model problem: $y' = \Theta y$, $y(0) = 1$ (Θ is a parameter)

integrator: N steps of Eulers Method $y_{n+1} = y_n + h f(y_n)$

data: $\{(T, \hat{y})\}$ ← for simplicity we only have one data point

For our problem:
$f(y) = \Theta y$
$h = \frac{T}{N}$

A. write down the error function $E(\Theta) =$

B. write down the constraints (these are the Euler steps – N total steps)

C. write down the Lagrangian:

$$L(y_1, \ldots, y_N, \lambda_1, \ldots, \lambda_n, \Theta) =$$

D. We can find our optimal Θ (i.e. the Θ that minimizes our error function) if we satisfy the conditions:

we will focus on these 2

$\frac{\partial L}{\partial y_j} = 0 \quad j = 1, \ldots, n$ ← this gives us a formula for λ_j. the iteration is solved backward in time.

$\frac{\partial L}{\partial \lambda_j} = 0 \quad j = 1, \ldots, n$ ← this tells us to solve for y_j using Euler.

$\frac{\partial L}{\partial \Theta} = 0$ ← almost always too difficult to solve directly

i. show that $\frac{\partial L}{\partial \lambda_j} = 0$ is equivalent to saying that $y_{j+1} = y_j + h f(y_j)$

ii. show that $\frac{\partial L}{\partial y_j} = 0$ gives an iterative method for computing λ_j. You will find λ_n explicitly by taking $\frac{\partial L}{\partial y_N} = 0$

All other λ_j can be expressed in terms of λ_{j+1} (this leads to an iteration that we solve in reverse order)

E. To use Gradient Descent to find the optimal θ, we require $\frac{\partial E}{\partial \theta}$. Notice that $L(y_1^*, \dots, y_n^*, \lambda_1^*, \dots, \lambda_n^*, \theta) = E(\theta)$, therefore $\frac{\partial L}{\partial \theta}(y_1^*, \dots, y_n^*, \lambda_1^*, \dots, \lambda_n^*, \theta) = \frac{\partial E}{\partial \theta}(\theta)$.

Compute the formula for $\frac{\partial L}{\partial \theta}$.

F. repeat the analysis for the generic ODE $\dot{y} = f(y;\theta)$. Since $f(y;\theta)$ is not specified, Your formulas will contain the terms $\frac{\partial f}{\partial \theta}$ and $\frac{\partial f}{\partial y}$