Chapter 20

Degradation Modeling and Destructive Degradation Data Analysis

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Based on Meeker, Escobar, and Pascual (2021): *Statistical Methods for Reliability Data, Second Edition*, John Wiley & Sons Inc.

May 24, 2021
11h 6min

Chapter 20
Accelerated Destructive Degradation Tests
Data, Models, and Data Analysis

Objectives and Overview

Topics discussed in this chapter are:

- Degradation data and degradation path models.
- Mechanistic motivation for degradation path models and parameter interpretation.
- **Destructive** degradation background and an example of destructive degradation field data analysis.
- Failure-time distributions induced from degradation models and failure-time inferences.
- Background and an example of **accelerated** destructive degradation testing (ADDT) and model building.
- Fitting an acceleration model to ADDT data.
- ADDT model checking.
- ADDT failure-time inferences.
- ADDT using an asymptotic model.

Chapter 20

Degradation Modeling
and Destructive Degradation Data Analysis

Segment 1

Degradation Reliability Data
and Degradation Path Models
Introduction and Background

Degradation Leading to Failure

- Most failures can be traced to an underlying degradation process.

- Degradation curves can have different shapes.

- A **soft failure** occurs when the observed degradation level crosses a threshold.

- Some applications have more than one degradation variable or more than one underlying degradation process.

- Examples here have only one degradation variable and underlying degradation process.

Degradation Data

- Degradation is natural response for some reliability applications.
- Degradation data can provide considerably more reliability information than censored failure-time data (especially with few or no failures). Reduction of degradation data to failure-time data loses information.
- There can be useful reliability inferences even with 0 failures.
- Direct observation of the degradation process allows direct modeling of the failure-causing mechanism.
- Degradation data provides better justification and credibility for extrapolative acceleration models. (Modeling is closer to the physics-of-failure.)

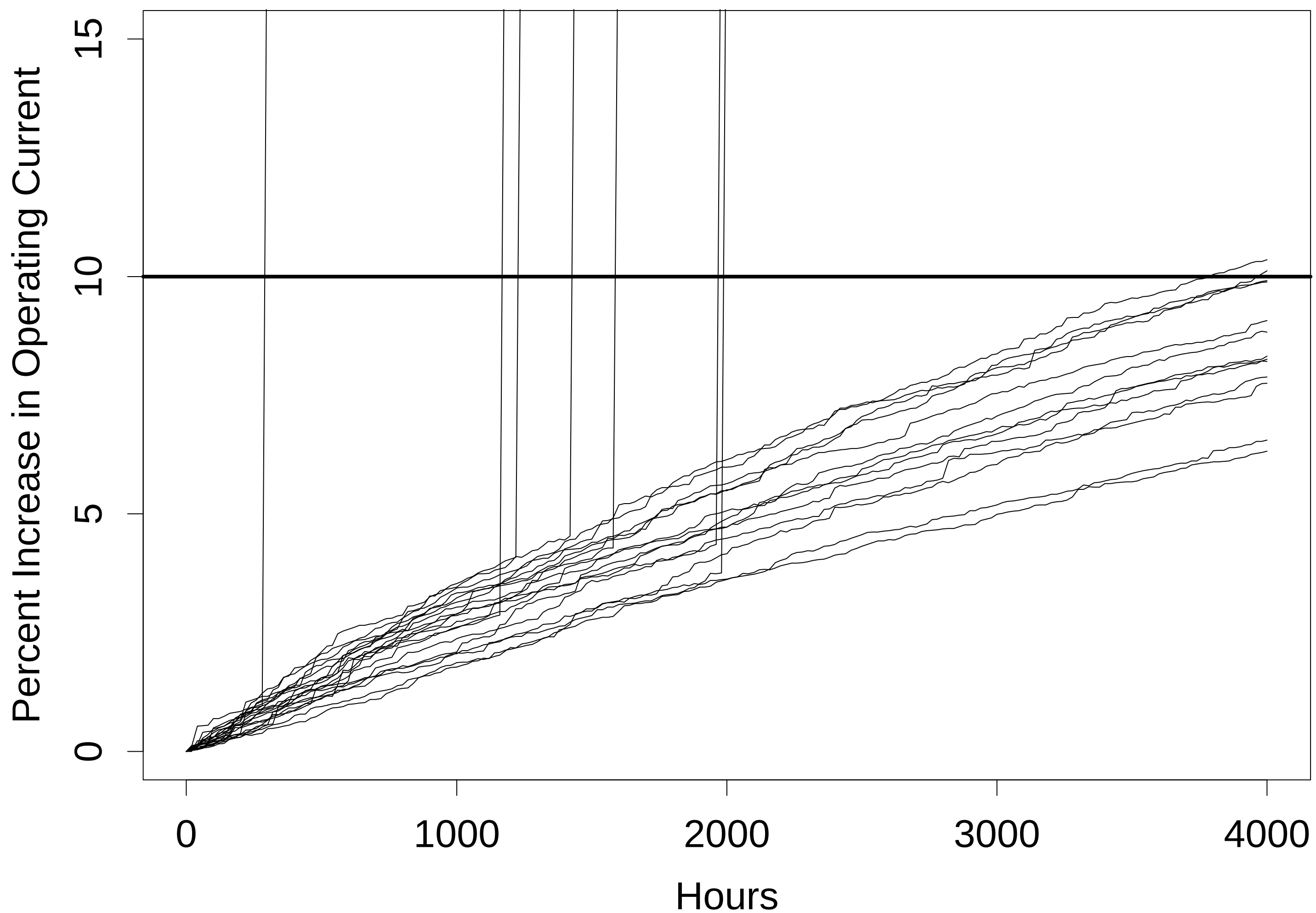
Limitations of Degradation Data

- Degradation data may be difficult or impossible to obtain.
- Obtaining degradation data may have an effect on future product degradation (e.g., taking apart a motor to measure wear).
- Substantial measurement error can diminish the information in degradation data.
- In some applications the **degradation** level may not correlate well with failure.

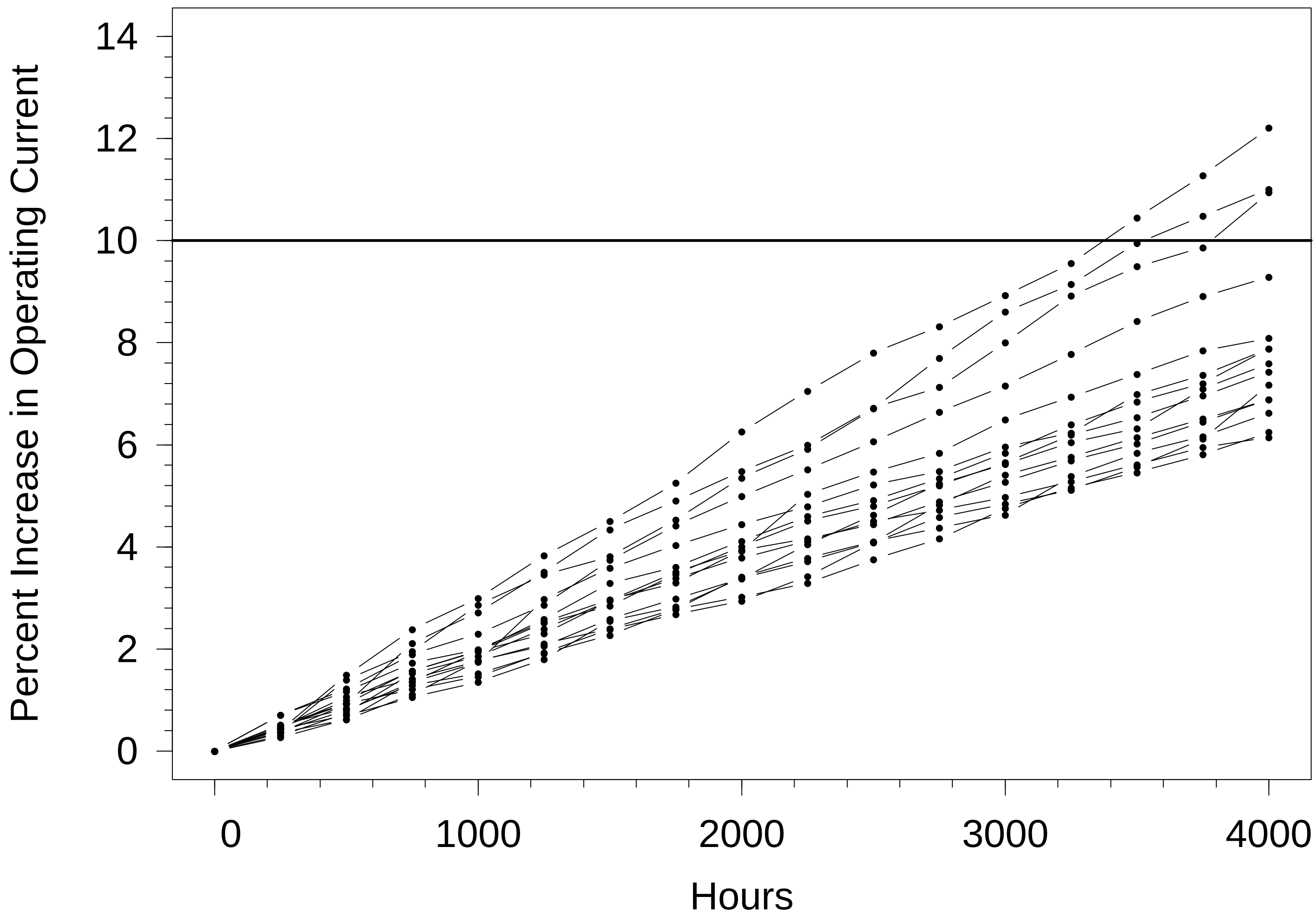
Types of Degradation Data

- Destructive degradation data (Chapter 20).
- Repeated-measures degradation data (Chapter 21).
- The underlying paths models will be the same for both types of data.
- In models for repeated measures degradation data, one or more of the parameters in assumed paths model will typically have random-parameter unit-to-unit variability.

Percent Increase in Operating Current for GaAs Lasers Tested at 80°C



Plot of Laser Operating Current as a Function of Time



Laser Repeated Measures Degradation Data

- Percentage increase in operating current for GaAs lasers tested at 80°C.

- Fifteen (15) devices each measured every 250 hours up to 4000 hours of operation.

- For these devices and the corresponding application, a $\mathcal{D}_{\mathsf{f}} =$ 10% increase in current was the specified failure level.

General Degradation Path Models

- When there are no explanatory variables, the general degradation path models has the form

$$Y = h_d[\mathcal{D}(t)] = \xi(t) + \epsilon.$$

- Transformations are often used to linearize or otherwise simplify the form of a degradation model and may be suggested by physics of failure or from the data.
- $h_d[\mathcal{D}(t)]$ is a monotone increasing transformation of the observed degradation $\mathcal{D}(t)$.
- $\xi(t)$ is a monotone function (either increasing or decreasing) of (possibly transformed) time $\tau = h_t(t)$.
- The error term ϵ will be described by a location-scale distribution (e.g., a normal distribution) with parameters ($\mu = 0$ and σ_ϵ (although technically, other distributions could also be used).

General Degradation Path Regression Models

- Explanatory variables $\boldsymbol{x}$ arise from
  - Accelerating variables (e.g., temperature, voltage, or pressure) in accelerated tests.
  - Covariates from field data.

  The regression model for degradation will be

$$Y = h_d[\mathcal{D}(t)] = \xi(t, x) + \epsilon.$$

- For a fixed value of $\boldsymbol{x}$, $\xi(t, \boldsymbol{x})$ is a monotone increasing function of (possibly transformed) time $\tau = h_t(t)$.
- The transformation for the $\boldsymbol{x}$ could be suggested from physics of failure (e.g., the Arrhenius and Power-rule models described in Chapters 18 and 19) or from the data.

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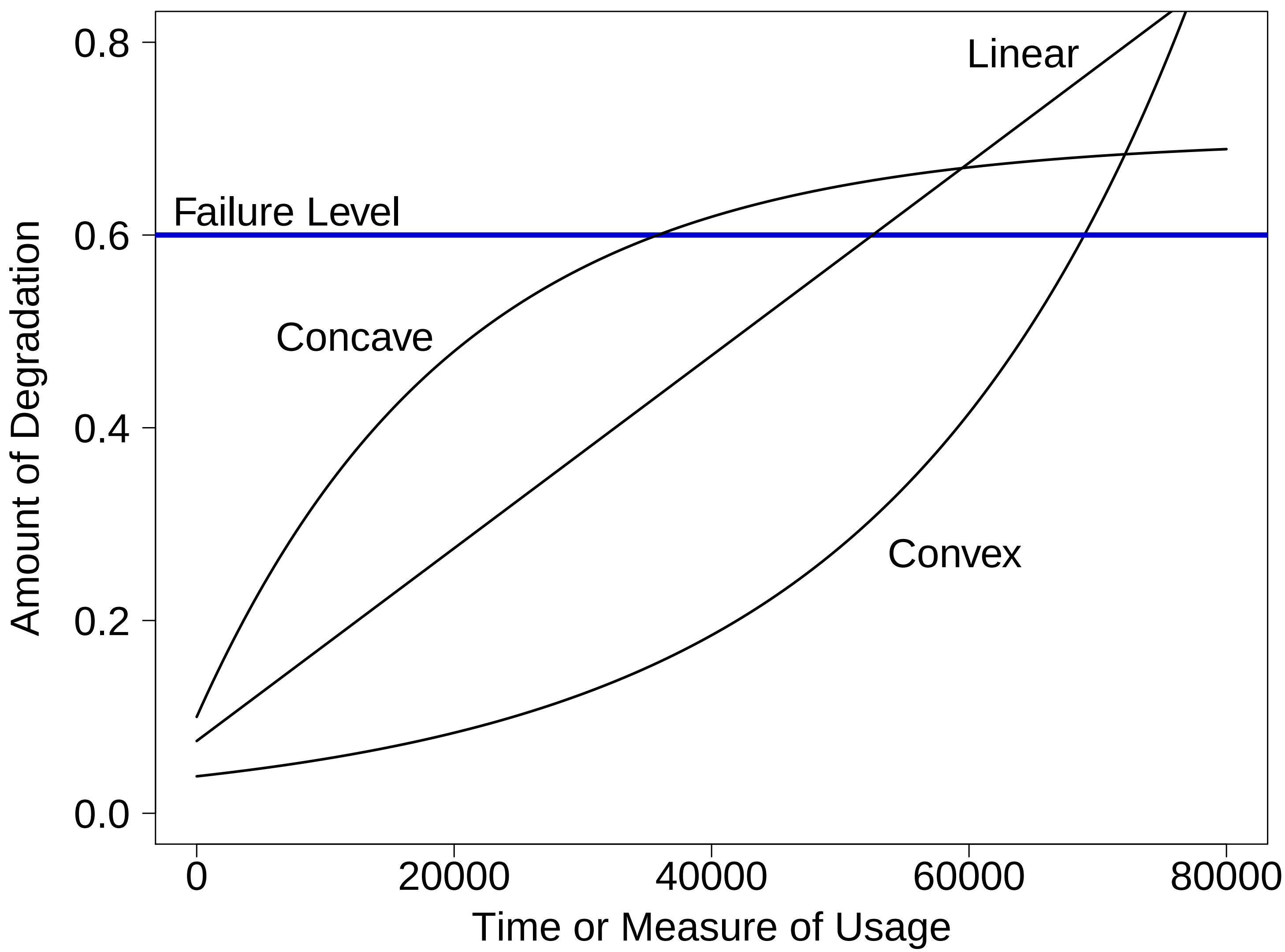
Degradation Modeling
and Destructive Degradation Data Analysis

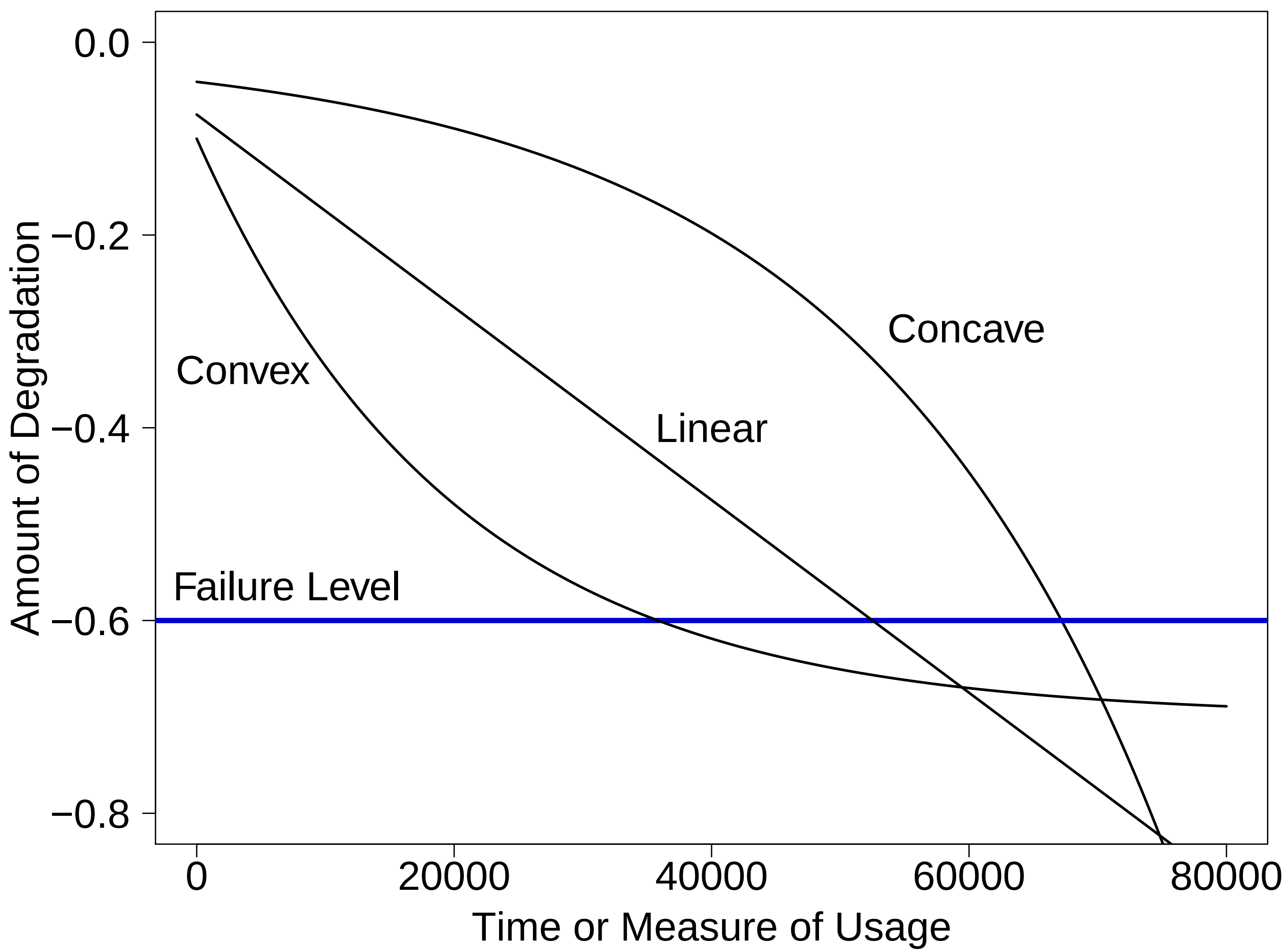
Segment 2

Mechanistic Motivation for Degradation Path Models
and Parameter Interpretation

Possible Shapes for Univariate Increasing Degradation Curves



Possible Shapes for Univariate Decreasing Degradation Curves



Possible Shapes for Univariate Degradation Curves

- **Linear degradation:** Degradation **rate**

$$\frac{d\,\xi(t)}{d\,t} = \beta_1$$

is constant over time. Degradation **level** at time t, $\xi(t) = \beta_0 + \beta_1 t$, is linear in t. Examples include the amount of automobile tire tread wear, mechanical wear of a bearing, or a zero-order chemical reaction.

- **Concave degradation:** Degradation rate decreasing in time. Degradation level increasing at a decreasing rate. Examples include chemical processes with a limited amount of material to react (e.g., a first-order chemical reaction).

- **Convex degradation:** Degradation rate increasing in time. Degradation level increasing at an increasing rate. Examples include the Paris-law crack growth model.

Motivation for the Asymptotic Degradation Path Model Simple One-Step Chemical Reaction Leading to Failure

- $A_1(t)$ is the amount of harmful material at time t that is available for reaction to failure-causing A_2.
- $A_2(t)$ is observable or proportional to an observable performance degradation $\mathcal{D}(t)$ at time t.
- Consider the chemical reaction:

$$A_1 \xrightarrow{k_1} A_2$$

- A soft failure occurs when $\mathcal{D}(t)$ exceeds the threshold $\mathcal{D}_f$
- The rate equations for this reaction are

$$\frac{dA_1}{dt} = -k_1 A_1 \quad \text{and} \quad \frac{dA_2}{dt} = k_1 A_1$$

  where k_1 is the **reaction rate constant**.

Asymptotic Degradation Path Model

- The solution to the system of differential equations is:

$$\begin{aligned} A_1(t) &= A_1(0)\exp(-k_1 t) \\ A_2(t) &= A_2(0) + A_1(0)[1 - \exp(-k_1 t)] \end{aligned} \tag{1}$$

  where $A_1(0)$ and $A_2(0)$ are initial conditions.

- The asymptote for A_2 is

$$\mathcal{D}_\infty = \lim_{t\to\infty} A_2(t) = A_2(0) + A_1(0).$$

- The expression in (1) is the basis for the statistical model

$$Y = \xi(t) + \epsilon = \beta_0 + \beta_3[1 - \exp(-\beta_1 \tau)] + \epsilon$$

  where $\tau = h_t(t)$ is (possibly) transformed time.

- Note that if $\mathcal{D}_f > \mathcal{D}_\infty$, there will never be a failure.
- A simple one-step diffusion process can be modeled in the same way.

Some Common Degradation Path Models

Model	$\xi(t)$	Description
1	$\beta_0 + \beta_1\tau$	$\uparrow$ Linear
2	$\beta_0 - \beta_1\tau$	$\downarrow$ Linear
3	$\beta_0 + \beta_3[1 - \exp(-\beta_1\tau)]$	$\uparrow$ Asymptotic
4	$\beta_0 - \beta_3[1 - \exp(-\beta_1\tau)]$	$\downarrow$ Asymptotic

Note that $\tau = h_t(t)$.

- Transformed time τ is a positive power transformation of t. Consequently, τ is a monotone increasing function of t.
- Note that $\beta_1 > 0$ and $\beta_3 > 0$ but β_0 is unrestricted in sign and may be constrained to be equal to 0 or some other value.
- Models 1 and 2 describe degradation that is **linear** in τ.
- Models 3 and 4 describe degradation that is **asymptotic** in τ.

Degradation Model Parameter Interpretation

- $\beta_0 = \xi(0)$ is the y intercept for all of the models.
- β_1 is the absolute value of the degradation rate (slope) for the linear models and the differential equation reaction rate constant for the asymptotic models.
- The asymptote of the **increasing** asymptotic degradation path Model 3 for large t is

$$\xi(\infty) = \lim_{t \to \infty} \xi(t) = \beta_0 + \beta_3.$$

- The asymptote of the **decreasing** asymptotic degradation path Model 4 for large t is

$$\xi(\infty) = \lim_{t \to \infty} \xi(t) = \beta_0 - \beta_3.$$

Some Common Degradation Path Regression Models

Model	$\xi(t, x, x_0)$	Description
5	$\beta_0 + \beta_1 \exp[-\beta_2(x - x_0)]\tau$	$\uparrow$ Linear
6	$\beta_0 - \beta_1 \exp[-\beta_2(x - x_0)]\tau$	$\downarrow$ Linear
7	$\beta_0 + \beta_3(1 - \exp\{-\beta_1 \exp[-\beta_2(x - x_0)]\tau\})$	$\uparrow$ Asymptotic
8	$\beta_0 - \beta_3(1 - \exp\{-\beta_1 \exp[-\beta_2(x - x_0)]\tau\})$	$\downarrow$ Asymptotic

Note that $\tau = h_t(t)$.

- Transformed time τ is a positive power transformation of t. Consequently, τ is a monotone increasing function of t.
- Models 5 and 6 describe **linear** degradation in τ.
- Models 7 and 8 describe **asymptotic** degradation in τ.
- The factor $AF = \exp[-\beta_2(x - x_0)]$ is a time-scaling acceleration factor (scaling transformed time τ) and $\beta_2 > 0$.
- If there are $p > 1$ explanatory variables, the factor $\exp[-\beta_2(x - x_0)]$ is replaced by

$$\exp\left[-\boldsymbol{\beta}_2'(\boldsymbol{x} - \bar{\boldsymbol{x}})\right] = \exp\left[-\sum_{i=1}^{p} \beta_{2i}(x_i - x_{0,i})\right].$$

Degradation Regression Model
Parameter Interpretation

- $\beta_0 = \xi(0, x, x_0)$ is the y intercept for all of the models, is unrestricted in sign and may be constrained to be equal to 0 or some other value.
- $\beta_1 > 0$ is the absolute value of the degradation rate (slope) at x_0 for the linear models and the differential equation reaction rate constant at x_0 for the asymptotic models.
- **Note** that instead of x_0, one can use any other value of x for this baseline value.
- For fixed x, the asymptote of the **increasing** asymptotic degradation path Model 7 for large t is

$$\xi(\infty, x, x_0) = \lim_{t \to \infty} \xi(t, x, x_0) = \beta_0 + \beta_3.$$

- For fixed x, the asymptote of the **decreasing** asymptotic degradation path Model 8 for large t is

$$\xi(\infty, x, x_0) = \lim_{t \to \infty} \xi(t, x, x_0) = \beta_0 - \beta_3.$$

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Degradation Modeling and Destructive Degradation Data Analysis

Segment 3

Destructive Degradation Background and an Example of Destructive Degradation Field Data Analysis

Destructive Degradation Data

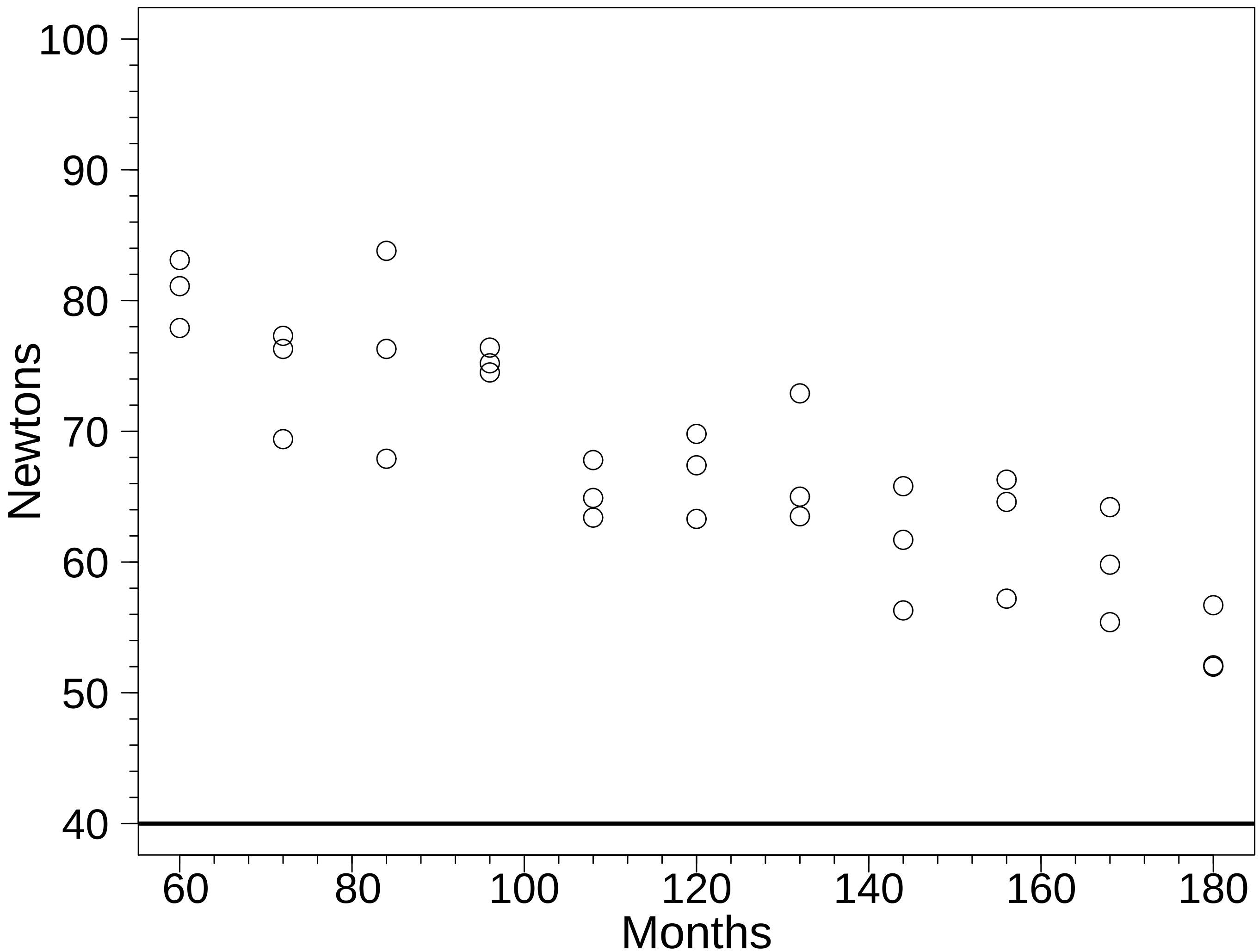
- Some degradation measurements are destructive.

- Examples include testing materials and components such as
  - ▶ Adhesive strength.
  - ▶ Dielectric strength of an insulating material.
  - ▶ Tensile strength of a polymer.
  - ▶ Strength of a seal.

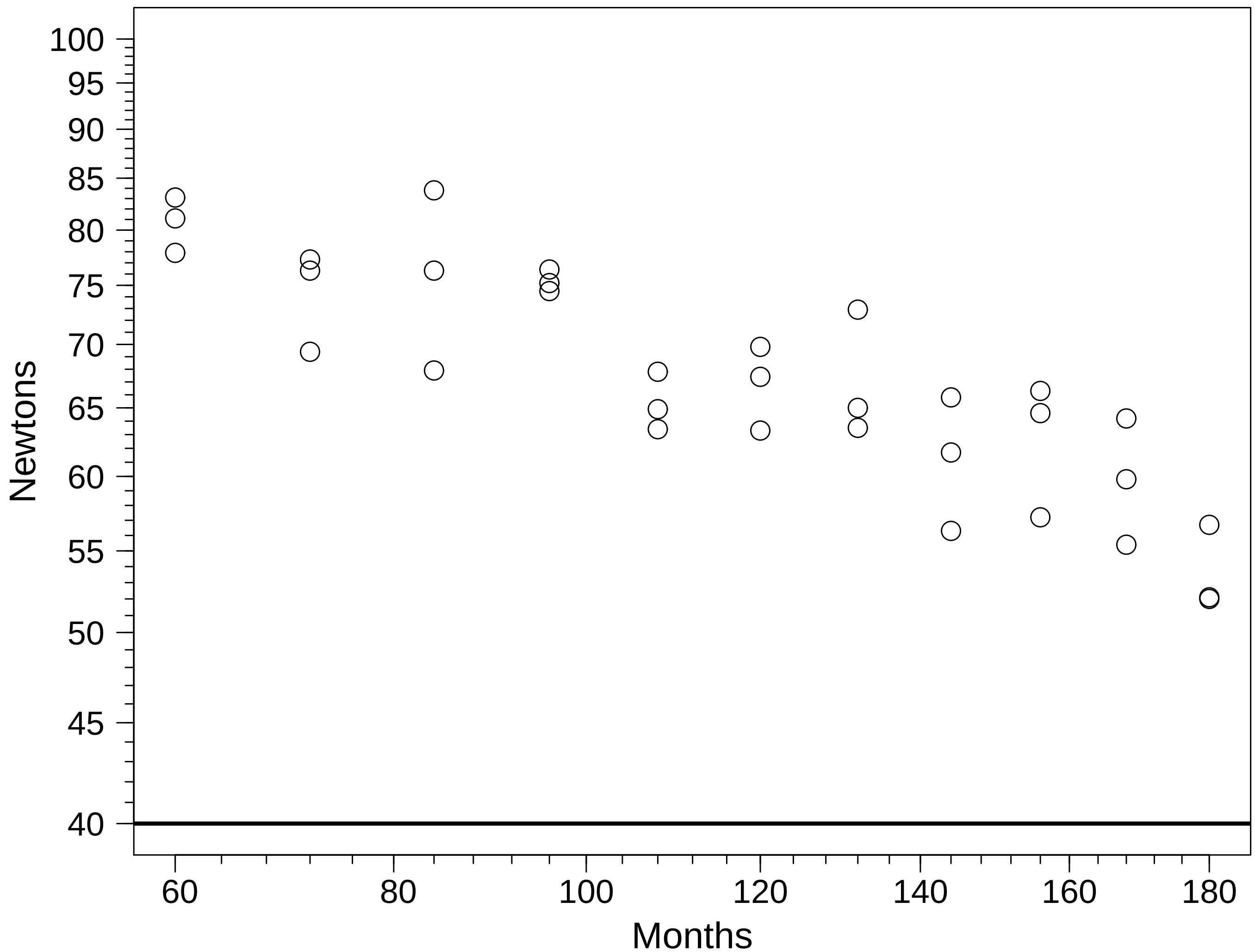
Adhesive Bond A Strength Field Data

- An accelerated test estimated that the 0.01 quantile of the failure time distribution of **Adhesive Bond A** would be at least 20 years.
- Over the next 15 years, tens of thousands of the systems using Adhesive Bond A had been deployed in the field.
- There was concern that the large amount of extrapolation (in both the time and temperature dimension) might have provided overly optimistic lifetime estimates.
- Could the systems (originally designed for 15-year life) safely stay in service for 20 or 30 years?
- Three units were randomly selected from each of 11 age groups of the deployed systems having ages between 5 and 15 years, returned to the laboratory, and strengths of the 33 adhesive bonds were measured destructively.
- Want an estimate of the fraction failing (strength falling below 40 Newtons) after both 20 and 30 years.

Adhesive Bond A Strength Field Data
Linear–Linear Axes



Adhesive Bond A Strength Field Data
Square Root–Log Axes



General Structure of Destructive Degradation Models

- Degradation model: $Y = \xi(t) + \epsilon$ where the path function $\xi(t)$ is monotone in t and ϵ has a location-scale distribution.
- Other forms could be used for $\xi(t)$.
- Time t can be viewed as a special kind of explanatory variable for Y.
- ϵ is an error term that describes unit-to-unit variability (and probably some measurement errors and model uncertainty that may not be independently estimable).
- The degradation distribution is:

$$G(y;t) = \Pr(Y \le y) = \Phi\left[\frac{y - \xi(t)}{\sigma}\right].$$

For given value of t the p quantile of the distribution of Y is

$$y_p(t) = \xi(t) + \Phi^{-1}(p)\,\sigma.$$

Degradation Model
Likelihood with No Explanatory Variables

- For the data with exact observations and right-censored observations, the likelihood is

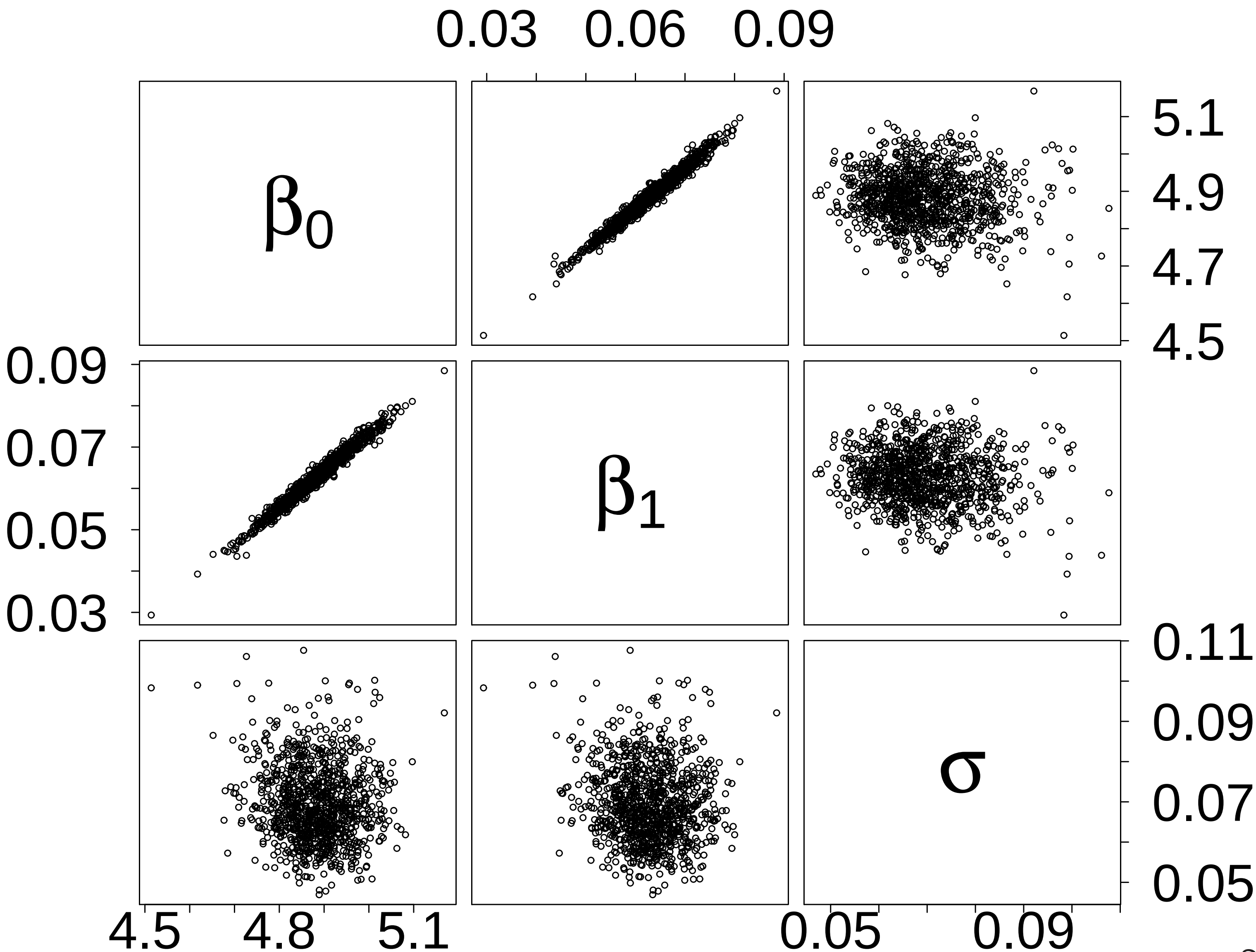
$$L(\boldsymbol{\theta}|\text{DATA}) = \prod_{i=1}^{n} \left[\frac{1}{\sigma}\phi\left(\frac{y_i - \xi(t_i)}{\sigma}\right)\right]^{\delta_i} \times \left[1 - \Phi\left(\frac{y_i - \xi(t_i)}{\sigma}\right)\right]^{1-\delta_i}.$$

- n is the number of observations.
- $\xi(t)$ is the chosen path model (say one of Models 1–4).
- The censoring indicator

$$\delta_i = \begin{cases} 1 & \text{if } y_i \text{ is an exact observation} \\ 0 & \text{if } y_i \text{ is a right-censored observation.} \end{cases}$$

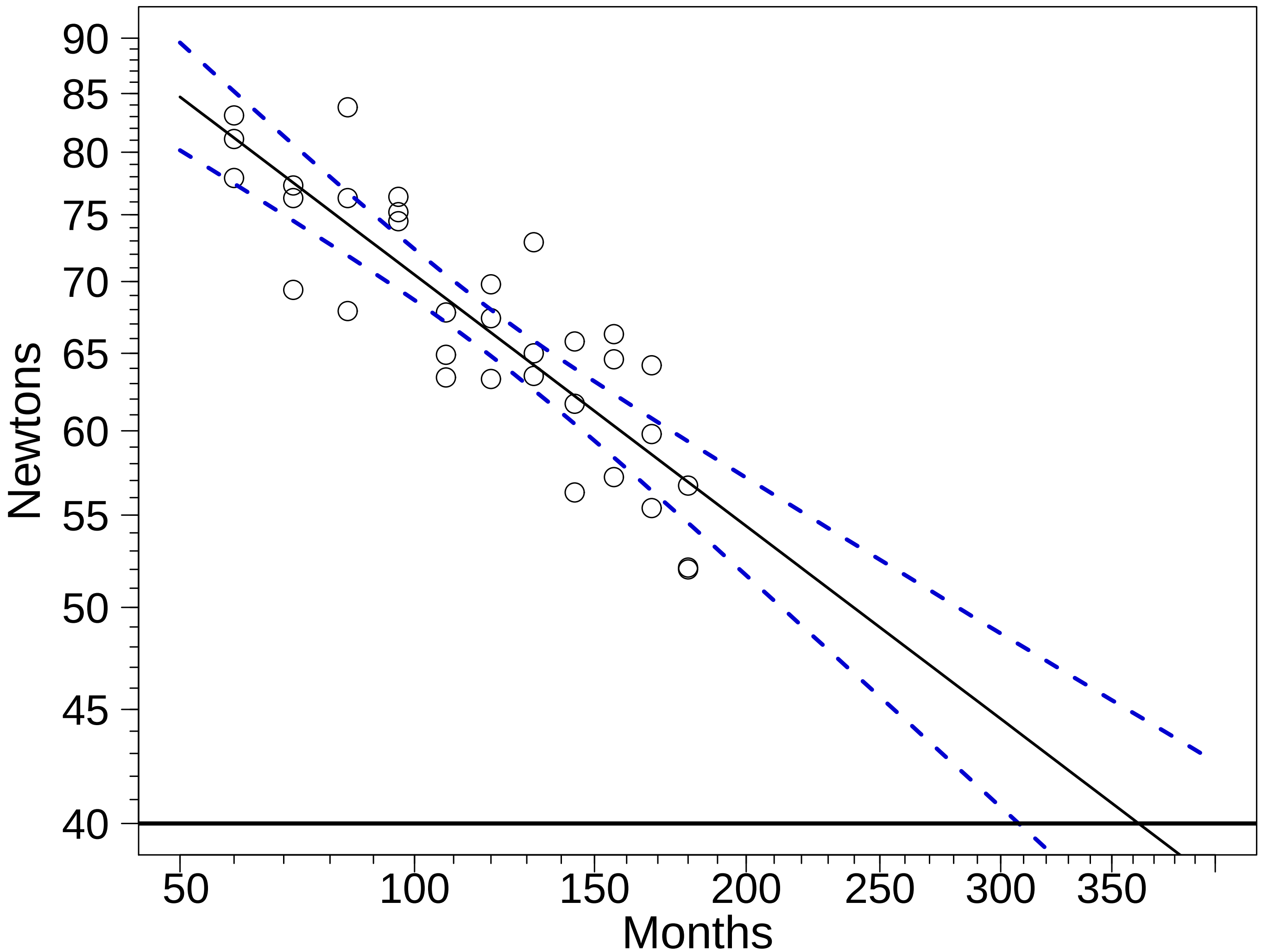
- $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)$ for the linear models.
- $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_3, \sigma)$ for the asymptotic models.

Adhesive Bond A Strength Field Data
Log/Square Root Transformation
Weakly Informative Prior Distribution
Posterior Pairs Plot $\widehat{\xi}(t) = \widehat{\beta}_0 - \widehat{\beta}_1 \tau$



Adhesive Bond A Strength Field Data and Fitted Model Normal Distribution Linear Path

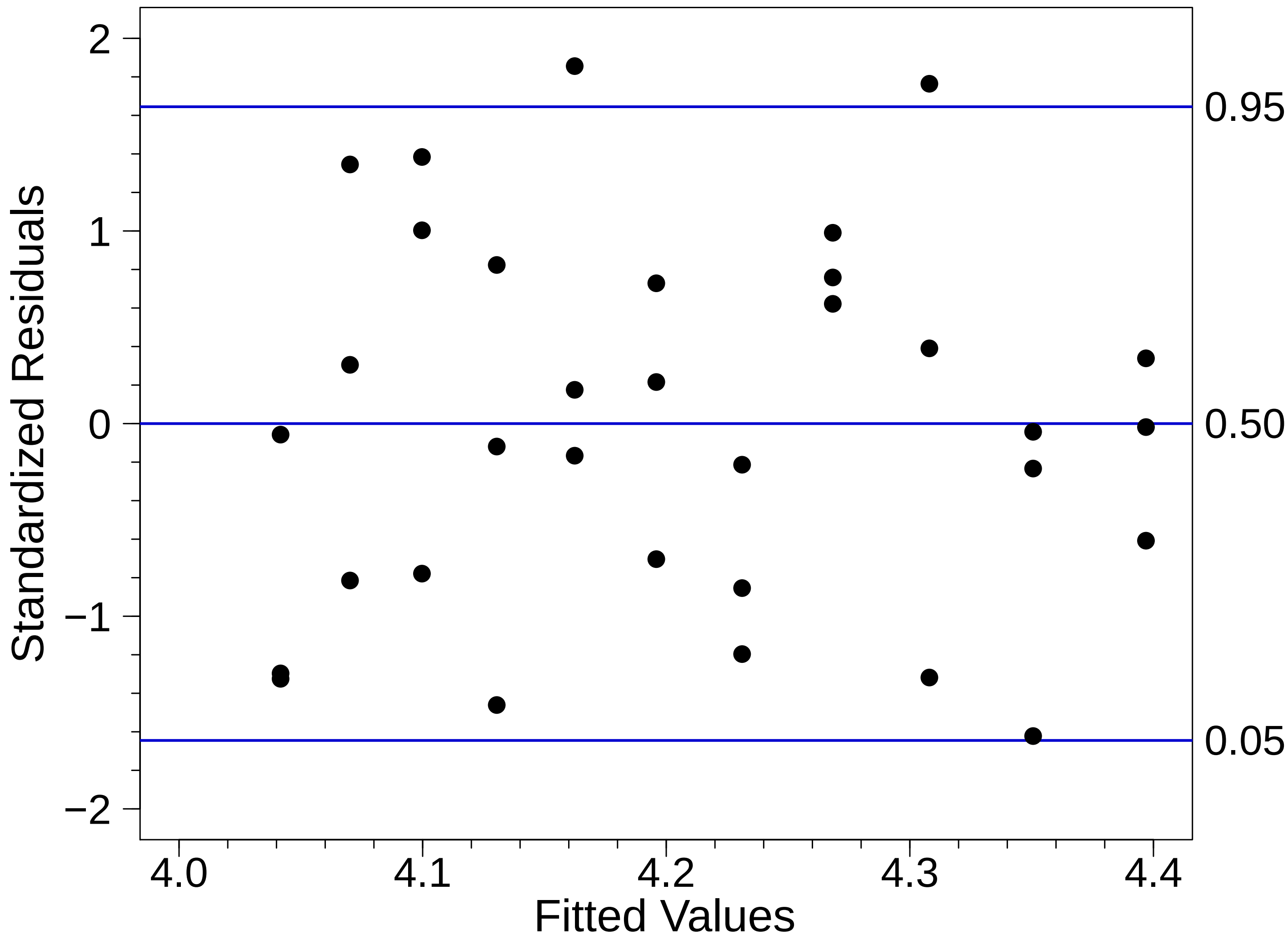
$$\widehat{\xi}(t) = \widehat{\beta}_0 - \widehat{\beta}_1 \tau$$



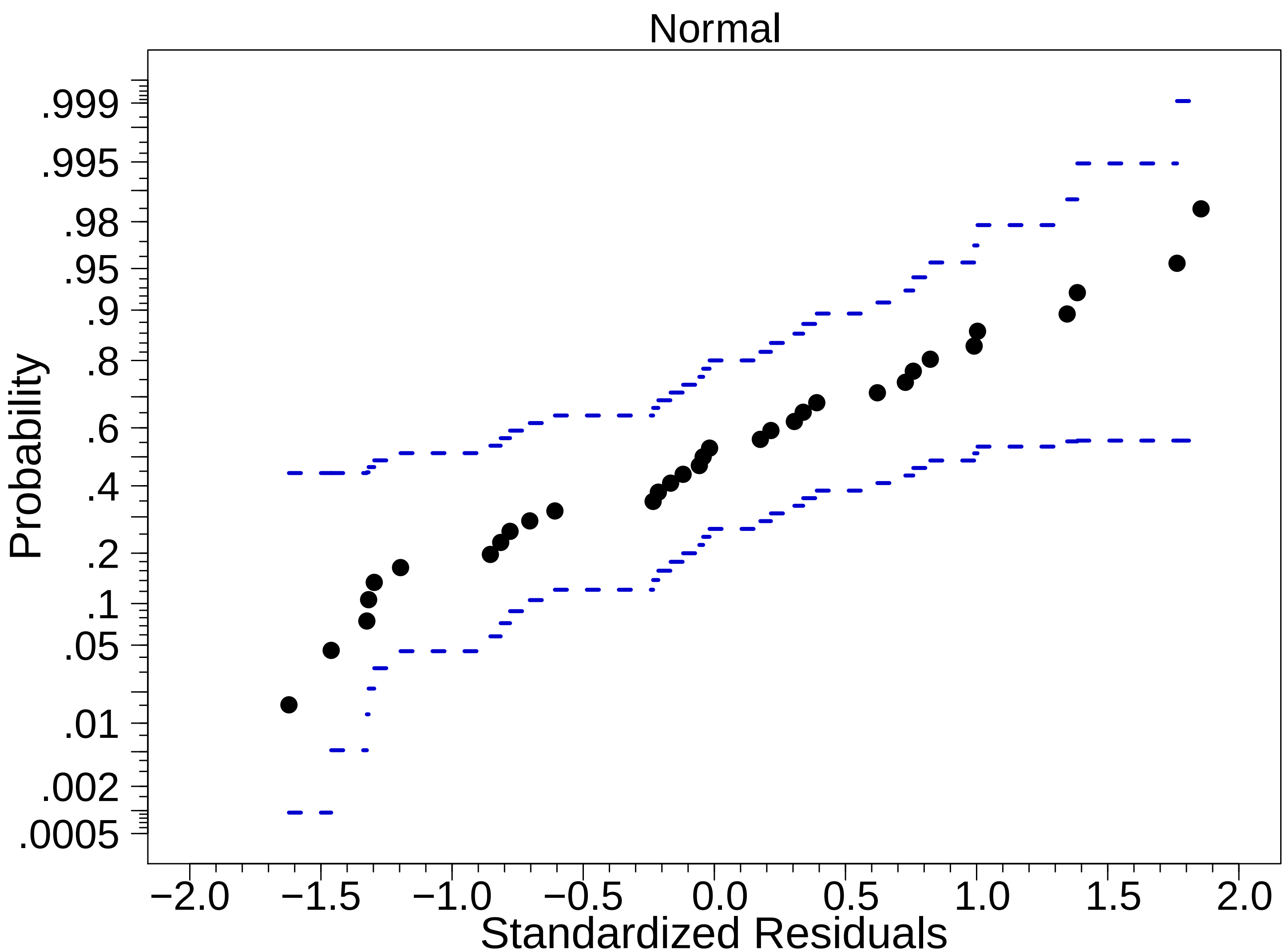
Adhesive Bond A Strength Field Data
Bayesian Parameter Estimates
Normal Distribution Linear Path Model

Parameter	Estimate	Standard Error	95% Credible Interval	
			Lower	Upper
β_0	4.49	0.01	4.46	4.51
β_1	0.37	0.02	0.33	0.40
σ	0.05	0.005	0.04	0.06

Adhesive Bond A Strength Field Data
Residuals Versus Fitted Values



Adhesive Bond A Strength Field Data
Normal Distribution Residual Probability Plot



Chapter 20

Degradation Modeling and Destructive Degradation Data Analysis

Segment 4

Failure-Time Distributions Induced from Destructive Degradation Models and Failure-Time Inferences

A General Approach to Obtaining the Failure Time Distribution for Increasing Destructive Degradation Models

For increasing degradation, the failure time T of a unit is defined to be the time that its observed degradation exceeds a critical value $\mathcal{D}_{\mathsf{f}}$. The event $T \leq t$ is equivalent to observed degradation being greater than or equal to $\mathcal{D}_{\mathsf{f}}$ [i.e., $Y \geq h_d(\mathcal{D}_{\mathsf{f}})$]. Then,

$$F(t,x) = \Pr(T \leq t) = 1 - \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \xi(t,x)}{\sigma}\right], \text{ for } t \geq 0.$$

$$t_p = \begin{cases} 0 & \text{if } p \leq F(0,x) \\ \xi^{-1}\left[h_d(\mathcal{D}_{\mathsf{f}}) - \sigma\Phi^{-1}(1-p)\right] & \text{if } F(0,x) < p < F(\infty,x) \\ \infty & \text{if } F(\infty,x) < p, \end{cases}$$

where for given x, $\xi^{-1}(w)$ is the unique solution for t in the equation $\xi(t,x) = w$. That is, $\xi[\xi^{-1}(w), x] = w$.

A General Approach to Obtaining the Failure Time Distribution for <u>Decreasing</u> Destructive Degradation Models

For decreasing degradation, $T \leq t$ is equivalent to observed degradation being less than or equal to $\mathcal{D}_{\mathsf{f}}$ [i.e., $Y \leq h_d(\mathcal{D}_{\mathsf{f}})$]. Then,

$$F(t,x) = \mathrm{Pr}(T \leq t) = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \xi(t,x)}{\sigma}\right], \text{ for } t \geq 0.$$

$$t_p = \begin{cases} 0 & \text{if } p \leq F(0,x) \\ \xi^{-1}\left[h_d(\mathcal{D}_{\mathsf{f}}) - \sigma\Phi^{-1}(p)\right] & \text{if } F(0,x) < p < F(\infty,x) \\ \infty & \text{if } F(\infty,x) < p. \end{cases}$$

Induced Failure Time Distribution for the Linear Degradation Model 2 (Decreasing Degradation)

- For Model 2 $T \le t$ is equivalent to observed degradation being less than or equal to $\mathcal{D}_{\mathsf{f}}$ [i.e., $Y \le h_d(\mathcal{D}_{\mathsf{f}})$]. Then

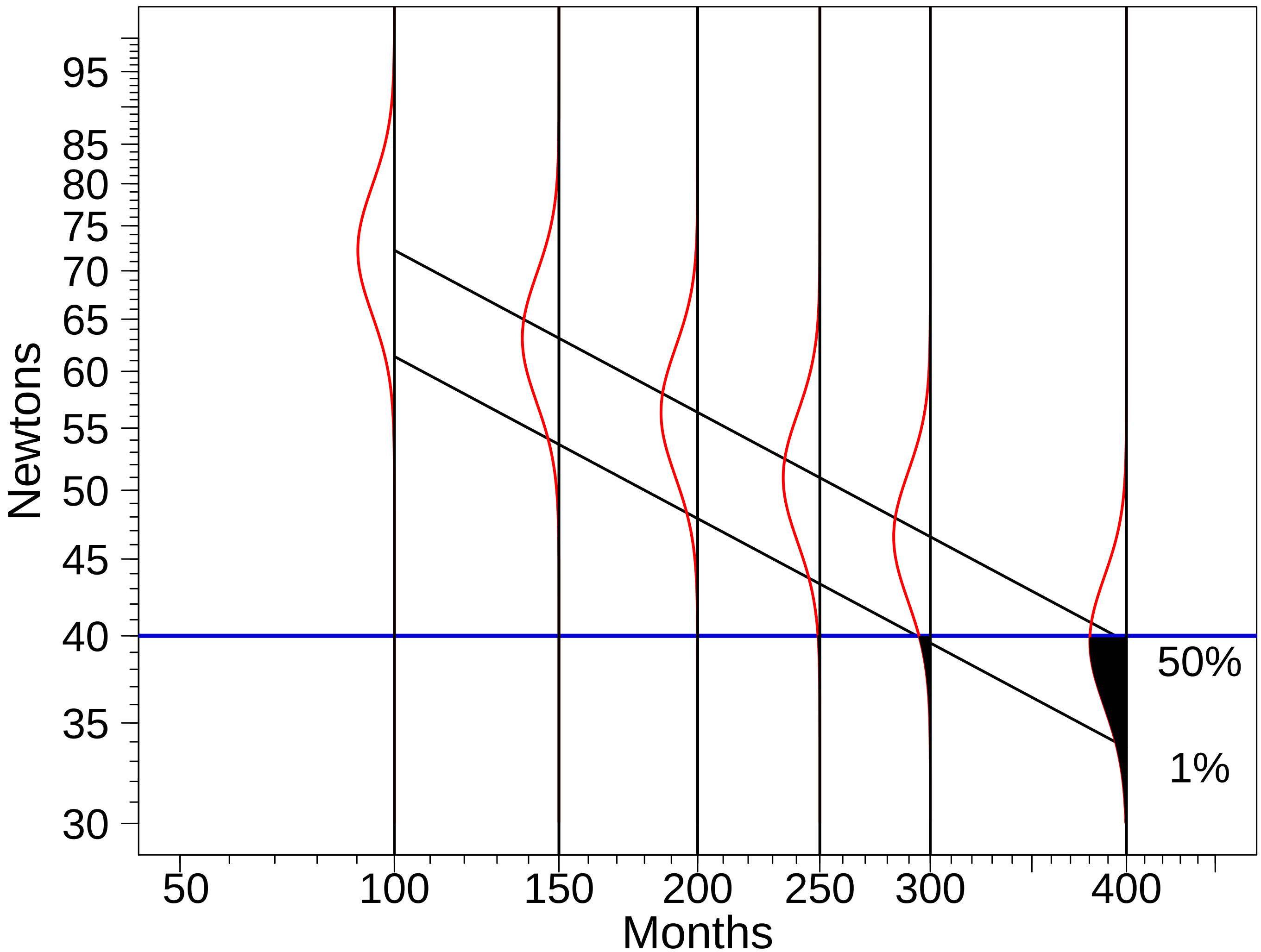
$$F(t) = \Pr[Y \le h_d(\mathcal{D}_{\mathsf{f}})] = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \xi(t)}{\sigma}\right], \text{ for } t \ge 0.$$

- This failure time distribution is a mixed distribution with a probability **atom** at $t = 0$ and probability

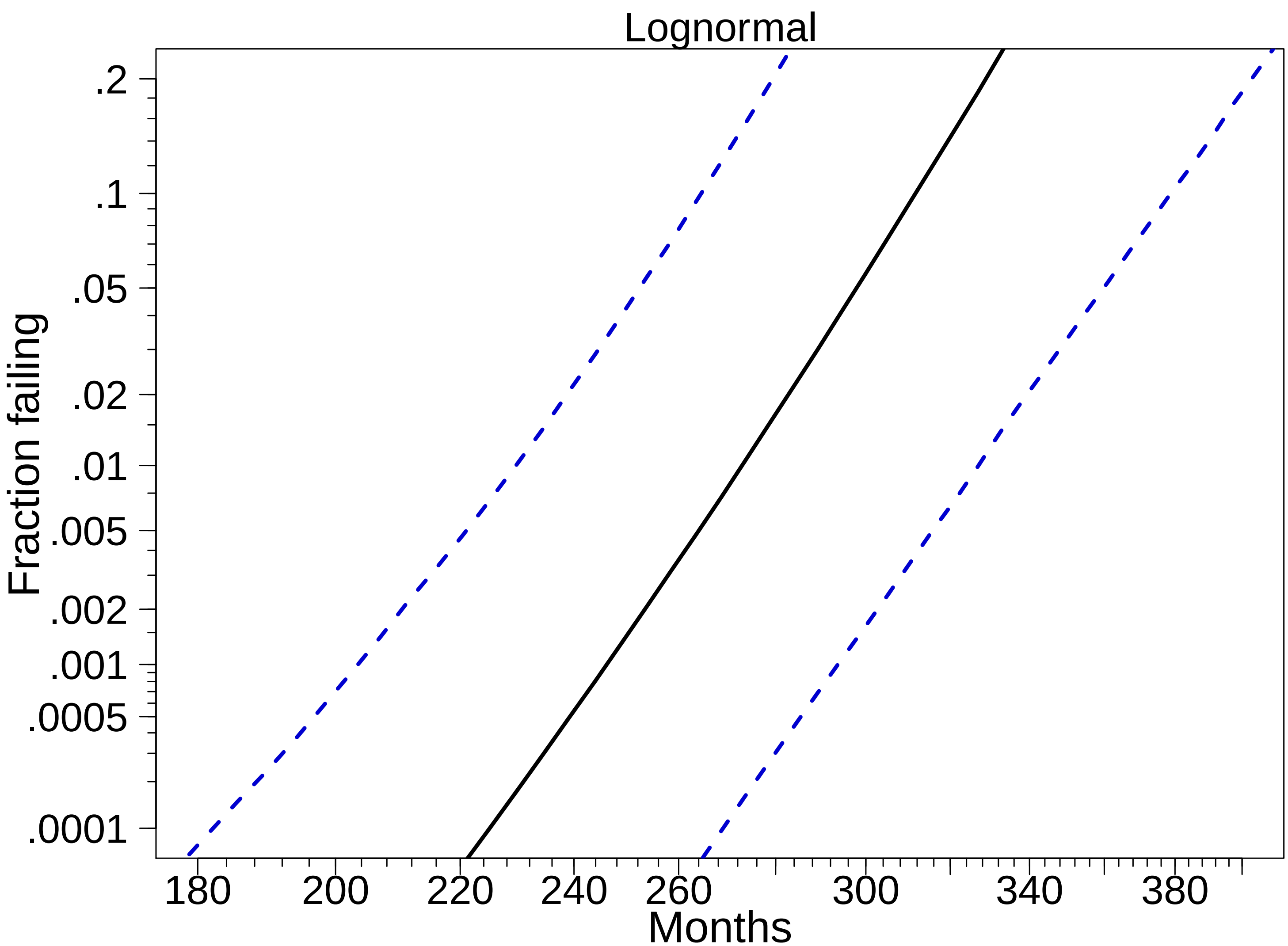
$$\Pr(T = 0) = F(0) = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0}{\sigma}\right].$$

Adhesive Bond A
Estimate of Fraction Failing as a Function of Time

$$\widehat{y}_p = \widehat{\beta}_0 - \widehat{\beta}_1 \tau + \widehat{\sigma} \Phi^{-1}_{\textbf{norm}}(p)$$



Adhesive Bond A
Lognormal Probability Plot of the Failure-Time cdf Estimate and 95% Credible Intervals



Quantiles for the Failure Time Distribution at Fixed Values of $\mathcal{D}_{\mathsf{f}}$ for Model 2

For Model 2, the p quantile is $t_p = h_t^{-1}(\tau_p)$, where

$$\tau_p = \begin{cases} 0 & \text{if } p \le F(0) \\ \dfrac{1}{\beta_1}\Big[\beta_0 - h_d(\mathcal{D}_{\mathsf{f}}) + \Phi^{-1}(p)\sigma\Big] & \text{if } p > F(0), \end{cases}$$

where $F(0) = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}})-\beta_0}{\sigma}\right]$.

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Degradation Modeling and Destructive Degradation Data Analysis

Segment 5

Background and an Example of <u>Accelerated</u> Destructive Degradation Testing (ADDT) and Model Building.

Accelerated Destructive Degradation Test of Adhesive Bond B

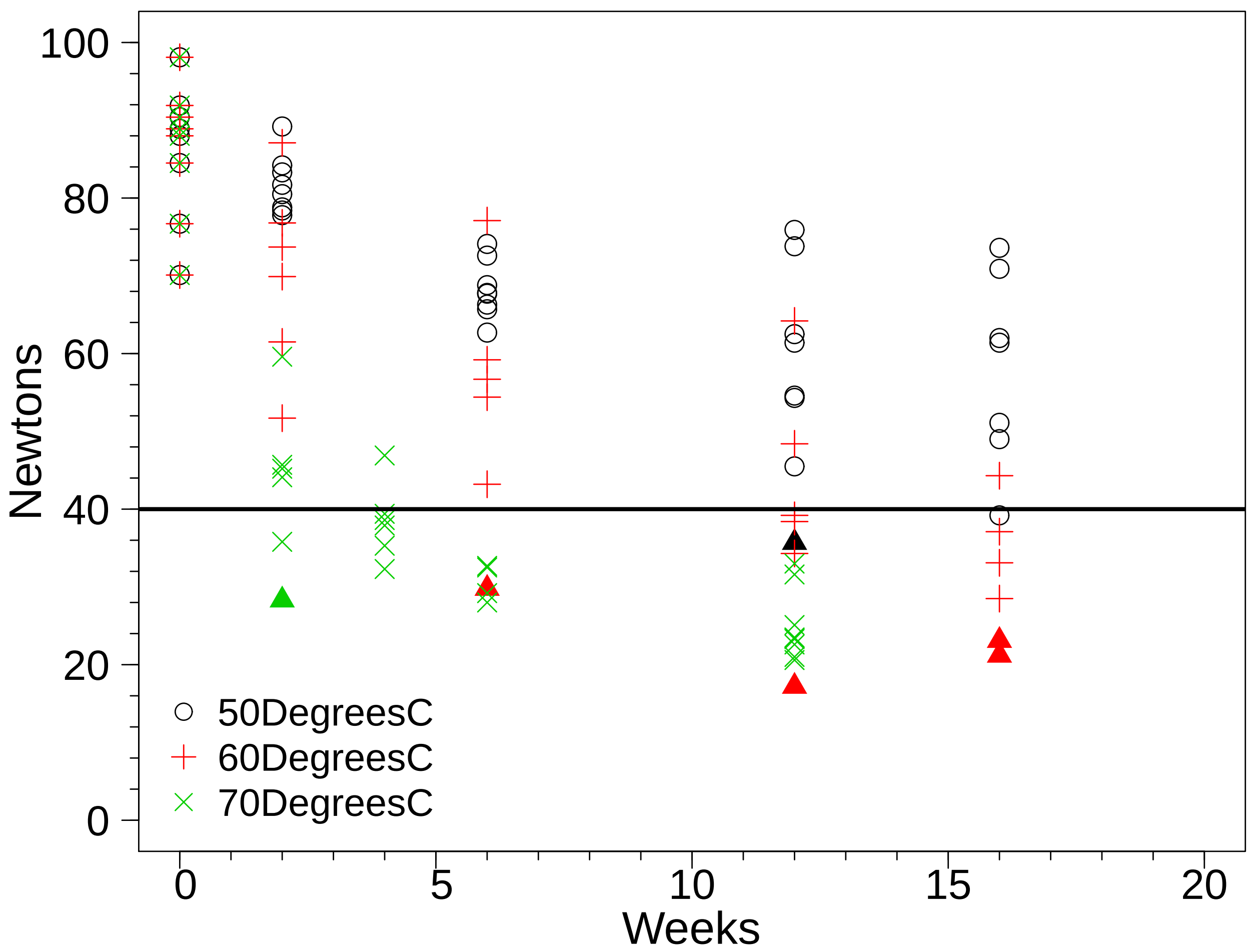
- **Objective:** Assess the strength of an **adhesive bond** as a function of time. Estimate the fraction of devices with a strength below 40 Newtons after 5 years of operation (approximately 260 weeks) at 25°C.
- The test is destructive; each unit can be measured only once.
- There were 6 right-censored observations.
- 8 units with no aging were measured at the start of the experiment.
- A total of 80 additional units were aged and measured according to a temperatures and time schedule.

Adhesive Bond B ADDT Test Plan

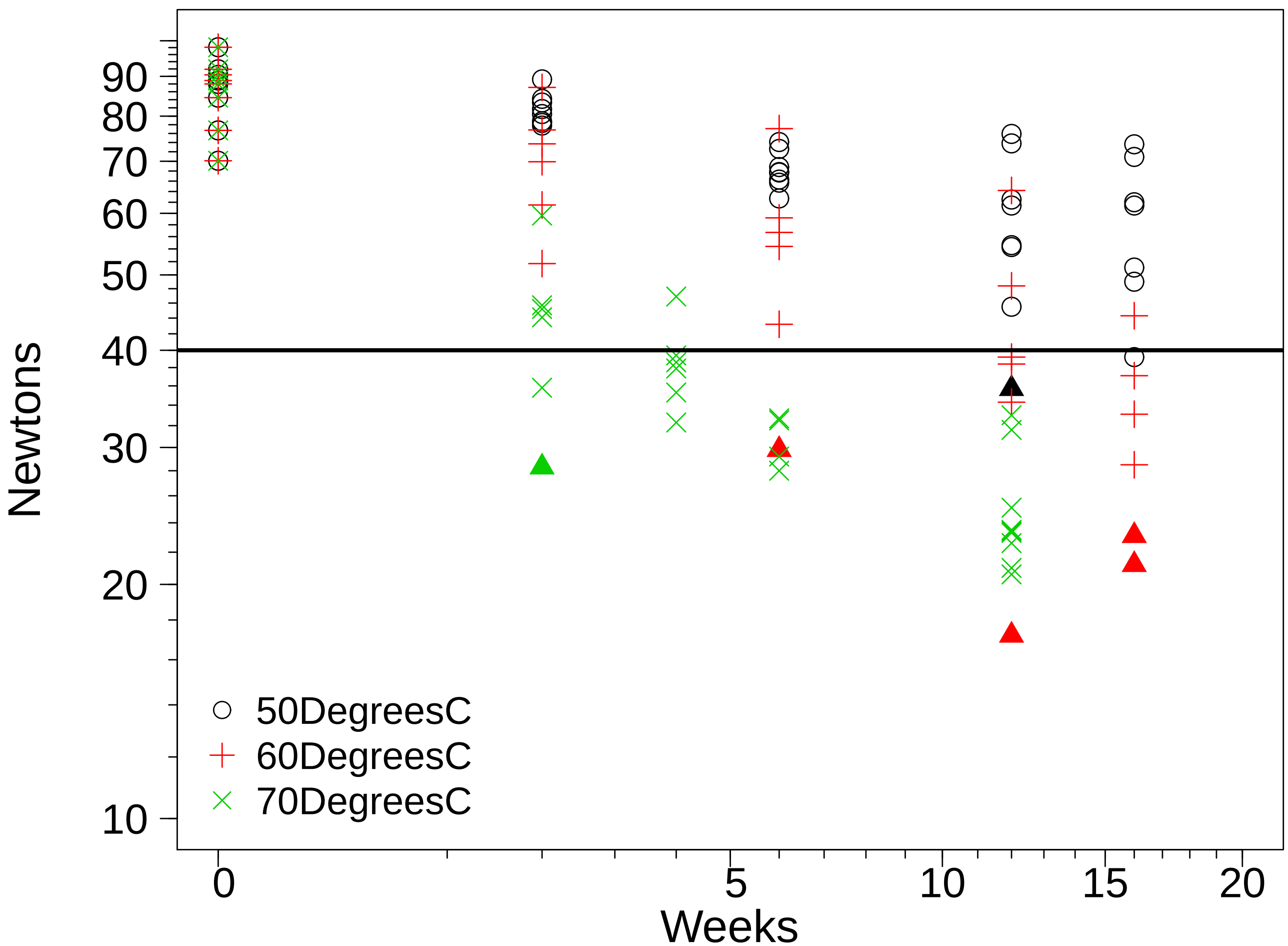
Number of Specimens Tested

Temp °C	Weeks Aged 0	2	4	6	12	16	Totals
—	8						8
50		8	0	8	8	7	31
60		6	0	6	6	6	24
70		6	6	4	9	0	25
Totals	8	20	6	18	23	13	88

Adhesive Bond B ADDT Data
Scatter Plot
Linear–Linear Axes

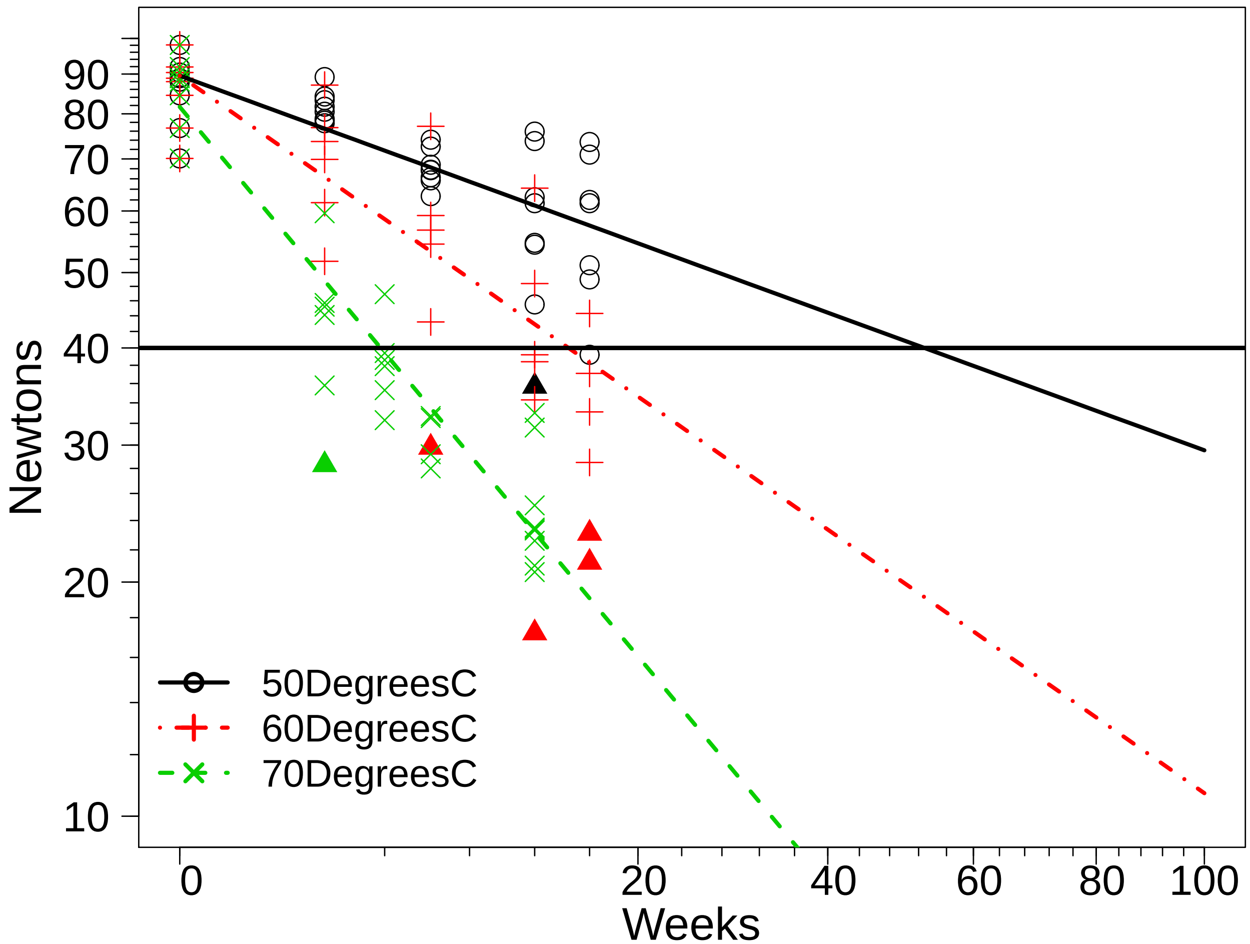


Adhesive Bond B ADDT Data
Scatter Plot
Square Root–Log Axes



Adhesive Bond B ADDT Data
Overlay of Individual Normal Distribution Fits
Square Root–Log Axes

$$\widehat{\xi}^{[j]}(t) = \widehat{\beta}_0^{[j]} + \widehat{\beta_1}^{[j]}\tau, \quad j = 50, 60, 70$$



General Structure of Destructive Degradation Regression Models

- Degradation model: $Y = \xi(t, x) + \epsilon$, where for fixed x, path $\xi(t, x)$ is monotone in t and ϵ has location-scale distribution with parameters $\mu = 0$ and σ.
- Other forms could be used for $\xi(t, x)$.
- Time t can be viewed as a special kind of explanatory variable for Y.
- ϵ is an error term that describes unit-to-unit variability (and probably some measurement errors and model uncertainty that may not be independently estimable).
- The degradation distribution and its quantile:

$$G(y; t, x) = \Pr(Y \leq y) = \Phi\left[\frac{y - \xi(t, x)}{\sigma}\right].$$

For given (t, x), the p quantile for the cdf $G(y; t, x)$ is

$$y_p(t, x) = \xi(t, x) + \Phi^{-1}(p)\,\sigma.$$

Adhesive Bond B ADDT Data
Bayesian Estimates
Linear Path Normal Distribution Individual Line Fits

- For each temperature level j three individual estimates are obtained: $\widehat{\beta}_0^{[j]}$, $\widehat{\beta}_1^{[j]}$, and $\widehat{\sigma}^{[j]}$.
- A summary of the linear path normal distribution estimates for individual temperatures for the Adhesive Bond B data is

	Estimates			95% Credible Interval for $\widehat{\beta}_1^{[j]}$	
Temperature	$\widehat{\beta}_0^{[j]}$	$\widehat{\beta}_1^{[j]}$	$\widehat{\sigma}^{[j]}$	$\underset{\sim}{\beta}_1^{[j]}$	$\widetilde{\beta}_1^{[j]}$
50°C	4.50	0.11	0.14	0.08	0.14
60°C	4.50	0.21	0.17	0.17	0.26
70°C	4.40	0.36	0.15	0.32	0.40

Individual Degradation Rate Estimates

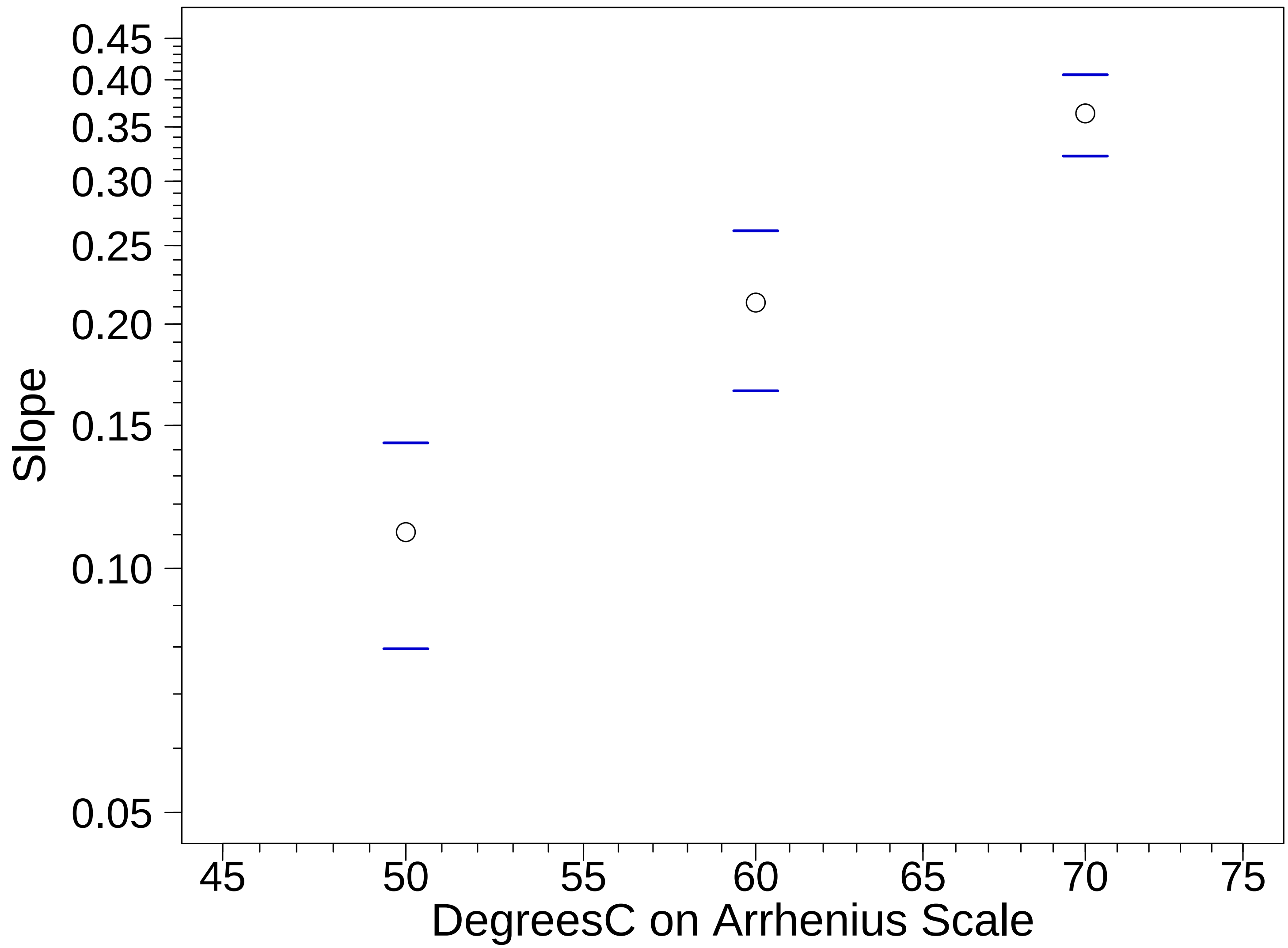
- The estimates $\widehat{\beta}_1^{[j]}$ (slopes of the individual lines at test condition j) can be used to identify the relationship between the degradation rate and the accelerating variables.
- Taking the log of the slope in Model 6 gives

$$\log(\beta_1^{[j]}) = \log(\beta_1) - \boldsymbol{\beta}_2'(\boldsymbol{x}_j - \bar{\boldsymbol{x}}_j)$$

  the surface $\log(\widehat{\beta}_1^{[j]})$ versus $\boldsymbol{x}_j$ should be approximately linear in the $\boldsymbol{x}_j$ if the model relating degradation rate and the accelerating variables is adequate. Then

  - For a single accelerating variable x, the plot of $\log(\widehat{\beta}_1^{[j]})$ versus x_j, for all values of j should be approximately linear.
  - For a vector $\boldsymbol{x}$ the plot of $\log(\widehat{\beta}_1^{[j]})$ versus any of the accelerating variables, conditional on fixed values of the remaining accelerating variables, should be approximately linear.

Adhesive Bond B ADDT Data Arrhenius Plot of Individual Degradation Rate Estimates $\widehat{\beta}_1^{[j]}$ versus °C Normal Distribution Estimates



Chapter 20

Degradation Modeling and Destructive Degradation Data Analysis

Segment 6

Fitting an <u>Acceleration Model</u> to ADDT Data

Linear-Path Acceleration Model for the Adhesive Bond B Data

For the Adhesive Bond B data, the strength of the adhesive as a function of time and temperature is modeled by

$$\begin{aligned} Y_i &= \xi(t_i, x_i) + \epsilon_i \\ &= \beta_0 - \beta_1 \exp[-\beta_2(x_i - x_0)]\tau_i + \epsilon_i \end{aligned}$$

where

$$\begin{aligned} Y_i &= \log(\text{Newtons}_i) \\ \tau_i &= \sqrt{t_i} = \sqrt{\texttt{Weeks}_i} \\ x_i &= 11604.52/(^\circ\text{C}_i + 273.15) \\ x_0 &= 50^\circ\text{C} \\ \epsilon_i &\sim \text{NORM}(0, \sigma), \quad i = 1, \dots, n. \end{aligned}$$

Likelihood for the ADDT Model with Right Censored Data

- For a sample of n units consisting of exact failure times and right-censored observations, the likelihood can be expressed as

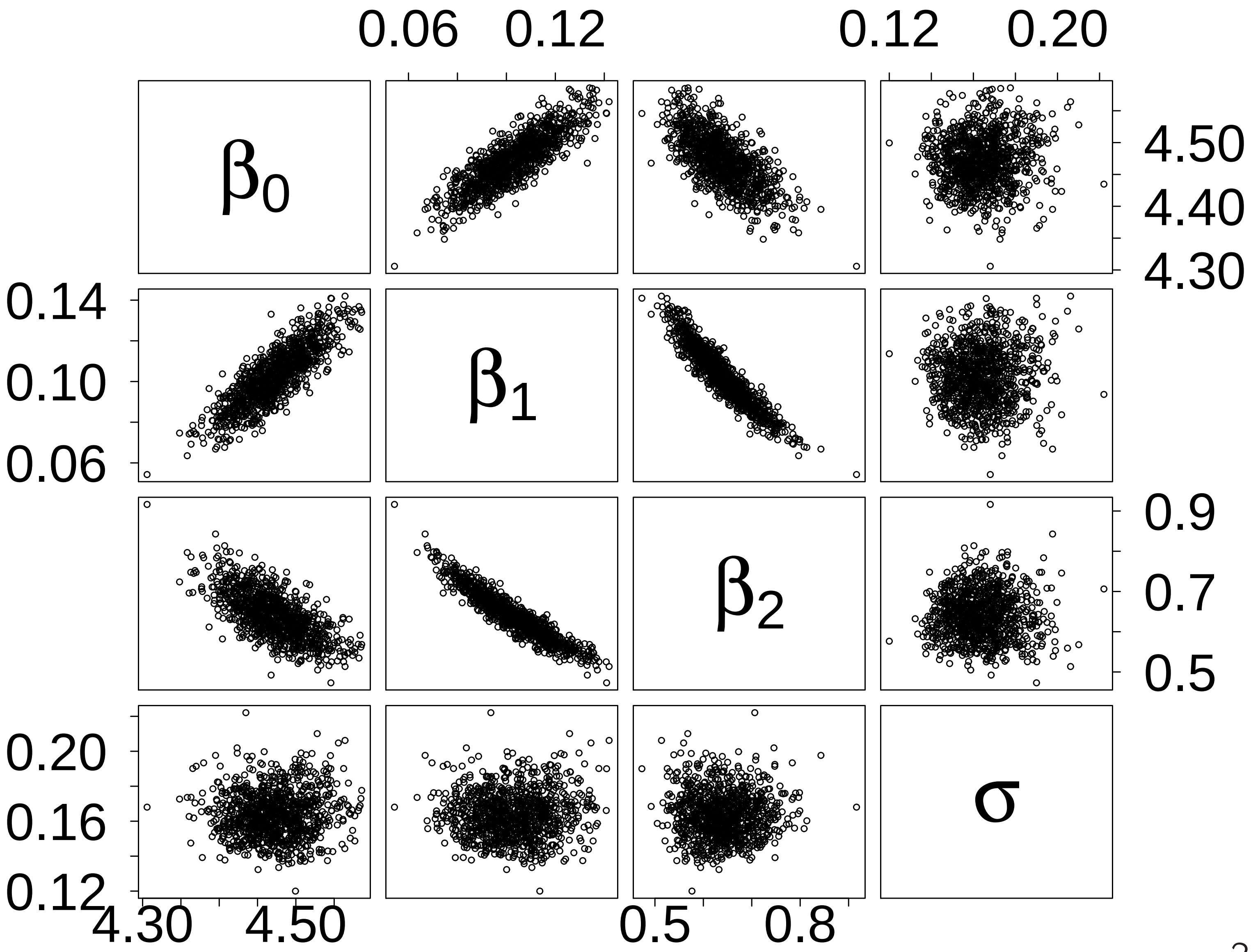
$$L(\boldsymbol{\theta}|\text{DATA}) = \prod_{i=1}^{n} \left[\frac{1}{\sigma}\phi\left(\frac{y_i - \xi(t_i, x_i)}{\sigma}\right)\right]^{\delta_i} \times \left[1 - \Phi\left(\frac{y_i - \xi(t_i, x_i)}{\sigma}\right)\right]^{1-\delta_i}.$$

- n is the number of observations.
- $\xi(t, x_i)$ is the chosen path model (say one of Models 5–8).
- The censoring indicator

$$\delta_i = \begin{cases} 1 & \text{if } y_i \text{ is an exact observation} \\ 0 & \text{if } y_i \text{ is a right-censored observation.} \end{cases}$$

- $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_2, \sigma)$ for the linear models.
- $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_2, \beta_3, \sigma)$ for the asymptotic models.

Adhesive Bond B Strength Data
Log/Square Root Transformation
Weakly Informative Prior Distribution
Posterior Pairs Plot



Adhesive Bond B ADDT Data
Bayesian Parameter Estimates
Normal Distribution Linear Path Arrhenius Model

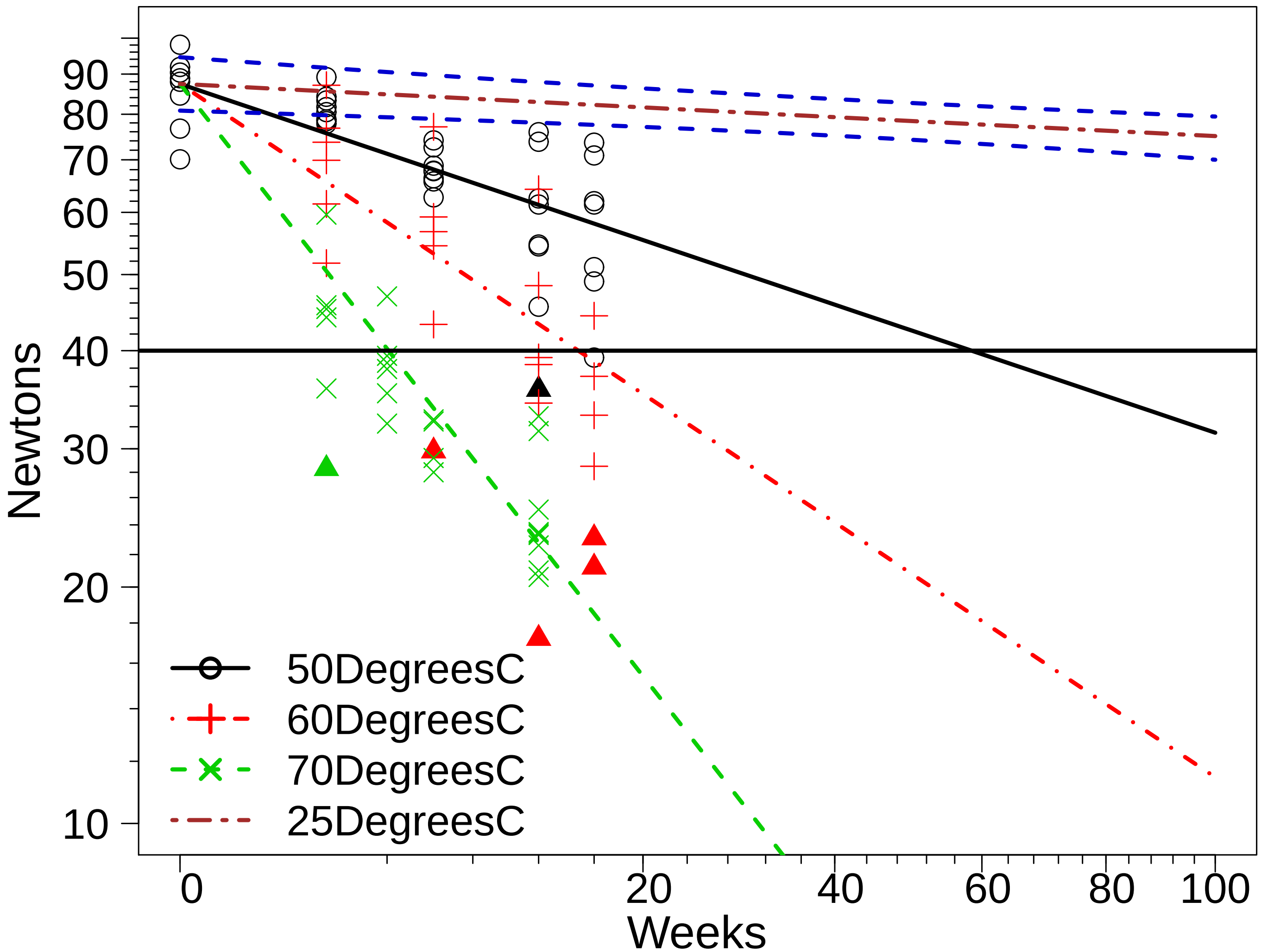
Parameter	Estimate	Standard Error	95% Credible Interval	
			Lower	Upper
β_0	4.47	0.04	4.39	4.55
β_1	0.10	0.01	0.08	0.13
β_2	0.64	0.06	0.54	0.77
σ	0.16	0.01	0.14	0.19

Estimates for the slopes (degradation rates) at each temperature are obtained from $\widehat{\beta}_1^{[j]} = \widehat{\beta}_1 \exp\left[-\widehat{\beta}_2(x - x_0)\right]$ where $x = 11604.52/(°\text{C} + 273.15)$ and $x_0 = 50°\text{C}$. In this case for the four temperatures of interest, the estimates are

$$\widehat{\beta}_1^{[25]} = 0.015, \quad \widehat{\beta}_1^{[50]} = 0.101$$
$$\widehat{\beta}_1^{[60]} = 0.202, \quad \widehat{\beta}_1^{[70]} = 0.388$$

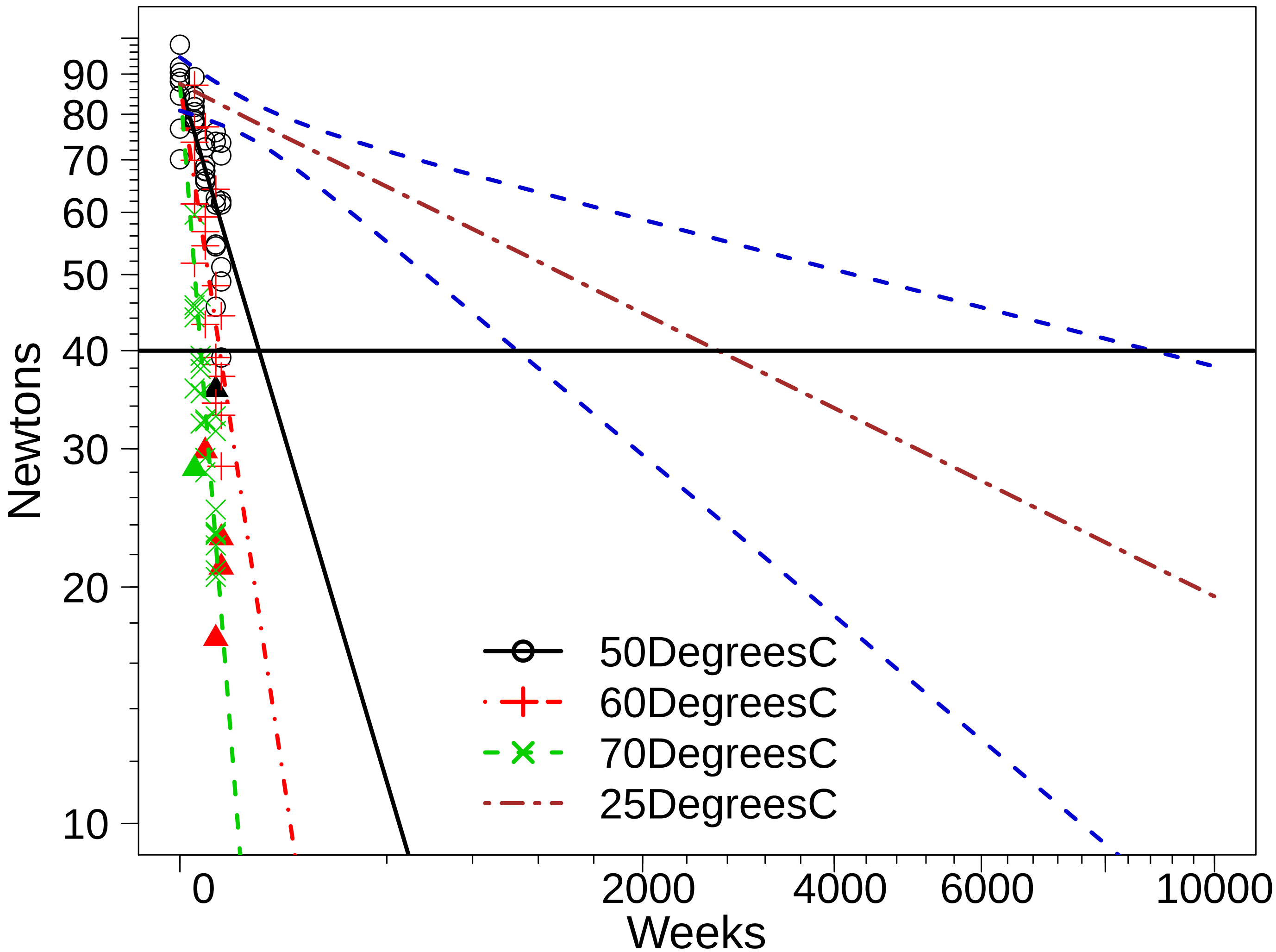
Adhesive Bond B ADDT Data and Fitted Model
Normal Distribution Linear Path Arrhenius Model

$$\widehat{\xi}(t,x) = \widehat{\beta}_0 - \widehat{\beta}_1 \exp[-\widehat{\beta}_2(x - x_0)]\tau$$



Adhesive Bond B ADDT Data and Fitted Model
Normal Distribution Linear Path Arrhenius Model

$$\widehat{\xi}(t, x) = \widehat{\beta}_0 - \widehat{\beta}_1 \exp[-\widehat{\beta}_2(x - x_0)]\tau$$



Chapter 20

Degradation Modeling and Destructive Degradation Data Analysis

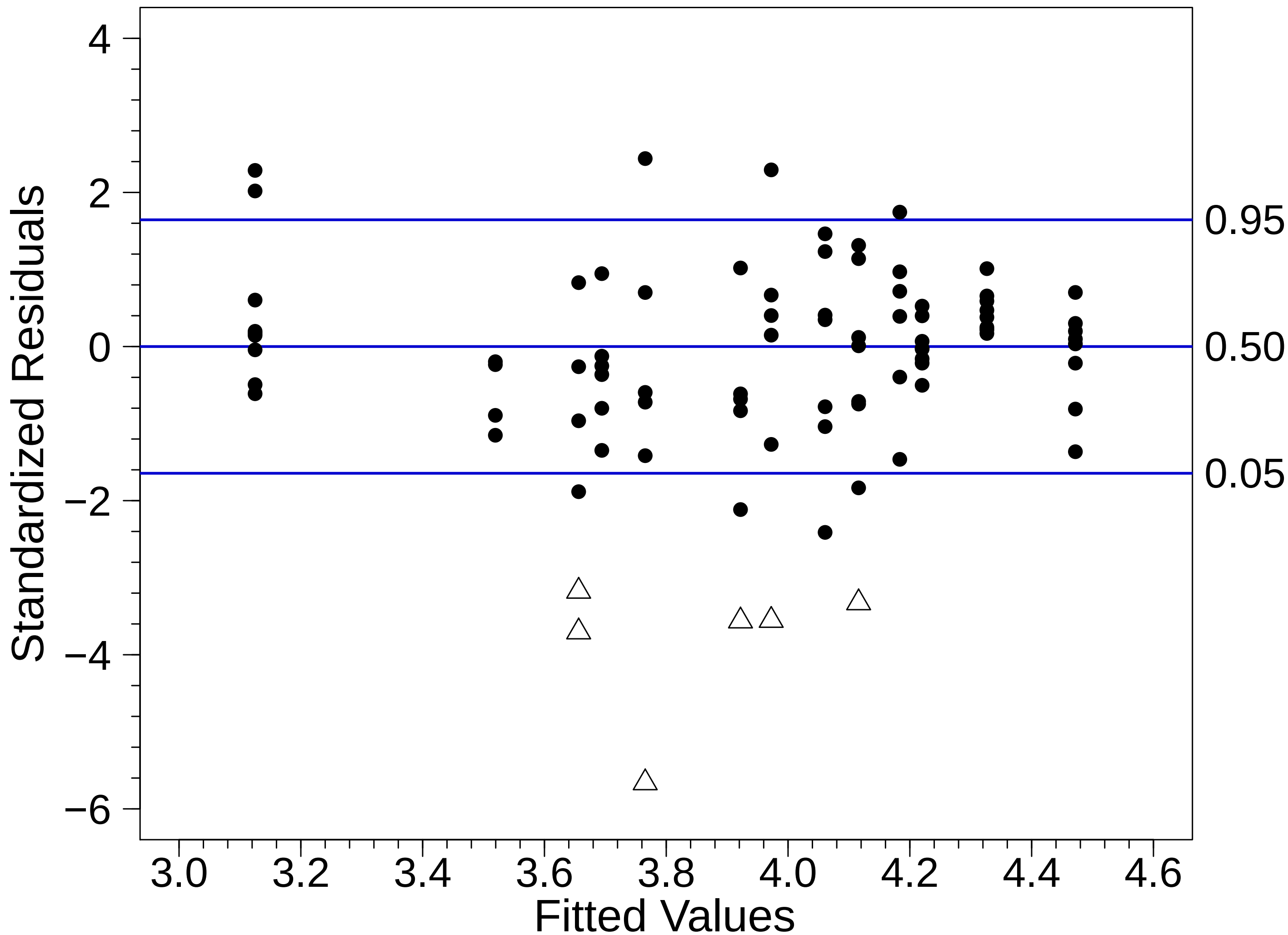
Segment 7

ADDT Model Checking

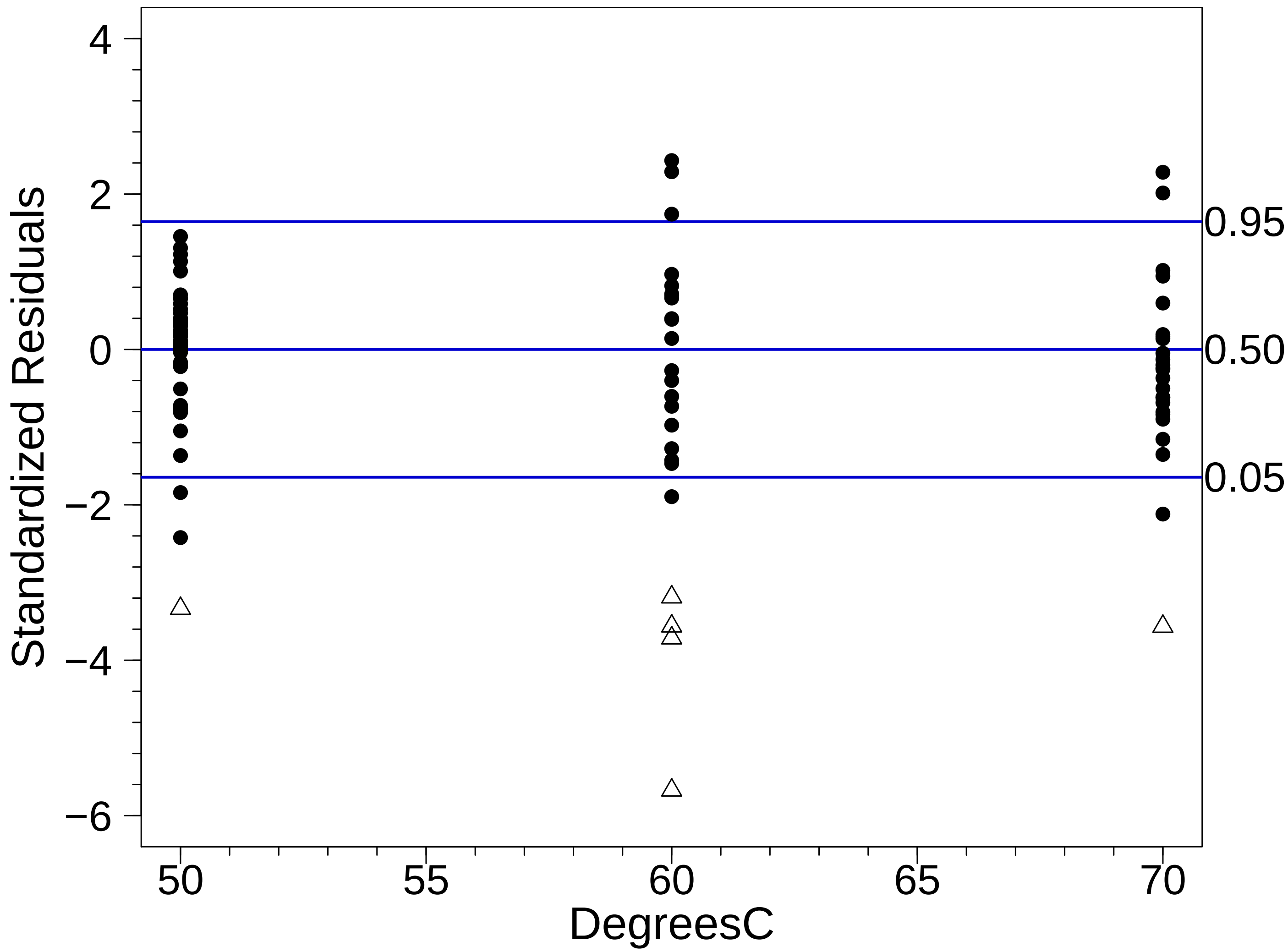
ADDT Model Checking
Residual Plots

- Residuals versus fitted values.
- Residuals versus accelerating variables.
- Residuals versus time of exposure.
- Residuals versus observation order is useful when observations are taken sequentially in time.
- Residual probability plot.

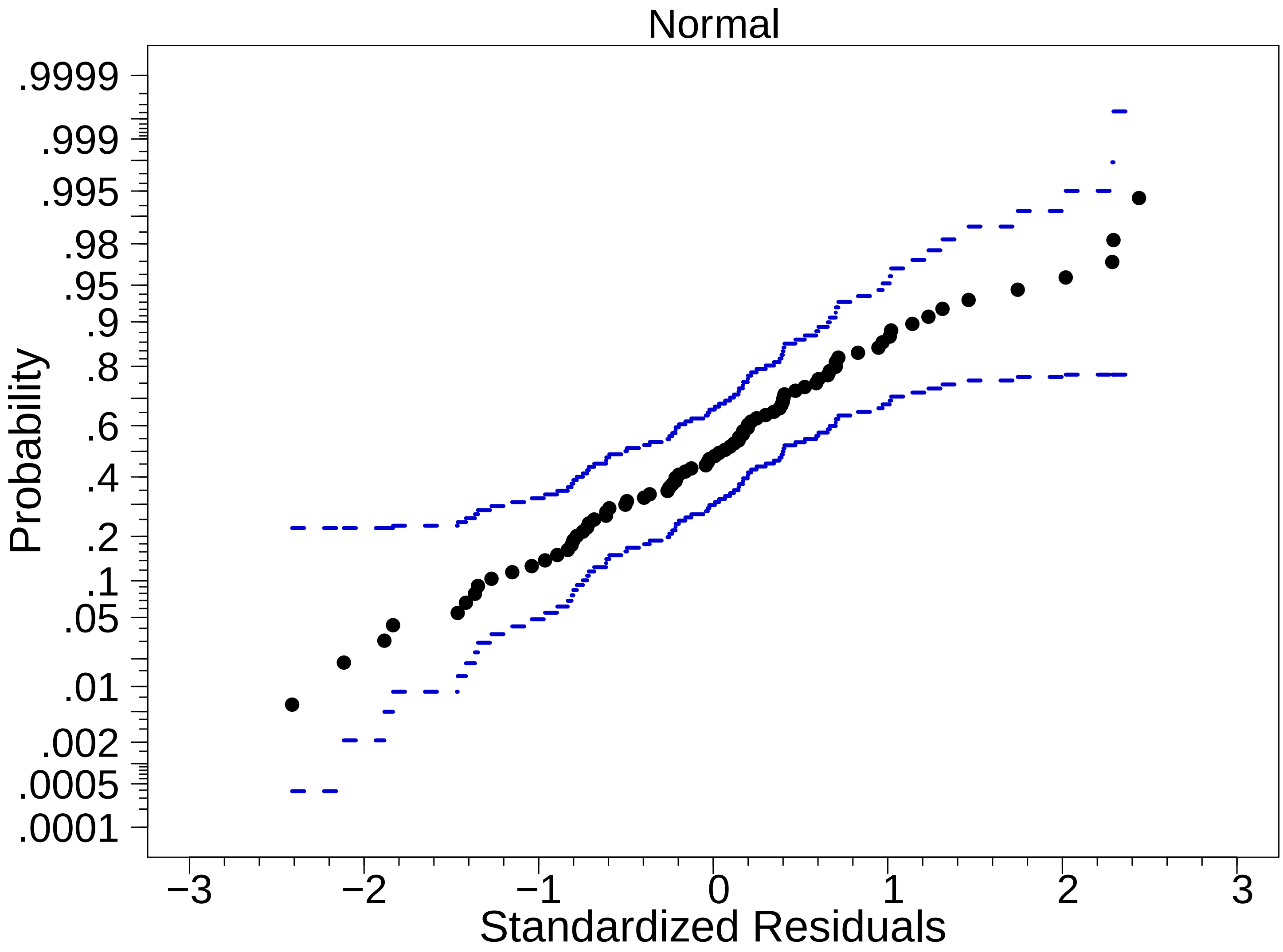
Adhesive Bond B ADDT Data
Residuals Versus Fitted Values



Adhesive Bond B ADDT Data
Residuals Versus Temperature Conditions



Adhesive Bond B ADDT Data
Residual Normal Distribution Probability Plot



Some Comments on the Adhesive Bond B Residuals

- The standardized residuals look approximately like a random sample from a $\mathrm{NORM}(0, 1)$ distribution.

- The horizontal line at 0 in the plot versus fitted values and versus temperature indicate the median of the standardized distribution under the fitted model. Then approximately 50% of the residuals should be above that line.

- There appears to be some evidence of nonconstant variance, but it is not systematic with temperature or times.

Chapter 20

Degradation Modeling and Destructive Degradation Data Analysis

Segment 8

ADDT Failure-Time Distribution Inferences

Induced Failure Time Distribution for the Linear Degradation Model 6 (Decreasing Degradation)

- For Model 6, $T \le t$ is equivalent to degradation being less than or equal to $\mathcal{D}_{\mathsf{f}}$ [i.e., $Y \le h_d(\mathcal{D}_{\mathsf{f}})$]. Then

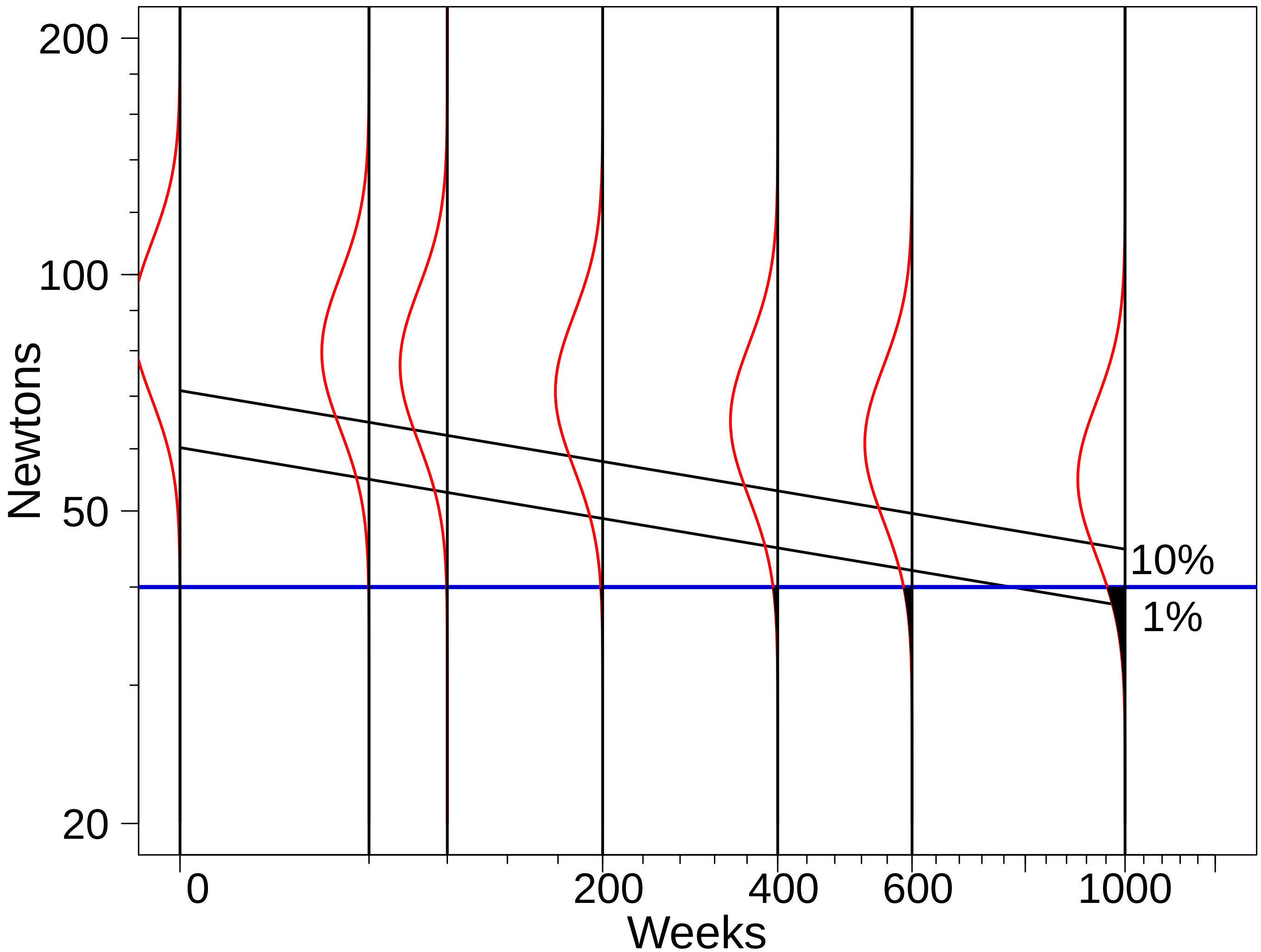
$$\begin{aligned} F(t,x) &= \Pr(T \le t) = \Pr[Y \le h_d(\mathcal{D}_{\mathsf{f}})] \\ &= \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \xi(t,x)}{\sigma}\right], \text{ for } t \ge 0. \end{aligned}$$

- This failure time distribution is a mixed distribution with a probability **atom** at $t = 0$ so

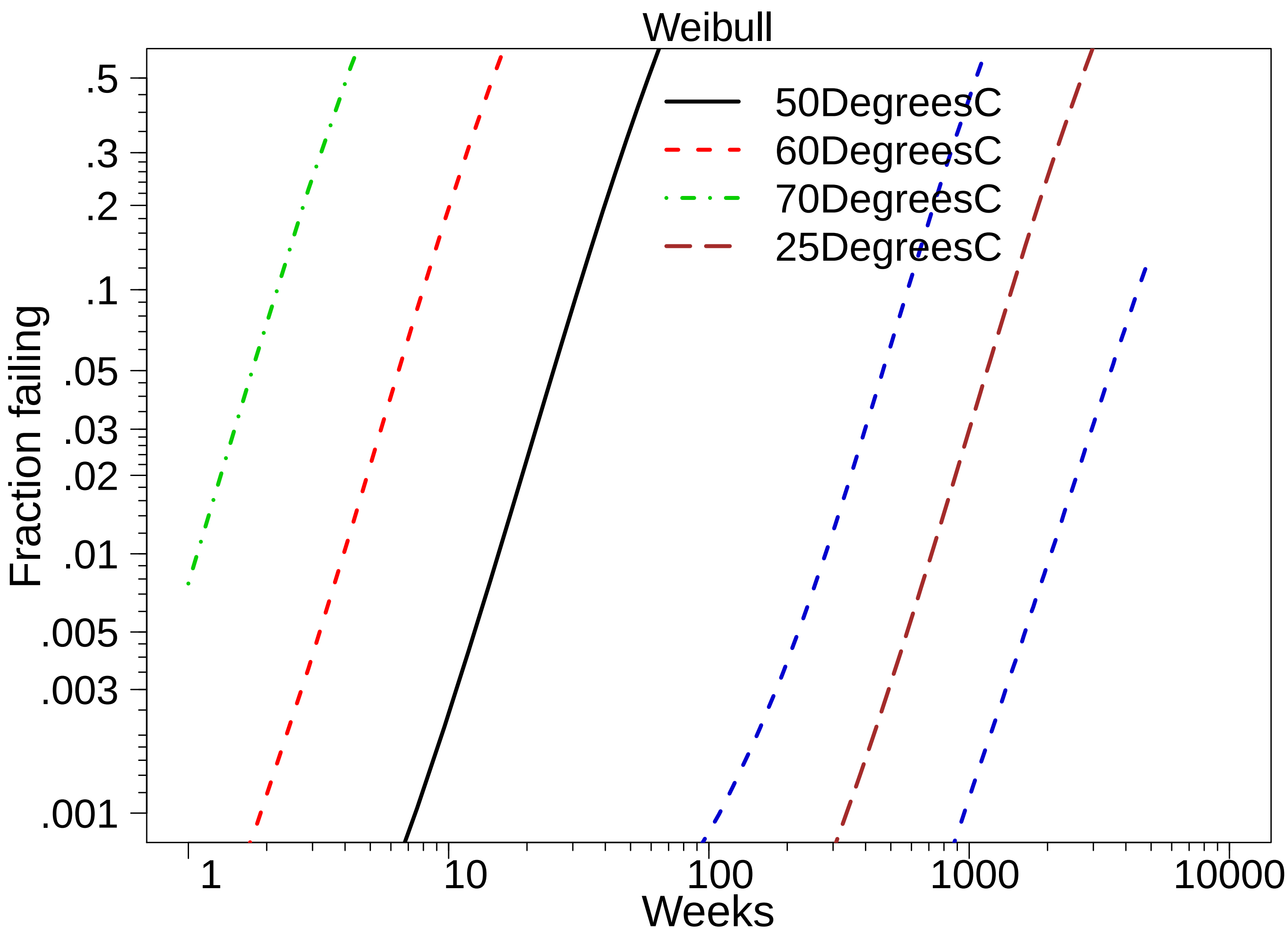
$$\begin{aligned} \Pr(T = 0, x) = F(0,x) &= \Pr(Y \le h_d(\mathcal{D}_{\mathsf{f}})) \\ &= \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0}{\sigma}\right]. \end{aligned}$$

Adhesive Bond B
Estimates of Fraction Failing as a Function of Time at 25°C

$$\widehat{y}_p = \widehat{\beta}_0 - \widehat{\beta}_1^{[25]}\tau + \widehat{\sigma}\Phi^{-1}_{\textbf{norm}}(p)$$



Adhesive Bond B
Weibull Multiple Probability Plot
cdf Estimates at Test Temperatures
and Use Conditions



Quantiles for the Failure Time Distribution at Fixed Values of x and $\mathcal{D}_{\mathsf{f}}$ for Model 6

For Model 6, the p quantile is $t_p = h_t^{-1}(\tau_p)$, where

$$\tau_p = \begin{cases} 0 & \text{if } p \le F(0, x) \\ \dfrac{1}{\beta_1 AF}\Big[\beta_0 - h_d(\mathcal{D}_{\mathsf{f}}) + \Phi^{-1}(p)\sigma\Big] & \text{if } p > F(0, x), \end{cases}$$

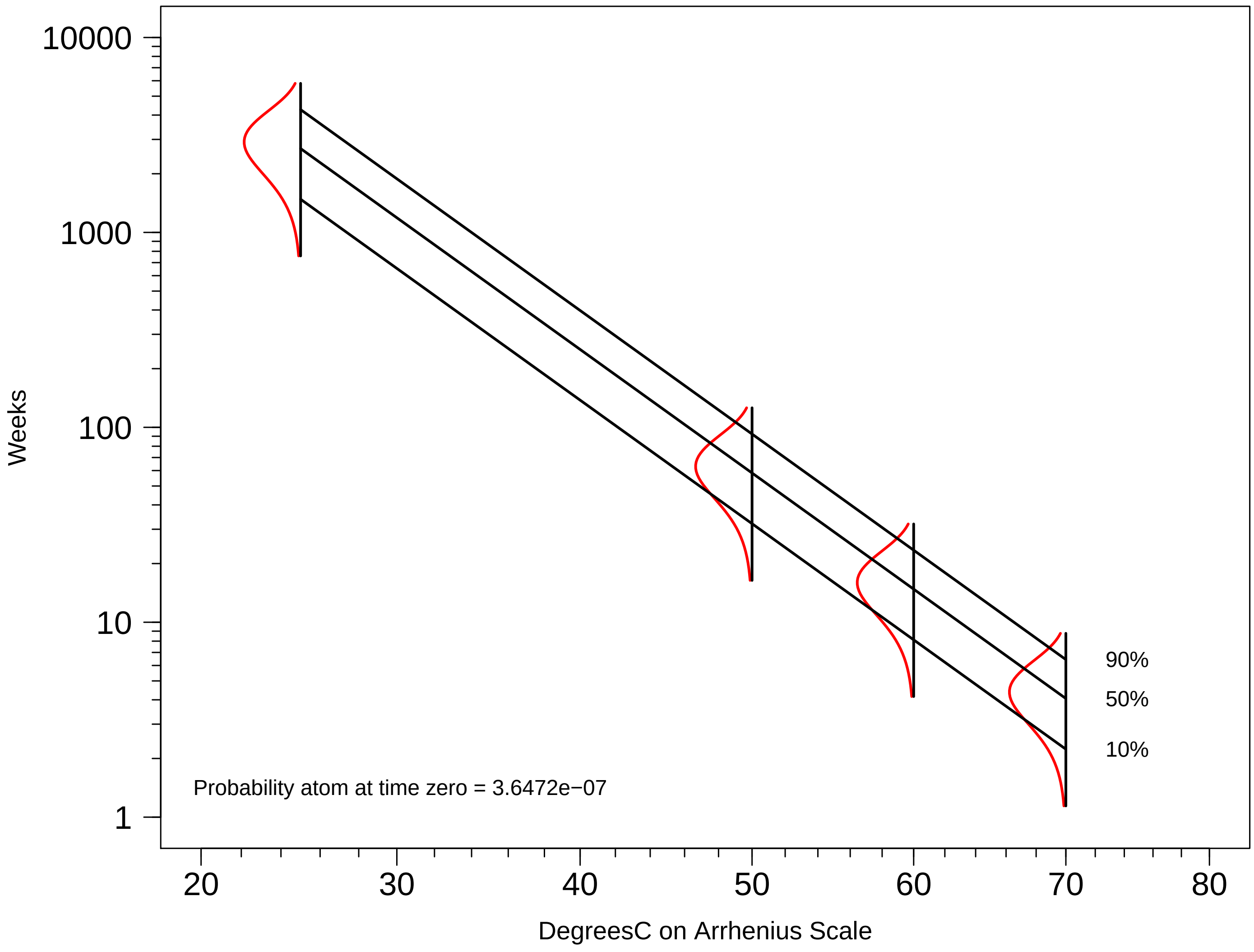
where

$$AF = \exp[-\beta_2(x - x_0)]$$

and

$$\begin{aligned} F(0, x) &= \Pr[Y \le h_d(\mathcal{D}_{\mathsf{f}})] \\ &= \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0}{\sigma}\right]. \end{aligned}$$

Adhesive Bond B Data
Model Plot Estimates of Failure-Time Distribution as a Function of Temperature



Chapter 20

Degradation Modeling and Destructive Degradation Data Analysis

Segment 9

ADDT with an Asymptotic Model Adhesive Formulation K

Accelerated Destructive Degradation Test of Adhesive Formulation K

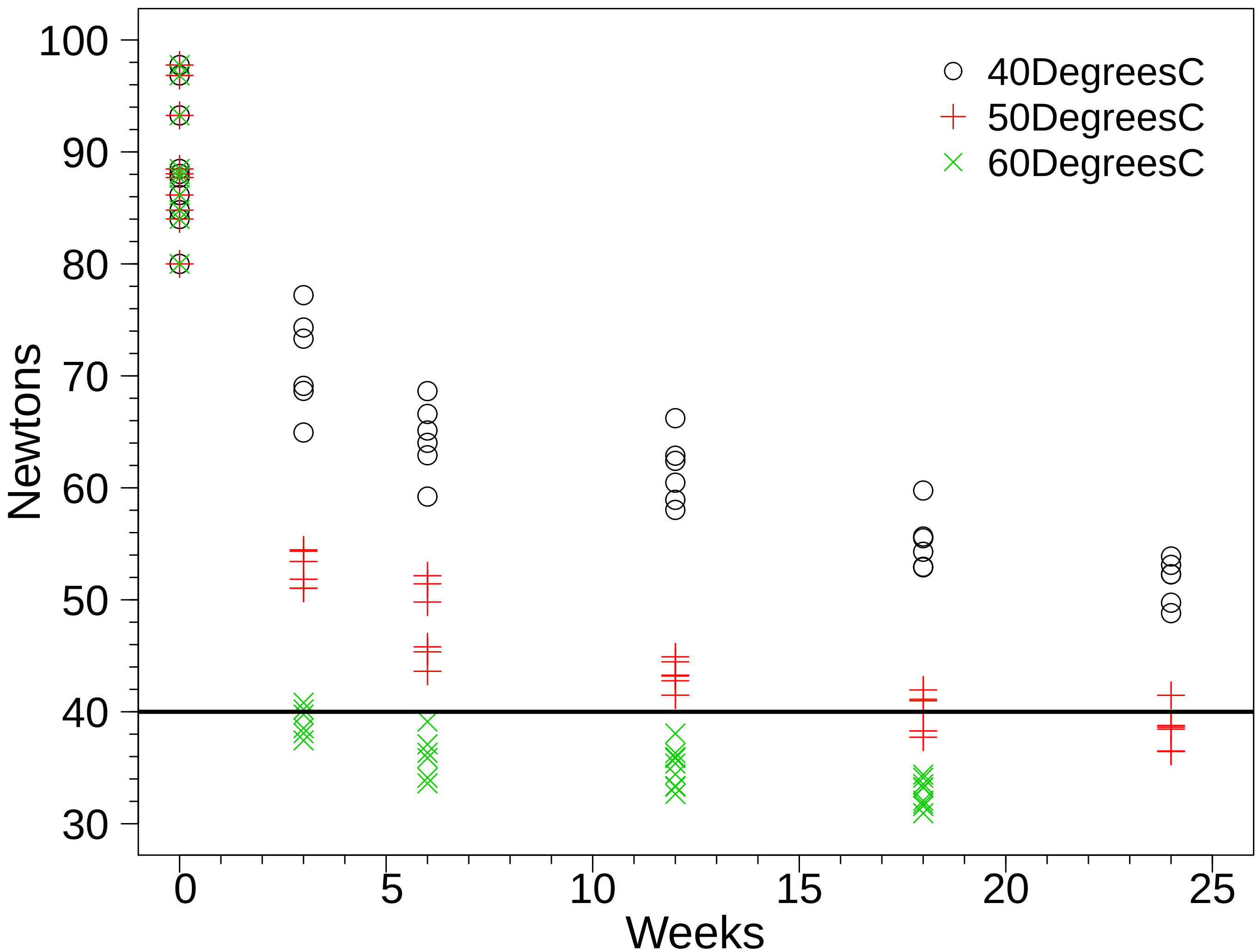
- Formulation K was a newly developed adhesive using a special additive compound that enhances performance.
- The additive degrades over time, through a diffusion process, reducing adhesive strength.
- **Objective:** Assess the strength of the adhesive as a function of time. Estimate the fraction of devices with a strength below 45 Newtons after 2 and 5 years of operation (approximately 104 and 260 weeks, respectively) at 25°C.
- 30 specimens were put into temperature-controlled chambers at 40, 50, and 60°C (total of 90 specimens).
- A specified number of units were removed and tested destructively after 3, 6, 12, 18 and 24 weeks of exposure.
- An additional 10 units with no aging were measured at the start of the experiment.

Adhesive Formulation K Test Plan

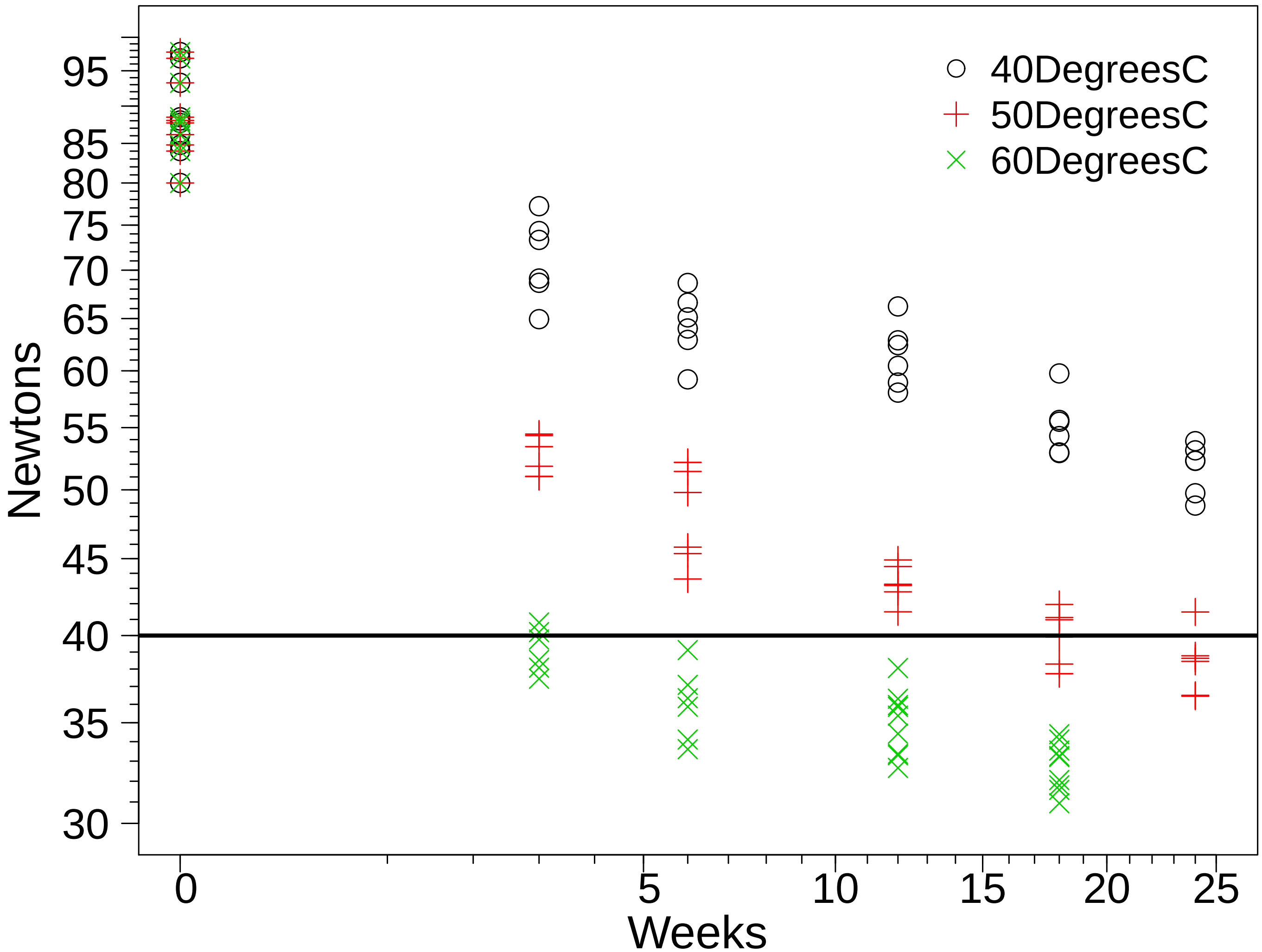
Number of Specimens Tested

Temp °C	Weeks Aged 0	3	6	12	18	24	Totals
—	10						10
40		6	6	6	6	6	30
50		6	6	6	6	6	30
60		6	6	9	9	0	30
Totals	10	18	18	21	21	12	106

Adhesive Formulation K ADDT Data as a Function of Temperature Linear–Linear Axes

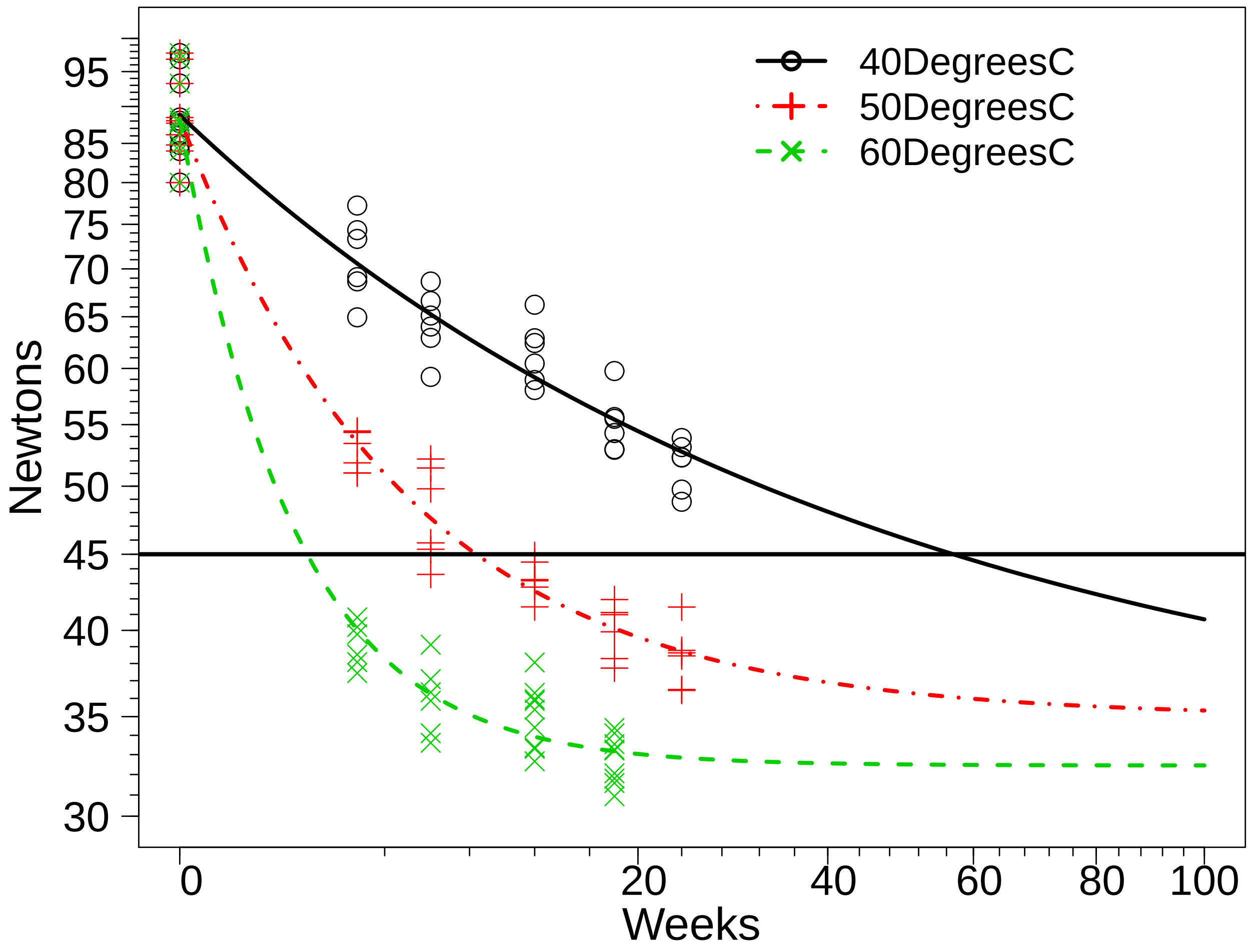


Adhesive Formulation K ADDT Data as a Function of Temperature Square Root–Log Axes



Adhesive Formulation K ADDT Data
Overlay of Individual Normal Distribution Fits
Square Root–Log Axes

$$\widehat{\xi}^{[j]}(t) = \widehat{\beta}_0^{[j]} - \widehat{\beta}_3^{[j]}[1 - \exp(-\widehat{\beta}_1^{[j]}\tau)], \quad j = 50, 60, 70$$

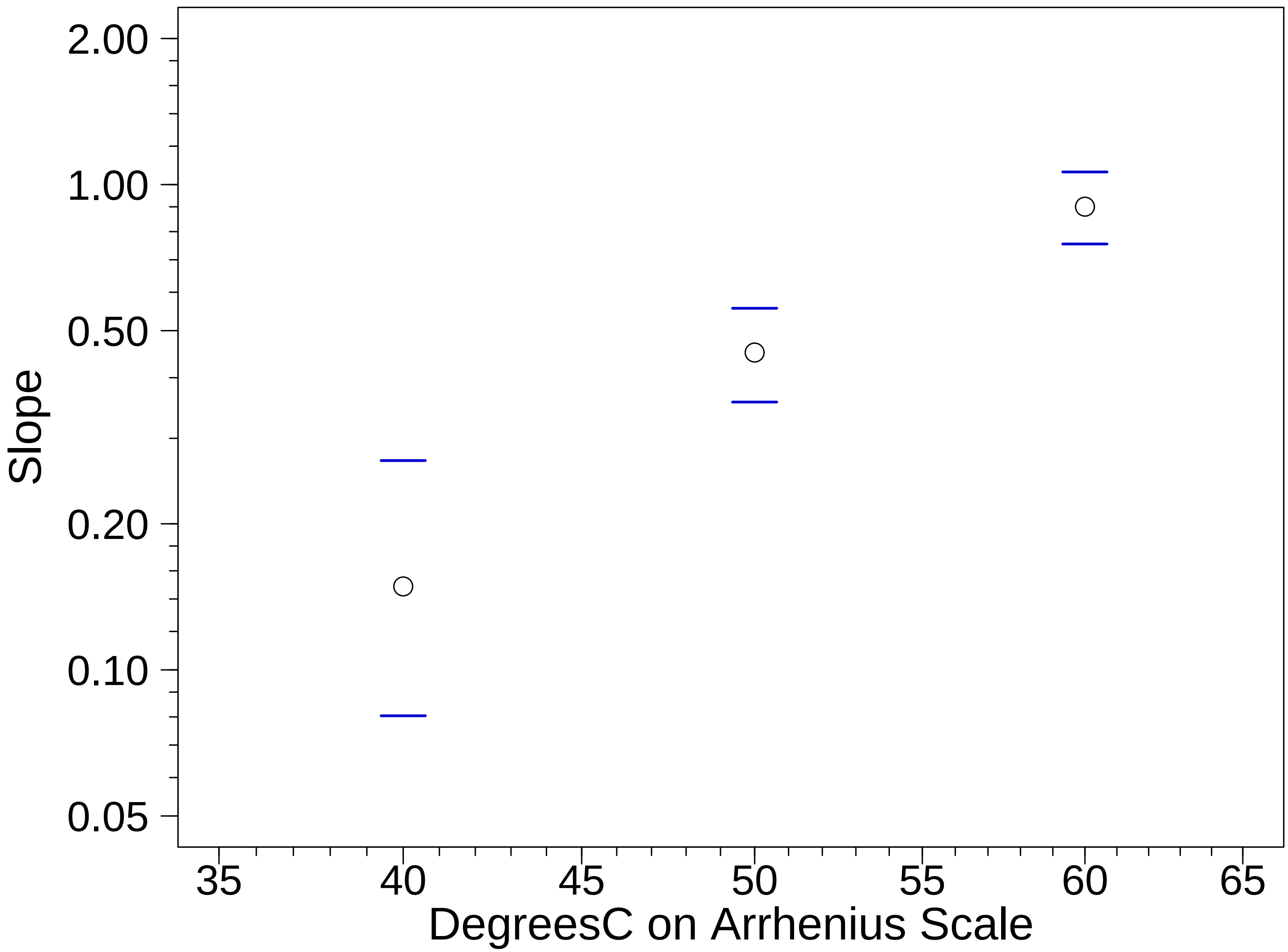


Adhesive Formulation K ADDT Data
Bayesian Parameter Estimates
Asymptotic Path Normal Distribution
Individual Line Fits

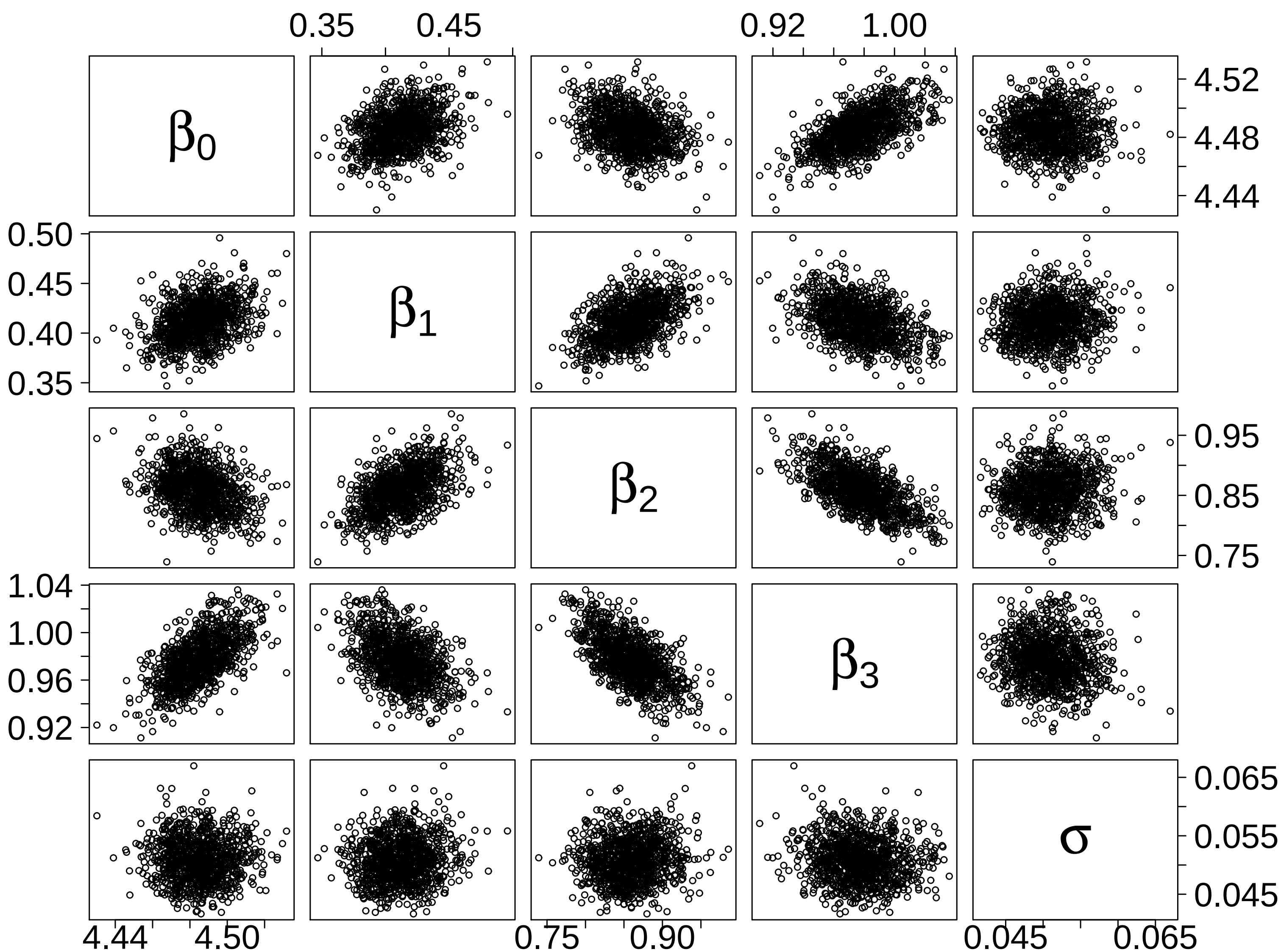
- For each temperature level three individual estimates are obtained: $\widehat{\beta}_0^{[j]}$, $\widehat{\beta}_1^{[j]}$, $\widehat{\beta}_3^{[j]}$, and $\widehat{\sigma}^{[j]}$.
- A summary of the asymptotic path normal distribution estimates for individual temperatures for the Adhesive Formulation K ADDT data is

	Estimates				95% Credible Interval for $\widehat{\beta}_1^{[j]}$	
Temperature	$\widehat{\beta}_0^{[j]}$	$\widehat{\beta}_1^{[j]}$	$\widehat{\beta}_3^{[j]}$	$\widehat{\sigma}^{[j]}$	$\underset{\sim}{\beta}_1^{[j]}$	$\widetilde{\beta}_1^{[j]}$
40°C	4.49	0.15	1.01	0.056	0.081	0.27
50°C	4.48	0.45	0.93	0.052	0.36	0.56
60°C	4.48	0.90	1.00	0.054	0.32	1.06

Adhesive Formulation K ADDT Data Arrhenius Plot
Individual Degradation Rate Estimates $\widehat{\beta}_1^{[j]}$ versus °C
Arrhenius Plot



Adhesive Formulation K ADDT Data
Log/Square Root Transformation
Weakly Informative Prior Distribution
Posterior Pairs Plot



Adhesive Formulation K ADDT Data
Bayesian Parameter Estimates
Normal Distribution Asymptotic Path Arrhenius Model

$$Y = \beta_0 - \beta_3[1 - \exp(-\beta_1 \exp[-\beta_2(x - x_0)]\tau)] + \epsilon$$

		Standard	95%Credible Interval	
Parameter	Estimate	Error	Lower	Upper
β_0	4.49	0.01	4.46	4.51
β_1	0.41	0.02	0.37	0.45
β_2	0.86	0.03	0.79	0.93
β_3	0.98	0.02	0.94	1.02
σ	0.05	0.005	0.04	0.06

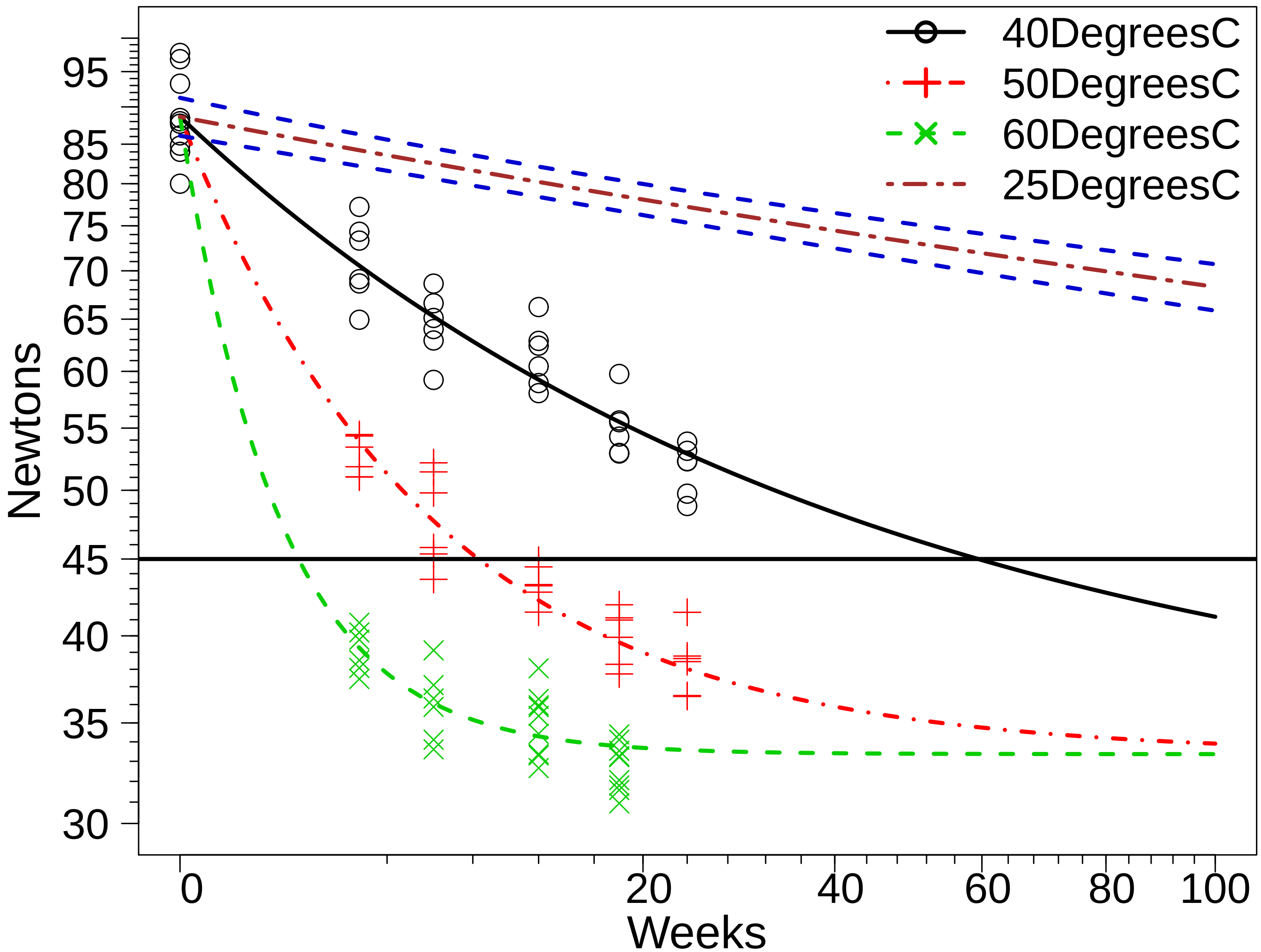
Estimates for the slopes (degradation rates) at each temperature are obtained from $\widehat{\beta}_1^{[j]} = \widehat{\beta}_1 \exp\left[-\widehat{\beta}_2(x - x_0)\right]$ where $x = 11604.52/(°\text{C} + 273.15)$. In this case for the four temperatures of interest, the estimates are

$$\widehat{\beta}_1^{[25]} = 0.031, \quad \widehat{\beta}_1^{[40]} = 0.154$$
$$\widehat{\beta}_1^{[50]} = 0.412, \quad \widehat{\beta}_1^{[60]} = 1.037$$

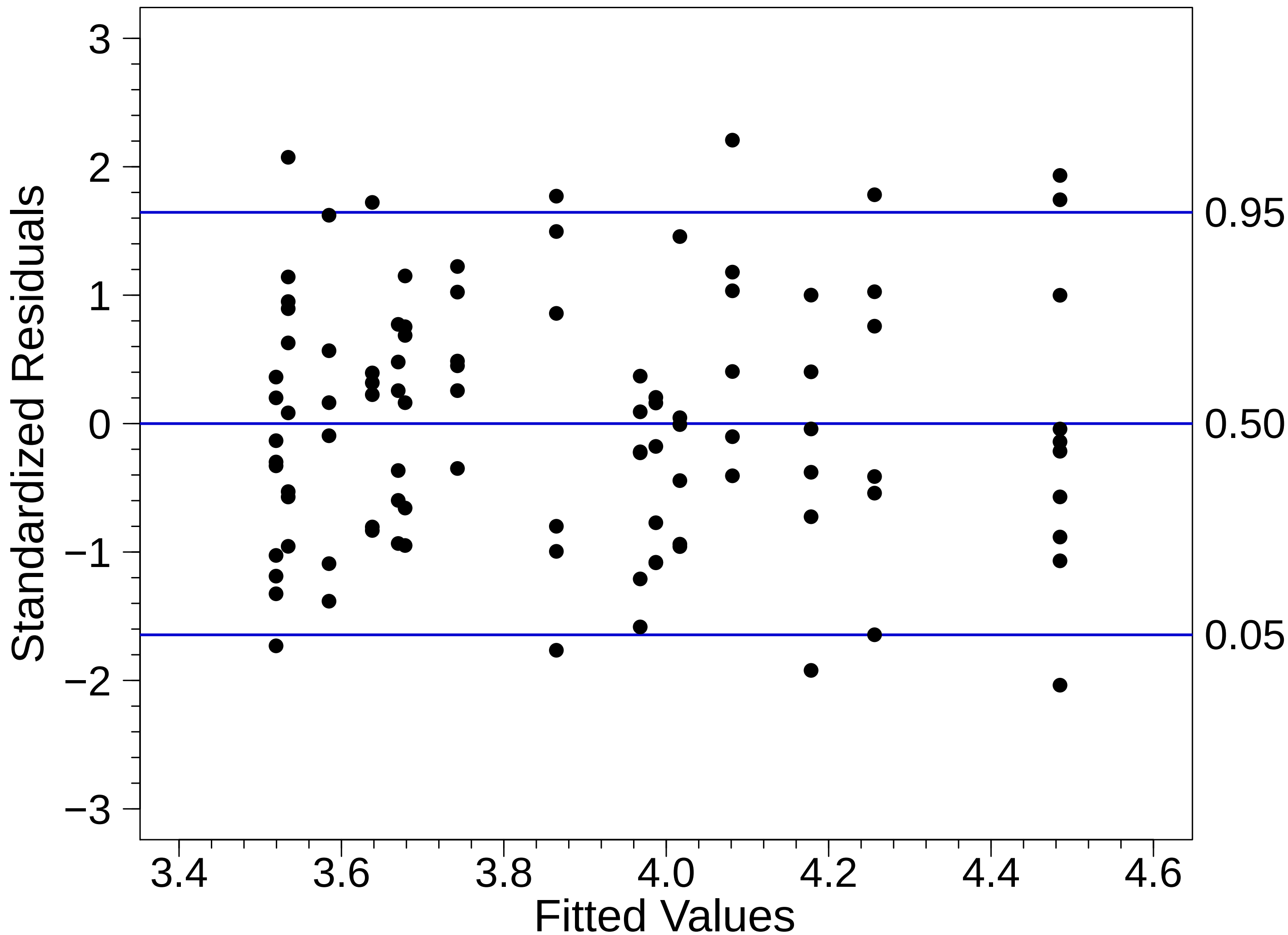
Adhesive Formulation K ADDT Data and Fitted Model

Normal Distribution Asymptotic Path Arrhenius Model

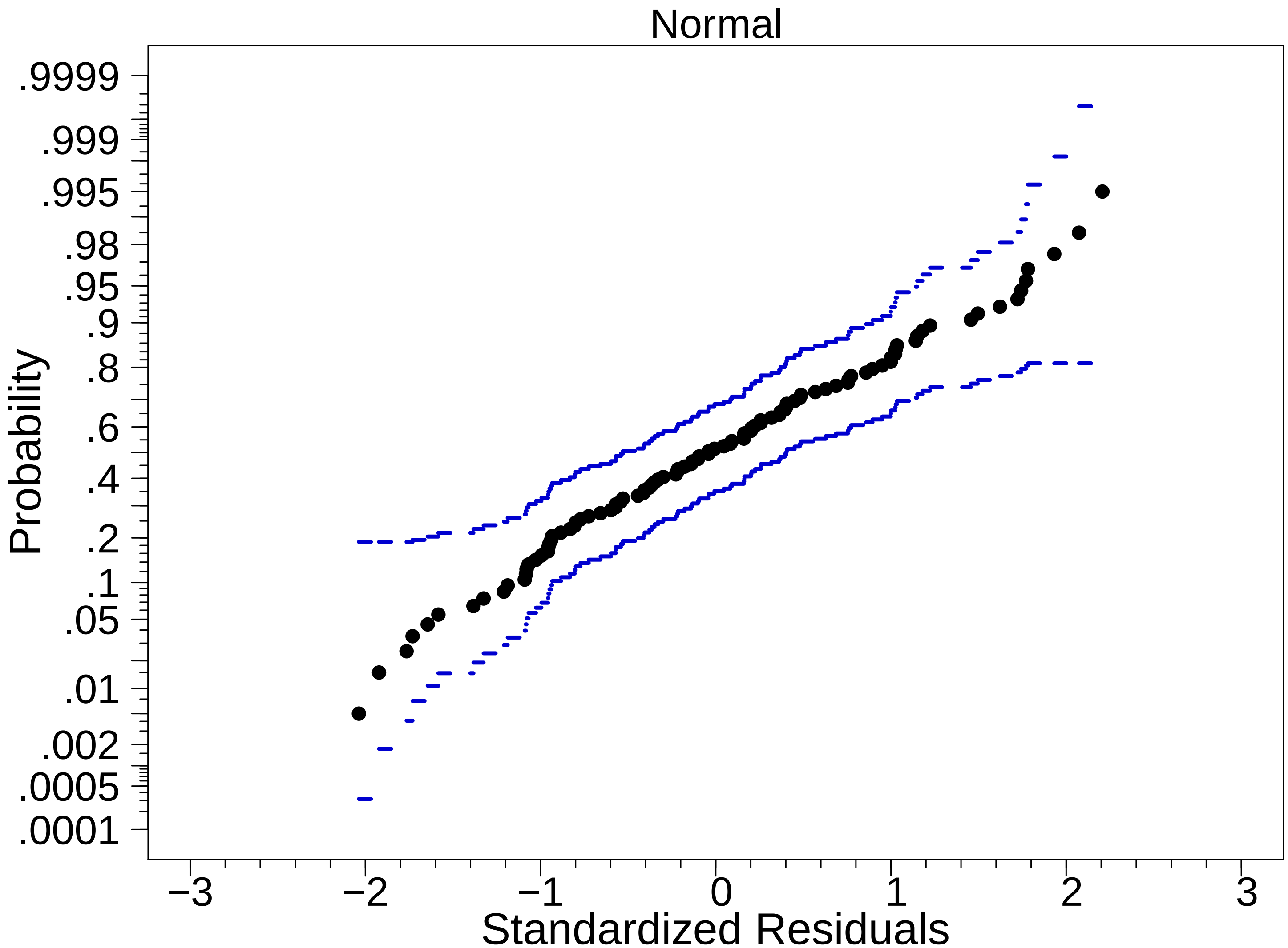
$$\widehat{\xi}(t) = \widehat{\beta}_0 - \widehat{\beta}_3\left[1 - \exp\left(-\widehat{\beta}_1 \exp[-\widehat{\beta}_2(x - x_0)]\tau\right)\right]$$



Adhesive Formulation K ADDT Data
Residuals Versus Fitted Values



Adhesive Formulation K ADDT Data
Normal Distribution Residual Probability Plot



Induced Failure Time Distribution for the Asymptotic Model 8 (Decreasing Degradation)

- For Model 8, $T \leq t$ is equivalent to observed degradation less than $\mathcal{D}_{\mathsf{f}}$ [i.e., $Y \leq h_d(\mathcal{D}_{\mathsf{f}})$]. Then

$$F(t,x) = \Pr[Y \leq h_d(\mathcal{D}_{\mathsf{f}})] = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \xi(t,x)}{\sigma}\right], \text{ for } t \geq 0.$$

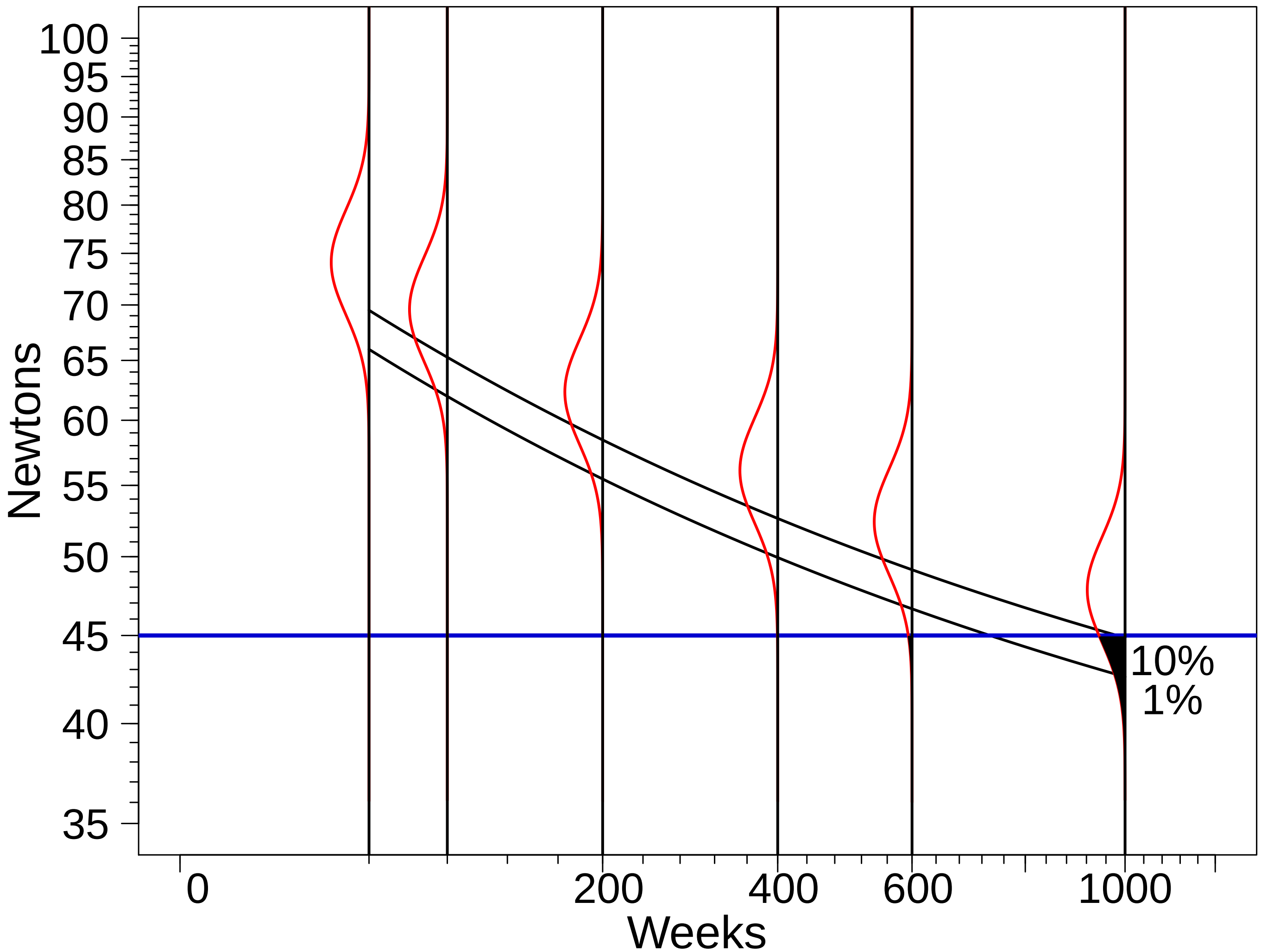
- This failure time distribution is a mixed distribution with probability **atoms** at $t = 0$ and $t = \infty$ with probabilities

$$\Pr(T = 0, x) = F(0, x) = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \xi(0,x)}{\sigma}\right] = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0}{\sigma}\right]$$

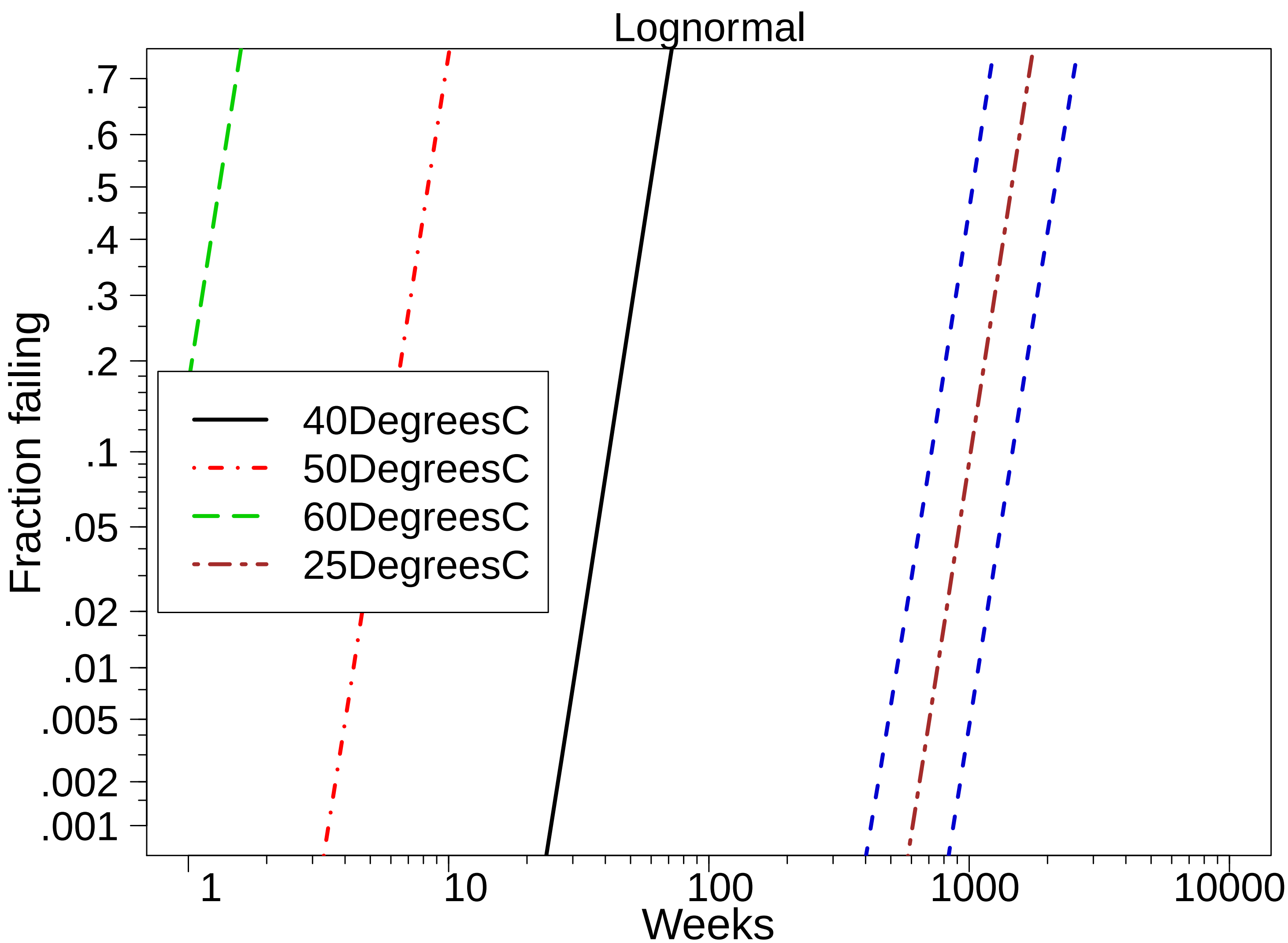
$$\Pr(T = \infty, x) = 1 - F(\infty, x) = 1 - \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - (\beta_0 - \beta_3)}{\sigma}\right].$$

Adhesive Formulation K
Estimate of Fraction Failing as a Function of Time

$$\widehat{y}_p = \widehat{\beta}_0 - \widehat{\beta}_3\left(1 - \exp\left\{-\widehat{\beta}_1^{[25]}\tau\right\}\right) + \widehat{\sigma}\Phi^{-1}_{\textbf{norm}}(p)$$



Adhesive Formulation K
Lognormal Multiple Probability Plot
cdf Estimates at Test Temperatures
and Use Conditions



Quantiles for the Failure Time Distribution at Fixed Values of x and $\mathcal{D}_{\mathsf{f}}$ for Model 8

- For Model 8, the p quantile is $t_p = h_t^{-1}(\tau_p)$, where

$$\tau_p = \begin{cases} 0 & \text{if } p \le F(0, x) \\ \dfrac{1}{\beta_1 AF} \log\left[\dfrac{\beta_3}{h_d(\mathcal{D}_{\mathsf{f}}) - \Phi^{-1}(p)\sigma - (\beta_0 - \beta_3)}\right] & \text{if } F(0, x) < p < F(\infty, x) \\ \infty & \text{if } p > F(\infty, x), \end{cases}$$

where

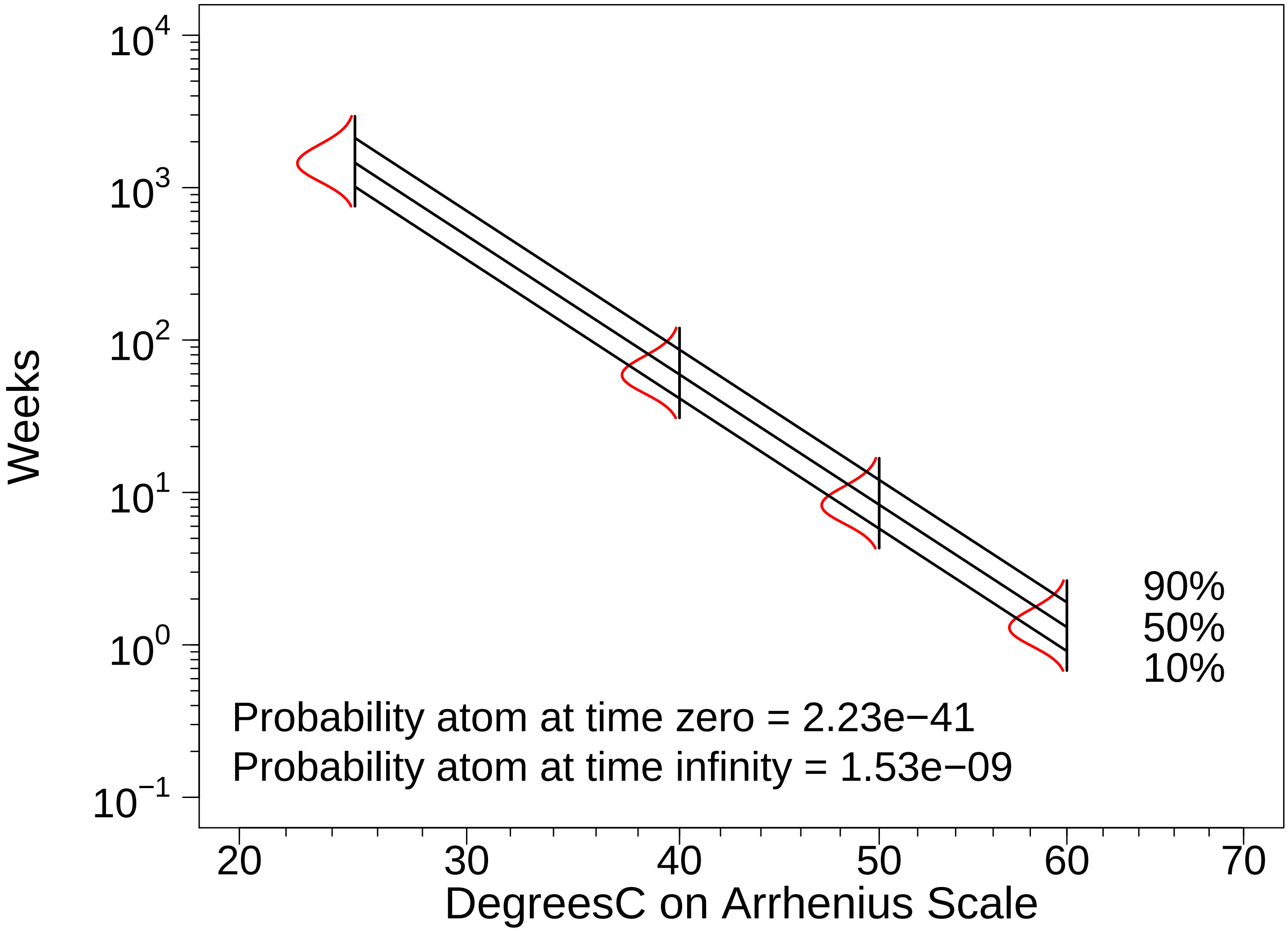
$$AF = \exp[-\beta_2(x - x_0)]$$

and

$$F(0, x) = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - \beta_0}{\sigma}\right]$$

$$F(\infty, x) = \Phi\left[\frac{h_d(\mathcal{D}_{\mathsf{f}}) - (\beta_0 - \beta_3)}{\sigma}\right].$$

Adhesive Formulation K
Model Plot Estimates of Failure Time Distribution as a Function of Temperature



References

Meeker, W. Q., L. A. Escobar, and F. G. Pascual (2021). *Statistical Methods for Reliability Data* (Second Edition). Wiley. [1]