Chapter 21

Repeated Measures Degradation Analysis

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Chapter 21
Repeated Measures Degradation Data, Models, and Data Analysis

Topics discussed in this chapter are:

- Repeated measures degradation models.
- Fitting repeated measures degradation reliability models.
- The relationship between degradation reliability models and failure-time reliability models.
- Accelerated repeated measures degradation tests, modeling and analysis and Device-B reliability assessment.
- Accelerated repeated measures degradation modeling and analysis with multiple accelerating variables and LED-A reliability assessment.

Chapter 21

Repeated Measures Degradation Analysis

Segment 1

Repeated Measures Degradation Models

Repeated Measures Degradation Models

- The structural form of the repeated measures degradation models is the same as that introduced in Chapter 20 for destructive degradation.

    - With no explanatory variables

$$Y = h_d[\mathcal{D}(t)] = \xi(t) + \epsilon.$$

    - With one or more explanatory variables

$$Y = h_d[\mathcal{D}(t)] = \xi(t, x) + \epsilon.$$

- As in Chapter 20, $\xi(t)$ [or $\xi(t, x)$] is a monotone function (either increasing or decreasing) of (possibly transformed) time $\tau = h_t(t)$ and will depend on parameters $\beta_1, \ldots, \beta_k$.

- In repeated-measures degradation, it is possible to allow the model parameters in $\xi(t, x)$ to vary from unit-to-unit.

Model for Repeated-Measures Degradation Data

- **Observed degradation path model:** Observed path of unit i at time t_{ij} is

$$\mathcal{D}_{ij} = \mathcal{D}(t_{ij}, \beta_0, \ldots, \beta_k, \epsilon_{ij}), \;\; i = 1, \ldots, n, \;\; j = 1, \ldots, m_i.$$

- **Path parameters:** $\beta_0, \ldots, \beta_k$ may be random from unit-to-unit or fixed in the population/process.

- **Sample path model:** Sample degradation path of unit i at time t_{ij} (the inspection time j for unit i) is

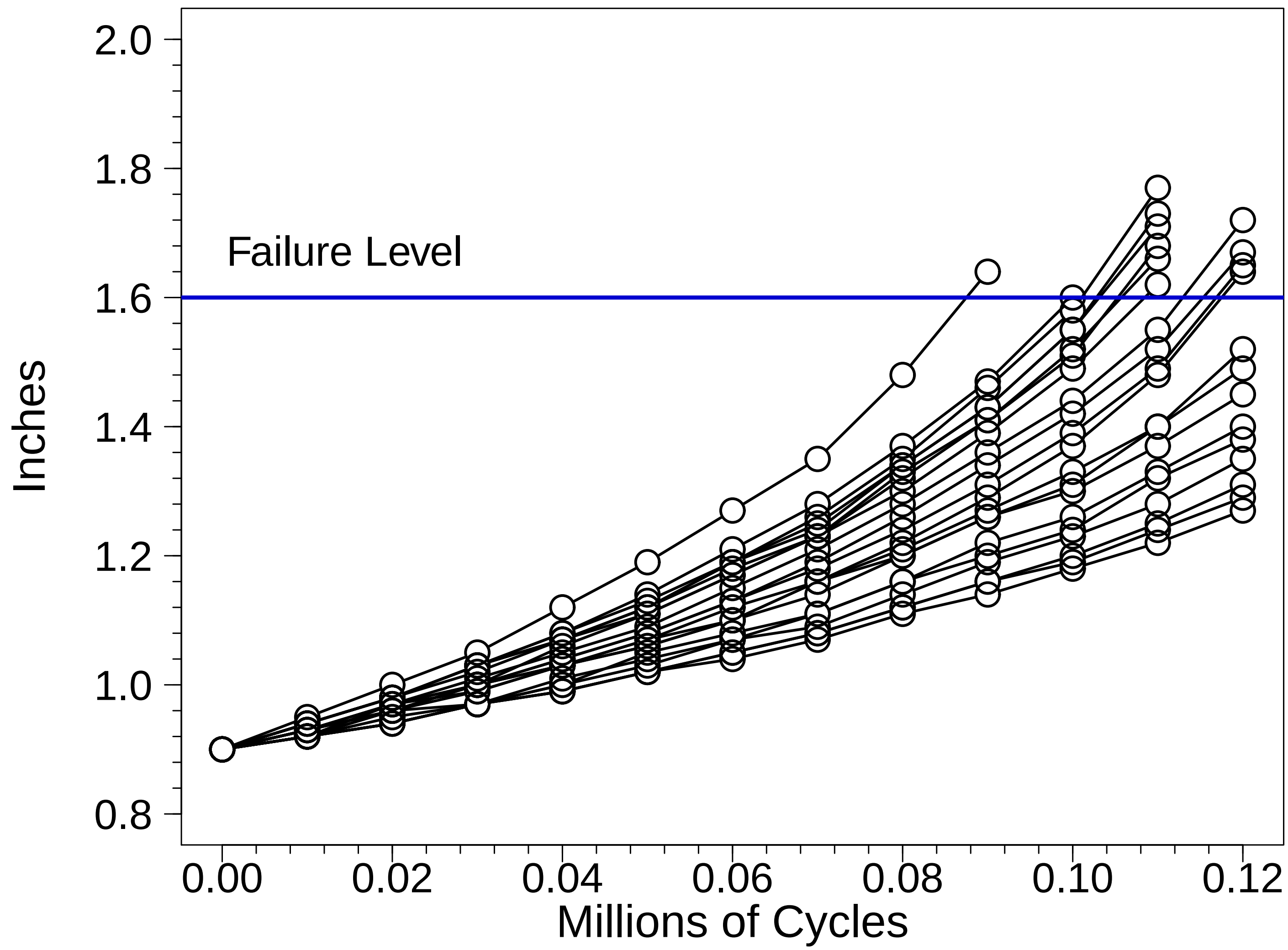
$$Y_{ij} = h_d(\mathcal{D}_{ij}) = \xi(t_{ij}, x_i) + \epsilon_{ij}, \;\; \epsilon_{ij} \sim \mathrm{NORM}(0, \sigma_\epsilon),$$

  where the ϵ_{ij} are independent distributed and independent of any random parameters in $\xi(t, x)$.

- Can use transformations on the response, time, or the x variables to simplify the model structure, as suggested by physical/chemical theory, past experience, or the data.

- Note that the **real sample size** in repeated-measures degradation data is the number of units or specimens (and **not** the number of readings per specimen).

Fatigue Crack Size Observations for Alloy-A



Alloy-A Fatigue Crack-Size Data

- 21 notched specimens were tested using cyclic stress loading.
- Fatigue cracks grow due to cyclic stress loading.
- Suppose investigators wanted to:
  - Estimate materials-related crack-growth parameters.
  - Estimate time (measured in millions of cycles) at which 50% of the cracks would reach 1.6 inches.
  - Assess adequacy of the **Paris** crack-growth model.
- Data from Hudak et al. (1978) and Bogdanoff and Kozin (1985, page 242).

Paris Crack-Growth Model

- The Paris crack-growth model is
$$\frac{da(t)}{dt} = C \times [\Delta K(a)]^m$$
is a commonly used model to describe the growth of fatigue cracks over some range of size.

- $C > 0$ and $m > 0$ are materials properties

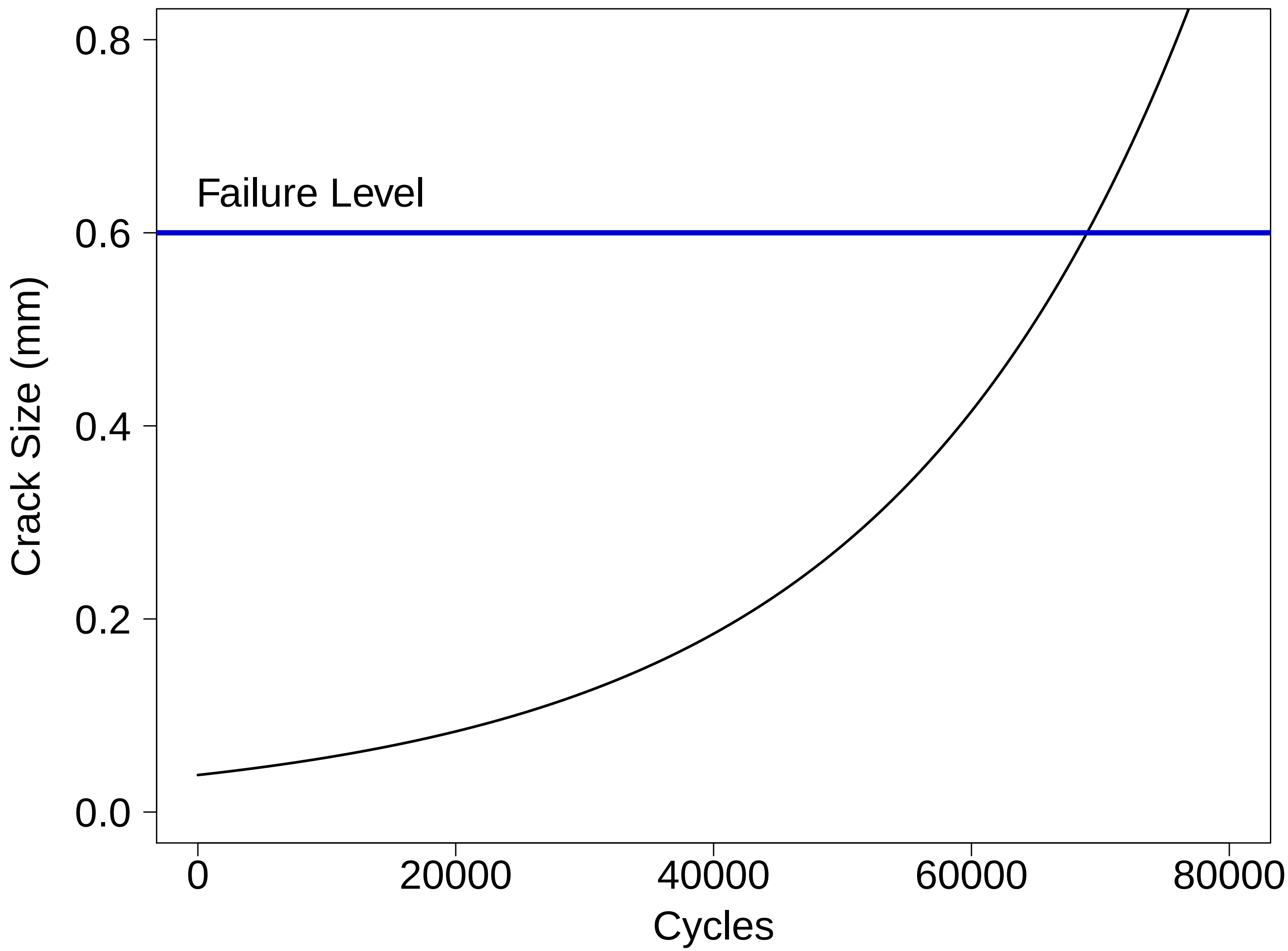
- $\Delta K(a)$, the range of the stress intensity factor and a function of a. Form of $K(a)$ depends on applied stress, part dimensions, and geometry.

- To model a two-dimensional edge-crack in a plate with a crack that is small relative to the width of the plate (say less than 3%), $K(a) = \text{Stress}\sqrt{\pi a}$ and the solution to the resulting differential equation is

$$a(t) = \begin{cases} \left[\{a(0)\}^{1-\frac{m}{2}} + \left(1 - \frac{m}{2}\right) \times C \times (\text{Stress}\sqrt{\pi})^m \times t\right]^{\frac{2}{2-m}}, & m \neq 2 \\ a(0) \times \exp\left[C \times (\text{Stress}\sqrt{\pi})^2 \times t\right], & m = 2. \end{cases}$$

Paris Crack-Growth Model with no Variability

$$\frac{da(t)}{dt} = C \times [\Delta K(a)]^m$$



Models for Variation in Degradation and Failure Time

- Need to **identify and model** important sources of variability in the degradation process.

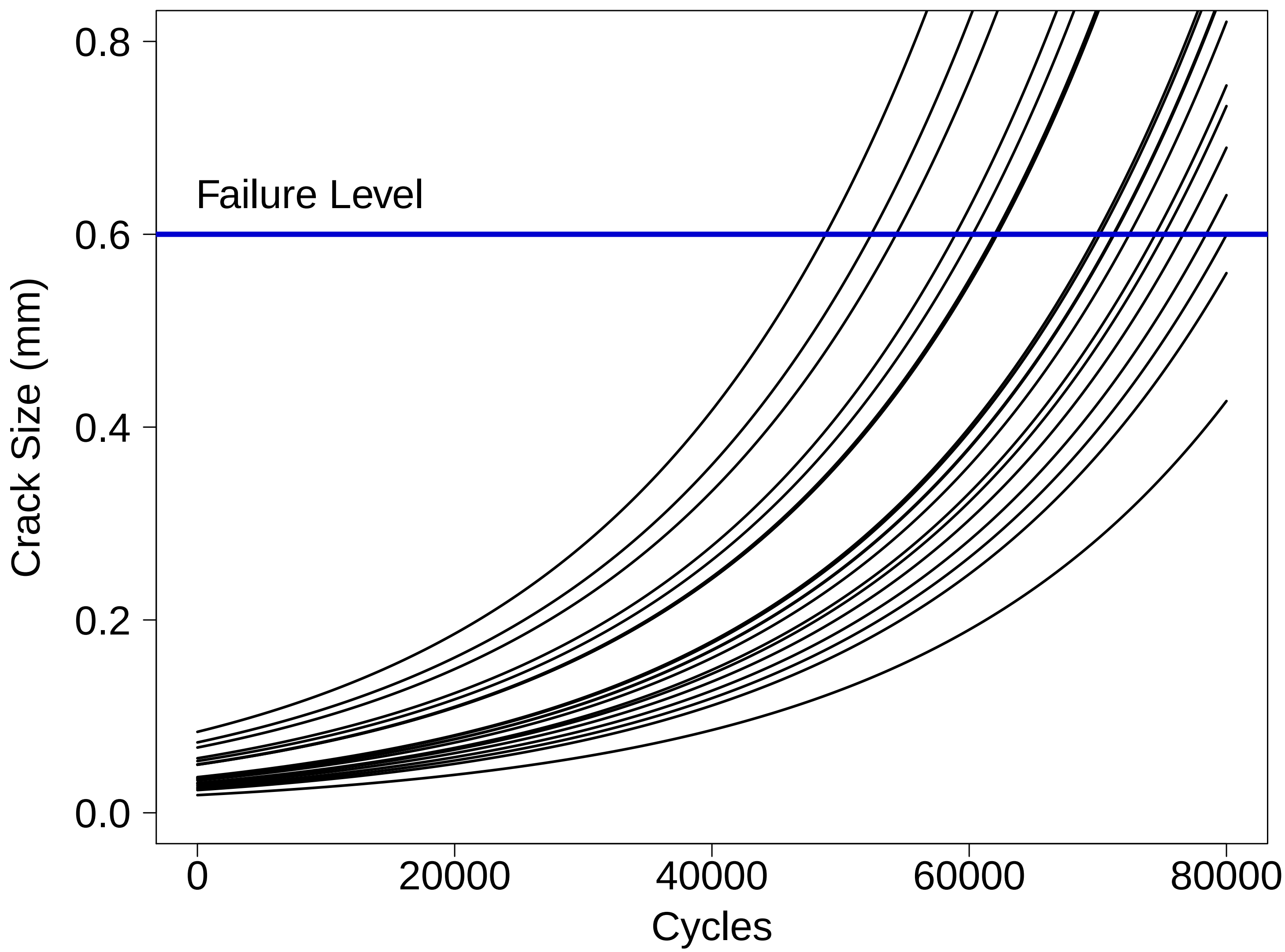
  If all manufactured units were identical, under exactly the same conditions, in exactly the same environment, and if every unit failed as it reached a particular **critical** level of degradation, then all units would fail after exactly the same amount of operating time.

- Quantities that might be modeled as random include:

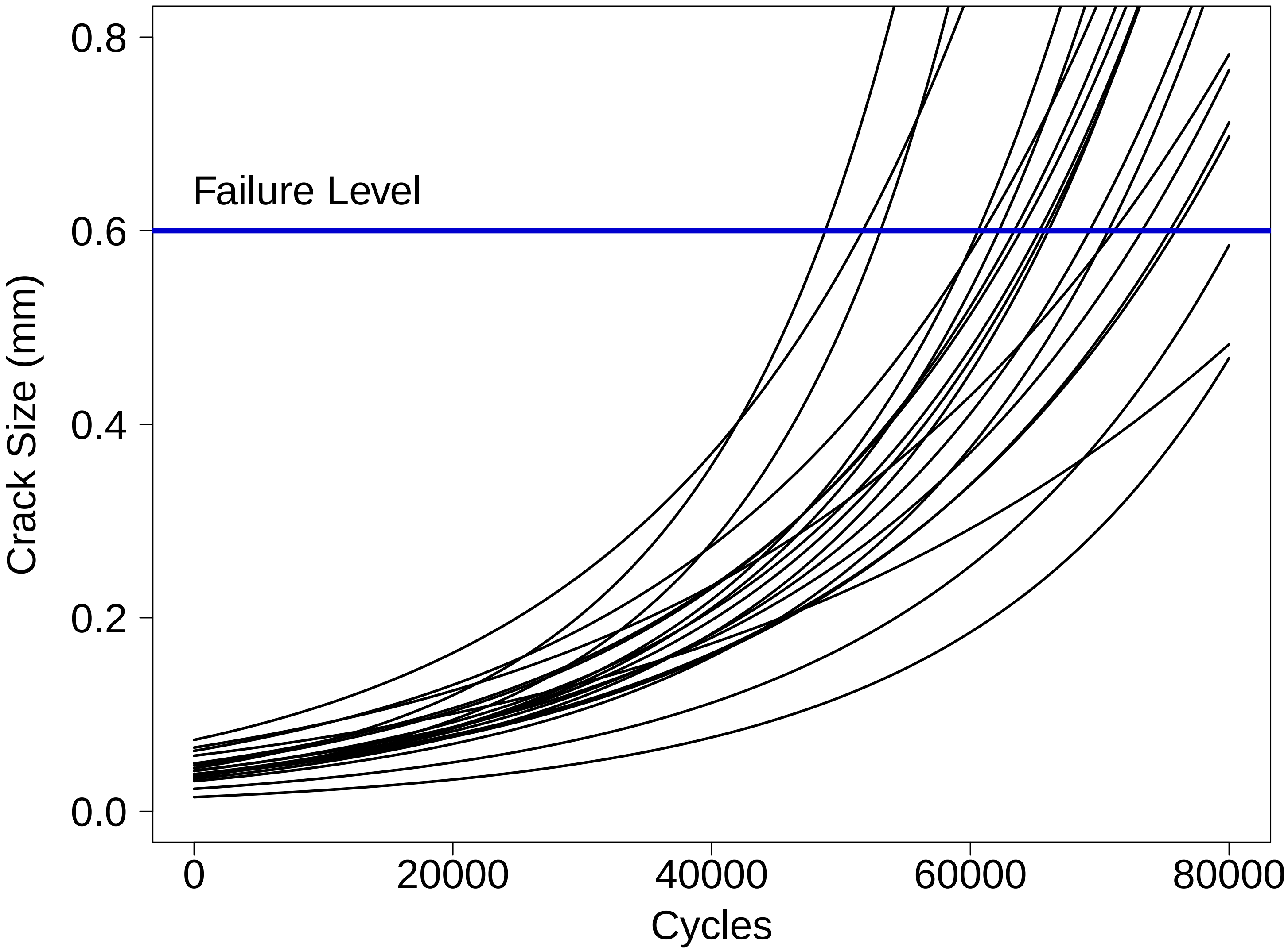
  - Initial conditions (flaw size, amount of material).
  - Material properties (related to **degradation rate**).
  - Level of degradation at which unit will fail.

- Stochastic process variability (e.g., stress of other environmental variables changing over time).

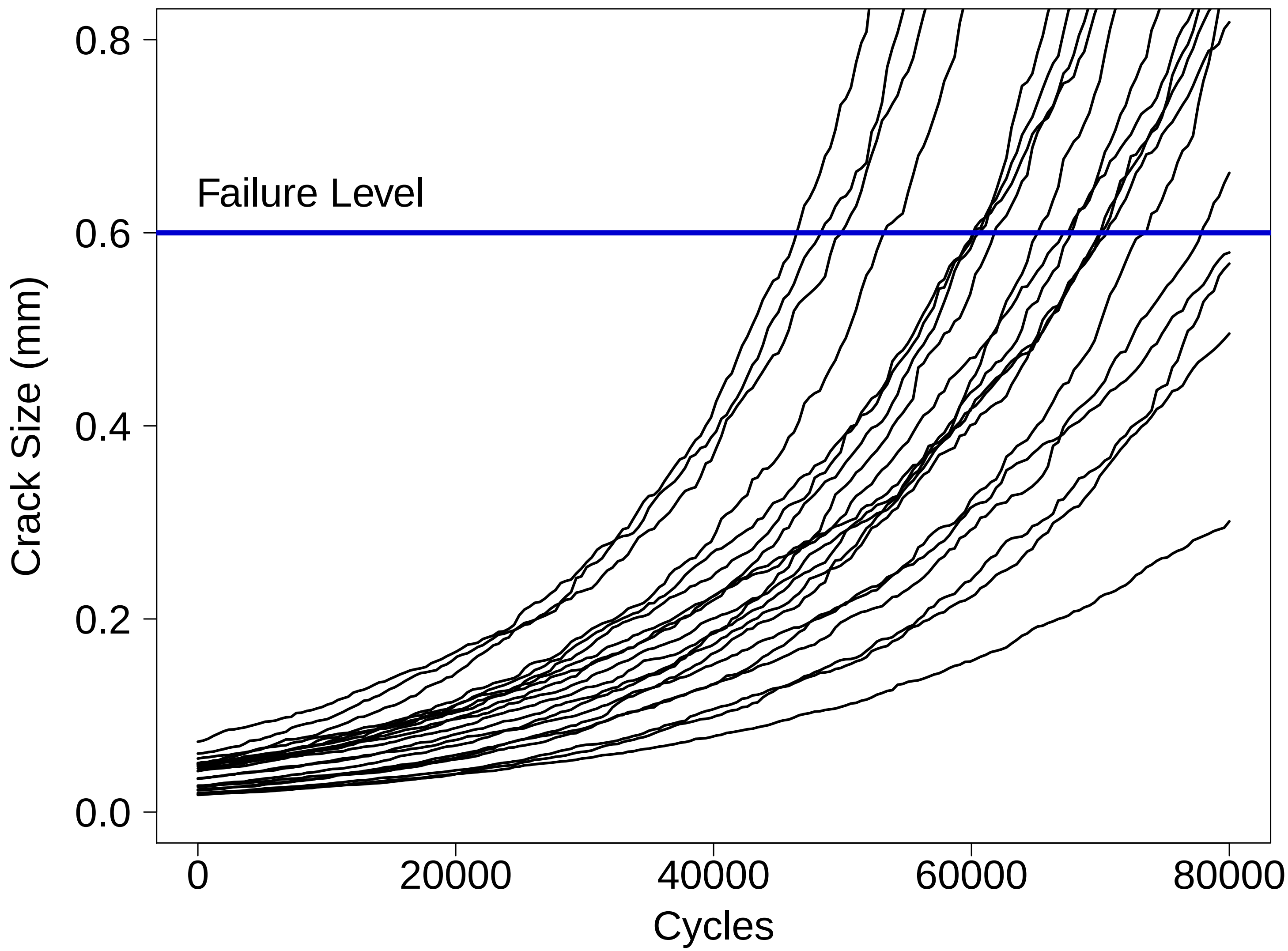
Paris Crack-Growth Model with Simulated Unit-to-Unit Variability in Initial Crack Size but with Fixed Material Properties but Constant Stress



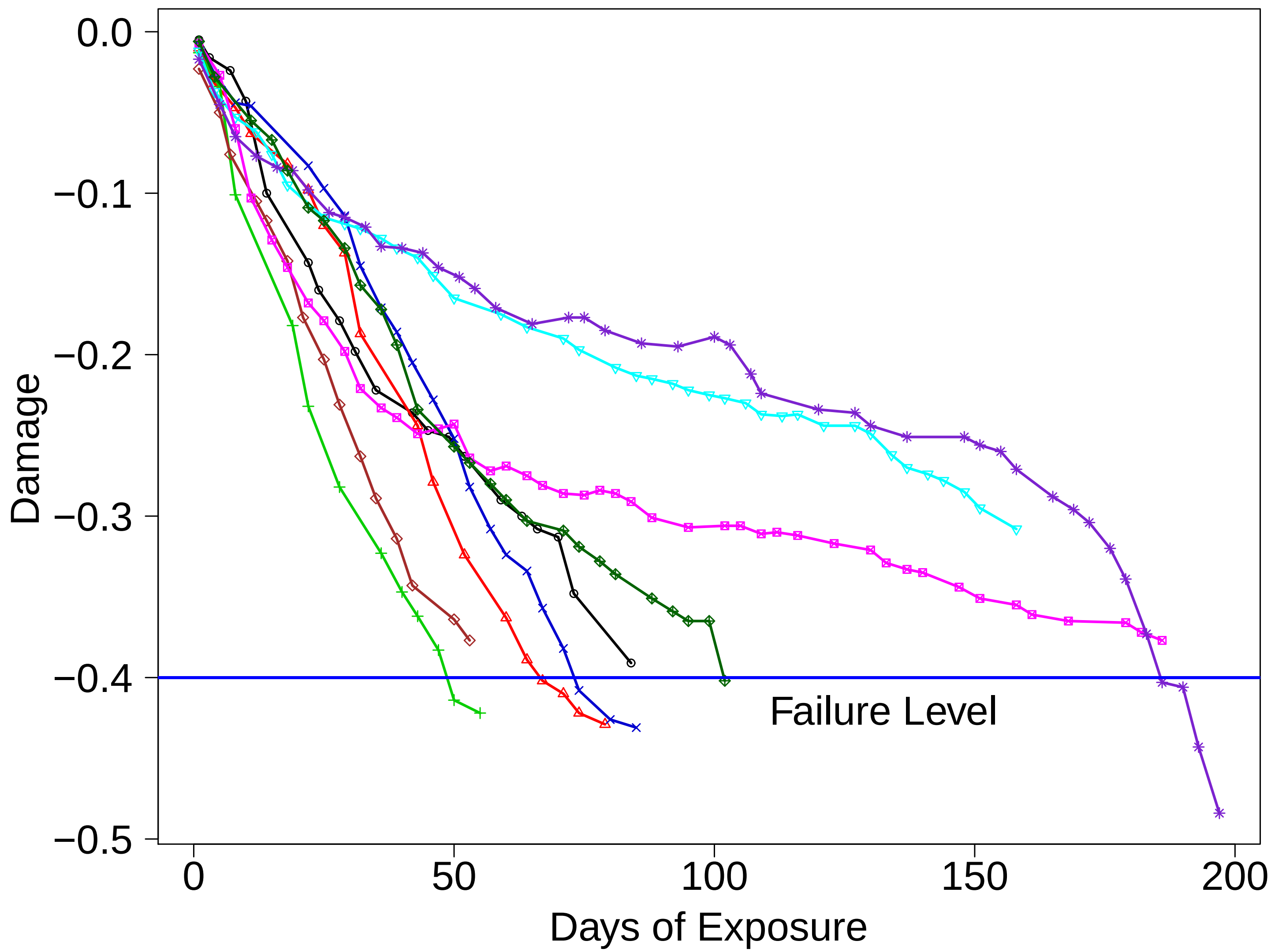
Paris Crack-Growth Model with Simulated Unit-to-Unit Variability in the Initial Crack Size and Material Properties but Constant Stress



Paris Model with Simulated Unit-to-Unit Variability in the Initial Crack Size and Material Properties and Stochastic Stress



Degradation of
Units Exposed to UV in an Outdoor Experiment



Chapter 21

Repeated Measures Degradation Analysis

Segment 2

Repeated Measures Degradation Estimation

Alloy-A Fatigue Crack-Growth Degradation Model

Likelihood for Random-Parameter Models

The likelihood for the random-parameter degradation model is

$$L(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}, \sigma_\epsilon|\text{DATA}) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_{j=1}^{m_i} \frac{1}{\sigma_\epsilon} \phi_{\text{norm}}(\zeta_{ij}) \right] \quad (1)$$
$$\times f_{\boldsymbol{\beta}}(\beta_{1i}, \ldots, \beta_{ki}; \boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}) d\beta_{1i}, \ldots, d\beta_{ki},$$

- $\zeta_{ij} = [y_{ij} - \xi(t_{ij}, \beta_{1i}, \ldots, \beta_{ki})]/\sigma_\epsilon$
- $f_{\boldsymbol{\beta}}(\beta_{1i}, \ldots, \beta_{ki}; \boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}})$ is the multivariate normal distribution density function with mean vector $\boldsymbol{\mu}_{\boldsymbol{\beta}}$ and covariance matrix $\Sigma_{\boldsymbol{\beta}}$.
- $L(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}, \sigma_\epsilon|\text{DATA})$ can be maximized with modern software (e.g., using the `R` packages `lme4` and `nlme`).
- Confidence intervals can be computed by using Wald, likelihood, or bootstrap methods.

Motivation for Using Bayesian Estimation

- Bayesian inference with weakly-informative prior distributions generally provide estimation results (point estimates and credible intervals) that are close to ML estimation results.
- When informative prior information is available (e.g., from knowledge of the physics of failure and/or previous experience with similar failure modes) it can be readily incorporated into the analysis.
- Available software for non-Bayesian methods for random parameter models based on an assumed multivariate normal distribution. Modern Bayesian analysis software is more flexible.
- Bayesian statistical methods are justified on probability theory instead of large-sample approximations.
- When there is limited information in the data about certain quantities of interest, Bayesian inferences can be sensitive to the manner in which a weakly-informative prior distribution is specified.

Bayesian Estimation of the (Random Parameter) Repeated-Measures Degradation Model Parameters

- Specify the statistical model for the data.
- Generally use weakly-informative prior distributions for the unknown model parameters, unless there is solid prior information about one or more parameters.
- Use an MCMC method to generate draws from the joint posterior distribution of
  - Random parameters.
  - Error term standard deviation.
  - Parameters for the individual units.
- Compute the marginal posterior distribution of the induced failure-time distribution, providing point estimates and credible (confidence) intervals.

Alloy-A
Repeated Measures Degradation Model

- The crack-growth degradation path model is $Y = \mathcal{D}(t) = \xi(t) + \epsilon$ where

$$\xi(t) = \begin{cases} \left[(0.90)^{1-\frac{m}{2}} + \left(1 - \frac{m}{2}\right) \times C \times t\right]^{\frac{2}{2-m}}, & m \neq 2 \\ 0.90 \times \exp(C \times t), & m = 2, \end{cases}$$

  is the solution to the Paris crack-growth model differential equation.

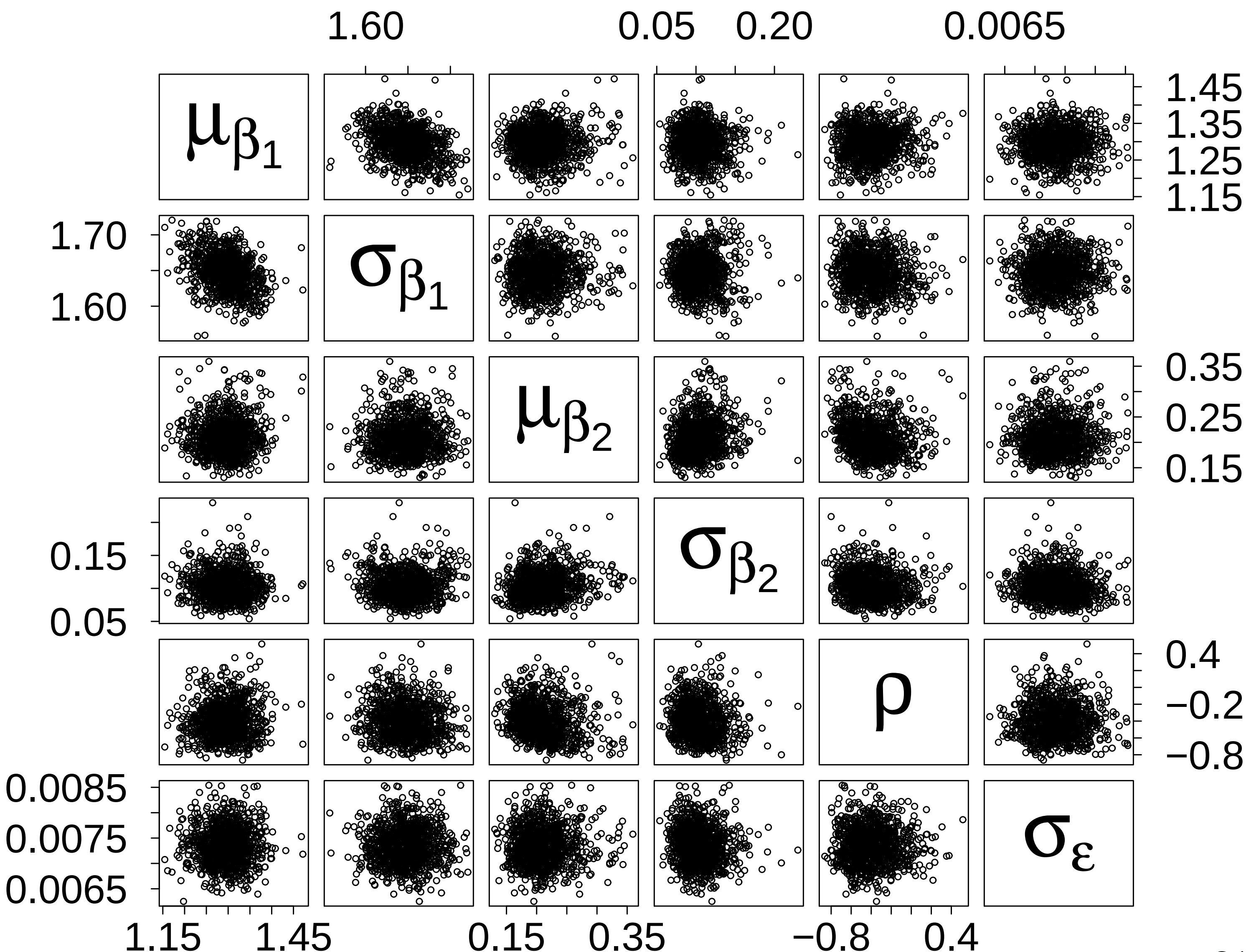
- $\beta_1 = C(\sqrt{\pi})^m$ and $\beta_2 = m$ have a joint bivariate lognormal (BVLN) distribution with parameters.

- The BVLN parameters are $(\mu_{\log(\beta_1)}, \mu_{\log(\beta_2)}, \sigma_{\log(\beta_1)}, \sigma_{\log(\beta_2)}, \rho)$.

- The error term $\epsilon \sim \text{NORM}(0, \sigma_\epsilon)$ is independent of the random parameters.

- Weakly informative prior distributions were used.

Alloy-A Fatigue Crack Size
Bivariate Lognormal Distribution Paris Model
Weakly Informative Prior Distribution
Posterior Pairs Plot



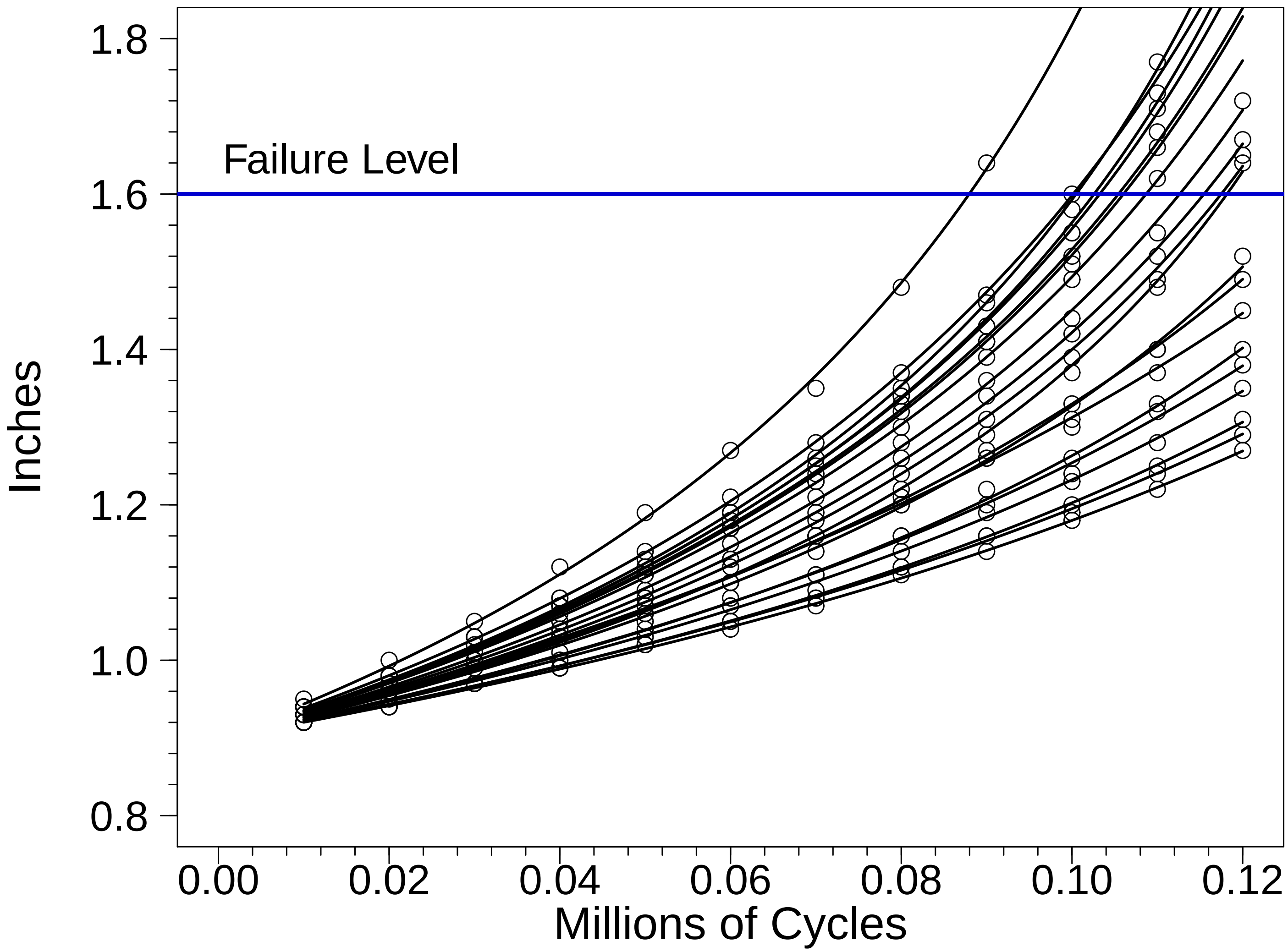
Estimation Results for the Alloy-A Data BVLN Repeated Measures Degradation Model Medians of the Marginal Posterior Distributions

- Point estimates of the parameters are: $\widehat{\mu}_{\log(\beta_1)} = 3.72$, $\widehat{\mu}_{\log(\beta_2)} = 5.21$, $\widehat{\sigma}_{\log(\beta_1)} = 0.75$, $\widehat{\sigma}_{\log(\beta_2)} = 0.52$, $\widehat{\rho} = -0.40$, and $\widehat{\sigma}_{\epsilon} = 0.007$.

- Statistical uncertainty for the parameters can be assessed by inspection of the pairs plot of the MCMC draws and computation of corresponding credible intervals.

- Draws from the joint posterior distribution of $\mu_{\log(\beta_1)}$, $\mu_{\log(\beta_2)}$, $\sigma_{\log(\beta_1)}$, $\sigma_{\log(\beta_2)}$, ρ and $\widehat{\sigma}_{\epsilon}$ can be used to estimate the failure-time distribution.

Alloy-A Fatigue Crack Size Observations and Fitted Paris-Rule Model



Chapter 21

Repeated Measures Degradation Analysis

Segment 3

Relationship Between the Degradation and Failure-Time Models

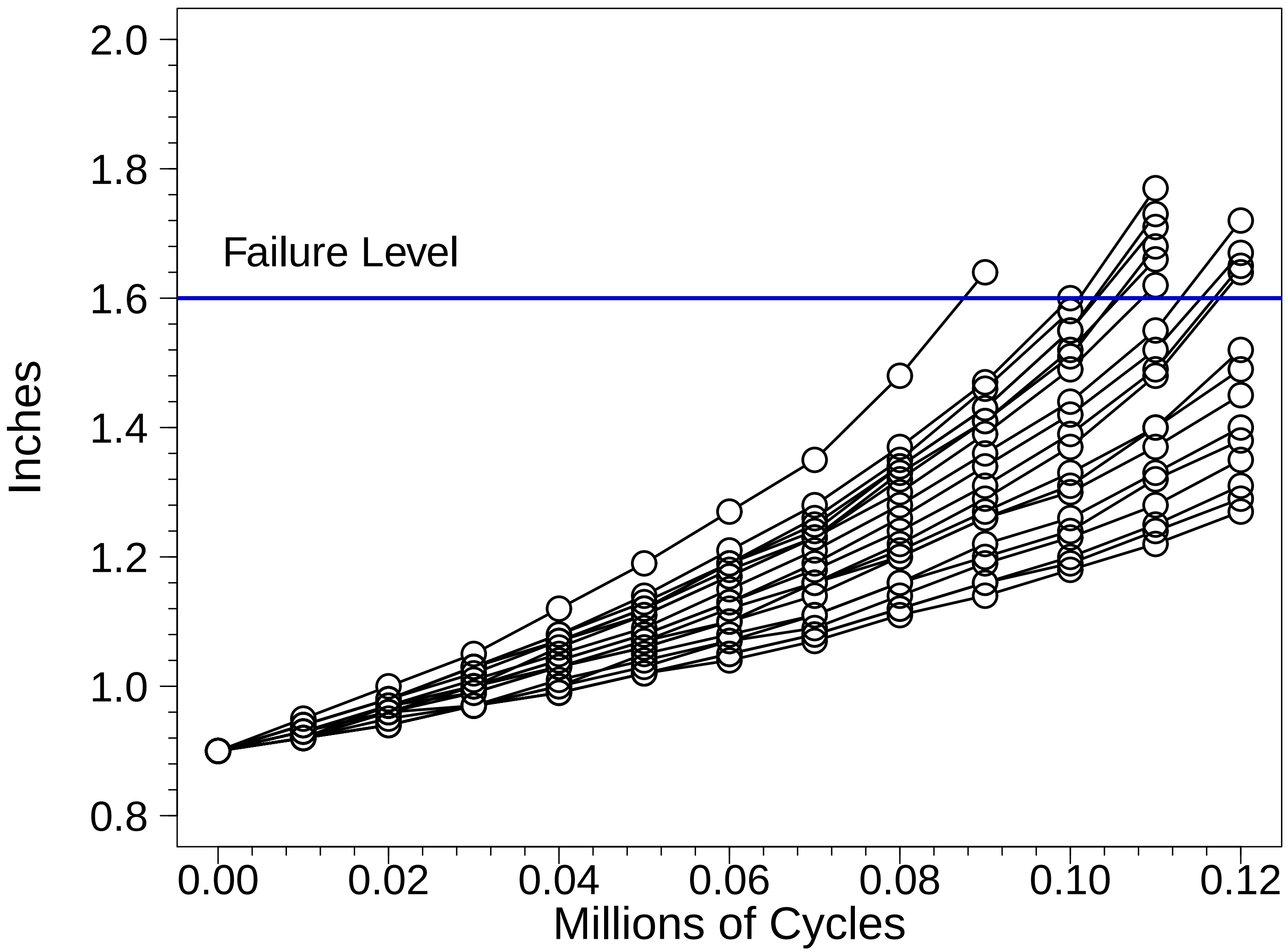
Time to First Crossing Distribution

Estimating $F(t)$ for Alloy-A

Models Relating Degradation and Failure

- **Soft failures and specified degradation level**
  - In some products there is a gradual loss of performance (e.g., decreasing light output from a fluorescent light bulb or LEDs).
  - Use fixed $\mathcal{D}_{\mathsf{f}}$ to denote the critical level for the degradation path.
  - Soft failures can be defined in terms of either actual or observed degradation.
- **Hard failures and correlation between failure and degradation level**
  - Immediate loss of functionality or other catastrophic failure.
  - Random $\mathcal{D}_{\mathsf{f}}$. Use a joint distribution of $\mathcal{D}_{\mathsf{f}}$ and other random parameters.

Fatigue Crack Size Observations for Alloy-A



General Methods for Obtaining Induced Time-to-First-Crossing Distributions $F(t) = \Pr(T \le t)$ from $Y = h_d[\mathcal{D}(t)] = \xi(t, x) + \epsilon$

Criterion	Increasing path model	Decreasing path model
Observed degradation $Y = h_d(\mathcal{D})$	$\Pr(T \le t) = \Pr[Y > h_d(\mathcal{D}_\mathsf{f})]$ $= 1 - \Pr[Y \le h_d(\mathcal{D}_\mathsf{f})]$ $= 1 - G[h_d(\mathcal{D}_\mathsf{f})]$	$\Pr(T \le t) = \Pr[Y \le h_d(\mathcal{D}_\mathsf{f})]$ $= G[h_d(\mathcal{D}_\mathsf{f})]$
Actual degradation $\xi(t, x)$	$\Pr(T \le t) = \Pr[\xi(t, x) > h_d(\mathcal{D}_\mathsf{f})]$ $= 1 - \Pr[\xi(t, x) \le h_d(\mathcal{D}_\mathsf{f})]$ $= 1 - H[h_d(\mathcal{D}_\mathsf{f})]$	$\Pr(T \le t) = \Pr[\xi(t, x) \le h_d(\mathcal{D}_\mathsf{f})]$ $= H[h_d(\mathcal{D}_\mathsf{f})]$

- $G(x)$ is the cdf of **observed** degradation $Y = h_d(\mathcal{D}_{ij})$.
- $H(x)$ is the cdf of **actual** degradation $\xi(t, x)$.
- In many applications the difference between these failure-time definitions will be small because the random-parameters variability (e.g., σ_{β_1} and σ_{β_2}) will dominate $\sigma_\epsilon = 0$.
- Generally, evaluating $F(t)$ and corresponding quantiles with **actual** degradation is easier because there is one less source of variability in the failure-time model.

Evaluation of $F(t)$

$$F(t) = F(t; \boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}) = \mathsf{Pr}(T \le t) = \mathsf{Pr}[\xi(t, \beta_0, \ldots, \beta_k) \ge \mathcal{D}_{\mathsf{f}}].$$

- **Direct evaluation of $F(t)$:** Closed forms available for simple problems (e.g., a single random variable and other special case models).
- **Numerical integration:** Useful for a small number of random variables (e.g., 2 or 3).
- **Monte Carlo simulation:** General method. Needs much computer time to evaluate small probabilities. Can use **importance sampling**.

  Estimate failure probabilities and obtain credible (confidence) intervals by obtaining the marginal posterior distribution of $F(t)$ at selected values of time t.

When is Direct Evaluation of $F(t)$ Possible?

- Failure defined by **observed** degradation:
  - ▶ $Y = \xi(t, x) + \epsilon$ has a normal distribution (e.g., linear $\xi(t, x)$ and bivariate normal distribution for the random slope and intercept).
- Failure defined by **actual** degradation:
  - ▶ There is only one random parameter in $\xi(t, x)$.
  - ▶ $\xi(t, x)$ has a normal distribution (e.g., linear $\xi(t, x)$ and bivariate normal distribution for the random slope and intercept).

Otherwise numerical methods (numerical integration or Monte Carlo simulation) are required.

Example of Direct Evaluation of $F(t)$ for a Single Lognormal Distribution Random Parameter

- Suppose the actual degradation path of a particular unit is given by $\xi(t) = \beta_0 + \beta_1\tau$, where β_0 is fixed and $\beta_1 > 0$ varies from unit to unit according to a LNORM$(\mu_{\log(\beta_1)}, \sigma_{\log(\beta_1)})$ distribution and failure occurs when **actual** degradation $\xi(t) \geq h_d(\mathcal{D}_{\mathsf{f}})$.

- Then because $\beta_1 > 0$ (**increasing** degradation path)

$$
\begin{aligned}
F(t; \beta_0, \mu_{\log(\beta_1)}, \sigma_{\log(\beta_1)}) &= \Pr[\beta_0 + \beta_1\tau > h_d(\mathcal{D}_{\mathsf{f}})] = \Pr\{\beta_1 > [h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0]/\tau\} \\
&= 1 - \Phi_{\text{norm}}\left[\frac{\log[(h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0)/\tau] - \mu_{\log(\beta_1)}}{\sigma_{\log(\beta_1)}}\right] \\
&= \Phi_{\text{norm}}\left[\frac{\log(\tau) - \left\{\log[h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0] - \mu_{\log(\beta_1)}\right\}}{\sigma_{\log(\beta_1)}}\right], \quad t > 0.
\end{aligned}
$$

- Similarly, if $\xi(t) = \beta_0 - \beta_1\tau$ (**decreasing** degradation path) and failure occurs when $\xi(t) \leq h_d(\mathcal{D}_{\mathsf{f}})$, it can be shown that

$$
F(t; \beta_0, \mu_{\log(\beta_1)}, \sigma_{\log(\beta_1)}) = \Phi_{\text{norm}}\left[\frac{\log(\tau) - \left\{\log[\beta_0 - h_d(\mathcal{D}_{\mathsf{f}})] - \mu_{\log(\beta_1)}\right\}}{\sigma_{\log(\beta_1)}}\right], \quad t > 0.
$$

Example of Direct Evaluation of $F(t)$ for a Single Normal Distribution Random Parameter

- Here $\xi(t) = \beta_0 + \beta_1 \tau$ and failure occurs when **actual** degradation $\xi(t) \geq h_d(\mathcal{D}_{\mathsf{f}})$ where β_0 is fixed and $\beta_1 \sim \text{NORM}(\mu_{\beta_1}, \sigma_{\beta_1})$. It is required that $\mu_{\beta_1}/\sigma_{\beta_1}$ be large enough such that $\Pr(\beta_1 < 0) \approx 0$ (increasing degradation path). Then

$$F(t; \beta_0, \mu_{\beta_1}, \sigma_{\beta_1}) = \Pr[\beta_0 + \beta_1 \tau > h_d(\mathcal{D}_{\mathsf{f}})] = \Pr[\beta_1 > [h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0]/\tau]$$

$$= 1 - \Phi_{\text{norm}}\left[\frac{[h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0]/\tau - \mu_{\beta_1}}{\sigma_{\beta_1}}\right], \ \ t > 0.$$

  This $F(t)$ has an atom at $t = \infty$ equal to $\Pr(\beta_1 < 0)$.

- Similarly, if $\xi(t) = \beta_0 - \beta_1 \tau$ (decreasing degradation path), failure occurs when $\xi(t) \leq h_d(\mathcal{D}_{\mathsf{f}})$, and requiring that $\mu_{\beta_1}/\sigma_{\beta_1}$ be large enough such that $\Pr(\beta_1 < 0) \approx 0$, it can be shown that

$$F(t; \beta_0, \mu_{\beta_1}, \sigma_{\beta_1}) = \Pr[\beta_0 - \beta_1 \tau < h_d(\mathcal{D}_{\mathsf{f}})] = \Pr[\beta_1 > [\beta_0 - h_d(\mathcal{D}_{\mathsf{f}})]/\tau]$$

$$= 1 - \Phi_{\text{norm}}\left[\frac{[\beta_0 - h_d(\mathcal{D}_{\mathsf{f}})]/\tau - \mu_{\beta_1}}{\sigma_{\beta_1}}\right], \ \ t > 0.$$

  This $F(t)$ has an atom at $t = \infty$ equal to $\Pr(\beta_1 < 0)$.

Induced Failure-Time Distribution Model 5 with Bivariate Normal Random Parameters Using Observed Degradation and an Increasing Degradation Path Model

- Using **observed** degradation, $T \leq t$ is equivalent to observed degradation exceeding $h_d(\mathcal{D}_\mathsf{f})$ [i.e., $h_d[\mathcal{D}(t)] > h_d(\mathcal{D}_\mathsf{f})$]. Then

$$F(t,x) = 1-\Phi\left[\frac{h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0} - \mu_{\beta_1} AF\tau}{(\sigma^2_{\beta_0} + \sigma^2_{\beta_1}(AF)^2\tau^2 + 2\sigma_{\beta_0}\sigma_{\beta_1}\rho AF\tau + \sigma^2_\epsilon)^{1/2}}\right], \quad t \geq 0,$$

  where Φ stands for Φ_{norm}.

- For all x, this $F(t,x)$ has two probability **atoms**:

  - At $t = 0$

$$\Pr(T = 0) = F(0) = \Pr[\beta_0 + \epsilon > h_d(\mathcal{D}_\mathsf{f})] = 1 - \Phi\left[\frac{h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0}}{(\sigma^2_{\beta_0} + \sigma^2_\epsilon)^{1/2}}\right].$$

  - At $t = \infty$

$$\Pr(T = \infty) = \Pr(\beta_1 < 0) = 1 - F(\infty, x) = \Phi\left(-\frac{\mu_{\beta_1}}{\sigma_{\beta_1}}\right).$$

Failure-Time Quantiles
Model 5 with Bivariate Normal Random Parameters
Using Observed Degradation
and an Increasing Degradation Path Model

- The p quantile t_p must satisfy the relationship $p = F(t_p, x)$. Using the expression for the cdf, the solutions for τ_p are

$$\tau_p = \frac{1}{AF}\left\{\frac{\sigma_{\beta_0}\sigma_{\beta_1}\rho z^2_{(1-p)} + [h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0}]\mu_{\beta_1} \mp \sqrt{W}}{\mu^2_{\beta_1} - \sigma^2_{\beta_1} z^2_{(1-p)}}\right\}, \qquad (2)$$

  where

$$W = \left\{\sigma_{\beta_0}\sigma_{\beta_1}\rho z^2_{(1-p)} + [h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0}]\mu_{\beta_1}\right\}^2 - \left[\mu^2_{\beta_1} - \sigma^2_{\beta_1} z^2_{(1-p)}\right]\left\{\left[h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0}\right]^2 - (\sigma^2_{\beta_0} + \sigma^2_{\epsilon}) z^2_{(1-p)}\right\},$$

  and $F(0) \leq p < F(\infty)$.

- In (2), we use the term $-\sqrt{W}$ when $F(0) < p \leq 0.5$ and the term $+\sqrt{W}$ when $0.5 \leq p < F(\infty)$. When $p = 0.5$, $W = 0$.

- Then the p quantile for $F(t; x)$ is given by $t_p = h_t^{-1}(\tau_p)$.

Induced Failure-Time Distribution
Model 6 with Bivariate Normal Random Parameters
Using Observed Degradation
and a Decreasing Degradation Path Model

- Using **observed degradation,** $T \leq t$ is equivalent to observed degradation not exceeding $h_d(\mathcal{D}_{\mathrm{f}})$ [i.e., $h_d[\mathcal{D}(t)] \leq h_d(\mathcal{D}_{\mathrm{f}})$]. Then

$$F(t, x) = \Phi\left[\frac{h_d(\mathcal{D}_{\mathrm{f}}) - \mu_{\beta_0} + \mu_{\beta_1} AF\tau}{\left[\sigma^2_{\beta_0} + \sigma^2_{\beta_1}(AF)^2\tau^2 - 2\sigma_{\beta_0}\sigma_{\beta_1}\rho AF\tau + \sigma^2_{\epsilon}\right]^{1/2}}\right], \ \ t \geq 0.$$

- For all x, this $F(t, x)$ has two probability **atoms**:

  - At $t = 0$

$$\Pr(T = 0) = F(0) = \Pr[\beta_0 + \epsilon < h_d(\mathcal{D}_{\mathrm{f}})] = \Phi\left[\frac{h_d(\mathcal{D}_{\mathrm{f}}) - \mu_{\beta_0}}{(\sigma^2_{\beta_0} + \sigma^2_{\epsilon})^{1/2}}\right].$$

  - At $t = \infty$

$$\Pr(T = \infty) = \Pr(\beta_1 < 0) = 1 - F(\infty, x) = 1 - \Phi\left(\frac{\mu_{\beta_1}}{\sigma_{\beta_1}}\right).$$

Failure-Time Quantiles Model 6 with Bivariate Normal Random Parameters Using Observed Degradation and a Decreasing Degradation Path Model

- The p quantile t_p must satisfy the relationship $p = F(t_p, x)$. Using the expression for the cdf, the solutions for τ_p are

$$\tau_p = \frac{1}{AF}\left\{\frac{-\sigma_{\beta_0}\sigma_{\beta_1}\rho z_p^2 - \left[h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0}\right]\mu_{\beta_1} \mp \sqrt{W}}{\mu_{\beta_1}^2 - \sigma_{\beta_1}^2 z_p^2}\right\}, \qquad (3)$$

  where

$$W = \left\{\sigma_{\beta_0}\sigma_{\beta_1}\rho z_p^2 + \left[h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0}\right]\mu_{\beta_1}\right\}^2 - \left(\mu_{\beta_1}^2 - \sigma_{\beta_1}^2 z_p^2\right)\left\{\left[h_d(\mathcal{D}_\mathsf{f}) - \mu_{\beta_0}\right]^2 - (\sigma_{\beta_0}^2 + \sigma_\epsilon^2) z_p^2\right\},$$

  and $F(0) \leq p < F(\infty)$.

- In (3), we use the term $-\sqrt{W}$ when $F(0) < p \leq 0.5$ and the term $+\sqrt{W}$ when $0.5 < p < F(\infty)$. When $p = 0.5$, $W = 0$.

- Then the p quantile for $F(t; x)$ is given by $t_p = h_t^{-1}(\tau_p)$.

Evaluation of $F(t)$ by Numerical Integration

- The failure (crossing probability) can be expressed as

$$\Pr(T \le t) = F(t) = F(t; \boldsymbol{\theta}_{\boldsymbol{\beta}}) = \Pr[\xi(t, \beta_1, \ldots, \beta_k) > h_d(\mathcal{D}_{\mathsf{f}})].$$

- If (β_1, β_2) follows a bivariate normal distribution with parameters $\mu_{\beta_1}, \mu_{\beta_2}, \sigma_{\beta_1}, \sigma_{\beta_2}, \rho$, then

$$\Pr(T \le t) = \int_{-\infty}^{\infty} \Phi_{\text{norm}}\left[-\frac{g[h_d(\mathcal{D}_{\mathsf{f}}), t, \beta_1] - \mu_{\beta_2|\beta_1}}{\sigma_{\beta_2|\beta_1}}\right] \frac{1}{\sigma_{\beta_1}} \phi_{\text{norm}}\left(\frac{\beta_1 - \mu_{\beta_1}}{\sigma_{\beta_1}}\right) d\beta_1$$

where $g[h_d(\mathcal{D}_{\mathsf{f}}), t, \beta_1]$ is the value of β_2 for given β_1, that gives $\xi(t) = h_d(\mathcal{D}_{\mathsf{f}})$ and where

$$\mu_{\beta_2|\beta_1} = \mu_{\beta_2} + \rho\sigma_{\beta_2}\left(\frac{\beta_1 - \mu_{\beta_1}}{\sigma_{\beta_1}}\right),$$
$$\sigma_{\beta_2|\beta_1} = \sigma_{\beta_2}(1 - \rho^2)^{1/2}.$$

- This method generalizes to bivariate lognormal and multivariate normal/lognormal distributions.

Estimation of $F(t)$

- For non-Bayesian inference
  - ▶ Point estimates are obtained by evaluating

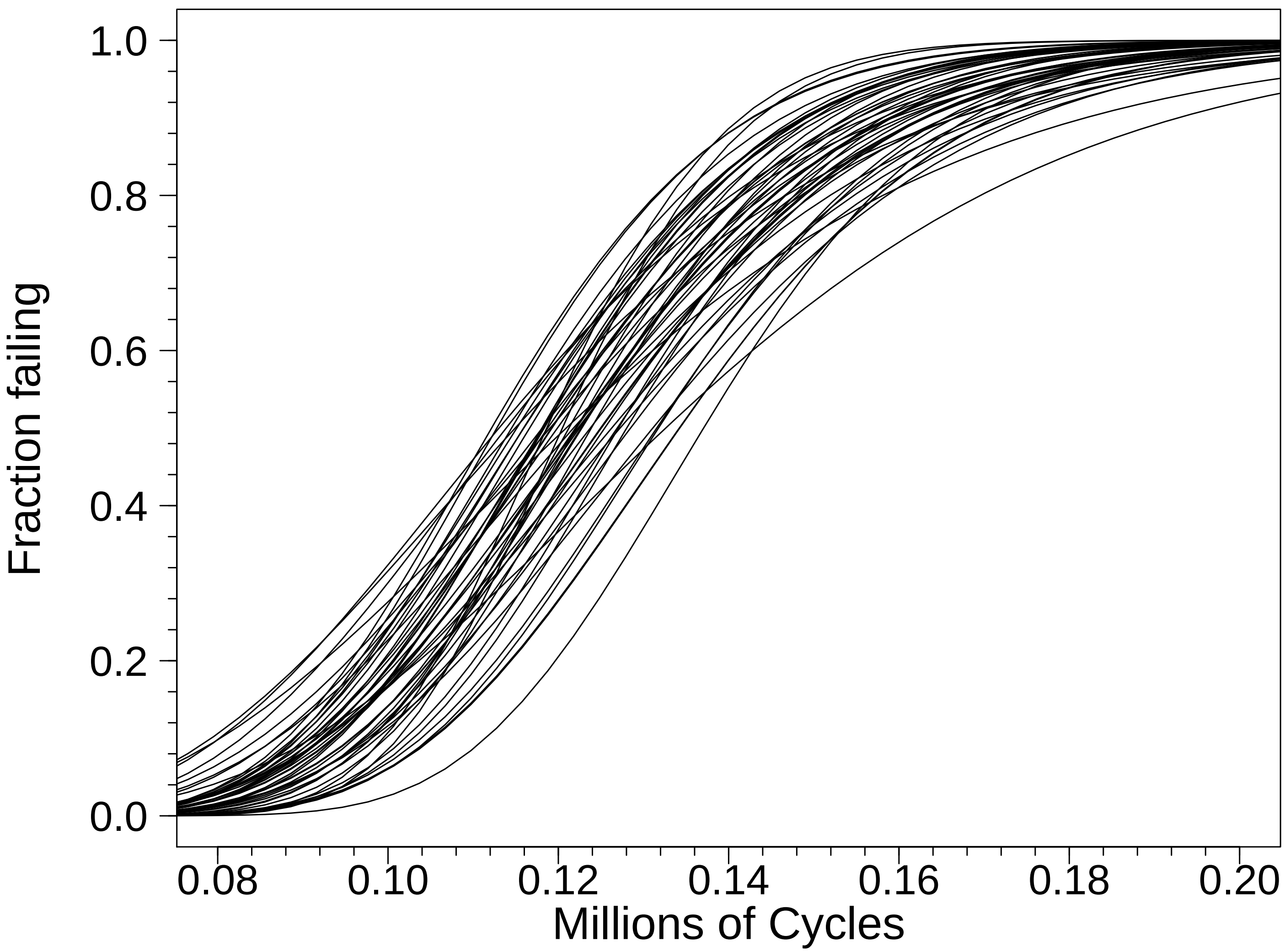
    $$F(t) = F(t; \boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}) = \Pr(T \leq t) = \Pr[\xi(t, \beta_0, \ldots, \beta_k) \geq \mathcal{D}_{\mathsf{f}}].$$

    at the ML estimates of the model parameters.
  - ▶ Confidence intervals can be obtained by using Wald, likelihood, or bootstrap methods.
- For Bayesian inference
  - ▶ Use draws from the joint posterior distribution of $(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}})$ to compute draws from the marginal posterior distribution of $F(t)$.
  - ▶ Compute credible intervals in the usual way.

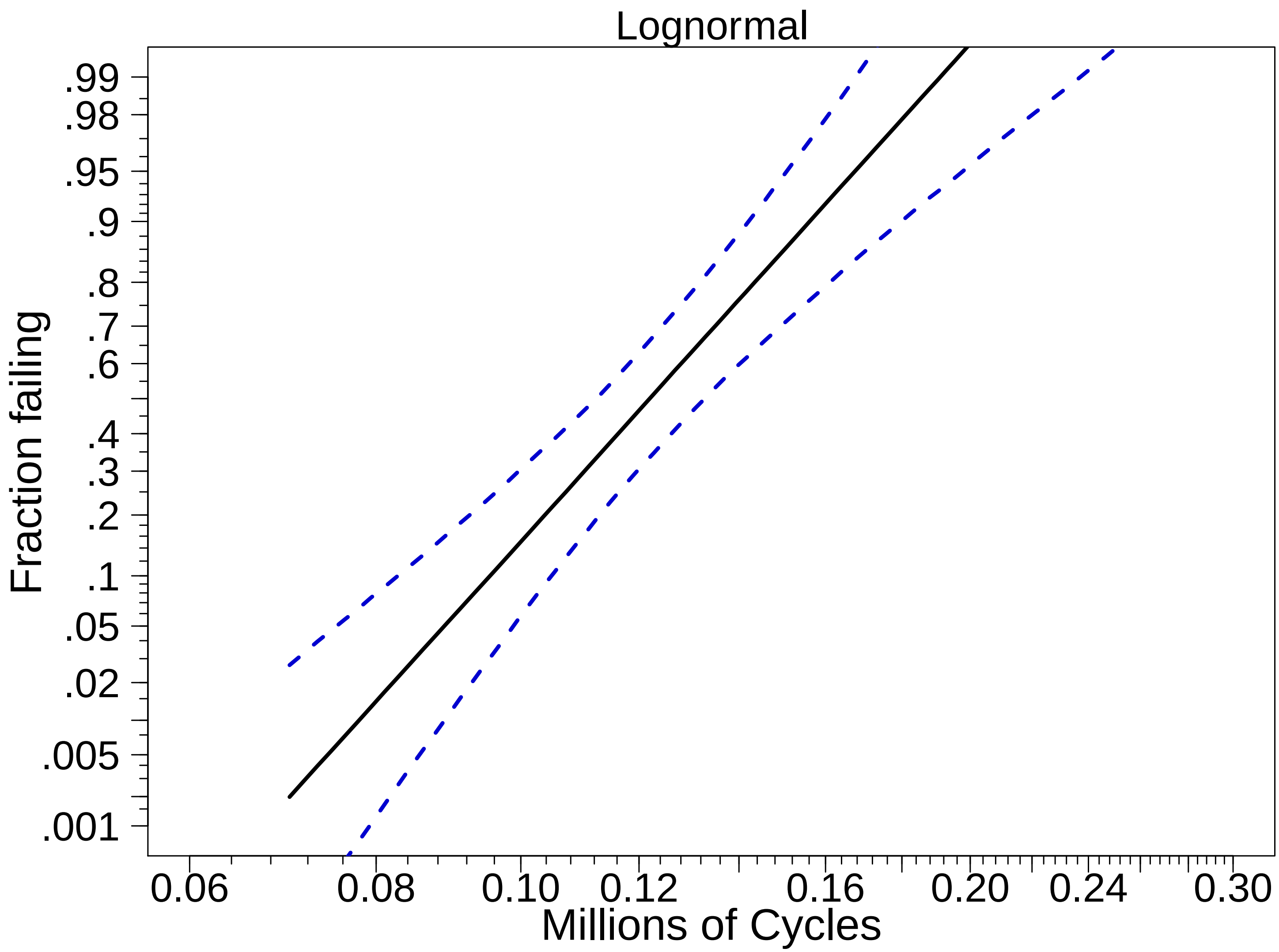
Posterior Distribution Estimates of $F(t)$ for Alloy A

Bivariate Lognormal Model for β_1 and β_2

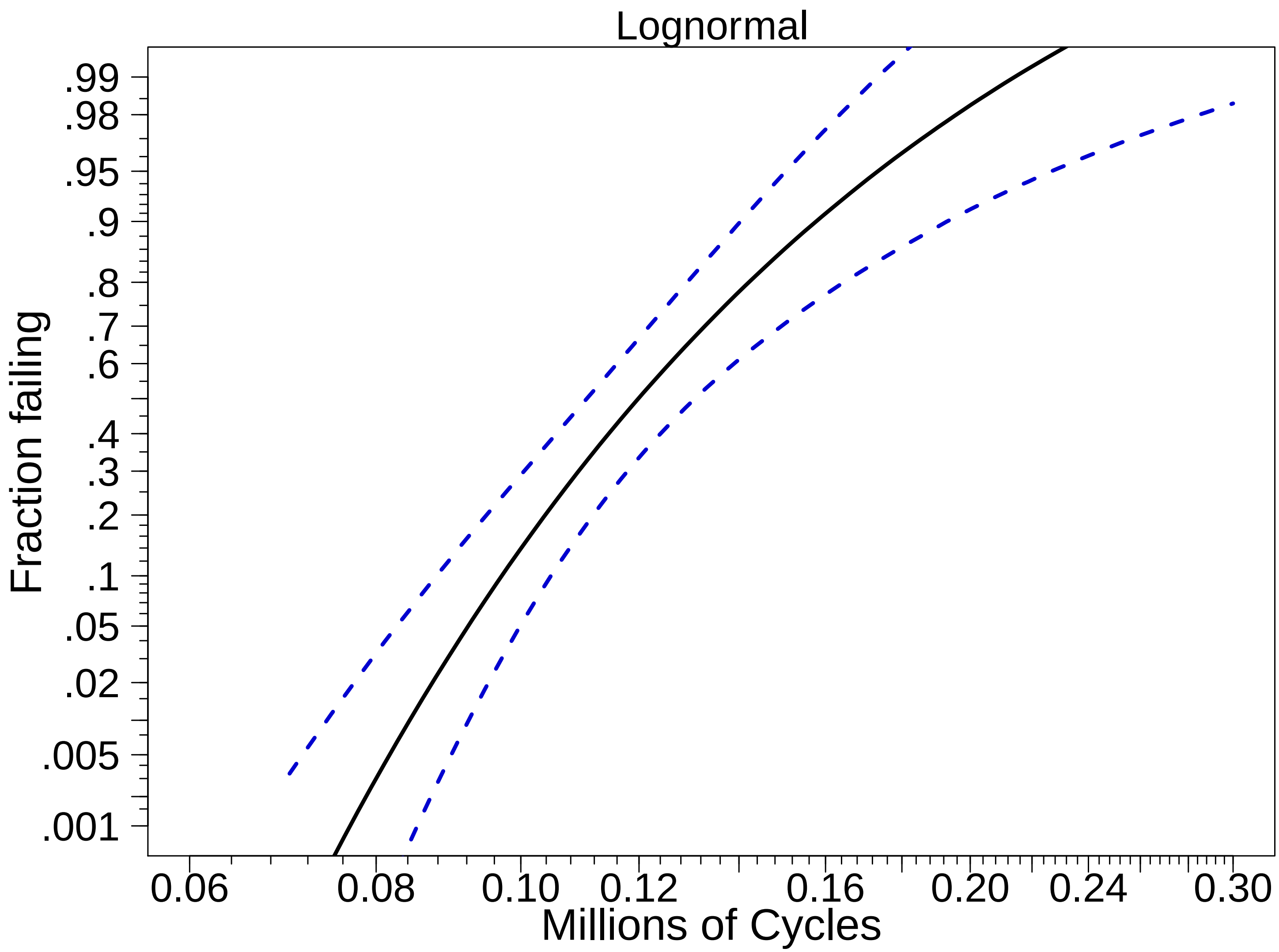
y



Lognormal Probability Plot of the Alloy-A Estimated cdf and 95% Two-Sided Credible Intervals under the Bivariate <u>Lognormal</u> Model for β_1 and β_2.



Lognormal Probability Plot of the Alloy-A Estimated cdf and 95% Two-Sided Credible Intervals under the Bivariate Normal Model for β_1 and β_2



Chapter 21

Repeated Measures Degradation Analysis

Segment 4

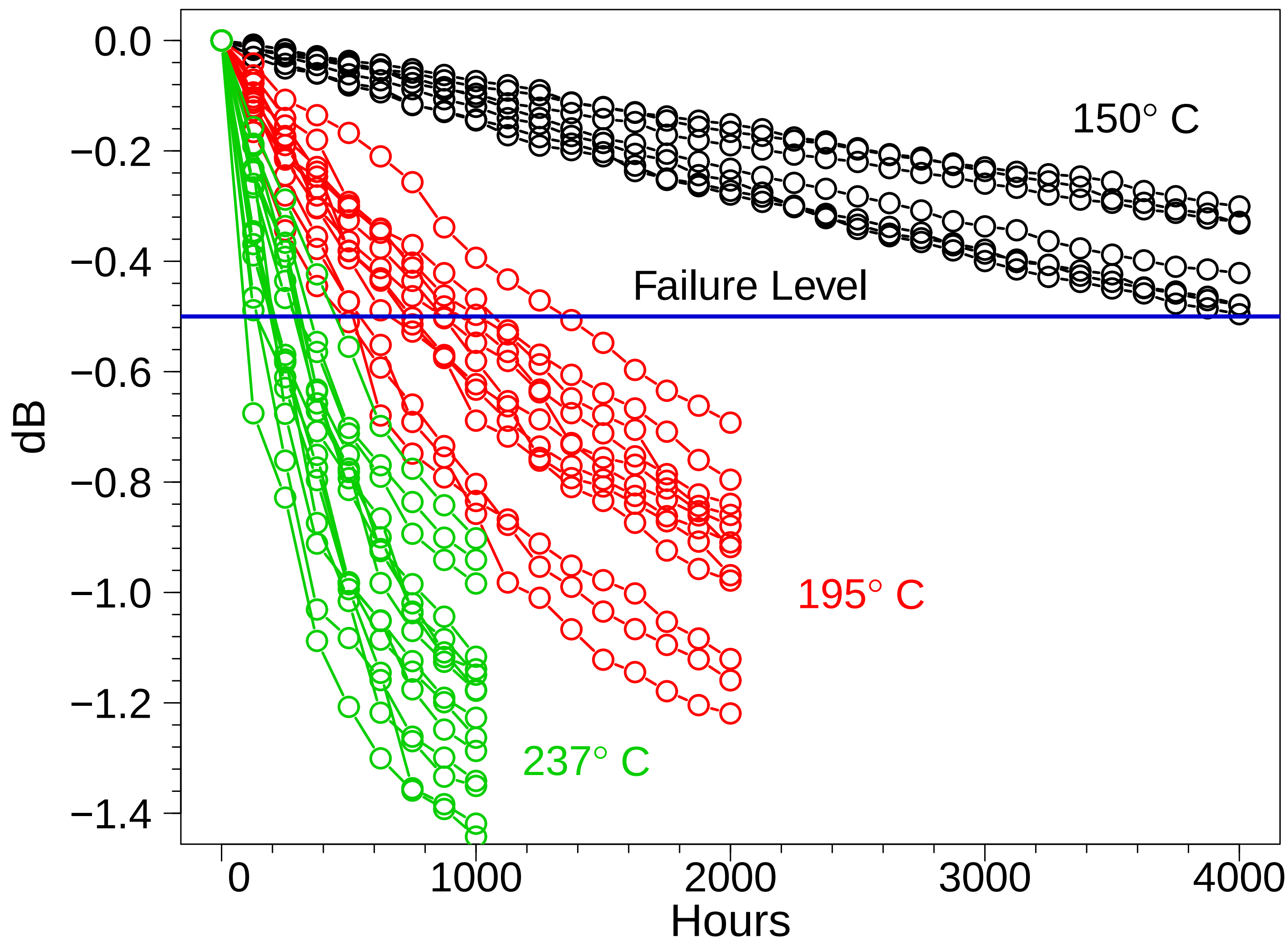
Accelerated Repeated Measures Degradation Modeling and Analysis

Device-B Reliability Assessment

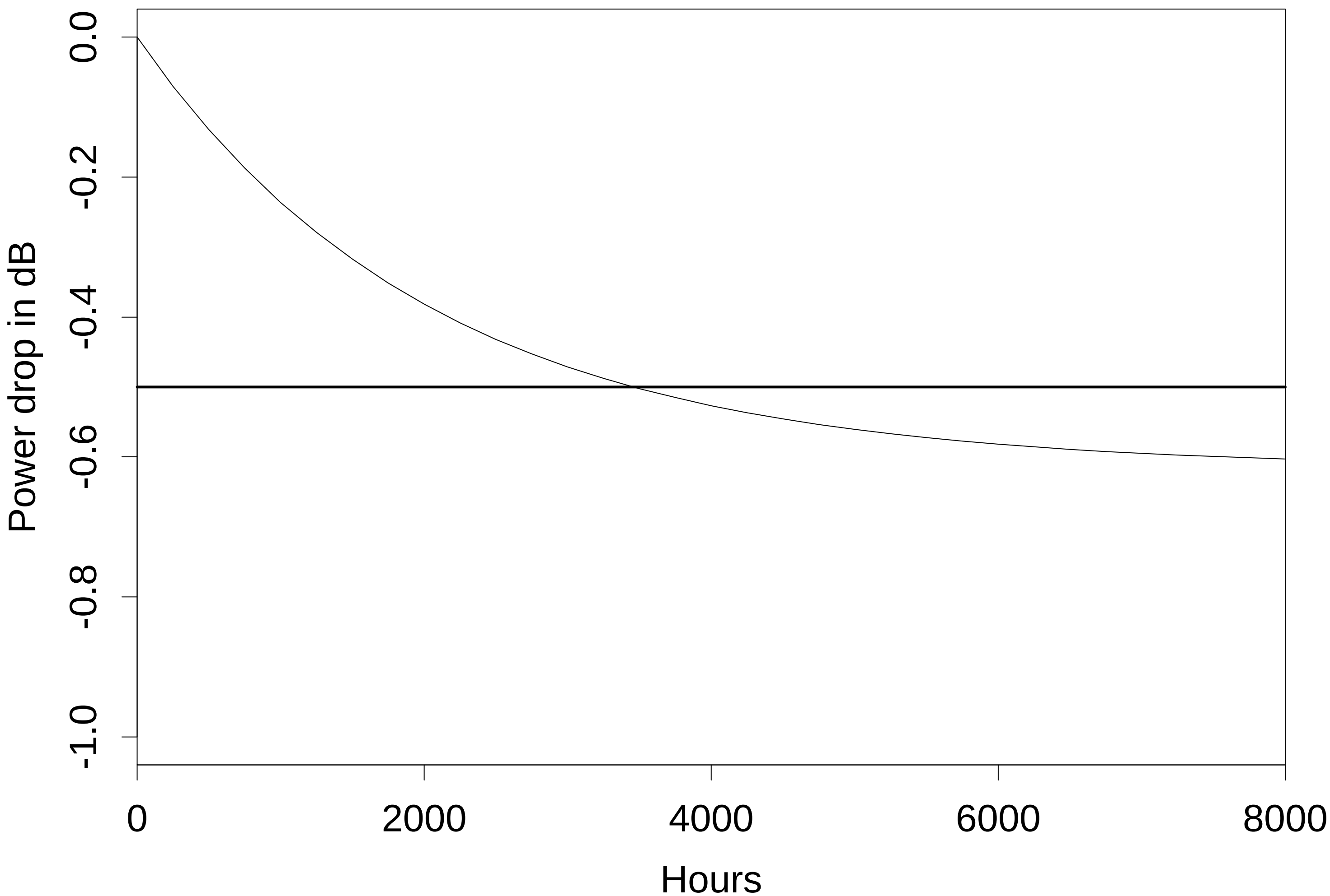
Device-B RF Power Amplifier

- Device-B was to be used in a communications satellite.
- Device-B is said to have failed if the power output drops by more than 0.50 dB.
- The manufacturer of the satellite would make a profit if the satellite remains operational for ten years.
- An accelerated life test was conducted to decide how much redundancy to build into the satellite.
- Units were tested at 150, 195, and 237°C junction temperature.
- No failures at 150°C.
- The use condition is 80°C junction temperature.

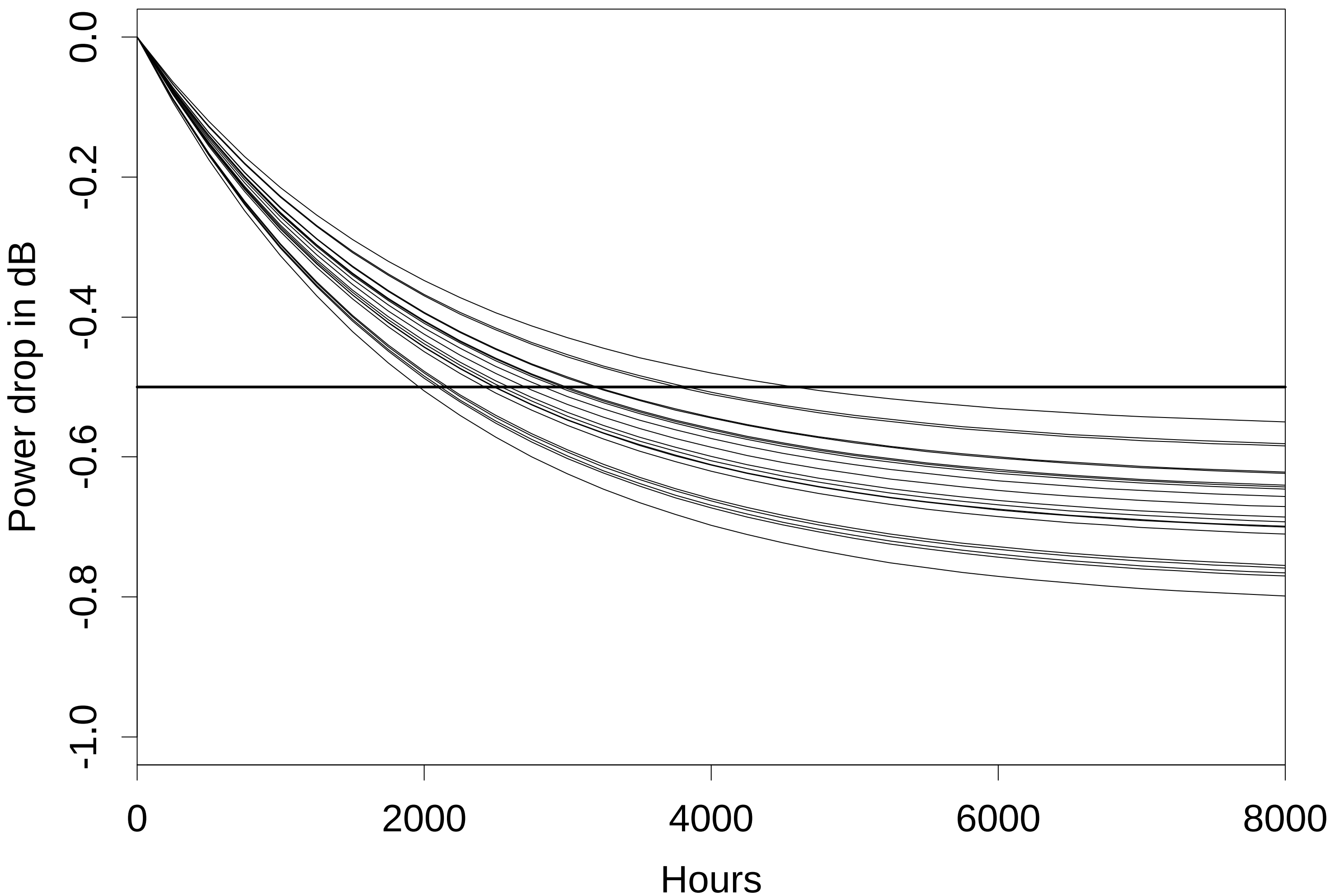
Device-B Power Drop
Accelerated Degradation Test Data
at 150°C, 195°C, and 237°C
(Use Conditions is 80°C)



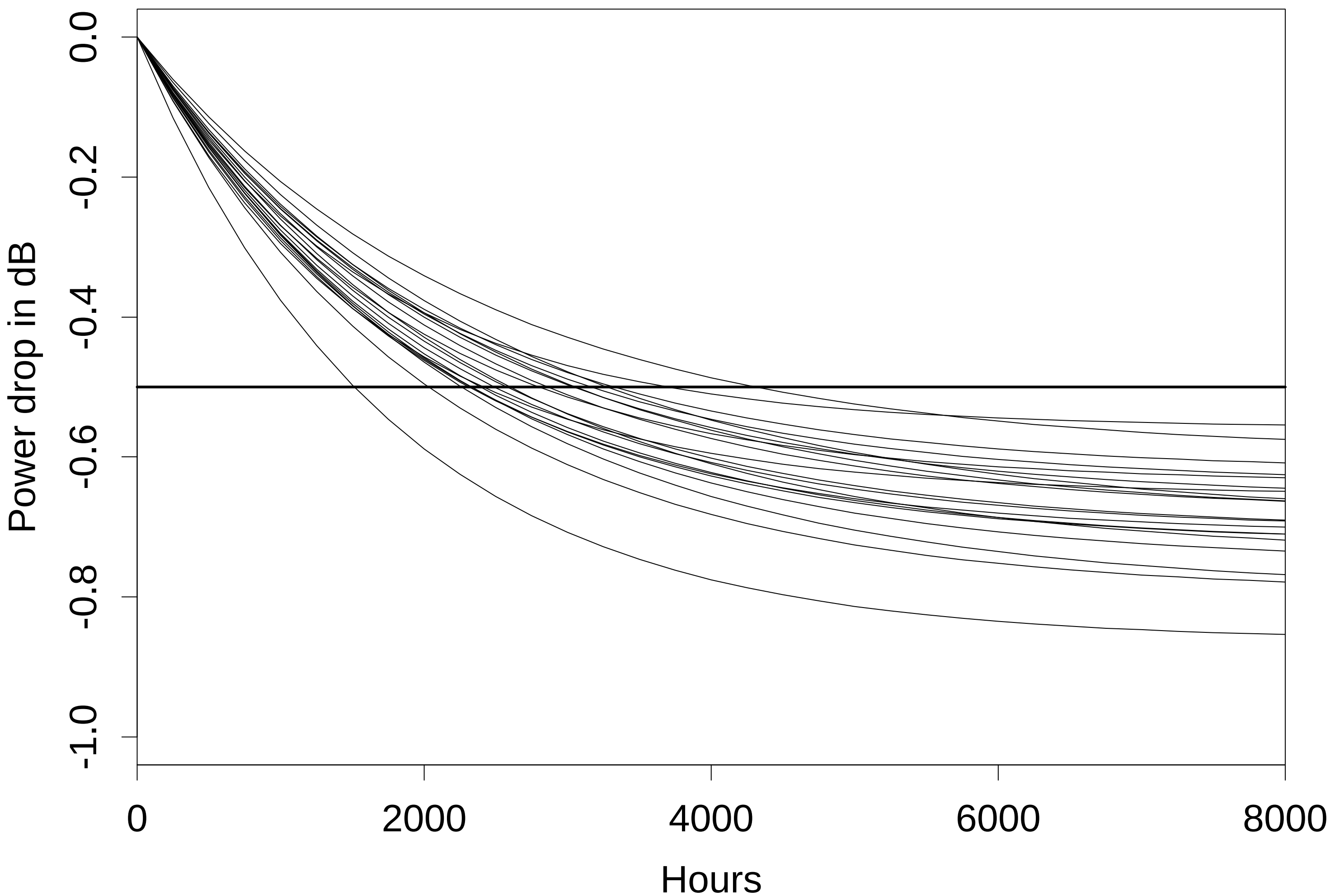
Simulated Device-B Power Drop
Model 8 $\xi(t) = \xi_\infty[1 - \exp(-\mathcal{R}\, t)]$
Fixed ξ_∞ and Rate $\mathcal{R}$



Simulated Device-B Power Drop
Model 8 $\xi(t) = \xi_\infty[1 - \exp(-\mathcal{R}\,t)]$
Variability in Asymptote ξ_∞



Simulated Device-B Power Drop
Model 8 $\xi(t) = \xi_{\infty}[1 - \exp(-\mathcal{R}\, t)]$
Variability in Asymptote ξ_{∞} and Rate $\mathcal{R}$



Device-B Power Drop
Repeated Measures Degradation Model

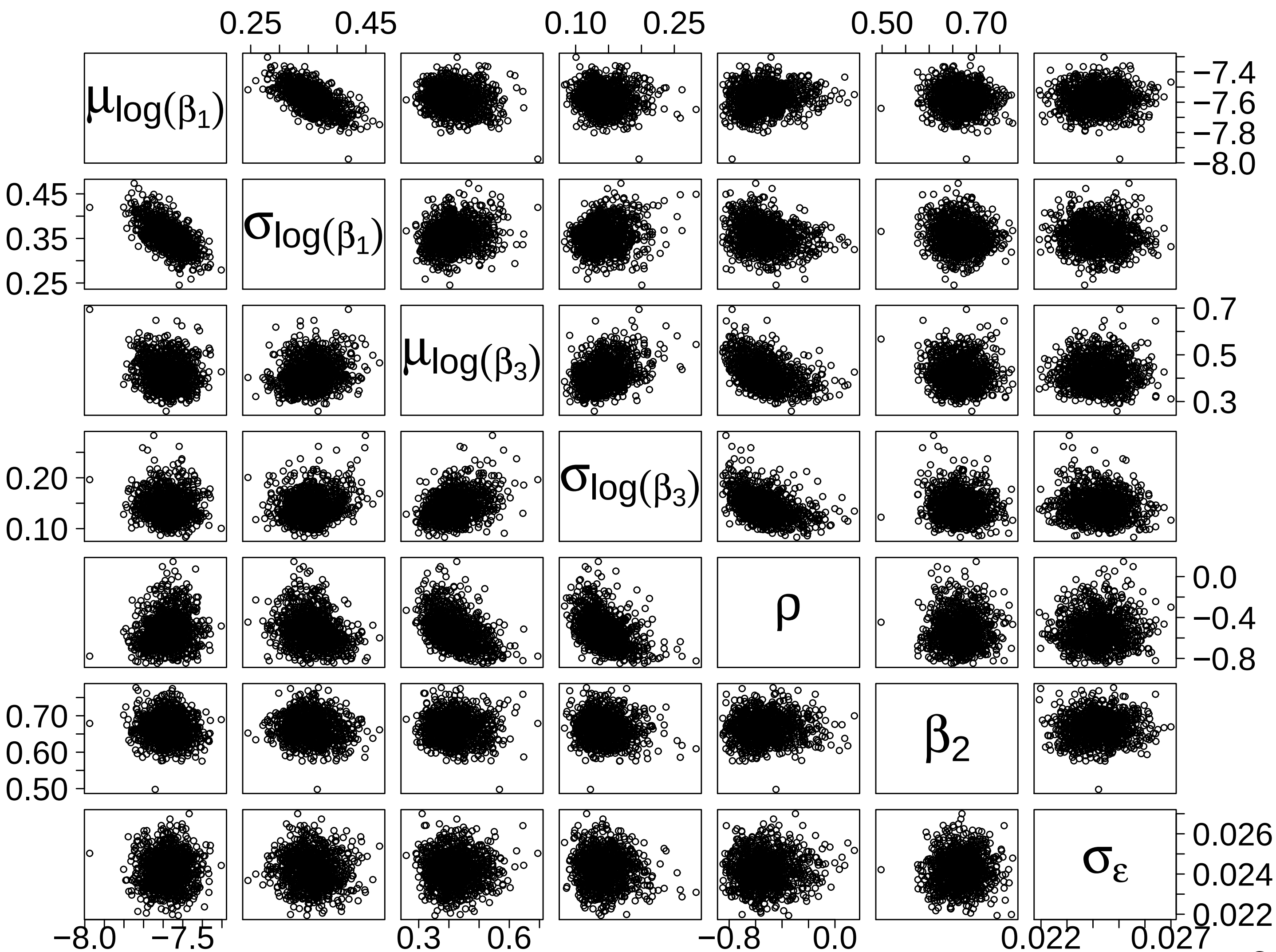
- The degradation path model is

$$Y = \xi(t) + \epsilon = -\beta_3(1 - \exp\{-\beta_1 \exp[-\beta_2(x - x_0)]t\}) + \epsilon$$

Note that $\xi(0) = 0$.

- $x = 11604.52/(\text{Temp}\,^\circ\text{C} + 273.15)$ and $x_0 = 25.433$ is the sample mean of all of the x values (Arrhenius transformation of temperatures).
- β_1 and β_3 are random parameters with a joint bivariate lognormal (BVLN) distribution.
- The BVLN parameters are $(\mu_{\log(\beta_1)}, \mu_{\log(\beta_3)}, \sigma_{\log(\beta_1)}, \sigma_{\log(\beta_3)}, \rho)$.
- β_2, the effective activation energy, is fixed but unknown.
- The error term $\epsilon \sim \text{NORM}(0, \sigma_\epsilon)$ is independent of the random parameters.
- Weakly informative prior distributions were used.

Device-B Power Drop ARMDT
Asymptotic Path Model
Bivariate Lognormal Distribution Random Parameters
Weakly Informative Prior Distribution
Posterior Pairs Plot



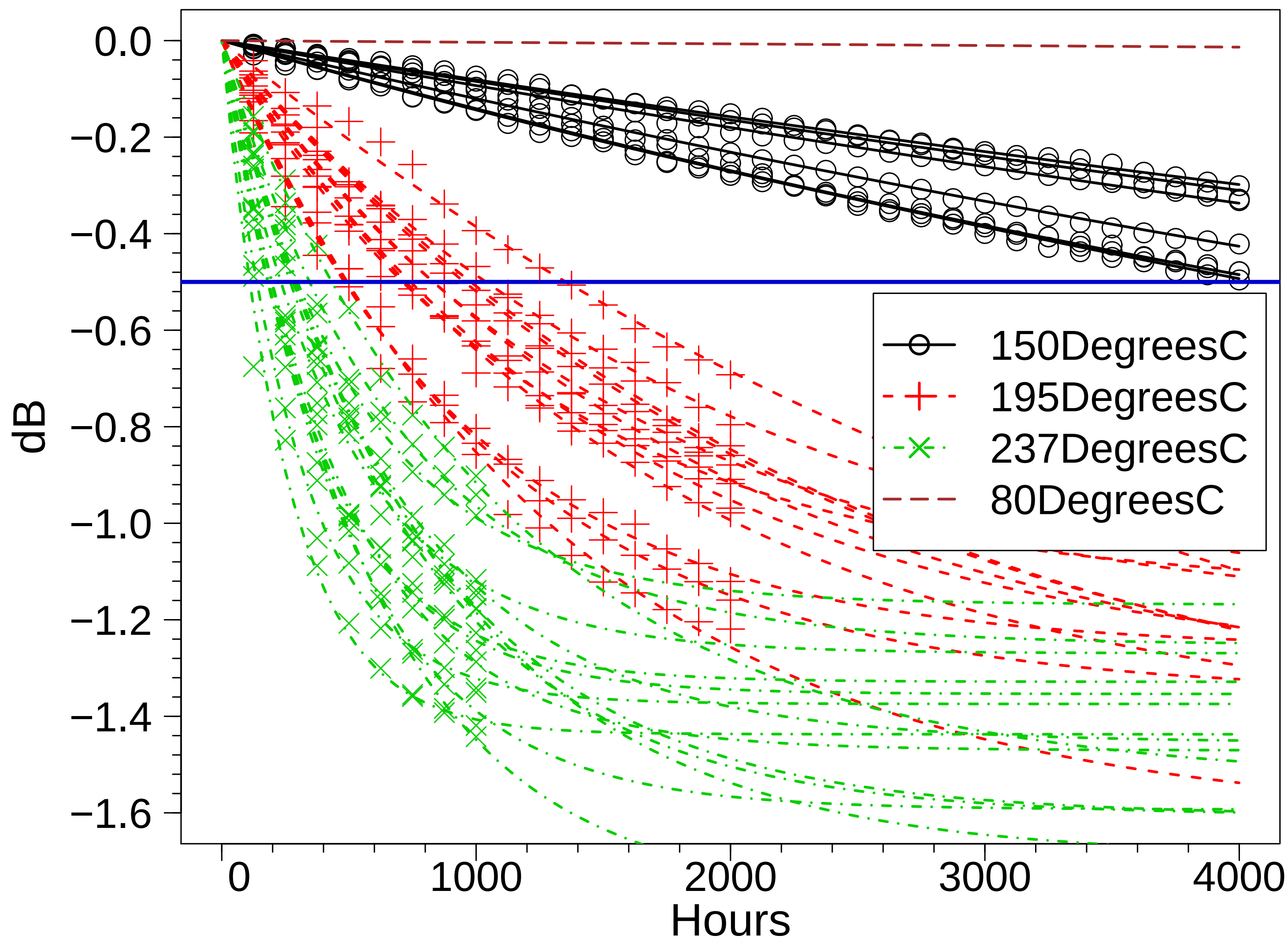
Estimation Results for the Device-B Power Drop Data BVLN Repeated Measures Degradation Model Medians of the Marginal Posterior Distributions

- Point estimates for the parameters are: $\widehat{\mu}_{\log(\beta_1)} = -7.6$, $\widehat{\mu}_{\log(\beta_3)} = 0.36$, $\widehat{\sigma}_{\log(\beta_1)} = 0.41$, $\widehat{\sigma}_{\log(\beta_3)} = 0.14$, $\widehat{\rho} = -0.51$, $\widehat{\beta}_2 = 0.66$, and $\widehat{\sigma}_\epsilon = 0.024$.

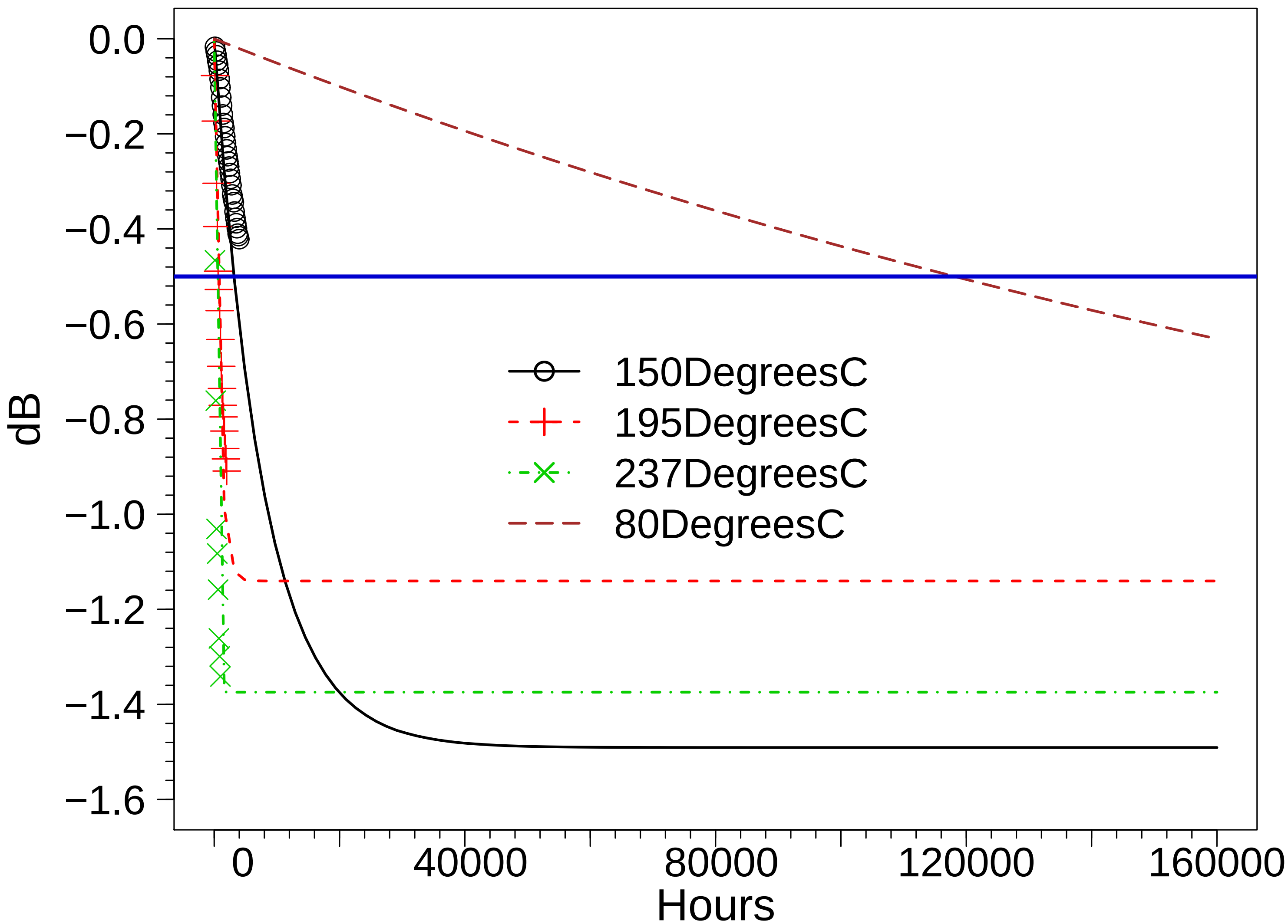
- Statistical uncertainty for the parameters can be assessed by inspection of the pairs plot of the MCMC draws and computation of corresponding credible intervals.

- Draws from the joint posterior distribution of $\mu_{\log(\beta_1)}$, $\mu_{\log(\beta_3)}$, $\sigma_{\log(\beta_1)}$, $\sigma_{\log(\beta_3)}$, β_2, ρ and $\widehat{\sigma}_\epsilon$ can be used to estimate the failure-time distribution.

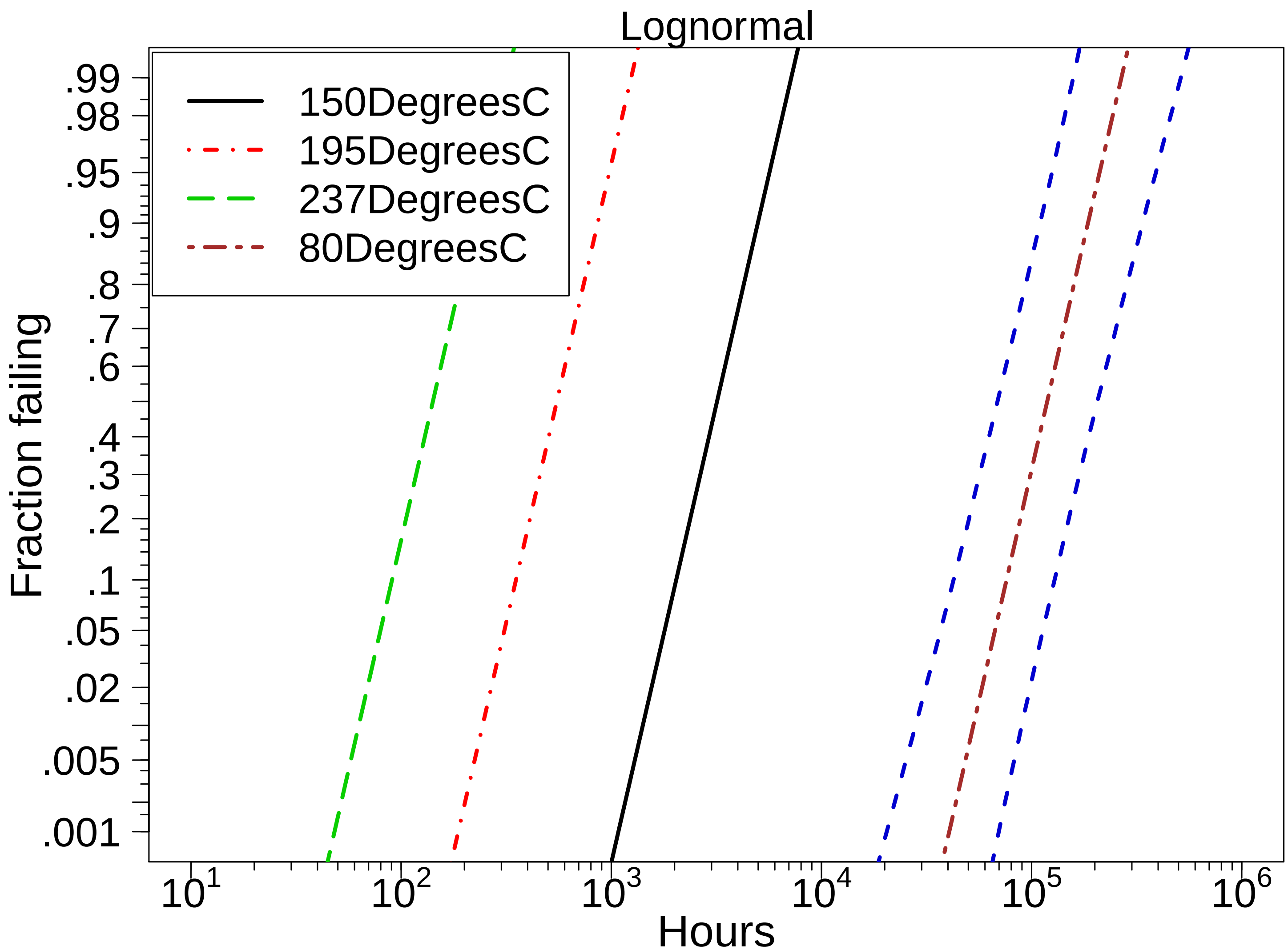
Device-B Power Drop Observations and Fitted Degradation Model for the $i = 1, \ldots, 34$ Sample Paths



Device-B Power Drop Observations and Fitted Degradation Model Showing Extrapolation in Time and Temperature



Device-B Degradation Model Estimates of the Failure-Time Distributions on Lognormal Probability Paper 95% Credible Intervals Given for 80°C



Chapter 21

Repeated Measures Degradation Analysis

Segment 5

Accelerated Repeated Measures Degradation Modeling and Analysis with Multiple Accelerating Variables

Modeling the LED-A Repeated-Measures Degradation Data

Accelerated Repeated Measures Degradation Test for LED-A

- Test was conducted to evaluate whether a certain kind of LED would be suitable for use in a high-performance flashlight.

- Accelerated tests run with 30 LEDs at each of six combinations of junction temperature and current (two accelerating variables).

- Measured light output on each unit periodically until 900 hours.

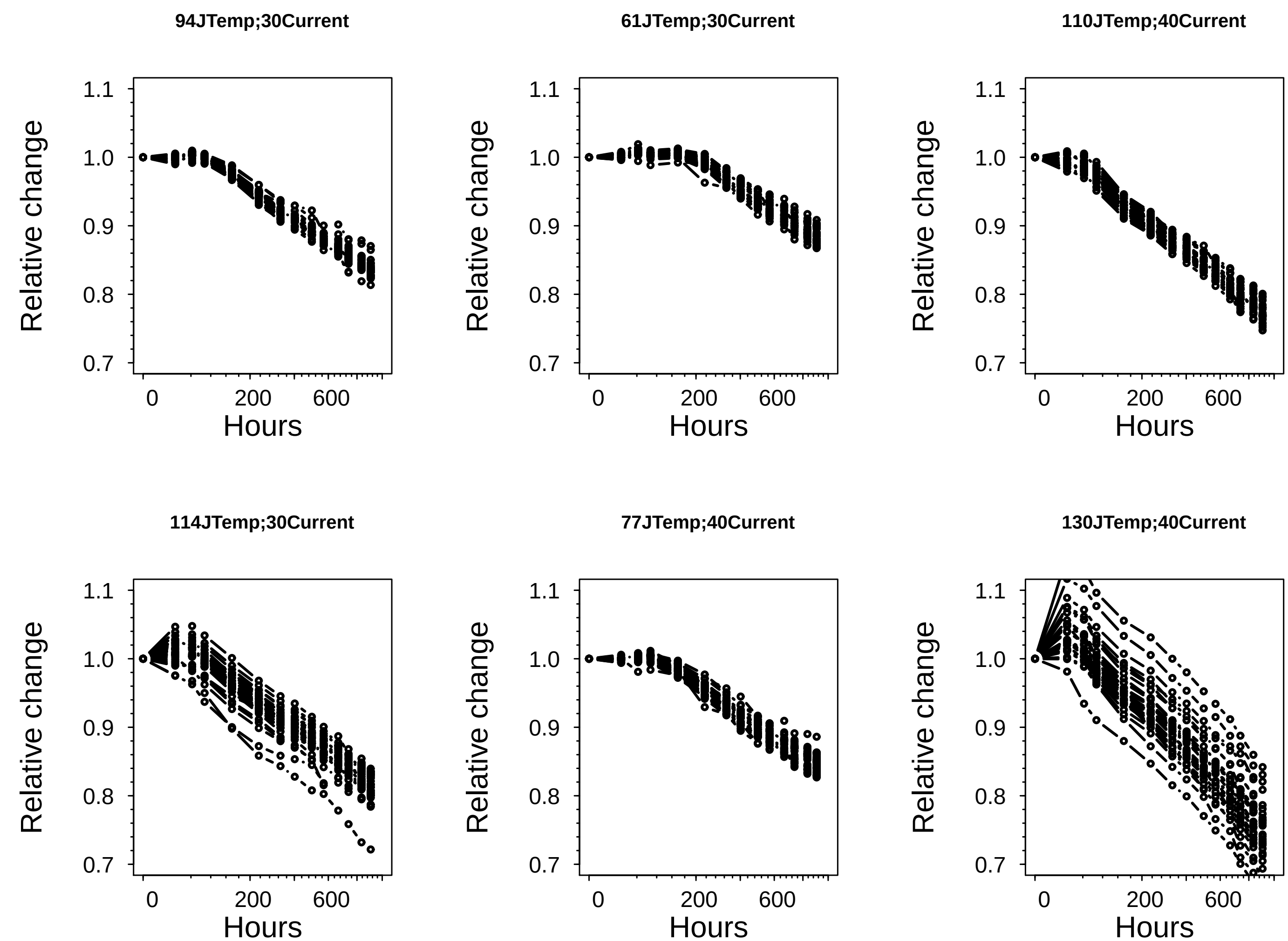
- Failure was defined as the time when light output fell below 60% of an LEDs initial output.

- Engineers wanted to estimate the LED failure-time distribution at the use conditions of 40°C junction temperature and 20 mA of current.

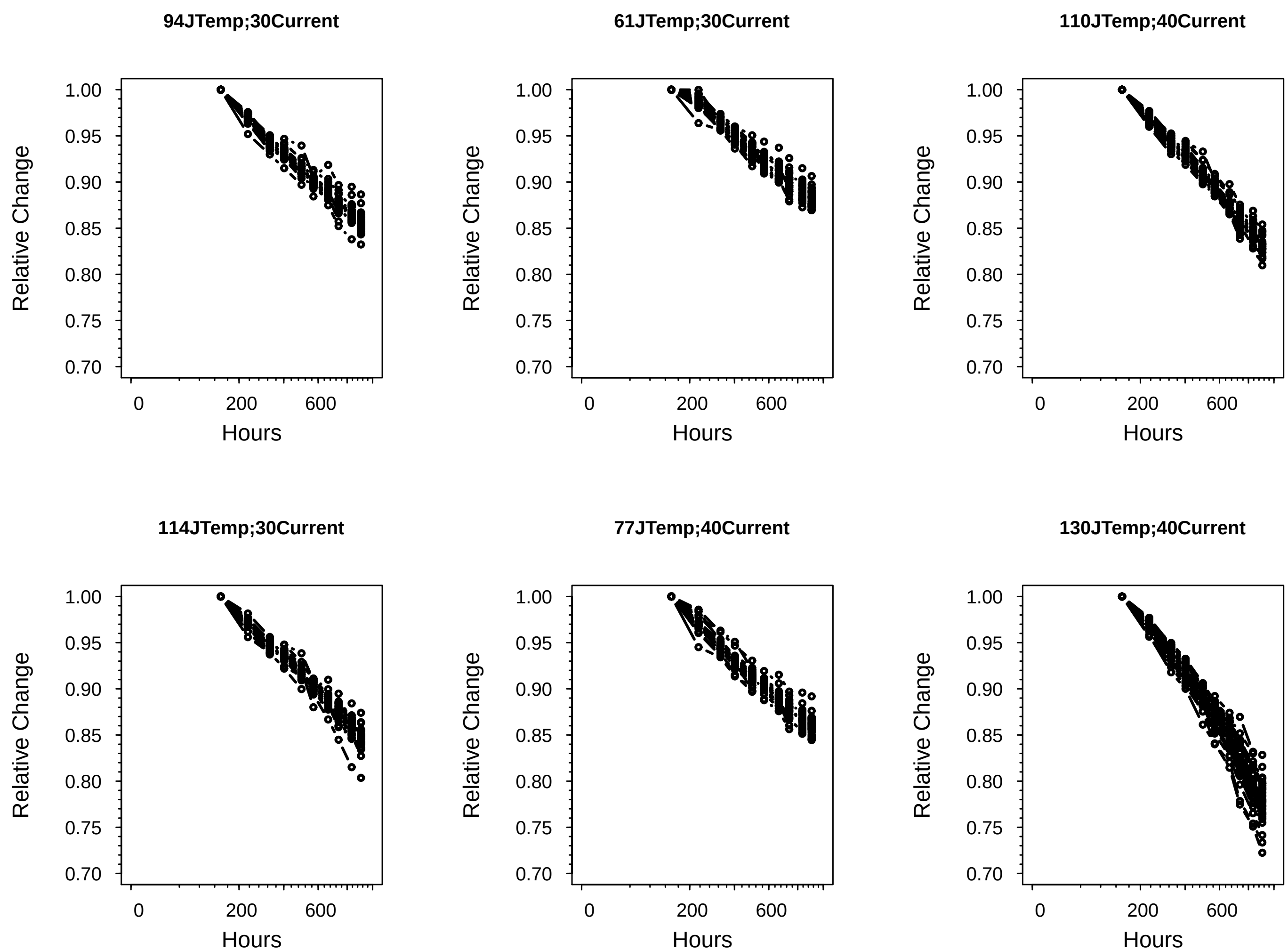
LED-A Data Pre-Processing and Cleaning

- The data were messy and required pre-processing and cleaning before analysis.
- In the first 138 hours of the test, light output actually increased.
- Data before 138 hours were discarded and the output for all units were re-normalized to be equal to one at that time (divide all readings for an LED by the light output at 138 hours).
- The data at 130°C junction temperature and 40 mA current resulted in a different of failure mode (thought to be due to thermal runaway) and had to be discarded.

LED-A
Plot of Relative Light Output at Six Conditions on Square Root-Linear Axes Before Cleaning



LED-A
Plot of Relative Light Output at Six Conditions on Square Root-Linear Axes After Cleaning



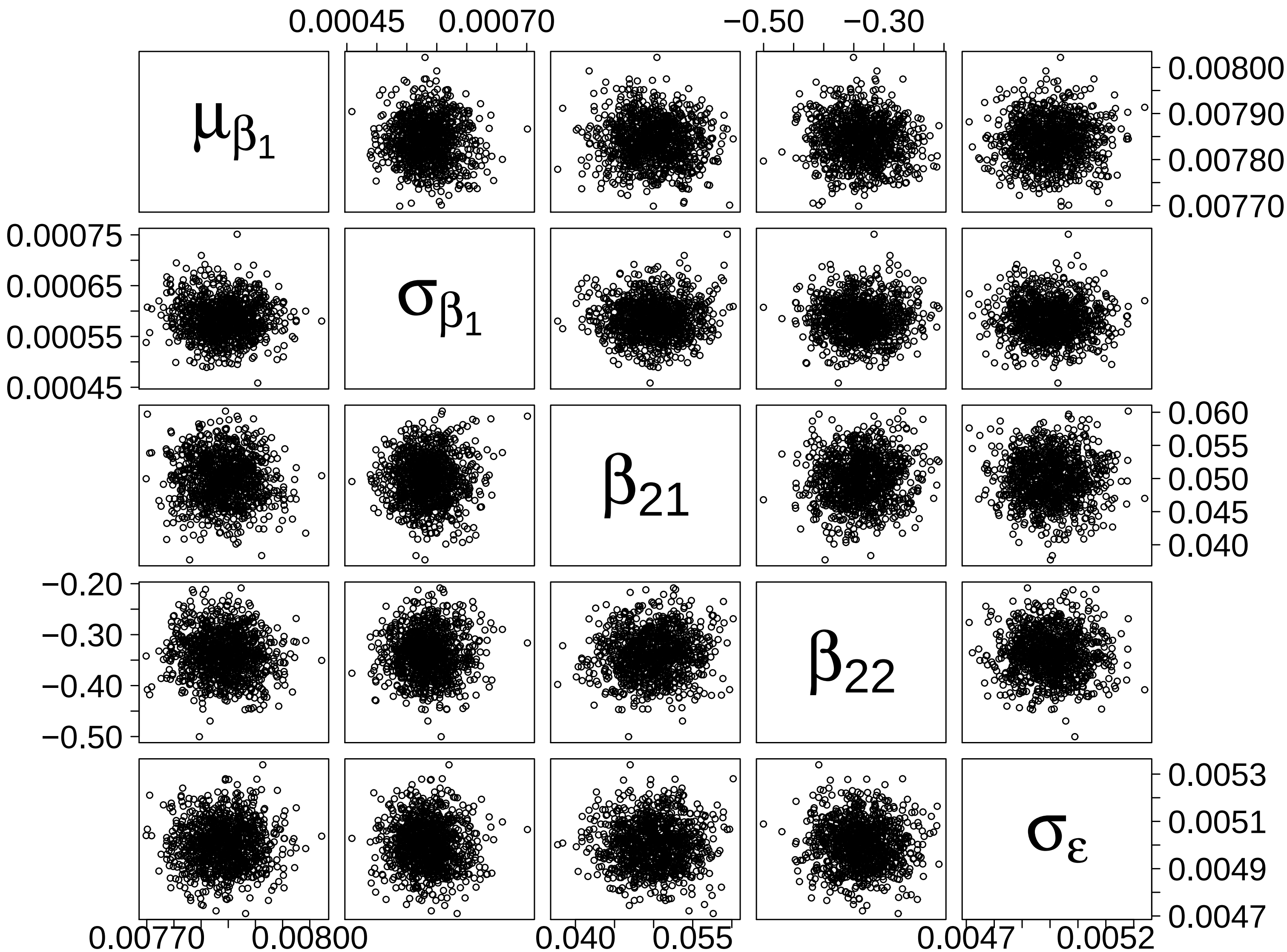
LED-A ARMDT Light Output Model
Repeated Measures Degradation Model

- The degradation model is

$$Y = 1 - \beta_1 \exp\Big\{-[\beta_{21}(x_1 - x_{0,1}) + \beta_{22}(x_2 - x_{0,2})]\Big\}\tau + \epsilon$$

- The intercept at 138 hours was fixed at 1, corresponding to the data.
- $\tau = \sqrt{\text{Hours}} - \sqrt{138}$.
- The random slope is $\beta_1 \sim \text{TNORM}(\mu_{\beta_1}, \sigma_{\beta_1})$.
- $x_1 = 11604.52/(\text{Temp}\,^\circ\text{C} + 273.15)$, $x_2 = \log(\text{Current})$, $x_{0,1} = 31.949$ and $x_{0,2} = 3.5163$ are the sample means of the corresponding x variables.
- The error term $\epsilon \sim \text{NORM}(0, \sigma_\epsilon)$ is independent of the random parameters.
- Weakly informative prior distributions were used.

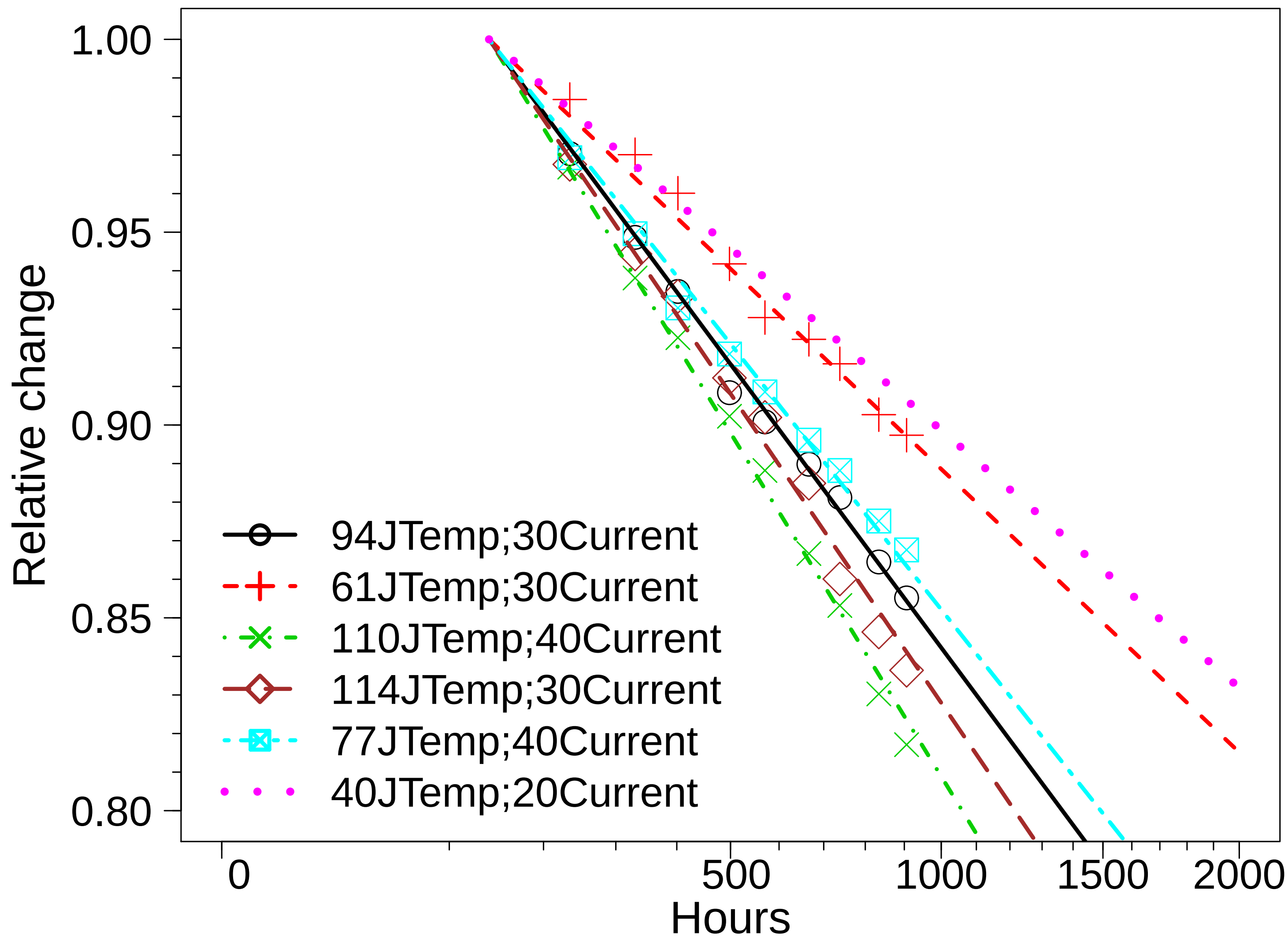
LED-A ARMDT
Linear Model Normal Distribution Slope
Weakly Informative Prior Distribution
Posterior Pairs Plot



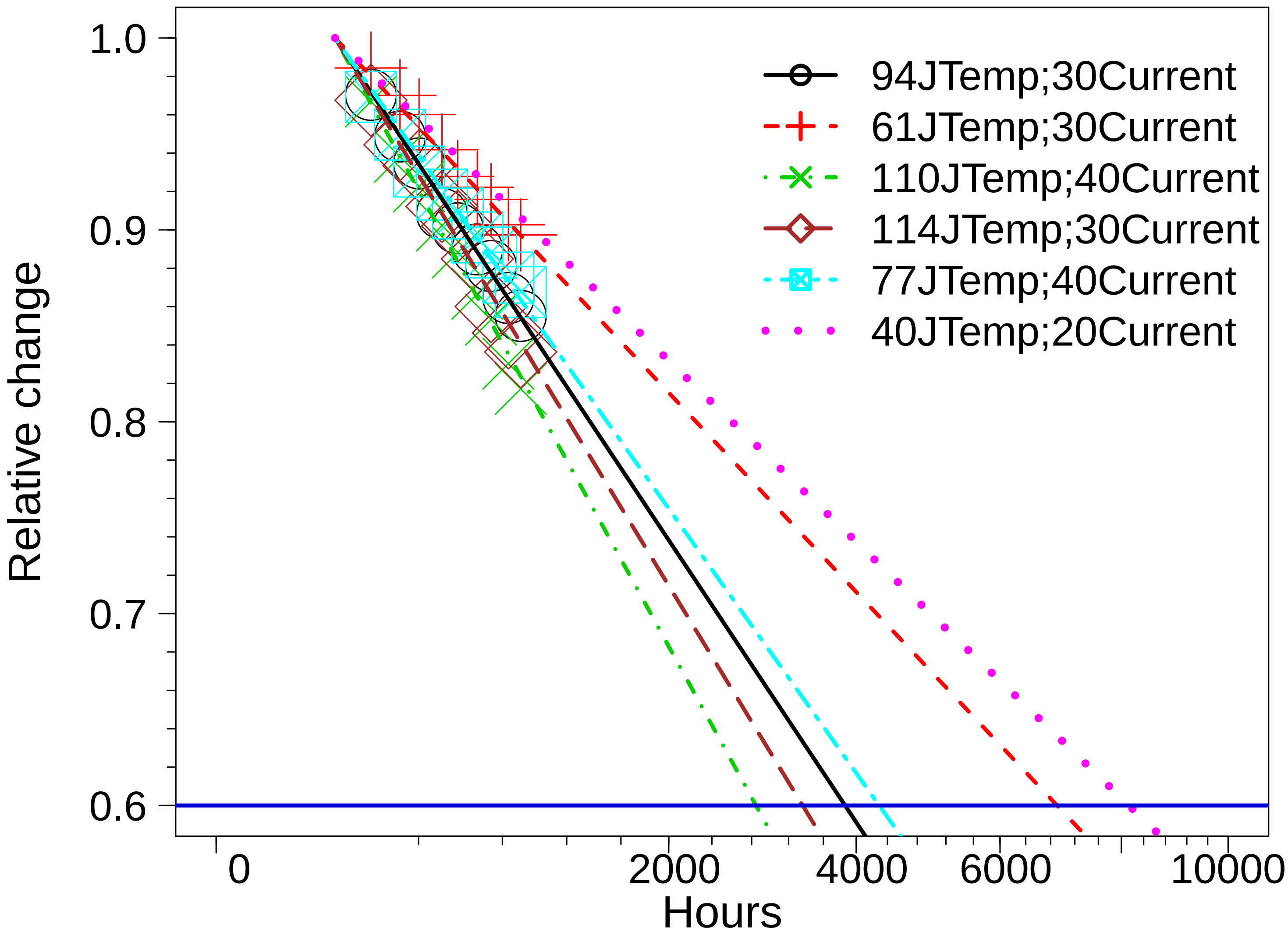
Estimation Results for the LED-A ARMDT Light Output Data Random Slope Repeated Measures Degradation Model Medians of the Marginal Posterior Distributions

- The parameter point estimates are $\widehat{\mu}_{\beta_1} = 0.0078$, $\widehat{\sigma}_{\beta_1} = 0.0006$, $\widehat{\beta}_{21} = 0.050$, $\widehat{\beta}_{22} = -0.33$, and $\widehat{\sigma}_{\epsilon} = 0.0050$.
- Statistical uncertainty for the parameters can be assessed by inspection of the pairs plot of the MCMC draws and computation of corresponding credible intervals.
- Draws from the joint posterior distribution of μ_{β_1}, σ_{β_1}, β_{21}, β_{22}, and $\widehat{\sigma}_{\epsilon}$ can be used to estimate the failure-time distribution.

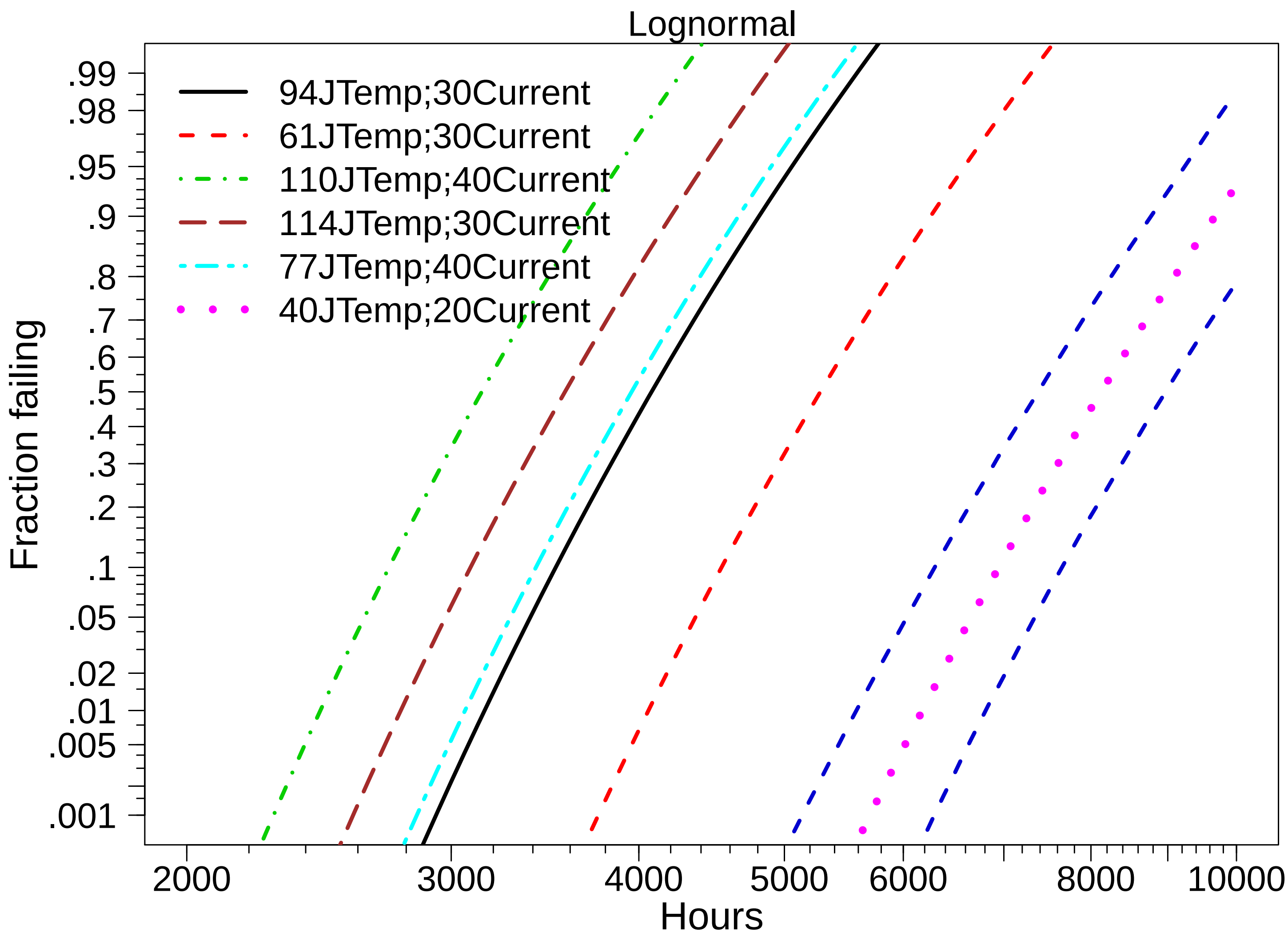
LED-A ARMDT Decrease in Light Output and Fitted Degradation Model (for the First Path in Each Group)



LED-A ARMDT Decrease in Light Output and Fitted Degradation Model (for the First Path in Each Group) Showing Extrapolation in Time



LED-A Degradation Model Estimates of the Failure-Time Distributions on Lognormal Probability Paper 95% Credible Intervals Given for 40°C and 20 mA



Chapter 21

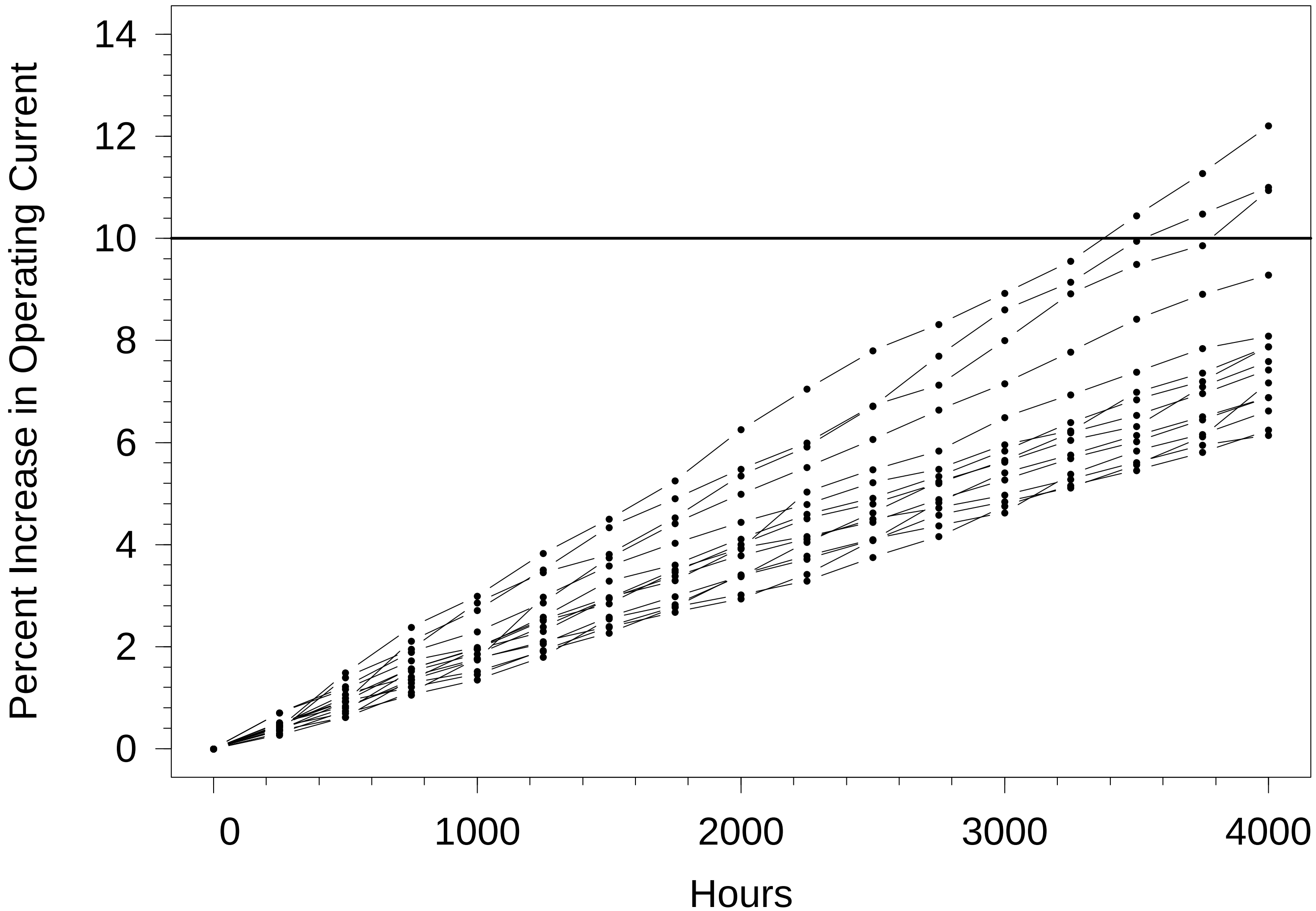
Repeated Measures Degradation Analysis

Segment 6

Accelerated Repeated Measures Degradation Data

Other Examples

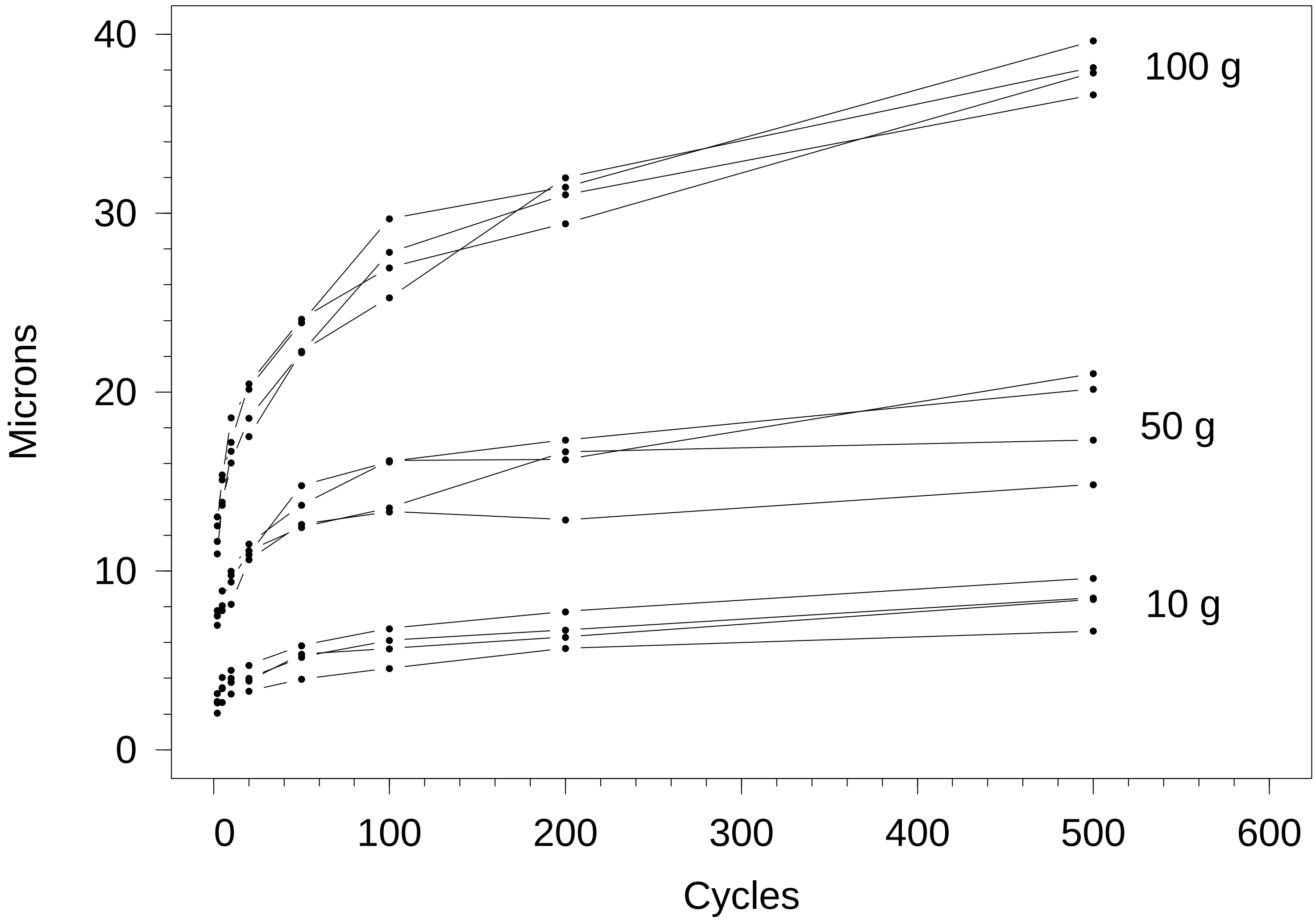
Plot of Laser Operating Current as a Function of Time



Laser Life Data

- Percentage increase in operating current for GaAs lasers tested at 80°C.
- Fifteen (15) devices each measured every 250 hours up to 4000 hours of operation.
- For these devices and the corresponding application, a $\mathcal{D}_{\mathsf{f}} =$ 10% increase in current was the specified **failure criterion**.

Scar Width Resulting from a Metal-to-Metal Sliding Test for Different Applied Weights



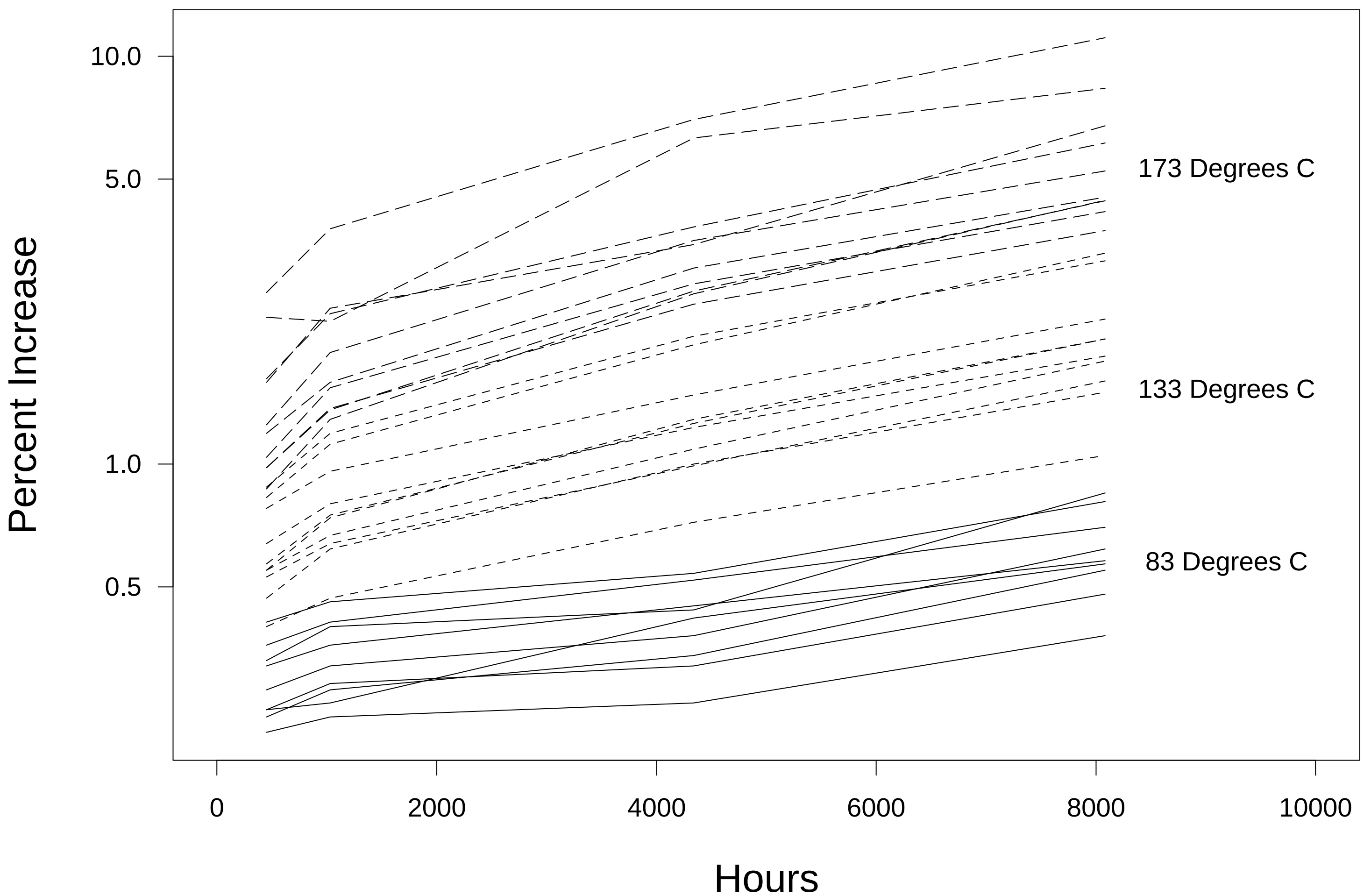
Sliding Metal Wear Data Analysis

- An experiment was conducted to test the wear resistance of a particular metal alloy.

- The sliding test was conducted over a range of different applied weights in order to study the effect of weight and to gain a better understanding of the wear mechanism.

- The predicted pseudo failure times were obtained by using ordinary least squares to fit a line through each sample path on the log-log scale and extrapolating to the time at which the scar width would be 50microns.

Percent Increase in Resistance Over Time for Carbon-Film Resistors (Shiomi and Yanagisawa 1979)



Percent Increase in Resistance Over Time for Carbon-Film Resistors

- Carbon-film resistors are components in electronic circuits.
- Carbon is an organic compound that will chemically degrade with time.
- The chemical degradation tends to increase resistance over time.
- The need to evaluate the rate of degradation and the failure-time distribution quickly required an accelerated test.

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Hudak, Jr., S. J., A. Saxena, R. J. Bucci, and R. C. Malcolm (1978). Development of standard methods of testing and analyzing fatigue crack growth rate data. Technical report, AFML-TR-78-40. Westinghouse R & D Center, Westinghouse Electric Corporation, Pittsburgh, PA 15235. []

Meeker, W. Q., L. A. Escobar, and F. G. Pascual (2021). *Statistical Methods for Reliability Data* (Second Edition). Wiley. [1]