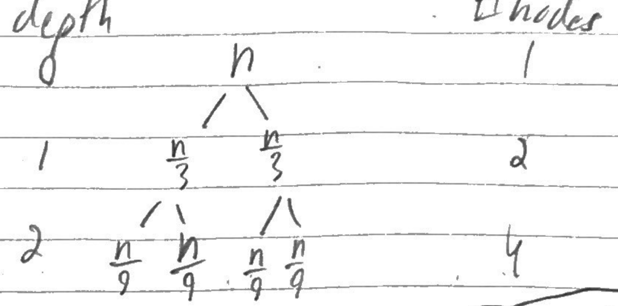
Assignment 2

1a. $T(n) = 2T(n/3) + 1$



level 0: c_1

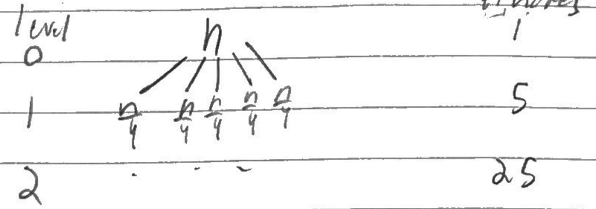
level 1: $2(c_1) = 2c_1$

level 2: $4(c_1) = 4c_1$

$\hookrightarrow 2^i c_1$, lgn nodes

$$T(n) = \sum_{i=0}^{\lg n} 2^i c_1 \Rightarrow c_1 \sum_{i=0}^{\lg n} 2^i < c_1 \cdot \frac{2}{2-1} 2^{\lg n} = c_1 \cdot 2 \cdot 2^{\log_3 n} \Rightarrow O(n^{\log_3 2})$$

1b. $T(n) = 5T(n/4) + n$



level 0: $c_1 n + c_2$

level 1: $5(c_1 \frac{n}{4} + c_2) = \frac{5c_1}{4} n + 5c_2$

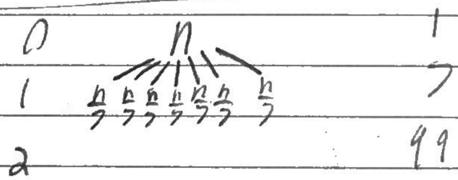
level 2: $25(c_1 \frac{n}{16} + c_2) = \frac{25}{16} c_1 n + 25 c_2$

$\frac{5^i}{4^i} c_1 n + 5^i c_2$, lgn levels

$$T(n) = \sum_{i=0}^{\lg n} \frac{5^i}{4^i} c_1 n + \sum_{i=0}^{\lg n} 5^i c_2 \Rightarrow c_1 n \sum_{i=0}^{\lg n} \left(\frac{5}{4}\right)^i + c_2 \sum_{i=0}^{\lg n} 5^i$$

$$< c_1 n \cdot \frac{\frac{5}{4}}{\frac{5}{4} - 1} \left(\frac{5}{4}\right)^{\lg n} + c_2 \cdot \frac{5}{5-1} \cdot 5^{\lg n} = c_1 n \cdot 5 \cdot \frac{5^{\lg n}}{n} + c_2 \cdot \frac{5}{4} \cdot 5^{\lg n} \Rightarrow O(n^{\log_4 5})$$

1c. $T(n) = 7T(n/7) + n$



level 0: $c_1 \cdot n + c_2$

level 1: $7(c_1 \cdot \frac{n}{7} + c_2) = c_1 \cdot n + 7c_2$

level 2: $49(c_1 \cdot \frac{n}{49} + c_2) = c_1 \cdot n + 49 c_2$

$\Rightarrow c_1 \cdot n + 7^i \cdot c_2$, lgn levels

$$T(n) = \sum_{i=0}^{\lg n} c_1 \cdot n + \sum_{i=0}^{\lg n} 7^i \cdot c_2 = c_1 \cdot n \cdot \sum_{i=0}^{\lg n} 1 + c_2 \sum_{i=0}^{\lg n} 7^i$$

$$< c_1 \cdot n \cdot \lg n + c_2 \cdot n \Rightarrow O(n \lg n)$$

Assignment 2 Cont.

1d. $T(n) = 9T(n/3) + n^2$

level 0: $c_1 \cdot n^2 + c_2$

level 1: $9(c_1(\frac{n}{3})^2 + c_2) = c_1 \cdot n^2 + 9c_2$

level 2: $81(c_1(\frac{n}{9})^2 + c_2) = c_1 \cdot n^2 + 81c_2$

$\Rightarrow c_1 \cdot n^2 + 9^i c_2$, lgn levels

$$T(n) = \sum_{i=0}^{\lg n} c_1 \cdot n^2 + \sum_{i=0}^{\lg n} 9^i \cdot c_2 \Rightarrow c_1 n^2 \sum_{i=0}^{\lg n} 1 + c_2 \sum_{i=0}^{\lg n} 9^i$$

$$< c_1 \cdot n^2 \cdot \lg n + c_2 \cdot n \Rightarrow O(n^2 \lg n)$$

0	n	1
1	$\frac{n}{3}$	9
2	$\frac{n}{9}$	81

1e. $T(n) = 8T(n/2) + n^3$

level 0: $c_1 \cdot n^3 + c_2$

level 1: $8(c_1 \cdot (\frac{n}{2})^3 + c_2) = c_1 \cdot n^3 + 8c_2$

level 2: $64(c_1 \cdot (\frac{n}{4})^3 + c_2) = c_1 \cdot n^3 + 64c_2$

$\Rightarrow c_1 \cdot n^3 + 8^i c_2$, lgn levels

$$T(n) = \sum_{i=0}^{\lg n} c_1 \cdot n^3 + \sum_{i=0}^{\lg n} 8^i c_2 \Rightarrow c_1 \cdot n^3 \sum_{i=0}^{\lg n} 1 + c_2 \sum_{i=0}^{\lg n} 8^i$$

$$< c_1 \cdot n^3 \cdot \lg n + c_2 \cdot n \Rightarrow O(n^3 \lg n)$$

0	n	1
1	$\frac{n}{2}$	8
2	$\frac{n}{4}$	64

1f. $T(n) = 49T(n/25) + n^{3/2} \lg n$

level 0: $c_1 \cdot n^{3/2} \lg n + c_2$

level 1: $49(c_1(\frac{n}{25})^{3/2} \lg(\frac{n}{25}) + c_2) = 49 \cdot c_1 \cdot \frac{n^{3/2}}{125} \cdot \lg(\frac{n}{25}) + 49c_2$

level 2: $2401(c_1 \cdot (\frac{n}{625})^{3/2} \lg(\frac{n}{625}) + c_2) = 2401 \cdot c_1 \cdot \frac{n^{3/2}}{15625} \cdot \lg(\frac{n}{625}) + 2401c_2$

$\Rightarrow 49^i \cdot c_1 \cdot (\frac{n}{25^i})^{3/2} \cdot \lg(\frac{n}{25^i}) + 49^i c_2$, lgn levels

$$\Rightarrow T(n) = \sum_{i=0}^{\lg n} 49^i \cdot c_1 \cdot \frac{n^{3/2}}{25^{3i/2}} \cdot \lg(\frac{n}{25^i}) + \sum_{i=0}^{\lg n} 49^i \cdot c_2$$

$$\Rightarrow c_1 \cdot n^{3/2} \cdot \left(\sum_{i=0}^{\lg n} \frac{49^i}{(25^{3/2})^i} \cdot (\lg n - \lg 25^i)\right) + c_2 \sum_{i=0}^{\lg n} 49^i \Rightarrow n$$

$\alpha < 1$ $\alpha < 1$

$$\Rightarrow c_1 \cdot n^{3/2} \left(\sum_{i=0}^{\lg n} \frac{49^i \cdot \lg n}{(25^{3/2})^i} - \frac{49^i \cdot i \lg 25}{(25^{3/2})^i}\right) \Rightarrow c_1 \cdot n^{3/2} \left(\lg n \cdot \sum_{i=0}^{\lg n} \left(\frac{49}{25^{3/2}}\right)^i - \lg 25 \sum_{i=0} \left(\frac{49}{25^{3/2}}\right)^i \cdot i\right.$$

$$< c_1 \cdot n^{3/2} \left(\lg n - \sum_{i=0}^{\lg n} i\right) \Rightarrow c_1 \cdot n^{3/2} \left(\lg n - \frac{\lg n(\lg n + 1)}{2}\right.$$

$\Rightarrow$

0	n	1
1	$\frac{n}{25}$	49
2	$\frac{n}{625}$	2401

Assignment 2 Cont.

1g. $T(n) = T(n-1) + 2$

level 0: $2c_1$

level 1: $2c_1$

level 2: $2c_1$

$\Rightarrow 2c_1$, n levels

$\Rightarrow T(n) = \sum_{i=0}^{n} 2c_1 = 2c_1 \cdot \sum_{i=0}^{n} 1 = 2c_1 \cdot n \Rightarrow O(n)$

0	n	1
1	$n-1$	1
2	$n-2$	1
3	$n-3$	1
	⋮	

1h. $T(n) = T(n-1) + n^c$, $c \geq 1$

level 0: $n^c \cdot c_1 + c_2$

level 1: $(n-1)^c \cdot c_1 + c_2$

level 2: $(n-2)^c \cdot c_1 + c_2$

$\Rightarrow (n-i)^c \cdot c_1 + c_2$, n levels

$\Rightarrow T(n) = \sum_{i=0}^{n} (n-i)^c c_1 + \sum_{i=0}^{n} c_2$

0	n	1
1	$n-1$	1
2	$n-2$	1
3	$n-3$	1
⋮	⋮	⋮
n	0	1

Brick

root: n^c

level 1: $(n-1)^c =$

1

level $n-1$: 1^c

↳ balanced

depth: n

cost at each level: $(n-i)^c$

$\Rightarrow n \cdot (n-i)^c < n \cdot n^c = n^{c+1}$

$\Rightarrow O(n^{c+1})$

1f. $T(n) = 49T(n/25) + n^{3/2} \lg n$

brick

root cost: $n^{3/2} \lg n$

children cost: $49\left(\left(\frac{n}{25}\right)^{3/2} \lg\left(\frac{n}{25}\right)\right) = \frac{49n^{3/2}}{125} \cdot \lg\left(\frac{n}{25}\right) = \frac{49n^{3/2}}{125}(\lg n - \lg 25)$

$\frac{49}{125} \cdot n^{3/2} \lg n - \frac{49}{125} \lg 25 < \frac{49}{125} n^{3/2} \lg n$ (b/c getting rid of a term making it less. also $\alpha < 1$) $\Rightarrow O(n^{3/2} \lg n)$

Assignment 2 Cont.

1i. $T(n) = T(\sqrt{n}) + 1$

level 0: c_1

level 1: c_1

level 2: c_1

$\Rightarrow c_1,$

$\Rightarrow$ balanced

cost at each level: 1

depth: $\lg\lg n$

$\Rightarrow 1 \cdot \lg\lg n \Rightarrow O(\lg\lg n)$

for $n > 1$,

$\sqrt{n} > 1$

$\Rightarrow$ base case: 2

$n^{\frac{1}{2^i}} = 2$

$n = 2^{2^i}$

$\lg n = 2^i$

$\lg\lg n = i$

0	n	1
1	$n^{1/2}$	1
2	$n^{1/4}$	1
3	$\vdots$	
4	$\vdots$	
i	$n^{\frac{1}{2^i}}$	

$\lg_2 2^{2^i} = 2^i \lg_2 2$

2a. $T(n) = 5T(n/2) + n$

root: n

children: $5(\frac{n}{2}) = \frac{5}{2}n$

$n \leq \frac{5}{2}n \Rightarrow$ leaf dom.

branching factor: 5

depth: $\lg n$

$\Rightarrow 5^{\lg n} = n^{\lg 5} \Rightarrow O(n^{\lg 5})$

2b. $T(n) = 2T(n-1) + 1$

root: 1

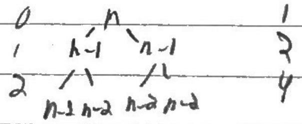
children: $1 + 1 = 2$

$1 \leq 2 \Rightarrow$ leaf dom

branching factor: 2

depth: n

$\Rightarrow 2^n \Rightarrow O(2^n)$



level 0: c_1

level 1: $2 \cdot c_1$

level 2: $4 \cdot c_1$

$\Rightarrow 2^i \cdot c_1$, n levels

$$\Rightarrow T(n) = \sum_{i=0}^{n} 2^i c_1 \Rightarrow c_1 \sum_{i=0}^{n} 2^i < c_1 \cdot \frac{2}{2-1} 2^n \Rightarrow O(2^n)$$

Assignment 2 Cont.

2c. $T(n) = 9T(n/3) + n^2$

root cost: n^2

children cost: $9C\left(\left(\frac{n}{3}\right)^2\right) = 9\left(\frac{n^2}{9}\right) = n^2$ } $\Rightarrow$ balanced

depth: $\lg n$ (base 3)

cost/level: n^2 } $\Rightarrow \boxed{O(n^2 \lg n)}$

~~I would choose algorithm A as it has the lowest~~

run~~time overall as $n \to \infty$. (verified by Desmos graph)~~

$$\lim_{n\to\infty} \frac{n^2 \lg n}{n^{\lg_2 5}} = \lim_{n\to\infty} \frac{n^2 \lg n}{n^2 \cdot n^{\lg_2 5 - 2}} = \lim_{n\to\infty} \frac{\lg n}{n^{\lg 5 - 2}} \overset{LH}{\Rightarrow} \lim_{n\to\infty} \frac{\frac{1}{\ln(2)\cdot n}}{(\lg 5 - 2)n^{\lg 5 - 3}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n\ln(2)} \cdot \frac{1}{(\lg 5 - 2)n^{\lg 5 - 3}} \Rightarrow 0$$ $\Rightarrow$ bottom grows faster $\Rightarrow$ bottom >> top

$\Rightarrow O(n^2 \lg n)$ is more efficient $\Rightarrow$ $\boxed{\text{Algorithm C}}$

Assignment 2 Cont.

3. SPARC

```
let
   type BinaryNumber = decimal_val = int
                       binary_vec = list[str]
   f(x,y) = if |x.binary_vec| == 1 and |y.binary_vec| == 1
       then (x.decimal_val * y.decimal_val)
   else
         (x_L, x_R) = splitMid(x)
         (y_L, y_R) = splitMid(y)
         (a,b) = (quad_mult(x_L, y_L), quad_mult(x_L, y_R))
         (c,d) = (quad_mult(x_R, y_L), quad_mult(x_R, y_R))
         n = len(x, y)
in
      int((2**n)*a + (2**(n/2))*(b+c) + d)
end
```