A Dynamic System is a system that chances over time

Markov Chain

Dynamic Systems evolving according to the Markov property: future evolution only depends on the current state X_t

$$x_0 \longrightarrow x_1 \dashrightarrow x_t \dashrightarrow x_T$$

dynamic Bayesian network that models a random process in which state variables are defined in terms of random variables

Bayesian network WITHOUT the Markov property $\longrightarrow$

$$\begin{tikzcd} x_0 \arrow[r] \arrow[rr, bend left] & x_1 \arrow[r, dashed] & x_t \arrow[r, dashed] \arrow[r, bend right] & x_T \end{tikzcd}$$

Markov Decision Process (MDP)

MDP is defined as a stochastic decision-making process that uses a mathematical framework to model the decision-making of a dynamic system in scenarios where the results are either random or controlled by a decision maker which makes sequential decisions over time. MDP evaluate which actions the decision maker should take considering the current state and environment of the system. In Artificial Intelligence they are used to design intelligent machines or agents that need to function longer in an environment where actions can yield uncertain results.

$$\begin{tikzcd} & x_1 \\ x_0 \arrow[ur, "a_1"] \arrow[dr, "a_2"'] & \\ & x_2 \end{tikzcd}$$

System that evolves over time and can be driven towards a goal state

We will instead consider a system that evolves over time but that cannot be driven towards some evolution: its evolution is not under our control.

Hidden Markov Models (HMM)

Another interesting case is when the state is not observable.

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x_1 \arrow[r, dashed] \arrow[d] & x_t \arrow[r, dashed] \arrow[d] & x_T \arrow[d] \\ z_0 & z_1 & z_t & z_T \end{tikzcd}$$

real state of the system NOT observable

the agent can observe these variables $\longrightarrow$ observations

hidden

We can't directly observe the state of the system but we can observe something that only depends on it.

Observations can be either discrete or continuous.
Evolution is not controlled by our system.

We don't have actions
We have observations

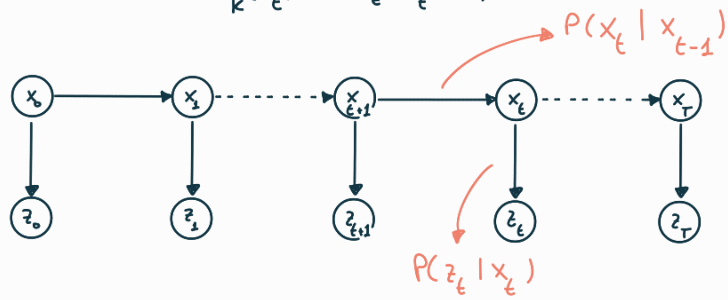
HMM formal definition

HMM = $\langle X, Z, \pi_0 \rangle$

- π_0 → initial probability distribution over the states
- Z → set of observations
- X → set of states

two functions :

- **transition model** — models the transition between states $\qquad A_{ij} = P(x_t = j \mid x_{t-1} = i)$
- **observation model** $\qquad b_k(z_t) = P(z_t \mid x_t = k)$



If we have a finite number of states, the transition model can be expressed by a discrete function.

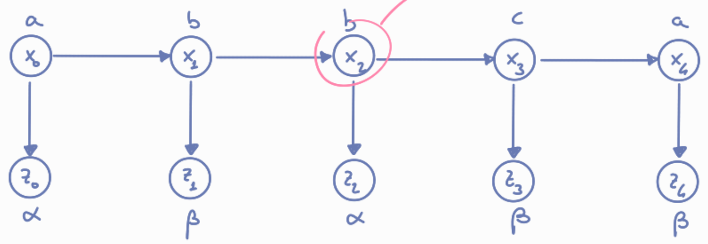
$$P(x_t = \underbrace{\{\dots\}}_{n} \mid x_{t-1} = \underbrace{\{\dots\}}_{n})$$

In this case P can be expressed by a matrix

e.g.

States $X : \{a, b, c\}$

→ at time $t=2$ the system has value b



$P(X_t = \{a, b, c\} \mid X_{t-1} = \{a, b, c\}$

	$x_t = a$	$x_t = b$	$x_t = c$
$x_t = a$	·	·	·
$x_t = b$	·	0.5	·
$x_t = c$	·	·	·

→ the probability that the system transitions from b to b is 0.5

The matrix has as many rows and as many columns as the number of states.

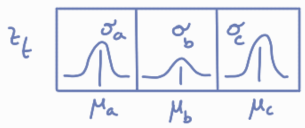
$Z = \{\alpha, \beta\}$

$P(z_t = \{\alpha, \beta\} \mid X_t = \{a, b, c\})$

$z_t = \alpha$	·	0.3	·
$z_t = \beta$	·	0.7	·

z can be a continuous variable

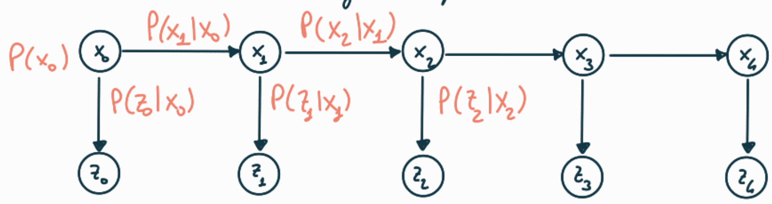
$$P(z_t \mid x_t = \{a, b, c\}) = N(x \mid \mu, \sigma)$$



The transition model is always represented as a matrix; The observation model can be represented either as a matrix or with a continuous function, possibly in a parametric form.

How do we solve the problem?

The application of the chain rule is very simple



$$P(x_{0:T}, z_{1:T}) = P(x_0)P(z_0|x_0)P(x_1|x_0)P(z_1|x_1)P(x_2|x_1)P(z_2|x_2)\ \dots$$

When you know, in addition to the set of states and the set of observations, also the transition function and the observation function, then applying the chain rule you can compute all the information

We have two problems (use cases) that can be easily modeled with HMM:

- filtering ⟶ given the past observations we are interested in estimating the current state
- smoothing ⟶ given all the observations we are interested in estimating a state in the past

Filtering: estimating the state of the system at time t given all the past observations

$$P(X_T = k \mid z_{1:T}) = \frac{\alpha_T^k}{\sum_j \alpha_T^j}$$

Smoothing: given all the observations collected so far we want to estimate what was the value of one state in the past.

$$P(x_t = k \mid z_{1:T}) = \frac{\alpha_t^k \beta_t^k}{\sum_j \alpha_t^j \beta_t^j}$$

Computing α is done with the forward algorithm:

$$\alpha_t^k = P(x_t = k, z_{1:t})$$

- for each state k do:

$$\alpha_0^k = \pi_0 b_k(z_0)$$

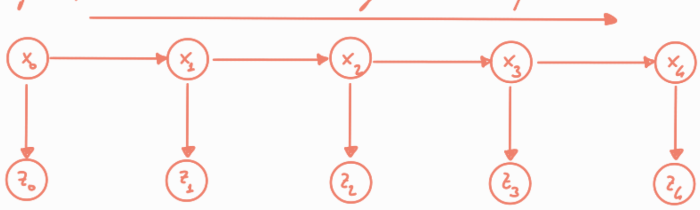
- for each time $t = 1, \dots, T$ do:

  for each state k do:

$$\alpha_t^k = b_k(z_t) \sum_j \alpha_{t-1}^j A_{jk}$$

transition matrix

this algorithm goes from one state to the subsequent one, starting from the initial state, and computes these terms for the current state given the previous



Computing β is done with the backward algorithm:

$$\beta_t^k = P(z_{t+1:T} \mid x_t = k)$$

- for each state k do:

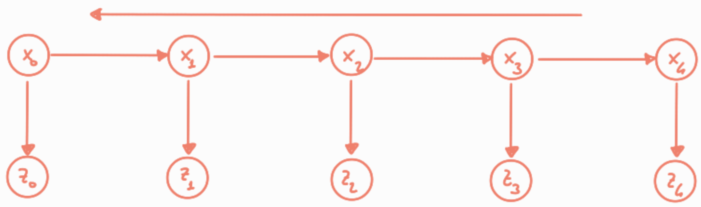
$$\beta_T^k = 1$$

- for each time $t = T-1, \dots, 1$ do:

  for each state k do:

$$\beta_t^k = \sum_j \beta_{t+1}^j A_{kj} b_j(z_{t+1})$$

observation model



Combining α and β we can solve the filtering and the smoothing problems under the assumption that we know the transition matrix and the observation model

What can we do when these information (transition matrix and observation model) are not known?

We have two cases:

- States can be observed at training time
  transition and observation models can be estimated with statistical analysis

probability of transition from i to j

$$A_{ij} = \frac{|\{i \to j \text{ transitions}\}|}{|\{i \to * \text{ transitions}\}|}$$

number of transitions from i to j

total number of transitions from i (to $*$)

probability of observing v given the state k

$$b_k(v) = \frac{|\{\text{observe } v \wedge \text{state } k\}|}{|\{\text{observe } * \wedge \text{state } k\}|}$$

number of events in which we observe v from state k

total number of events in which we observe anything and we are in state k

- States cannot be observed at training time
  Compute a local maximum likelihood with an Expectation-Maximization (EM) method (e.g. Baum-Welch algorithm)