Chapter 14

Planning Reliability Demonstration Tests

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Chapter 14
Planning Reliability Demonstration Tests

Topics discussed in this chapter are:

- The basic ideas behind reliability demonstration tests.
- The tradeoff between sample size and test length.
- How to compute probability of successful demonstration.

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Segment 1

Criteria and Other Basic Ideas Behind Reliability Demonstration Tests

Possible Criteria for Doing a Demonstration

- Consider the following three demonstrations:
  - Demonstrate that $S(t_e)$, the reliability at time t_e is at least $S^{\dagger}$. The demonstration is successful if $\underset{\sim}{S}(t_e) \geq S^{\dagger}$.
  - Demonstrate that $F(t_e)$, the proportion failing at time t_e is less than $F^{\dagger}$. The demonstration is successful if $\widetilde{F}(t_e) \leq F^{\dagger}$.
  - Demonstrate that t_p, the p quantile of the failure-time distribution, is at least $t_p^{\dagger}$. The demonstration is successful if $\underset{\sim}{t_p} \geq t_p^{\dagger}$.
- With appropriate choice of t_e and p, these are all equivalent.
- Following tradition and common use, we will discuss the reliability demonstration that $S(t_e) > S^{\dagger}$.

Basic Ideas

- Want to demonstrate reliability $S(t_e) = \Pr(T > t_e)$ is at least $S^{\dagger}$ (e.g., $S^{\dagger} = 0.99$ or $S^{\dagger} = 0.999$) for a given log-location-scale distribution (e.g., Weibull).
- Test a **small number of units** for a **long time** (e.g., continuous testing for a large number of operations hours).
- Denote the sample size by n and the censoring time by t_c.
- Pass the test if there are r_c or fewer failures up to t_c.
- **Required:** Specification of the log-location-scale distribution shape parameter σ (or Weibull shape parameter)—generally done in a conservative way. Larger (smaller) values of σ (β) are conservative.

Data and Distribution

- The number of units **surviving** until t_c is X. The realized value of X is x. The observed number of **failures** in the demonstration test is $r = (n - x)$.
- To simplify test plan specification, ignore the failure times and use the fact that X has a binomial distribution with parameters n and $S(t_c)$.
- Because σ is given, little information is lost by ignoring the failure times.
- The ML estimate of $S(t_c)$ is $\widehat{S}(t_c) = x/n$.

Important Relationship Between $S(t_e)$ and $S(t_c)$

- Let $k = t_c/t_e$ be the test-length factor (i.e., $t_c = kt_e$). Typically, $t_c > t_e$ so $k > 1$.
- Then, using the assumed failure-time distribution,
$$S(t_e) = 1 - \Phi\left\{\Phi^{-1}[1 - S(t_c)] - \log(k^{1/\sigma})\right\}$$
and
$$S(t_c) = 1 - \Phi\left\{\Phi^{-1}[1 - S(t_e)] + \log(k^{1/\sigma})\right\}.$$
- $S(t_e)$ and $S(t_c)$ are monotone increasing functions of each other.

Decision Rule

- A conservative lower $100(1-\alpha)\%$ confidence bound for $S(t_c) = \Pr(T > t_c)$ is (see Meeker, Hahn, and Escobar, 2017, page 103)
$$\begin{aligned}\underset{\sim}{S}(t_c) &= \texttt{qbeta}(\alpha; x, n - x + 1)\\ &= \texttt{qbeta}(\alpha; n - r, r + 1).\end{aligned}$$
- Using the assumed log-location-scale distribution, the given value of σ, and using $k = t_c/t_e$, a conservative lower $100(1-\alpha)\%$ confidence bound for $S(t_e) = \Pr(T > t_e)$ is
$$\underset{\sim}{S}(t_e) = 1 - \Phi\left\{\Phi^{-1}\left[1 - \underset{\sim}{S}(t_c)\right] - \log(k^{1/\sigma})\right\},$$
- The demonstration is successful if $\underset{\sim}{S}(t_e) > S^{\dagger}$.

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Segment 2

Required Sample Size n for a Given Test-Length Factor k

Required Test-Length Factor k for a Given Sample Size n

Required Sample Size n for a Given Test-Length Factor k

- For a given value of r, say r_c, α, and $k = t_c/t_e$, the required sample size is the smallest n for which $\underset{\sim}{S}(t_e) \geq S^{\dagger}$.
- The required sample size can obtained by rounding up to the next integer number the solution n to
$$\texttt{qbeta}(\alpha; n - r_c, r_c + 1) = 1 - \Phi\left[\Phi^{-1}\left(1 - S^{\dagger}\right) + \log(k^{1/\sigma})\right].$$
- There is a tradeoff between k and n (longer tests allow smaller sample size).
- For a given value of k, the required sample size is an increasing function of r_c.
- Taking $r_c = 0$ gives what is known as the "minimum-sample-size test."

Required Test-Length Factor k for a Given Sample Size n

- Using the previous result, for a given values of α, r_c and n, the required test-length factor is
$$\begin{aligned}k &= \exp\left(\sigma\left\{\Phi^{-1}\left[1 - \underset{\sim}{S}(t_c)\right] - \Phi^{-1}\left(1 - S^{\dagger}\right)\right\}\right)\\ &= \left(\frac{t_{[1-\underset{\sim}{S}(t_c)]}}{t_{[1-S^{\dagger}]}}\right)^{\sigma},\end{aligned}$$
confirming that k is unitless. Note that $\underset{\sim}{S}(t_c)$ is a function of the given values of α, r_c and n.
- For the Weibull distribution with $r_c = 0$ and $\beta = 1/\sigma$ there are simplifications giving
$$k = \left[\frac{\log(\alpha)}{n\log\left(S^{\dagger}\right)}\right]^{1/\beta}.$$

Special Results for the Minimum-Sample-Size Tests

- For the case of zero failures, $r_c = 0$ and
$$\underset{\sim}{S}(t_c) = \texttt{qbeta}(\alpha, n - r_c, r_c + 1) = \texttt{qbeta}(\alpha; n, 1) = \alpha^{1/n}.$$
- This leads to the simplification
$$n = \frac{\log(\alpha)}{\log\left\{1 - \Phi\left[\Phi^{-1}\left(1 - S^{\dagger}\right) + \log(k^{1/\sigma})\right]\right\}}.$$
- For the case $r_c = 0$ and the Weibull distribution, $\Phi(z) = 1 - \exp[-\exp(z)]$ and $\sigma = 1/\beta$. Then the required sample size simplifies to
$$n = \frac{\log(\alpha)}{k^{\beta}\log(S^{\dagger})}.$$
- Minimum-sample-size tests tend to have a small probability of successful demonstration, unless $S(t_e) >> S^{\dagger}$.

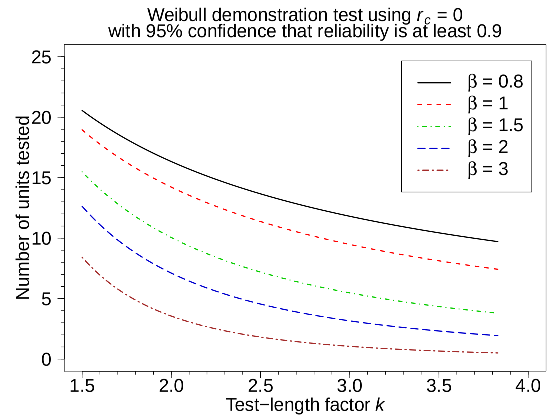
Common Implementation of the Zero-Failure Minimum Sample Size Test

Use a Weibull distribution with $\beta = 1$ (constant hazard, or exponential distribution), as this is **conservative** if we are **sure** that the failure mode is wearout

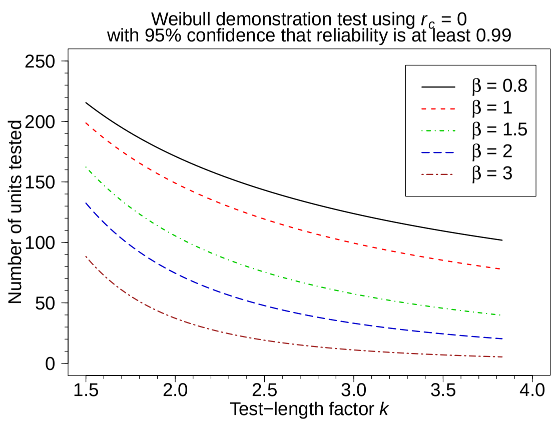
$$n = \frac{1}{k} \times \frac{\log(\alpha)}{\log(S^{\dagger})}.$$

- Test run until $k \times t_e$ for demonstration with $100(1-\alpha)\%$ confidence.
- Requires the assumption that there is no infant mortality.
- Is conservative if $\beta > 1$.
- Smaller sample sizes are possible if you can bound β higher.
- Probability of successful demonstration is small unless $S(t_e) >> S^{\dagger}$.
- It is generally better to use a test that allows a few failures.

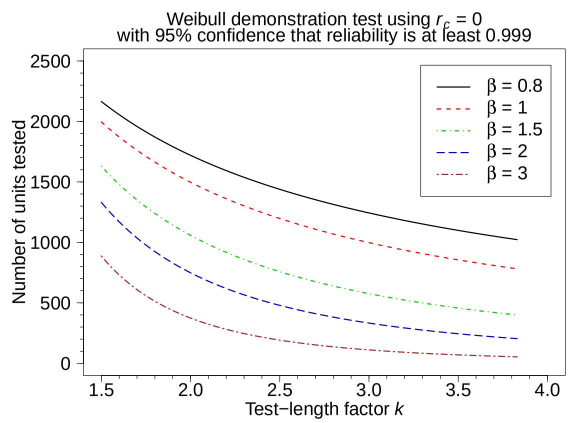
Zero-failure Weibull 95% Reliability Demonstration for $S^{\dagger} = 0.9$ with Given β



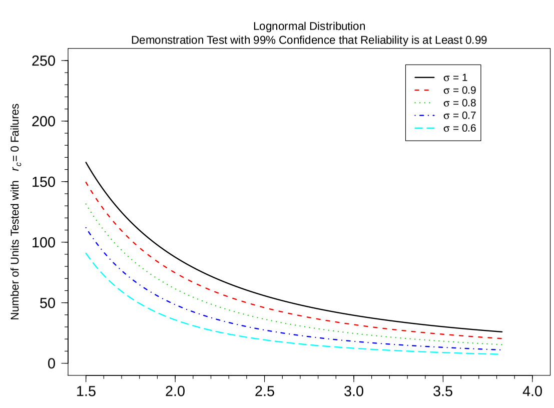
Zero-Failure Weibull 95% Reliability Demonstration for $S^{\dagger} = 0.99$ with Given β



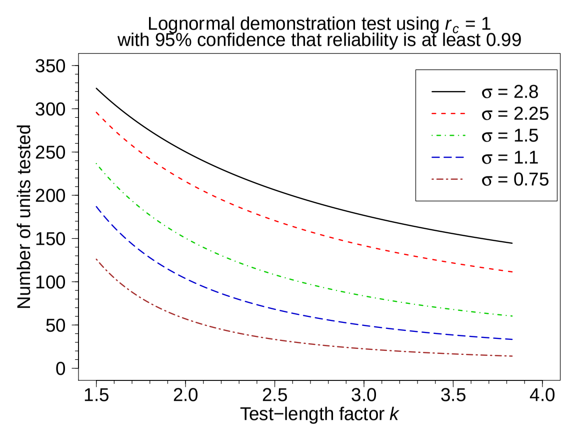
Zero-Failure Weibull 95% Reliability Demonstration for $S^{\dagger} = 0.999$ with Given β



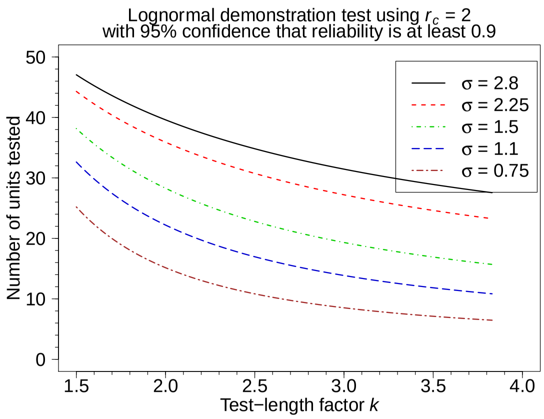
Zero-Failure Lognormal 95% Reliability Demonstration for $S^{\dagger} = 0.99$ with Given σ



One-Failure Lognormal 95% Reliability Demonstration for $S^{\dagger} = 0.99$ with Given σ



Two-Failure Lognormal 95% Reliability Demonstration for $S^{\dagger} = 0.99$ with Given σ



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Segment 3

Probability of Successful Demonstration

Probability of Successful Demonstration

- The probability of a successful demonstration (a.k.a., power) for a demonstration test allowing at most r_c failures, as a function of $S(t_e)$, is

$$\text{PrSD}(r_c) = \Pr(n - X \le r_c) = \texttt{pbinom}[r_c; n, 1 - S(t_c)]$$
$$= 1 - \texttt{pbeta}[1 - S(t_c); r_c + 1, n - r_c]$$

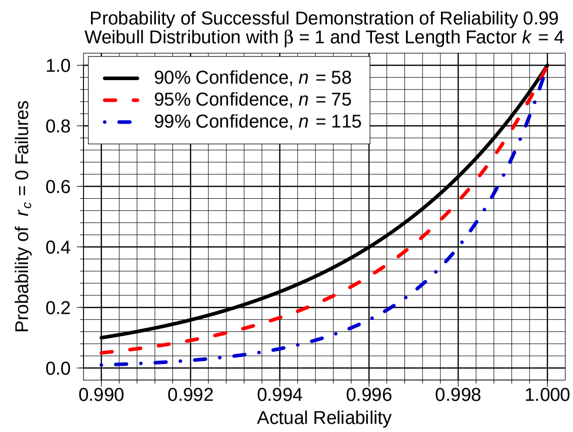
where $S(t_c) = 1 - \Phi\left\{\Phi^{-1}[1 - S(t_e)] + \log(k^{1/\sigma})\right\}$. The pbeta expression allow computation with non-integer n.

- $\text{PrSD}(r_c)$ is an increasing function of r_c.
- If $r_c = 0$, then $\text{PrSD}(0) = [S(t_c)]^n$.
- If $r_c = 0$, with the **Weibull** distribution and $\beta = 1/\sigma$, then

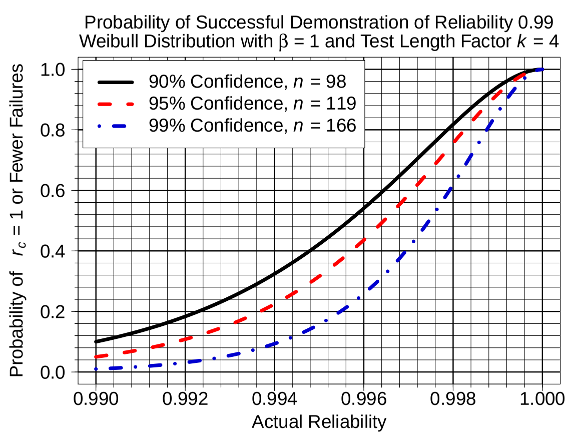
$$\text{PrSD}(0) = [S(t_e)]^{\log(\alpha)/\log(S^{\dagger})},$$

which, interestingly, does not depend on n, k, or β.

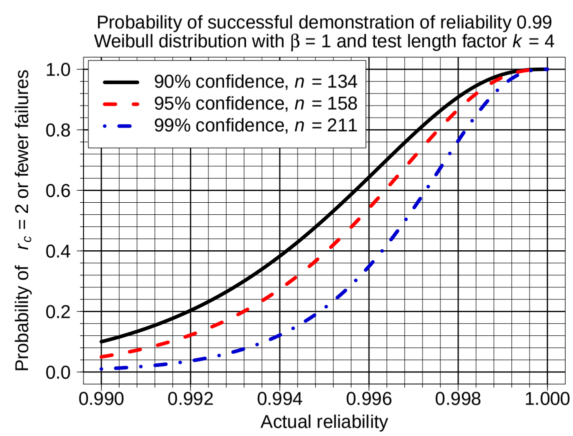
Weibull Reliability Demonstration with $k = 4$, $r_c = 0$, $\beta = 1$, and $S^{\dagger} = 0.99$ for Different Confidence Levels



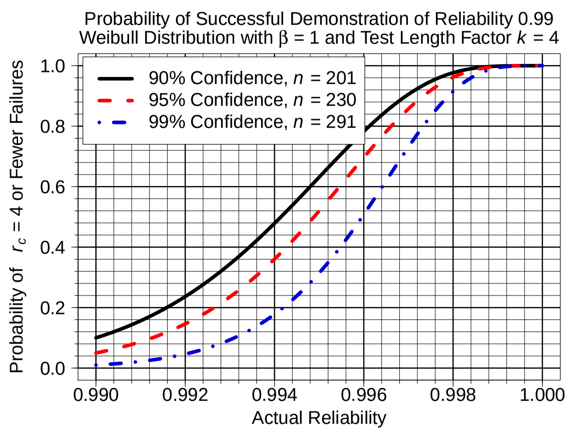
Weibull Reliability Demonstration with $k = 4$, $r_c = 1$, $\beta = 1$, and $S^{\dagger} = 0.99$ for Different Confidence Levels



Weibull Reliability Demonstration with $k = 4$, $r_c = 2$, $\beta = 1$, and $S^{\dagger} = 0.99$ for Different Confidence Levels



Weibull Reliability Demonstration with $k = 4$, $r_c = 4$, $\beta = 1$, and $S^{\dagger} = 0.99$ for Different Confidence Levels



Reliability Demonstration Tests
Summary

- Reliability demonstration tests are a useful alternative to life tests aimed at estimation because they generally require smaller sample sizes.
- There is a tradeoff between sample size n and test length, controlled by k. Generally, there is a need to test for a number of hours/cycles that is substantially larger than the design life of the product, usually by acceleration.
- Zero-failure minimum-sample-size tests may appear to be attractive, but tend to have a small probability of successful demonstration, unless $S(t_e) >> S^{\dagger}$.
- It is generally better to use a test that allows a few failures.

References

Meeker, W. Q., L. A. Escobar, and F. G. Pascual (2021). *Statistical Methods for Reliability Data* (Second Edition). Wiley. [1]