Chapter 6

Probability Plotting

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Chapter 6
Probability Plotting

Topics discussed in this chapter are:

- The **purposes** of probability plots.
- The basic **concepts** of probability plotting.
- How to **linearize** a cdf by using special plotting scales.
- How to plot a nonparametric estimate $\widehat{F}$ to judge the adequacy of a particular parametric distribution.
- Using probability plots to obtain **graphical** estimates of reliability characteristics like failure probabilities and quantiles.
- Methods of separating **useful** information from **noise** when interpreting a probability plot.

Chapter 6

Segment 1

Purposes of Probability Plots and Linearizing a cdf

Purposes of Probability Plots

Probability plots are used to:

- Assess the adequacy of a particular distributional model.
- Detect multiple failure modes or mixture of different populations.
- Display the results of a parametric maximum likelihood fit along with the data.
- Obtain, by drawing a smooth curve through the points, a semiparametric estimate of failure probabilities and distributional quantiles.
- Obtain graphical estimates of model parameters (e.g., by fitting a straight line through the points on a probability plot).

Probability Plotting Scales: Linearizing a cdf

Main Idea: For a given cdf, $F(t)$, one can **linearize** the $\{ \ t$ versus $F(t) \ \}$ plot by:

- Finding transformations of $F(t)$ and t such that the relationship between the transformed variables is linear.
- The transformed axes are relabeled in terms of the original probability and time variables.

  The resulting probability axis is generally nonlinear and is called the **probability** scale. The data axis is usually a linear axis or a log axis.

Linearizing the Exponential cdf

cdf: $\quad p = F(t;\theta,\gamma) = 1 - \exp\left[-\frac{(t-\gamma)}{\theta}\right], \quad t \geq \gamma.$

Quantiles : $\quad t_p = \gamma - \theta \log(1-p).$

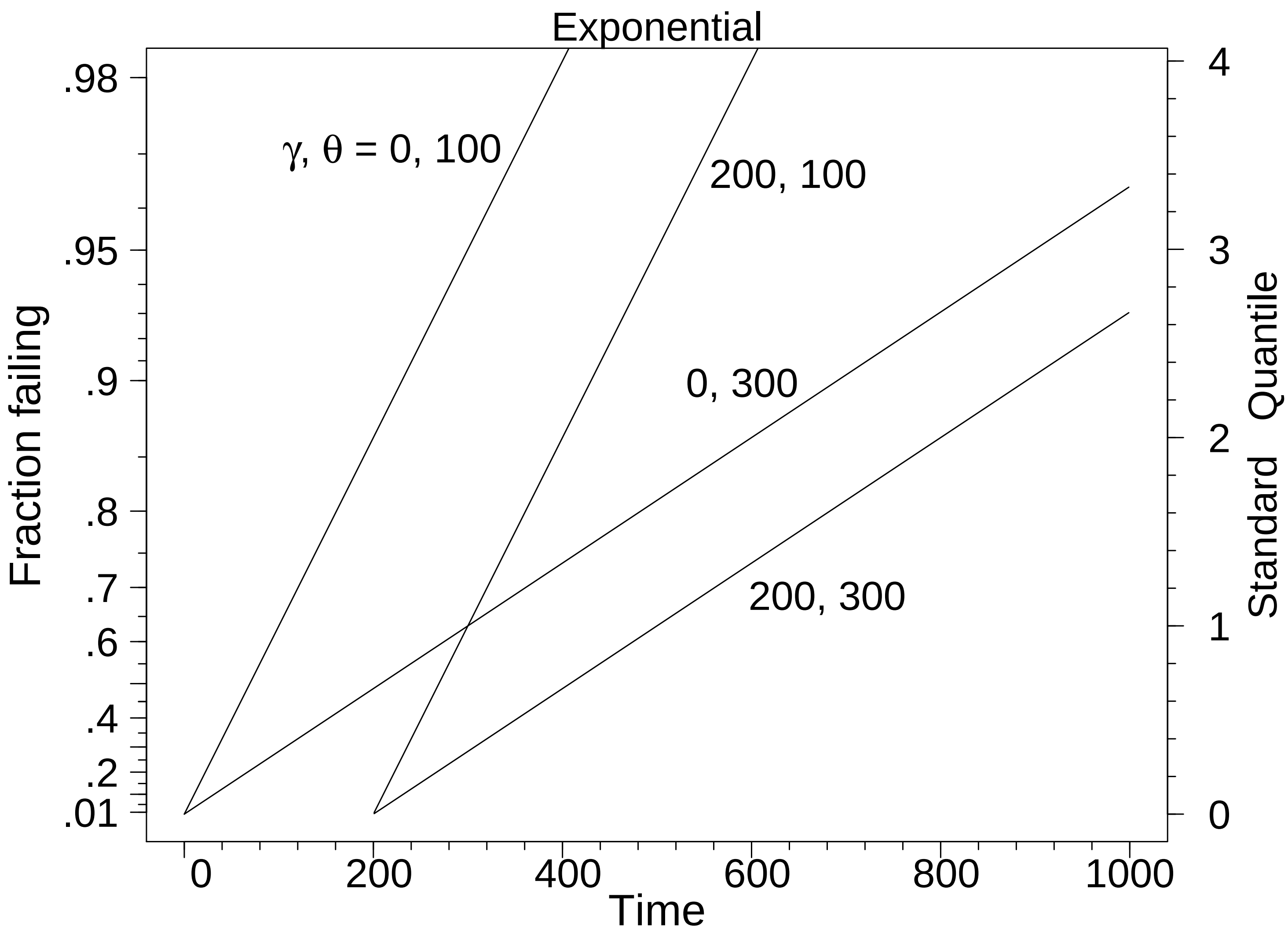
Conclusion:
The $\{\ t_p$ versus $-\log(1-p)\ \}$ plot is a straight line (the cdf line).

We plot t_p on the horizontal axis and p on the vertical axis. γ is the **intercept** on the time axis and $1/\theta$ is equal to the slope of the cdf line.

Note:
Changing θ changes the slope of the line and changing γ changes the position of the line.

Plot with Exponential Distribution Probability Scales Showing Exponential cdfs as Straight Lines for Combinations of Parameters $\theta = 100, 300$ and $\gamma = 0, 200$

$$t_p = \gamma - \theta \log(1 - p)$$



Linearizing the Normal Distribution cdf

cdf: $p = F(y; \mu, \sigma) = \Phi_{\text{norm}}\left(\frac{y-\mu}{\sigma}\right), \quad -\infty < y < \infty.$

Quantiles : $y_p = \mu + \sigma\Phi^{-1}_{\text{norm}}(p).$

$\Phi^{-1}_{\text{norm}}(p)$ is the p quantile of the standard normal distribution.

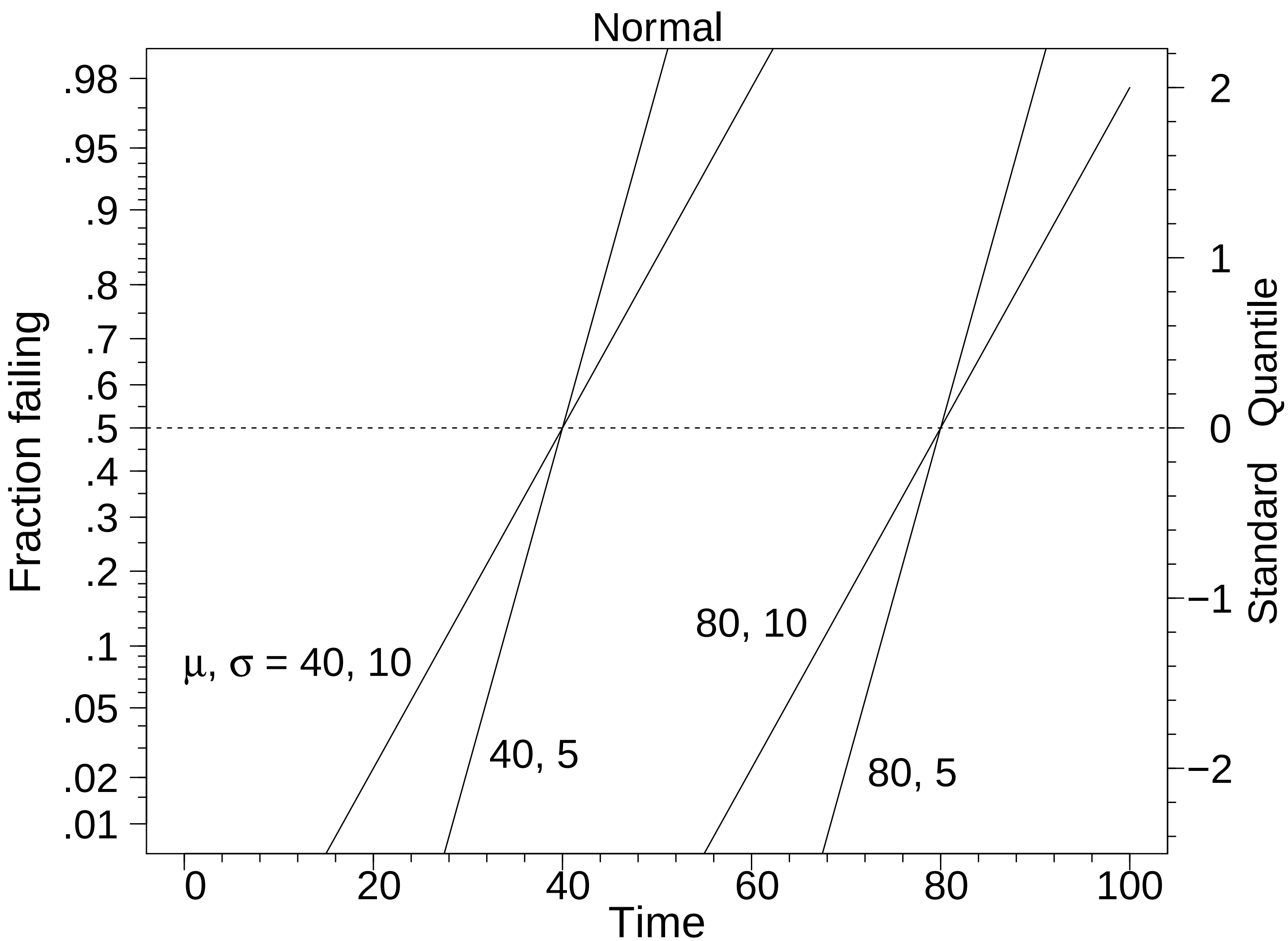
Conclusion:
$\{\ y_p$ versus $\Phi^{-1}_{\text{norm}}(p)\ \}$ will plot as a straight line.

μ is the point at the time axis where the cdf intersects the $\Phi^{-1}(p) = 0$ line (i.e., $p = 0.5$). The slope of the cdf line on the graph is $1/\sigma$.

Note:
Any normal distribution cdf plots as a straight line with a positive slope. Also, any straight line with positive slope corresponds to a normal cdf.

Plot with Normal Distribution Probability Scales Showing Normal distribution cdfs as Straight Lines for Combinations of Parameters $\mu = 40, 80$ and $\sigma = 5, 10$

$$y_p = \mu + \sigma \Phi^{-1}_{\textbf{norm}}(p)$$



Linearizing the Lognormal Distribution cdf

cdf: $$p = F(t;\mu,\sigma) = \Phi_{\text{norm}}\left[\frac{\log(t)-\mu}{\sigma}\right], \quad t > 0.$$

Quantiles : $$t_p = \exp\left[\mu + \sigma\Phi^{-1}_{\text{norm}}(p)\right].$$

Then $\log(t_p) = \mu + \sigma\Phi^{-1}_{\text{norm}}(p)$

Conclusion:

$\{ \log(t_p)$ versus $\Phi^{-1}_{\text{norm}}(p) \}$ will plot as a straight line.

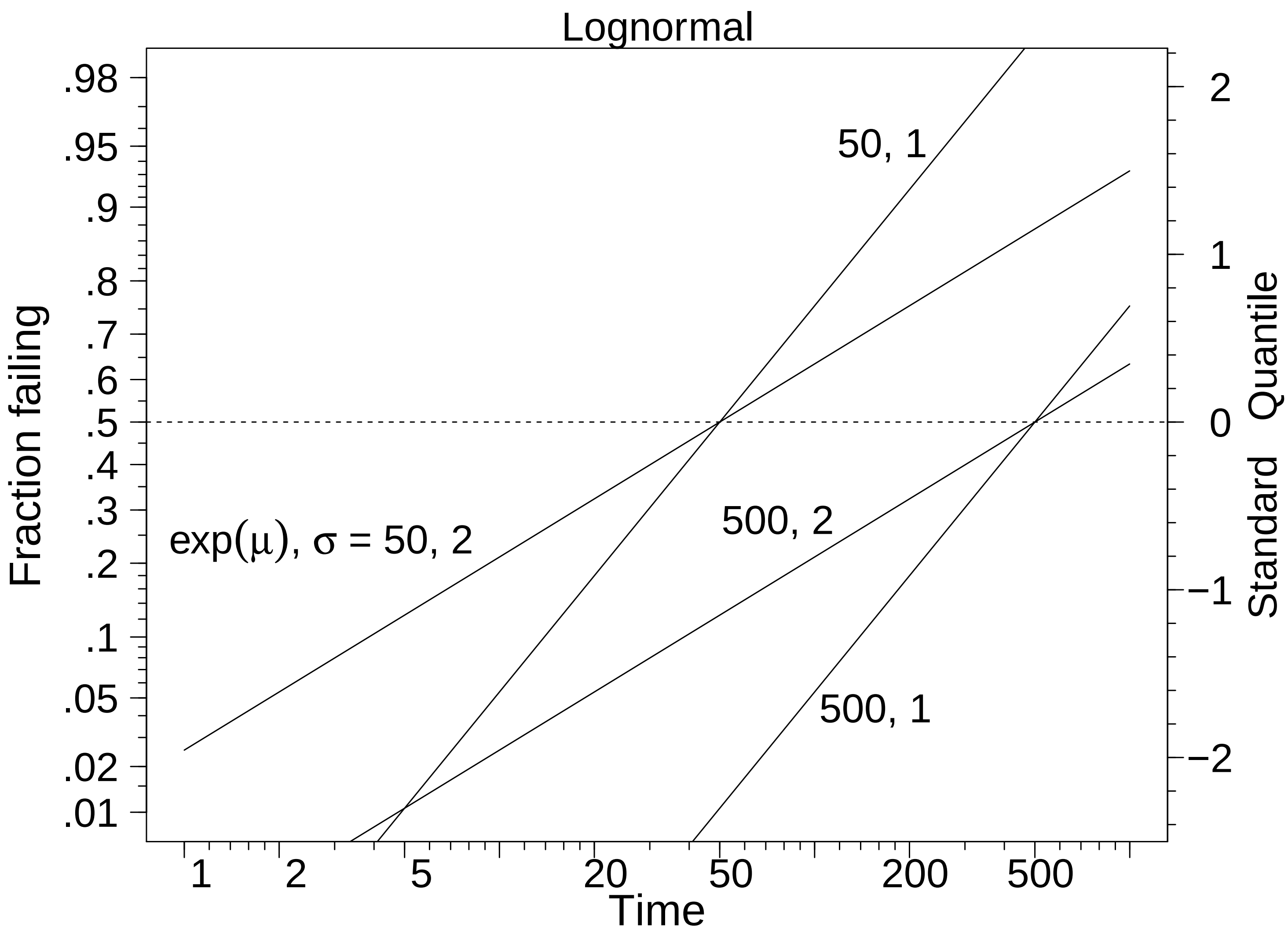
The median $\exp(\mu)$ can be read from the time axis at the point where the cdf intersects the horizontal line $\Phi^{-1}_{\text{norm}}(p) = 0$ (which corresponds to $p = 0.50$). The slope of the cdf line on the graph is $1/\sigma$ (but in the computations use base e logarithms for the times rather than the base 10 logarithms shown on the figures).

Note:

Any lognormal distribution cdf plots as a straight line with a positive slope. Also, any straight line with positive slope corresponds to a lognormal distribution.

Plot with Lognormal Distribution Probability Scales Showing Lognormal Distribution cdfs as Straight Lines for Combinations of $\exp(\mu) = 50, 500$ and $\sigma = 1, 2$

$$\log(t_p) = \mu + \sigma \Phi^{-1}_{\textbf{norm}}(p)$$



Linearizing the Weibull Distribution cdf

cdf: $\quad p = F(t;\mu,\sigma) = \Phi_{\text{sev}}\left[\frac{\log(t)-\mu}{\sigma}\right], \quad t > 0.$

Quantiles : $\quad t_p = \exp\left[\mu + \sigma \Phi_{\text{sev}}^{-1}(p)\right] = \eta[-\log(1-p)]^{1/\beta},$

where $\Phi_{\text{sev}}^{-1}(p) = \log[-\log(1-p)]$, $\eta = \exp(\mu)$, $\beta = 1/\sigma$.

This leads to

$$\log(t_p) = \mu + \sigma \log[-\log(1-p)] = \log(\eta) + \frac{1}{\beta}\log[-\log(1-p)]$$

Conclusion:

$\{\ \log(t_p)$ versus $\log[-\log(1-p)]\ \}$ will plot as a straight line.

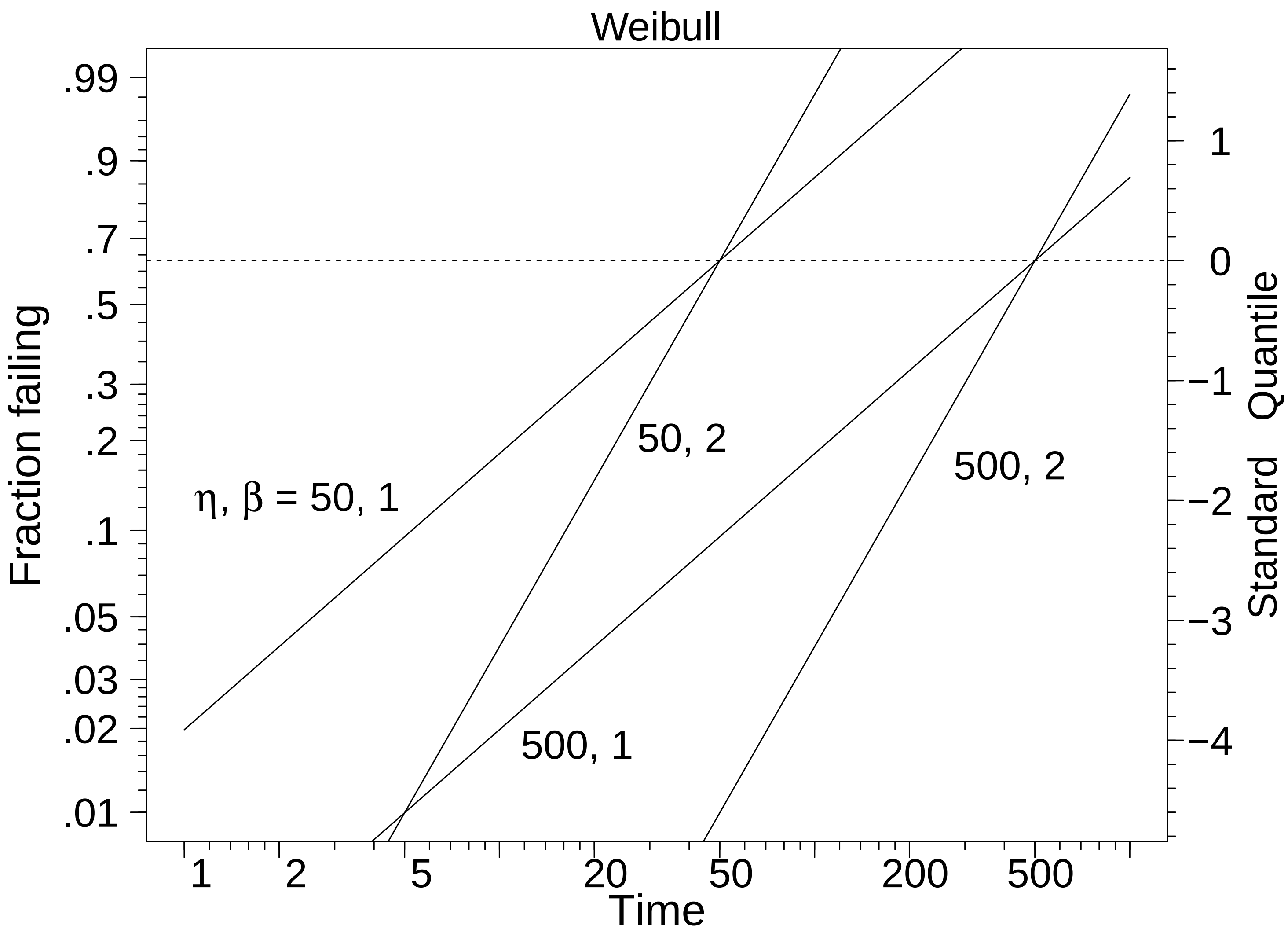
Linearizing the Weibull Distribution cdf-Continued

Comments:

- $\eta = \exp(\mu)$ can be read from the time axis at the point where the cdf intersects the horizontal $\log[-\log(1-p)] = 0$ line, which corresponds to $p \approx 0.632$.
- The slope of the cdf line on the graph is $\beta = 1/\sigma$ (but in the computations use base e logarithms for the times rather than the base 10 logarithms used for the figures).
- Any Weibull distribution cdf plots as a straight line with a positive slope. And any straight line with positive slope corresponds to a Weibull distribution cdf.
- Exponential distribution cdfs plot as straight lines with slopes equal to 1.

Plot with Weibull Distribution Probability Scales Showing Weibull cdfs as Straight Lines for Combinations of $\eta = 50, 500$ and $\beta = 1, 2$

$$\log(t_p) = \log(\eta) + \frac{1}{\beta}\log[-\log(1-p)]$$



Chapter 6

Segment 2

Choice of Plotting Positions

Choosing Plotting Positions to Plot the Nonparametric Estimate of F

- The **discontinuity** and **randomness** of $\widehat{F}(t)$ make it difficult to choose a definition for pairs of points $(t, \widehat{F})$ to plot.
- **General Idea:** Plot an estimate of F at some specified set of points in time and define **plotting** positions consisting of a corresponding estimate of F at these points in time.
- With times reported as **exact**, it has been traditional to plot $\{\ t_i$ versus $\widehat{F}(t_i)\ \}$ at the observed failure times.

Criteria for Choosing Plotting Positions

Criteria for choosing plotting positions should depend on the **application** or **purpose** for constructing the probability plot.

Some applications that suggest criteria:

- Checking distributional assumptions.
- Display and comparison of maximum likelihood estimates of a parametric distribution with the data.
- Estimation of parameters.

Plotting Positions: Continuous Inspection Data and Single Censoring

Let $t_{(1)}, t_{(2)}, \ldots$ be the ordered failure times with no ties. When there are not ties, $\widehat{F}(t)$ is a step function increasing by an amount $1/n$ until the last reported failure time.

Plotting Positions: $\left\{ t_i \text{ versus } \frac{i-0.5}{n} \right\}$.

- **Justification:**

$$\frac{i-0.5}{n} = \frac{1}{2}\left\{\widehat{F}\left[t_{(i)}\right] + \widehat{F}\left[t_{(i-1)}\right]\right\}$$

$$\mathrm{E}\left[t_{(i)}\right] \approx F^{-1}\left(\frac{i-0.5}{n}\right).$$

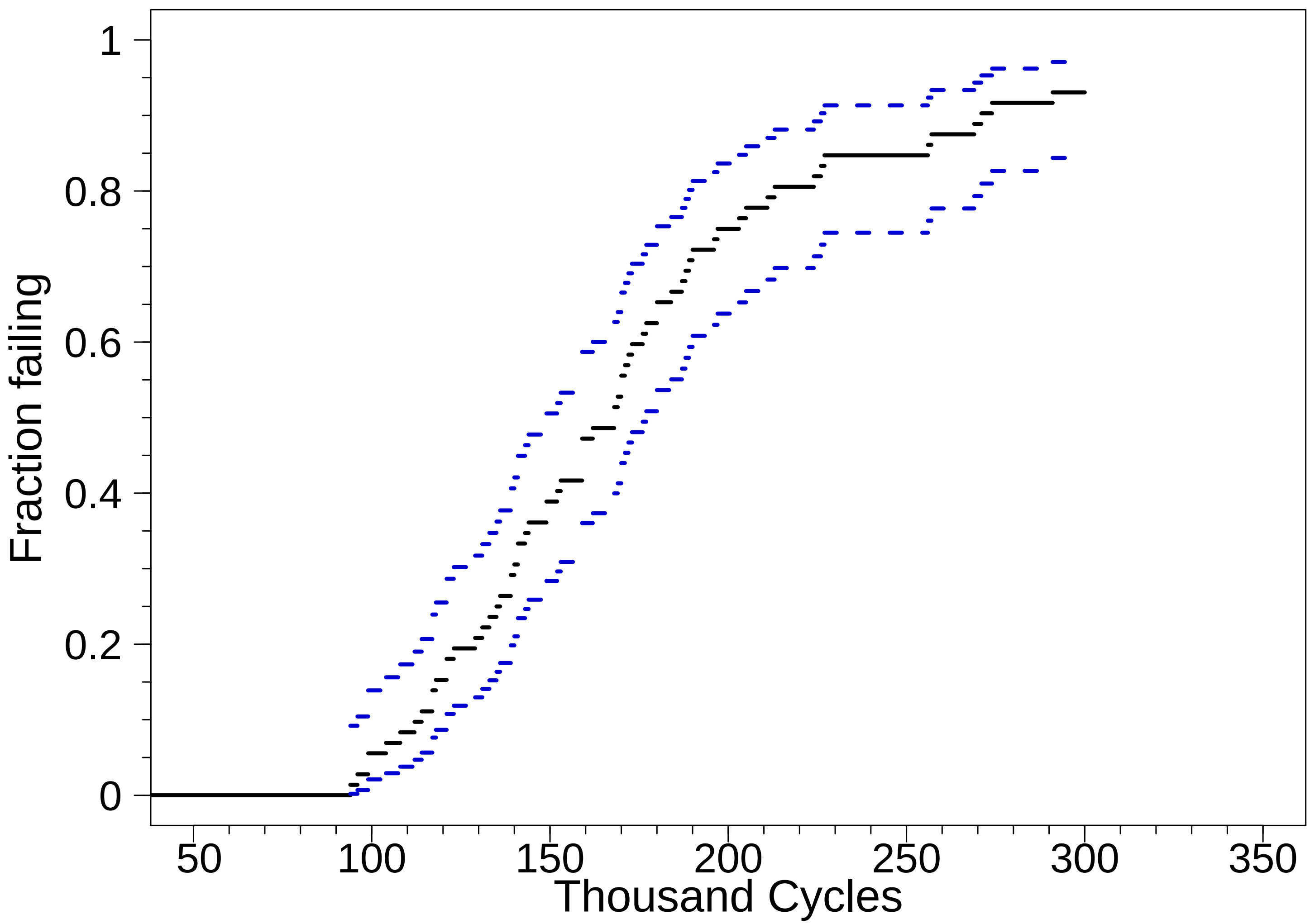
- A simple modification is required if there are ties in the reported failure times.

Chapter 6

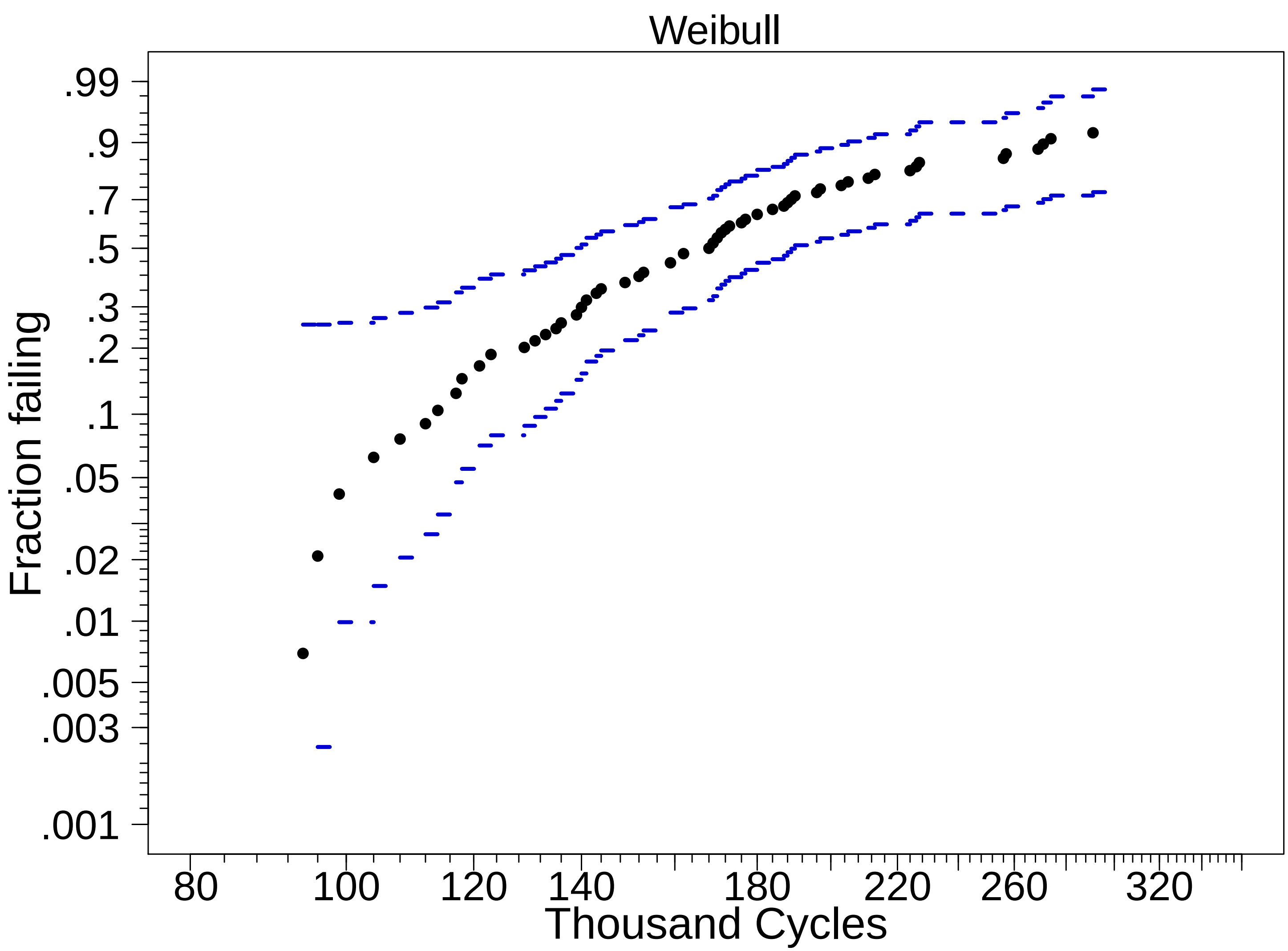
Segment 3

The Alloy T7987 and Heat-Exchanger Tube Crack Examples

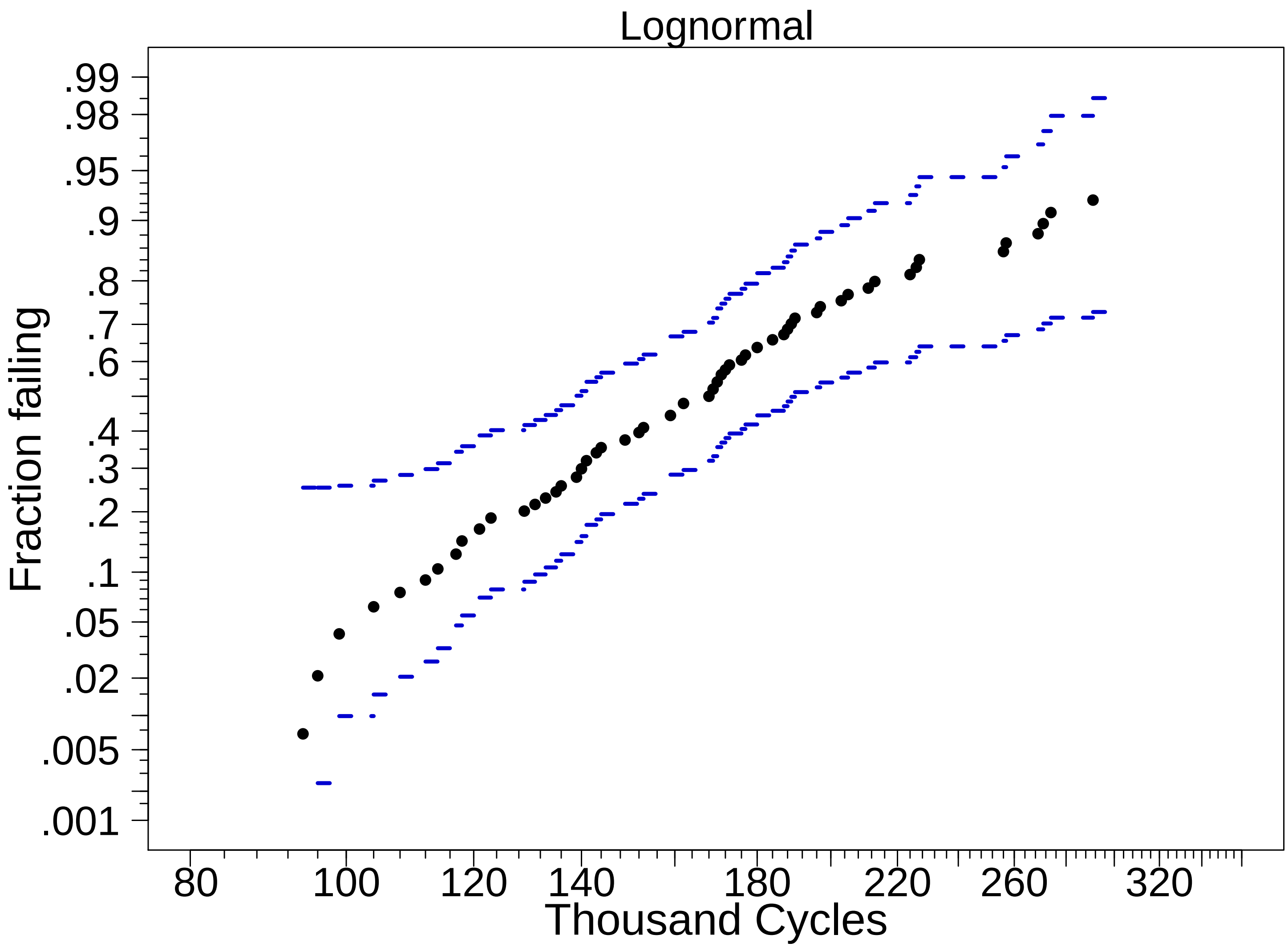
Plot of Nonparametric Estimate of $F(t)$ for the Alloy T7987 Fatigue Life and Simultaneous Approximate 95% Confidence Bands



Weibull Probability Plot for the Alloy T7987 Fatigue Life and Simultaneous Approximate 95% Confidence Bands for $F(t)$



Lognormal Probability Plot for the Alloy T7987 Fatigue Life and Simultaneous Approximate 95% Confidence Bands for $F(t)$



Plotting Positions: Continuous Inspection Data and Multiple Censoring

$\widehat{F}(t)$ is a step function until the last reported failure time, but the step increases may be different than $1/n$.

Plotting Positions: $\{\, t_{(i)}$ versus $p_i\}$ with

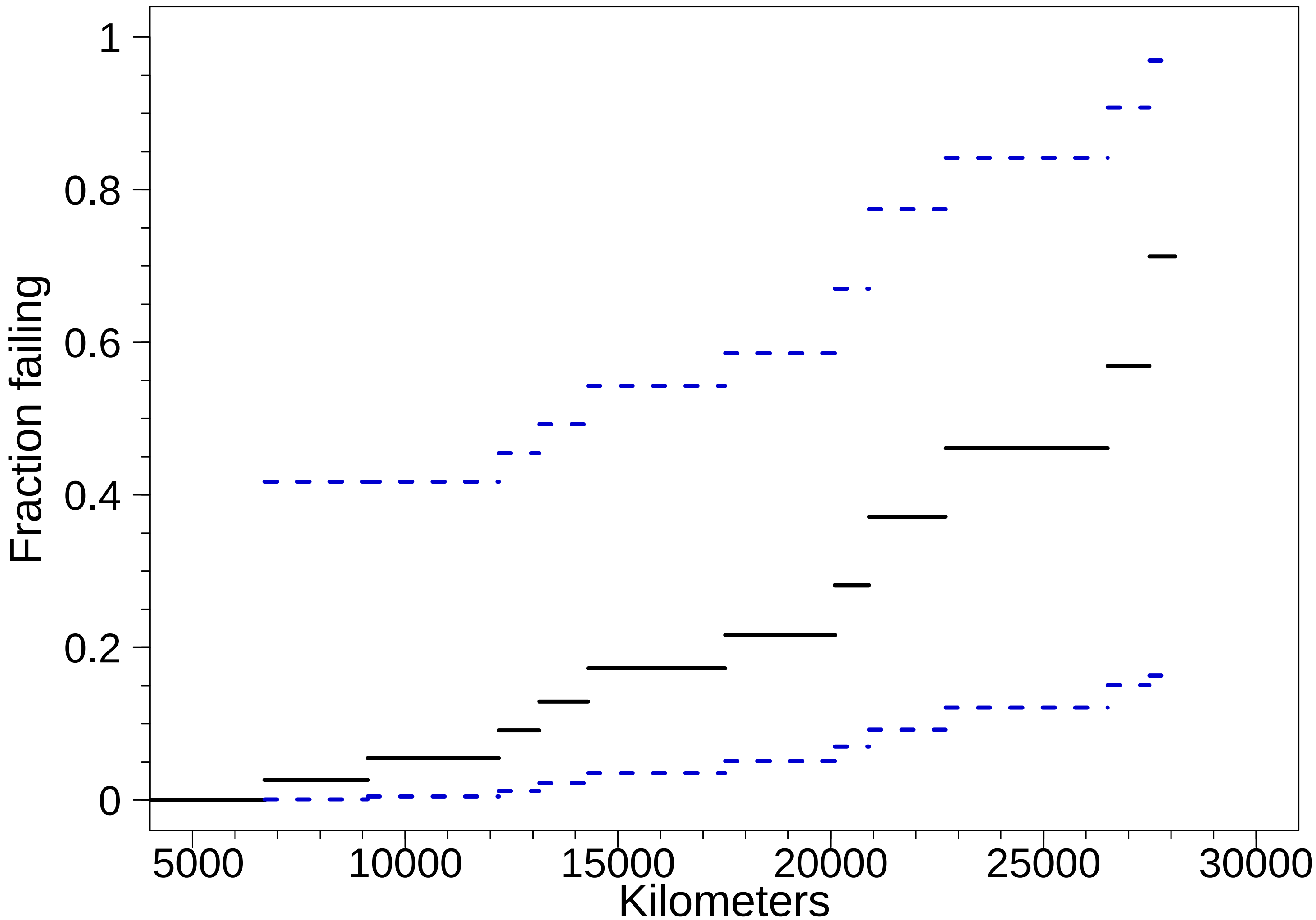
$$p_i = \frac{1}{2}\Big\{ \widehat{F}\Big[t_{(i)}\Big] + \widehat{F}\Big[t_{(i-1)}\Big]\Big\}.$$

- **Justification:** This is consistent with the commonly-used definition for single censoring.

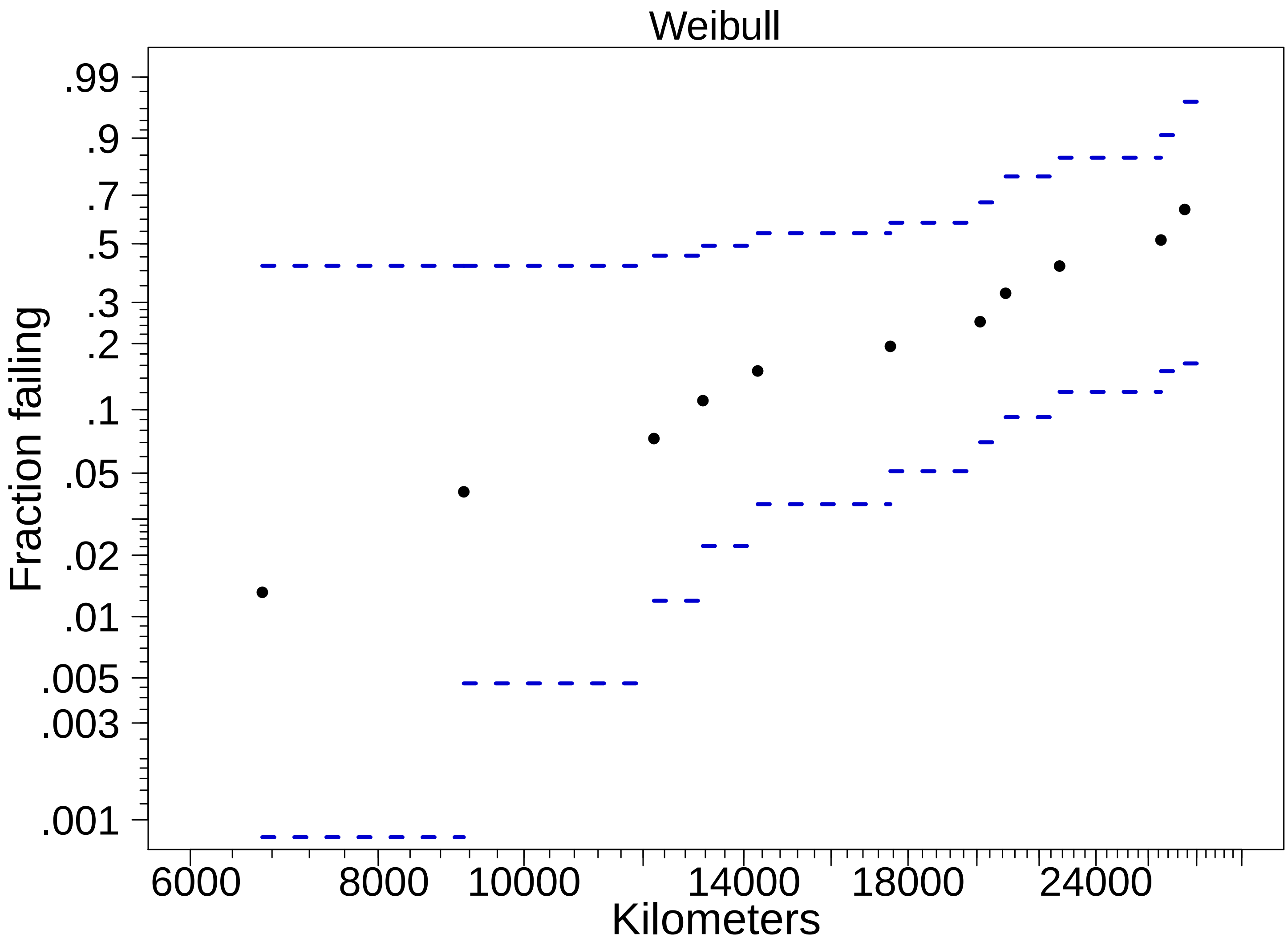
- When the model fits well, the ML line approximately goes through the points.

- Need to adjust these plotting positions when there are ties.

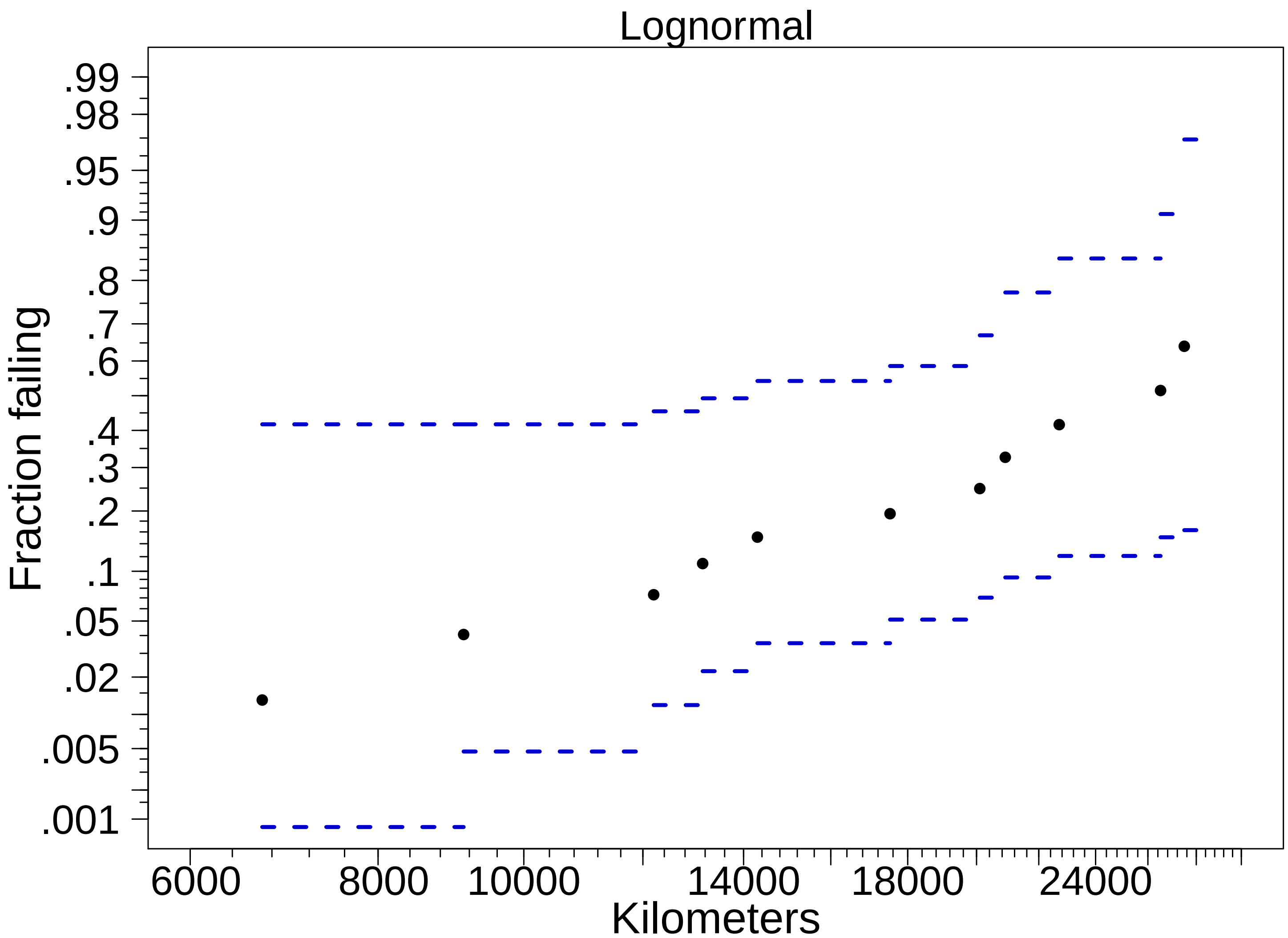
Nonparametric Estimate of $F(t)$ for the Shock Absorbers. Simultaneous Approximate 95% Confidence Bands for $F(t)$



Weibull Probability Plot of the Shock Absorber Data. Also Shown are Simultaneous Approximate 95% Confidence Bands for $F(t)$



Lognormal Probability Plot of the Shock Absorber Data. Also Shown are Simultaneous Approximate 95% Confidence Bands for $F(t)$



Plotting Positions: Interval-Censored Inspection Data

- Let $(t_0, t_1], \ldots, (t_{m-1}, t_m]$ be the inspection times.

- The upper endpoints of the inspection intervals t_i, $i = 1, 2, \ldots$, are convenient plotting times.

- **Plotting Positions:** $\{\, t_i$ versus $p_i \,\}$ with

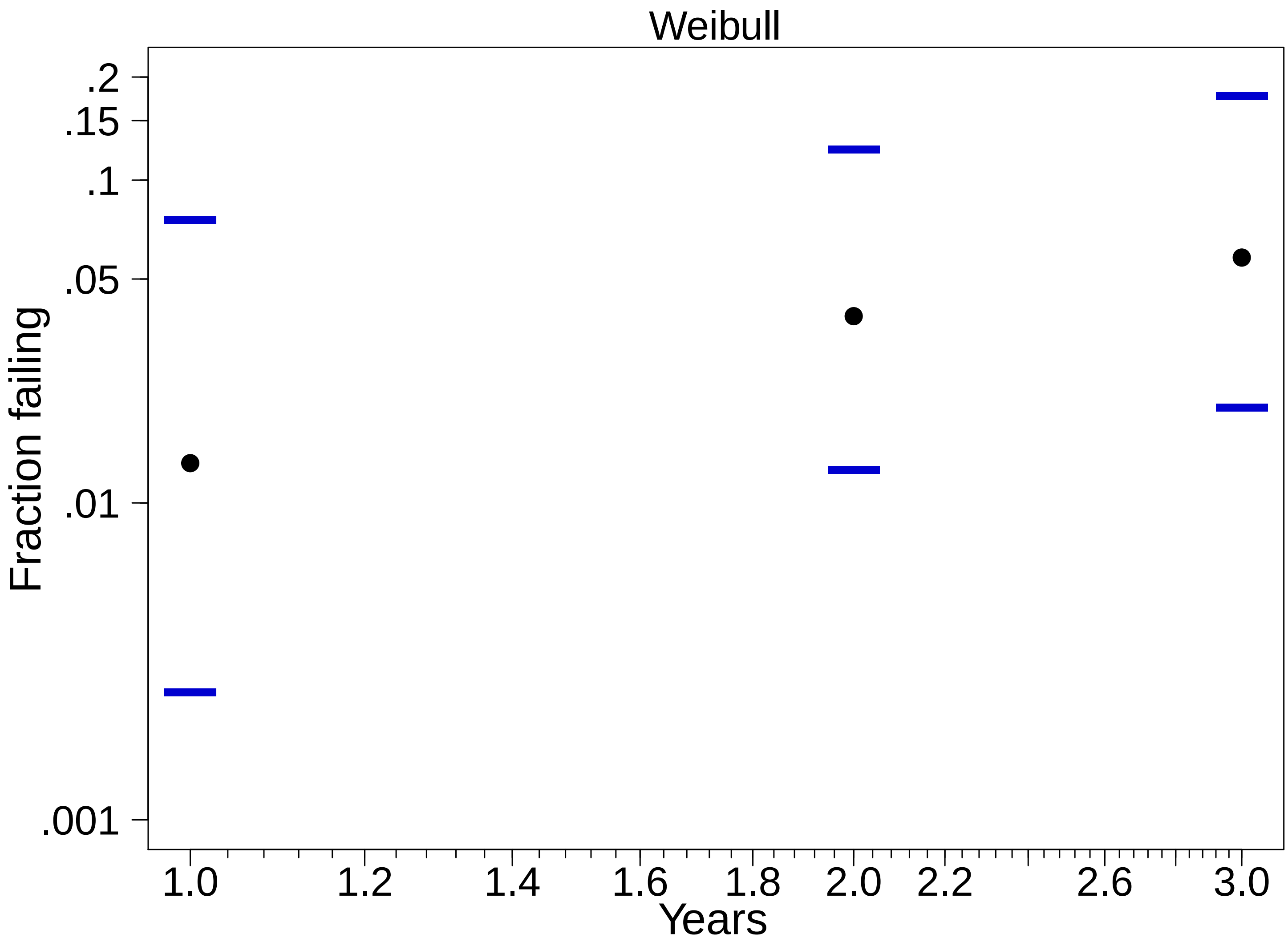
$$p_i = \widehat{F}(t_i)$$

- When there are no censored observations beyond t_m, $F(t_m) = 1$ and this point cannot be plotted on probability paper.

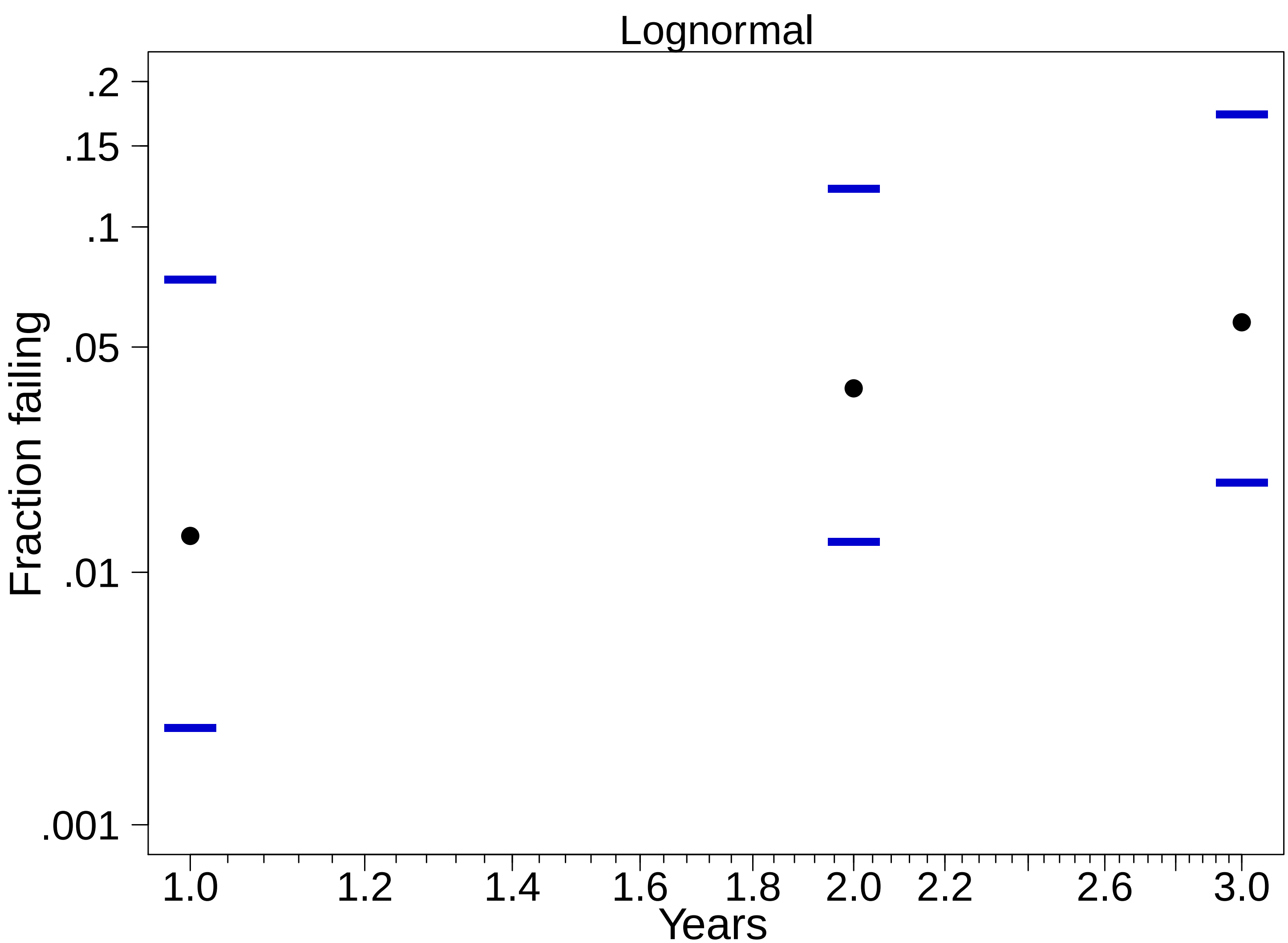
- **Justification:** with single censoring, from standard binomial theory,

$$\mathrm{E}[\widehat{F}(t_i)] = F(t_i).$$

Weibull Distribution Probability Plot of the Heat-Exchanger Tube Crack Data and Simultaneous Approximate 95% Confidence Bands for $F(t)$



Lognormal Distribution Probability Plot of the Heat-Exchanger Tube Crack Data and Simultaneous Approximate 95% Confidence Bands for $F(t)$

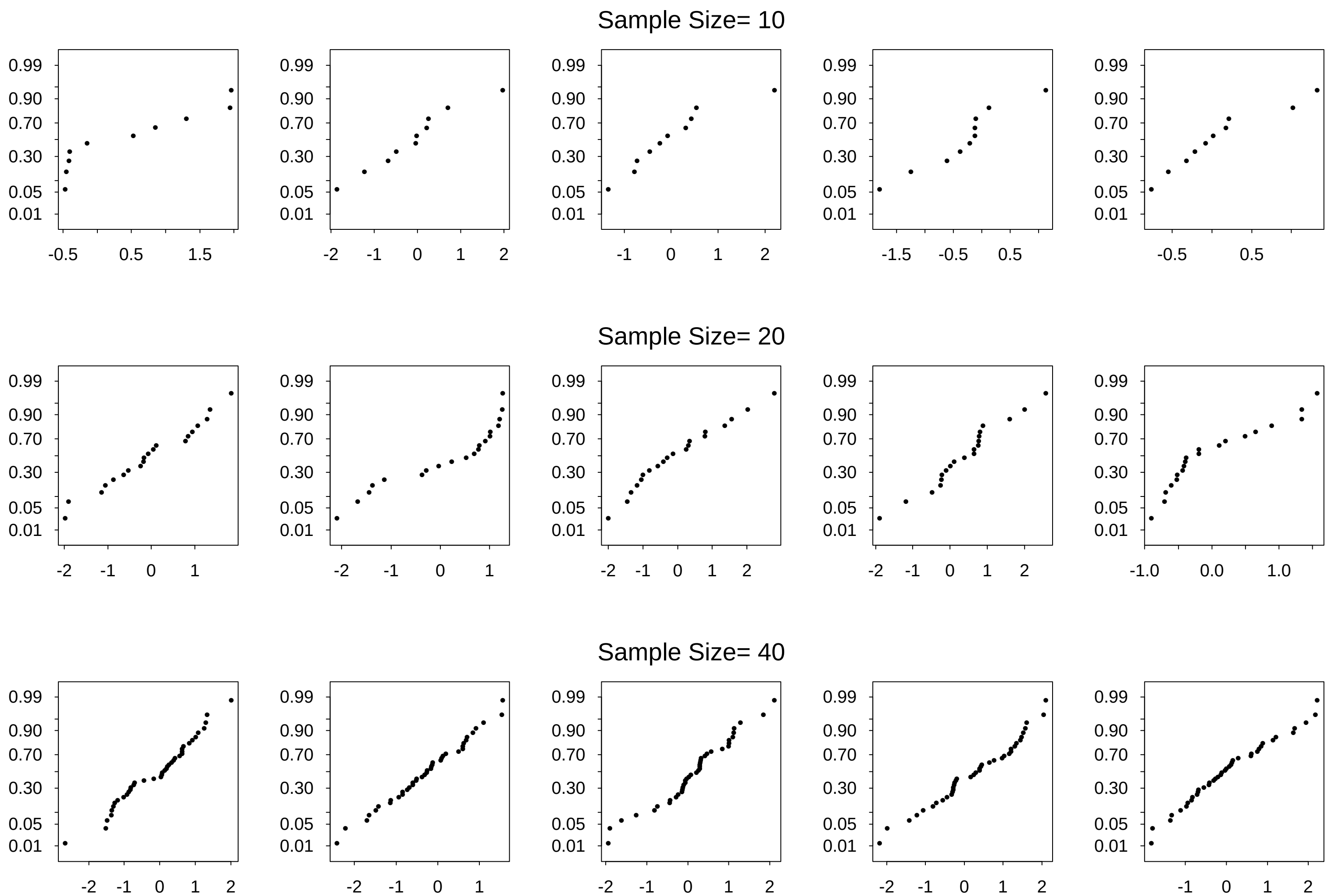


Chapter 6

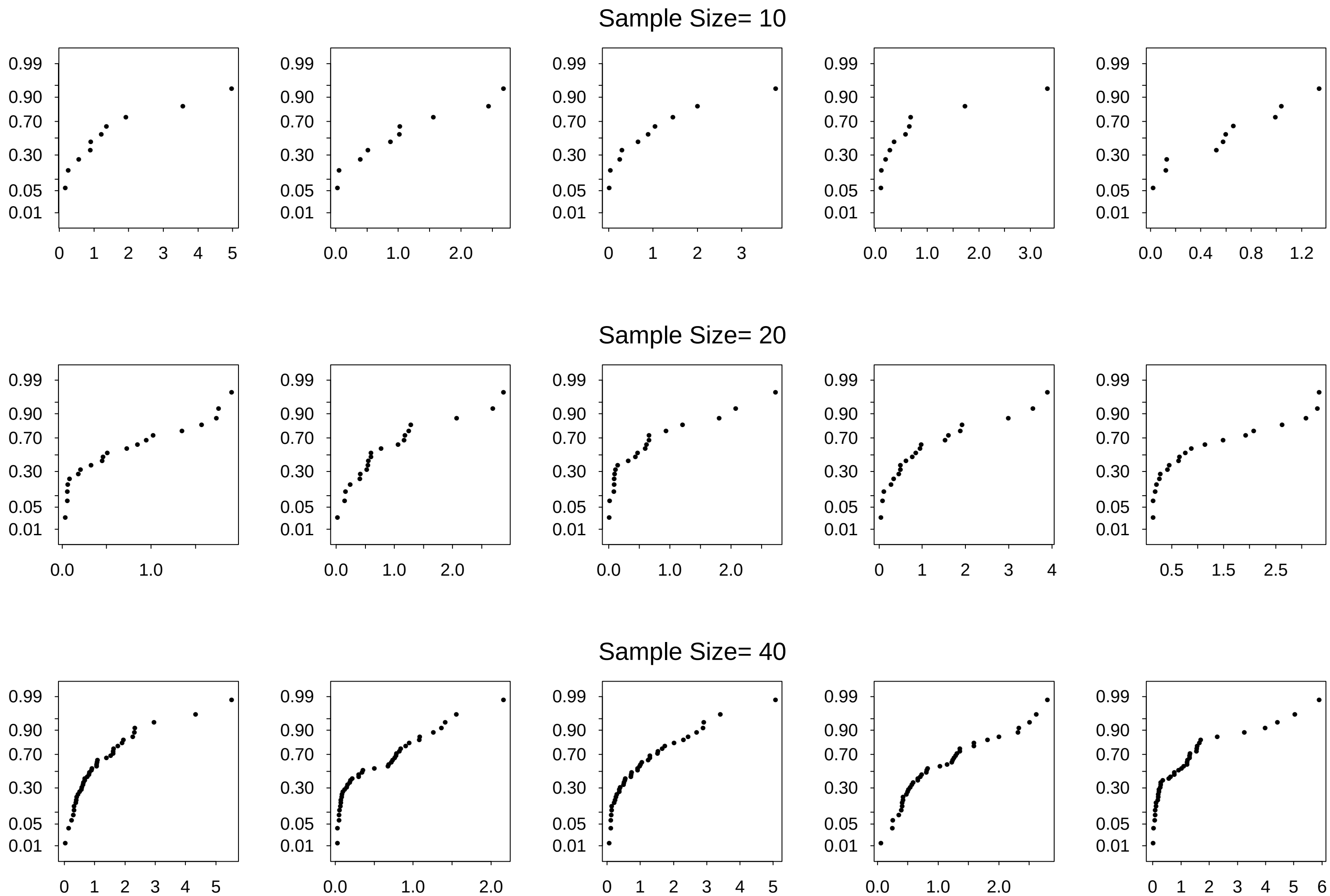
Segment 4

Using Simulation to Calibrate Interpretation of Probability Plots

Random Normal Variates Plotted on Normal Probability Plots with Sample Sizes of n=10, 20, and 40. Five Replications of Each Probability Plot



Random Exponential Variates Plotted on Normal Probability Plots with Sample Sizes of n=10, 20, and 40. Five Replications of Each Probability Plot



Notes on the Application of Probability Plotting

- Try different assumed distributions and compare the results.
- Assess linearity, allowing for more variability in the tails.
- To help calibrate, use
  - Simultaneous nonparametric confidence bands.
  - Simulation or bootstrap.
- A sharp bend or change in slope in a probability plot generally indicates the appearance of a different failure mode (different than the early failures).

Chapter 6

Segment 5

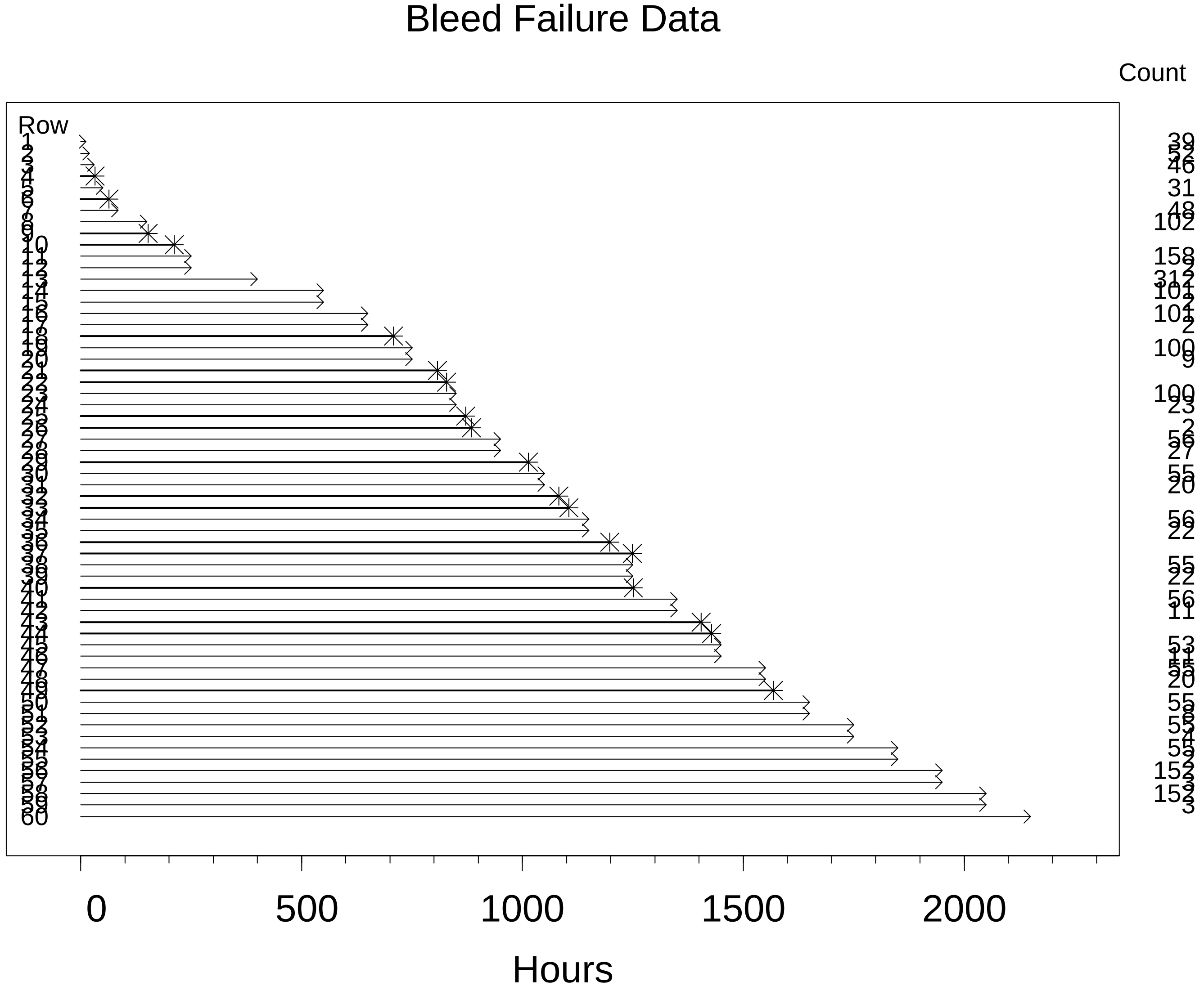
Bleed System Example
Segmenting Data to Explain Variability

Transmitter Vacuum Tube Example
Probability Plots with Enhancements

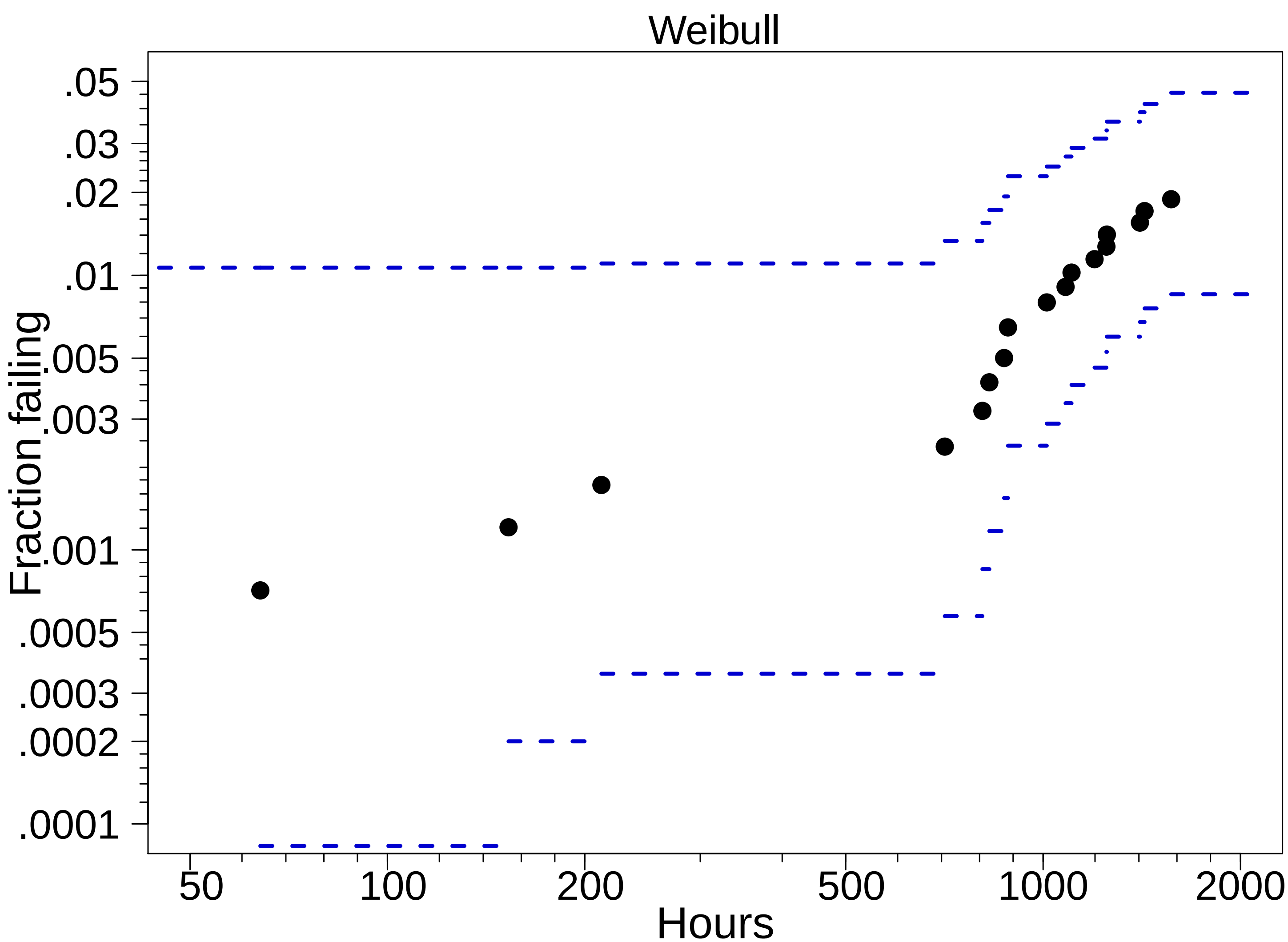
Jet Engine Bleed System Failures

- Data from the Weibull Handbook Abernethy et al. (1983).
- Field data from 2256 systems in the field; staggered entry–multiple censoring.
- Unexpected failures.
- What is going on?
- The Weibull probability plot suggests changes in the failure distribution after 600 hours.

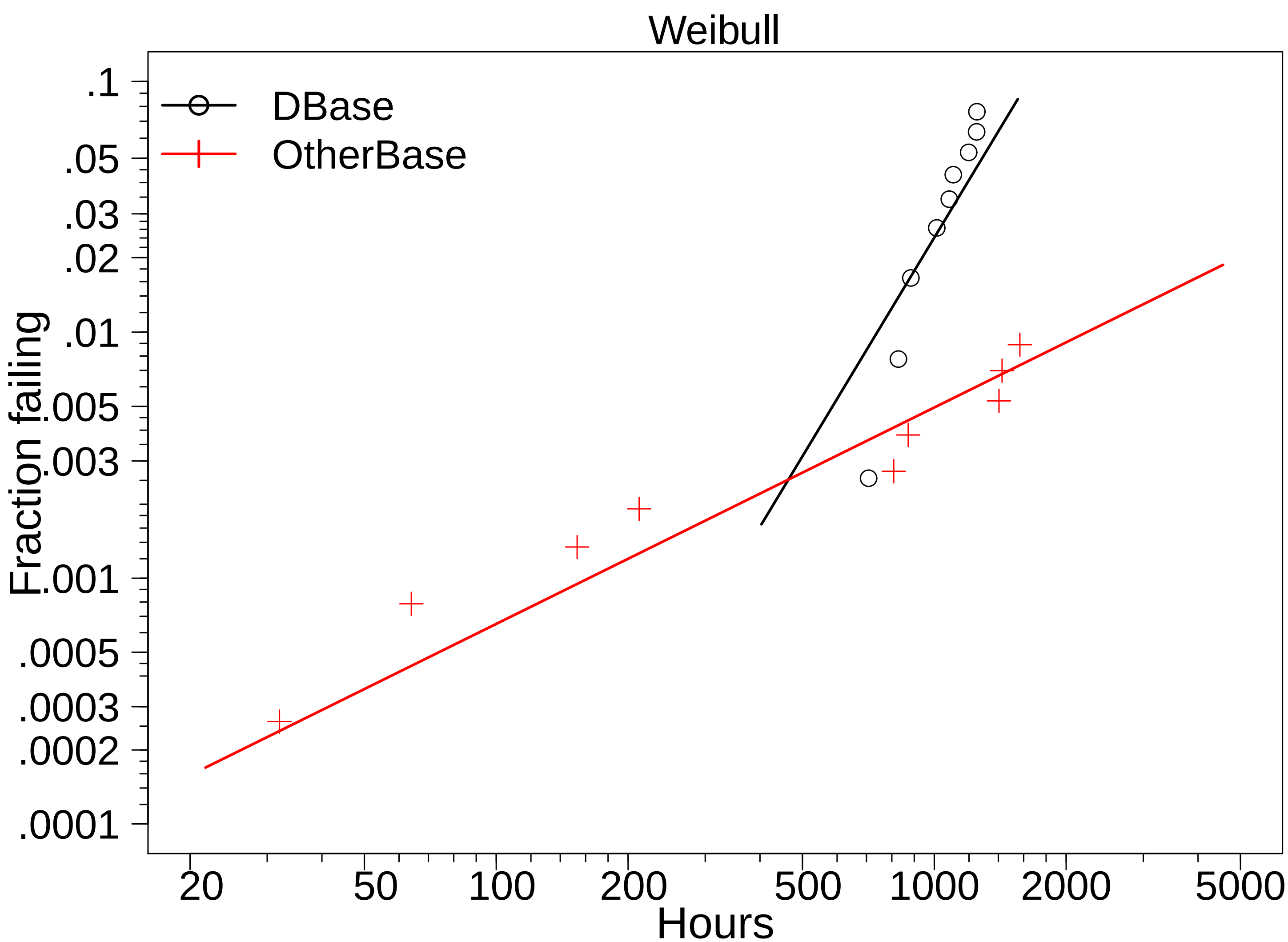
Bleed System
Event Plot
All Bases



Bleed System
Weibull Probability Plot
All Bases



Bleed System
Weibull Probability Plot
Separate Estimatess for Base D and Other Bases



Bleed System
Failure Data Analysis-Conclusions

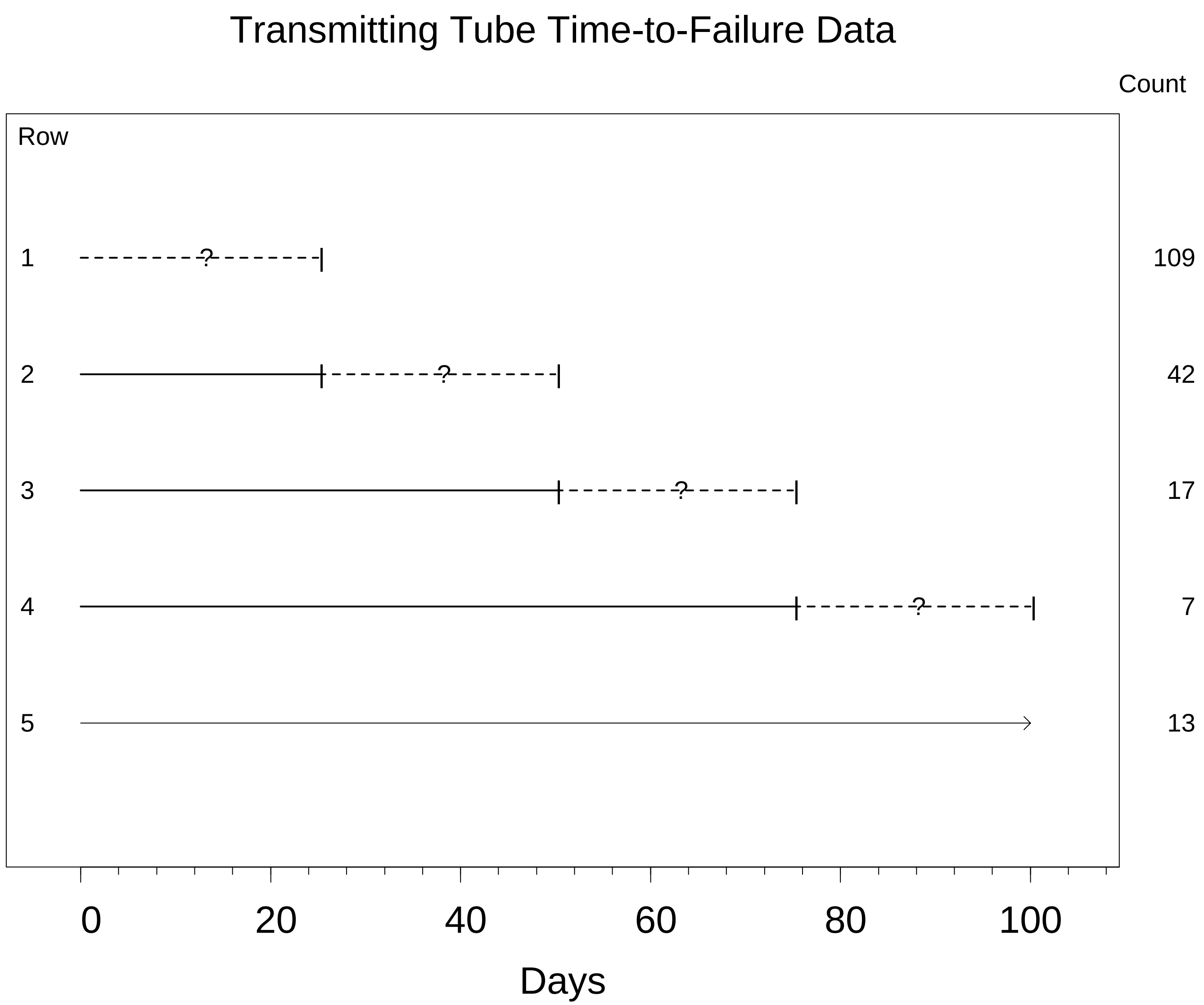
- A shift in the slope of a probability plot often indicates a different failure mode.
- Look for explanatory variables to help better understand data sources.
- Separate analyses of the Base D data and the data from the other bases indicated different failure distributions.
- The large slope ($\beta \approx 5$) for Base D indicated strong wearout.
- The relatively small slope for the other bases ($\beta \approx 0.85$) suggested a small proportion of bleed systems susceptible to failure.
- The problem at base D was caused by salt air. A change in maintenance procedures there solved the problem.

Transmitter Vacuum Tube Data
(Davis 1952)

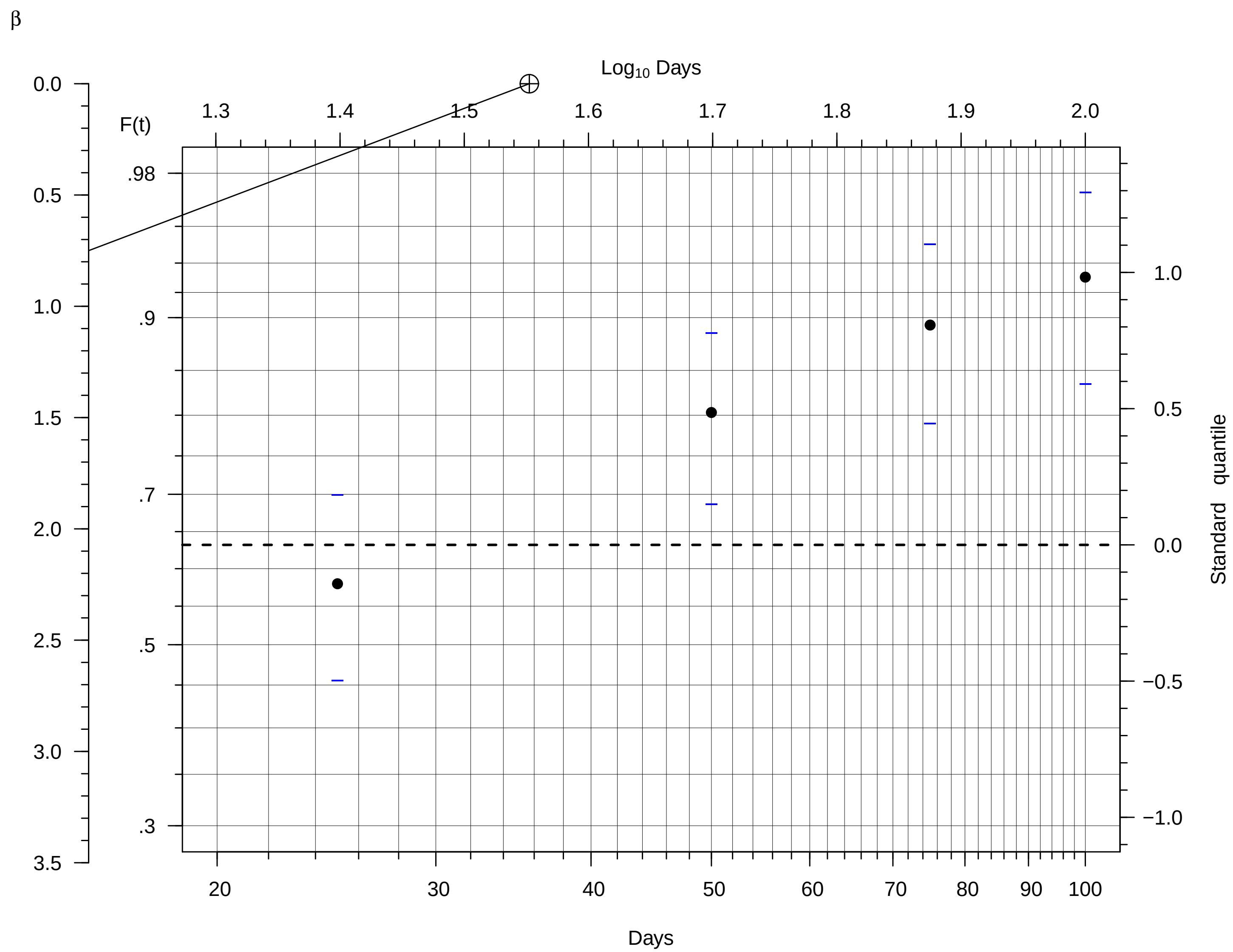
- Life data for a certain kind of transmitter vacuum tube used in the output stage of high-power transmitters.
- The data are read-out (interval-censored) data.

Days		
Interval Endpoint		Number
Lower	Upper	Failing
0	25	109
25	50	42
50	75	17
75	100	7
100	∞	13

V7 Transmitter Tube Failure Data
Event Plot



Weibull Probability Plot of the V7 Transmitter Tube Failure Data with Simultaneous Approximate 95% Confidence Bands for $F(t)$



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Abernethy, R. B., J. E. Breneman, C. H. Medlin, and G. L. Reinman (1983). *Weibull Analysis Handbook*. Air Force Wright Aeronautical Laboratories Technical Report AFWAL-TR-83-2079. Available from: http://apps.dtic.mil/dtic/tr/fulltext/u2/a143100.pdf. []

Meeker, W. Q., L. A. Escobar, and F. G. Pascual (2021). *Statistical Methods for Reliability Data* (Second Edition). Wiley. [1]