prove: $F(x^*(\gamma))^{-1}/\lambda$ solves

minimize $\log\det Z^{-1}$

subject to $\operatorname{Tr} F_i Z = c_i$, $Z > 0$, $-\operatorname{Tr} F_0 Z = \gamma - \frac{n}{\lambda}$.

proof: write the Lagrangian of the upper problem.

$$\mathcal{L}(Z, \nu_i, \eta) = -\log\det Z + \sum_i \nu_i (\operatorname{Tr} F_i Z - c_i) + \eta\cdot(\operatorname{Tr} F_0 Z + \gamma - \frac{n}{\lambda})$$

o optimal condition of:

$$\frac{\partial \mathcal{L}(Z, \nu, \eta)}{\partial Z} = -Z^{*-1} + \sum_i \nu_i F_i + \eta F_0 = 0.$$

$$\therefore Z^{*-1} = \sum_i \nu_i F_i + \eta F_0$$

$$= \left(\sum_i \left(\frac{\nu_i}{\eta}\right) F_i + F_0\right)\eta = F\left(\frac{\nu}{\eta}\right)\eta.$$

$$\therefore Z^* = \left[F\left(\frac{\nu}{\eta}\right)\right]^{-1}/\eta.$$

(left to prove $\nu/\eta = x^*(\gamma)$ and $\eta = \lambda$)

o. we have the dual problem function

$$g(\nu, \eta) = \mathcal{L}(Z^*, \nu, \eta)$$

$$= \log\det\left[F\left(\frac{\nu}{\eta}\right)\eta\right] + \operatorname{Tr}\left[\underbrace{\left(\sum_i \nu_i F_i + \eta F_0\right) Z}_{I_m}\right] - \sum_i \nu_i c_i + \eta\left(\gamma - \frac{n}{\lambda}\right)$$

$$= \log\det\left[F\left(\frac{\nu}{\eta}\right)\eta\right] + m - \sum_i \nu_i c_i + \eta\left(\gamma - \frac{n}{\lambda}\right)$$

$$\frac{\partial g}{\partial v_0} = \mathrm{Tr}\left\{ \left[F(v/\eta)\,\eta \right]^{-1} F_0 \right\} - c_0 = 0.$$

$$\Rightarrow \quad \mathrm{Tr}\left[F(v/\eta)\, F_0 \right] = c_0 \eta.$$

$$\frac{\partial g}{\partial \eta} = \mathrm{Tr}\left\{ \left[F(v/\eta)\,\eta \right]^{-1} F_0 \right\} + \gamma - \frac{n}{\lambda} = 0.$$

$$\Rightarrow \frac{1}{\eta} \mathrm{Tr}\left[F(v/\eta) \underbrace{\left(F_0 + \sum \frac{v_i}{\eta} F_i \right)}_{I_n} \right] + \gamma - \frac{n}{\lambda} - \frac{1}{\eta} \mathrm{Tr}\left[F(v/\eta) \sum_i \frac{v_i}{\eta} F_i \right] = 0.$$

$$\therefore \quad \frac{n}{\eta} + \gamma - \frac{n}{\lambda} - \frac{1}{\eta} \sum_i \left(\mathrm{Tr}\left[F(v/\eta)\, F_i \right] \frac{v_i}{\eta} \right) = 0,$$

$$\frac{n}{\eta} + \gamma - \frac{n}{\lambda} - \sum c_i \frac{v_i}{\eta} = 0. \quad \leftarrow \text{satisfy}$$

$\eta = \lambda$, and $\gamma = c^T \frac{v}{\eta} \Rightarrow \left(\underline{x^*(\gamma) = \frac{v}{\eta}} \right)$ $\square$