6.13 $\mathbf{RCA}_0$ proves every linear order with its order topology has a covering relation, so $\mathbf{RCA}_0$ proves every well order with maximal element is effectively compact. Using the same proof, we can show that the upper limit topology on a well order with a maximal element is effectively compact.

Proposition 6.14 The following are equivalent over $\mathbf{RCA}_0$:

1. Π^1_1 comprehension.
2. For every sequence of $CSCS$ $(X^j)_{j\in\mathbb{N}}$ the set $\{j \in \mathbb{N} : X^j \text{ is compact}\}$ exists.

Proof: $(1 \to 2)$ follows from the fact that being a compact is a Π^1_1 definable property. We first show that 2 implies arithmetic comprehension. Let A be a set, we define for each $e \in \mathbb{N}$ the space X^e which is the set $\mathbb{N} \cup \{\infty\}$ and has as a basis of open sets $(U^e_i)_{i\in\mathbb{N}}$ where $U^e_{2t} = \{t\}$ and

$$U_{2t+1} = \begin{cases} \{\infty\} \cup \{n \in \mathbb{N} : n > t\} & \text{if} \quad \neg\Phi^A_e(e)\downarrow_{\leq t} \\ \{\infty\} & \text{if} \quad \Phi^A_e(e)\downarrow_{\leq t} \end{cases}$$

We define $k(t, a, b) = 2t$ and $k(\infty, a, b) = \max\{a, b\}$. We observe that X^e is compact if and only if $\Phi^A_e(e)\uparrow$. So $\mathbb{N} \setminus A' = \{e \in \mathbb{N} : X^e \text{ is compact}\}$. So 2 implies that the Turing jump of every set exists which implies arithmetic comprehension. It therefore suffices to show $(2 \to 1)$ over $\mathbf{ACA}_0$.

We show that 2 implies that for any sequence of trees $(T_i)_{i\in\mathbb{N}}$ the collection $\{i \in \mathbb{N} : T_i \text{ is well founded}\}$ exists, which by 2.8 is equivalent to Π^1_1 comprehension. It suffices to prove that a tree T is well founded if and only if T with the upper limit topology with respect to the Kleene Brouwer ordering is compact. If T is well founded, then $\mathrm{KB}(T)$ is well ordered, which implies that its order topology is equal to its upper limit topology. By the previous lemma, we have that $\mathrm{KB}(T)$ with the upper limit topology is compact. If T is not well founded then $\mathrm{KB}(T)$ has an infinite descending chain $(x_j)_{j\in\mathbb{N}}$. The space T admits a partition of open sets:

$$\{T\setminus \uparrow \{x_j : j \in \mathbb{N}\},]x_0, +\infty[\,\} \cup \{\,]x_{j+1}, x_j] : j \in \mathbb{N}\}$$

so T is not compact. □

Observation 6.15 In the previous proof, we saw that for every tree, there is a canonically defined $CSCS$ which is compact if and only if the tree is well founded. This implies that being a compact $CSCS$ is a universal Π^1_1 formula over $\mathbf{ACA}_0$, so in particular, compactness for $CSCS$ cannot be expressed by a Σ^1_1 formula.

7. Linear Orders

It is tempting to say that since we showed that over $\mathbf{ACA}_0$ every T_3 space is homeomorphic to a subset of the rationals, we have proved that over $\mathbf{ACA}_0$ every T_3 space is linearly ordered. The issue is that, in general, given X a subspace of a linear order $(Y, <)$, the subspace topology on X is not the same as the topology given by the order $<|_{X\times X}$. For example, the set $S = \{0\} \cup \{1 + \frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{Q}$ with the subspace topology will be a countable discrete space while with the order topology it will be homeomorphic to the 1 point compactification of a countable discrete space. The issue here is that $\{0\} = S \cap]-\frac{1}{2}, \frac{1}{2}[$ but there isn't a $b \in S$ such that $]-\infty, b[\, \cap S = \{0\}$ and so $\{0\}$ is not open in the order topology of S. In general topology, the subspaces of linearly ordered spaces with the order topology are called Generalized Ordered spaces or GO spaces. There are GO spaces that are not orderable, for example, the space $(0, 1) \cup \{2\} \subseteq \mathbb{R}$. However, in the countable case, we have that all GO spaces are orderable.

Observation 7.1 Let $(L, <_L)$ be a linear order and $S \subseteq L$. Then the order topology on S does not coincide with the subspace topology if and only if there is an $x \in S$ such that either:

1. There is a $y \in L \setminus S$ such that $y > x$ and $]x, y[\, \subseteq L \setminus S$ and $\{z \in S : x <_L z\}$ does not have a least element.
2. There is a $y \in L \setminus S$ such that $y < x$ and $]y, x[\, \subseteq L \setminus S$ and $\{z \in S : z <_L x\}$ does not have a greatest element.

In general, the subspace topology on S is finer than the topology induced by $<_L$. In the case where L is a well order, then the order topology on S coincides with the subspace topology if and only if every x which is isolated in the subspace topology is also isolated in the order topology.

Theorem 7.2 (Lynn [13]) Any separable zero dimensional metric space is orderable.

We modify Lynn's proof so that it can be carried out in $\mathbf{ACA}_0$.

Theorem 7.3 $\mathbf{ACA}_0$ proves that any T_3 space is homeomorphic to a linear order with its order topology.

Proof: Let $X = (x_i)_{i\in\mathbb{N}}$ be a T_3 space. By Theorem 5.17 every T_3 space is homeomorphic to a subset of the rationals, so without loss of generality, we may assume X is a subspace of $]0, \sqrt{2}[$. For each $\sigma \in 2^{<\mathbb{N}}$ we define I_σ to be the clopen subinterval of $\mathbb{Q}$:

$$\left]\sqrt{2}\left(\sum_{i<|\sigma|}\frac{\sigma(i)}{2^{i+1}}\right), \sqrt{2}\left(\sum_{i<|\sigma|}\frac{\sigma(i)}{2^{i+1}}+\frac{1}{2^{|\sigma|}}\right)\right[$$

By arithmetic comprehension the sequence $(I_\sigma)_{\sigma\in 2^{<\mathbb{N}}}$ exists. We observe that $\sigma \sqsubseteq \tau \leftrightarrow I_\tau \subseteq I_\sigma$ and that for all n the family $\{I_\sigma : |\sigma| = n\}$ is a partition of $]0, \sqrt{2}[$ into 2^n many clopen sets. Given an $x \in X$ and $n \in \mathbb{N}$ we write $seq_n(x)$ to be the unique sequence of length n such that $x \in I_{seq_n(x)}$.

The set $T = \{\sigma \in 2^{<\mathbb{N}} : \exists x \in X\ x \in I_\sigma\}$ exists by arithmetic comprehension. Inductively we define a function a partial function $a_{(\cdot)} : 2^{<\mathbb{N}} \to X$. We set $a_\emptyset = a_{(0)} = x_0$.

Assume a_σ is defined. Let $i = \sigma(|\sigma| - 1)$, that is i is the last digit in the sequence σ, then we defined $a_{\sigma^\frown(i)} = a_\sigma$. For ease of notation, let:

$$\tau = seq_n(a_\sigma) = seq_n(a_{\sigma^\frown(i)}) \quad \text{and} \quad j = seq_{n+1}(a_\sigma)(n)$$

If $\tau^\frown(1-j) \in T$ let $a_{\sigma^\frown(1-i)}$ be the least element of $X \cap I_{\tau^\frown(1-j)}$.

We note that for a sequence σ in the domain of $a_{(\cdot)}$ if $i = \sigma(|\sigma| - 1)$ then $a_\sigma = a_{\sigma^\frown(i)} = \cdots = a_{\sigma^\frown(i,\ldots,i)}$.

By construction we have that for all $\sigma \in T$ exists a unique τ such that $|\tau| = |\sigma|$ and $a_\tau \in I_\sigma$, let $f : T \to 2^{\mathbb{N}}$ denote the map that sends σ to τ. We have $f(seq_n(a_\sigma)) = \sigma$ and f is total, order preserving, level preserving, and injective. The function f is arithmetically definable relative to X, so it exists by arithmetic comprehension.

We define an order $<_f$ where:

$$a <_f b \leftrightarrow \exists n \in \mathbb{N}\ (f(seq_n(a)) <_{lex} f(seq_n(b)))$$

It is easy to verify that $<_f$ defines a total order on X. Under this order, we have:

$$I_{f^{-1}(\sigma)} = I_{seq_{|\sigma|}(a_\sigma)} = [a_{\sigma^\frown(0)}, a_{\sigma^\frown(1)}]_{<_f}$$

We show that the order topology induced by $<_f$ on X is the same topology as the subspace topology. Given $x \in]a, b[_{<_f}$ we have that by definition there exists an n such that:

$$f(seq_n(a)) <_{lex} f(seq_n(x)) <_{lex} f(seq_n(b))$$

So $I_{seq_n(x)} \subseteq]a, b[_{<_f}$ which implies that the topology on X is finer than the topology induced by $<_f$.

We now show the converse. We have that the set $\{I_\sigma \cap X : \sigma \in T\}$ generates the subspace topology on X, so it suffices to prove that they are open in the topology generated by $<_f$. Given

$x \in I_\sigma$ then let τ_0 and τ_1 be such that $|\tau_0| = |\sigma| = |\tau_1|$ and $f(\tau_0)$ is the lexicographic predecessor of $f(\sigma)$ and $f(\tau_1)$ the lexicographic successor of $f(\sigma)$ in the set $\{f(\rho) : \rho \in T \wedge |\rho| = |\sigma|\}$. By definition of $<_f$ we have that:

$$I_{\tau_0} = [a_{f(\tau_0)^\frown(0)}, a_{f(\tau_0)^\frown(1)}]_{<f} \quad \text{and} \quad I_{\tau_1} = [a_{f(\tau_1)^\frown(0)}, a_{f(\tau_1)^\frown(1)}]_{<f}$$

So in particular $\min I_{\tau_1} = a_{f(\tau_1)^\frown(0)}$ and $\max I_{\tau_0} = a_{f(\tau_0)^\frown(1)}$. By definition of $<_f$ we have:

$$I_\sigma \subseteq]a_{f(\tau_0)^\frown(1)}, a_{f(\tau_1)^\frown(0)}[_{<_f}$$

Let $x \in]a_{f(\tau_0)^\frown(1)}, a_{f(\tau_1)^\frown(0)}[_{<_f}$, setting $n = |\sigma|$ we have:

$$f(\tau_0) <_{lex} f(seq_n(x)) <_{lex} f(\tau_1)$$

Since $f(\sigma)$ is the only sequence of length n that is lexicographically between $f(\tau_0)$ and $f(\tau_1)$ we have that $f(seq_n(x)) = f(\sigma)$ which by injectivity of f means that $x \in I_\sigma$. So $I_\sigma =]a_{f(\tau_0)^\frown(1)}, a_{f(\tau_1)^\frown(0)}[_{<_f}$ which implies the topology induced by $<_f$ is the same as the subspace topology of X. □

For compact T_2 $CSCS$, we can do better. Friedman and Hirst essentially showed that over $\mathbf{ATR}_0$ any compact T_2 $CSCS$ is homeomorphic to a well order with the order topology [6, Lemma 4.7]. We will show that this characterization can be carried out in $\mathbf{ACA}_0$. To do so, we will give a characterization of the effectively T_2 effectively compact $CSCS$ over $\mathbf{RCA}_0$.

Observation 7.4 Over $\mathbf{RCA}_0$ let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a compact effectively zero dimensional $CSCS$ with covering relation C and G the function witnessing the effective zero dimensionality of X. We have that the inclusion relation on indices of basic open sets $\{(i,j) : U_i \subseteq U_j\}$ is defined by:

$$\forall x \in X \ (x \in U_i \to x \in U_j)$$

which is a Π^0_1 formula, and by:

$$\exists F \subseteq X \setminus U_i \ (F \text{ is finite} \wedge \{G(x,i) : x \in F\} \cup \{j\} \in C)$$

which is a Σ^0_1 formula. That is the inclusion relation $\{(i,j) : U_i \subseteq U_j\}$ is Δ^0_1 definable relative to G and C and therefore exists by Δ^0_1 comprehension. If X is an effectively compact space with an algebra of clopen sets, then we have:

$$\{i \in \mathbb{N} : U_i \neq \emptyset\} = \{i : Comp(i) \notin C\}$$

that is, being an empty basic open subset is decidable relative to $Comp$ and C. Similarly we have that for effectively compact $CSCS$ with an algebra of clopen sets, strict inclusion will be decidable relative to the additional structure.

Theorem 7.5 $\mathbf{RCA}_0$ proves that for every $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ the following are equivalent:

(1) X is effectively T_2 and effectively compact.
(2) X is effectively homemorphic to a well order with a maximum element with the upper limit topology.

Proof: $(2 \to 1)$ The upper limit topology on a well order with maximum element is effectively T_2 and effectively compact and being effectively T_2, and effectively compact is preserved under effective homeomorphism .

$(1 \to 2)$ Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be an effectively compact effectively T_2 $CSCS$ and let C be the cover relation of X. The case in which $X = \emptyset$ is trivial, so we may assume $X \neq \emptyset$. By Theorem 5.13 and 6.10 we may assume without loss of generality that X has an algebra of clopen sets and for every $x \in X$ there are infinitely many $j \in \mathbb{N}$ such that $x \in U_j$. Let Int and $Comp$ code the intersection and complement, respectively. As observed, we can effectively determine inclusion and being empty relative to the cover relation and the functions $Comp$ and Int.

Let $F : X \times \mathbb{N} \to \mathbb{N}$ be a partial function such that for all x in X we have that

$$F(x,n) = \min\{s \in \mathbb{N} : \forall m < n\ U_s \subseteq U_{F(x,m)}\}$$

We have that F exists by Δ^0_1 comprehension. Informally, F lists out a weakly descending sequence of neighborhoods for every point. For ease of notation, we will write $U(x,n) = U_{F(x,n)}$. We observe that the sets of the form $U(x,n)$ form a basis of clopen sets for X.

We construct inductively a tree $\mathcal{A}$ and we assign to each $\sigma \in \mathcal{A}$ a label $(x,i) \in X \times \mathbb{N}$ where x is the $<_{\mathbb{N}}$ least element of U_i. For the first step, we add the empty sequence to $\mathcal{A}$ with label (x,i), where i is such that $X = U_i$ and x is the $<_{\mathbb{N}}$ least element of X. Let σ be a sequence in $\mathcal{A}$ with label (y,j) and let m be the least number such that such that $U(y,m) \subseteq U_j$. For each $k \geq m$, such that $U(x,k) \setminus U(x,k+1)$ is non empty, we add $\sigma^\frown(k+1)$ to $\mathcal{A}$ with label (z,l) where l is an index for $U(y,k) \setminus U(y,k+1)$ and z is the $<_{\mathbb{N}}$ least element of U_l. If $U(y,m) \neq U_j$ then we add $\sigma^\frown(0)$ to $\mathcal{A}$ with label (z,l) where l an index for $U_j \setminus U(y,m)$ and z is the $<_{\mathbb{N}}$ least element of U_l. We have that $\mathcal{A}$ is Δ^0_1 definable relative to $(X,(U_i)_{i\in\mathbb{N}},k), C$, and *Comp* so it exists by Δ^0_1 comprehension.

Observation 7.6 Given $\sigma,\tau \in \mathcal{A}$ with labels (y,j) and (z,l) respectively, we have:

(1) $\tau \sqsubseteq \sigma$ if and only if $U_l \subseteq U_j$.
(2) If τ properly extends σ then $y \notin U_l$.
(3) If τ properly extends σ then $y <_{\mathbb{N}} z$.
(4) If $\sigma \in \mathcal{A}$ is terminal then y is isolated.
(5) τ and σ are incomparable if and only if U_j and U_l are disjoint.
(6) for all $w \in U_j \setminus \{y\}$ there exists exactly one $k \in \mathbb{N}$ such $\sigma^\frown(k) \in \mathcal{A}$, $\sigma^\frown(k)$ has label (u,r) for some $u \in X$, and $w \in U_r$.

We prove that $\mathcal{A}$ is well founded. Seeking a contradiction, assume that $\mathcal{A}$ is not well founded. Let f be a branch of $\mathcal{A}$ and let (x_n,i_n) be the label on $f|_{<n}$. We have that $(U_{i_n})_{n\in\mathbb{N}}$ is a descending sequence of non empty clopen sets, so its intersection is also non empty otherwise $(X \setminus U_{i_n})_{n\in\mathbb{N}}$ will be a covering of X without a finite subcovering. Let x be an element of $\bigcap_{n\in\mathbb{N}} U_{i_n}$, then for all $n \in \mathbb{N}$ we have that $x_n <_{\mathbb{N}} x$. This is absurd since the sequence $(x_n)_{n\in\mathbb{N}}$ is strictly increasing and therefore is unbounded in $\mathbb{N}$.

Since we are working over $\mathbf{RCA}_0$, we cannot conclude that the Kleene Brouwer order on $\mathcal{A}$ is a well order. We will show that X is homeomorphic to $\mathcal{A}$ with the upper limit topology induced by the Kleene Brouwer order. By 6.14, we have that the upper limit topology of a linear order L is compact if and only if L is well order with maximum element.

We show that every $x \in X$ appears in the label of exactly one sequence in $\mathcal{A}$. By the observations above, we have that x cannot appear in the label of more than one sequence in $\mathcal{A}$. Given an $x \in X$, define:

$$S = \{\sigma \in \mathcal{A} : (y,j) \text{ is the label on } \sigma \text{ and } x \in U_j\}$$

$S \neq \emptyset$ since it contains the empty sequence. By the observations made above, S is a chain, and since $\mathcal{A}$ is well founded, we have S must have a maximal element σ with respect to inclusion. Let (y,j) be the label on σ. If $x \neq y$ then by construction of $\mathcal{A}$ there will be an extension τ of σ with label (z,l) such that $x \in U_l$ contradicting the maximality of σ.

Let α_x denote the unique sequence of $\mathcal{A}$ that has x in its label. We show the map $G : X \to \mathcal{A}$ given by $x \mapsto \alpha_x$ is a homeomorphism between X and $\mathcal{A}$ with the upper limit topology induced by the Kleene Brouwer order.

We show that G is effectively continuous. Given $x \in X$ then a basic neighborhood for α_x is of the form $]\alpha_y, \alpha_x]$ with $\alpha_y <_{\mathrm{KB}} \alpha_x$. Let (x, j) be the label of α_x and m the least number such that $U(x, m) \subseteq U_j$. Define:

$$l = \begin{cases} \min\{n \geq m : y \notin U(x, n)\} & \text{if} \quad \alpha_x \sqsubseteq \alpha_y \\ m & \text{otherwise} \end{cases}$$

If $y \in U(x, m)$ then $\alpha_x \sqsubseteq \alpha_y$ and $y \in U(x, l-1) \setminus U(x, l)$ so $\alpha^\frown(l) \in \mathcal{A}$. We have that $\alpha_y \leq_{\mathrm{KB}} \alpha_x^\frown(l) <_{\mathrm{KB}} \alpha_x$ by construction and so $G(U(x, l)) \subseteq]\alpha_y, \alpha_x]$. If $y \notin U(x, m)$ then for all $z \in U(x, m)$ we have:

$$\alpha_y <_{\mathrm{KB}} \alpha_x^\frown(m) \leq_{\mathrm{KB}} \alpha_z <_{\mathrm{KB}} \alpha_x$$

and so $G(U(x, m)) = G(U(x, l)) \subseteq]\alpha_y, \alpha_x]$ (note that $\alpha_x^\frown(m)$ may not be a member of $\mathcal{A}$). The function $(x, y) \mapsto l$ exists by Δ^0_1 comprehension and witnesses that G is effectively continuous.

We show that G is effectively open. Let $x \in X$ be a point and U_i a basic neighborhood of x. Let (x, j) be the label on α_x, m the least number such that $U(x, m) \subseteq U_j \cap U_i$, and (z, l) the label on $\alpha_x^\frown(m+1)$. We consider the following cases:

Case 1: If $U(x, m) \subsetneq U_j$, then set $s = \min\{t \geq m : U(x, t) \setminus U(x, t+1) \neq \emptyset\}$ and y the $<_{\mathbb{N}}$ least element of $U(x, t) \setminus U(x, t+1)$. By construction we have that $\alpha_y = \alpha_x^\frown(t+1)$ and for any $z \in X$ if $\alpha_y <_{KB} \alpha_z \leq_{\mathrm{KB}} \alpha_x$ then $z \in U(x, t+1) \subseteq U(x, m) \subseteq U_i$. We set $I =]\alpha_y, \alpha_x]$.

Case 2: If $U(x, m) = U_j$ and there are no sequences to the left of α_x we set $I =]-\infty, \alpha_x] \subseteq G(U_j) \subseteq G(U_i)$.

Case 3: If $U(x, m) = U_j$ and there are sequences to left of α_x in $\mathcal{A}$ then define:

$$s = \max\{t < |\alpha_x| : \exists v < \alpha_x(t) \ (\alpha_x|_{<s}^\frown(v) \in \mathcal{A})\}$$

$$u = \max\{v \in \mathbb{N} : v < \alpha_x(s) \wedge \alpha_x|_{<s}^\frown(v) \in \mathcal{A}\}$$

We have that $\alpha_x|_{<s}^\frown(u) \in \mathcal{A}$ will be equal to some α_y and for all $z \in U(x, m)$ we have $\alpha_y <_{KB} \alpha_z \leq_{KB} \alpha_x$ and so let $I =]\alpha_y, \alpha_x] = G(U(x, m))$. In all three cases we have that $I \subseteq G(U(x, m)$, we can also effectively find an index for I and so G is effectively open.

This proves G is an effective homeomorphism from X to a linear order with the upper limit topology. By 6.14 have that the upper limit topology on a linear order L is compact if and only if L is a well order with maximal element. So $(\mathcal{A}, <_{\mathrm{KB}})$ is a well order and X is effectively homeomorphic to $\mathcal{A}$ with the upper limit topology. □

Corollary 7.7 $\mathbf{ACA}_0$ proves that every sequence of compact T_2 $CSCS$ $(K_i)_{i \in \mathbb{N}}$ the disjoint sum $\coprod_{i \in \mathbb{N}} K_i$ is homeomorphic to a well order with the order or the upper limit topology.

Proof: Over $\mathbf{ACA}_0$ the upper limit topology on a well order is homeomorphic to the order topology. Following the proof of Theorem 7.5 we can uniformly define a sequence of well orders $(<_i)_{i \in \mathbb{N}}$ such that $<_i$ has K_i as its field and the order topology induced by $<_i$ is the same as the topology of K_i. Let $<_K$ be the order on $\coprod_{i \in \mathbb{N}} K_i$ given by:

$$(x, j) <_K (y, i) \leftrightarrow (j <_{\mathbb{N}} i \vee (j = i \wedge x <_i u))$$

It is straightforward to show that $<_K$ is a well order and its order topology is the same as the topology of $\coprod_{i \in \mathbb{N}} K_i$.

□

Corollary 7.8 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic comprehension.
(2) Every T_3 space is effectively homeomorphic to a linear order with the order topology.
(3) Every compact uniformly T_3 space is effectively homeomorphic to a well order with the order topology.
(4) Every compact uniformly T_3 space is effectively homeomorphic to a linear order with the order topology.

Proof: $(1 \to 3)$ is Theorem 7.5 and $(1 \to 2)$ is Theorem 7.3. $(2 \to 4)$ and $(3 \to 4)$ are immediate. To show $(4 \to 1)$, we have that by 6.6 and 6.2 any compact space which is effectively homeomorphic to a linear order will be effectively compact. In particular, 4 implies that every compact uniformly T_3 space is effectively compact which by Proposition 6.5 implies arithmetic comprehension. □

8. Summary Part I

Theorem 8.1 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic comprehension
(2) Every compact T_2 $CSCS$ is effectively T_2 (Dorais [3, Example 7.4]).
(3) Every well order with the upper limit topology is effectively homeomorphic to a well order with the order topology (Proposition 4.9).
(4) Every effectively T_2 scattered T_3 space is uniformly T_3 (Proposition 4.20 and Observation 4.21).
(5) Every T_3 $CSCS$ is effectively homeomorphic to a linear order with the order topology (Theorem 7.3).
(6) Every compact $CSCS$ is effectively compact (Dorais [3, Example 3, 5]).
(7) Every uniformly T_3 compact $CSCS$ is effectively compact (Theorem 6.5).
(8) Every T_2 compact $CSCS$ is effectively homeomorphic to a well order with the upper limit topology or the order topology (Corollary 7.7).
(9) Every effectively T_2 T_3 scattered $CSCS$ effectively embeds into a linear order (Theorem 4.22).

Along with the following results, one can produce many more statements equivalent to arithmetic comprehension over $\mathbf{RCA}_0$. (Proposition 5.12 and Theorem 5.13) $\mathbf{RCA}_0$ proves that for any $CSCS$ X the following are equivalent:

(1) X is uniformly T_3.
(2) X is metrizable.
(3) X is T_0 and effectively zero dimensional.
(4) X is T_0 effectively homemorphic to a space with an algebra of clopen sets.
(5) X is homeomorphic to a subspace of $\mathbb{Q}$.
(6) X is homeomorphic to a closed subspace of $\mathbb{Q}$.

(Theorem 7.3 and [3, Proposition 7.5]) $\mathbf{RCA}_0$ proves that for any $CSCS$ X the following are equivalent:

(1) X is effectively T_2 and effectively compact.
(2) X is effectively T_2 and effectively homemorphic to a linear order with the order topology.
(3) X is effectively homeomorphic to a linear order with the upper limit topology.
(4) X is effectively homeomorphic to a well order with the upper limit topology.

Part 2. Topological Characterizations of $\mathbf{ATR}_0$

We saw that T_3 $CSCS$ admit a nice characterization. Namely, they are all orderable, metrizable, and homeomorphic to a subspace of the rationals, and that these characterizations can be carried out in $\mathbf{ACA}_0$. We will restrict our attention to locally compact $CSCS$ and T_3 scattered $CSCS$. The well orderability of T_2 compact $CSCS$ was first proved by Sierpinski and Mazurkiewicz [14]. Milliet gives a proof of a stronger theorem, namely that any two T_2 locally compact $CSCS$ with same Cantor Bendixson rank and degree are homeomorphic [15]. We also have that T_3 scattered spaces are completely metrizable and are homeomorphic to subspaces of well orders. It is natural to ask what system can we carry out these characterizations. It turns out all these characterizations are equivalent to arithmetic transfinite recursion unless we restrict ourselves to spaces that have additional structure. We will also find a series of interesting intermediate principles that are all equivalent to arithmetic transfinite recursion.

9. Locally Compact and Scattered Spaces

The study of T_2 locally compact $CSCS$ was done indirectly by Hirst, who considered locally totally bounded closed sets of complete separable metric space [11]. He proved that over $\mathbf{ACA}_0$ any such space is the disjoint union of compact balls and used this decomposition to define the one point compactification. In this section, we will reformulate these theorems for $CSCS$ and we will give a definition for the one point compactification which is suitable for $\mathbf{RCA}_0$.

Definition 9.1 A topological space is said to be dense in itself if it does not contain isolated points. By Corollary 5.15 $\mathbf{ACA}_0$ proves that $\mathbb{Q}$ is the only non empty dense in itself T_3 $CSCS$ up to homeomorphism.

Definition 9.2 A $CSCS$ is said to be scattered if every subspace has an isolated point or, rather, it does not contain a non empty dense in itself subspace. We observe that over $\mathbf{ACA}_0$, being a T_3 scattered $CSCS$ is equivalent to not having a subspace homeomorphic to $\mathbb{Q}$. Furthermore, over $\mathbf{RCA}_0$, we have that a uniformly T_3 $CSCS$ is scattered if and only if it does not have a subspace which is effectively homeomorphic to $\mathbb{Q}$.

Definition 9.3 Let $(X,(U_i)_{i\in\mathbb{N}},k)$ be $CSCS$. We say that X is locally compact if, for every $x\in X$, there exists a compact neighborhood of x. We say that X is locally effectively compact if, for every $x\in X$, there exists an effectively compact neighborhood for x. We say that a $CSCS$ X has a choice of compact neighborhoods or CCN for short, if there exists a sequence $(K_x,i(x))_{x\in X}$ such that for all $x\in X$ we have $x\in U_{i(x)}\subseteq K_x$ and K_x is compact. We say that a $CSCS$ X has a choice of effective compact neighborhoods or an effective CCN if there is a sequence $(K_x,i(x),C_x)_{x\in X}$ such that $x\in U_{i(x)}\subseteq K_x$ and C_x is a covering relation for K_x.

Observation 9.4 In the case in which $(X,(U_i)_{i\in\mathbb{N}},k)$ is locally compact with a basis of clopen sets, then a choice of compact neighborhoods is equivalent to having a sequence of indices $(i(x))_{x\in X}$ such that $U_{i(x)}$ is compact. Since we will often be working with zero dimensional spaces we will usually use this definition of CCN.

Lemma 9.5 $\mathbf{ACA}_0$ proves that every T_2 locally compact $CSCS$ is scattered.

Proof: Let $(X,(U_i)_{i\in\mathbb{N}})$ be a T_2 locally compact $CSCS$, and let S be a dense in itself subset of X. Enumerate $X=(x_n)_{n\in\mathbb{N}}$, let $y_0\in S$, and let V_0 be a compact neighborhood of y_0. Since we are working in $\mathbf{ACA}_0$, we have that V_0 is also sequentially compact. Given y_n and V_n such that $y_n\in V_n$, let y_{n+1} be the least element in $S\cap V_n\setminus\{y_n\}$ which will be infinite since S does not have isolated points. Let V_{n+1} be the first basic open set such that $y_{n+1}\in V_{n+1}\subseteq V_n$ and $x_{n+1}\notin V_{n+1}$. The sequence $(y_n,V_n)_{n\in\mathbb{N}}$ exists by arithmetic comprehension. Assume that the subsequence $(y_{n_k})_{k\in\mathbb{N}}$ converges to $x_m \in X$, then by construction $(y_{n_k})_{k\in\mathbb{N}}$ is definitely in V_m which does not contain x_m by construction. This contradicts the fact that V_0 is sequentially compact. $\square$

Observation 9.6 The converse does not hold. For a counter example, take the set:

$$S = \omega^2 + 1 \setminus \{\omega \cdot n : n \in \omega\} \subseteq \omega^2 + 1$$

with the subspace topology. We observe that the point $\omega^2 \in S$ does not have a compact neighborhood in S. Since S is order isomorphic to $\omega^2 + 1$, its subspace topology cannot coincide with its order topology. In general, the subspaces of locally compact $CSCS$ may not be locally compact. This counter example can be easily formalized in $\mathbf{RCA}_0$. We will see later on that all scattered T_3 $CSCS$, over a sufficiently strong theory, are precisely the subspaces of well orders.

Observation 9.7 In general, locally compact T_2 second countable spaces are not scattered. For example, $\mathbb{R}$ is locally compact, but it is not scattered since it is connected.

Proposition 9.8 $\mathbf{RCA}_0$ proves that every effectively T_2 locally effectively compact $CSCS$ is T_3.

Proof: Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a T_2 locally effectively compact $CSCS$ and let $x \in X$ and $i \in \mathbb{N}$ be such that $x \in U_i$. Let K be an effectively compact neighborhood of x and let $j \in \mathbb{N}$ be such that $x \in U_j \subseteq K$. By Proposition 6.8 there is an f which codes K. We have that K is effectively T_2 since it is the subspace of an effectively T_2 space. By Theorem 6.10 K is uniformly T_3. Let R_0^K and R_1^K witness that K is uniformly regular and let $s = k(x, i, j)$ then :

$$x \in U_{R_0^K(x,s)} \subseteq K \setminus \bigcup_{y \notin U_s} U_{R_1^K(x,s,y)} \subseteq U_s \subseteq K \cap U_i$$

So, in particular:

$$x \in U_{R_0^K(x,s)} \subseteq X \setminus \left(\bigcup_{y \notin U_s} U_{R_1^K(x,s,y)} \cup \bigcup_{n\in\mathbb{N}} U_{f(n)} \right) \subseteq U_s \subseteq U_i$$

which shows that X is regular. $\square$

Using the same proof, we get:

Corollary 9.9 $\mathbf{RCA}_0$ proves that every effectively T_2 locally compact $CSCS$ with an effective CCN is uniformly T_3.

Corollary 9.10 $\mathbf{ACA}_0$ proves that every locally compact T_2 $CSCS$ is T_3.

Observation 9.11 Let $(W, <_W)$ be a well order, then we have that the collection $(]-\infty, w+1[_{<_w})_{w\in W}$ is a CCN for W with the order topology.

Proposition 9.12 $\mathbf{RCA}_0$ proves that having a CCN and an effective CCN are preserved under effective homeomorphism.

Proof: Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ and $(Y, (V_i)_{i\in\mathbb{N}}, k')$ be $CSCS$ and $f : X \to Y$ a homeomorphism and let v be the function witnessing that f is effective open. Assume that $(K_x, i(x))_{x\in X}$ is a CCN for X. Then $(f(K_{f^{-1}(y)}), v(f^{-1}(y), i(f^{-1}(y))))_{y\in Y}$ is a CCN for Y. If for there exists a sequence $(C_x)_{x\in X}$ such that for all x the set C_x is the covering relation for K_x then by 6.6 there exist a sequence $(D_y)_{y\in Y}$ such that for all $y \in Y$ D_y is the covering relation for $f(K_{f^{-1}(y)})$. $\square$

Observation 9.13 This definition of local compactness is quite weak. The standard definition of local compactness used in topology is that for every point x and open neighborhood of U of x, there is a compact neighborhood of x contained in U. It is a classic result in general topology that both definitions of local compactness coincide for T_2 spaces. In our case, this result is trivialized since we are working with zero dimensional spaces.

Proposition 9.14 $\mathbf{ACA}_0$ proves that any T_2 locally compact $CSCS$ with CCN is homeomorphic to a space with a basis of compact sets.

Proof: Let $(X,(U_i)_{i\in\mathbb{N}},k)$ to be a T_2 $CSCS$ with a CCN. By Propositions 9.8, 9.12, and Theorem 5.17 we may assume that X has a basis of clopen sets. Let $(i(x))_{x\in X}$ be a CCN for X and let $J=\{j\in\mathbb{N} : \exists x\in X\, U_j\subseteq U_{i(x)}\}$. We have that for all $j\in J$, the basic clopen set U_j is compact by 6.7. We have that for all $n\in\mathbb{N}$ the set U_n is open with respect to the basis $(U_j)_{j\in J}$ since for all $x\in U_n$ we have:

$$x\in U_{k(x,i(x),n)}\subseteq U_n\cap U_{i(x)}\subseteq U_n$$

and $k(x,i(x),n)\in J$ since $U_{k(x,i(x),n)}\subseteq U_{i(x)}$. So $(U_j)_{j\in J}$ is a basis for X consisting of compact sets. □

Proposition 9.15 $\mathbf{ACA}_0$ proves that every locally compact T_2 $CSCS$ with a CCN is the disjoint union of open compact sets and well orderable.

Proof: Let $(X,(U_i)_{i\in\mathbb{N}},k)$ be a locally compact T_2 $CSCS$ with a CCN. By 9.14, we may assume that X has a basis of compact sets. The sequence of sets:

$$V_i=U_j\setminus\left(\bigcup_{i<j}U_i\right)$$

defines a partition of X into clopen compact sets. In particular, we have that X is homeomorphic to the disjoint sum of $(V_i)_{i\in\mathbb{N}}$. By Corollary 7.7 we have that $\coprod_{i\in\mathbb{N}}V_i$ is well orderable and therefore X is well orderable. □

Observation 9.16 Since every well order has a CCN, we have that over $\mathbf{ACA}_0$ a T_2 $CSCS$ is well orderable if and only if it has a CCN. This means that the strength of the well orderability of T_2 locally compact $CSCS$ lies in being able to choose a compact neighborhood for every point of a T_2 locally compact $CSCS$.

Proposition 9.17 $\mathbf{ACA}_0$ proves that for any locally compact T_2 $CSCS$ $(X,(U_i)_{i\in\mathbb{N}})$ and $Y\subseteq X$, if Y is a locally compact subspace then Y has a CCN.

Proof: By Proposition 4.6 Y is T_2 since it is the subspace of a T_2 space. Since CCN are preserved under homeomorphism, we may assume without loss of generality that X has a basis of clopen sets. Let $(i(x))_{x\in X}$ be a CCN for X. For each $y\in Y$ and $j\in\mathbb{N}$ such that $U_j\cap Y$ is closed in X if $U_j\subseteq U_{i(y)}$ then $U_j\cap Y$ is compact by Proposition 6.7. Conversely, if $U_j\cap Y$ is compact, then by Proposition 6.8 $Y\cap U_j$ is closed in X since it is a compact set in a T_2 space. For each $y\in Y$ let $\widehat{i}(y)$ be the first i such that $y\in U_i\subseteq U_{i(x)}$ and $U_i\cap Y$ is closed in X. The sequence $(\widehat{i}(y))_{y\in Y}$ is arithmetically definable, so it exists by arithmetic comprehension and is a CCN for Y. □

For a locally compact $CSCS$, embedding into a compact space implies having a CCN over $\mathbf{ACA}_0$. We will show that the converse is also true. To do so, we will introduce the one point compactification of a $CSCS$.

Definition 9.18 Let X be a topological space, the Alexandrov compactification or one point compactification of X is the space $X\cup\{\infty\}$ where $U\subseteq X\cup\{\infty\}$ is open if $\infty\notin U$ and U is open in X or $\infty\in U$ and $X\setminus U$ is compact in X. It is a classical result in general topology that the Alexandrov compactification of X is T_2 if and only if X is T_2 and locally compact.

Observation 9.19 In general the one point compactification of a $CSCS$ may not be a $CSCS$. For example, the one point compactification of $\mathbb{Q}$ is not second countable. We would like to consider $CSCS$ that have one point compactification which is also a $CSCS$.

Definition 9.20 A space X is hemicompact if there exists an increasing sequence of compact sets $(K_n)_{n\in\omega}$ such that every compact set of X is contained in some K_n. Equivalently, a space is hemicompact if the poset of compact subsets has a countable cofinal sequence. The Alexandrov compactification of a space X is first countable if and only if X is first countable and hemicompact.

A space X has an exhaustion by compact sets if there is a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that for all n $K_n \subseteq int(K_{n+1})$ and $X = \bigcup_{n\in\mathbb{N}} K_n$. We observe that a space X with an exhaustion by compact sets is hemicompact and locally compact.

Definition 9.21 Over $\mathbf{RCA}_0$ we say that a $CSCS$ X has an exhaustion by compact sets if there exists a sequence $(K_i, V_i, f_i, g_i)_{i\in\mathbb{N}}$ such that:

- $\bigcup_{i\in\mathbb{N}} K_i = X$
- For each $i \in \mathbb{N}$ K_i is compact, effectively closed, and g_i is a code for K_i.
- For each $i \in \mathbb{N}$ V_i is an effectively open set and f_i is a code for V_i.
- For each $i \in \mathbb{N}$ $K_i \subseteq V_{i+1} \subseteq K_{i+1}$.

Proposition 9.22 $\mathbf{RCA}_0$ proves that every T_2 locally effectively compact $CSCS$ with an effective CCN has an exhaustion of compact sets $(\widehat{K}_i, V_i, f_i, g_i)_{i\in\mathbb{N}}$. Furthermore there exists a sequence $(C_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ C_n is the covering relation for $\widehat{K}_n$.

Proof: Let $(X, (U_i)_{i\in\mathbb{N}})$ be a $CSCS$ locally compact space and let $(K_x, i(x), C_x)_{x\in X}$ be an effective CCN. For each finite subset $a \subseteq X$ we have that $\widehat{C}_a = \bigcap_{i\in a} C_i$ is the covering relation for $\bigcup_{x\in a} K_x$. The sequence $(\widehat{C}_a)_{a\subseteq X}$ exists by Δ^0_1 comprehension. Instead of defining directly the sequence $(\widehat{K}_i)_{i\in\mathbb{N}}$ we instead define recursively a sequence of finite sets $(F_n)_{n\in\mathbb{N}}$ such that $\widehat{K}_n = \bigcup_{x\in F_n} K_x$. Define $F_0 = \{\min X\}$. Given F_n, define:

$$F_{n+1} = \min\{F \in \widehat{C}_{F_n} : \forall x \in F\, \exists y \in F_n\, (x \in U_{i(y)})\}$$

Equivalently F_{n+1} is the least finite subset of $\bigcup_{y\in F_n} K_y$ such that $(U_{i(x)})_{x\in F_{n+1}}$ is a covering of $\bigcup_{y\in F_n} K_y$. We have that the sequence $(F_n)_{n\in\mathbb{N}}$ is well defined since for each n $\bigcup_{y\in F_n} K_y$ is compact and it exists by Δ^0_1 comprehension.

Define $\widehat{K}_n = \bigcup_{x\in F_n} K_x$, we have that $\widehat{C}_{F_n}$ is the covering relation for $\widehat{K}_n$. By Proposition 6.9, we have that the sequence $(\widehat{K}_n)_{n\in\mathbb{N}}$ will be uniformly effectively closed and so there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ g_n is a closed code for $\widehat{K}_n$.

Let f_n be the partial function that enumerates F_n increasingly. The sequences $(\widehat{K}_n, \bigcup_{x\in F_n} U_{i(x)}, f_n, g_n)_{n\in\mathbb{N}}$ and $(C_{F_n})_{n\in\mathbb{N}}$ are Δ^0_1 definable relative to $(F_n)_{n\in\mathbb{N}}$ and so they exist by Δ^0_1 comprehension. By construction we have that $(\widehat{K}_n, \bigcup_{x\in F_n} U_{i(x)}, f_n, g_n)_{n\in\mathbb{N}}$ is an exhaustion by compact set of X and for all $n \in \mathbb{N}$ C_{F_n} is a covering relation for $\widehat{K}_n$. □

Definition 9.23 Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a T_2 $CSCS$ with an exhaustion of compact sets $(K_n, A_n, f_n, g_n)_{n\in\mathbb{N}}$ we define it's one point compactification as the space:

$$(X \cup \{\infty\}, (V_i)_{F,n\in\mathbb{N}}\, \widehat{k})$$

where $V_{2i} = U_i$ and $V_{2i+1} = (X \cup \{\infty\}) \setminus K_i$.

(1) $\widehat{k}(x, 2i, 2j) = k(x, i, j)$.
(2) $\widehat{k}(x, 2i, 2j+1) = k(x, i, g(r))$ where $r = \min\{n \in \mathbb{N} : x \in U_{g(r)}\}$.
(3) $\widehat{k}(x, 2i+1, 2j+1) = \max\{2i+1, 2j+1\}$.

We show that $X \cup \{\infty\}$ is compact. Let $I \subseteq \mathbb{N}$ be such that $(V_i)_{i\in I}$ is a covering of $X \cup \{\infty\}$ then there exists some $2j+1 \in I$ such that $\infty \in V_{2j+1} = (X \cup \{\infty\}) \setminus K_j$. Since K_j is compact and $(U_i)_{i\in I}$ covers K_j we have that there exists a finite set $a \subseteq I$ such that $K_j \subseteq \bigcup_{i\in a} U_i$ and so $(U_i)_{i\in a\cup\{2j+1\}}$ is a finite subcovering of $X \cup \{\infty\}$.

Assume X is effectively T_2, and H_0, H_1 the functions that witness that X is effectively T_2 then the functions $\widehat{H}_0, \widehat{H}_1$ such that for all $x, y \in X$ we have:

$$\widehat{H}_0(x,y) = 2 \cdot H_0(x,y) \qquad \text{and} \qquad \widehat{H}_1(x,y) = 2 \cdot H_1(x,y)$$

For all $x \in X$, ket $j = \min\{i : x \notin K_j\}$, then we define:

$$H_0(\infty, x) = 2 \cdot k + 1$$

and:

$$\widehat{H}_1(\infty, x) = 2 \cdot f_j(r) \qquad \text{where} \qquad r = \min\{t : x \in f_(t)\}$$

The functions $\widehat{H}_0, \widehat{H}_1$ exists by Δ^0_1 comprehension and witness that $X \cup \{\infty\}$ is effectively T_2.

Assume that X is effectively T_2 and there exists a sequence $(C_n)_{n\in\mathbb{N}}$ such that for all n C_n is the covering relation of K_n. Given a finite set $F \subseteq I$, let $j = \max\{i : 2i+1 \in F\}$, then $\bigcup_{i\in F} V_i = X \cup \{\infty\}$ if and only if $\{i : 2i \in F\} \in C_j$, so the covering relation for $X \cup \{\infty\}$ is Δ^0_1 definable relative to $(C_n)_{n\in\mathbb{N}}$ and so $X \cup \{\infty\}$ is effectively compact.

Proposition 9.24 Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a T_2 $CSCS$ and let $(K^0_i, U^0_i, f^0_i, g^0_i)_{i\in\mathbb{N}}$ and $(K^1_i, U^1_i, f^1_i, g^1_i)_{i\in\mathbb{N}}$ be two exhaustions by compact sets of X. The $CSCS$:

$$(X \cup \{\infty\}, (U_i)_{i\in\mathbb{N}} \cup (X \cup \{\infty\} \setminus K^0_n)_{n\in\mathbb{N}}, \widehat{k}^0)$$

and

$$(X \cup \{\infty\}, (U_i)_{i\in\mathbb{N}} \cup (X \cup \{\infty\} \setminus K^1_n)_{n\in\mathbb{N}}, \widehat{k}^1)$$

are effectively homeomorphic. That is, the topology of the one point compactification does not depend on the choice of the exhaustion by compact sets.

Proof: We show that the identity $Id : X \cup \{\infty\} \to X \cup \{\infty\}$ is an effective homeomorphism. Let $v : X \times \mathbb{N} \to \mathbb{N}$ be the function given by:

(1) $v(x, 2i) = 2i$.
(2) $v(x, 2i+1) = 2 \cdot g(r)$ where $r = \min\{s : x \in U_{g(s)}\}$
(3) $v(\infty, 2j+1) = \min\{r : \{f_i(m) : i, m < r\} \in C_i\}$

We have that v exists by Δ^0_1 comprehension and witnesses that the identity $Id : X \cup \{\infty\} \to X \cup \{\infty\}$ is effectively continuous. By symmetry, we have that the identity is also effectively open, and so the identity is an effective homeomorphism. $\square$

Proposition 9.25 $\mathbf{RCA}_0$ proves that every locally compact effectively T_2 $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}, k)$ with a effective CCN has a one point compactification $X \cup \{\infty\}$ which effectively compact effectively T_2 and the inclusion $X \to X \cup \{\infty\}$ is an effective embedding.

Proof: By Proposition 9.22 we have that X has an exhaustion by compact sets $(K_i, U_i, f_i, g_i)_{i\in\mathbb{N}}$ and there is a sequence $(C_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ we have that C_i is the covering relation for K_i. By the observations made in Definition 9.23, we have that the one point compactification of X exists and is an effectively T_2 effectively compact $CSCS$. We show that the inclusion $X \to X \cup \{\infty\}$ is an effective homeomorphism. We define $v : X \times \mathbb{N} \to \mathbb{N}$ where $v(x, 2i) = i$ and $v(x, 2i+1) = g(r)$ where $r = \min\{s \in \mathbb{N} : x \in U_{g_i(s)}\}$. The function v exists by Δ^0_1 comprehension and witnesses that the inclusion $X \to X \cup \{\infty\}$ is effectively continuous. We have that the map $(x, i) \mapsto 2i$

witnesses that the inclusion is effectively open, so the inclusion of X in $X \cup \{\infty\}$ is an effective homeomorphism. $\square$

Proposition 9.26 $\mathbf{RCA}_0$ proves that for any $CSCS$ $(X, (U_i)_{i\in\mathbb{N}}$ if the one point compactification of X is effectively T_2 and the inclusion is an effective embedding then X has a CCN.

Proof: Let H_0, H_1 be the functions that witness that $X \cup \{\infty\}$ is effectively T_2 then we have that the sequence:

$$(X \setminus U_{H_0(\infty,x)}, H_1(\infty, x))_{x\in X}$$

will be a CCN for X. $\square$

10. Moduli for Tree Arrays

$\mathbf{ATR}_0$ proves the existence of CCN for locally compact $CSCS$ by the following lemma:

Lemma 10.1 (Simpson [20, Lemma VIII.4.7]) For any Π^0_1 formula $\varphi(x, i, X)$ where X is the only set variable in φ then $\mathbf{ATR}_0$ proves:

$$\forall X\, (\forall x\, \exists i\, \varphi(x, i, X) \to \exists f\, \forall x\, \varphi(x, f(x), X))$$

We introduce a special case of 10.1 to prove the equivalence between arithmetic transfinite recursion and the existence of CCN for locally compact $CSCS$.

Definition 10.2 An eventually well founded tree array is a collection $(T^j_i)_{i,j\in\mathbb{N}}$ of trees such that for all $j \in \mathbb{N}$ the sequence $(T^j_i)_{i\in\mathbb{N}}$ is eventually well founded. A modulus for an eventually well founded array of trees $(T^j_i)_{i,j\in\mathbb{N}}$ is a sequence $(n_j)_{j\in\mathbb{N}}$ such that for all j and all $i \geq n_j$ the tree T^j_i is well founded. By TAM, we mean the statement that every eventually well founded array of trees admits a modulus. By $1TAM$, we mean TAM restricted to arrays of trees that have at most 1 branch each.

Proposition 10.3 $\mathbf{RCA}_0 + 1TAM$ implies arithmetic comprehension.

Proof: We show that $1TAM$ implies the existence of the Turing jump. Fix a set $A \subseteq \mathbb{N}$, for each $e, s \in \mathbb{N}$ let T^s_e be the tree containing only sequences of the form $(t, t, \ldots, t)$ where $t \leq s$ and:

$$t = \min\{r \in \mathbb{N} : \Phi^A_e(e) \downarrow_{\leq r}\}$$

otherwise, T^s_e is empty. By construction, the trees T^s_e can have at most one branch, and the sequence $(T^s_e)_{s\in\mathbb{N}}$ is eventually well founded. Let $(m_e)_{e\in\mathbb{N}}$ be a modulus for the tree array $(T^s_e)_{s,e\in\mathbb{N}}$. We have that $e \in A'$ if and only if $\Phi^A_e(e) \downarrow_{\leq m_e}$ and therefore the jump of A is computable relative to $(m_e)_{e\in\mathbb{N}}$. $\square$

Lemma 10.4 (**(Simpson [20, Exercise VIII.4.25])**) $\mathbf{RCA}_0 + TAM$ proves that for every sequence of trees $(T_i)_{i\in\mathbb{N}}$ such that:

$$\forall i \in \mathbb{N}\, \exists g\, \forall f \in [T_i]\, \forall n\, (f(n) < g(n))$$

or rather that the branches of each T_i are dominated by some function g, then the set $\{i \in \mathbb{N} : [T_i] \neq \emptyset\}$.

Proof: Since $1TAM$, and therefore TAM, imply arithmetic comprehension, we may work over $\mathbf{ACA}_0$. Let $(T^i)_{i\in\mathbb{N}}$ be a sequence of trees such that for each i there is a g that dominates the branches of $[T_i]$ we define the following array of trees:

$$T^i_{n,m} = \{\sigma \in T^i : |\sigma| \leq n \vee \sigma(n) \geq m\}$$

Since T^i has branches that are dominated by some function g, we have that for any $i, n \in \mathbb{N}$ sequence of trees $(T^i_{n,m})_{m\in\mathbb{N}}$ will be eventually well founded. Let $(m^i_j)_{j\in\mathbb{N}}$ be such that for all $i, j \in \mathbb{N}$ and all $m \geq m^i_j$ the tree $T^i_{j,m}$ is well founded then we have that T^i is ill founded if and only there is a branch is in the finitely branching tree:

$$\widehat{T^i} = \{\sigma \in T^i : \forall j < |\sigma|\ \sigma(j) \leq m^i_j\}$$

by König's lemma, which is a consequence of $\mathbf{ACA}_0$, $\widehat{T^i}$ has a branch if and only if it is infinite. Since being infinite is arithmetically definable, we have the collection:

$$\{i \in \mathbb{N} : [\widehat{T_i}] \neq \emptyset\} = \{i \in \mathbb{N} : [T_i] \neq \emptyset\}$$

is arithmetically defined relative to $(m^i_j)_{i,j\in\mathbb{N}}$, so it exists by arithmetic comprehension. □

Observation 10.5 Using essentially the same proof we can show that $\mathbf{RCA}_0 + 1TAM$ proves that for every sequence of trees $(T_i)_{i\in\mathbb{N}}$ such that $|[T_i]| \leq 1$ the set $\{i \in \mathbb{N} : [T_i] \neq \emptyset\}$ exists.

Observation 10.6 The proof of Proposition 10.3 and Lemma 10.4 also show that TAM is equivalent over $\mathbf{RCA}_0$ to the statement that for any sequence of trees $(T_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ the branches of T_i are dominated by some f then there exists a sequence $(f_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ f_i dominates the branches of T_i.

Lemma 10.7 The following are equivalent over $\mathbf{RCA}_0$:

1. TAM
2. For any sequence of sequences of linear orders $(L^j_i)_{i,j\in\mathbb{N}}$ such that for each j the sequence $(L^j_i)_{i\in\mathbb{N}}$ is decreasing with respect to inclusion and eventually well ordered then there is a sequence $(n_j)_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$ and all $i \geq n_j$ L^j_i is a well order.

Proof: We have that TAM implies $\mathbf{ACA}_0$ and using a similar argument usend Proposition 10.3 we have that 2 also implies arithmetic comprehension over $\mathbf{RCA}_0$. So it suffices to show that TAM is equivalent to 2 over $\mathbf{ACA}_0$.

$(1 \to 2)$ follows from the fact that a linear order L is a well order if and only if:

$$T(L) = \{\sigma \in L^{<\mathbb{N}} : \forall j < |\sigma|\ \sigma(j+1) <_L \sigma(j)\}$$

is well founded.

Assume 2. Let $(T^j_i)_{i,j\in\mathbb{N}}$ be a sequence of trees such that for all $j \in \mathbb{N}$ the sequence $(T^j_i)_{i\in\mathbb{N}}$ is eventually well founded. Define:

$$S^j_n = \{\emptyset\} \cup \{(m)^\frown\sigma : m \geq n \wedge \sigma \in T^j_m\} = \coprod_{m\geq n} T^j_m$$

We observe that T^j_n is well founded if and only $\mathrm{KB}(S^j_n)$ is a well order and that for all $m \geq n$ $S^j_m \subseteq S^j_n$. The family of linear orders $(\mathrm{KB}(S^j_n))_{n,j\in\mathbb{N}}$ satisfies the conditions of 2 so there exists by assumption a sequence $(m_j)_{j\in\mathbb{N}}$ such that $\forall i \geq m_j \mathrm{KB}(S^j_i)$ is a well order. So $\forall i \geq m_j$ T^j_i is well founded. □

Proposition 10.8 $\mathbf{ATR}_0$ proves TAM.

Proof: Let $(L^j_i)_{i,j\in\mathbb{N}}$ be as in the previous lemma. Fix a $j \in \mathbb{N}$, we wish to show that there exists m_j such that $\forall k \in \mathbb{N}$ $L^j_{m_j} \cong L^j_{m_j+k}$. By assumption, there is an n_j such that L^j_i is well ordered for all $i \geq n_j$. Let $L = L^j_{n_j} + 1$, by using the comparability of well orders and that $\forall i\ L^j_{n_j+i} \subseteq L^j_{n_j}$ we have that every $L^j_{n_j+i}$ is isomorphic to a proper initial segment of L. So:

$$\forall i\ \exists a \in L\ \exists f : L^j_{n_j+i} \xrightarrow{\sim} \{b \in L : b < a\}$$

By Σ^1_1 choice, we have that:

$$\exists (a_i)_{i \geq n_j}\ \exists (f_i)_{i \geq n_j}\ \forall i \geq n_j\ a_i \in L \wedge f_i : L^j_i \xrightarrow{\sim} \{b \in L : b < a_i\}$$

We have that for all $i_0 < i_1$ that $a_{i_0} \geq a_{i_1}$ and since L is a well order the sequence $(a_i)_{i \geq n_j}$ is eventually constant. Let m_j be such that for all $i \in \mathbb{N}\ a_{m_j} = a_{m_j+i}$ then we have for all $i \geq m_j$ that $L^j_{m_j} \cong L^j_{m_j+i}$. In particular we have that for all $i \geq m_j$ the function $f_i^{-1} \circ f_{m_j}$ is an isomorphism between $L^j_{m_j}$ and $L^j_{m_j+i}$.

We have, therefore, that:

$$\forall j\ \exists m\ \forall k\ \exists f : L^j_m \xrightarrow{\sim} L^j_{m+k}$$

Using Σ^1_1 choice, we have:

$$\forall j\ \exists (f_k)_{k \in \mathbb{N}}\ \exists m\ \forall k\ f_k : L^j_m \xrightarrow{\sim} L^j_{m+k}$$

Using Σ^1_1 choice again, we get:

$$\exists (f^j_k)_{j,k \in \mathbb{N}}\ \ \exists (m_j)_{j \in \mathbb{N}}\ \ \forall j\ \forall k\ f^j_k : L^j_m \xrightarrow{\sim} L^j_{m+k}$$

We observe that for all $j \in \mathbb{N}\ L^j_{m_j}$ must be a well order so $(m_j)_{j \in \mathbb{N}}$ is a sequence with the desired property. □

Theorem 10.9 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic transfinite recursion.
(2) TAM
(3) For every sequence of trees $(T_i)_{i \in \mathbb{N}}$ such that for all i the branches of T_i are dominated by some function then the set $\{i \in \mathbb{N} : [T_i] \neq \emptyset\}$ exists.
(4) For every sequence of trees $(T_i)_{i \in \mathbb{N}}$ such that for all i $|[T_i]| \leq 1$ then the set $\{i \in \mathbb{N} : [T_i] \neq \emptyset\}$ exists.

Proof: Proposition 10.8 shows that $(1 \to 2)$, Observation 10.4 shows that $(2 \to 3)$, $(3 \to 4)$ is immediate, and the proof of $(4 \to 1)$ can be found in [20]. □

11. Topological Equivalents to $\mathbf{ATR}_0$

In this section, we will show that every locally compact $CSCS$ has a CCN is equivalent to TAM over $\mathbf{RCA}_0$. We will also introduce the Cantor Bendixson rank for $CSCS$ and show that over $\mathbf{ACA}_0$ locally compact T_2 $CSCS$ with a Cantor Bendixson rank have a CCN. These results will give us some equivalences between arithmetic transfinite recursion and topological principles.

Proposition 11.1 Over $\mathbf{RCA}_0$ every T_2 locally compact $CSCS$ having a basis of compact neighborhoods implies arithmetic comprehension.

Proof: Let $A \subseteq \mathbb{N}$ be a set, we show that its Turing jump exists. Set:

$$X = \{(e,t,s) : (\Phi^A_e(e)\downarrow_{\leq s} \wedge \neg \Phi^A_e(e)\downarrow_{\leq t}) \vee (s = 0 \wedge \Phi^A_e(e)\downarrow_{\leq t})\} \cup \{(e,\infty) : e \in \mathbb{N}\}$$

and define the open sets of X to be:

$$U_{2(e,t,s)} = \begin{cases} \{(e,t,s)\} \text{ if } (e,t,s) \in X \\ \emptyset \text{ otherwise} \end{cases}$$

and

$$U_{2(e,m)+1} = \{(e,t,s) \in X : t \geq m\} \cup \{(e,\infty)\}$$

The space X can be viewed as the disjoint union of the spaces $X_e = \{(e,s,t) \in X : s,t \in \mathbb{N}\}$. The function k, in this case, is straightforward to define effectively. We prove that X is locally compact. Any point $\{(e,t,s)\} \in X$ will be isolated, and so it has a compact neighborhood. For a fixed $e \in \mathbb{N}$ if $\Phi_e^A(e) \uparrow$ then $U_{2(e,m)+1} = \{(e,\infty)\}$ for all $m \in \mathbb{N}$ otherwise we have that there exists an m such that $\Phi_e^A(e) \downarrow_{\leq m}$ and in such case $U_{2(e,m)+1}$ will be a compact neighborhood for (e,∞). We observe that all of the basic sets are clopen and that X is T_3.

For all $t < s$, if $\Phi_e^A(e) \downarrow_{\leq s}$, but $\neg\Phi_e^A(e) \downarrow_{\leq t}$ then $U_{2(e,t)+1}$ is not compact since:

$$\{\{(e,t,m)\} : \Phi_e^A(e) \downarrow_{\leq m}\} \cup \{U_{2(e,t+1)+1}\}$$

is an infinite partition of $U_{2(e,t)+1}$ into clopen sets. So if $\Phi_e^A(e) \downarrow$ then for all $m \in \mathbb{N}$ $\Phi_e^A(e) \downarrow_{\leq m}$ if and only if $U_{2(e,m)+1}$ is compact.

Let $(i(x))_{x\in X}$ be a CCN for X. We have by construction of X that for every $e \in \mathbb{N}$ there is a unique $m_e \in \mathbb{N}$ where $i(x) = 2(e,m_e)+1$. Therefore $\Phi_e^A(e) \downarrow \leftrightarrow \Phi_e^A(e) \downarrow_{\leq m(e)}$ and so $A' \leq_T (m_e)_{e\in\mathbb{N}}$ will exist by Δ_1^0 comprehension. □

Proposition 11.2 $\mathbf{RCA}_0 + TAM$ proves that every locally compact $CSCS$ has a choice of compact neighborhoods.

Proof: Since TAM implies arithmetic comprehension we will work over $\mathbf{ACA}_0$. Let $(X,(U_i)_{i\in\mathbb{N}},k)$ be a locally compact space and let $X = (x_n)_{n\in\mathbb{N}}$. Let $f : X \times \mathbb{N} \to \mathbb{N}$ be the function given by:

$$f(x,i) = \begin{cases} \min\{j > f(x,i-1) : x \in U_j\} & \text{if such } j \text{ exists} \\ \max\{j : x \in U_j\} & \text{otherwise} \end{cases}$$

Intuitively, $f(x,\cdot)$ lists out the indices basic neighborhoods of x. For each $x \in X$ let $(T_i^x)_{i\in\mathbb{N}}$ be the sequence of trees where for all $i \in \mathbb{N}$ T_i^x is the set of strictly increasing sequences σ such that:

$$\forall\tau \sqsubseteq \sigma(\tau \neq \sigma \to \overline{U_{f(x,i)}} \not\subseteq \bigcup_{n<|\tau|} U_{\tau(n)}) \wedge \forall n < |\sigma|\ (x_n \in \overline{U_{f(x,i)}} \to x_n \in U_{\sigma(n)})$$

That is, all sequences σ such that none of its proper initial segments defines a covering for $\overline{U_{f(x,i)}}$ and for each $n < |\sigma|$ if $x_n \in \overline{U_{f(x,i)}}$ then $x_n \in U_{\sigma(n)}$. An infinite branch in T_i^x defines a covering of $\overline{U_{f(x,i)}}$, which does not have finite subcovering. Similarly, an infinite cover of $\overline{U_{f(x,i)}}$ that does not admit a finite subcover defines a branch in T_i^x. Therefore, $\overline{U_{f(x,i)}}$ is compact if and only if T_i^x is well founded. Since X is locally compact, we have that for each $x \in X$, the sequence $(T_i^x)_{i\in\mathbb{N}}$ is eventually well founded. Thus the modulus $(n_x)_{x\in X}$ given by TAM is such that for all $x \in X$ the neighborhood $\overline{U_{f(x,n_x)}}$ is a compact. This means that $(n_x, \overline{U_{f(x,i)}})_{x\in X}$ is a CCN for X. □

Proposition 11.3 Over $\mathbf{RCA}_0$ the statement that every T_2 locally compact $CSCS$ has a CCN implies arithmetic transfinite recursion.

Proof: By Proposition 11.1 we may work over $\mathbf{ACA}_0$. We show that every T_2 locally compact $CSCS$ has a CCN implies $1TAM$ which is equivalent to $\mathbf{ATR}_0$ over $\mathbf{RCA}_0$. Let $(T_j^i)_{i,j\in\mathbb{N}}$ be an eventually well founded array of trees as in the condition of $1TAM$ and let X be equal to the disjoint union of the spaces $X^j = \mathrm{KB}(\coprod_{i\in\mathbb{N}} T_i^j) \cong \sum_{i\in\mathbb{N}} \mathrm{KB}(T_i^j) + 1$ with the upper limit topology where:

$$\coprod_{i\in\mathbb{N}} T_i^j = \{\emptyset\} \cup \bigcup \{(m)^\frown T_m^j : m \in \mathbb{N}\}$$

We show that for every $j \in \mathbb{N}$, the space X^j is locally compact. To do this, it suffices to show that a tree with at most one branch is locally compact with the upper limit topology induced by the Kleene Brouwer order. If T is well founded, then by Proposition 6.12 T with the topology induced by the Kleene Brouwer order is compact and therefore locally compact. Let T be a tree with one branch f and $x \in T$. If $\forall n \in \mathbb{N}\ x <_{\mathrm{KB}} f_{\leq n}$ then $]-\infty, x]$ is well ordered with a maximal element and therefore is a compact neighborhood of x. Otherwise, if there is an n such that $f_{\leq n} <_{\mathrm{KB}} x$ then the interval $]f_{\leq n}, +\infty[$ is well ordered with respect to the Kleene Brouwer order and so it is a compact neighborhood of x.

By assumption, there is a choice of compact neighborhoods for the space X. This implies that we have a choice for a compact neighborhood of $\emptyset \in X_j$, which we denote by K_j. We define:

$$m_j = \min\{m \in \mathbb{N} :](m), \emptyset] \subseteq K_j\} + 1$$

Recall that the upper limit topology on a linear order with maximal element is compact if and only if it is well ordered. So for all $i \geq m_j$, the $\mathrm{KB}(T^i_j)$ is a well order, and so T^i_j is well founded. □

We have that $\mathbf{ATR}_0$ proves that every locally compact T_2 $CSCS$ has a CCN and therefore embeds into a compact $CSCS$. We would like to show that we can embed T_3 scattered $CSCS$ into a well order. For scattered linear orders, we have the following results.

Theorem 11.4 (Clote [2]) Arithmetic transfinite recursion is equivalent to every countable scattered linear order having a countable set of initial segments. That is for any scattered linear order $(L, <_L)$ there is a sequence of sets $(I_n)_{n \in \mathbb{N}}$ such that:

$$\forall X \subseteq L\ [(\forall x \in X\ \forall y <_L x (y \in X)) \to \exists n\ X = I_n]$$

In particular, we consider the empty set and all of L to be initial segments.

Theorem 11.5 (Shafer [17]) $\mathbf{WKL}_0$, and in particular $\mathbf{ACA}_0$, proves the order topology of every complete linear order is compact.

Corollary 11.6 $\mathbf{ATR}_0$ proves that the order topology of any scattered linear order embeds into a T_2 compact $CSCS$.

Proof: By Theorem 11.4 we have that the Dedekind completion of X is countable, and so X embeds into a complete linear order which by Theorem 11.5, it will be compact. □

The proof of Theorem 11.4 requires some results that use the Hausdorff rank, which is not a topological invariant. We also have that non scattered linear order may be homeomorphic to a scattered $CSCS$. So Corollary 11.6 does not ensure us that every T_3 scattered $CSCS$ will embed into a T_2 compact $CSCS$. We will need to use Cantor-Bendixson rank instead. This will allow us to show that every T_3 scattered $CSCS$ is homeomorphic to a scattered linear order with its order topology. The Cantor-Bendixson derivative and rank of $CSCS$ in reverse mathematics has already been studied by Montálban and Greenberg [8], Friedman [5] in the case of countable metric spaces, and by Friedman and Hirst [6] in the form of characteristic systems.

Observation 11.7 $\mathbf{RCA}_0$ proves that any uniformly T_3 scattered $CSCS$ is the closure of its isolated points, in the sense that any open set will contain an isolated point. This follows from the definition of scattered. The converse does not hold. In general, there are non scattered T_3 spaces, which are the closure of their isolated points. This fact can be proven directly by constructing an explicit example. Alternatively, one can use the fact that being scattered is a Π^1_1 universal formula, and being a $CSCS$ equal to the closure of its isolated points is expressible by an arithmetic formula.

Definition 11.8 Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be a $CSCS$ then we write:

$$D(X) = \{x \in X : x \text{ is not isolated}\}$$

We call $D(X)$ the sets of limit points of X. We observe that $D(X)$ is arithmetical relative to X, however, in general it will not be recursive.

Definition 11.9 Over $\mathbf{ACA}_0$ we say that a Cantor Bendixson rank or rank for a $CSCS$ X is a well order R and a sequence $(X_r)_{r\in R}$ such that:

(1) $X_0 = X$.
(2) For all $r \in R$ that is not maximal $X_{r+1} = D(X_r)$.
(3) For every $r \in R$ which is limit $X_r = \bigcap_{s<_R r} X_s$.
(4) $D(\bigcap_{r\in R} X_r) = \bigcap_{r\in R} X_r$.

we call a well order such that $1, 2, 3$ hold, but not necessarily 4, a partial rank. A rank R of X is minimal if none of its initial segments are a rank for X. Given a rank R on X we write

$$\operatorname{rank}_R(x) = \operatorname{rank}(x) = \min\{r \in R : x \notin X_{r+1}\}$$

It is straightforward that for any x of rank r, we have $x \in X_r$.

Observation 11.10 If X is a T_3 space and R is a rank for X then by definition of rank we have $D(\bigcap_{r\in R} X_r) = \bigcap_{r\in R} X_r$. So $\bigcap_{r\in R} X_r$ is invariant under the D and, therefore, does not have any isolated points. This means that $\bigcap_{r\in R} X_r$ is either homeomorphic to $\mathbb{Q}$ or it is empty. In particular, we have that if X is scattered then $\bigcap_{r\in R} X_r = \emptyset$.

Observation 11.11 Since over $\mathbf{ACA}_0$ we have arithmetic transfinite induction, we have that any rank R for a T_3 scattered $CSCS$ X has an initial segment that is a minimal rank. Over $\mathbf{ACA}_0$, any two minimal ranks R_0 and R_1 will be order isomorphic. We may say, over $\mathbf{ACA}_0$, that R is the Cantor-Bendixson rank of X if R is a minimal rank.

Theorem 11.12 (Friedman [5, Lemma 15]) $\mathbf{ATR}_0$ proves that every T_3 scattered space is ranked.

Proof: Let $(X, (U_i)_{i\in\mathbb{N}}, k)$ be T_3 and scattered. We observe that over $\mathbf{ATR}_0$, every well order is a partial rank for X. Seeking a contradiction, we assume that no well order is a rank for X. Consider the Σ^1_1 formula $\varphi(Y)$ given by:

$Y = (L, <_L)$ is a linear order and there exists $(X_j)_{j\in L}$ such that for all $i \in L$ $X_i \neq \emptyset$ and $\forall j <_L i$ $X_i \subseteq D(X_j)$.

We have that φ is true for any well order. Since being a well order is a universal Π^1_1 formula, there exists an $(L, <_L)$ that is not a well order and $\varphi((L, <_L))$ holds. Let $(X_j)_{j\in L}$ be the sequence given by $\varphi((L, <_L))$ then we have since L is not well ordered there exists a descending chain $(n_j)_{j\in\mathbb{N}}$ in L. For all $i <_L j$ if $x \in X_j$ then x is not isolated in X_i. Let $Z = \bigcup_{j\in\mathbb{N}} X_{n_j} \subseteq X$. Since every point of Z is contained in some X_j, it cannot be isolated in Z. So Z with the subspace topology is homeomorphic to $\mathbb{Q}$, which contradicts the assumption that X is scattered. □

Proposition 11.13 (See [8]) Over $\mathbf{RCA}_0$ every well order has a rank implies arithmetic comprehension.

Proof: Let $A \subseteq \mathbb{N}$ be a set. Let $L_e = \{t \in \mathbb{N} : \neg\Phi^A_e(e) \downarrow_{\leq t}\} + 1$ and $L = \sum_{e\in\mathbb{N}} L_e$ which is the sum of well orders and so it is a well order. We can compute A' from the set of non isolated points of L, which is computable from the rank of L. □

We now will show that over $\mathbf{ACA}_0$ scattered T_3 $CSCS$ with a rank embed into a well order. To show these results, we make use of a construction similar to that done in Theorem 7.5.

Construction 11.14 Working over $\mathbf{ACA}_0$, let $(X, (U_i)_{i\in\mathbb{N}})$ be a T_3 scattered $CSCS$, which, without loss of generality, we assume it has an algebra of clopen sets. Let R be a rank for X. We now lay out an arithmetic procedure to associate to each point x a unique sequence α_x, which we call the address of x. We will see later that the Kleene Brouwer order on the set of addresses is a well order and that the map $x \mapsto \alpha_x$ will be an embedding.

Let $F : X \times \mathbb{N}$ be a partial function such that for all x in X if x is isolated $U_{F(x,0)} = \{x\}$ otherwise we define $F(x, n)$ to be the least $s \in \mathbb{N}$ such that:

(1) $U_s \cap X_{\mathrm{rank}(x)} = \{x\}$, or rather, all of the elements in U_s besides x are of lower rank. Since x is isolated in $X_{\mathrm{rank}(x)}$, there must be a neighborhood of x containing x and points of rank strictly lower than x.
(2) $\forall m < n\ U_s \subsetneq U_{F(x,m)}$.

We have that F exists by arithmetic comprehension. Informally, F lists out a descending sequence of neighborhoods for every point. For ease of notation, we will write $U(x, n) = U_{F(x,n)}$. We observe that the sets of the form $U(x, n)$ form a basis of clopen sets for X.

For each $x \in X$ and $n \in \mathbb{N}$ let $A^{(x,n)} = U(x, n) \setminus U(x, n+1)$ and $A^{-1} = X$. For $h = -1$ or $h = (x, n)$ we define inductively two sequence $\sigma^h \in X^{<\mathbb{N}}$ and $\tau^h \in \mathbb{N}^{<\mathbb{N}}$. If $A^h \setminus \bigcup_{i<k} U(\sigma^h(i), \tau^h(i))$ is empty we terminate the construction, otherwise let $\sigma^h(k)$ be the first element in $A^h \setminus \bigcup_{i<k} U(\sigma^h(i), \tau^h(i))$. Define:

$$\tau^h(k) = \min\left\{ s \in \mathbb{N} : U(\sigma^h(k), s) \subseteq A^h \setminus \bigcup_{i<k} U(\sigma^h(i), \tau^h(i)) \right\}$$

If A^h is compact then there exists a k such that $A^h = \bigcup_{i<k} U(\sigma^h(i), \tau^h(i))$ and at such k we halt the construction. Otherwise, the construction might go on forever, and σ^h and τ^h will be functions. We observe that the sets $(U(\sigma^h(i), \tau^h(i)))_{i\in\mathrm{dom}(\sigma)}$ form a partition of A^h into clopen sets. Since σ^h and τ^h are uniformly arithmetically defined with respect to h we have that:

$$((\sigma^h, \tau^h))_{h=-1\vee(h=(x,n)\wedge x\in X\wedge n\in\mathbb{N})}$$

exists by arithmetic comprehension.

For each point $x \in X$ we define $\alpha_x, \beta_x \in \mathbb{N}^{<\mathbb{N}}$, and $\gamma_x \in \mathbb{N}$ as:

(1) $\alpha_x(0)$ is the unique i such that $x \in U(\sigma^{(-1)}(i), \tau^{(-1)}(i))$ and $\beta_x(0) = \sigma^{(-1)}(i)$.
(2) $\alpha_x(2n + 1)$ is the unique m such that $x \in U(\beta_x(n), m) \setminus U(\beta_x(n), m + 1)$.
(3) $\alpha_x(2n + 2)$ is the unique i such that:

$$x \in U(\sigma^{(\beta_x(n),\alpha_x(2n+1))}(i), \tau^{(\beta_x(n),\alpha_x(2n+1))}(i))$$

and let:

$$\beta_x(n + 1) = \sigma^{(\beta_x(n),\alpha_x(2n+1))}(i)$$

If $x = \beta_x(n + 1)$, then we set $\gamma_x = 0$ if x is isolated and $\gamma_x = \tau^{(\beta_x(n),\alpha_x(2n+1))}(i)$ otherwise and the construction terminates.

Since for all n $\mathrm{rank}(\beta_x(n)) > \mathrm{rank}(\beta_x(n + 1))$, we have that for all x the construction above must eventually terminate. The set of all $(\alpha_x, \beta_x, \gamma_x)_{x\in X}$ is arithmetically definable, and so it exists by arithmetic comprehension.

For each $x \in X$ we call α_x the address of x. Since X is Hausdorff, we have that the map $x \mapsto \alpha_x$ is injective. The collection of addresses $\mathcal{A}$ will not be a tree since every address has odd length.

But for any $\alpha_x \in \mathcal{A}$, every odd length initial segment of α_x will be an address. In fact, for each $n < |\beta_x|$ $\alpha_{\beta_x(n)}$ is the initial segment of α_x of length $2n+1$. We observe that for all $x, y \in X$ that $y \in U(x, \gamma_x)$ if and only if $\alpha_x \sqsubseteq \alpha_y$. By construction, we have that if $\alpha_x \sqsubseteq \alpha_y$ then $\beta_x \sqsubseteq \beta_y$. In particular, since the rank of $\beta_x(n)$ is strictly decreasing, there cannot exist an infinite increasing sequence of addresses.

Let $<_X$ be the order given by:

$$x <_\alpha y \leftrightarrow \alpha_x <_{\mathrm{KB}} \alpha_y$$

Since the Kleene Brouwer ordering on a well founded tree is a well order, we have that $<_\alpha$ defines a well order on X. In general, we have that $<_\alpha$ induces a coarser topology on X. We show that the map $x \mapsto \alpha_x$ defines an embedding from X to $\downarrow \mathcal{A}$. That is, the map $x \mapsto \alpha_x$ is a homeomorphism between $(X, (U_i)_{i\in\mathbb{N}})$ and $\mathcal{A}$ with the subspace topology.

If x is isolated in X, then we have that α_x does not have any extensions in $\mathcal{A} \downarrow$, so it has a predecessor with respect to the Kleene Brouwer order on $\downarrow \mathcal{A}$. So x is isolated in $\mathcal{A}$ with the subspace topology. If x is not isolated in X, then we have that $U(x, \gamma_x)$ will be infinite. In particular, for all $m \geq \gamma_x$, the set $U(x, m) \setminus U(x, m+1)$ is non empty. Given a $\sigma \in \downarrow \mathcal{A}$ such that $\sigma <_{\mathrm{KB}} \alpha_x$ then there exists an $m \geq \gamma_x$ such that $\sigma <_{\mathrm{KB}} \alpha_x^\frown(m)$. For any $y \in U(x, m+1) \setminus U(x, m+2)$ we have $\alpha_x^\frown(m) <_{\mathrm{KB}} \alpha_y <_{\mathrm{KB}} \alpha_x$. This implies that α_x is not isolated in $\mathcal{A}$ with the subspace topology. So x is isolated in X if and only if α_x is isolated in the subspace topology of $\mathcal{A}$. It suffices to show that $x \mapsto \alpha_x$ is continuous and open on the non isolated points of X.

Let $x, y \in X$ such that $\alpha_y <_{\mathrm{KB}} \alpha_x$ and α_x is not isolated in $\mathcal{A}$ with the subspace topology. If $y \notin U(x, \gamma_x)$ then $U(x, \gamma_x) \subseteq (y, x]_{<_\alpha}$. Otherwise there exists a unique $n \geq \gamma_x$ such that $y \in U(x, n) \setminus U(x, n+1)$, so $\alpha_x^\frown(n) \sqsubseteq \alpha_y$. Since:

$$\forall z \in U(x, n+1)\ (\alpha_x^\frown(n+1) \sqsubseteq \alpha_z)$$

we have that $\forall z \in U(x, n+1)$ $\alpha_y <_{\mathrm{KB}} \alpha_z <_{\mathrm{KB}} \alpha_x$ and so $U(x, n+1) \subseteq (y, x]_{<_\alpha}$. This proves that the map $x \mapsto \alpha_x$ is continuous.

Let $x \in X$ be a non isolated point and $n \in \mathbb{N}$. Since x is not isolated there exists a $y \in U(x, \max\{n, \gamma_x\}) \setminus \{x\}$. We have that $\alpha_x \sqsubseteq \alpha_y$ and so $(y, x]_{<_\alpha} \subseteq U(x, n)$ by construction. This proves the map $x \mapsto \alpha_x$ is open with its image. So X is homeomorphic to $\mathcal{A}$ with the subspace topology.

Theorem 11.15 $\mathbf{ACA}_0$ proves that every T_3 scattered $CSCS$ with rank is homeomorphic the subspace of a well order.

Proof: Using the construction before we have that X embeds into $\downarrow \mathcal{A}$ with the Kleene Brouwer order and so X embeds into a well order. □

Theorem 11.16 $\mathbf{ACA}_0$ proves that every T_2 locally compact $CSCS$ with a rank has a CCN.

Proof: Fix a point $x \in X$, we have show that for $n \geq \gamma_x$ that $U(x, n)$ is compact if and only if for all $y \in U(x, n)$ such that $\alpha_x \sqsubseteq \alpha_y$ and all $s \in \mathbb{N}$ then $\alpha_y^\frown(s)$ finitely branches. If $U(x, n)$ is compact and $y \in U(x, n)$ then $U(y, s) \setminus U(y, s+1)$ will also be compact and so $\alpha_y^\frown(s)$ has finitely many extensions of length $|\alpha_y| + 2$. If instead for all y such that $\alpha_x \sqsubseteq \alpha_y$ and all s $\alpha_y^\frown(s)$ is finitely branching, then by the previous result we have that the topology on $U(x, n)$ is the same topology as the topology induced by $<_\alpha$, and so, in particular, $U(x, n)$ will homeomorphic to a well order with maximal element and so it will be compact. We can, therefore, verify if $U(x, n)$ is compact arithmetically, so by arithmetic comprehension, there exists a CCN for X. □

Theorem 11.17 The following are equivalent over $\mathbf{RCA}_0$:

1. Arithmetic transfinite recursion.
2. Every locally compact T_2 $CSCS$ is well orderable.
3. Every T_3 scattered $CSCS$ embeds into a well order.
4. Every T_3 scattered $CSCS$ has a rank.

Proof: $(4 \to 3)$ By Proposition 11.13, we have that every well order is ranked implies arithmetic comprehension; so we may work over $\mathbf{ACA}_0$. By Proposition 11.15 every T_3 scatter $CSCS$ with rank embeds into a well order.

$(3 \to 1)$ By Theorem 4.22 we have that every T_3 scattered $CSCS$ embeds into a linear order implies arithmetic comprehension. So we may work over $\mathbf{ACA}_0$. By Lemma 9.5 and Proposition 9.8 T_2 locally compact $CSCS$ are T_3 and scattered. Since every well order has a CCN, we have that every T_2 locally compact $CSCS$ embeds into a $CSCS$ with a CCN. So by Proposition 9.12 and 9.17 every T_2 locally compact $CSCS$ has a CCN which by Proposition 11.3 implies arithmetic transfinite recursion.

$(4 \to 2)$ Every locally compact T_2 space with Cantor Bendixson rank has a CCN and by Proposition 9.15 every T_2 $CSCS$ with a CCN is well orderable.

$(2 \to 1)$ If every T_2 locally compact $CSCS$ is well orderable, then the limit topology of every well order is effectively homeomorphic to the order topology of a well order which by 4.9 implies arithmetic comprehenesion. Furthermore, if every T_2 locally compact $CSCS$ is well orderable then every T_2 locally compact $CSCS$ has a CCN which by Theorem 11.3 implies arithmetic transfinite recursion.

$(1 \to 4)$ is Theorem 11.12. $\square$

Theorem 11.18 $\mathbf{ACA}_0$ proves every subspace of a well order is homeomorphic to a scattered linear order with countably many cuts.

Proof: Let $(W, <_W)$ be a well order and let $S \subseteq W$ be a subspace. Let $L \subseteq W \setminus S$ be the collection of all $w \in W \setminus S$ such that $]-\infty, w[_{<_W} \cap S$ is unbounded. Since $L \subseteq W$ is well ordered every element is maximal or has a successor. For each $l \in L \cup \{-\infty\}$, let $l^+ \in L \cup \{+\infty\}$ denote the successor of l. We observe that for every $l \in L$, the point $\min\{s \in S : l <_W s\}$ will be isolated with respect to the subspace topology but will be a limit point with respect to the order topology (see Observation 7.1).

The sequence $(]l, l^+]_{<_W})_{l \in L \cup \{-\infty\}}$ defines a partition of W into clopen sets. So we have:

$$W \cong \coprod_{l \in L \cup \{-\infty\}}]l, l^+]_{<_W} \quad \text{and} \quad S \cong \coprod_{l \in L \cup \{-\infty\}}]l, l^+]_{<_W} \cap S$$

We would like to define a new order $<_n$ on W by flipping and rearranging the intervals $]l, l^+]$ so that the subspace topology on S will be the same as the order topology of the new order. Let $(l_n)_{n \in \mathbb{N}}$ be an enumeration of the elements of $L \cup \{-\infty\}$ (we note that in general $l_n^+ \neq l_{n+1}$). We then define on W the order $<_n$ where $a <_n b$ if and only if one of the following occurs:

1. $a \in]l_n, l_n^+]_{<_W} \wedge b \in]l_m, l_m^+]_{<_W} \wedge n < m$
2. $\exists n \ (a, b \in]l_{2n}, l_{2n}^+]_{<_W} \wedge a <_W b)$
3. $\exists n \ (a, b \in]l_{2n+1}, l_{2n+1}^+]_{<_W} \wedge b <_W a)$

It is routine to show that $<_n$ has the same order topology as $<_W$. We show that the subspace topology on S is the same as the order topology of $<_n$. In general, the subspace topology is finer or equal to the order topology, so it suffices to show that the order topology is finer or equal to the subspace topology.

Let $x \in S$ be a point and $a, b \in W$ such that $a <_n x <_n b$, iwe show that there are $c, d \in S$ such that $x \in]c, d[_{<_n} \cap S \subseteq]a, b[_{<_n} \cap S$. There exists a unique n such that $x \in]l_n, l_n^+]_{<_W}$. We consider the case in which n is even, the odd case is proved similarly. Since n is even we have by definition that $<_n$ restricted to $]l_n, l_n^+]_{<_W}$ is equal to $<_W$. By definition of l_n^+ we have that $]l_n, l_n^+]_{<_W} \cap S$ is unbounded. Let $d = \min\{s \in]l_n, l_n^+]_{<_W} \cap S : x <_n s\}$, that is, d is the successor of x in $]l_n, l_n^+]_{<_W} \cap S$. If x is the $<_W$ minimum element of $]l_n, l_n^+]_{<_W} \cap S$ then let $c = -\infty$ if $n = 0$ otherwise let $c = \min_{<_W}]l_{n-1}, l_{n-1}^+]_{<_n} \cap S$. If x is not the $<_W$ minimum in $]l_n, l_n^+] \cap S$ then if there is an $s \in [a, x[_{<_W} \cap S$ we set $c = s$. Otherwise, the set:

$$\{l \in]l_n, l_n^+] : l < a \wedge \forall j\ (l \leq j < x \to l \notin S\}$$

is non empty since it contains a and has a least element u since W is a well order. By definition, the only element of L in $]l_n, l_n^+]_{<_W}$ is l_n^+, so we have that $u \notin L$. By definition of L, $]l_n, u[_{<_W} \cap S$ is bounded in $]l_n, u[_{<_W}$. So u is a $<_W$ successor and its predecessor must be in S. We set c to be the predecessor of u. It is straightforward to verify that $x \in]c, d[_{<_n} \cap S \subseteq]a, b[_{<_n} \cap S$.

Any cut of $(S, <_n)$ will either be of the form $]-\infty, l]_{<_n} \cap S$ or of the form $\bigcup_{i \leq 2n+1}]l_i, l_i^+]_{<_W} \cap S$. Since $(S, <_n)$ has countably many cuts, it is a scattered linear order. □

Corollary 11.19 $\mathbf{ACA}_0$ proves every ranked T_3 scattered *CSCS* is effectively homeomorphic to the order topology of a scattered linear order with countably many cuts.

Corollary 11.20 Over $\mathbf{RCA}_0$ the following are equivalent:

1. Arithmetic transfinite recursion.
2. Every T_3 scattered *CSCS* is effectively homeomorphic to the order topology of a scattered linear order with countably many cuts.

Proof: $\mathbf{ATR}_0$ proves that every scattered T_3 *CSCS* embeds into a well order and so by Theorem 11.18 $\mathbf{ATR}_0$ proves every scattered T_3 *CSCS* is effectively homeomorphic to a scattered linear order with countably many cuts.

For the converse, assume every T_3 scattered *CSCS* is effectively homeomorphic to the order topology of a scattered linear order with countably many cuts. In particular, we have that every T_3 scattered *CSCS* embeds into a linear order, which by Theorem 4.22 implies arithmetic comprehension. So we may work over $\mathbf{ACA}_0$. We also have that every scattered T_3 space embeds into the order topology of a complete linear order. By Theorem 11.5, we have that the order topology of a complete linear order is compact. So every T_3 scattered *CSCS*, and therefore every T_2 locally compact *CSCS* embeds into a compact *CSCS*. This implies that every T_2 locally compact *CSCS* has a *CCN* which by Theorem 11.3 implies arithmetic transfinite recursion. □

12. Complete Metrizability

We have shown that $\mathbf{ATR}_0$ proves that T_3 scattered *CSCS* can embed into a well order with a maximal element, and in particular, they can embed into a compact metric space. From general topology, we know that compact metric spaces are complete and the G_δ subspaces of a complete metric space are completely metrizable. For countable T_1 spaces, all of the subspaces are G_δ, so all T_3 scattered *CSCS* are completely metrizable. We will show that this proof can be carried out in $\mathbf{ATR}_0$.

Proposition 12.1 $\mathbf{RCA}_0$ proves every complete countable metric space is T_3 and scattered.

Proof: Let (X, d) be a complete metric space, by Proposition 3.9 we have that for some $a \in \mathbb{R}_{>0}$ the balls $(B(x, q \cdot a))_{x \in X, q \in \mathbb{Q}}$ are clopen. Without loss of generality, we may assume $a = 1$. By Theorem 5.13 $\mathbf{RCA}_0$ proves that $(X, (B(x,q)_{x \in X, q \in \mathbb{Q}}, k)$, is uniformly T_3. If X were not scattered, then it would contain a non empty dense in itself subset S that will be a uniformly T_3 space that does not have isolated points. We note that any neighborhood of a point in S will contain infinitely many points. Let $(q_n)_{n \in \mathbb{N}}$ enumerate the elements of $\mathbb{Q}_{>0}$. Let x_0 be the $<_{\mathbb{N}}$ first element of S and set $r_0 = 1$. Assume that for all $i \leq n$ $y_i \in X$ and $r_i \in \mathbb{Q}_{>0}$ are defined and for all $i < n$ we have $x_i \notin B(x_{i+1}, r_{i+1}) \subseteq B(x_i, r_i)$. Let x_{n+1} be the $<_{\mathbb{N}}$ least element of $S \cap B(x_n, r_n) \setminus \{x_n\}$, which is not empty since S is dense in itself, and let r_{n+1} be the first element in the enumeration of $\mathbb{Q}$ such that:

$$r_{n+1} \leq \min\{d(x_{n+1}, x_n), r_n - d(x_{n+1}, x_n), \frac{1}{n+1}\}$$

We have by the triangle inequality that $x_n \notin B(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n)$. By construction we have that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy and, therefore, must converge to some $x \in X$. Since for all n we have that x_{n+1} is the $<_{\mathbb{N}}$ least element of $B(x_n, r_n) \cap S \setminus \{x_n\}$ and $x_{n+2} \in B(x_n, r_n) \cap S \setminus \{x_n\}$ we have that $x_{n+1} <_{\mathbb{N}} x_{n+2}$. In particular the sequence $(x_n)_{n \in \mathbb{N}}$ is strictly $<_{\mathbb{N}}$ increasing. For every $n \in \mathbb{N}$ we have that $(x_n)_{n \in \mathbb{N}}$ is definitely in $B(x_n, r_n)$ and so $x \in \overline{B(x_n, r_n)} = B(x_n, r_n)$, so in particular we have that $x_{n+1} \leq_{\mathbb{N}} x$. But this is absurd since the sequence $(x_n)_{n \in \mathbb{N}}$ is strictly increasing and therefore is unbounded. So X is a uniformly T_3 scattered $CSCS$.

□

Definition 12.2 A metric space (X, d) is said to be totally bounded if for every $n \in \mathbb{N}$ there exists a finite set $F \subseteq X$ such that $X = \bigcup_{x \in F} B(x, \frac{1}{n})$. Being a totally bounded space is arithmetically definable, as noted by Hirst [10].

Lemma 12.3 $\mathbf{RCA}_0$ proves that if (X, d) is totally bounded then any subset of X is totally bounded.

Proof: Let (X, d) be a totally bounded metric space and $Y \subseteq X$ be a subset. Given $n > 0$, we show that we can cover Y with finitely many balls of radius $\frac{1}{n}$. Since X is totally bounded, there exists a finite $F \subseteq X$ such that $X = \bigcup_{x \in F} B(x, \frac{1}{2n})$. Define:

$$G = \left\{ y \in Y : \exists x \in F \; y \in B(x, \frac{1}{2n}) \wedge \forall z < y \; z \notin S \cap B(x, \frac{1}{2n}) \right\}$$

we have that G exists by Δ^0_1 comprehension and is a finite set. For all $x \in F$ if $B(x, \frac{1}{2n}) \cap Y \neq \emptyset$ then there is a $y \in G$ such that $B(x, \frac{1}{2n}) \subseteq B(y, \frac{1}{n})$ and so $Y \subseteq \bigcup_{y \in G} B(y, \frac{1}{n})$. Thus, Y is totally bounded. □

Theorem 12.4 $\mathbf{ACA}_0$ proves that a metric space is compact if and only if it is complete and totally bounded.

Proof: Let (X, d) be a compact metric space. Since over $\mathbf{ACA}_0$ sequential compactness is equivalent to compactness, we have that X is sequentially compact. Therefore, every Cauchy sequence in X will have a convergent subsequence, so X is complete. Since X is compact, for every $n \in \mathbb{N}$, it will have a finite covering of balls of radius $\frac{1}{n}$, so X is totally bounded.

For the converse, let X be a complete, non-empty, and totally bounded metric space. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in X, we define inductively a subsequence $(z_n)_{n \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ and a sequence points $(x_n)_{n \in \mathbb{N}}$. Let $z_0 = y_0$ and $x_0 = z_0$. Assume that we have defined z_i and x_i for $i \leq k$ such that for infinitely many n $y_n \in \bigcap_{i\leq k} B(x_i, \frac{1}{2i+1})$. Since X is totally bounded, we have that $\bigcap_{i\leq k} B(x_i, \frac{1}{2i+1})$ is totally bounded, and so there is a least finite set F such that:

$$\bigcap_{i\leq k} B(x_i, \frac{1}{2i+1}) \subseteq \bigcup_{x\in F} B(x, \frac{1}{2k+1})$$

By pigeon hole principle let x_{k+1} be the least $x \in F$ such that for infinitely many n $y_n \in B(x, \frac{1}{2k+1}) \cap \bigcap_{i\leq k} B_i$ and we define z_{k+1} to be the first $y_n \in B(x_n, \frac{1}{2k+1})$.

For every $m \in \mathbb{N}$ the sequence $(z_n)_{n\in\mathbb{N}}$ is definitely in $B(x_m, \frac{1}{2m+1})$ which has diameter $\frac{2}{2m+1} < \frac{1}{m}$ and so we have that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy subsequence of $(y_n)_{n\in\mathbb{N}}$. Since (X, d) is complete we have that $(z_n)_{n\in\mathbb{N}}$ converges. □

Corollary 12.5 $\mathbf{RCA}_0$ proves every effectively T_2 effectively compact $CSCS$ is completely metrizable.

Proof: By Corollary 6.11 we have that $\mathbf{ACA}_0$ proves that every T_2 compact $CSCS$ is T_3 and by Theorem 5.17 it is metrizable. Any metric on a compact space must be complete by the previous theorem. □

Definition 12.6 (Simpson [20, Exercise VI.1.8]) Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree then we define:

$$T^+ = \{\tau \in \mathbb{N}^{<\mathbb{N}} : \exists \sigma \in T\ (|\tau| = |\sigma| \wedge \forall n < |\tau|\ (\sigma(n) \leq \tau(n)))\}$$

which is the tree of all sequences τ that dominate some sequence of T of equal length.

Proposition 12.7 Over $\mathbf{ACA}_0$ we have that T is well founded if and only if $\mathrm{KB}(T^+)$ with the order topology is scattered.

Proof: The Kleene Brouwer order of any tree will have the empty sequence as a maximal element. If $\mathrm{KB}(T^+)$ is not scattered, then it is not compact, so it cannot be well ordered. This means T^+ is not well founded. Given $f \in [T^+]$ we define the subtree:

$$\{\sigma \in T : \forall n < |\sigma|\, (\sigma(n) \leq f(n))\}$$

which is a finitely branching infinite subtree of T. By weak König's lemma, we have that the subtree, and therefore T, also has a branch.

Assume instead T is not well founded and let $f \in [T]$ be a branch. We show that:

$$S = \{\sigma \in T^+ : \forall n < |\sigma|\ (\sigma(n) \geq f(n))\} \cong \mathbb{Q}$$

to do this, it suffices to show that S is dense in itself. Let $\rho \in S$ and $\tau, \sigma \in T^+$ such that $\rho \in]\sigma, \tau[_{<_{\mathrm{KB}}}$ then if $\rho \sqsubseteq \sigma$ then we have $\sigma <_{\mathrm{KB}} \rho^\frown(\sigma(|\rho|) + 1) <_{\mathrm{KB}} \rho$, and so $\rho^\frown(\sigma(|\rho| + 1)) \in]\sigma, \tau[_{<_{\mathrm{KB}}} \cap S$. Otherwise, if $\rho \not\sqsubseteq \sigma$ then we have $\sigma <_{\mathrm{KB}} \rho^\frown(f(|\rho|) + 1) <_{\mathrm{KB}} \rho$. So S with the subspace topology is dense in itself and so T^+ is not scattered. □

Observation 12.8 The proof above can be modified to show that if T is well founded then T^+ with the topology given by being a subspace of $(\mathbb{N}^{<\mathbb{N}}, <_{\mathrm{KB}})$ is compact. Similarly, if T is ill founded then T^+ with the topology given by being a subspace of $(\mathbb{N}^{<\mathbb{N}}, <_{\mathrm{KB}})$ is scattered.

Clote [2] used a similar construction to show that being a scattered linear order is a Π^1_1 universal formula. However, as noted before, the notion of scatteredness for linear orders does not, in general, coincide with the topological notion of scatteredness.

Proposition 12.9 Over $\mathbf{ACA}_0$ being a complete metric space is a Π^1_1 complete formula.

Proof: We can express (M, d) being complete as:

$$\forall Y \subseteq M \ (Y \text{ is a Cauchy sequence} \to \exists x \in M \ (Y \text{ converges to } x))$$

Which is a Π^1_1 formula since converging to x and being a Cauchy sequence are arithmetically defined.

Fix any arithmetically definable metric d that is compatible with the order topology on $\mathrm{KB}(\mathbb{N}^{<\mathbb{N}})$. Over $\mathbf{ACA}_0$ being a well founded tree is Π^1_1 universal. Over $\mathbf{ACA}_0$ we have that:

$$T \text{ is well founded} \leftrightarrow \mathrm{KB}(T^+) \text{ is compact} \leftrightarrow (\mathrm{KB}(T^+), d) \text{ is complete}$$

and

$$T \text{ is ill founded} \leftrightarrow \mathrm{KB}(T^+) \text{ is not scattered} \leftrightarrow (\mathrm{KB}(T^+), d) \text{ is not complete}$$

So over $\mathbf{ACA}_0$ we have that for any tree T:

$$T \text{ is well founded} \leftrightarrow (\mathrm{KB}(T^+), d) \text{ is complete}$$

which implies that being a complete metric space is a Π^1_1 universal formula. $\square$

Corollary 12.10 Over $\mathbf{RCA}_0$ the following are equivalent:

1. Π^1_1 comprehension.
2. For any sequence of $CSCS$ $(X^i)_{i\in\mathbb{N}}$ the set $\{i \in \mathbb{N} : X^i \text{ is scattered}\}$ exists.
3. For any sequence of metric spaces $(X^i, d^i)_{i\in\mathbb{N}}$ the set $\{i \in \mathbb{N} : (X^i, d^i) \text{ is complete}\}$ exists.

Proof: Since being a complete metric space and being scattered are both Π^1_1 universal formulas over $\mathbf{ACA}_0$, it suffices to show that 1 and 2 both imply arithmetic comprehension. Using Theorem 2.2, we show that 1 and 2 both imply that every set has a Turing jump.

Assume 2 and let $A \subseteq \mathbb{N}$ be a set. Let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Define:

$$X^e = \{q_n : \neg\Phi^A_e(e)\downarrow_{\leq n}\} \subseteq \mathbb{Q}$$

with the subspace topology. We have that $X^e = \mathbb{Q}$ if and only if $\Phi^A_e(e)\uparrow$ and X^e is a discrete finite space, and therefore scattered, if and only if $\Phi^A_e(e)\downarrow$. So $A' \leq_T \{e \in \mathbb{N} : X^e \text{ is scattered}\}$ and so A' exists by Δ^0_1 comprehension.

Assume 3 and let $A \subseteq \mathbb{N}$ be a set. Define $X^e = \{\frac{1}{n+1} : \neg\Phi^A_e(e)\downarrow_{\leq n}\}$ with the metric $d^e(x, y) = |x - y|$. We have that X^e is complete if and only if $\Phi^A_e(e)\downarrow$ and so $A' \leq_T \{e \in \mathbb{N} : (X^e, d^e) \text{ is complete}\}$ exists by Δ^0_1 comprehension. $\square$

Definition 12.11 In general topology, a subset of a topological space X is said to be G_δ if it is the countable intersection of open sets. If X is countable and T_1, that is, all singletons are closed, then all subsets of X are G_δ.

Lemma 12.12 $\mathbf{ACA}_0$ proves every subspace of a complete metric is completely metrizable.

Proof: Let (X, d) be a complete metric space and $Y \subseteq X$ and $(x_n)_{n\in\mathbb{N}}$ an enumeration of $X \setminus Y$. We define:

$$d'(x, y) = d(x, y) + \sum_{n\in\mathbb{N}} \min\left\{2^{-n-1}, \left|\frac{1}{d(x, x_n)} - \frac{1}{d(y, x_n)}\right|\right\}$$

The metric d' is arithmetically definable with respect to (X, d), and so it exists by arithmetic comprehension. It is standard to check that the metrics d and d' induce the same topology on Y.

We show now that d' is a complete metric on Y. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (Y, d'), then $(y_n)_{n\in\mathbb{N}}$ is also Cauchy in (X,d). Since X is complete there exists $y \in X$ such that $(y_n)_{n\in\mathbb{N}}$ converges to y with respect to d or rather $d(y_i, y) \to 0$. If $y = x_m \in X \setminus Y$ for some $m \in \mathbb{N}$ then since $(y_n)_{n\in\mathbb{N}}$ is Cauchy in (Y, d') we have that:

$$\lim_{i,j\to\infty} \left| \frac{1}{d(y_i, x_m)} - \frac{1}{d(y_j, x_m)} \right| = 0$$

which means $(\frac{1}{d(y_j,x_m)})_{j\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ and therefore convergent since over $\mathbf{ACA}_0$ $\mathbb{R}$ is complete. This implies that $(d(y_j, x_m))_{j\in\mathbb{N}}$ diverges which contradicts our assumption that $(y_j)_{j\in\mathbb{N}}$ converges to $y = x_m$ with respect to d. So $y \in Y$ which implies that (Y, d') is a complete metric space. □

The previous result is essentially a modification of the proof that every G_δ subset of a Polish space is completely metrizable (See [12, Theorem 3.11]).

Theorem 12.13 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic transfinite recursion.
(2) Every T_3 scattered $CSCS$ can be embedded into a completely metrizable space.
(3) Every T_3 scattered $CSCS$ is completely metrizable.

Proof: $(1 \to 2)$ By Lemma 11.15 every T_3 scattered $CSCS$ embeds into a T_2 compact space and by Corollary 12.5 every T_2 compact space is completely metrizable.

2, 3 both imply arithmetic comprehension since they imply that every T_3 scattered $CSCS$ is metrizable and by Theorem 5.13 they are uniformly T_3 and by Corollary 4.21 this implies arithmetic comprehension. So we may prove the remaining implications over $\mathbf{ACA}_0$.

$(2 \to 3)$ Let X be a T_3 scattered $CSCS$. If X can be embedded into a complete metric space, then by Lemma 12.12 X is completely metrizable.

$(3 \to 1)$ By Lemma 9.8 and Proposition 9.5 we have that every T_2 locally compact space is T_3 and scattered. So 3 implies every T_2 locally compact $CSCS$ is completely metrizable. Let X be a T_2 locally compact $CSCS$ and let d be a complete metric on X. For each $x \in X$ let $i(x)$ be the least number such that $(\overline{B(x, \frac{1}{2^{i(x)}})})_{i\in\mathbb{N}}$ is totally bounded. Since being totally bounded is arithmetical, the sequence $(i(x))_{x\in X}$ exists by arithmetic comprehension. Since closed subsets of complete metric spaces are complete, we have that for all x, the set $\overline{B(x, \frac{1}{2^{i(x)}})}$ is compact by Theorem 12.4. So the sequence $(\overline{B(x, \frac{1}{2^{i(x)}})})_{i\in\mathbb{N}}$ is a CCN. We have that every T_2 locally compact space has a CCN, which is equivalent to arithmetic transfinite recursion. □

13. Comparability of Locally compact T_2 Spaces

A classic result in reverse mathematics is that over $\mathbf{RCA}_0$ arithmetic transfinite recursion is equivalent to the comparability of well orders. By comparability of well orders, we mean that for any pair of well orders, there is an order preserving map from one to the other. It turns out there is a similar topological equivalent to arithmetic transfinite recursion. It was shown by Friedman [5] that topological comparability of well orders is equivalent to arithmetic transfinite recursion. We will give an alternative proof to a weaker theorem of Hirst [10], namely that arithmetic transfinite recursion is equivalent over $\mathbf{ACA}_0$ to the topological comparability of T_2 locally compact $CSCS$.

Definition 13.1 We say a pair $CSCS$ X and Y are topologically comparable if there is an effective embedding from X to Y or from Y to X.

Proposition 13.2 Arithmetic transfinite recursion is equivalent over $\mathbf{ACA}_0$ to every T_2 locally compact $CSCS$ can be embedded into a compact space.

Proof: By Theorem 11.2 we have that $\mathbf{ATR}_0$ proves that every T_2 locally compact space has a CCN. By Proposition 9.25 every locally compact T_2 $CSCS$ with a CCN embeds into its one point compactification. On the other hand, every compact space trivially has CCN. By Proposition 9.12, if every locally compact T_2 $CSCS$ can be embedded into a compact $CSCS$, then it will have a CCN, which by Proposition 11.3 implies arithmetic transfinite recursion. □

Lemma 13.3 $\mathbf{ACA}_0$ proves that for any $CSCS$ space $(X, (U_i)_{i\in\mathbb{N}}, k)$ if every T_2 compact $CSCS$ embeds into X then X is not scattered.

Proof: Let X be a $CSCS$ such that every compact T_2 $CSCS$ space C embeds into X. For every well founded tree T, we have that T^+ is well founded and therefore, $\mathrm{KB}(T^+)$ (see Definition 12.6) embeds into X since it is compact. So we have that:

$$\forall T \subseteq \mathbb{N}^{<\mathbb{N}}(\mathrm{WO}(\mathrm{KB}(T)) \to \mathrm{KB}(T^+) \text{ embeds into } X)$$

But $\mathrm{KB}(T^+)$ embeds into X is Σ^1_1 relative to X. Since $\mathrm{WO}(\mathrm{KB}(T))$ is a universal Π^1_1 formula we have that there exists a non well founded T such that $\mathrm{KB}(T^+)$ embeds into X. Since T is not well founded by the Proposition 12.7, we have that $\mathbb{Q}$ embeds into $\mathrm{KB}(T^+)$ and so $\mathbb{Q}$ embeds into X contradicting our assumption that X was scattered. □

Theorem 13.4 Over $\mathbf{ACA}_0$ the following are equivalent:

(1) Arithmetic transfinite recursion.
(2) All pairs of locally compact T_2 $CSCS$ are topologically comparable.

Proof: $(1 \to 2)$ follows immediately from the fact that $\mathbf{ATR}_0$ proves that every T_2 locally compact $CSCS$ is homeomorphic to some well order and that for every pair of well orders, X and Y either X is isomorphic to an initial segment of Y or Y isomorphic to an initial segment of X.

$(2 \to 1)$. By Corollary 13.2, arithmetic transfinite recursion is equivalent to every T_2 locally compact $CSCS$ can be embedded into a T_2 compact $CSCS$. Let X be a T_2 locally compact $CSCS$. If X does not embed into a T_2 compact $CSCS$, we have that every compact T_2 $CSCS$ is homeomorphic to a subspace of X. By the previous lemma, this implies that X is not scattered, which by Lemma 9.5 contradicts our assumption that X is locally compact. □

Observation 13.5 We might wonder if T_3 scattered spaces are topologically comparable. This is not the case; we can consider the ordinal $\omega \cdot 2 + 1$ and a point with infinitely many sequences converging to it. These two spaces are both T_3 and scattered. However, they are not topologically comparable. This observation can be carried out easily in $\mathbf{RCA}_0$.

For completeness, we will lay out Friedman's proof that topological comparability for well orders is equivalent to arithmetic transfinite induction. To do so, we need to define well order exponentiation.

Definition 13.6 (Hirst [11, Definition 2.1]) Given two well orders L and W we define W^L to be the set of sequences which includes the empty sequence and all sequences of the form:

$$((a_0, b_0), \dots (a_n, b_n))$$

such that $\forall i \leq n$ $b_i \in L$ and $a_i \in W \setminus \{\min W\}$ and for all $j < i \leq n$ $b_i <_L b_j$.

Given $\sigma, \tau \in W^L$ we define set $\sigma <_{W^L} \tau$ if and only if either $\sigma \subsetneq \tau$ or given that:

$$j = \min\{j < \min\{|\tau|, |\sigma|\}, \sigma(j) \neq \tau(j)\}$$

and

$$(a_j, b_j) = \sigma(j) \wedge (c_j, d_j) = \tau(j)$$

then:

$$(b_j < d_j \vee (d_j = b_j \wedge a_j < c_j))$$

Intuitively, the elements W^L can be viewed as being ordinals less than W^L in their Cantor normal form in base W and ordered in the standard way.

Theorem 13.7 (Hirst [11, Theorem 2.6]) Over $\mathbf{RCA}_0$ arithmetic comprehension is equivalent to ordinal exponentiation being well defined. That is, for any pair of well orders W and L, the set W^L is well ordered.

Observation 13.8 $\mathbf{ACA}_0$ proves that the isolated points of $\mathbb{N}^L$ are the empty sequence and all sequences of the form:

$$((a_0, b_0), \dots (a_n, 0))$$

since it has as successor:

$$((a_0, b_0), \dots (a_n + 1, 0))$$

and a predecessor:

$$\begin{cases} ((a_0, b_0), \dots (a_n - 1, 0)) & \text{if} \quad a_n > 1 \\ ((a_0, b_0), \dots (a_{n-1}, b_{n-1})) & \text{if} \quad a_n = 1 \end{cases}$$

On the other hand, sequences of the form:

$$((a_0, b_0), \dots (a_n, b_n))$$

where $b_n > 0$ will not have a predecessor and are not isolated.

Lemma 13.9 $\mathbf{ACA}_0$ proves that for ever well order L $\mathbb{N}^L + 1$ is a well order and has L as a rank.

Proof: For each $l \in L$ let X_l be the set set of all sequences in $\mathbb{N}^L$ of the form:

$$((a_0, b_0), \dots (a_n, b_n))$$

such that for all $i \leq n$ we have $b_i \geq l$. We have that the sequence $(X_l)_{l \in L}$ exists by arithmetical comprehension. By Observation 13.8, we have that X_{l+1} are the limit points of X_l. So we have that the sequence $(X_l)_{l \in L}$ witnesses that L is a rank for $\mathbb{N}^L$. □

Lemma 13.10 Let X be a compact space with rank W and Y with rank L and assume that X embeds homeomorphically into Y then there exists an order embedding of W into L.

Proof: Let $f : X \to Y$ be an embedding, then we define $\phi : W \to L$ where given $a \in \alpha$:

$$\phi(w) = \min\{l \in L : \exists x \in X(\mathrm{rank}_W(x) = w \wedge \mathrm{rank}_L(f(x)) = l\}$$

We show that f preserves the strict order. Seeking a contradiction assume that there exists a least $w \in W$ and $w_0 <_W w$ such that $\phi(w) \leq_L \phi(w_0)$. Let $x \in X$ be of rank w such that $f(x)$ has rank $\phi(w)$. In X_{w_0}, x is a limit point of the elements of rank w_0 and so $f(x)$ must be the limit point of the image of the elements of rank w_0 which are contained in $Y_{\phi(w_0)}$, so $f(x)$ is a limit point of $Y_{\phi(w_0)}$. Since $\mathrm{rank}_L(f(x)) = \phi(w) \leq_L \phi(w_0)$ either $f(x) \notin Y_{\phi(w_0)}$ or $f(x)$ is isolated in $Y_{\phi(w_0)}$. Both cases contradict the fact that $f(x)$ is a limit point of $Y_{\phi(w_0)}$ □

Friedman proved a special case of the previous theorem, where X and Y are of the form $\mathbb{N}^{W_0}$ and $\mathbb{N}^{W_1}$, where W_0 and W_1 are well orders (Friedman [5, Lemma 25]).

Theorem 13.11 (Friedman [5, Lemma 20]) Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic transfinite recursion.
(2) Every pair of T_2 locally compact *CSCS* is topologically comparable.
(3) Every pair of well orders are topologically comparable.

Proof: We already showed that $(1 \to 2)$, and $(2 \to 3)$ is obvious.

We show first that 3 implies the image of every injective function exists which by Theorem 2.2 implies arithmetic comprehension. Let $h : \mathbb{N} \to \mathbb{N}$ be an injective function. Consider the space:

$$X = \{(x, y, t) : (y = 0 \wedge t = 0) \vee (h(y) = x \wedge t \leq y)\} \cup \{\infty\}$$

with the lexicographic ordering and ∞ is the maximal element. By Σ^0_1 induction, we have that any $(x, 0, 0) \in X \setminus \{\infty\}$ has finitely many predecessors, so X is well ordered. Let $Y = \{0, 1\} \times \mathbb{N}$ with the lexicographic ordering. Both X and Y are well orders, but Y cannot effectively embed into X since Y has an infinite closed set of isolated points while X does not. By assumption, there exists an $f : X \to Y$ that is an effective embedding. We have that $f(\infty) = (1, 0)$ since f sends limit points to limit points. Furthermore there exists an $m \in X \setminus \{\infty\}$ such that $f(]m, \infty]_X) \subseteq](0, 0), (1, 0)]_Y$. Up to changing finitely many values for f may assume that $m = 0$. Since f is effectively open with respect to its image, there is a function $v : \mathbb{N} \to \mathbb{N}$ such that for all $m \in \mathbb{N}$:

$$](0, v(m)), (1, 0)]_Y \cap \operatorname{rng}(f) \subseteq f(\,](m, 0, 0), \infty]_X)$$

If $h(y) = x$ let $F = \{(a, b, c) <_X (x + 1, 0, 0)\} \subseteq X$, since for all $t \leq y$ $(x, y, t) <_X (x + 1, 0, 0)$ we have that $|F| \geq y$. We have that $f(F) \subseteq](0, 0), (0, v(x))[_Y$ and since g is injective and $|F| \geq y$ we have that $v(x) \geq y$. So:

$$x \in \operatorname{rng}(h) \leftrightarrow \exists y \leq v(x)\ h(y) = x$$

and so the range of h is Δ^0_0 definable relative to v and so it exists by Δ^0_1 comprehension.

We now show over $\mathbf{ACA}_0$ that 3 implies that for any two well orders W and L, either there is a strictly increasing function from W to L or from L to W, which is equivalent to arithmetic transfinite recursion. We have by Lemma 13.9 that $\mathbb{N}^L + 1$ has L as a rank and $\mathbb{N}^W + 1$ has W as a rank. $\mathbf{ACA}_0$ proves that $\mathbb{N}^L + 1$ and $\mathbb{N}^W + 1$ are well orders and so there exists either an embedding from $\mathbb{N}^L + 1$ to $\mathbb{N}^W + 1$ or an embedding from $\mathbb{N}^W + 1$ to $\mathbb{N}^L + 1$. Without loss of generality, assume there is an embedding from $\mathbb{N}^L + 1$ to $\mathbb{N}^W + 1$. By Lemma 13.10, this means that there is a strictly increasing function from L to W. $\square$

Montalbán and Greenberg [8] showed that $\mathbf{ATR}_0$ is equivalent to every T_2 compact $CSCS$ having a rank. We give an alternative proof.

Theorem 13.12 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic transfinite recursion.
(2) Every scattered T_3 space has Cantor Bendixson rank.
(3) Every well order with the order topology has Cantor Bendixson rank.

Proof: $(1 \to 2)$ follows from Theorem 11.12 and $(2 \to 3)$ is obvious.

By proposition 11.13 3 implies arithmetic comprehension, so we may work over $\mathbf{ACA}_0$. Let W and L be well orders. We show that one embeds as an initial segment of the other, which over $\mathbf{RCA}_0$ is equivalent to arithmetic transfinite recursion. Let $X = (\mathbb{N}^L + 1)$ and $Y = (\mathbb{N}^W + 1)$. The space Z, which is the disjoint union of X and Y, is homeomorphic to the well order $(\mathbb{N}^W + 1) + (\mathbb{N}^L + 1)$ and so by assumption Z has a rank R. We have that $(\mathbb{N}^W + 1)$ has W as rank and $(\mathbb{N}^L + 1)$ has L as rank. By transfinite induction on L, we have that the map $L \to R$ given by $\operatorname{rank}_R(x) \mapsto \operatorname{rank}_L(x)$ is well defined, and it defines an order isomorphism from L to an initial segment of R. Similarly, we can embed W as an initial segment of R. So either W or L is order isomorphic to an initial segment of the other. $\square$

14. Π^1_1 Comprehension

Question: (Chan [1]) Are the following equivalent over $\mathbf{RCA}_0$?

1. Π^1_1 comprehension.
2. For every sequence $((X^j,(U^j_i)_{i\in\mathbb{N}}))_{j\in\mathbb{N}}$ of $CSCS$ the set $\{j\in\mathbb{N} : X^j \text{ is connected}\}$ exists.

Where a $CSCS$ $(X,(U_i)_{i\in\mathbb{N}})$ is said to be connected for any open set A and B such that $X = A\cup B$ then $A\cap B \neq \emptyset$.

Chan showed that 2 implies arithmetic comprehension, so the following result gives a positive answer to the question.

Theorem 14.1 Being a connected $CSCS$ is Π^1_1 universal over $\mathbf{ACA}_0$.

Proof: Let L be a linear order. Up to adding a new element, we may assume that L has a minimal element. For each $l\in L$ let V_l be the set of all $j\in L$ such that either $j\leq_L l$ or there exists a sequence σ such that $\sigma(0)=l$ for all $n<|\sigma|$ we have that $\sigma(n+1)$ is the smallest element greater than $\sigma(n)$ and that $\sigma(|\sigma|-1)=j$. That is, V_l is the smallest set containing l that is closed under successor when it's defined and downwards closed. Define the topology $Top(L)$ on L to be generated by the sets $(V_l)_{l\in L}\cup(\,]a,+\infty[\,)_{l\in L}$. We observe that sets of the form $V_l\cap\,]j,+\infty[$ form a basis for this topology and that all open sets are closed under successor when it's defined.

We show that L is well ordered if and only if it is connected with respect to $Top(L)$. If L has a descending sequence $(a_i)_{i\in\mathbb{N}}$ then the set $\uparrow\{a_i : i\in\mathbb{N}\}$ is clopen since its complement is downwards closed and closed under successor. We have that $L\neq\uparrow\{a_i : i\in\mathbb{N}\}$ since L is assumed to have a minimal element.

If L is a well order, let A be a non empty clopen set of L which contains the least element of L. If $A\neq L$, then let x be the least element of $L\setminus A$. Since $L\setminus A$ is open, we have that x must be in a basic open set of the form $V_j\cap\,]l,+\infty[\,\subseteq L\setminus A$. We have that $l<x$, so by minimality of x, we have that $l\in A$. Since A is closed under successor we have $l+1\in A$ but $l+1\in\,]l,+\infty[$ and since V_j is downwards closed and $l+1\leq x$ we have that $l+1\in V_j\cap\,]l,+\infty[\,\subseteq L\setminus A$ which is absurd since $l+1\in A$. So L must be connected as it has only trivial clopen sets.

So we have that L is a well order if and only if $Top(L)$ is a connected topology on L. Since being a connected space is Π^1_1 definable, it is a Π^1_1 universal formula. □

Theorem 14.2 Over $\mathbf{RCA}_0$ the following are equivalent:

1. Π^1_1 comprehension.
2. Every T_3 $CSCS$ is the disjoint union of a scattered space and a dense in itself space (this will not, in general, be a topological disjoint union as the dense in itself subspace may not be open).
3. Every T_3 space is ranked.

Proof: 1 implies arithmetic comprehension since arithmetic formulas are Π^1_1 and 3 implies arithmetic comprehension by 11.13. We show that 2 implies arithmetic comprehension. Let $A\subseteq\mathbb{N}$ be a set and $(q_n)_{n\in\mathbb{N}}$ be an enumeration of $]0,1[_{<\mathbb{Q}}\subseteq\mathbb{Q}$ and consider:

$$X=\{(e,q_n) : e\in\mathbb{N}\wedge\neg\Phi^A_e(e)\downarrow_{\leq n}\}\cup\{(e,0) : e\in\mathbb{N}\}\subseteq\mathbb{N}\times\mathbb{Q}$$

with the subspace topology. Let $D\subseteq X$ be a maximal dense in itself subspace of X then we have that $(e,0)\notin D\leftrightarrow\Phi^A_e(e)\downarrow$ and so $A'\leq D$. So 2 implies the Turing jump of every set exists, which by 2.2 implies arithmetic comprehension. Since $1,2$ and 3 all imply arithmetic comprehension, it suffices to show that they are equivalent over $\mathbf{ACA}_0$.

$(1 \to 2)$ Let X be a T_3 space and define $U \subseteq X$ to be the set of points $x \in X$ such that there exists an infinite dense in itself subspace of X containing x. The set U exists by Σ^1_1 comprehension. We have that U is a maximal dense in itself subspace of X, and therefore, $X \setminus U$ must be scattered.

$(2 \to 1)$ Since being scattered is Π^1_1 universal, we show that for any sequence $(X_i)_{i\in\mathbb{N}}$ of T_3 $CSCS$ space the set of indices i such that X_i is scattered exists. Let $X = \coprod X_i$ be the disjoint union of all the X_i and let D be the maximal dense in itself set of X. We have that X_i is scattered if and only if $X_i \cap D = \emptyset$ and so the set:

$$\{i \in \mathbb{N} : X_i \text{ is scattered }\}$$

exists by arithmetic comprehension.

$(1 + 2 \to 3)$ Let X be a T_3 $CSCS$, then we have by 2 that $X = P \sqcup Y$ where D is dense in itself and Y is scattered. Since $\mathbf{ATR}_0$ proves T_3 scattered $CSCS$ is ranked we have that 1 implies that Y has a rank R, let $(Y_r)_{r\in R}$ witness that R is a rank for Y. We have that $(Y_r \sqcup P)_{r\in R}$ witnesses that R is a rank for X.

$(3 \to 2)$ Let X be a T_3 $CSCS$. By assumption X has a rank R, let $(X_r)_{r\in R}$ witness that R is a rank for X. By definition of rank 11.9, we have that $\bigcap_{r\in R} X_r = P$ doesn't have limit points. So P is either empty or is a maximal dense in itself subspace of X. So $X \setminus P$ is scattered and $X = P \sqcup (X \setminus P)$ can be written as the disjoint union between a dense in itself subspace and a scattered subspace. □

Note: The fact that a subset of $\mathbb{R}^n$ can be written as the sum of a scattered set and a dense in itself set is due to Sierpinski [19]. A similar proof is used to show that Π^1_1 comprehension is equivalent to the Cantor-Bendixson theorem for complete metric spaces [20, Theorem VI.1.3].

15. Summary Part II

We summarize the various results covered in the previous section.

$$\begin{tikzcd}
ATR \arrow[r, "10.8"] & TAM \arrow[d, "10.4"] \arrow[rd, "11.2"] & \\
1TC \arrow[u] & BTC \arrow[l, "\text{trivial}"] & \text{All } LCCSCS \text{ have a } CCN \arrow[d, "\text{trivial}"] \\
& 1TAM \arrow[lu, "10.5"] & \text{All } T_2 \ LCCSCS \text{ have a } CCN \arrow[l, "11.3"]
\end{tikzcd}$$

Theorem 15.1 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Arithmetic transfinite recursion.
(2) $1TAM$ (Theorem 2.3 and Observation 10.5).
(3) TAM (Proposition 10.3 and Theorem 10.8).
(4) For any sequence of trees $(T_i)_{i\in\mathbb{N}}$ such that for each i the branches of T_i are dominated by some function, then the set $\{i \in \mathbb{N} : [T_i] = \emptyset\}$ exists (Lemma 10.4).
(5) For any sequence of trees $(T_i)_{i\in\mathbb{N}}$ such that for each i the branches of T_i are dominated by some function, then there exists a sequence of functions $(f_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ we have f_i dominates the branches of T_i (Observation 10.6).
(6) Every locally compact $CSCS$ has a CCN (Proposition 11.1 and 11.2).
(7) Every T_2 locally compact $CSCS$ has a CCN(Proposition 11.3).
(8) Every T_2 locally compact $CSCS$ is the disjoint union of open compact sets (Proposition 9.15).
(9) Every T_2 locally compact $CSCS$ has an exhaustion of compact sets (Proposition 9.22).
(10) Every T_2 locally compact $CSCS$ has an effectively T_2 1 point compactification (Proposition 9.26).
(11) Every T_2 locally compact $CSCS$ is homeomorphic to a well order with the order topology (Proposition 9.15).
(12) Every well order has Cantor Bendixson rank (Frieman [5]).
(13) Every scattered T_3 space has Cantor Bendixson rank (Friedman [5]).
(14) Every pair of T_2 locally compact $CSCS$ are topologically comparable (Hirst [10]).
(15) Every pair of well orders with the order topology are topologically comparable (Friedman [5]).
(16) Every T_3 scattered $CSCS$ effectively embeds into a well order (Theorem 11.15).
(17) Every T_3 scattered $CSCS$ is effectively homeomorphic to a scattered linear order with countably many cuts (Theorem 11.20).
(18) Every T_3 scattered $CSCS$ is completely metrizable (Theorem 12.13).

Theorem 15.2 Over $\mathbf{RCA}_0$ the following are equivalent:

(1) Π^1_1 comprehension.
(2) For any sequence of $CSCS$ $(X_i)_{i\in\mathbb{N}}$ then the set $\{i \in \mathbb{N} : X_i \text{ is connected}\}$ exists (Theorem 14.1).
(3) For any sequence of $CSCS$ $(X_i)_{i\in\mathbb{N}}$ then the set $\{i \in \mathbb{N} : X_i \text{ is compact}\}$ exists (Proposition 6.14).
(4) For any sequence of $CSCS$ $(X_i)_{i\in\mathbb{N}}$ then the set $\{i \in \mathbb{N} : X_i \text{ is scattered}\}$ exists (Corollary 12.10).
(5) For any sequence of countable metric spaces $((X_i, d_i))_{i\in\mathbb{N}}$ then the set $\{i \in \mathbb{N} : (X_i, d_i) \text{ is Complete}\}$ exists (Corollary 12.10).
(6) Every T_3 $CSCS$ is the disjoint union of a scattered space and a space homeomorphic to $\mathbb{Q}$ (Theorem 14.2).

(7) Every T_3 $CSCS$ is ranked (Theorem 14.2).

Question 15.3 Does arithmetic comprehension follow from every uniformly T_3 scattered $CSCS$ having a complete metric?

Question 15.4 Does arithmetic comprehension follow from every T_3 scattered $CSCS$ having an effective embedding into a T_2 compact $CSCS$?

Question 15.5 We have that:

$$\text{Compact} \to \text{Locally Compact} \to \text{Scattered}$$

and being scattered and being compact are both Π^1_1 universal over $\mathbf{ACA}_0$. Is being locally compact Π^1_1 expressible over $\mathbf{ACA}_0$ or even $\mathbf{ATR}_0$? We observe that over $\mathbf{ATR}_0$ being locally compact is equivalent to:

$$\exists (K_x, i(x))_{x \in X}\ \forall x\ (K_x \text{ is compact } X \wedge x \in U_{i(x)} \subseteq K_x)$$

which is Σ^1_2 and:

$$\forall (Y, <_Y)\ ((Y, <_Y) \text{ is not a well order} \vee \exists y \in Y\ (X \cong \{z <_Y y\}) \vee Y \text{ embeds into } X)$$

Which is Π^1_2. So being locally compact is Δ^1_2 over $\mathbf{ATR}_0$.

Question 15.6 We have that $\mathbf{ATR}_0$ proves that any T_3 scattered $CSCS$ is homeomorphic to the order topology of a scattered linear order with countably many cuts. Can $\mathbf{ACA}_0$ prove that any T_3 scattered $CSCS$ is homeomorphic to a scattered linear order with the order topology?

Acknowledgements: Thanks to my supervisor, Paul Shafer, for many helpful suggestions and discussions and for reviewing my work.

References

[1] William Chan. "Reverse Mathematics of Topology". In: 2011. URL: https://api.semanticscholar.org/CorpusID:14914112.

[2] Peter Clote. "The metamathematics of scattered linear orderings". In: *Arch. Math. Logic* 29 (1989). DOI: 10.1007/BF01630807.

[3] François G. Dorais. *Reverse mathematics of compact countable second-countable spaces*. 2011. arXiv: 1110.6555 [math.LO].

[4] Damir D. Dzhafarov and Carl Mummert. "Reverse Mathematics: Problems, Reductions, and Proofs". In: *Reverse Mathematics* (2022).

[5] Harvey M. Friedman. *Metamathematics of Comparability*. 2002. URL: https://ssrn.com/abstract=3133675.

[6] Harvey M. Friedman and Jeffry L. Hirst. "Reverse mathematics and homeomorphic embeddings". In: *Annals of Pure and Applied Logic* 54.3 (1991), pp. 229–253. ISSN: 0168-0072. DOI: https://doi.org/10.1016/0168-0072(91)90048-Q. URL: https://www.sciencedirect.com/science/article/pii/016800729190048Q.

[7] Harvey M. Friedman and Jeffry L. Hirst. "Weak comparability of well orderings and reverse mathematics". In: *Annals of Pure and Applied Logic* 47.1 (1990), pp. 11–29. ISSN: 0168-0072. DOI: https://doi.org/10.1016/0168-0072(90)90014-S. URL: https://www.sciencedirect.com/science/article/pii/016800729090014S.

[8] Noam Greenberg and Antonio Montalbán. "Ranked Structures and Arithmetic Transfinite Recursion". In: *Transactions of the American Mathematical Society* 360 (2008), pp. 1265–1307.

[9] Denis R. Hirschfeldt. "Slicing the Truth - On the Computable and Reverse Mathematics of Combinatorial Principles". In: *Lecture Notes Series / Institute for Mathematical Sciences / National University of Singapore*. 2014.

[10] Jeffry L. Hirst. "Embeddings of countable closed sets and reverse mathematics". In: *Archive for Mathematical Logic* 32 (1993), pp. 443–449. URL: https://api.semanticscholar.org/CorpusID:41793206.

[11] Jeffry L. Hirst. "Reverse Mathematics and Ordinal Exponentiation". In: *Ann. Pure Appl. Log.* 66 (1994), pp. 1–18. URL: https://api.semanticscholar.org/CorpusID:5365115.

[12] Alexander S. Kechris. "Classical descriptive set theory". In: 1987. URL: https://api.semanticscholar.org/CorpusID:118957819.

[13] I. L. Lynn. "Linearly orderable spaces". In: *Proceedings of the American Mathematical Society* 13 (1962).

[14] Stefan Mazurkiewicz and Waclaw Sierpinski. "Contribution à la topologie des ensembles dénombrables". In: *Fundamenta Mathematicae* 1 (1920), pp. 17–27.

[15] Cédric Milliet. *A remark on Cantor derivative*. 2011. arXiv: 1104.0287 [math.LO].

[16] Carl Mummert. "Reverse Mathematics of MF Spaces". In: *J. Math. Log.* 6 (2006). URL: https://api.semanticscholar.org/CorpusID:53609771.

[17] Paul Shafer. "The strength of compactness for countable complete linear orders". In: *Comput.* 9 (2019), pp. 25–36. URL: https://api.semanticscholar.org/CorpusID:91184246.

[18] Wacław Sierpiński. "Sur une propriété topologique des ensembles dénombrables denses en soi". fre. In: *Fundamenta Mathematicae* 1.1 (1920), pp. 11–16.

[19] Wacław Sierpiński. "Une démonstration du théorème sur la structure des ensembles de points". In: *Fundamenta Mathematicae* 1 (1920), pp. 1–6.

[20] Stephen G. Simpson. "Subsystems of Second Order Arithmetic". In: *Perspectives in Mathematical Logic*. 1999. URL: https://api.semanticscholar.org/CorpusID:116890781.