Chapter 4

Location-Scale and Log-Location-Scale Parametric Distributions

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Chapter 4
Location-Scale and Log-Location-Scale Parametric Distributions

Topics discussed in this chapter are:

- Motivation, definition, metrics, and importance of location-scale and log-location-scale parametric distributions.
- Description and properties of the exponential, normal, log-normal, smallest extreme value, and Weibull distributions.
- Description and properties of other location-scale and log-location-scale parametric distributions including the largest extreme value, Fréchet, logistic, and loglogistic distributions.
- Description and properties of the generalized gamma distribution.
- Description and properties of threshold log-location-scale distributions.

Chapter 4

Segment 1

Location-Scale and Log-Location-Scale Parametric Distributions: Motivation, Definition, Metrics, and Importance

Motivation for Parametric Models

- Complement nonparametric models.

- Parametric models can be described concisely with just a few parameters, instead of having to report an entire curve.

- It is possible to use a parametric model to extrapolate (in time) to the lower or upper tail of a distribution.

- Parametric models provide smooth estimates of failure-time distributions.

In practice, it is often useful to compare various parametric estimates with nonparametric estimates for a given data set.

Location-Scale Distributions

The distribution of the random variable Y belongs to the location-scale family of distributions if the cdf of Y can be expressed as

$$F(y;\mu,\sigma) = \Pr(Y \le y) = \Phi\left(\frac{y-\mu}{\sigma}\right), \quad -\infty < y < \infty$$

where $-\infty < \mu < \infty$ is a location parameter and $\sigma > 0$ is a scale parameter.

Φ is the cdf of Y when $\mu = 0$ and $\sigma = 1$ and Φ does not depend on any unknown parameters.

Note: $Z = (Y-\mu)/\sigma$ is a standardized random variable and $Z \sim \Phi(z)$. Thus the distribution of Z does **not** depend on any unknown parameters.

Log-Location-Scale Distributions

The distribution of the random variable T belongs to the log-location-scale family of distributions if the cdf of T can be expressed as

$$F(t;\mu,\sigma) = \Pr(T \le t) = \Phi\left(\frac{\log(t)-\mu}{\sigma}\right), \quad 0 < t < \infty$$

where $0 < \exp(\mu) < \infty$ is a scale parameter and $\sigma > 0$ is a shape parameter.

Φ is the cdf of $\log(T)$ when $\mu = 0$ and $\sigma = 1$ and Φ does not depend on any unknown parameters.

Note: $Z = (\log(T) - \mu)/\sigma$ is a standardized random variable and $Z \sim \Phi(z)$. Thus the distribution of Z does **not** depend on any unknown parameters.

Metrics/Functions of the Parameters

- Cumulative distribution function (cdf) of T

$$F(t; \boldsymbol{\theta}) = \Pr(T \leq t), \quad t > 0.$$

- The p quantile of T is the smallest value t_p such that

$$F(t_p; \boldsymbol{\theta}) \geq p.$$

- Hazard function of T

$$h(t; \boldsymbol{\theta}) = \frac{f(t; \boldsymbol{\theta})}{1 - F(t; \boldsymbol{\theta})}, \quad t > 0.$$

Metrics/Functions of the Parameters-Continued

- The mean time to failure, MTTF, of T (also known as the expectation of T)

$$\mathsf{E}(T) = \int_0^\infty t f(t;\boldsymbol{\theta})\, dt = \int_0^\infty [1 - F(t;\boldsymbol{\theta})]\, dt.$$

If $\int_0^\infty t f(t;\boldsymbol{\theta})\, dt = \infty$, we say that the mean of T **does not exist**.

- The variance (or the second central moment) of T and the standard deviation

$$\mathsf{Var}(T) = \int_0^\infty [t - \mathsf{E}(T)]^2 f(t;\boldsymbol{\theta})\, dt$$
$$\mathsf{SD}(T) = \sqrt{\mathsf{Var}(T)}.$$

Importance of (Log)-Location-Scale Distributions

- The most commonly used statistical distributions are members of these classes of distributions: exponential, normal, Weibull, lognormal, Fréchet, logistic, loglogistic, smallest extreme value and largest extreme value distributions.

- Methods of inference, statistical theory, and computer software generated for the general family can be applied to this large, important class of models.

- Theory for (log)-location-scale distributions is relatively simple.

Chapter 4

Segment 2

Exponential, Normal, Lognormal, Smallest Extreme Value, and Weibull Distributions

Exponential Distribution

For $T \sim \text{EXP}(\theta, \gamma)$,

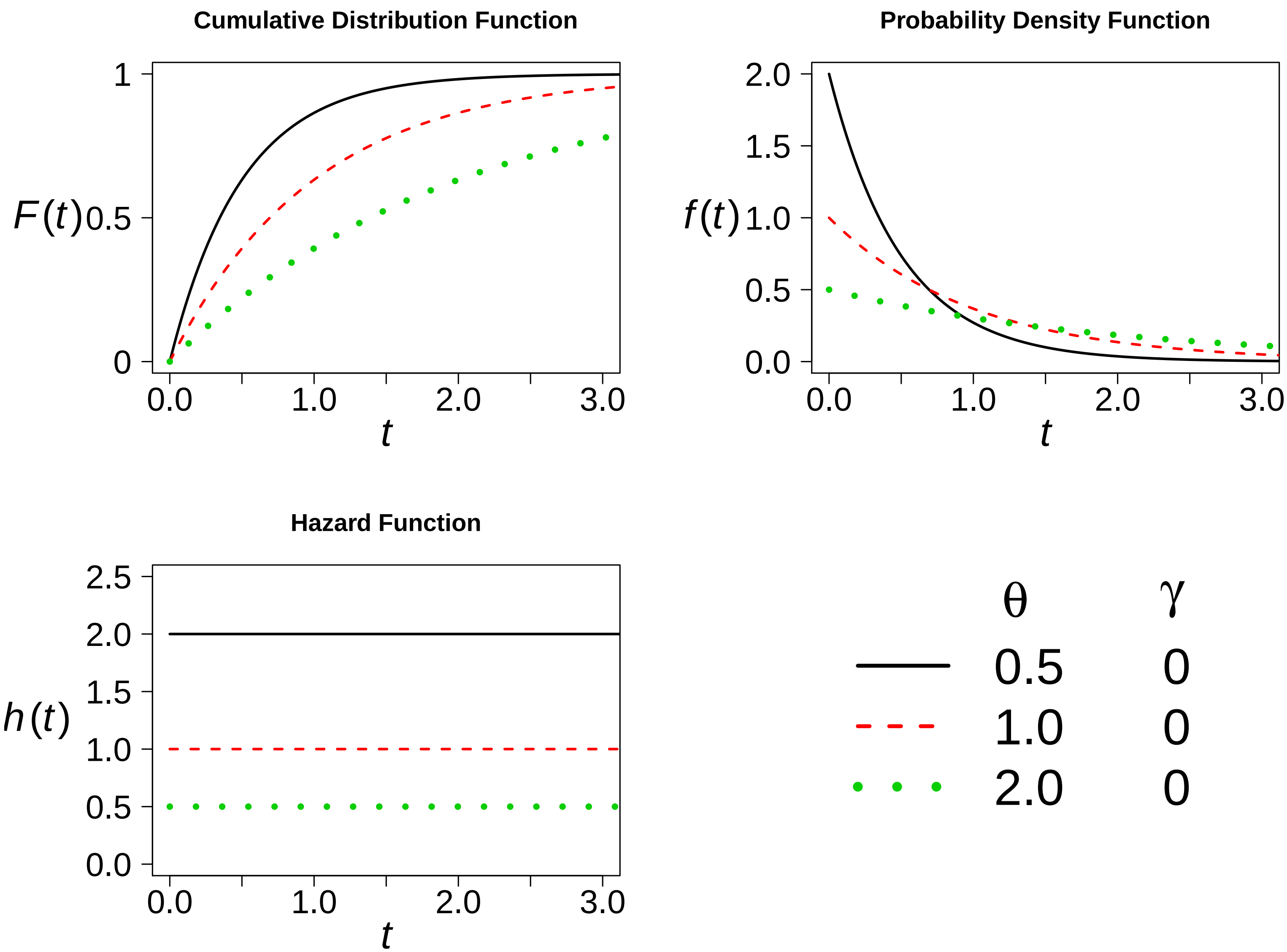
$$F(t;\theta,\gamma) = 1 - \exp\left(-\frac{t-\gamma}{\theta}\right)$$
$$f(t;\theta,\gamma) = \frac{1}{\theta}\exp\left(-\frac{t-\gamma}{\theta}\right)$$
$$h(t;\theta,\gamma) = \frac{f(t;\theta,\gamma)}{1-F(t;\theta,\gamma)} = \frac{1}{\theta}, \quad t > \gamma,$$

where $\theta > 0$ is a scale parameter and γ is both a location and a threshold parameter. When $\gamma = 0$, one gets the well-known one-parameter exponential distribution.

Quantiles: $t_p = \gamma - \theta \log(1-p)$.
Moments: For integer $m > 0$, $\text{E}[(T-\gamma)^m] = m!\,\theta^m$. Then

$$\text{E}(T) = \gamma + \theta, \quad \text{Var}(T) = \theta^2.$$

Examples of Exponential Distributions



Motivation for the Exponential Distribution

- Simplest distribution used in the analysis of reliability data.
- Has the important characteristic that its hazard function is constant (does not depend on time t).
- A popular distribution for some kinds of electronic components (e.g., capacitors with a chemically-stable dielectric or robust, high-quality integrated circuits).
- This distribution would **not** be appropriate for a population of electronic components having failure-causing quality-defects (infant mortality).
- Might be useful to describe failure times for components that exhibit physical wearout only after expected technological life of the system in which the component would be installed (e.g., some components of a personal computer have an increasing hazard function after four or five years of operation, but most such computers are retired by then).

Normal (Gaussian) Distribution

For $Y \sim \text{NORM}(\mu, \sigma)$,

$$F(y; \mu, \sigma) = \Phi_{\text{norm}}\left(\frac{y-\mu}{\sigma}\right)$$
$$f(y; \mu, \sigma) = \frac{1}{\sigma}\phi_{\text{norm}}\left(\frac{y-\mu}{\sigma}\right), \quad -\infty < y < \infty.$$

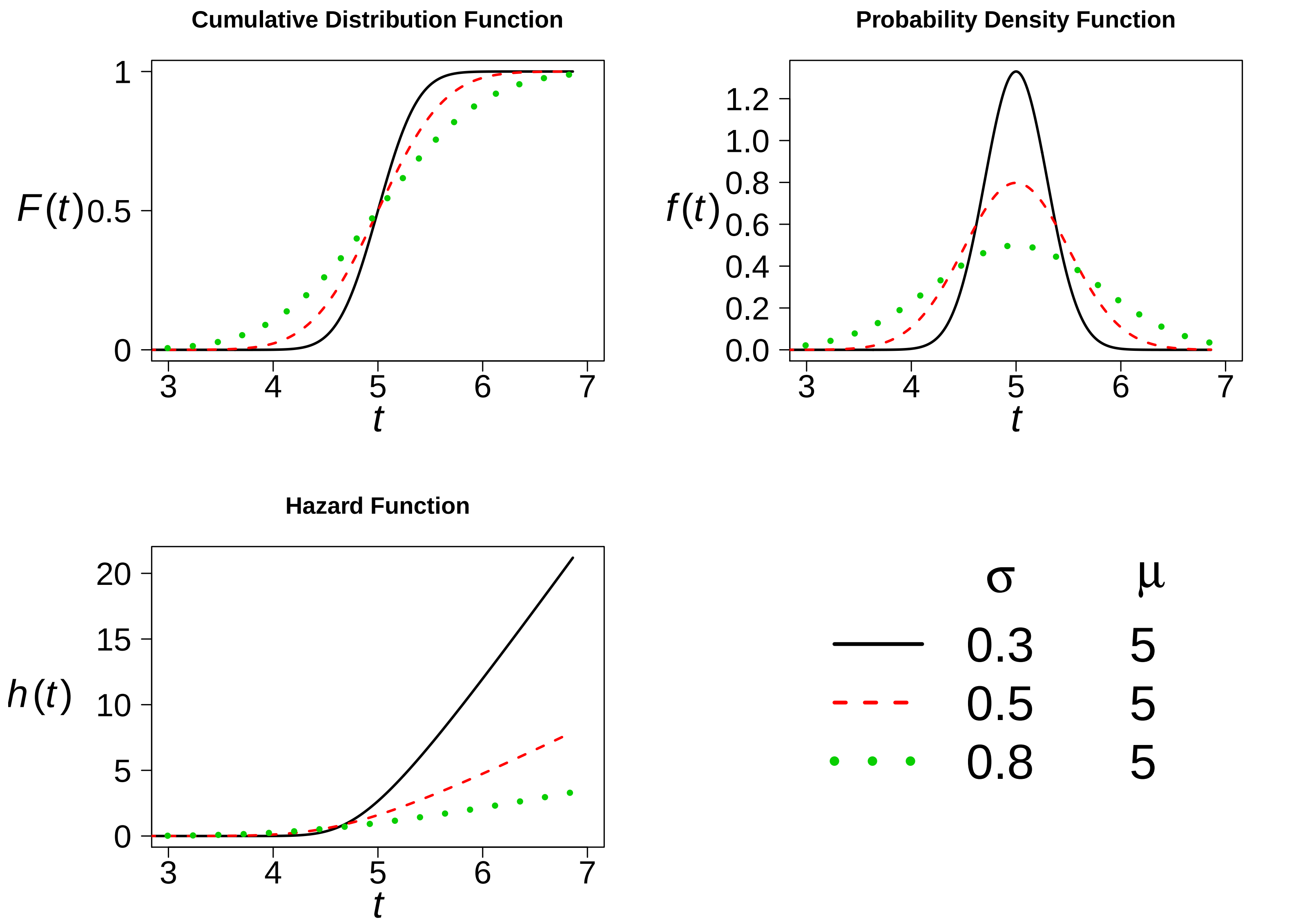
where $\phi_{\text{norm}}(z) = (1/\sqrt{2\pi})\exp(-z^2/2)$ and $\Phi_{\text{norm}}(z) = \int_{-\infty}^{z} \phi_{\text{norm}}(w)dw$ are pdf and cdf for the standard normal ($\mu = 0, \sigma = 1$). $-\infty < \mu < \infty$ is a location parameter and $\sigma > 0$ is a scale parameter.

Quantiles: $y_p = \mu + \sigma\Phi_{\text{norm}}^{-1}(p)$ where $\Phi_{\text{norm}}^{-1}(p)$ is the p quantile for the standard normal distribution.

Moments: For integer $m > 0$, $E[(Y-\mu)^m] = 0$ if m is odd, and $E[(Y-\mu)^m] = (m)!\sigma^m/[2^{m/2}(m/2)!]$ if m is even. Thus

$$\text{E}(Y) = \mu, \quad \text{Var}(Y) = \sigma^2.$$

Examples of Normal Distributions



Lognormal Distribution

If $T \sim \text{LNORM}(\mu, \sigma)$, then $\log(T) \sim \text{NORM}(\mu, \sigma)$ with

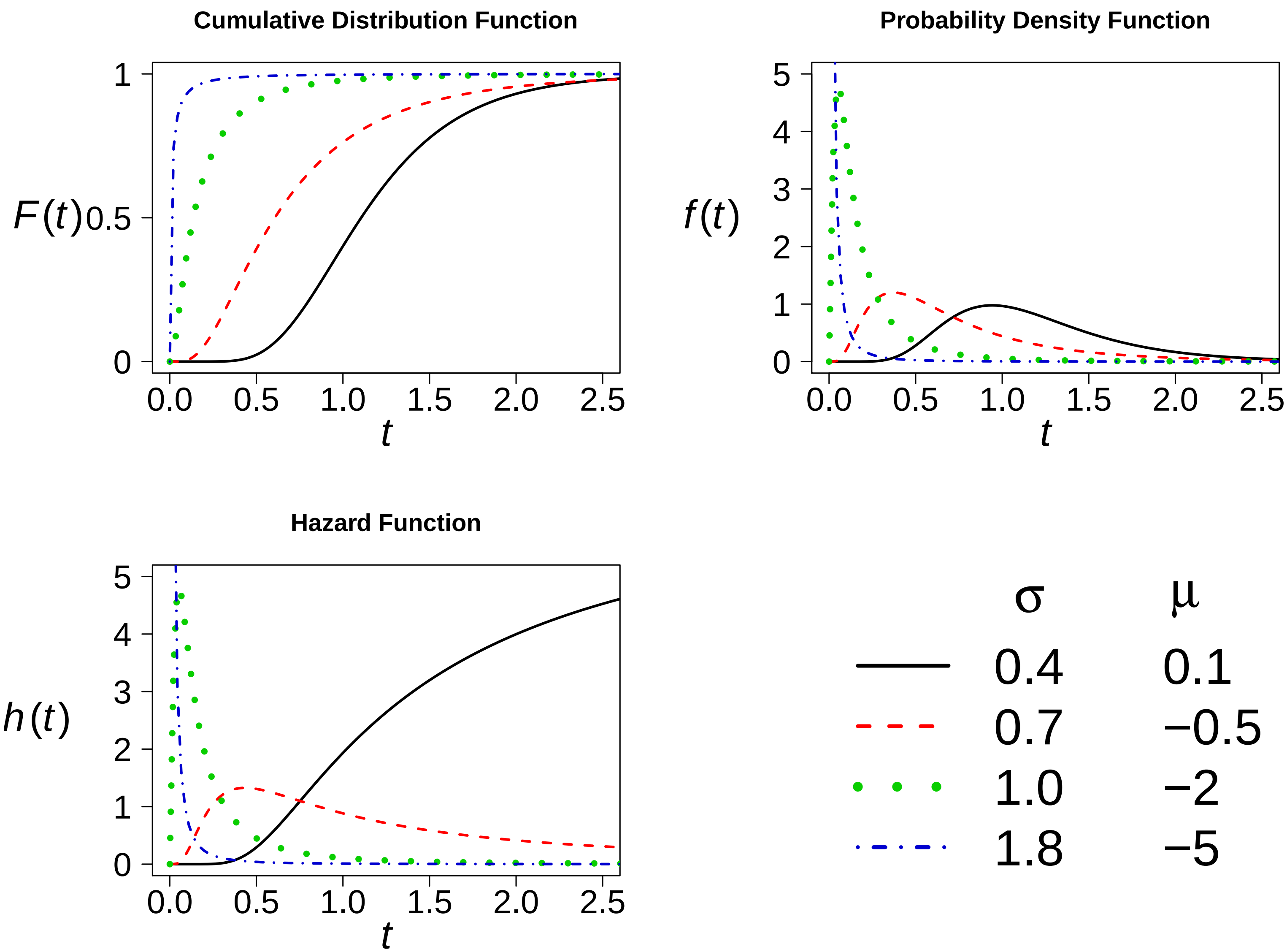
$$F(t; \mu, \sigma) = \Phi_{\text{norm}}\left[\frac{\log(t) - \mu}{\sigma}\right]$$
$$f(t; \mu, \sigma) = \frac{1}{\sigma t}\phi_{\text{norm}}\left[\frac{\log(t) - \mu}{\sigma}\right], \quad t > 0.$$

ϕ_{norm} and Φ_{norm} are the pdf and cdf for the standard normal distribution. $\exp(\mu)$ is a scale parameter; $\sigma > 0$ is a shape parameter.

Quantiles: $t_p = \exp\left[\mu + \sigma\Phi^{-1}_{\text{norm}}(p)\right]$, where $\Phi^{-1}_{\text{norm}}(p)$ is the p quantile for the standard normal distribution.
Moments: For integer $m > 0$, $E(T^m) = \exp\left(m\mu + m^2\sigma^2/2\right)$.

$\text{E}(T) = \exp\left(\mu + \sigma^2/2\right)$, $\text{Var}(T) = \exp\left(2\mu + \sigma^2\right)\left[\exp(\sigma^2) - 1\right]$.

Examples of Lognormal Distributions



Motivation for the Lognormal Distribution

- The lognormal distribution is a common model for failure times.
- It can be justified for a random variable that arises from the product of a number of identically distributed independent positive random quantities.
- It has been suggested as an appropriate model for failure time caused by a degradation process with combinations of random rates that combine multiplicatively.
- Commonly used to describe time to fracture from fatigue crack growth in metals.
- Useful in modeling failure time of a population of electronic components with a decreasing hazard function (due to a small proportion of defects in the population).

Smallest Extreme Value Distribution

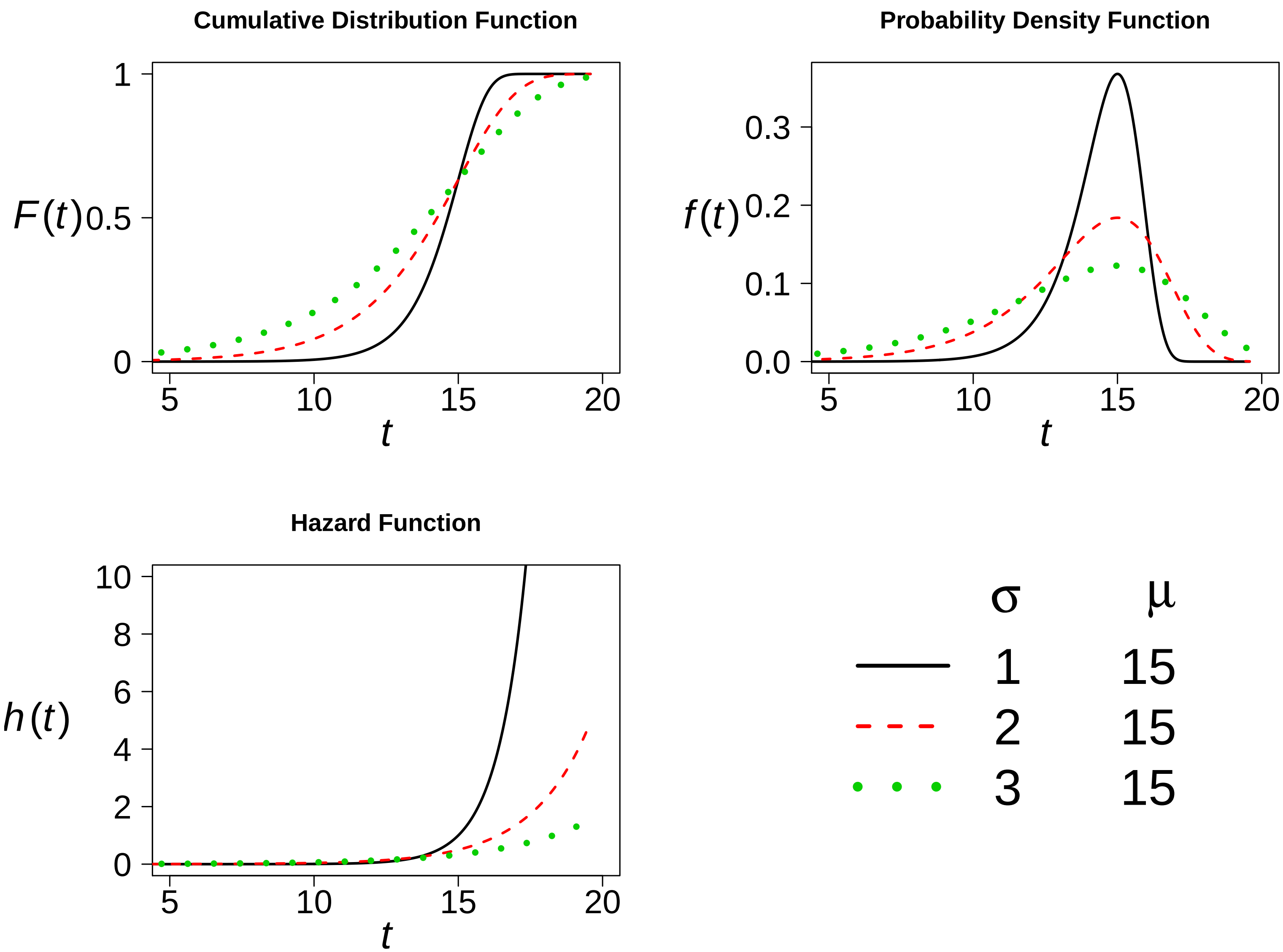
For $Y \sim \text{SEV}(\mu, \sigma)$,

$$
\begin{aligned}
F(y;\mu,\sigma) &= \Phi_{\text{sev}}\left(\frac{y-\mu}{\sigma}\right) \\
f(y;\mu,\sigma) &= \frac{1}{\sigma}\phi_{\text{sev}}\left(\frac{y-\mu}{\sigma}\right) \\
h(y;\mu,\sigma) &= \frac{1}{\sigma}\exp\left(\frac{y-\mu}{\sigma}\right), \quad -\infty < y < \infty.
\end{aligned}
$$

$\Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)]$, $\phi_{\text{sev}}(z) = \exp[z - \exp(z)]$ are cdf and pdf for the standard SEV ($\mu = 0, \sigma = 1$) distribution. $-\infty < \mu < \infty$ is a location parameter and $\sigma > 0$ is a scale parameter.

Quantiles: $y_p = \mu + \Phi_{\text{sev}}^{-1}(p)\sigma = \mu + \log[-\log(1-p)]\sigma$.
Mean and Variance: $\text{E}(Y) = \mu - \sigma\gamma$, $\text{Var}(Y) = \sigma^2\pi^2/6$, where $\gamma \approx 0.5772, \pi \approx 3.1416$.
Note: the hazard function is unbounded and increasing.

Examples of Smallest Extreme Value Distributions



Weibull Distribution

For $T \sim \text{WEIB}(\eta, \beta)$,

$$F(t;\eta,\beta) = \Pr(T \le t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$$

$$f(t;\eta,\beta) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$$

$$h(t;\eta,\beta) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}, \quad t > 0$$

where $\eta > 0$ is a scale parameter and $\beta > 0$ is a shape parameter.

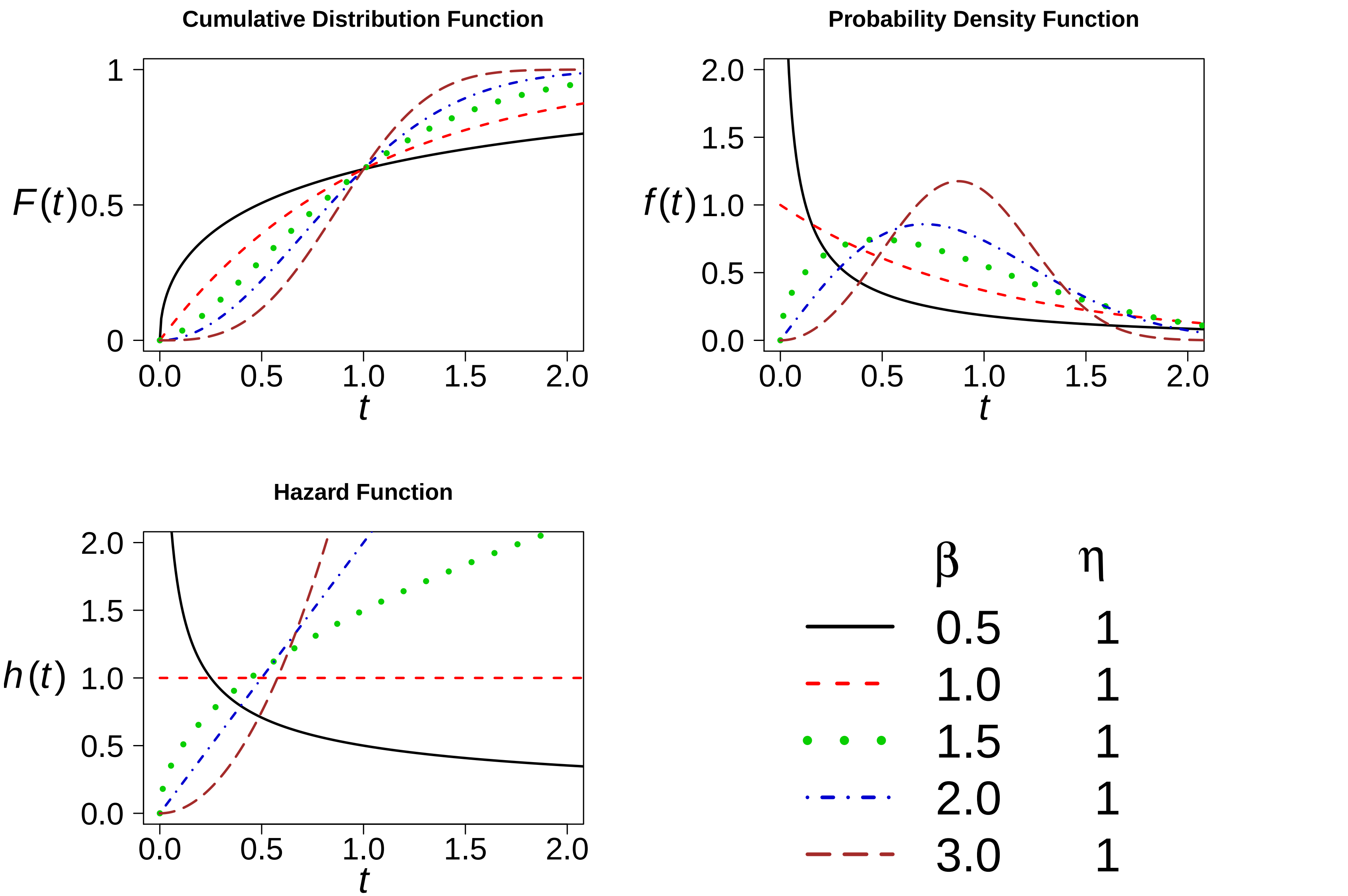
Quantiles: $t_p = \eta[-\log(1-p)]^{1/\beta}$.
Moments: For integer $m > 0$, $\text{E}(T^m) = \eta^m \Gamma(1+m/\beta)$. Then

$$\text{E}(T) = \eta\Gamma\left(1 + \frac{1}{\beta}\right), \quad \text{Var}(T) = \eta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]$$

where $\Gamma(\kappa) = \int_0^\infty w^{\kappa-1}\exp(-w)dw$ is the gamma function.

Note: When $\beta = 1$, then $T \sim \text{EXP}(\eta)$.

Examples of Weibull Distributions



Alternative Weibull Parameterization

Note: If $T \sim \text{WEIB}(\eta, \beta)$, then $Y = \log(T) \sim \text{SEV}(\mu, \sigma)$. Then the Weibull cdf and pdf can be written as

$$
\begin{aligned}
F(t; \mu, \sigma) &= \Pr(T \leq t) = \Phi_{\text{sev}}\left[\frac{\log(t) - \mu}{\sigma}\right] \\
f(t; \mu, \sigma) &= \frac{1}{\sigma t}\,\phi_{\text{sev}}\left[\frac{\log(t) - \mu}{\sigma}\right]
\end{aligned}
$$

where $\sigma = 1/\beta$ is a scale parameter for $\log(T)$ (shape parameter for T), $\mu = \log(\eta)$ is a location parameter for $\log(T)$, and

$$
\begin{aligned}
\phi_{\text{sev}}(z) &= \exp[z - \exp(z)] \\
\Phi_{\text{sev}}(z) &= 1 - \exp[-\exp(z)].
\end{aligned}
$$

Quantiles:

$$
t_p = \exp\left[\mu + \sigma \Phi_{\text{sev}}^{-1}(p)\right], \quad t > 0
$$

where $\Phi_{\text{sev}}^{-1}(p)$ is the p quantile for the standard SEV (i.e., $\mu = 0, \sigma = 1$) distribution.

Motivation for the Weibull Distribution

- The theory of extreme values shows that the Weibull distribution can be used to model the minimum of a large number of independent positive random variables from certain classes of distributions.

  - ▶ Failure of the weakest link in a chain with many links with failure mechanisms (e.g., creep or fatigue) in each link acting approximately independent.
  - ▶ Failure of a system with a large number of components in series and with approximately independent failure mechanisms in each component.

- A common justification for its use is empirical: the Weibull distribution can be used to model failure-time data with a decreasing or an increasing hazard function.

- When compared with the lognormal distribution, Weibull inferences are conservative when extrapolating into either tail.

Chapter 4

Segment 3

Other (Log)-Location-Scale Distributions: Largest Extreme Value, Fréchet, LOGISTIC, and Loglogistic

Largest Extreme Value Distribution

When $Y \sim \text{LEV}(\mu, \sigma)$,

$$F(y; \mu, \sigma) = \Phi_{\text{lev}}\left(\frac{y-\mu}{\sigma}\right)$$

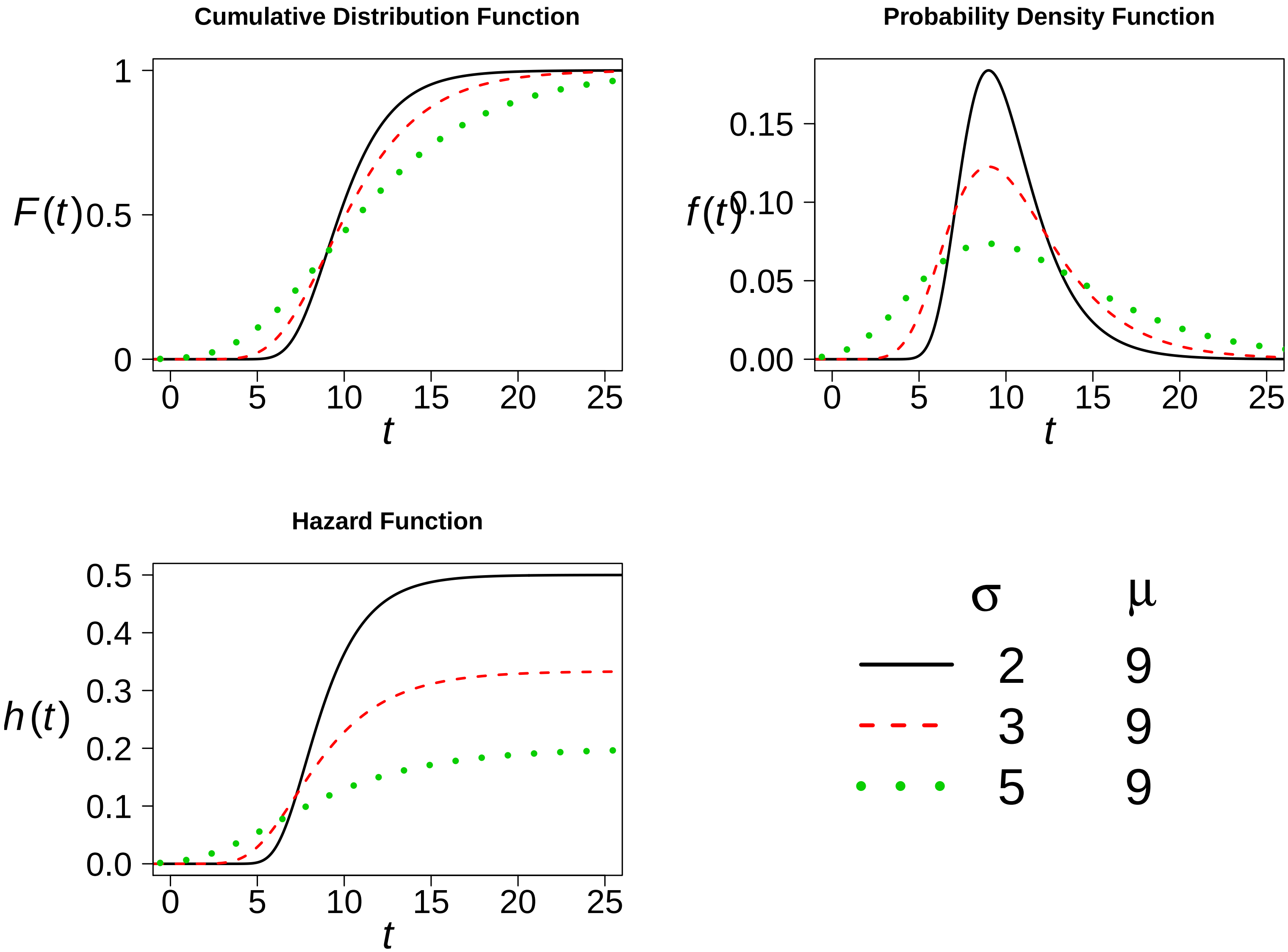
$$f(y; \mu, \sigma) = \frac{1}{\sigma}\phi_{\text{lev}}\left(\frac{y-\mu}{\sigma}\right)$$

$$h(y; \mu, \sigma) = \frac{\exp\left(-\frac{y-\mu}{\sigma}\right)}{\sigma\left\{\exp\left[\exp\left(-\frac{y-\mu}{\sigma}\right)\right]-1\right\}}, \quad -\infty < y < \infty.$$

where $\Phi_{\text{lev}}(z) = \exp[-\exp(-z)]$ and $\phi_{\text{lev}}(z) = \exp[-z-\exp(-z)]$ are the cdf and pdf for the standard LEV ($\mu = 0, \sigma = 1$) distribution.

$-\infty < \mu < \infty$ is a location parameter and $\sigma > 0$ is a scale parameter.

Examples of Largest Extreme Value Distributions



Largest Extreme Value Distribution - Continued

Quantiles: $y_p = \mu - \sigma \log[-\log(p)]$.
Mean and Variance: $\mathrm{E}(Y) = \mu + \sigma\gamma$, $\mathrm{Var}(Y) = \sigma^2\pi^2/6$, where $\gamma \approx 0.5772, \pi \approx 3.1416$.

Notes:

- The hazard is increasing but is bounded in the sense that $\lim_{y\to\infty} h(y;\mu,\sigma) = 1/\sigma$.

- If $Y \sim \mathrm{LEV}(\mu,\sigma)$ then $-Y \sim \mathrm{SEV}(-\mu,\sigma)$.

Fréchet Distribution

When $T \sim \text{FREC}(\eta, \beta)$,

$$F(t; \eta, \beta) = \Pr(T \le t) = \exp\left[-\left(\frac{\eta}{t}\right)^{\beta}\right],$$

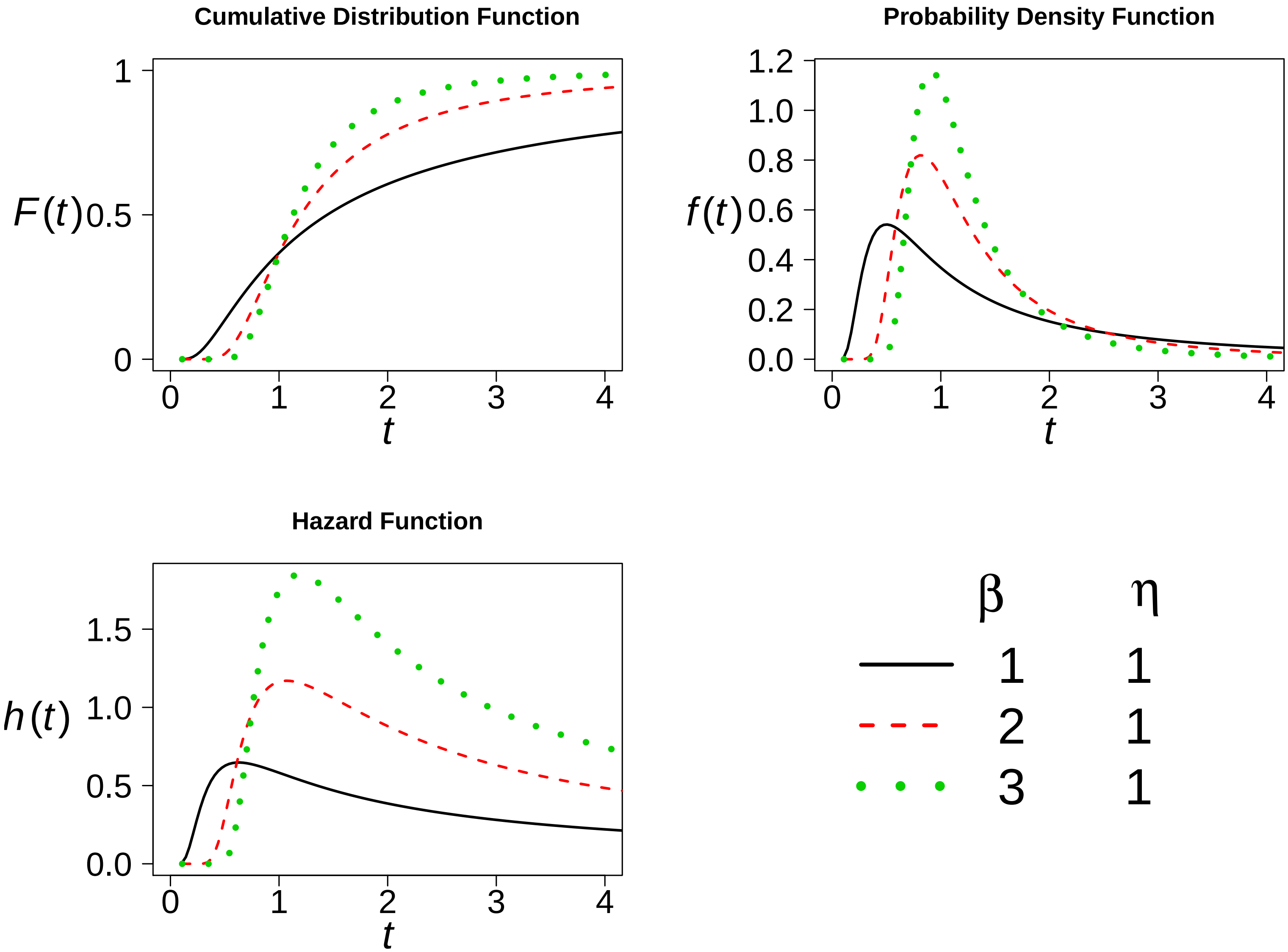
$$f(t; \eta, \beta) = \frac{\beta}{t}\left(\frac{\eta}{t}\right)^{\beta} \exp\left[-\left(\frac{\eta}{t}\right)^{\beta}\right]$$

$$h(t) = \frac{\frac{\beta}{t}\left(\frac{\eta}{t}\right)^{\beta}}{\exp\left[\left(\frac{\eta}{t}\right)^{\beta}\right] - 1}, \quad t > 0$$

where $\eta > 0$ is a scale parameter and $\beta > 0$ is a shape parameter. **Quantiles:** $t_p = \eta[-\log(p)]^{-1/\beta}$.

- If $T \sim \text{FREC}(\eta, \beta)$, then $1/T \sim \text{WEIB}(1/\eta, \beta)$
- The Fréchet distribution hazard function has a shape similar to that of the lognormal distribution.

Examples of Fréchet Distributions



Fréchet Distribution-Continued

Moments: For integer $m > 0$, the mth Fréchet moment $\mathrm{E}(T^m)$ exists if and only if $\beta > m$. In particular,

$$\mathrm{E}(T^m) = \begin{cases} \eta^m \Gamma\left(1 - \dfrac{m}{\beta}\right) & \text{for } \beta > m \\ \infty & \text{otherwise} \end{cases}$$

where $\Gamma(\kappa)$ is the gamma function. Then

$$\begin{aligned} \mathrm{E}(T) &= \eta \Gamma(1 - 1/\beta), \quad \beta > 1 \\ \mathrm{Var}(T) &= \eta^2 \Big[\Gamma(1 - 2/\beta) - \Gamma^2(1 - 1/\beta)\Big], \quad \beta > 2. \end{aligned}$$

- The Fréchet distribution can be used to model the maximum of a large number of independent identically distributed (iid) positive random variables.
- When extrapolating into either tail of a failure-time distribution, the Fréchet distribution provides more **optimistic** (anti-conservative) estimates relative to the lognormal or the Weibull distribution.

Alternative Fréchet Parameterization

If $T \sim \text{FREC}(\eta, \beta)$, then $Y = \log(T) \sim \text{LEV}(\mu, \sigma)$. Then

$$F(t; \mu, \sigma) = \Phi_{\text{lev}}\left[\frac{\log(t) - \mu}{\sigma}\right]$$

$$f(t; \mu, \sigma) = \frac{1}{\sigma t}\,\phi_{\text{lev}}\left[\frac{\log(t) - \mu}{\sigma}\right], \quad t > 0$$

where $\sigma = 1/\beta$, $\mu = \log(\eta)$, and

$$\Phi_{\text{lev}}(z) = \exp[-\exp(-z)]$$

$$\phi_{\text{lev}}(z) = \exp[-z - \exp(-z)].$$

Quantiles:

$$t_p = \exp\left[\mu + \sigma \Phi^{-1}_{\text{lev}}(p)\right]$$

where $\Phi^{-1}_{\text{lev}}(p)$ is the p quantile for the standard LEV (i.e., $\mu = 0, \sigma = 1$).

Logistic Distribution

For $Y \sim \text{LOGIS}(\mu, \sigma)$,

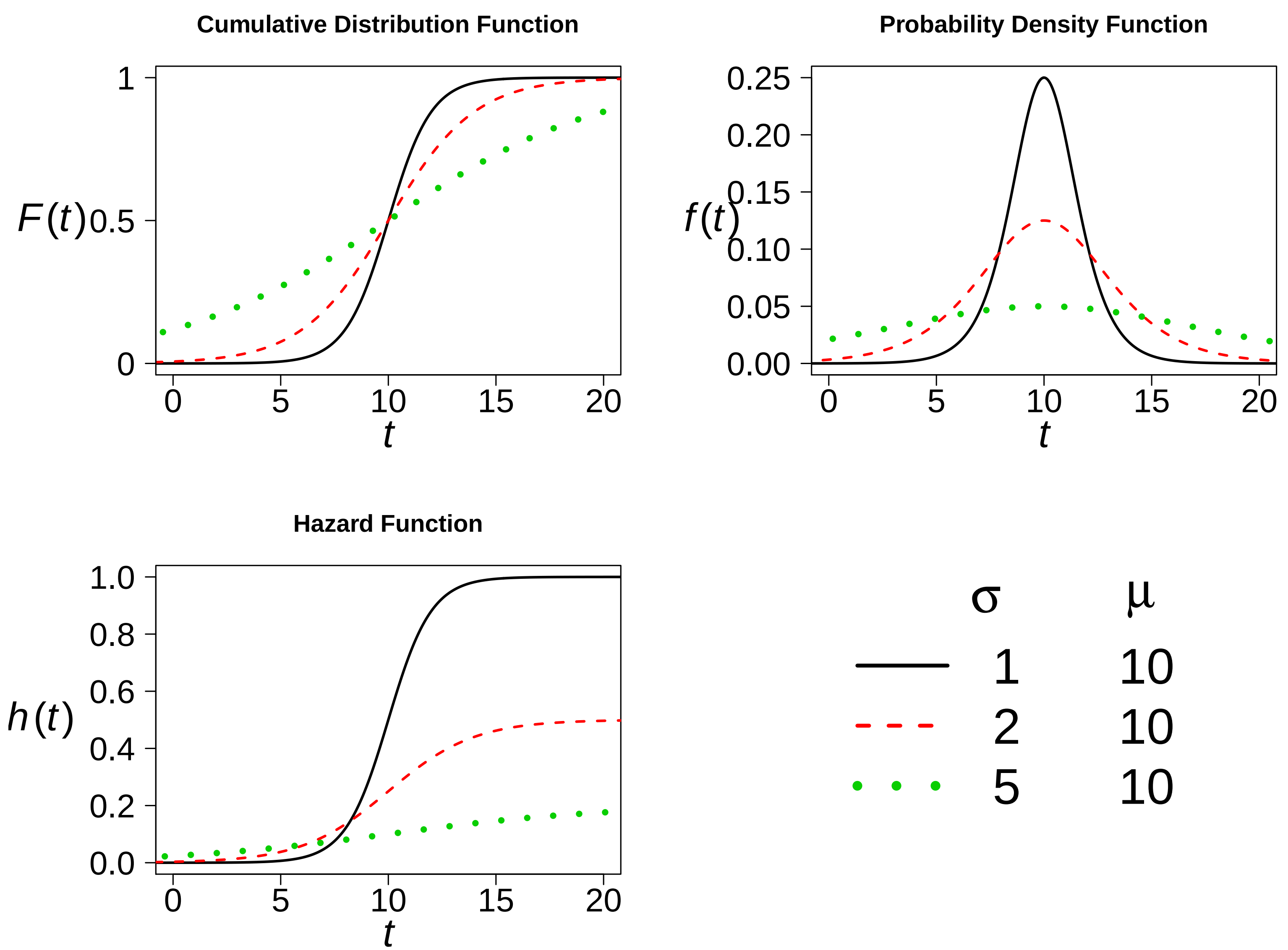
$$
\begin{aligned}
F(y;\mu,\sigma) &= \Phi_{\text{logis}}\left(\frac{y-\mu}{\sigma}\right)\\
f(y;\mu,\sigma) &= \frac{1}{\sigma}\phi_{\text{logis}}\left(\frac{y-\mu}{\sigma}\right)\\
h(y;\mu,\sigma) &= \frac{1}{\sigma}\Phi_{\text{logis}}\left(\frac{y-\mu}{\sigma}\right), \qquad -\infty < y < \infty.
\end{aligned}
$$

where $-\infty < \mu < \infty$ is a location parameter and $\sigma > 0$ is a scale parameter.

ϕ_{logis} and Φ_{logis} are pdf and cdf for the standard logistic distribution defined by

$$
\begin{aligned}
\phi_{\text{logis}}(z) &= \frac{\exp(z)}{[1+\exp(z)]^2}\\
\Phi_{\text{logis}}(z) &= \frac{\exp(z)}{1+\exp(z)}.
\end{aligned}
$$

Examples of Logistic Distributions



Logistic Distribution-Continued

Quantiles: $y_p = \mu + \sigma \Phi^{-1}_{\text{logis}}(p) = \mu + \sigma \log[p/(1-p)]$, where $\Phi^{-1}_{\text{logis}}(p) = \log[p/(1-p)]$ is the p quantile for the standard logistic distribution.

Moments: For integer $m > 0$, $E[(Y-\mu)^m] = 0$ if m is odd, and $E[(Y-\mu)^m] = 2\sigma^m\,(m!)\left[1-(1/2)^{m-1}\right]\sum_{i=1}^{\infty}(1/i)^m$ if m is even. Thus

$$\text{E}(Y) = \mu \quad \text{and} \quad \text{Var}(Y) = \frac{\sigma^2\pi^2}{3}.$$

Note: the hazard function is always increasing but bounded by $1/\sigma$, as shown in the plots.

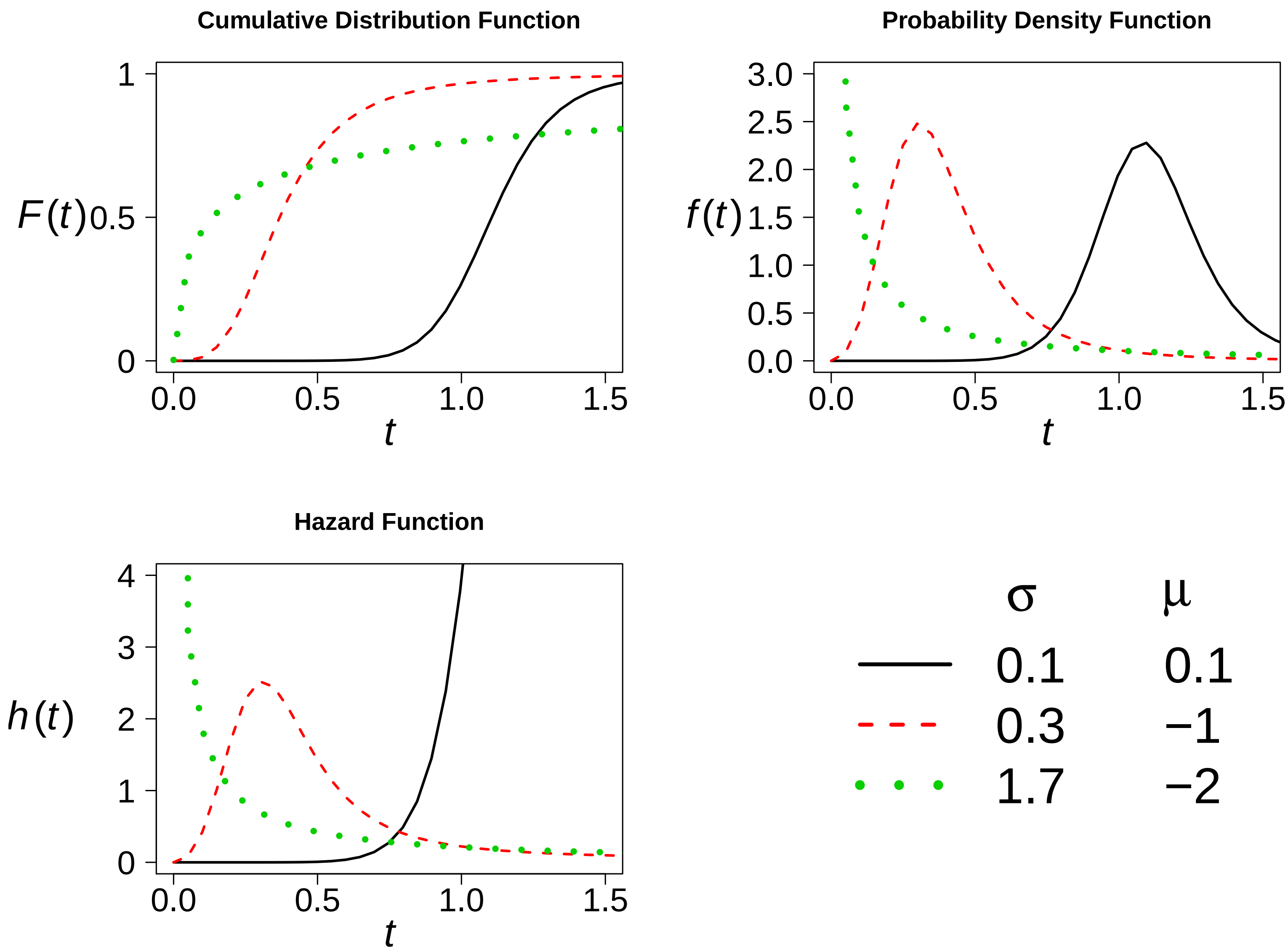
Loglogistic Distribution

If $Y \sim$ LOGIS(μ, σ), then $T = \exp(Y) \sim$ LOGLOGIS(μ, σ) with

$$
\begin{aligned}
F(t; \mu, \sigma) &= \Phi_{\text{logis}}\left[\frac{\log(t) - \mu}{\sigma}\right] \\
f(t; \mu, \sigma) &= \frac{1}{\sigma t}\phi_{\text{logis}}\left[\frac{\log(t) - \mu}{\sigma}\right] \\
h(t; \mu, \sigma) &= \frac{1}{\sigma t}\Phi_{\text{logis}}\left[\frac{\log(t) - \mu}{\sigma}\right], \quad t > 0.
\end{aligned}
$$

$\exp(\mu)$ is a scale parameter; $\sigma > 0$ is a shape parameter. Φ_{logis} and ϕ_{logis} are cdf and pdf for a LOGIS$(0, 1)$.

Examples of Loglogistic Distributions



Loglogistic Distribution-Continued

Quantiles: $t_p = \exp\Big[\mu + \sigma \Phi^{-1}_{\text{logis}}(p)\Big] = \exp(\mu)[p/(1-p)]^{\sigma}$.
Moments: For integer $m > 0$,

$$\mathrm{E}(T^m) = \exp(m\mu)\,\Gamma(1 + m\sigma)\,\Gamma(1 - m\sigma).$$

The m moment is not finite when $m\sigma \geq 1$.

For $\sigma < 1$,

$$\mathrm{E}(T) = \exp(\mu)\,\Gamma(1 + \sigma)\,\Gamma(1 - \sigma),$$

and for $\sigma < 1/2$,

$$\mathrm{Var}(T) = \exp(2\mu)\left[\Gamma(1 + 2\sigma)\,\Gamma(1 - 2\sigma) - \Gamma^2(1 + \sigma)\,\Gamma^2(1 - \sigma)\right].$$

Chapter 4

Segment 4

The Generalized Gamma Distribution

Generalized Gamma Distribution

For $T \sim \text{GENG}(\mu, \sigma, \lambda)$,

$$F(t;\mu,\sigma,\lambda) = \begin{cases} \Phi_{\text{lg}}\Big[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\Big] & \text{if } \lambda > 0 \\ \Phi_{\text{norm}}(\omega) & \text{if } \lambda = 0 \\ 1 - \Phi_{\text{lg}}\Big[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\Big] & \text{if } \lambda < 0 \end{cases}$$

$$f(t;\mu,\sigma,\lambda) = \begin{cases} \frac{|\lambda|}{\sigma t}\,\phi_{\text{lg}}\Big[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\Big] & \text{if } \lambda \neq 0 \\ \frac{1}{\sigma t}\phi_{\text{norm}}(\omega) & \text{if } \lambda = 0 \end{cases}$$

where $t > 0$, $\omega = [\log(t) - \mu]/\sigma$ and

$$\Phi_{\text{lg}}(z;\kappa) = \Gamma_{\text{I}}[\exp(z);\kappa],$$
$$\phi_{\text{lg}}(z;\kappa) = \frac{1}{\Gamma(\kappa)}\exp[\kappa z - \exp(z)].$$

$-\infty < \mu < \infty$, $\exp(\mu)$ is a scale parameter and $-\infty < \lambda < \infty$ and $\sigma > 0$ are shape parameters and $\Gamma_{\text{I}}(v;\kappa)$ is the incomplete gamma function defined by

$$\Gamma_{\text{I}}(v;\kappa) = \frac{\int_0^v x^{\kappa-1}\exp(-x)\,dx}{\Gamma(\kappa)}, \quad v > 0.$$

Generalized Gamma Distribution Moments

- **Moments:** For integer m and $\lambda \neq 0$,

$$\mathsf{E}(T^m) = \begin{cases} \dfrac{\exp(m\mu)\left(\lambda^2\right)^{m\sigma/\lambda}\Gamma\left[\lambda^{-1}(m\sigma+\lambda^{-1})\right]}{\Gamma(\lambda^{-2})} & \text{if } \; m\lambda\sigma + 1 > 0 \\ \infty & \text{if } \; m\lambda\sigma + 1 \leq 0. \end{cases}$$

  When $\lambda = 0$, $\mathsf{E}(T^m)$ is the same as that for the lognormal distribution.

- Thus when the mean and the variance are finite and $\lambda \neq 0$,

$$\mathsf{E}(T) = \frac{\theta\,\Gamma\left[\lambda^{-1}(\sigma + \lambda^{-1})\right]}{\Gamma(\lambda^{-2})}$$

$$\mathsf{Var}(T) = \theta^2\left[\frac{\Gamma\left[\lambda^{-1}(2\sigma + \lambda^{-1})\right]}{\Gamma(\lambda^{-2})} - \frac{\Gamma^2\left[\lambda^{-1}(\sigma + \lambda^{-1})\right]}{\Gamma^2(\lambda^{-2})}\right].$$

- When $\lambda = 0$, $\mathsf{E}(T)$ and $\mathsf{Var}(T)$ are the same as that for the lognormal distribution.

Generalized Gamma Distribution Quantiles

The GENG quantile function is

$$t_p = \exp[\mu + \sigma\,\omega(p;\lambda)],$$

where $\omega(p;\lambda)$ is the p quantile of $[\log(T) - \mu]/\sigma$ given by

$$\omega(p;\lambda) = \begin{cases} \left(\dfrac{1}{\lambda}\right)\log\left[\lambda^2 \Gamma_{\mathrm{I}}^{-1}(p;\lambda^{-2})\right] & \text{if } \lambda > 0 \\ \Phi_{\text{norm}}^{-1}(p) & \text{if } \lambda = 0 \\ \left(\dfrac{1}{\lambda}\right)\log\left[\lambda^2 \Gamma_{\mathrm{I}}^{-1}(1-p;\lambda^{-2})\right] & \text{if } \lambda < 0. \end{cases}$$

Generalized Gamma Distribution Special Cases

- If $\lambda = 1$, $T \sim \text{WEIB}(\mu, \sigma)$.

- If $\lambda = 0$, $T \sim \text{LNORM}(\mu, \sigma)$.

- If $\lambda = -1$, $T \sim \text{FREC}(\mu, \sigma)$.

- When $\lambda = \sigma$, T is has a Gamma distribution with a scale parameter $\lambda^2 \exp(\mu)$ and a shape parameter λ^{-2}.

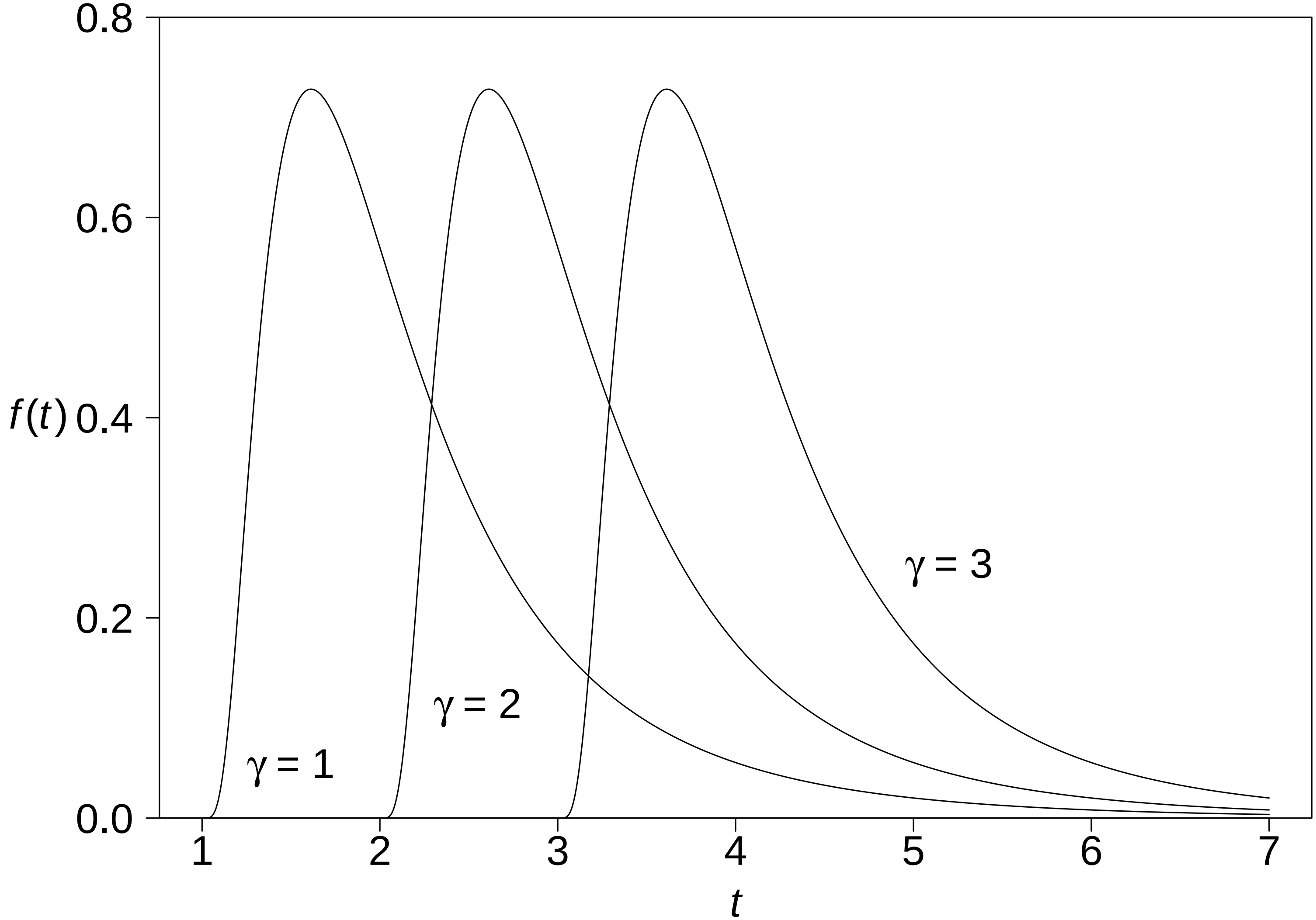
- When $\lambda = \sigma = 1$, $T \sim \text{EXP}(\theta)$, where $\theta = \exp(\mu)$.

Chapter 4

Segment 4

Threshold Log-Location-Scale Distributions

pdfs for Threshold (Three-Parameter) Lognormal Distributions for $\mu = 0$ and $\sigma = 0.5$ with $\gamma =$ 1,2,3.



Distributions with a Threshold Parameter

- So far we have discussed nonnegative random variables with cdfs that begin increasing at $t = 0$.
- One can generalize these and similar distributions by adding a **threshold** parameter, γ, to shift the beginning of the distribution away from 0.
- Distributions with a threshold are particularly useful for fitting skewed distributions that are shifted far to the right of 0.
- The cdf for log-location-scale threshold distributions is

$$
\begin{aligned}
F(t; \mu, \sigma, \gamma) &= \Phi\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right] \\
&= \Phi\left\{\log\left[\left(\frac{t-\gamma}{\exp(\mu)}\right)^{1/\sigma}\right]\right\}, \quad t > \gamma
\end{aligned}
$$

where $-\infty < \gamma < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$, and $\Phi(z)$ is a completely specified cdf (i.e., no unknown parameters).

Examples of Distributions with a Threshold Parameter

- Three-parameter lognormal distribution

$$F(t;\mu,\sigma,\gamma) = \Phi_{\text{norm}}\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right],\ t>\gamma.$$

- Three-parameter Weibull distribution

$$\begin{aligned} F(t;\eta,\beta,\gamma) &= 1-\exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right] \\ &= \Phi_{\text{sev}}\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right],\ t>\gamma \end{aligned}$$

where $\sigma = 1/\beta$ and $\mu = \log(\eta)$.

Properties of Distributions with a Threshold

- When the distribution of T has a threshold, γ, then the distribution of $W = T - \gamma$ has a distribution with 0 threshold.
- The properties of the distribution of T are **closely** related to the properties of the distribution of W.
- In general, $\mathrm{E}(T) = \gamma + \mathrm{E}(W)$ and $t_p = \gamma + w_p$, where w_p is the p quantile of the distribution of W.
- Changing γ simply shifts the distribution on the time axis, there is no effect on the distribution's spread or shape. Thus $\mathrm{Var}(T) = \mathrm{Var}(W)$.
- There are, however, some very specific issues in the estimation of γ because the points at which the cdf is positive depends on γ.

References

Meeker, W. Q., L. A. Escobar, and F. G. Pascual (2021). *Statistical Methods for Reliability Data* (Second Edition). Wiley. [1]