|HW1 - Q4|

1. NCut (normalized cut) $= \frac{cut(S)}{vol(S)} + \frac{cut(\bar{S})}{vol(\bar{S})}$

define $x = \begin{cases} \sqrt{\frac{vol(\bar{S})}{vol(S)}} & i \in S \\ -\sqrt{\frac{vol(S)}{vol(\bar{S})}} & i \in \bar{S} \end{cases}$

(1) $\mathcal{L} = \sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T$

proof: ($\mathcal{L} = D - A$)

$$\mathcal{L} = \sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T$$

$$= \sum_{(i,j)\in E} (e_i e_i^T - e_j e_i^T - e_i e_j^T + e_j e_j^T)$$

$$= \sum_{(i,j)\in E} \left((e_i e_i^T + e_j e_j^T) - (e_i e_j^T + e_j e_i^T) \right)$$

(note $A_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \cdots A_{ij} & \cdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$ all zeros except (i,j) element)

$$= \sum_{(i,j)\in E} \left[(diag(\vec{e_i}) + diag(\vec{e_j})) - (A_{ij} + A_{ji}) \right]$$

$$= \sum_{(i,j)\in E} \left[diag(\vec{e_i}) + diag(\vec{e_j}) \right] - \sum_{(i,j)\in E} (A_{ij} + A_{ji})$$

$$= D - A. \qquad \square$$

(iii) $x^T \mathcal{L} x = \sum_{(i,j)\in E} (x_i - x_j)^2$

proof 1: use the proof in the course.

proof 2: use (i) for help.

$$x^T \mathcal{L} x = \sum_{(i,j)\in E} x^T (e_i - e_j)(e_i - e_j)^T x$$

$$= \sum_{(i,j)\in E} \| (e_i - e_j)^T x \|_2$$

$$= \sum_{(i,j)\in E} (x_i - x_j)^2. \qquad \square$$

(iv) $x^T \mathcal{L} x = c \cdot \mathrm{NCut}(S)$

proof: $x^T \mathcal{L} x = \sum_{(i,j)\in E} (x_i - x_j)^2$

$$= \sum_{\substack{(i,j)\in E \\ (i,j)\in S}} \overbrace{(x_i - x_j)^2}^{0} + \sum_{\substack{(i,j)\in E \\ (i,j)\in \bar S}} \overbrace{(x_i - x_j)^2}^{0}$$

$$+ \sum_{\substack{(i,j)\in E \\ i\in S,\, j\in \bar S}} (x_i - x_j)^2$$

$$= \left(\sqrt{\frac{vol(\bar S)}{vol(S)}} + \sqrt{\frac{vol(S)}{vol(\bar S)}} \right)^2 cut(S)$$

$$= \left(\frac{vol(\bar S)}{vol(S)} + \frac{vol(S)}{vol(\bar S)} + 2 \right) cut(S)$$

$$\left(\text{set } c = vol(S) + vol(\bar S) \right)$$

$$= \left(\frac{c - \cancel{vol(S)}}{vol(S)} + \frac{c - \cancel{vol(\bar S)}}{vol(\bar S)} \cancel{+2} \right) cut(S)$$

$$= c \left(\frac{cut(S)}{vol(S)} + \frac{cut(\bar S)}{vol(\bar S)} \right)$$

$cut(S) = cut(\bar S)$

$\square$

(iv) $x^T D e = 0$.

proof. $x^T D e = x^T \operatorname{diag}(k) e$

$= x^T k$

$= \sum_i x_i k_i$ ← k_i is the degree of node i

$\cancel{= \sum_i x_i \sum_j A_{ij} = \sum_{ij} A_{ij} x_i}$

$= \sum_{i \in S} \sqrt{\frac{vol(\bar{S})}{vol(S)}} \cdot k_i - \sum_{i \in \bar{S}} \sqrt{\frac{vol(S)}{vol(\bar{S})}} k_i$

$= \sqrt{\frac{vol(\bar{S})}{vol(S)}} \sum_{i \in S} k_i - \sqrt{\frac{vol(S)}{vol(\bar{S})}} \sum_{i \in \bar{S}} k_i$

$= \sqrt{\frac{vol(\bar{S})}{vol(S)}} vol(S) - \sqrt{\frac{vol(S)}{vol(\bar{S})}} vol(\bar{S})$

$= \sqrt{vol(\bar{S}) vol(S)} - \sqrt{vol(S) vol(\bar{S})} = 0$ ☐

(v) $x^T D x = 2m$.

proof: $x^T D x = \sum_i x_i^2 k_i$

$= \sum_{i \in S} \frac{vol(\bar{S})}{vol(S)} k_i + \sum_{i \in \bar{S}} \frac{vol(S)}{vol(\bar{S})} k_i$

$= vol(\bar{S}) + vol(S) = 2m$. ☐

2. $\boxed{\begin{aligned}&\text{minimize } \frac{x^T L x}{x^T D x}\\ &\text{subject to } x^T D e = 0,\ x^T D x = 2m.\end{aligned}}$

the relaxed problem.

- take $y = D^{1/2} x$, the upper problem would be reformed into:
- minimize $\frac{y^T D^{-1/2} L D^{-1/2} y}{y^T y}$

  subject to $y^T y = 2m$, $y^T D^{1/2} e = 0$.
- equivalent to the problem in lecture if we take $\tilde{L} = D^{-1/2} L D^{-1/2}$ $\square$

3. relating modularity to cuts and volumes

- $Q(y) = \frac{1}{2m} \sum_{1 \le i, j \le n} \left[A_{ij} - \frac{d_i d_j}{2m}\right] \mathbb{I}_{y_i = y_j}$.

  ← indicator to same group

  where $y_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \in \bar{S} \end{cases}$ ← indicator function

- $Q(y) = \frac{1}{2m}\left(\sum_{i,j \in S}\left[A_{ij} - \frac{d_i d_j}{2m}\right] + \sum_{i,j \in \bar{S}}\left[A_{ij} - \frac{d_i d_j}{2m}\right]\right)$ ☆

  - $\sum_{i,j \in S} A_{ij} + \sum_{i,j \in \bar{S}} A_{ij} = 2(m - \mathrm{cut}(S))$
  - $\sum_{i,j \in S} \frac{d_i d_j}{2m} + \sum_{i,j \in \bar{S}} \frac{d_i d_j}{2m} = \frac{1}{2m}\left[\left(\sum_i d_i + \sum_j d_j\right)^2 - 2\sum_i d_i \sum_j d_j\right]$

    $= \frac{1}{2m}\left[(2m)^2 - 2\,\mathrm{vol}(S)\,\mathrm{vol}(\bar{S})\right]$

  → take summary: $Q(y) = \frac{1}{2m}\left[-2\,\mathrm{cut}(S) + \frac{1}{m}\mathrm{vol}(S)\,\mathrm{vol}(\bar{S})\right]$ $\square$