Chapter 3

Nonparametric Estimation of a Failure-Time Distribution

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10h 50min

Chapter 3
Nonparametric Estimation

Topics discussed in this chapter are:

- Use of the binomial distribution to estimate $F(t)$ from interval and singly right-censored failure-time data, without assumptions about the form of $F(t)$. This is called **nonparametric** estimation.
- Methods for computing confidence intervals for $F(t)$ with singly right-censored failure-time data.
- Nonparametric estimation with multiply-censored and interval-censored failure-time data
- The **Kaplan-Meier** nonparametric estimator for multiply-censored failure-time data and exact failure times.
- Nonparametric simultaneous confidence bands for $F(t)$
- Nonparametric estimation of $F(t_i)$ with current-status data or arbitrary censoring

Chapter 3

Segment 1

Nonparametric Estimation with Singly-Censored Failure-Time Data

Data for Plant 1 of the Heat Exchanger Tube Crack Data

Cracked tubes

100 tubes at start	Year 1	Year 2	Year 3	Uncracked tubes
Plant 1	1	2	2	95
Unconditional Failure Probability	π_1	π_2	π_3	π_4

Likelihood: $L(\boldsymbol{\pi}) = \mathcal{C} \times [\pi_1]^1 \times [\pi_2]^2 \times [\pi_3]^2 \times [\pi_4]^{95}$

$$\sum_{i=1}^{4} \pi_i = 1.$$

A Nonparametric Estimator of $F(t_i)$ Based on Binomial Methods for Interval Singly-Censored Data

Consider the nonparametric estimator of $F(t_i)$ for data situations illustrated by the Heat Exchanger Tube Crack from Plant 1:

- The data are:

$$n : \text{sample size}$$
$$d_i : \text{\# of failures (deaths) in interval } i$$

for $t_i, i = 1, 2, 3$.

- Application of simple binomial methods gives

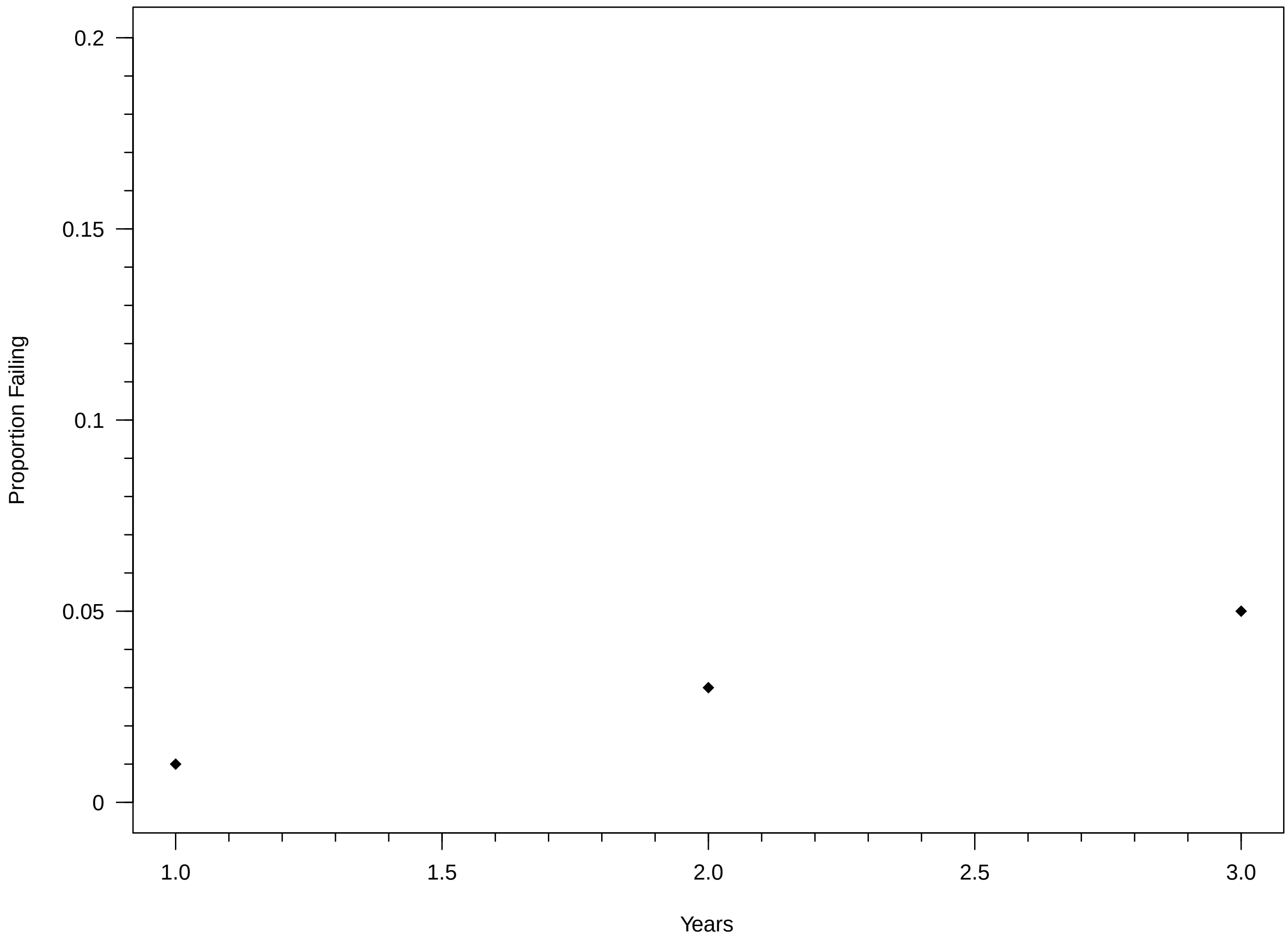
$$\widehat{F}(t_i) = \frac{\text{\# of failures up to time } t_i}{n} = \frac{\sum_{j=1}^{i} d_j}{n}$$

$$\text{se}_{\widehat{F}} = \sqrt{\frac{\widehat{F}(t_i)\left[1 - \widehat{F}(t_i)\right]}{n}}.$$

- For Plant 1 ($n = 100, d_1 = 1, d_2 = 2, d_3 = 2$),

$$\widehat{F}(1) = 1/100, \quad \widehat{F}(2) = 3/100, \quad \widehat{F}(3) = 5/100.$$

Nonparametric Estimate for Plant 1 from the Heat Exchanger Tube Crack Data



Comments on the Nonparametric Estimate of $F(t_i)$ from Interval-Censored and Singly-Right-Censored Failure-Time Data

- $\widehat{F}(t)$ is defined only at the upper ends of the intervals $(t_{i-1}, t_i]$.

- $\widehat{F}(t_i)$ is the ML estimator of $F(t_i)$.

- The increase in $\widehat{F}$ at each value of t_i is

$$\widehat{F}(t_i) - \widehat{F}(t_{i-1}) = \frac{d_i}{n}.$$

Chapter 3

Segment 2

Nonparametric Confidence Intervals for a Failure-Time Distribution based on Singly-Censored Failure-Time Data

Confidence Intervals

A point estimate can be misleading. It is important to quantify statistical uncertainty in point estimates.

- Confidence intervals are used to quantify statistical uncertainty in point estimates due to sampling error arising from limited data.
- Confidence intervals **do not** quantify deviations arising from incorrectly specified model assumptions.

Features of Confidence Intervals

- The level of **confidence** expresses one's confidence (not probability) that a specific interval contains the quantity of interest.
- The actual **coverage** probability is the probability that the **procedure** will result in an interval containing the quantity of interest.
- A confidence interval is **approximate** if the specified level of confidence is not equal to the actual coverage probability.
- With censored data most confidence interval procedures are approximate.
- Some confidence intervals procedures are conservative (coverage probability is larger than the confidence level).

Pointwise Wald (Normal Approximate) Confidence Interval for $F(t_i)$

- For a specified value of t_i, an approximate $100(1-\alpha)\%$ confidence interval for $F(t_i)$ is

$$[\underset{\sim}{F}(t_i), \quad \widetilde{F}(t_i)] = \widehat{F}(t_i) \mp z_{(1-\alpha/2)}\text{se}_{\widehat{F}},$$

where $z_{(1-\alpha/2)}$ is the $1-\alpha/2$ quantile of the standard normal distribution and $\text{se}_{\widehat{F}} = \sqrt{\widehat{F}(t_i)\left[1-\widehat{F}(t_i)\right]/n}$ is an estimate of the standard error of $\widehat{F}(t_i)$.

- This confidence interval is based on

$$Z_{\widehat{F}} = \frac{\widehat{F}(t_i) - F(t_i)}{\text{se}_{\widehat{F}}} \,\dot{\sim}\, \text{NORM}(0,1).$$

- This method is computationally simple but is **not recommended** because of its poor coverage probability properties (see Meeker, Hahn, and Escobar 2017, Section 6.2.6). Instead, use the Jeffreys or the Conservative method.

Pointwise Jeffreys Confidence Interval for $F(t_i)$

- A $100(1-\alpha)\%$ approximate confidence interval for $F(t_i)$ is

$$\underset{\sim}{F}(t_i) = \texttt{qbeta}(\alpha/2; n\widehat{F} + 0.5, n - n\widehat{F} + 0.5)$$
$$\widetilde{F}(t_i) = \texttt{qbeta}(1-\alpha/2; n\widehat{F} + 0.5, n - n\widehat{F} + 0.5)$$

  where $\widehat{F} = \widehat{F}(t_i)$ and $\texttt{qbeta}(p; a, b)$ is the p quantile of the beta distribution with shape parameters a and b.

- This method is based on a Bayesian interval using a Jeffreys prior distribution and has coverage probability properties that are much better than the Wald method (see Meeker et al. 2017, Section 6.2.6).

Pointwise Conservative Confidence Interval for $F(t_i)$

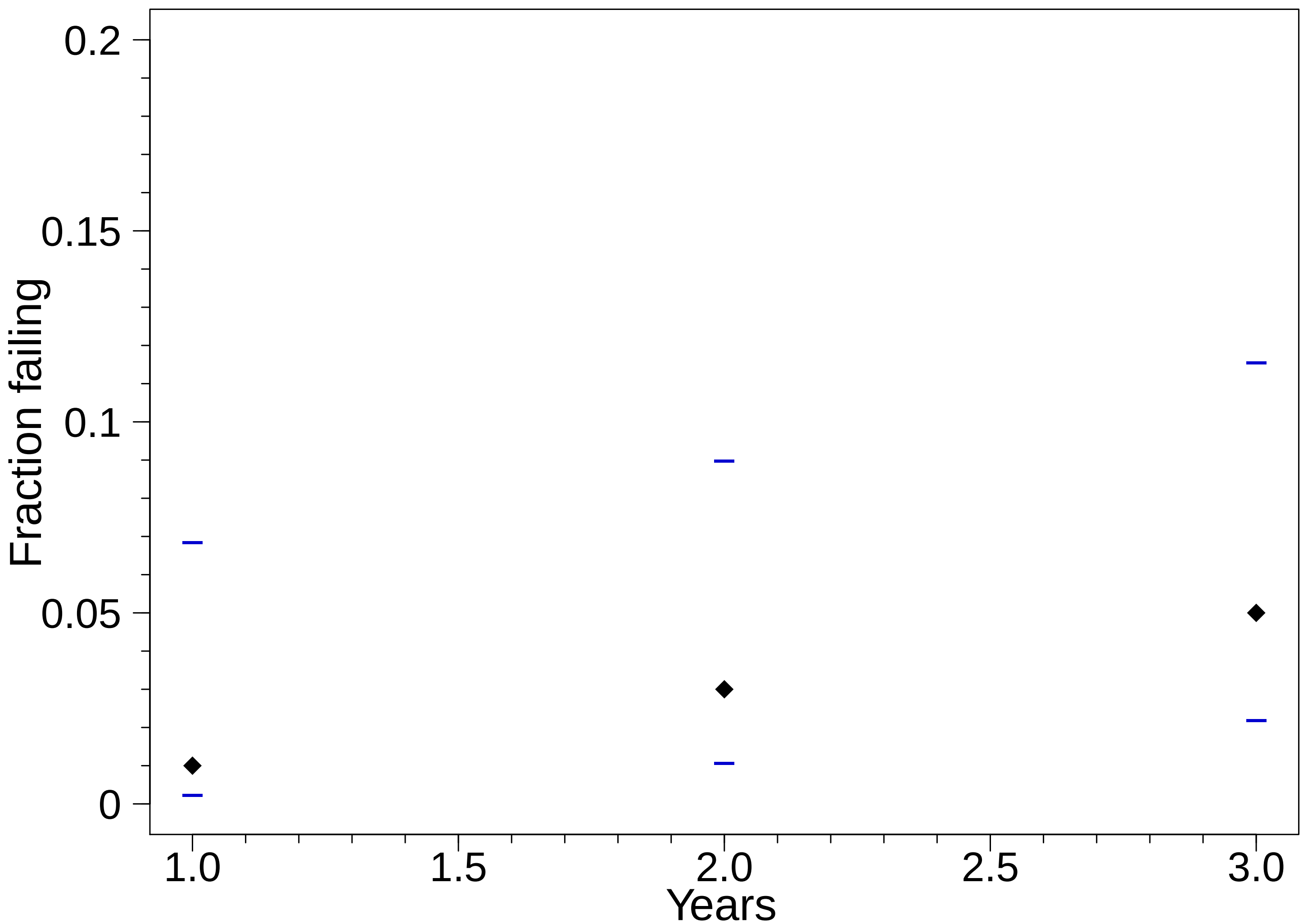
- A $100(1-\alpha)\%$ conservative confidence interval for $F(t_i)$ based on binomial sampling (see Meeker et al. 2017, Chapter 6) is

$$\underset{\sim}{F}(t_i) = \texttt{qbeta}(\alpha/2; n\widehat{F}, n - n\widehat{F} + 1)$$
$$\widetilde{F}(t_i) = \texttt{qbeta}(1-\alpha/2; n\widehat{F} + 1, n - n\widehat{F})$$

  where $\widehat{F} = \widehat{F}(t_i)$ and $\texttt{qbeta}(p; a, b)$ is the p quantile of the beta distribution with shape parameters a and b.

- This confidence interval is conservative in that the actual coverage probability is greater than or equal to $1-\alpha$.

Plant 1 Heat Exchanger Tube Crack Nonparametric Estimate with Pointwise Wald 95% Confidence Intervals Based on $Z_{\mathrm{logit}(\widehat{F})}$



Summary of the Nonparametric Estimate of $F(t_i)$ for Plant 1 from the Heat Exchanger Tube Crack Data

Year	t_i	d_i	$\widehat{F}(t_i)$	$\text{se}_{\widehat{F}}$	Pointwise Confidence Interval $\underset{\sim}{F}(t_i)$	$\widetilde{F}(t_i)$
$(0-1]$	1	1	0.01	0.00995		
95% Confidence Intervals for $F(1)$						
Wald					[−0.0095,	0.0295]
Jeffreys					[0.0011,	0.0458]
Conservative					[0.0003,	0.0545]
$(1-2]$	2	2	0.03	0.01706		
95% Confidence Intervals for $F(2)$						
Wald					[−0.0034,	0.0634]
Jeffreys					[0.0085,	0.0779]
Conservative					[0.0062,	0.0852]
$(2-3]$	3	2	0.05	0.02179		
95% Confidence Intervals for $F(3)$						
Wald					[0.0073,	0.0927]
Jeffreys					[0.0193,	0.1061]
Conservative					[0.0164,	0.1128]

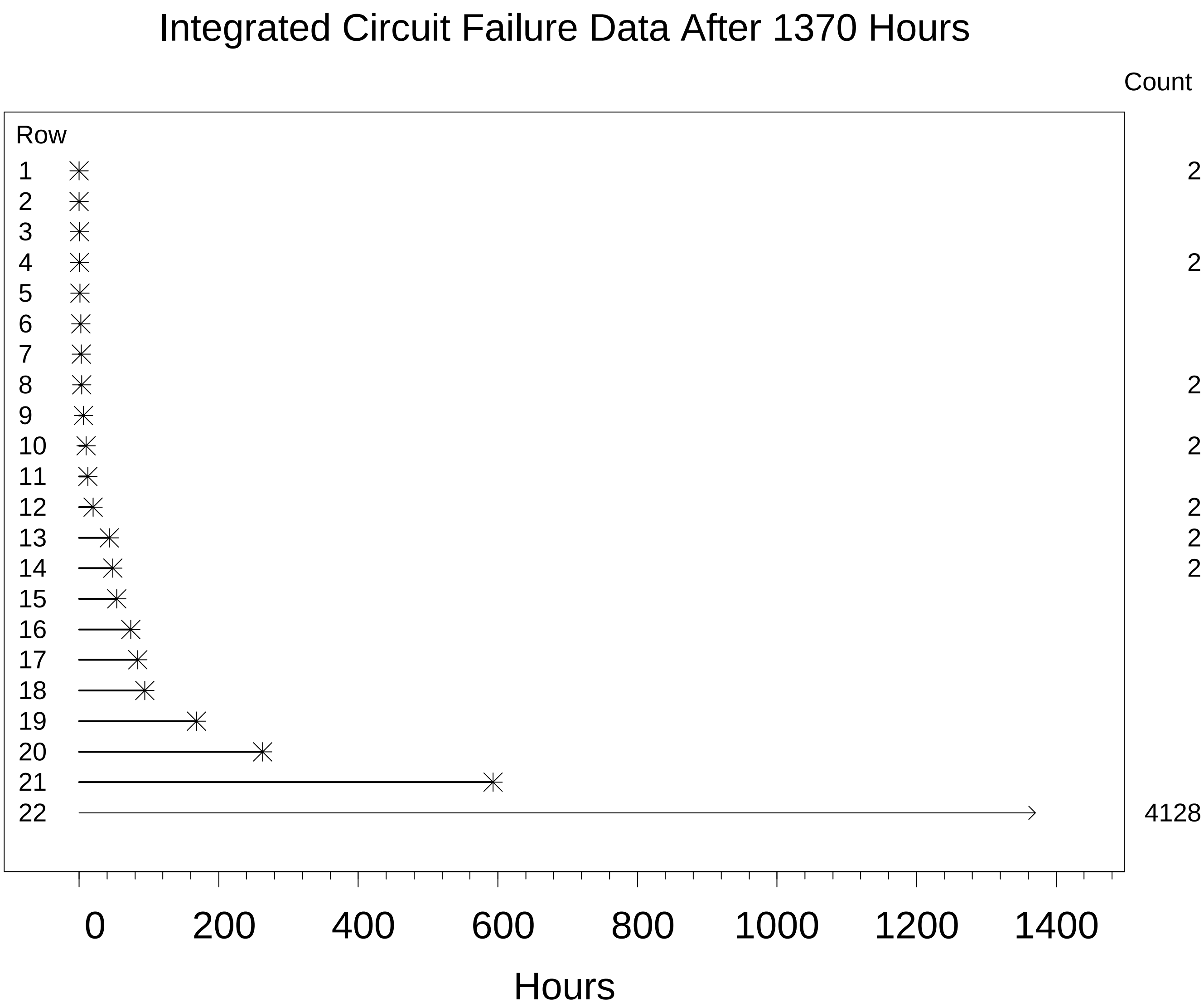
Integrated Circuit (IC) Failure Times in Hours (Data from Meeker 1987)

0.10	0.10	0.15	0.60	0.80	0.80
1.20	2.50	3.00	4.00	4.00	6.00
10.00	10.00	12.50	20.00	20.00	43.00
43.00	48.00	48.00	54.00	74.00	84.00
94.00	168.00	263.00	593.00		

When the test ended at 1370 hours, there were 28 observed failures and 4128 unfailed units.

Note: Ties in the data. Reason?

Event Plot
Integrated Circuit Life Test Data



Nonparametric Estimator of $F(t)$ Based on Binomial Methods for Exact Failures and Singly Right-Censored Data

When the number of inspections increases, the width of the time intervals $(t_{i-1}, t_i]$ approaches zero and the failure times are exact.

- For the **integrated circuit life test data**, we have: $n = 4156$ with 28 exact failures in 1370 hours.

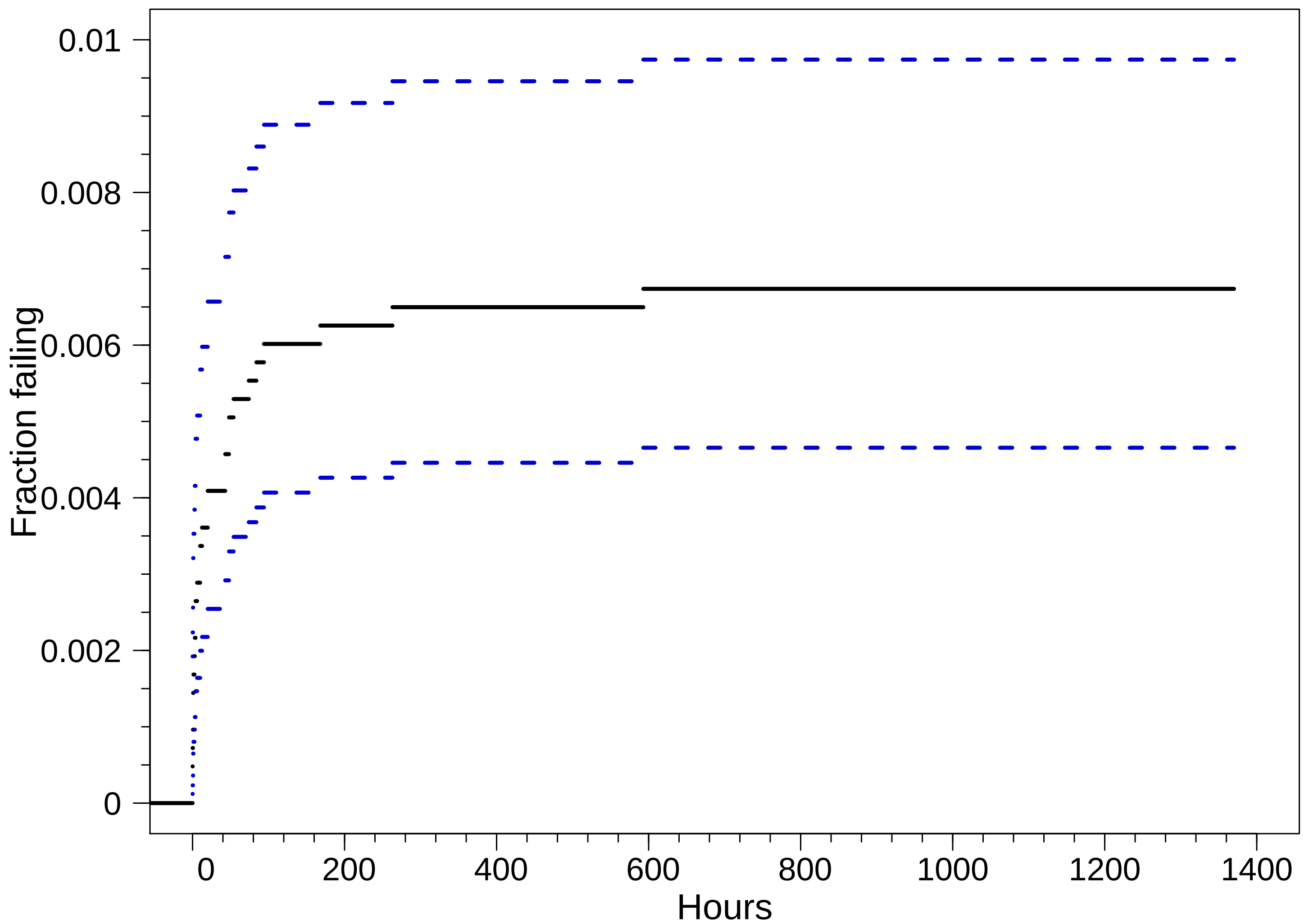
  For any particular t_e, $0 < t_e \leq 1370$, simple binomial methods give

$$\widehat{F}(t_e) = \frac{\text{\# of failures up to time } t_e}{n}$$

$$\text{se}_{\widehat{F}} = \sqrt{\frac{\widehat{F}(t_e)\left[1 - \widehat{F}(t_e)\right]}{n}}.$$

- Confidence interval methods for $F(t_e)$ are the same as the methods described for interval data.

Nonparametric Estimate with Pointwise Wald 95% Confidence Intervals Based on $Z_{\text{logit}(\widehat{F})}$ for the IC Data



Comments on the Nonparametric Estimate of $F(t)$

- $\widehat{F}(t)$ is defined for all t in the interval $(0, t_c]$ where t_c is the single censoring time.
- $\widehat{F}(t)$ is the ML estimator of $F(t)$.
- The estimate $\widehat{F}(t)$ is a step-up function with a step of size $1/n$ at each exact failure time (unless there are ties).

  Sometimes the step size is an integer multiple of $1/n$ because there are ties in the failure times.
- When there is no censoring, $\widehat{F}(t)$ is the well known empirical cdf.

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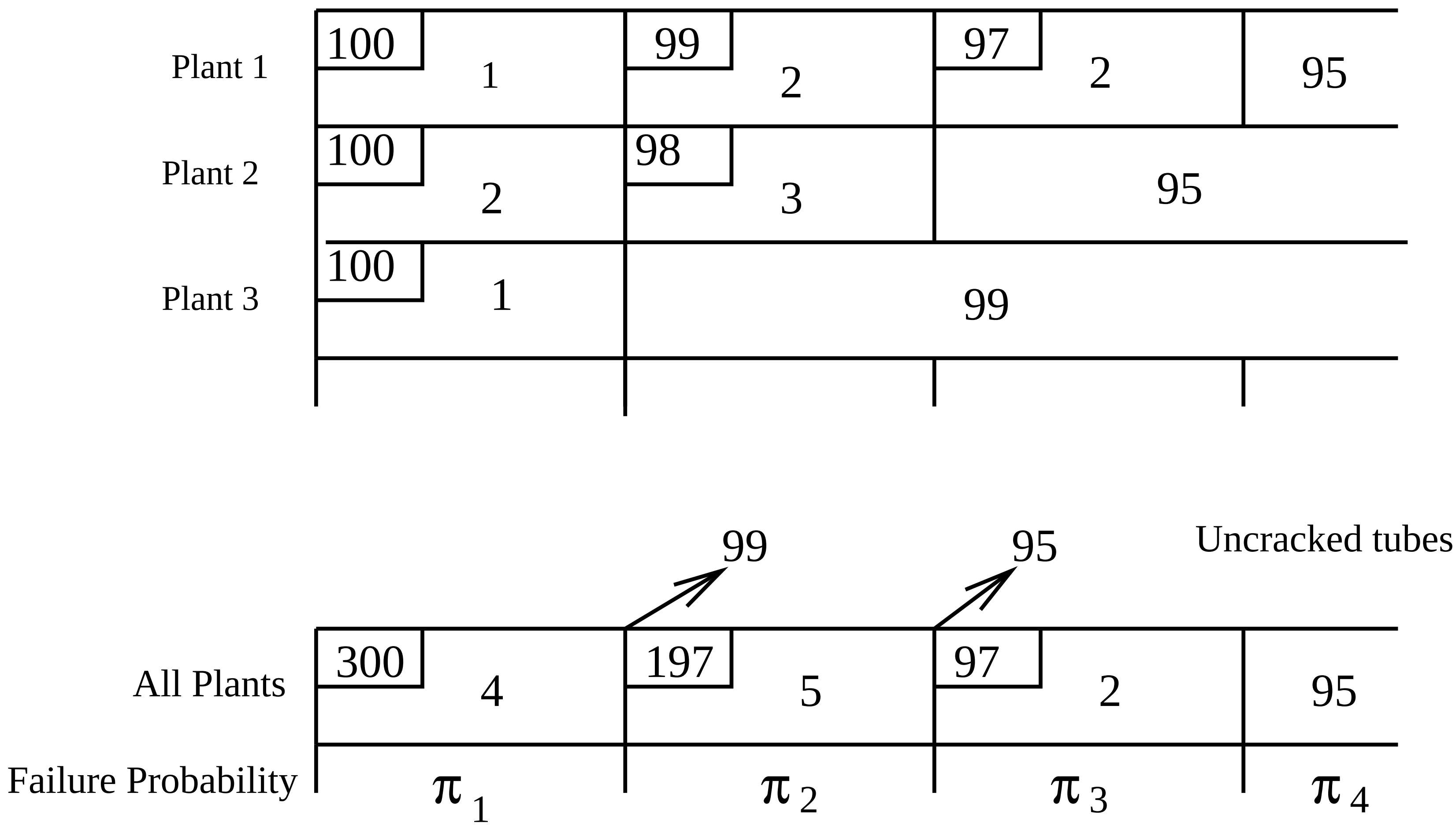
Segment 3

Nonparametric Estimation with Multiply-Censored and Interval-Censored Failure-Time Data

Pooling of the Heat Exchanger Tube Crack Data



Likelihood: $L(\underline{\pi}) = C\ [\pi_1]^4 [\pi_2]^5 [\pi_3]^2 [\pi_4]^{95} [\pi_3+\pi_4]^{95} [\pi_2+\pi_3+\pi_4]^{99}$

A Nonparametric Estimator of $F(t_i)$ Based on Interval Data and Multiple Right Censoring

The pooled data heat exchanger tube crack data are multiply censored and the simple binomial method to estimate $F(t_i)$ cannot be used.

Consider the more general nonparametric estimator of $F(t_i)$ based on the probability model introduced in Chapter 2:

$$\widehat{F}(t_i) = 1 - \widehat{S}(t_i)$$

$$\text{where} \quad \widehat{S}(t_i) = \prod_{j=1}^{i} \left(1 - \widehat{p}_j\right) \quad \text{with} \quad \widehat{p}_j = \frac{d_j}{n_j}$$

n : size of the risk set size at time 0

d_i : # of failures (deaths) in interval i

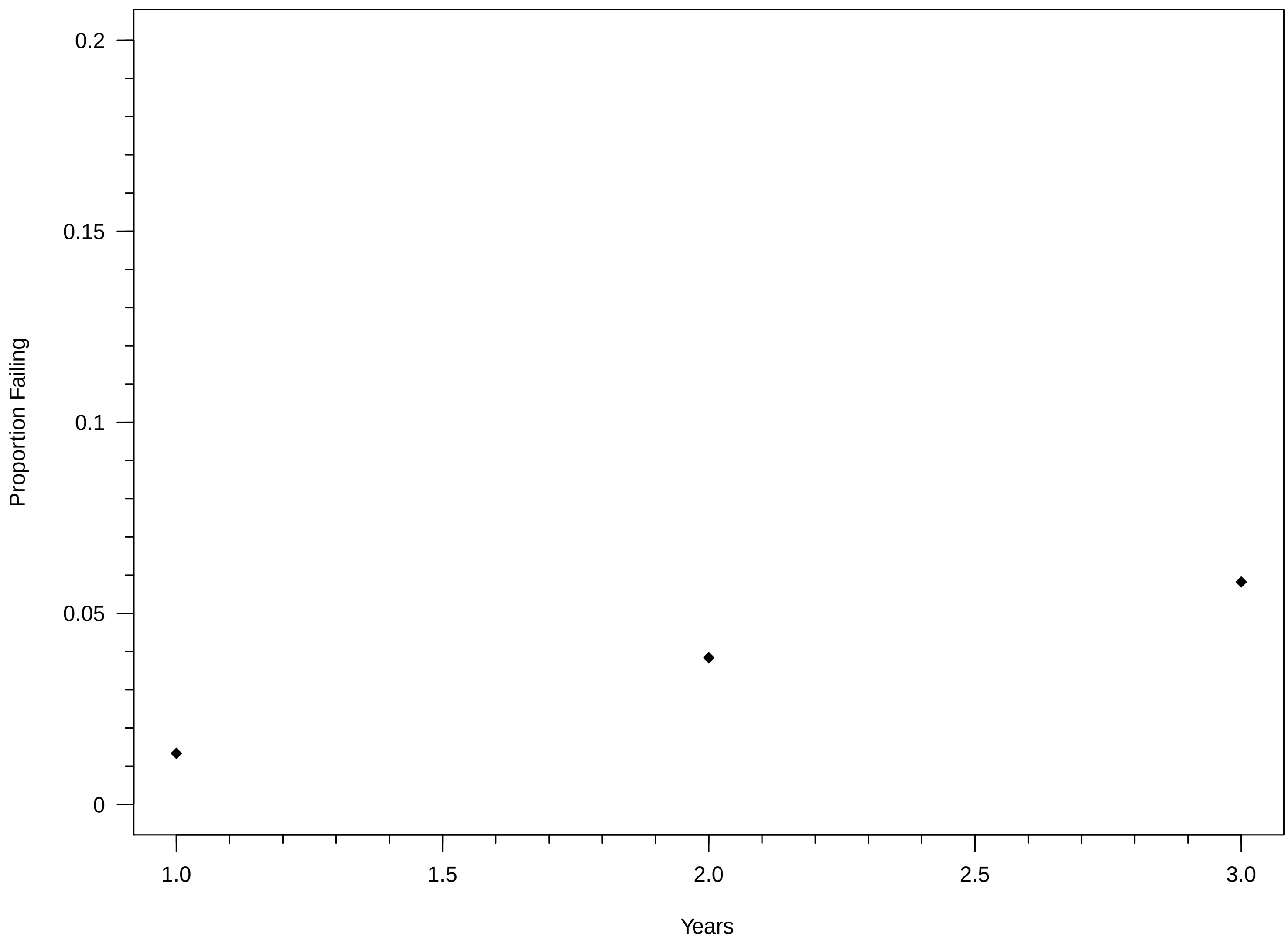
$n_i = n - \sum_{j=0}^{i-1} d_j - \sum_{j=0}^{i-1} r_j,$ the size of the risk set just after t_{i-1}

r_i : # of right censored observations at t_i

Summary of the Nonparametric Estimate of $F(t_i)$ for the Pooled Heat Exchanger Tube Crack Data

Year	t_i	n_i	d_i	r_i	$\widehat{p}_i$	$1 - \widehat{p}_i$	$\widehat{S}(t_i)$	$\widehat{F}(t_i)$
$(0 - 1]$	1	300	4	99	4/300	296/300	0.9867	0.0133
$(1 - 2]$	2	197	5	95	5/197	192/197	0.9616	0.0384
$(2 - 3]$	3	97	2	95	2/97	95/97	0.9418	0.0582

Nonparametric Estimate for the Heat Exchanger Tube Crack Data



Approximate Variance of $\widehat{F}(t_i)$

- Recall, $\widehat{F}(t_i) = 1 - \widehat{S}(t_i)$ and $\widehat{S}(t_i) = \prod_{j=1}^{i}\Big(1 - \widehat{p}_j\Big)$.
- Then $\text{Var}\Big[\widehat{F}(t_i)\Big] = \text{Var}\Big[\widehat{S}(t_i)\Big]$.
- A Taylor series first-order approximation of $\widehat{S}(t_i)$ is

$$\widehat{S}(t_i) \approx S(t_i) + \sum_{j=1}^{i} \left.\frac{\partial S}{\partial q_j}\right|_{q_j} \Big(\widehat{q}_j - q_j\Big)$$

where $q_j = 1 - p_j$.

- Then it follows that

$$\text{Var}\Big[\widehat{S}(t_i)\Big] \approx S^2(t_i) \sum_{j=1}^{i} \frac{p_j}{n_j(1 - p_j)}.$$

Estimating the Standard Error of $\widehat{F}(t_i)$

- Using the variance formula, one gets

$$\widehat{\mathsf{Var}}\Big[\widehat{F}(t_i)\Big] = \widehat{\mathsf{Var}}\Big[\widehat{S}(t_i)\Big] = \widehat{S}^2(t_i) \sum_{j=1}^{i} \frac{\widehat{p}_j}{n_j(1-\widehat{p}_j)}$$

  which is known as **Greenwood's** formula.

- An estimate of the standard error of $\widehat{F}(t_i)$ is

$$\mathsf{se}_{\widehat{F}_i} = \sqrt{\widehat{\mathsf{Var}}\Big[\widehat{F}(t_i)\Big]} = \widehat{S}(t_i) \sqrt{\sum_{j=1}^{i} \frac{\widehat{p}_j}{n_j(1-\widehat{p}_j)}}.$$

Pointwise Wald Confidence Interval for $F(t_i)$ Based on the Logit Transformation

- Better confidence intervals can be obtained by using the logit transformation ($\text{logit}(p) = \log[p/(1-p)]$) and basing the confidence intervals on

$$Z_{\text{logit}(\widehat{F})} = \frac{\text{logit}[\widehat{F}(t_i)] - \text{logit}[F(t_i)]}{\text{se}_{\text{logit}(\widehat{F})}} \dot{\sim} \text{NORM}(0,1).$$

- A pointwise Wald $100(1-\alpha)\%$ confidence interval for $\text{logit}[F(t_i)]$ is

$$\left[\underset{\sim}{\text{logit}(F)},\ \ \widetilde{\text{logit}(F)}\right] = \text{logit}(\widehat{F}) \mp z_{(1-\alpha/2)}\text{se}_{\text{logit}(\widehat{F})}$$
$$= \text{logit}(\widehat{F}) \mp z_{(1-\alpha/2)}\text{se}_{\widehat{F}}/[\widehat{F}(1-\widehat{F})]$$

because $\text{se}_{\text{logit}(\widehat{F})} = \text{se}_{\widehat{F}}/[\widehat{F}(1-\widehat{F})]$.

Pointwise Wald Confidence Interval for $F(t_i)$-Based on the Logit Transformation

- The confidence interval for $F(t_i)$ is obtained from the interval for logit(F) and using the inverse logit transformation

$$\text{logit}^{-1}(v) = \frac{1}{1+\exp(-v)}.$$

- Then

$$\begin{aligned}
[\underset{\sim}{F}(t_i), \quad \widetilde{F}(t_i)] &= \text{logit}^{-1}\left[\text{logit}(\widehat{F}) \mp z_{(1-\alpha/2)}\text{se}_{\text{logit}(\widehat{F})}\right] \\
&= \frac{1}{1+\exp\left[-\text{logit}(\widehat{F}) \pm z_{(1-\alpha/2)}\text{se}_{\text{logit}(\widehat{F})}\right]} \\
&= \left[\frac{\widehat{F}}{\widehat{F}+(1-\widehat{F})\times w}, \quad \frac{\widehat{F}}{\widehat{F}+(1-\widehat{F})/w}\right]
\end{aligned}$$

where $w = \exp\{z_{(1-\alpha/2)}\text{se}_{\widehat{F}}/[\widehat{F}(1-\widehat{F})]\}$.

- The endpoints $\underset{\sim}{F}(t_i)$ and $\widetilde{F}(t_i)$ will always lie between 0 and 1.

Pointwise Wald Confidence Intervals for the Heat Exchanger Tube Crack Data

- Computation of standard errors:

$$\widehat{\mathrm{Var}}\Big[\widehat{F}(t_i)\Big] = \widehat{S}^2(t_i) \sum_{j=1}^{i} \frac{\widehat{p}_j}{n_j(1-\widehat{p}_j)}$$

$$\widehat{\mathrm{Var}}\Big[\widehat{F}(t_1)\Big] = (0.9867)^2 \left[\frac{0.0133}{300(0.9867)}\right] = 0.0000438$$

$$\mathrm{se}_{\widehat{F}(t_1)} = \sqrt{0.0000438} = 0.00662$$

$$\widehat{\mathrm{Var}}\Big[\widehat{F}(t_2)\Big] = (0.9616)^2 \left[\frac{0.0133}{300(0.9867)} + \frac{0.0254}{197(0.9746)}\right] = 0.0001639$$

$$\mathrm{se}_{\widehat{F}(t_2)} = \sqrt{0.0001639} = 0.0128$$

Pointwise Wald Confidence Intervals for the Heat Exchanger Tube Crack Data

Computation of approximate 95% confidence intervals:

- For $F(1)$ with $\widehat{F}(t_1) = 0.0133$, $\text{se}_{\widehat{F}(t_1)} = \sqrt{0.0000438} = 0.00662$

  Based on: $Z_{\widehat{F}} = [\widehat{F}(t_1) - F(t_1)]/\text{se}_{\widehat{F}} \dot{\sim} \text{NORM}(0,1)$.

  $$[\underset{\sim}{F}(t_1),\ \widetilde{F}(t_1)] = 0.0133 \mp 1.96(0.00662) = [0.0003,\ 0.0263].$$

  Based on: $Z_{\text{logit}(\widehat{F})} = [\text{logit}[\widehat{F}(t_1)] - \text{logit}[F(t_1)]/\text{se}_{\text{logit}(\widehat{F})} \dot{\sim} \text{NORM}(0,1)$.

  $$[\underset{\sim}{F}(t_1),\ \widetilde{F}(t_1)] = \left[\frac{0.0133}{0.0133 + (1-0.0133)\times w},\ \frac{0.0133}{0.0133+(1-0.0133)/w}\right] = [0.0050,\ 0.0350].$$

  where $w = \exp\{1.96(0.00662)/[0.0133(1-0.0133)]\} = 2.687816$.

- For $F(2)$ with $\widehat{F}(t_2) = 0.0384$, $\text{se}_{\widehat{F}(t_2)} = \sqrt{0.0001639} = 0.0128$

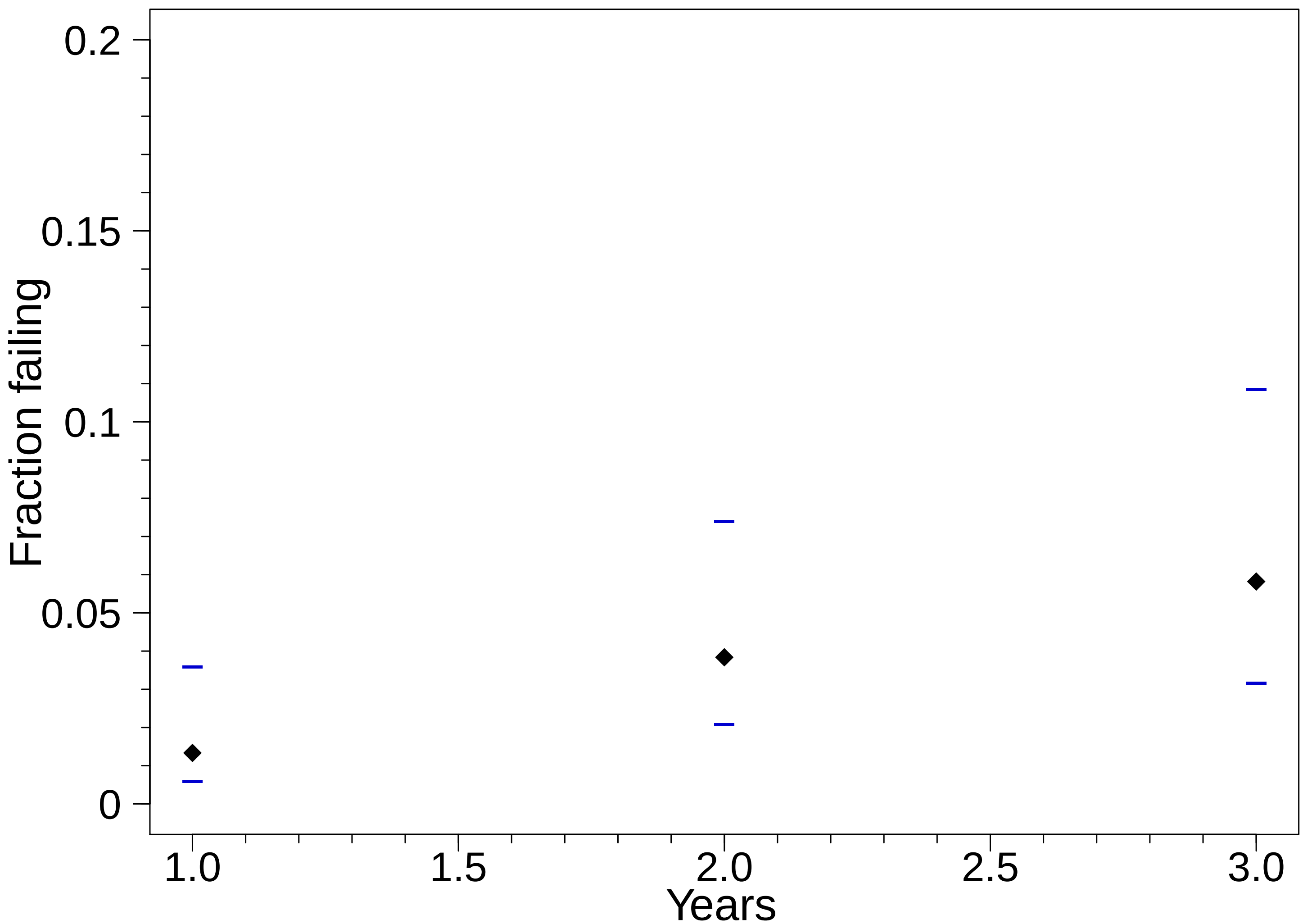
  Based on: $Z_{\widehat{F}}$, $\quad [\underset{\sim}{F}(t_2),\ \widetilde{F}(t_2)] = [0.0133,\ 0.0635]$.

  Based on: $Z_{\text{logit}(\widehat{F})}$, $\quad [\underset{\sim}{F}(t_2),\ \widetilde{F}(t_2)] = [0.0198,\ 0.0730]$.

Summary of the Nonparametric Pointwise Confidence Intervals for $F(t_i)$ Based on the Heat Exchanger Tube Crack Data

Year	t_i	$\widehat{F}(t_i)$	$\mathrm{se}_{\widehat{F}}$	Pointwise Confidence Intervals
$(0-1]$	1	0.0133	0.00662	
95% Confidence Intervals for $F(1)$				
Based on $Z_{\mathrm{logit}(\widehat{F})} \dot\sim \mathrm{NORM}(0,1)$				[0.0050, 0.0350]
Based on $Z_{\widehat{F}} \dot\sim \mathrm{NORM}(0,1)$				[0.0003, 0.0263]
$(1-2]$	2	0.0384	0.0128	
95% Confidence Intervals for $F(2)$				
Based on $Z_{\mathrm{logit}(\widehat{F})} \dot\sim \mathrm{NORM}(0,1)$				[0.0198, 0.0730]
Based on $Z_{\widehat{F}} \dot\sim \mathrm{NORM}(0,1)$				[0.0133, 0.0635]
$(2-3]$	3	0.0582	0.0187	
95% Confidence Intervals for $F(3)$				
Based on $Z_{\mathrm{logit}(\widehat{F})} \dot\sim \mathrm{NORM}(0,1)$				[0.0307, 0.1076]
Based on $Z_{\widehat{F}} \dot\sim \mathrm{NORM}(0,1)$				[0.0216, 0.0949]

Nonparametric Estimate with Pointwise Wald 95% Confidence Intervals Based on $Z_{\text{logit}(\widehat{F})}$ for the Heat Exchanger Tube Crack Failure-Time Distribution



Chapter 3

Segment 4

The Kaplan-Meier Nonparametric Estimator for Multiply-Censored Failure-Time Data and Exact Failure Times

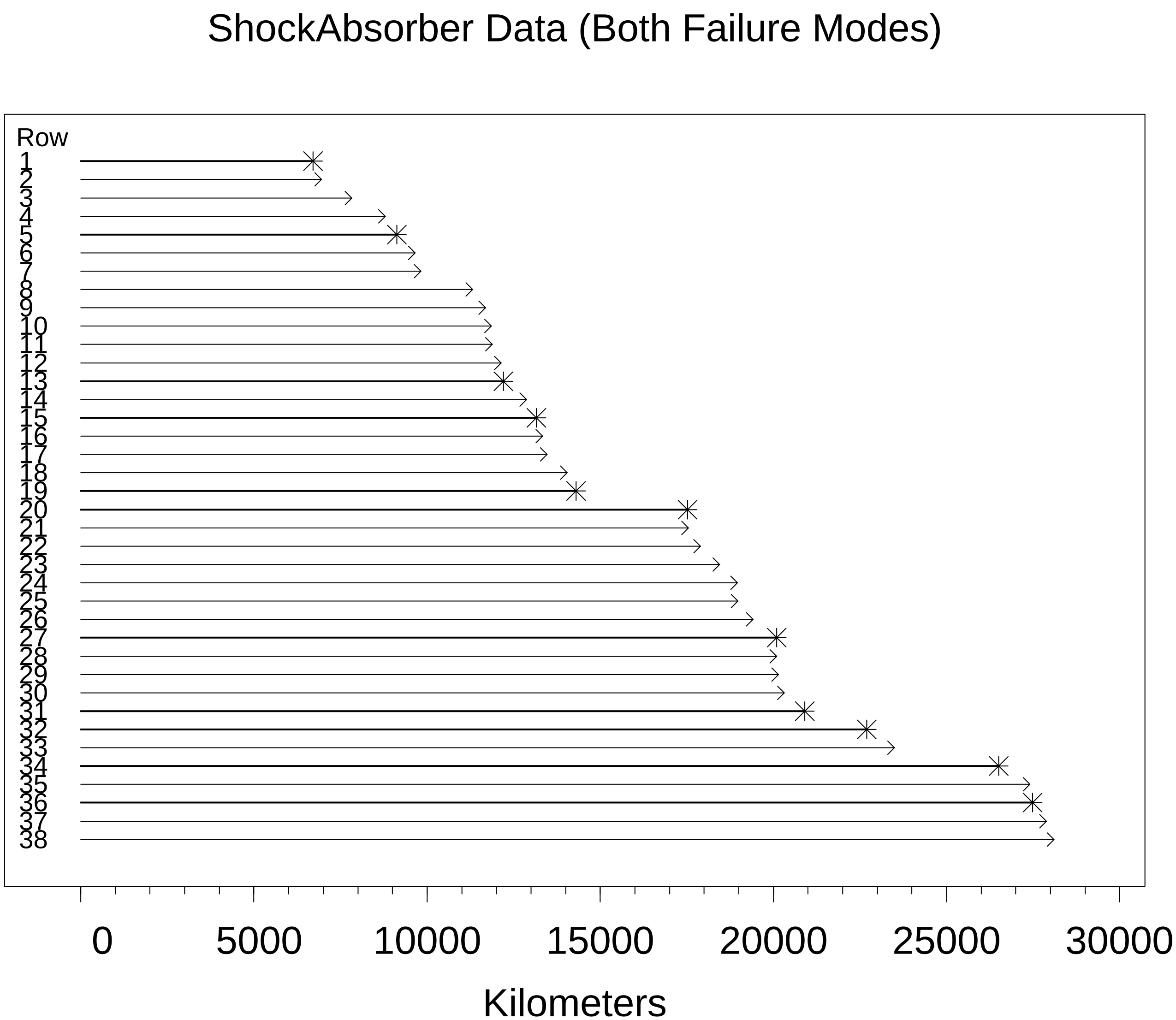
Shock Absorber Failure Data

First reported in O'Connor (1985).

- Failure times, in number of kilometers of use, of vehicle shock absorbers.
- Two failure modes, denoted by M1 and M2.
- One might be interested in the distribution of time to failure for mode M1, mode M2, or in the overall failure-time distribution of the part.

  Here, we do not differentiate between modes M1 and M2. We will estimate the distribution of time to failure by either mode M1 or M2.

Event Plot
Shock Absorber Field-Failure Data
Failure Mode Information Ignored



Nonparametric Estimation of $F(t)$ with Exact Failures Using the Kaplan-Meier Estimator

In the limit, as the number of inspections increases and the width of the inspection intervals approaches zero, we get the **product-limit** or **Kaplan-Meier** estimator:

- Failures are concentrated in a small number of intervals of infinitesimal length.
- $\widehat{F}(t)$ will be **constant** over all intervals that have no failures.
- $\widehat{F}(t)$ is a step function with **jumps** at each reported failure time.

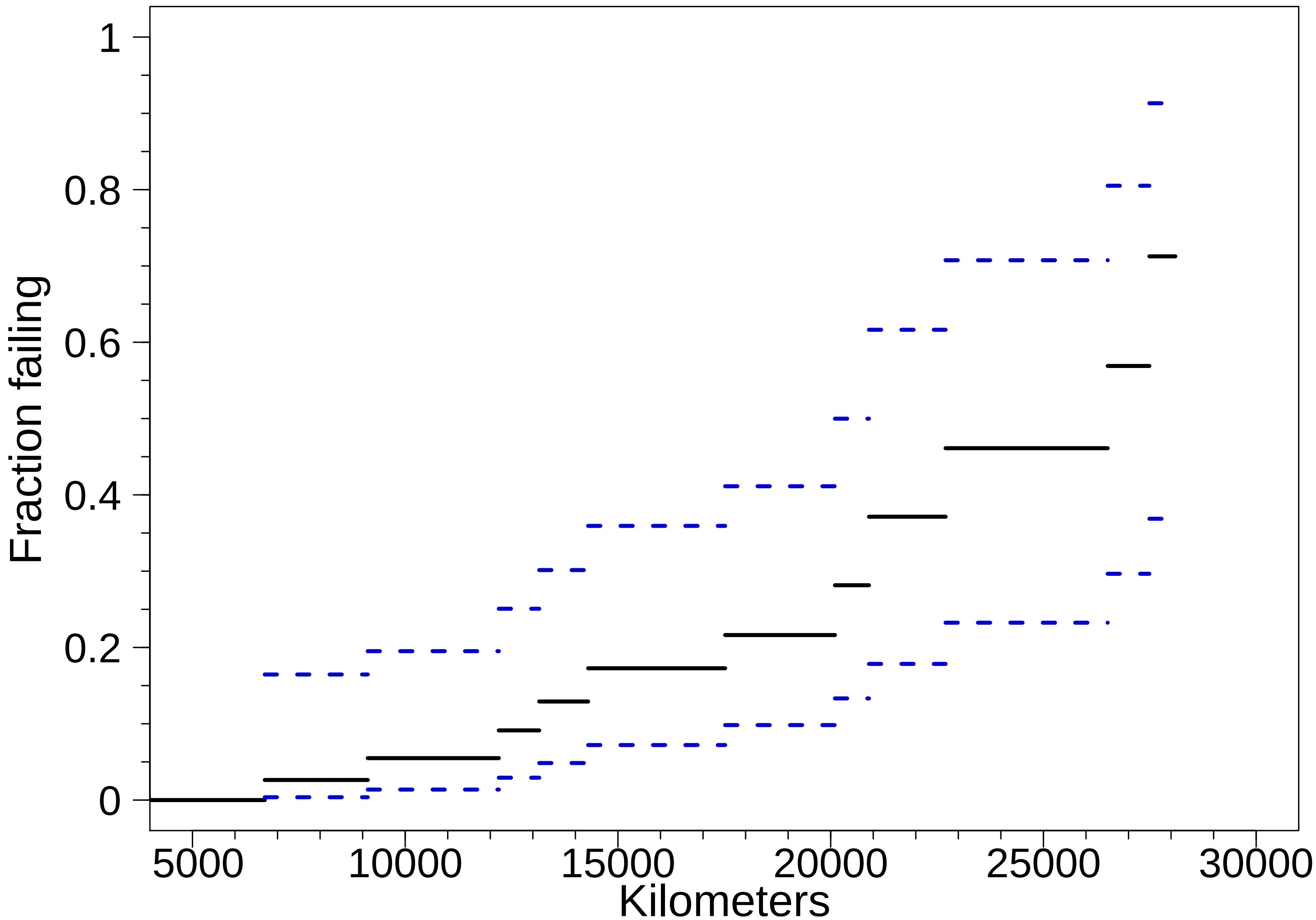
Confidence intervals are computed in a manner similar to that used with interval censoring.

Note: The binomial estimator for exact failures and singly right-censored data is a special case of the Kaplan-Meier estimator.

Kaplan-Meier Estimates for the Shock Absorber Data up to 12,220 km

				Conditional		Unconditional	
t_j (km)	n_j	d_j	r_j	$\widehat{p}_j$	$1-\widehat{p}_j$	$\widehat{S}(t_j)$	$\widehat{F}(t_j)$
6,700	38	1	0	1/38	37/38	0.97368	0.02632
6,950	37	0	1				
7,820	36	0	1				
8,790	35	0	1				
9,120	34	1	0	1/34	33/34	0.94505	0.05495
9,660	33	0	1				
9,820	32	0	1				
11,310	31	0	1				
11,690	30	0	1				
11,850	29	0	1				
11,880	28	0	1				
12,140	27	0	1				
12,200	26	1	0	1/26	25/26	0.90870	0.09130
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Kaplan-Meier Estimate with Pointwise 95% Confidence Intervals Based on $Z_{\text{logit}(\widehat{F})}$ for the Shock Absorber Data



Chapter 3

Segment 5

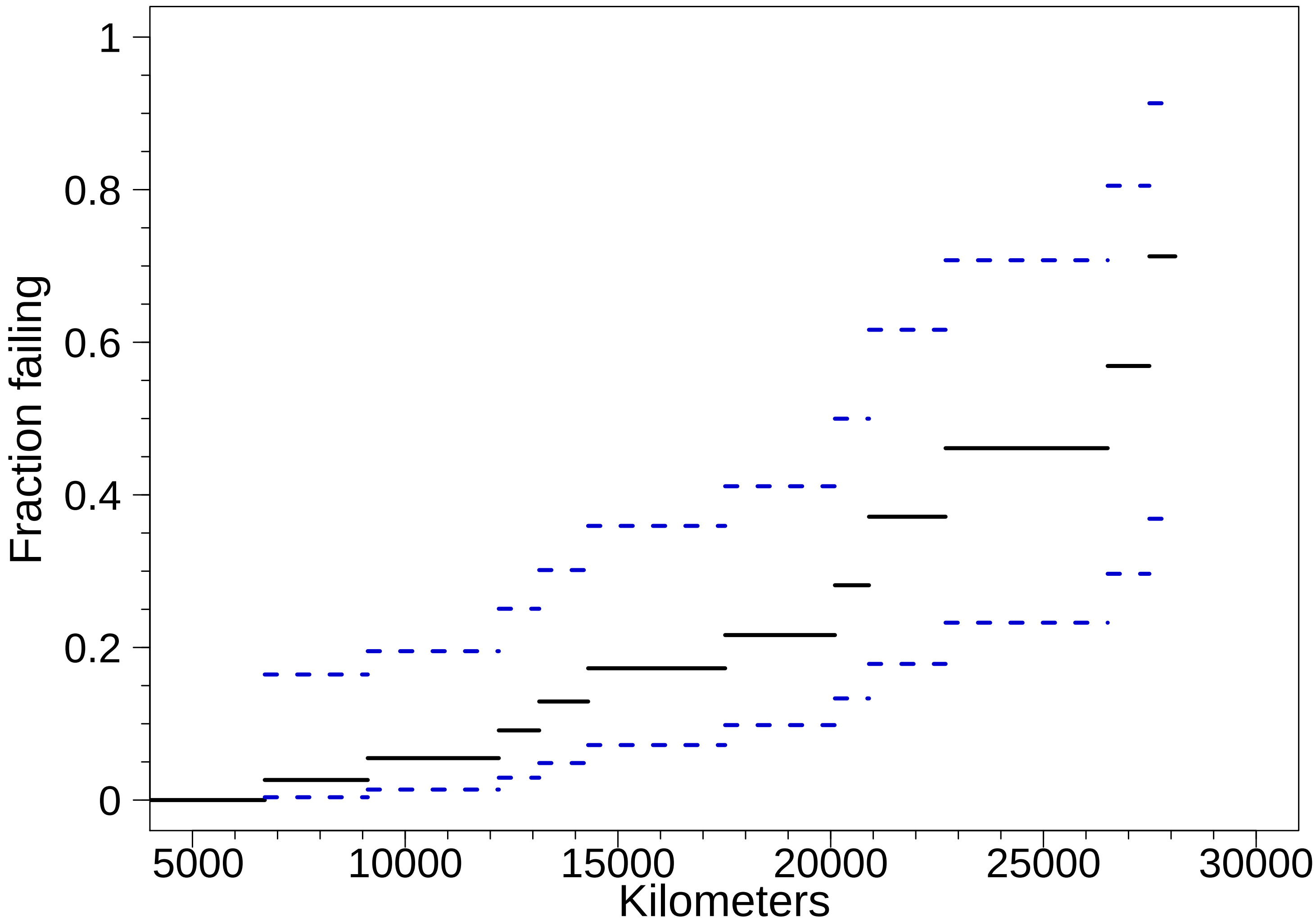
Nonparametric Simultaneous Confidence Bands for $F(t)$

Need for Nonparametric Simultaneous Confidence Bands for $F(t)$

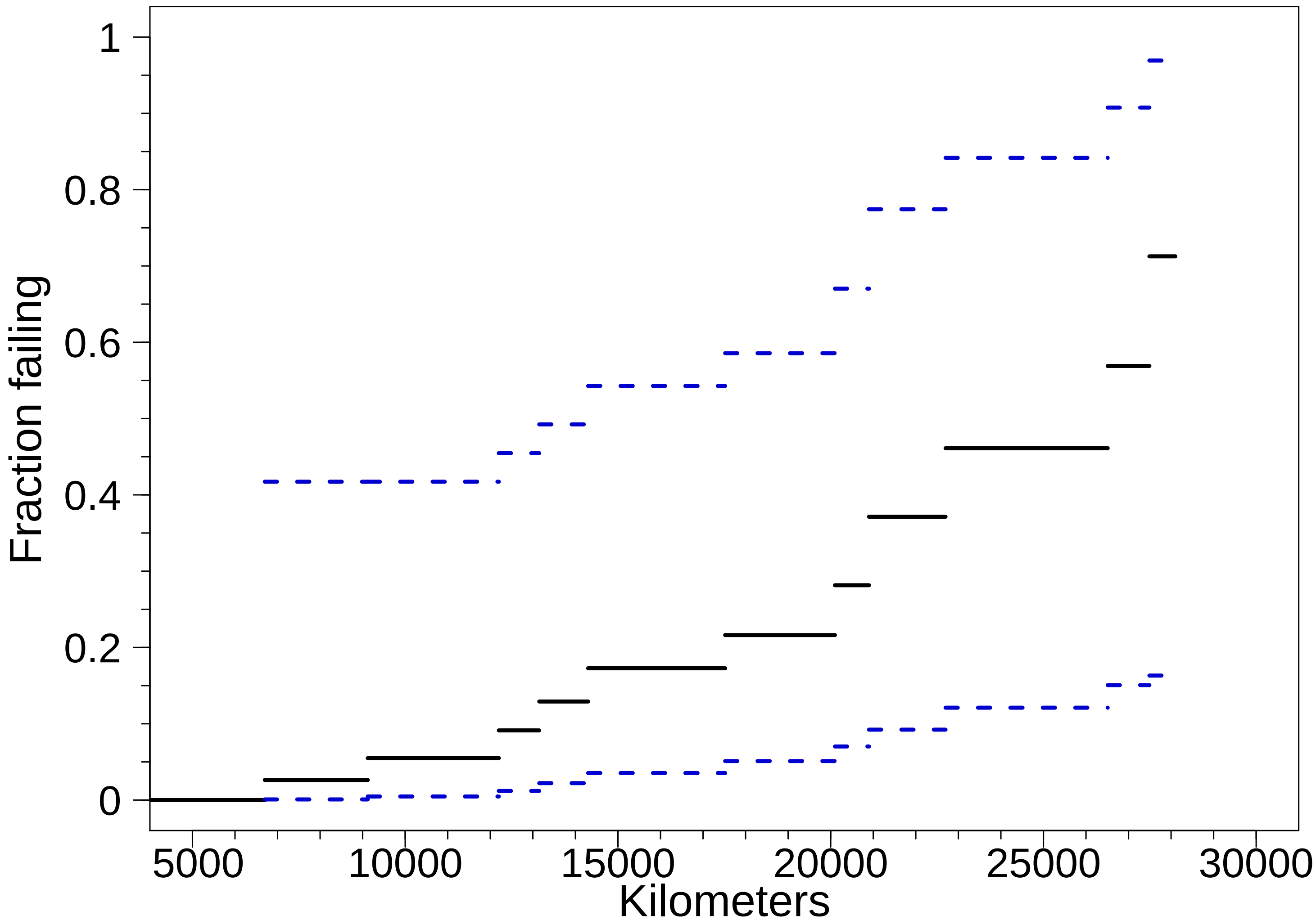
- **Pointwise confidence intervals** for $F(t)$ are useful for quantifying the statistical uncertainty in $F(t)$ at one particular value of t.

- **Simultaneous confidence bands** for $F(t)$ are necessary to quantify the the statistical uncertainty over a range of values of t.

Kaplan-Meier Estimate with Pointwise 95% Confidence Intervals Based on $Z_{\text{logit}(\widehat{F})}$ for the Shock Absorber Data



Kaplan-Meier Estimate with Simultaneous 95% Confidence Intervals Based on $Z_{\text{logit}(\widehat{F})}$ for the Shock Absorber Data



Nonparametric Simultaneous Confidence Bands for $F(t)$

Simultaneous approximate $100(1-\alpha)\%$ confidence bands for $F(t)$ can be obtained from

$$\left[\underset{\sim}{F}(t),\ \widetilde{F}(t)\right] = \widehat{F}(t) \mp e_{(a,b,1-\alpha/2)}\text{se}_{\widehat{F}}(t) \quad \text{for all } t \in [t_L(a), t_U(b)]$$

where $t_L(a)$ and $t_U(b)$ are complicated functions of the censoring pattern in the data.

Comments:

- These particular simultaneous confidence bands are known as "equal precision" or "EP" bands.
- The approximate factors $e_{(a,b,1-\alpha/2)}$ can be computed from a large-sample approximation given in Nair (1984).
- $e_{(a,b,1-\alpha/2)}$ is the same for all values of t.
- The factors $e_{(a,b,1-\alpha/2)}$ are larger than the corresponding $z_{(1-\alpha/2)}$ factors.

Factors $e_{(a,b,1-\alpha/2)}$ for Computing the EP Nonparametric Simultaneous Approximate Confidence Bands

Limits		Confidence Level			
a	b	0.80	0.90	0.95	0.99
0.005	0.999	2.92	3.17	3.41	3.88
0.01	0.999	2.90	3.15	3.39	3.87
0.05	0.999	2.84	3.10	3.34	3.82
0.001	0.995	2.92	3.17	3.41	3.88
0.005	0.995	2.86	3.12	3.36	3.85
0.01	0.995	2.84	3.10	3.34	3.83
0.05	0.995	2.76	3.03	3.28	3.77
0.001	0.99	2.90	3.15	3.39	3.87
0.005	0.99	2.84	3.10	3.34	3.83
0.01	0.99	2.81	3.07	3.31	3.81
0.05	0.99	2.73	3.00	3.25	3.75
0.001	0.95	2.84	3.10	3.34	3.82
0.005	0.95	2.76	3.03	3.28	3.77
0.01	0.95	2.73	3.00	3.25	3.75
0.05	0.95	2.62	2.91	3.16	3.68
0.001	0.9	2.80	3.07	3.31	3.80
0.005	0.9	2.72	3.00	3.25	3.75
0.01	0.9	2.68	2.96	3.21	3.72
0.05	0.9	2.56	2.85	3.11	3.64

Better Nonparametric Simultaneous Confidence Bands for $F(t)$

- The approximate $100(1-\alpha)\%$ simultaneous confidence bands

$$\left[\underset{\sim}{F}(t),\ \ \widetilde{F}(t)\right] = \widehat{F}(t) \mp e_{(a,b,1-\alpha/2)}\mathrm{se}_{\widehat{F}}(t) \ \text{ for all } t \in [t_L(a), t_U(b)]$$

are based on the approximate distribution of

$$Z_{\max \widehat{F}} = \max_{t \in [t_L(a), t_U(b)]} \left[\frac{\widehat{F}(t) - F(t)}{\mathrm{se}_{\widehat{F}(t)}}\right].$$

- It is generally better to compute the simultaneous confidence bands based on the logit transformation of $\widehat{F}$. This gives

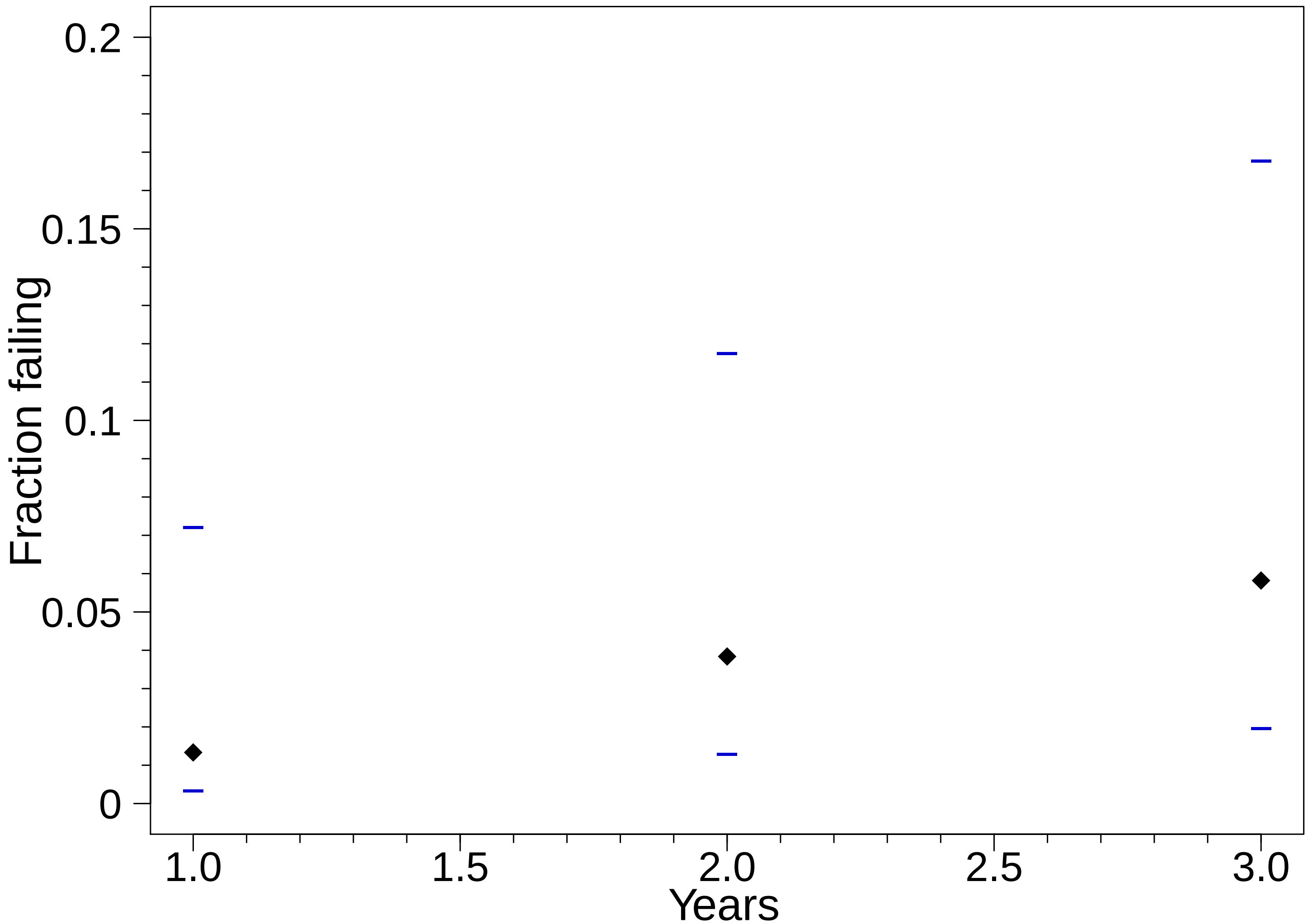
$$[\underset{\sim}{F}(t),\ \ \widetilde{F}(t)] = \left[\frac{\widehat{F}(t)}{\widehat{F}(t) + [1 - \widehat{F}(t)] \times w},\ \ \frac{\widehat{F}(t)}{\widehat{F}(t) + [1 - \widehat{F}(t)]/w}\right]$$

where $w = \exp\{e_{(a,b,1-\alpha/2)}\mathrm{se}_{\widehat{F}}/[\widehat{F}(1 - \widehat{F})]\}$.

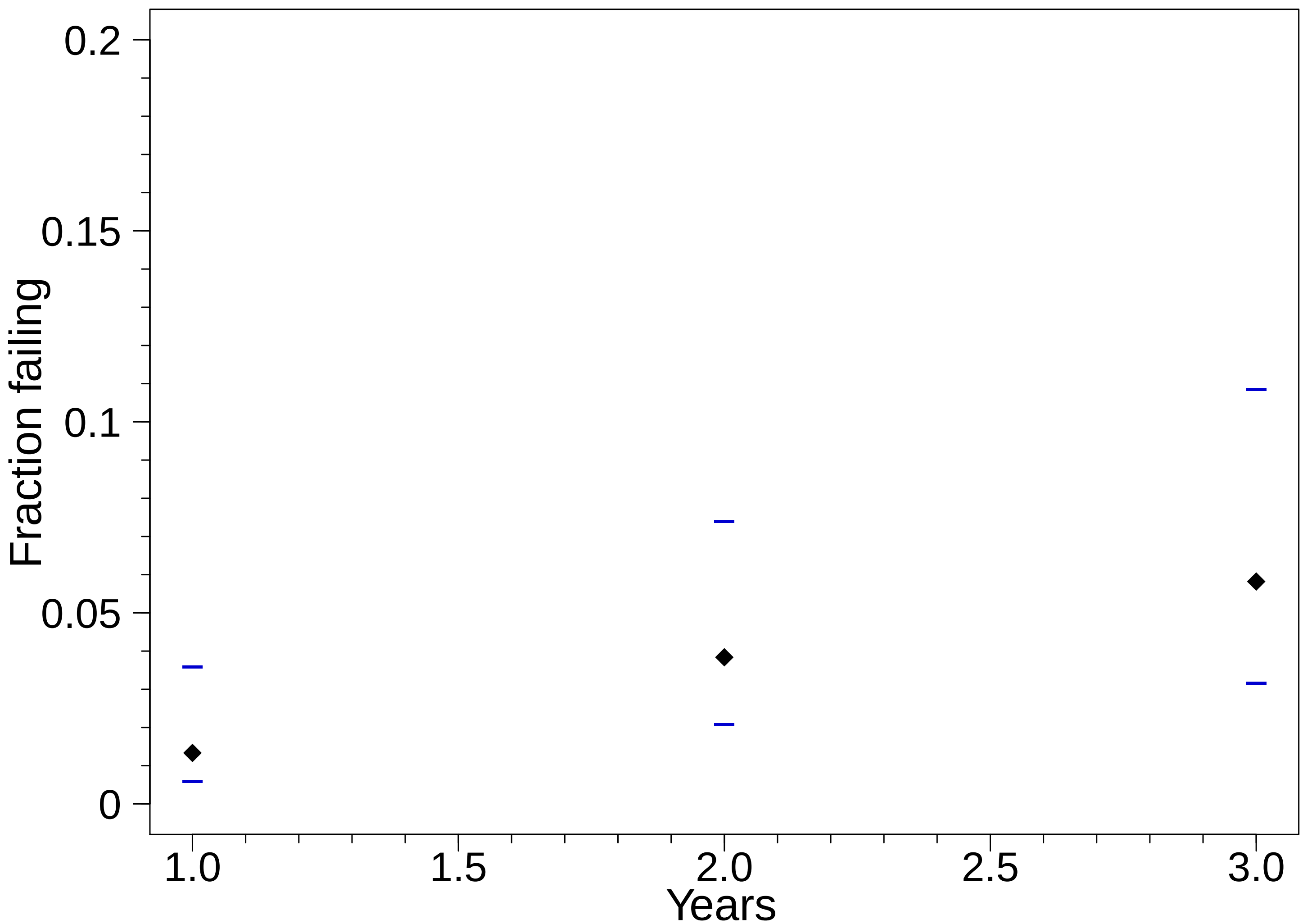
These are based on the approximate distribution of

$$Z_{\max \mathrm{logit}(\widehat{F})} = \max_{t \in [t_L(a), t_U(b)]} \left[\frac{\mathrm{logit}[\widehat{F}(t)] - \mathrm{logit}[F(t)]}{\mathrm{se}_{\mathrm{logit}[\widehat{F}(t)]}}\right].$$

Nonparametric Estimate with <u>Simultaneous</u> 95% Confidence Bands Based on $Z_{\mathbf{max}\,\mathbf{logit}(\widehat{F})}$ for the Heat Exchanger Tube Crack Data



Nonparametric Estimate with Pointwise Wald 95% Confidence Intervals Based on $Z_{\text{logit}(\widehat{F})}$ for the Heat Exchanger Tube Crack Data



Chapter 3

Segment 6

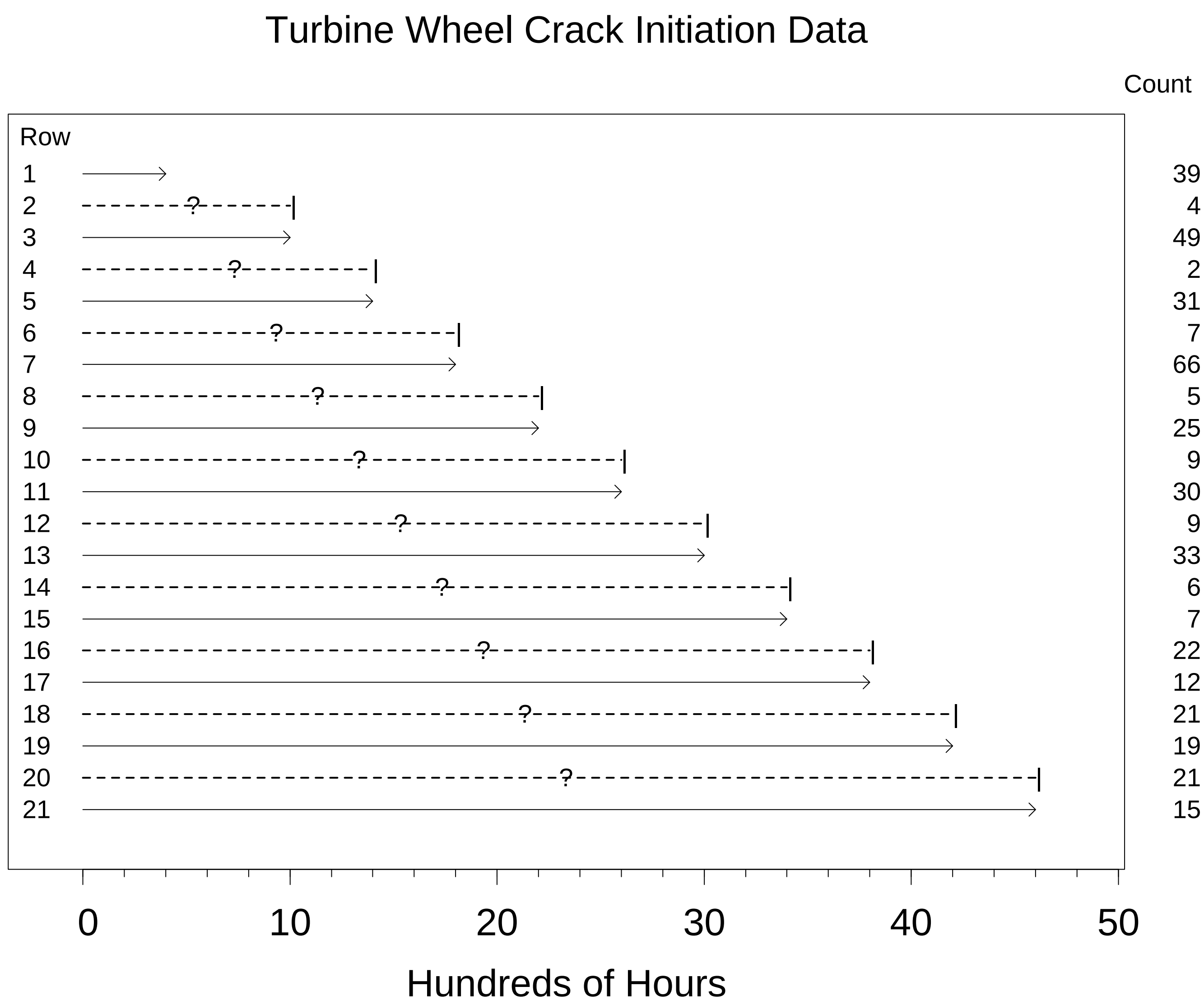
Nonparametric Estimation of $F(t_i)$ with Current-Status Data or Arbitrary Censoring

Nonparametric Estimation of $F(t_i)$ with Arbitrary Censoring

- The methods described so far work only for some kinds of censoring patterns (multiple right censoring, interval censoring with intervals that do not overlap, and some other very special censoring patterns.)

- The nonparametric maximum likelihood generalizations provided by the **Peto/Turnbull** estimator can be used for
  - Current-status data (e.g., both left- and right-censored, overlapping).
  - Interval censoring with overlapping intervals.
  - Arbitrary censoring—combinations of the above possibly with exact failures
  - Truncated data.

Event Plot
Turbine Wheel Current-Status Data



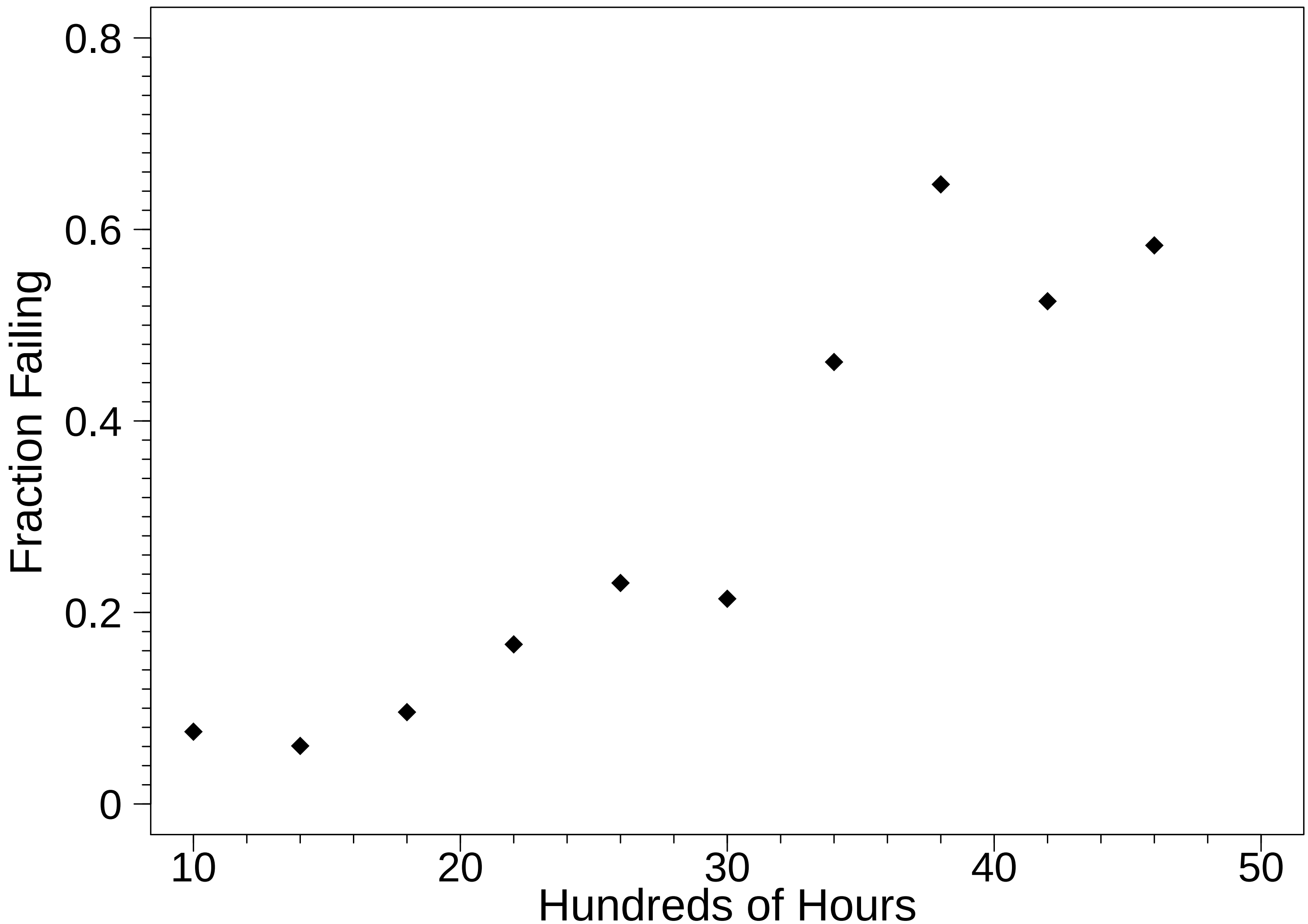
Turbine Wheel Inspection Data Summary

100-hours of Exposure t_i	# Cracked Left Censored	# Not Cracked Right Censored	Proportion Cracked Crude Estimate of $F(t)$
4	0	39	$0/39 = 0.000$
10	4	49	$4/53 = 0.075$
14	2	31	$2/33 = 0.060$
18	7	66	$7/73 = 0.096$
22	5	25	$5/30 = 0.167$
26	9	30	$9/39 = 0.231$
30	9	33	$9/42 = 0.214$
34	6	7	$6/13 = 0.462$
38	22	12	$22/34 = 0.647$
42	21	19	$21/40 = 0.525$
46	21	15	$21/36 = 0.583$

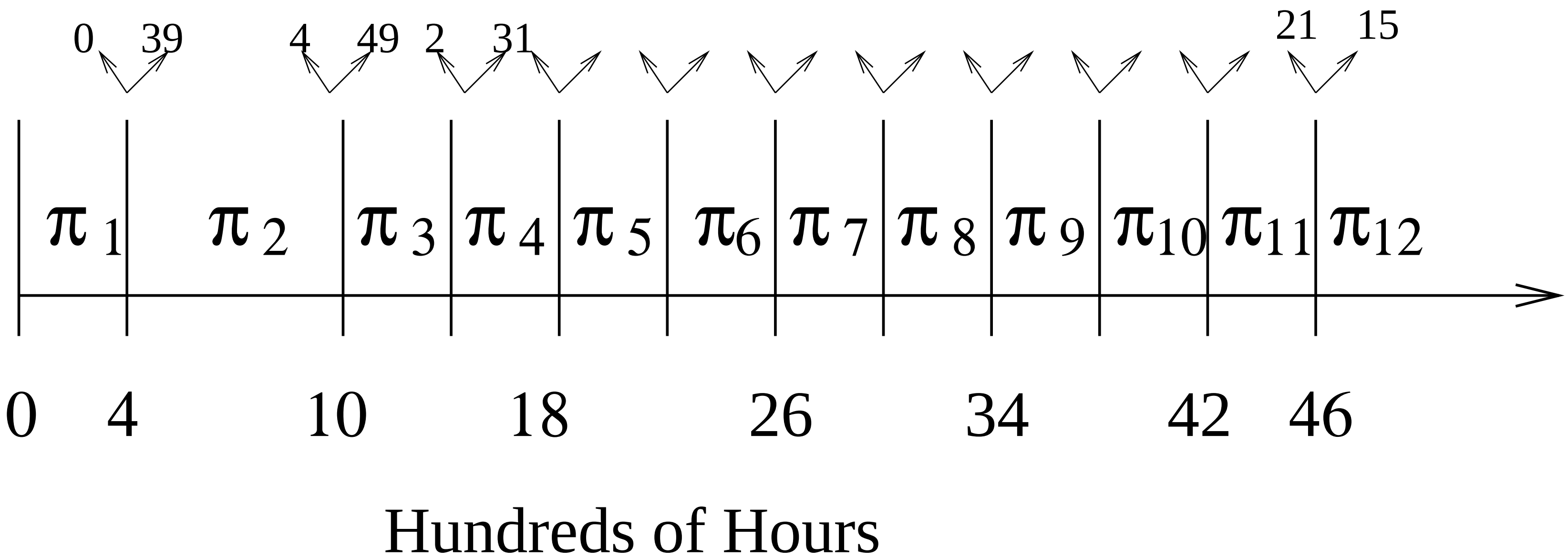
Data from Nelson (1982, page 409).

- The analysts did not know the initiation time for any of the wheels.
- All they knew about each wheel was its exposure time and whether a crack had initiated or not. For convenience and data compression, units were grouped by amount of exposure time.

Plot of Crude Estimates of the Proportions Failing Versus Hours of Exposure for the Turbine Wheel Current-Status Data



Basic Parameters Used in Computing the Nonparametric ML Estimate of $F(t)$ for the Turbine Wheel Data



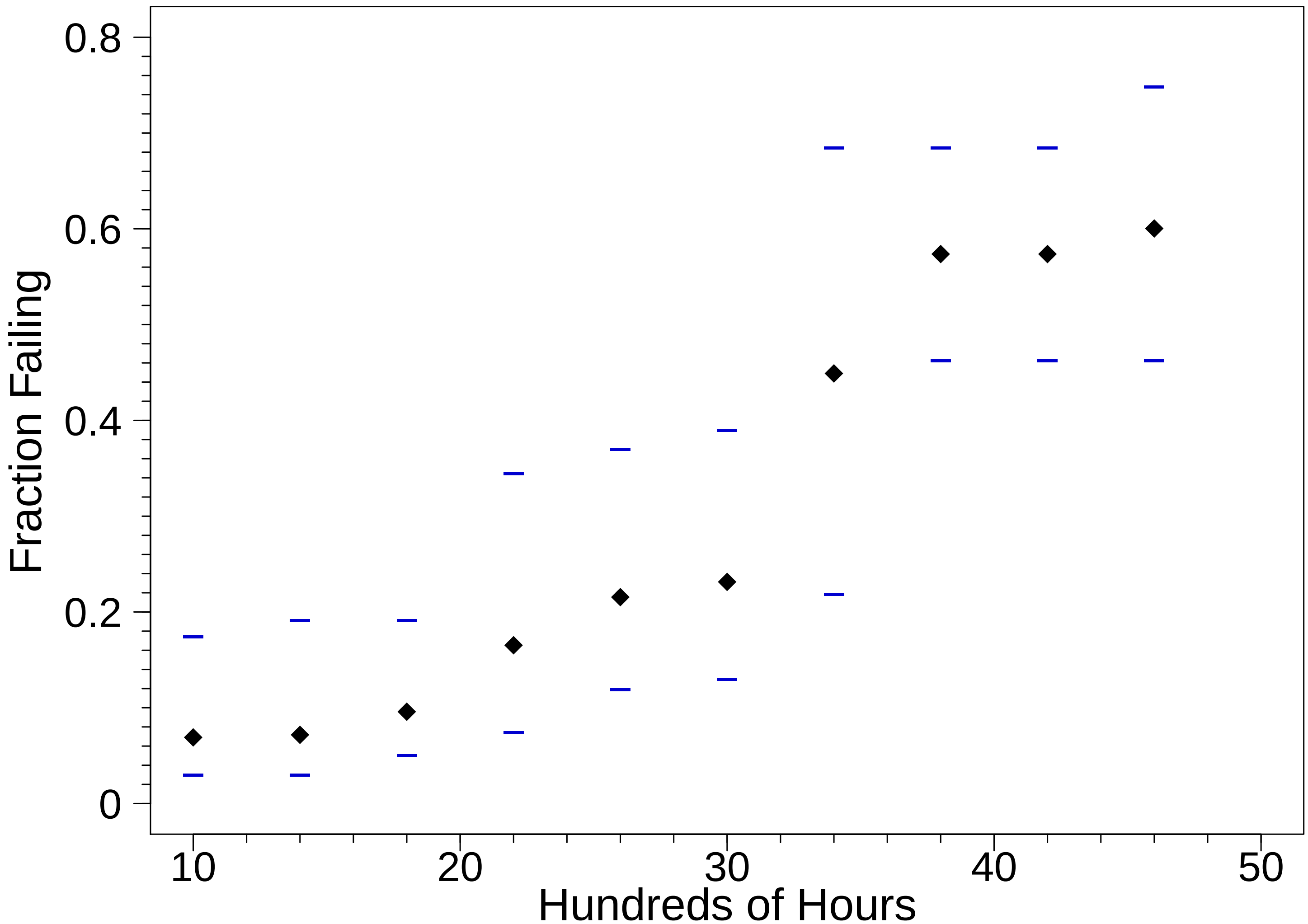
Nonparametric Estimation of $F(t)$ with Current-Status Data

- **Basic idea:** Write the likelihood (probability of the data) and maximize to obtain $\widehat{\boldsymbol{p}}$ or $\widehat{\boldsymbol{\pi}}$ from which one can compute $\widehat{F}(t_i)$ (Peto 1973).
- **Illustration:** The likelihood for the turbine wheel current-status data is

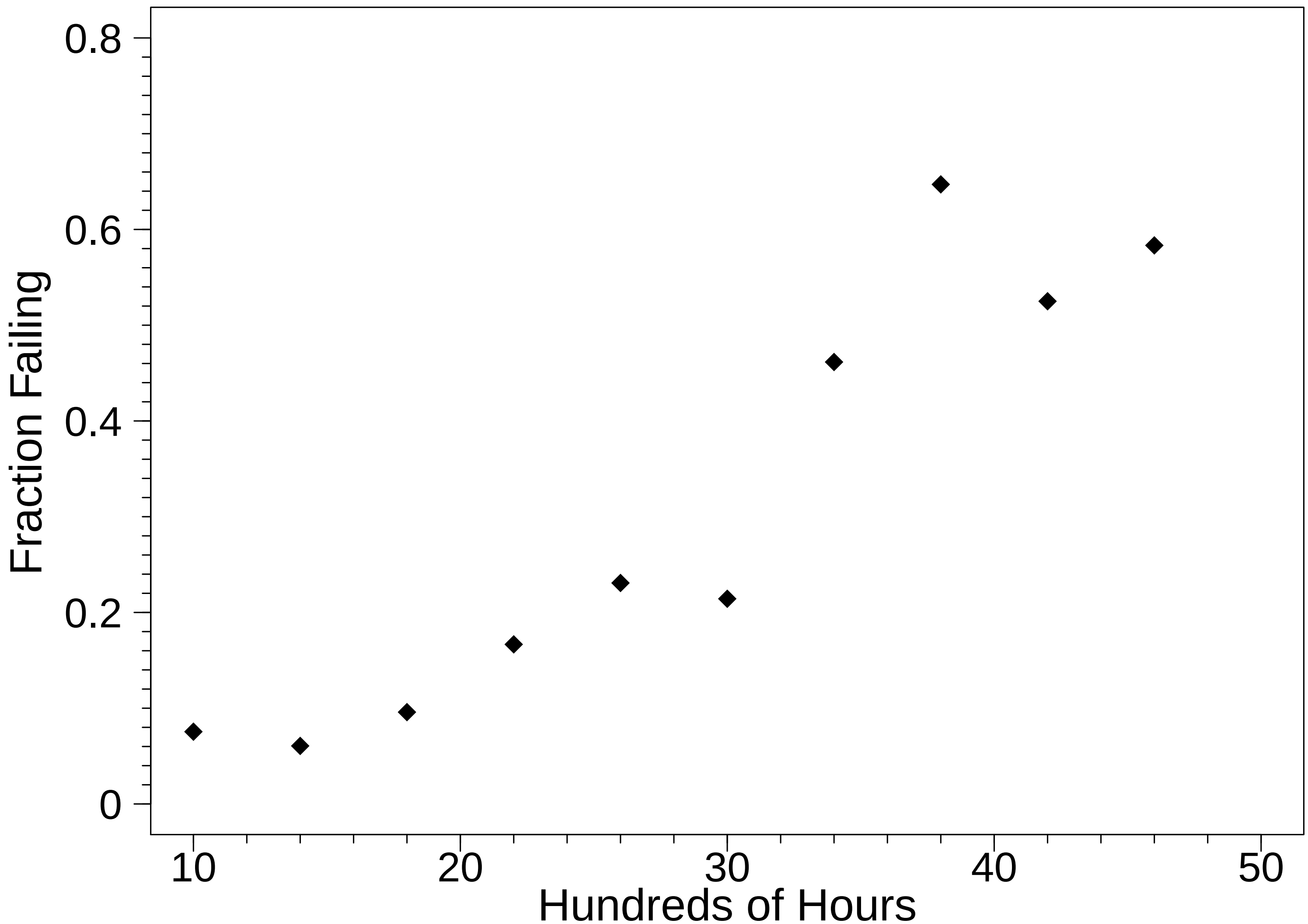
$$
\begin{aligned}
L(\boldsymbol{\pi}) = L(\boldsymbol{\pi}; \mathrm{DATA}) = \mathcal{C} \times [\pi_1]^0 \times [\pi_2 + \cdots + \pi_{12}]^{39} \times \\
\times [\pi_1 + \pi_2]^4 \times [\pi_3 + \cdots + \pi_{12}]^{49} \\
\times [\pi_1 + \cdots + \pi_3]^2 \times [\pi_4 + \cdots + \pi_{12}]^{31} \\
\vdots \\
\times [\pi_1 + \cdots + \pi_{11}]^{21} \times [\pi_{12}]^{15}
\end{aligned}
$$

where $\pi_{12} = 1 - \sum_{i=1}^{11} \pi_i$. The values of $\pi_1, \ldots, \pi_{11}$ that maximize $L(\boldsymbol{\pi})$ gives $\widehat{\boldsymbol{\pi}}$, the ML estimator of $\boldsymbol{\pi}$. Then, $\widehat{F}(t_i) = \sum_{j=1}^{i} \widehat{\pi}_j$, $i = 1, \ldots, m$.

Nonparametric ML Estimate with Pointwise Approximate 95% Confidence Intervals for $F(t_i)$ Based on $Z_{\textbf{logit}(\widehat{F})}$ for the Turbine Wheel Data



Plot of Crude Estimates of the Proportions Failing Versus Hours of Exposure for the Turbine Wheel Data



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Peto, R. (1973). Experimental survival curves for interval-censored data. *Journal of the Royal Statistical Society: Series C (Applied Statistics) 22*, 86–91. []