Chapter 8

Maximum Likelihood for Log-Location-Scale Distributions

W. Q. Meeker, L. A. Escobar, and F. G. Pascual
Iowa State University, Louisiana State University, and Washington State University.

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Chapter 8
Maximum Likelihood
for Log-Location-Scale Distributions

Topics discussed in this chapter are:

- How to express the likelihood for location-distributions like the normal distribution and log-location-scale distributions like the Lognormal and Weibull distributions.

- How to construct and interpret likelihood confidence intervals for parameters and for **functions** of parameters of log-location-scale distributions.

- The construction of Wald (normal-approximation) confidence intervals for parameters and functions of parameters of log-location-scale distributions.

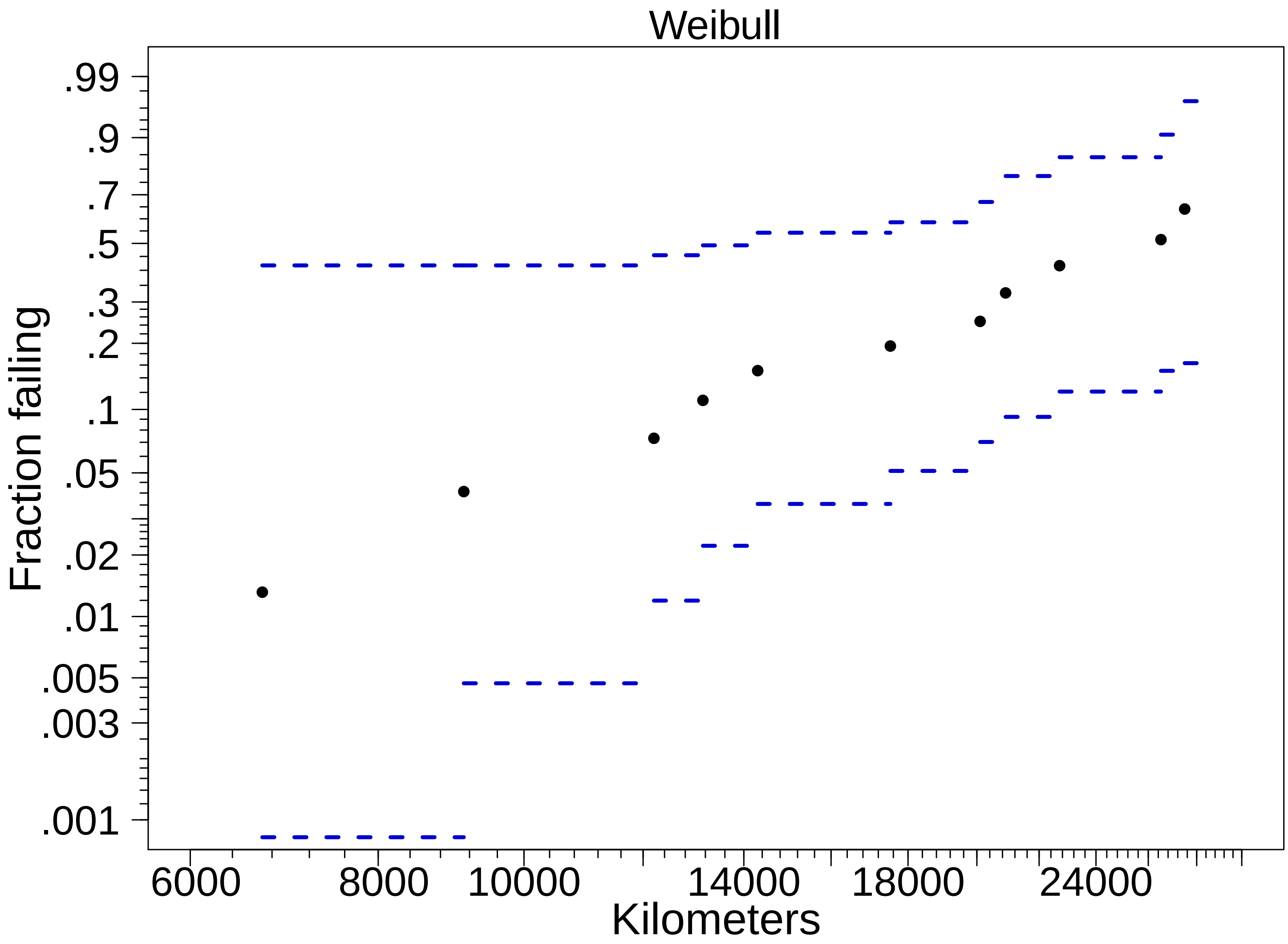
- Inference for log-location-scale distribution parameters and functions of parameters with a given shape parameter.

Chapter 8

Segment 1

Likelihood for Log-Location-Scale Distributions

Weibull Probability Plot of the Shock Absorber Data



Weibull Distribution Likelihood for Right-Censored Data

- The Weibull distribution model is

$$\Pr(T \le t) = F(t;\mu,\sigma) = \Phi_{\text{sev}}\{[\log(t)-\mu]/\sigma\}.$$

- The likelihood has the form

$$\begin{aligned} L(\mu,\sigma) &= \prod_{i=1}^{n} L_i(\mu,\sigma;\text{data}_i) \\ &= \prod_{i=1}^{n} [f(t_i;\mu,\sigma)]^{\delta_i}[1-F(t_i;\mu,\sigma)]^{1-\delta_i} \\ &= \prod_{i=1}^{n} \left[\frac{1}{\sigma t_i}\phi_{\text{sev}}\left(\frac{\log(t_i)-\mu}{\sigma}\right)\right]^{\delta_i} \times \left[1-\Phi_{\text{sev}}\left(\frac{\log(t_i)-\mu}{\sigma}\right)\right]^{1-\delta_i} \end{aligned}$$

$$\delta_i = \begin{cases} 1 & \text{if } t_i \text{ is an exact observation} \\ 0 & \text{if } t_i \text{ is a right-censored observation} \end{cases}$$

$\phi_{\text{sev}}(z)$ and $\Phi_{\text{sev}}(z)$ are the standardized smallest extreme value density and distribution functions, respectively.

Lognormal Distribution Likelihood for Right-Censored Data

- The lognormal distribution model is

$$\Pr(T \le t) = F(t;\mu,\sigma) = \Phi_{\text{norm}}\{[\log(t) - \mu]/\sigma\}.$$

- The likelihood has the form

$$\begin{aligned} L(\mu,\sigma) &= \prod_{i=1}^{n} L_i(\mu,\sigma;\text{data}_i) \\ &= \prod_{i=1}^{n} [f(t_i;\mu,\sigma)]^{\delta_i}[1 - F(t_i;\mu,\sigma)]^{1-\delta_i} \\ &= \prod_{i=1}^{n} \left[\frac{1}{\sigma t_i}\phi_{\text{norm}}\left(\frac{\log(t_i) - \mu}{\sigma}\right)\right]^{\delta_i} \times \left[1 - \Phi_{\text{norm}}\left(\frac{\log(t_i) - \mu}{\sigma}\right)\right]^{1-\delta_i} \end{aligned}$$

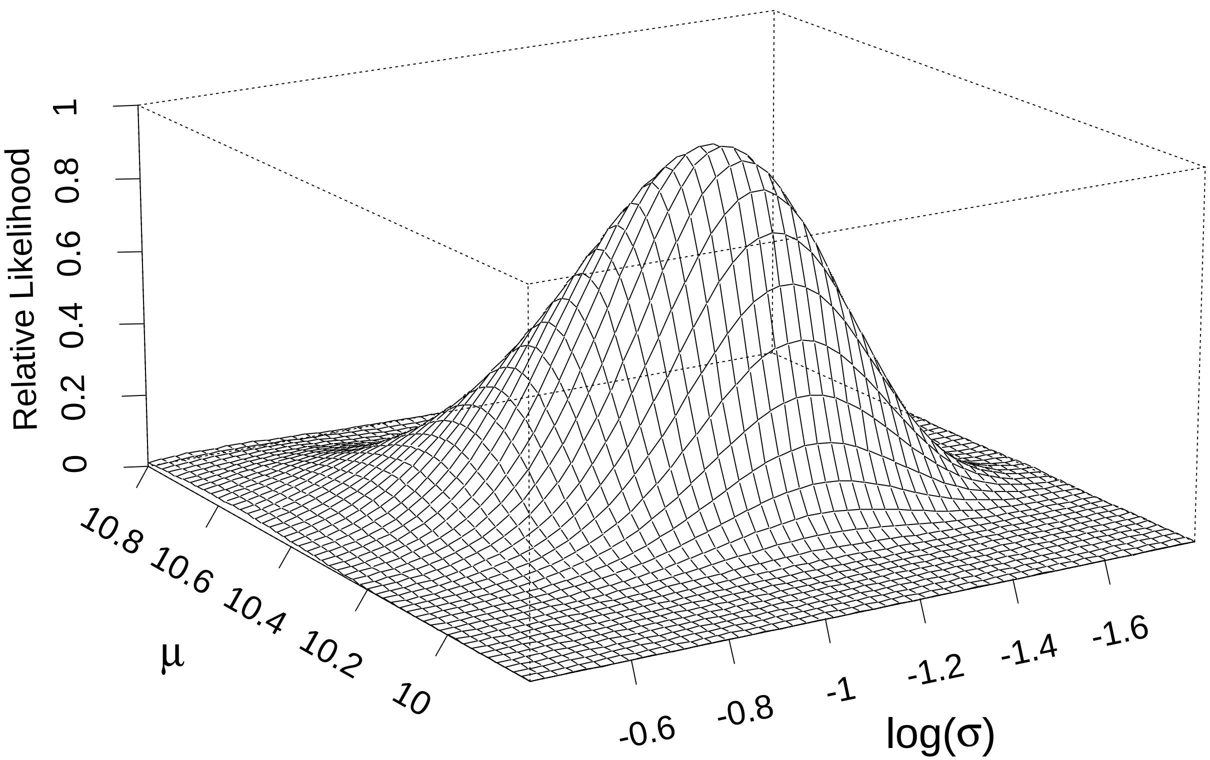
$$\delta_i = \begin{cases} 1 & \text{if } t_i \text{ is an exact observation} \\ 0 & \text{if } t_i \text{ is a right-censored observation} \end{cases}$$

$\phi_{\text{norm}}(z)$ and $\Phi_{\text{norm}}(z)$ are the standardized normal density and distribution functions, respectively.

Weibull Relative Likelihood for the Shock Absorber Data

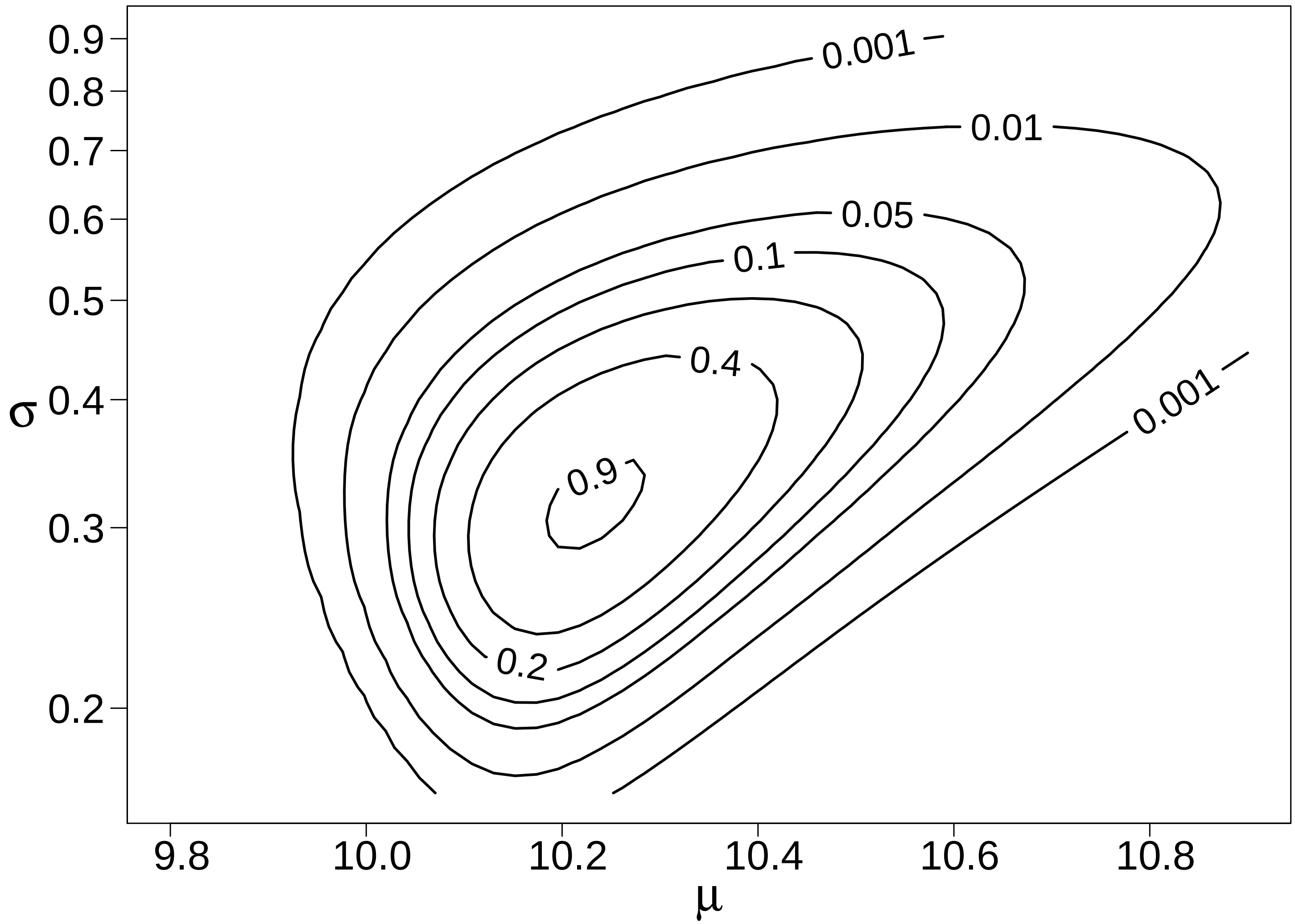
ML Estimates: $\widehat{\mu} = 10.23$ and $\widehat{\sigma} = 0.3164$

$$R(\mu, \log(\sigma)) = L(\mu, \log(\sigma))/L(\widehat{\mu}, \log(\widehat{\sigma}))$$

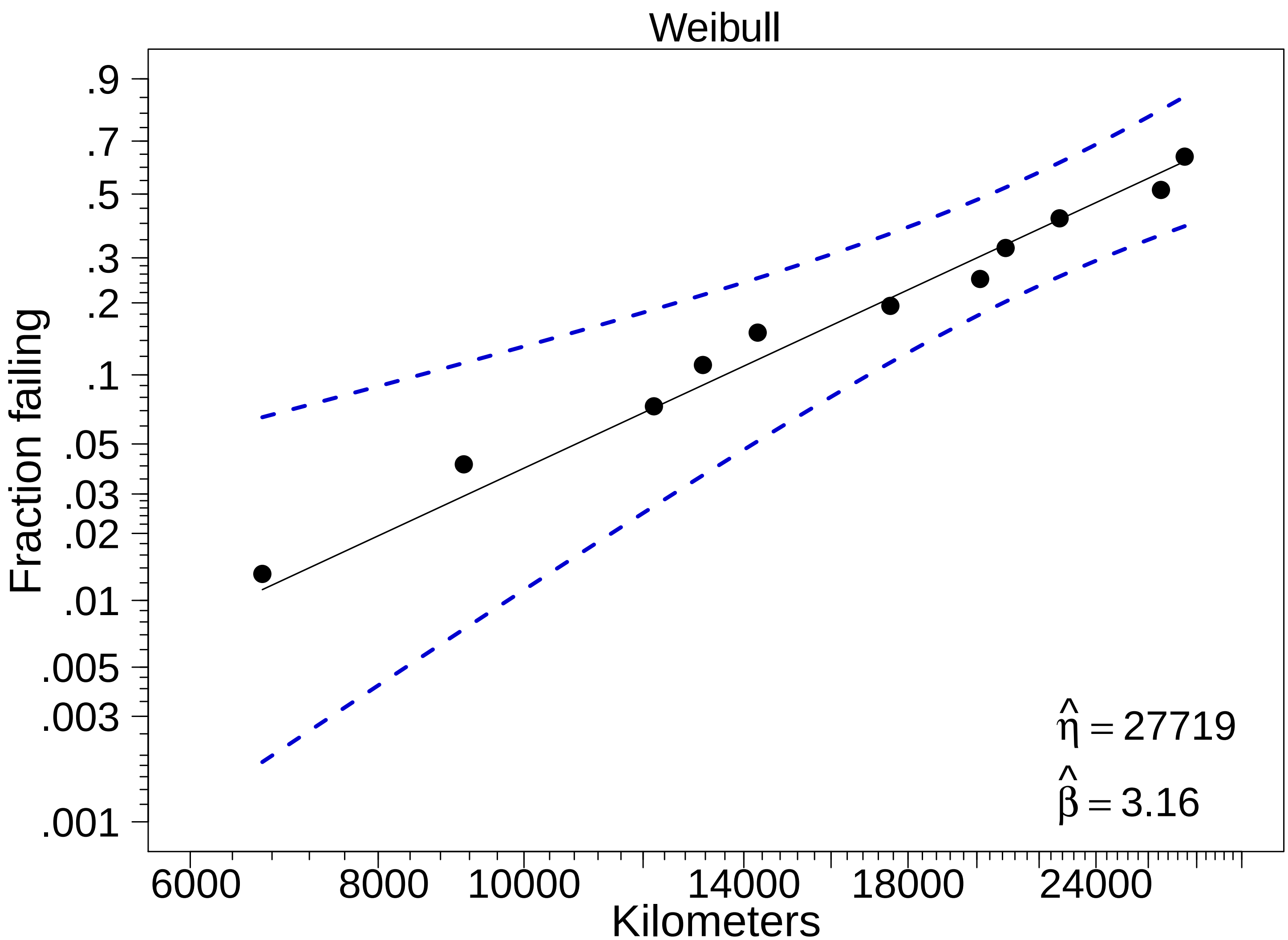


Weibull Relative Likelihood for the Shock Absorber Data ML Estimates: $\widehat{\mu} = 10.23$ and $\widehat{\sigma} = 0.3164$

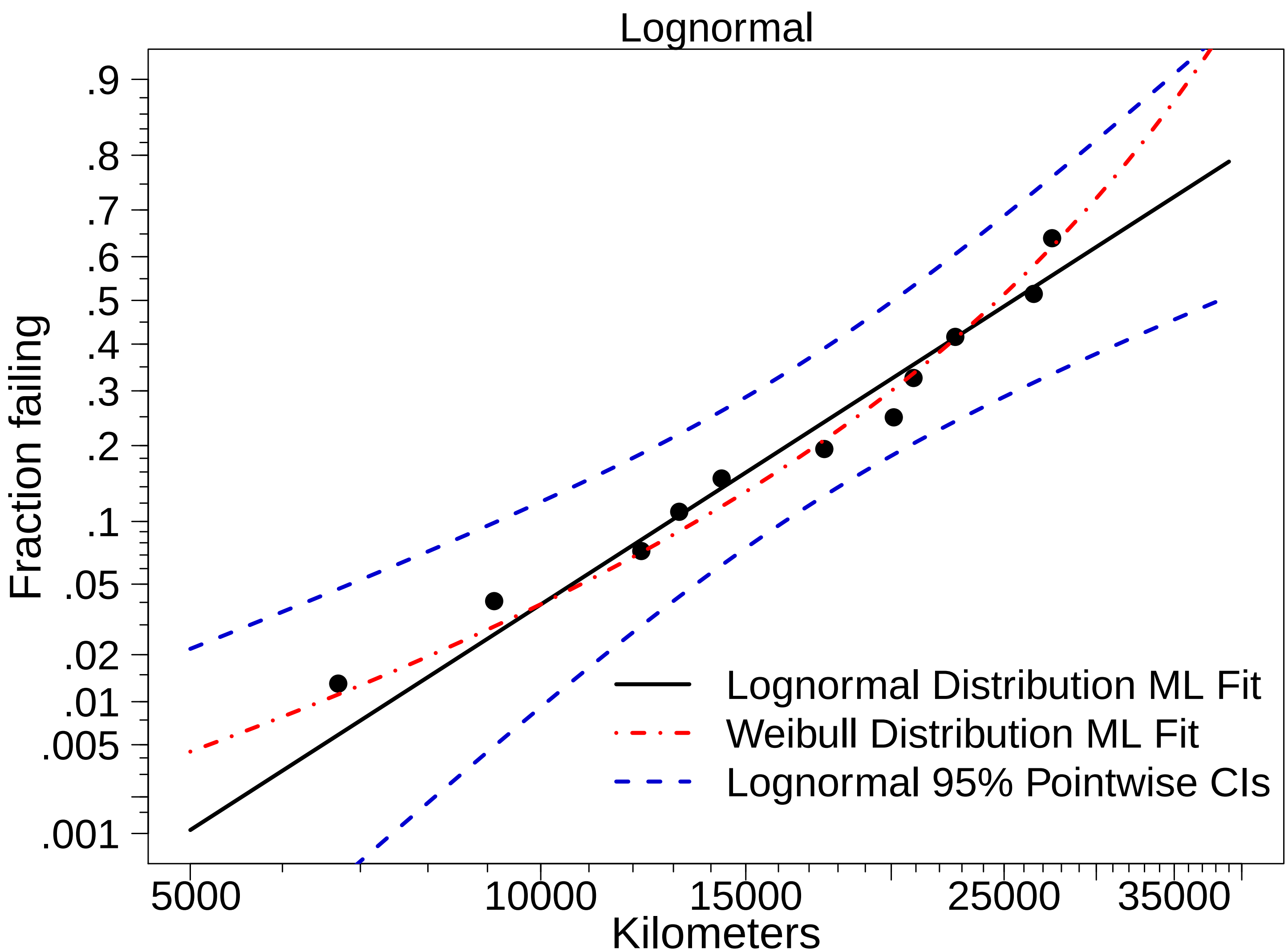
$$R(\mu, \sigma) = L(\mu, \sigma)/L(\widehat{\mu}, \widehat{\sigma})$$



Weibull Probability Plot of Shock Absorber Failure Times (Both Failure Modes) with Maximum Likelihood Estimates and Wald 95% Pointwise Confidence Intervals for $F(t)$



Lognormal Probability Plots of Shock Absorber Data with ML Estimates and Wald 95% Pointwise Confidence Intervals for $F(t)$. The Curved Line Is the Weibull ML Estimate



Chapter 8

Segment 2

Likelihood-Based Confidence Intervals for Log-Location-Scale Distribution Parameters μ and σ

Large-Sample Approximate Theory for Likelihood Ratios for Parameter Vector

- Relative likelihood for (μ, σ) is

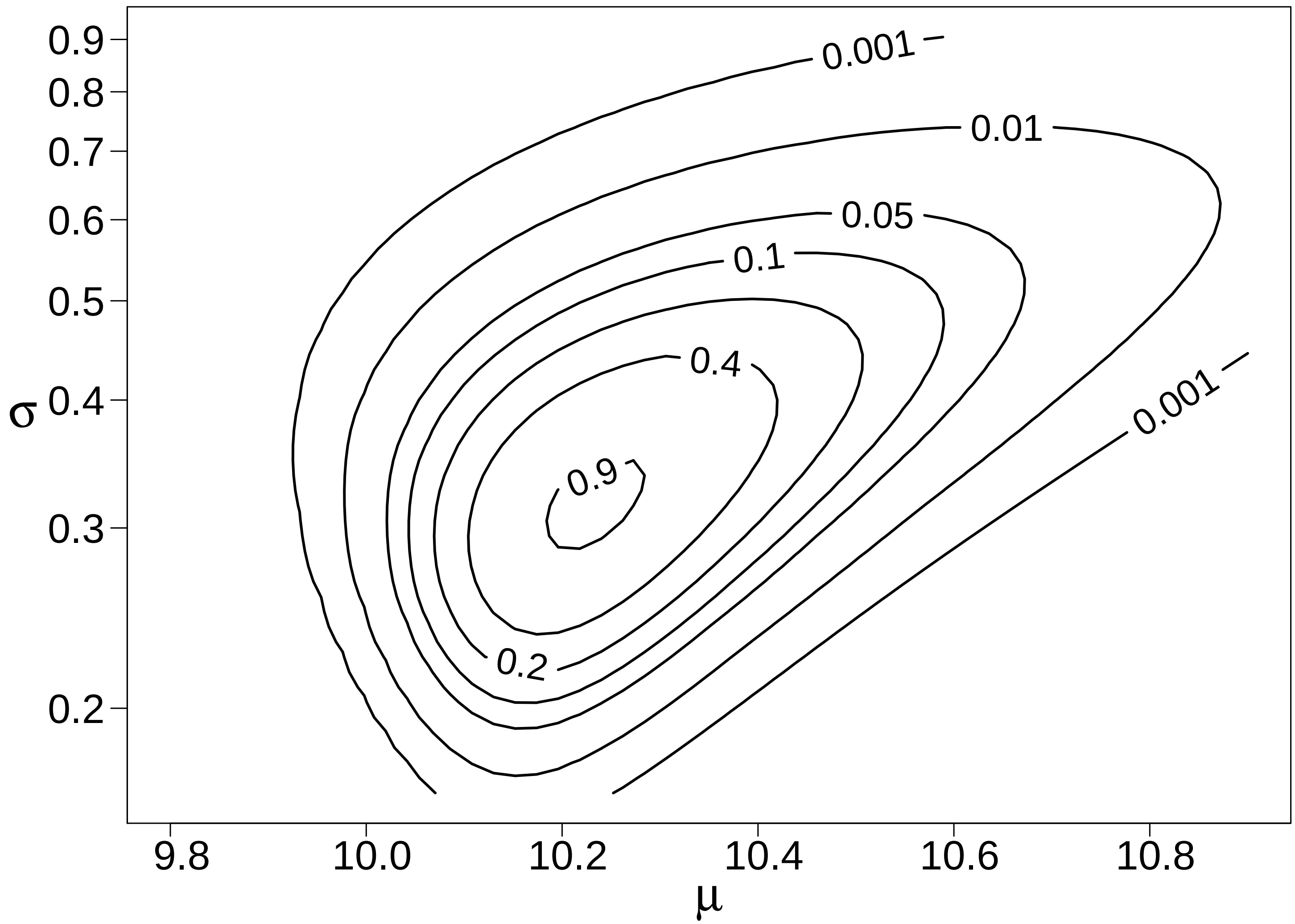
$$R(\mu, \sigma) = \frac{L(\mu, \sigma)}{L(\widehat{\mu}, \widehat{\sigma})}.$$

- If evaluated at the true (μ, σ), then, asymptotically, $-2 \log[R(\mu, \sigma)]$ has a chi-square distribution with 2 degrees of freedom.

- General theory in the Appendix.

Weibull Relative Likelihood for the Shock Absorber Data ML Estimates: $\widehat{\mu} = 10.23$ and $\widehat{\sigma} = 0.3164$

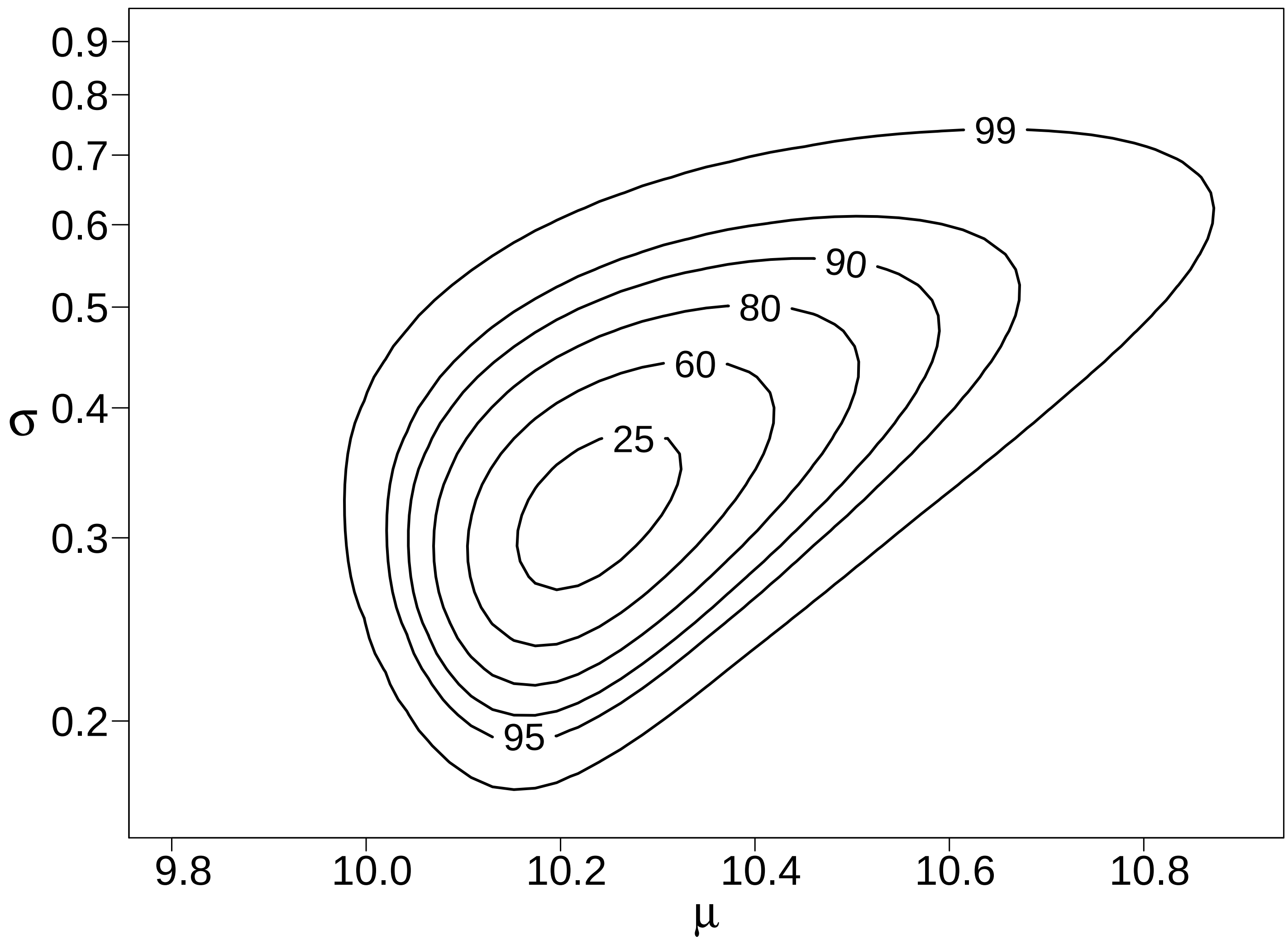
$$R(\mu, \sigma) = L(\mu, \sigma)/L(\widehat{\mu}, \widehat{\sigma})$$



Weibull Likelihood-Based Joint Confidence Regions for μ and σ for the Shock Absorber Data

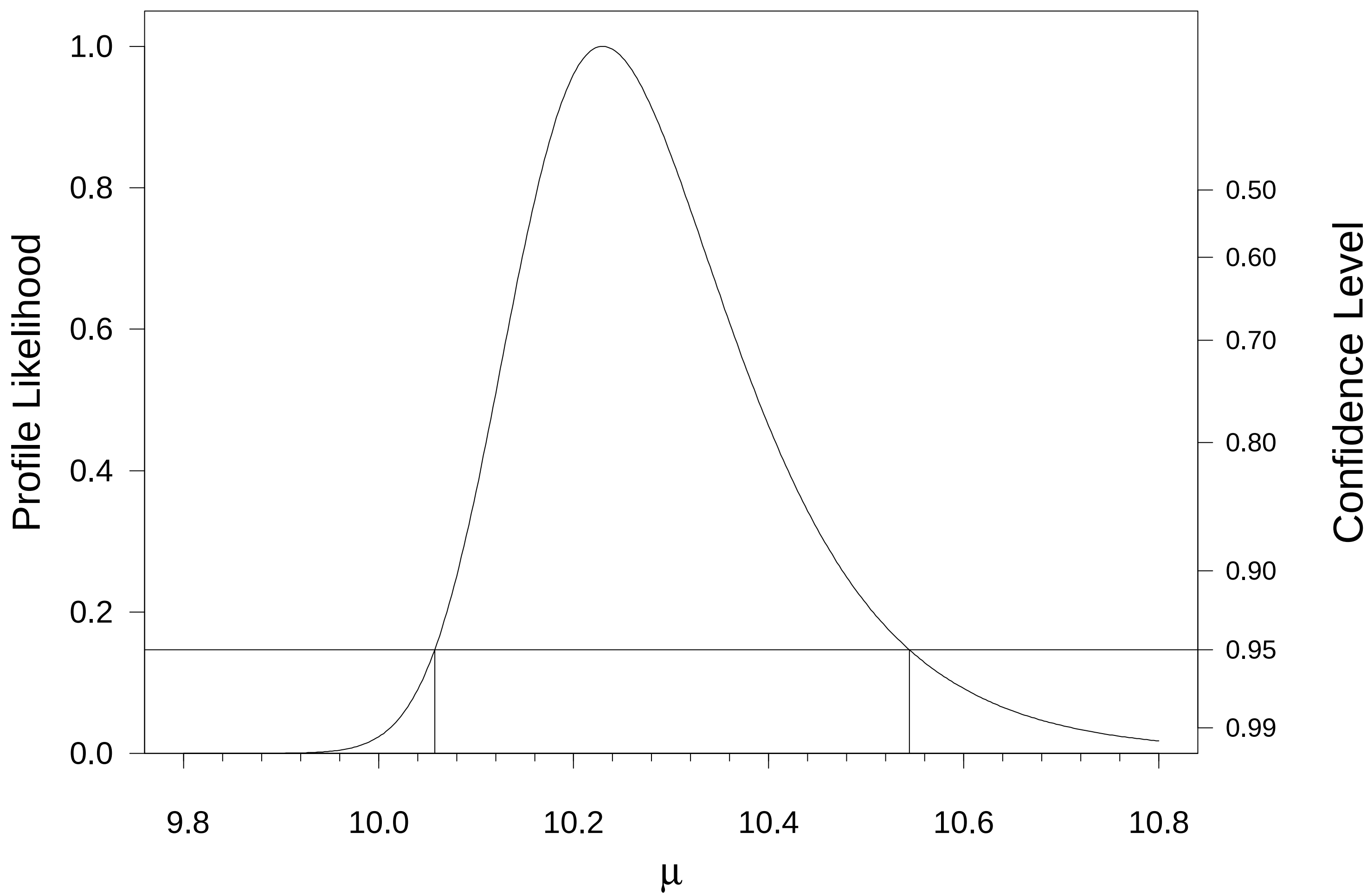
ML Estimates: $\widehat{\mu} = 10.23$ **and** $\widehat{\sigma} = 0.3164$

$100(1-\alpha)\%$ **Region:** $R(\mu, \sigma) > \exp\left[-\chi^2_{(1-\alpha;2)}/2\right] = \alpha$



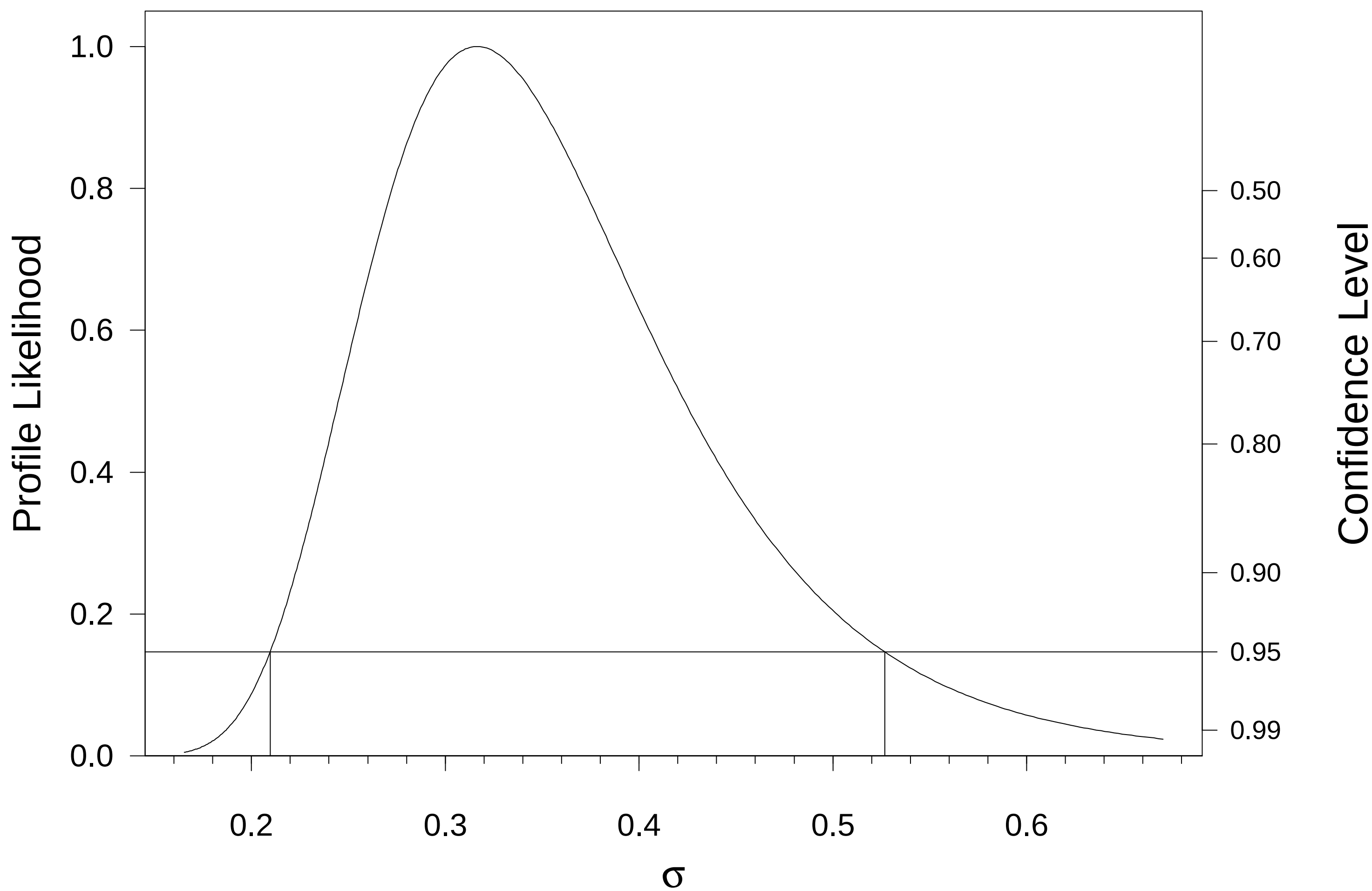
Weibull Profile Likelihood $R(\mu)$ ($\exp(\mu) = \eta \approx t_{0.63}$) for the Shock Absorber Data

$$R(\mu) = \max_{\sigma} \left[\frac{L(\mu,\sigma)}{L(\widehat{\mu},\widehat{\sigma})} \right]$$



Weibull Profile Likelihood $R(\sigma)$ ($\sigma = 1/\beta$) for the Shock Absorber Data

$$R(\sigma) = \max_{\mu} \left[\frac{L(\mu,\sigma)}{L(\widehat{\mu},\widehat{\sigma})}\right]$$



Large-Sample Approximate Theory for Likelihood Ratios for a Parameter Vector Subset

Need: Inferences on subset $\boldsymbol{\theta}_1$, from the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)'$.

- The parameter(s) in $\boldsymbol{\theta}_2$ are known as "nuisance parameters."
- $k_1 = \text{length}(\boldsymbol{\theta}_1)$.
- When $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)' = (\mu, \sigma)'$, profile likelihood for $\boldsymbol{\theta}_1 = \mu$ is
$$R(\mu) = \max_{\sigma} \left[\frac{L(\mu, \sigma)}{L(\widehat{\mu}, \widehat{\sigma})} \right].$$
- If evaluated at the true $\boldsymbol{\theta}_1 = \mu$, then, asymptotically, $-2\log[R(\mu)]$ follows, a chi-square distribution with $k_1 = 1$ degrees of freedom.
- General theory in the Appendix.

Large-Sample Approximate Theory of Likelihood Ratios – Continued

- An approximate $100(1-\alpha)\%$ likelihood-based confidence region for $\boldsymbol{\theta}_1$ is the set of all values of $\boldsymbol{\theta}_1$ such that

$$-2\log[R(\boldsymbol{\theta}_1)] < \chi^2_{(1-\alpha;k_1)}$$

or, equivalently, the set defined by

$$R(\boldsymbol{\theta}_1) > \exp\left[-\chi^2_{(1-\alpha;k_1)}/2\right].$$

- Transformation of $\boldsymbol{\theta}_1$ will not affect the confidence statement.

- Can improve the asymptotic approximation with simulation (only small effect except in very small samples).

Chapter 8

Segment 3

Likelihood-Based Confidence Intervals for Functions of μ and σ

Confidence Regions and Intervals for Functions of μ and σ

- The likelihood approach can be applied to functions of parameters. For monotone functions of a single parameter (e.g., $\beta = 1/\sigma$), the interval translates directly.

- Otherwise, define the function of interest as one of the parameters, replacing one of the original parameters giving one-to-one reparameterization $\boldsymbol{g}(\mu, \sigma) = [g_1(\mu, \sigma), g_2(\mu, \sigma)]$.

- Then use a profile likelihood, as with the original parameters.

- Simple to implement if the function and its inverse are easy to compute.

Reparameterization to Make t_p a Parameter

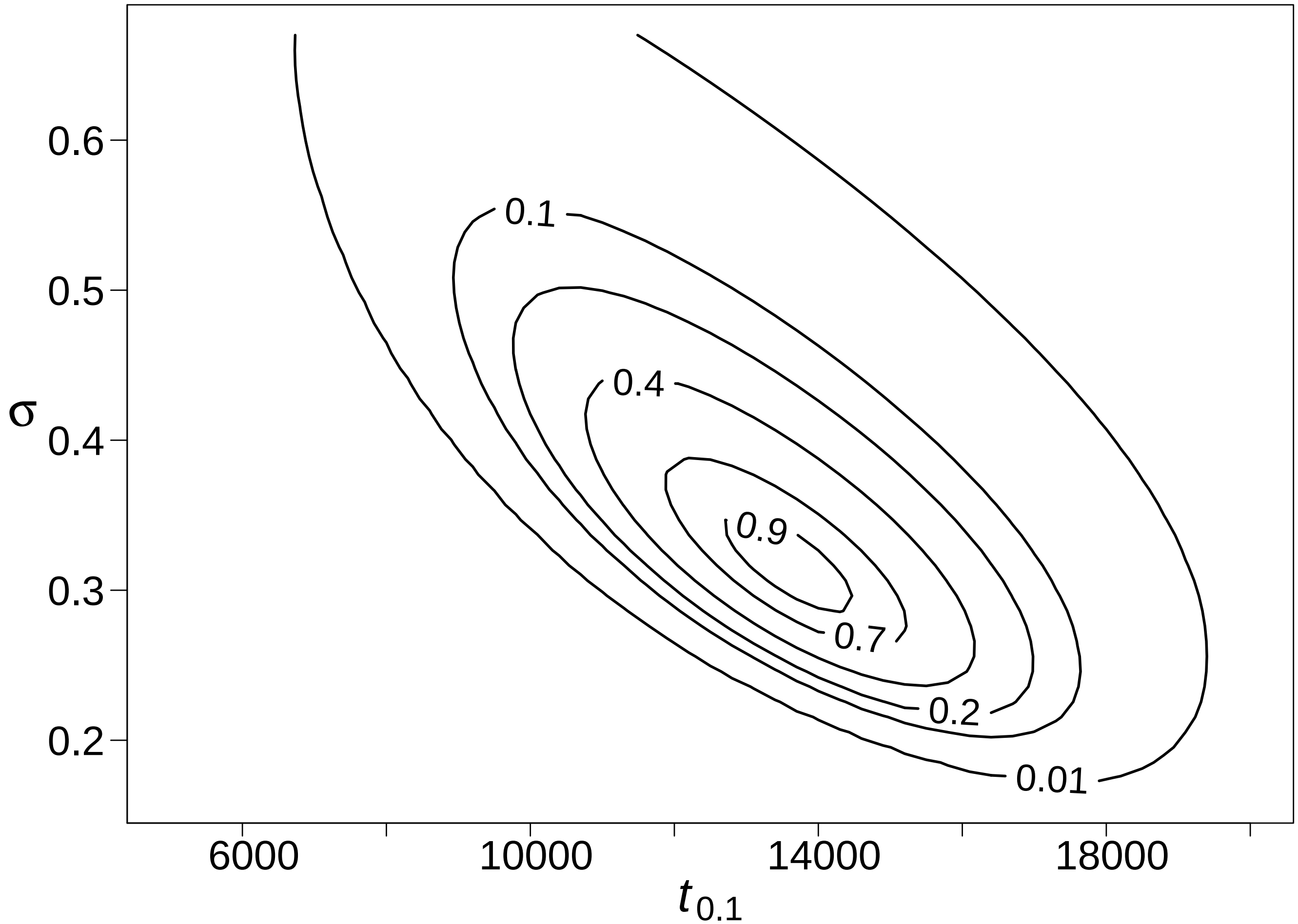
- We want to re-express the likelihood so that $t_p = \exp[\mu + \sigma\Phi^{-1}(p)]$ replaces μ in the likelihood.

- This can be done by substituting $\mu = \log(t_p) - \sigma\Phi^{-1}(p)$ for μ in the (log)-likelihood expression, giving an expression for $L(t_p, \sigma)$.

- A similar reparameterization is possible for writing the likelihood as a function of $F(t_e)$ and σ for a given t_e.

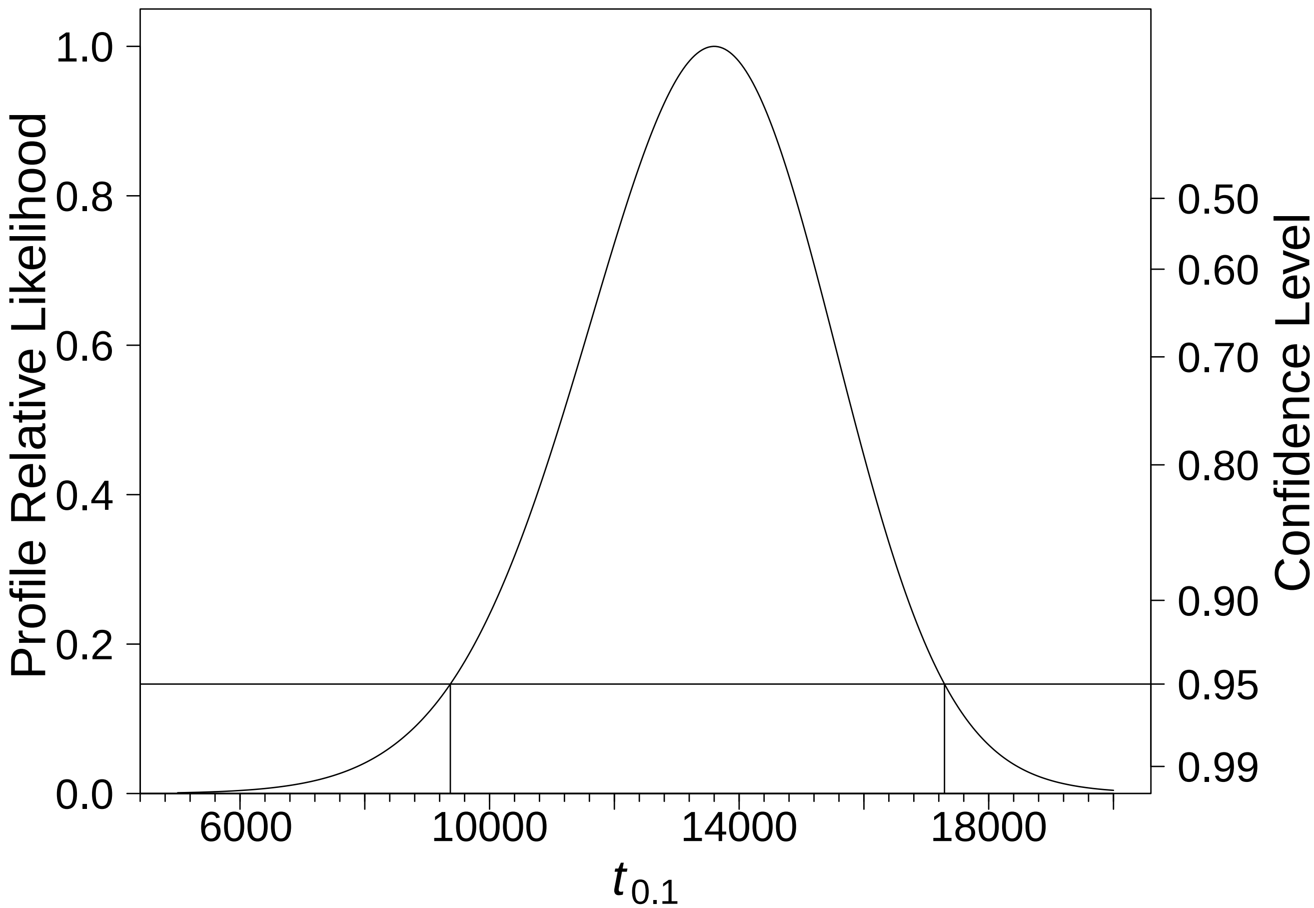
Contour Plot of Weibull Relative Likelihood $R(t_{0.1}, \sigma)$ for the Shock Absorber Data (Parameterized with $t_{0.1}$ and σ)

$$R(t_{0.1}, \sigma) = L(t_{0.1}, \sigma)/L(\widehat{t}_{0.1}, \widehat{\sigma})$$



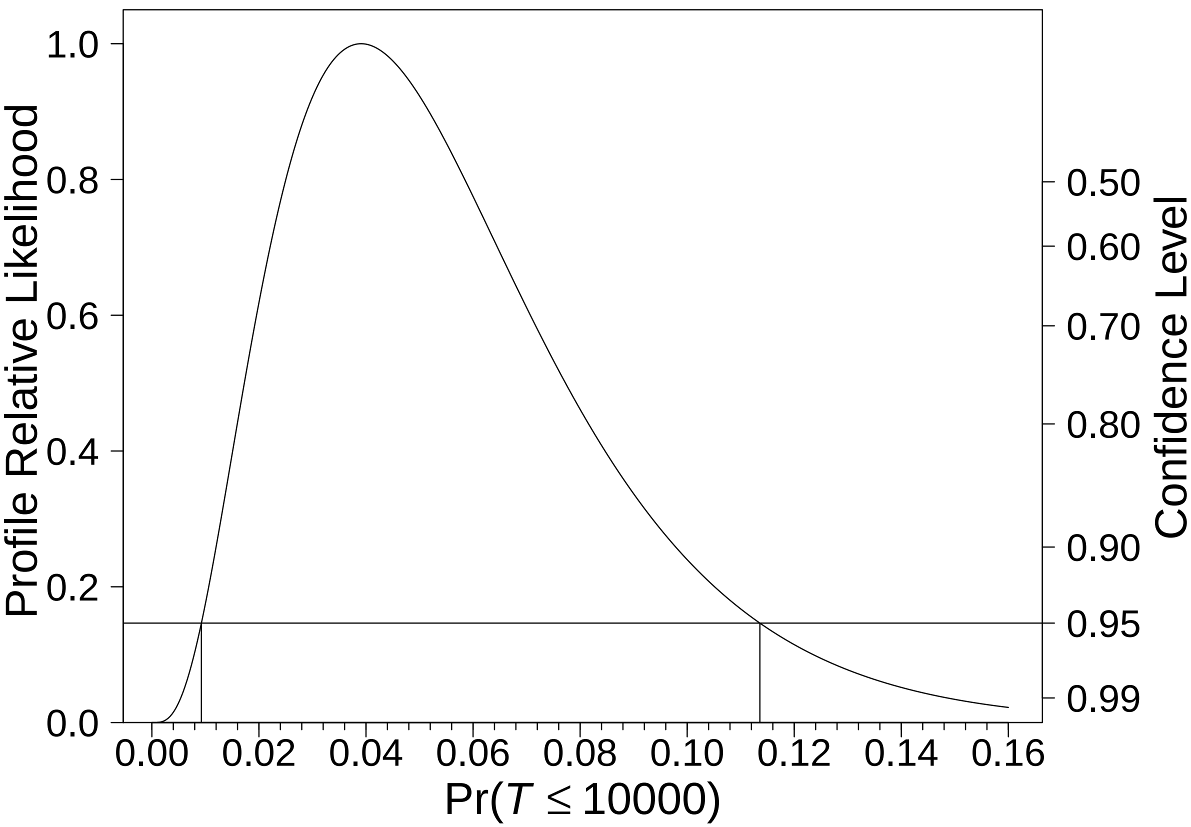
Weibull Profile Likelihood $R(t_{0.1})$ for the Shock Absorber Data

$$R(t_{0.1}) = \max_{\sigma} \left[\frac{L(t_{0.1}, \sigma)}{L(\widehat{t}_{0.1}, \widehat{\sigma})} \right]$$



Weibull Profile Likelihood $R[F(10000)]$ for the Shock Absorber Data

$$R[F(10000)] = \max_{\sigma} \left\{ \frac{L[F(10000), \sigma]}{L\left[\widehat{F}(10000), \widehat{\sigma}\right]} \right\}$$



Chapter 8

Segment 4

Wald Approximate Confidence Intervals for Log-Location-Scale Distribution Parameters μ and σ and Functions of μ and σ

Large-Sample Approximation Theory of ML Estimation

Let $\widehat{\boldsymbol{\theta}}$ denote the ML estimator of $\boldsymbol{\theta}$.

- If evaluated at the true value of $\boldsymbol{\theta}$, then asymptotically, (large samples) $\widehat{\boldsymbol{\theta}}$ has a MVN$(\boldsymbol{\theta}, \Sigma_{\widehat{\boldsymbol{\theta}}})$ and thus the **<u>Wald</u>** statistic

$$(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \Big[\Sigma_{\widehat{\boldsymbol{\theta}}}\Big]^{-1} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$$

  has a chi-square distribution with k degrees of freedom, where k is the length of $\boldsymbol{\theta}$.

- Here, $\Sigma_{\widehat{\boldsymbol{\theta}}} = I_{\boldsymbol{\theta}}^{-1}$ is the large-sample approximate covariance matrix where the Fisher information matrix for $\boldsymbol{\theta}$ is

$$I_{\boldsymbol{\theta}} = \mathsf{E}\left[- \frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right].$$

- For (log)-location-scale distributions, $\boldsymbol{\theta} = (\mu, \sigma)'$.

Large-Sample Approximation Theory for Wald's Statistic

- Alternative asymptotic theory is based on the large-sample distribution of quadratic forms (Wald's statistic).

- Let $\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}}$ be a consistent estimator of $\Sigma_{\widehat{\boldsymbol{\theta}}}$, the asymptotic covariance matrix of $\widehat{\boldsymbol{\theta}}$. For example,

$$\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}} = \left[-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right]^{-1}$$

  where the derivatives are evaluated at $\widehat{\boldsymbol{\theta}}$.

- Asymptotically, the Wald statistic

$$w(\boldsymbol{\theta}) = (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \left[\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}} \right]^{-1} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$$

  when evaluated at the true $\boldsymbol{\theta}$, follows a chi-square distribution with k degrees of freedom, where k is the length of $\boldsymbol{\theta}$.

Large-Sample Approximation Theory for Wald's Statistic – Continued

- A Wald approximate $100(1-\alpha)\%$ confidence region for $\boldsymbol{\theta}$ is the set of all values of $\boldsymbol{\theta}$ in the ellipsoid

$$(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \left[\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}}\right]^{-1} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \chi^2_{(1-\alpha;k)}.$$

- This is sometimes known as the normal-approximation confidence region.

- Can specialize to functions or subsets of $\boldsymbol{\theta}$.

- Wald confidence intervals are not transformation invariant. Thus there are multiple ways to compute a Wald interval.

- Can try to find a transformation that results in a log-likelihood with approximate quadratic shape.

Wald Confidence Intervals for μ and σ

- Estimated variance matrix for the shock absorber data

$$\widehat{\Sigma}_{\widehat{\mu},\widehat{\sigma}} = \begin{bmatrix} \widehat{\mathrm{Var}}(\widehat{\mu}) & \widehat{\mathrm{Cov}}(\widehat{\mu},\widehat{\sigma}) \\ \widehat{\mathrm{Cov}}(\widehat{\mu},\widehat{\sigma}) & \widehat{\mathrm{Var}}(\widehat{\sigma}) \end{bmatrix} = \begin{bmatrix} 0.01208 & 0.00399 \\ 0.00399 & 0.00535 \end{bmatrix}$$

- Assuming that $Z_{\widehat{\mu}} = (\widehat{\mu} - \mu)/\mathrm{se}_{\widehat{\mu}} \dot{\sim} \mathrm{NORM}(0,1)$ distribution, an approximate $100(1-\alpha)\%$ confidence interval for μ is

$$[\underset{\sim}{\mu}, \quad \widetilde{\mu}] = \widehat{\mu} \mp z_{(1-\alpha/2)}\mathrm{se}_{\widehat{\mu}}$$

where $\mathrm{se}_{\widehat{\mu}} = \sqrt{\widehat{\mathrm{Var}}(\widehat{\mu})}$.

- Assuming that $Z_{\log(\widehat{\sigma})} = [\log(\widehat{\sigma}) - \log(\sigma)]/\mathrm{se}_{\log(\widehat{\sigma})} \dot{\sim} \mathrm{NORM}(0,1)$ an approximate $100(1-\alpha)\%$ confidence interval for σ is

$$[\underset{\sim}{\sigma}, \quad \widetilde{\sigma}] = [\widehat{\sigma}/w, \quad \widehat{\sigma} \times w]$$

where $w = \exp\left[z_{(1-\alpha/2)}\mathrm{se}_{\widehat{\sigma}}/\widehat{\sigma}\right]$ and $\mathrm{se}_{\widehat{\sigma}} = \sqrt{\widehat{\mathrm{Var}}(\widehat{\sigma})}$.

Wald Confidence Intervals for a Function $g_1 = g_1(\mu, \sigma)$

- ML estimate $\widehat{g}_1 = g_1(\widehat{\mu}, \widehat{\sigma})$.
- Assuming $Z_{\widehat{g}_1} = (\widehat{g}_1 - g_1)/\text{se}_{\widehat{g}_1} \stackrel{.}{\sim} \text{NORM}(0, 1)$, an approximate $100(1-\alpha)\%$ confidence interval for g_1 is

$$[\underset{\sim}{g}_1, \quad \widetilde{g}_1] = \widehat{g}_1 \mp z_{(1-\alpha/2)}\text{se}_{\widehat{g}_1},$$

where

$$\begin{aligned} \text{se}_{\widehat{g}_1} &= \sqrt{\widehat{\text{Var}}(\widehat{g}_1)} \\ &= \left[\left(\frac{\partial g_1}{\partial \mu}\right)^2 \widehat{\text{Var}}(\widehat{\mu}) + \left(\frac{\partial g_1}{\partial \sigma}\right)^2 \widehat{\text{Var}}(\widehat{\sigma}) + 2\left(\frac{\partial g_1}{\partial \mu}\right)\left(\frac{\partial g_1}{\partial \sigma}\right) \widehat{\text{Cov}}(\widehat{\mu}, \widehat{\sigma})\right]^{1/2} \end{aligned}$$

- Partial derivatives evaluated at $\widehat{\mu}, \widehat{\sigma}$.
- General theory in the Appendix.

Wald Confidence Interval for $F(t_e; \mu, \sigma)$

Objective: Obtain a point estimate and a confidence interval for $\Pr(T \le t_e) = F(t_e; \mu, \sigma)$ at a given point t_e.

- The ML estimates $\widehat{\boldsymbol{\theta}} = (\widehat{\mu}, \widehat{\sigma})$ and $\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}}$ are available.
- The ML estimate for $F(t_e; \mu, \sigma)$ is

$$\widehat{F} = F(t_e; \widehat{\mu}, \widehat{\sigma}) = \Phi(\widehat{z}_e)$$

  where $\widehat{z}_e = [\log(t_e) - \widehat{\mu}]/\widehat{\sigma}$.

- There are many ways to obtain a Wald confidence interval for $F(t_e; \mu, \sigma)$.

Wald Confidence Interval for $F(t_e;\mu,\sigma)$–Continued

Note: Wald confidence intervals depend on the parameterization used to derive the intervals.

For example, an approximate $100(1-\alpha)\%$ confidence interval for $F(t_e;\mu,\sigma)$ can be obtained using:

- The asymptotic normality of $Z_{\widehat{F}} = (\widehat{F} - F)/\text{se}_{\widehat{F}}$

$$[\underset{\sim}{F}, \quad \widetilde{F}] = \widehat{F}(t_e) \mp z_{(1-\alpha/2)}\text{se}_{\widehat{F}}.$$

- The asymptotic normality of $\widehat{z}_e = [\log(t_e) - \widehat{\mu}]/\widehat{\sigma}$

$$[\underset{\sim}{z_e}, \ \widetilde{z_e}] = \widehat{z}_e \mp z_{(1-\alpha/2)}\text{se}_{\widehat{z}_e}.$$

  Then

$$[\underset{\sim}{F}(t_e), \ \widetilde{F}(t_e)] = [\Phi(\underset{\sim}{z_e}), \ \Phi(\widetilde{z_e})].$$

- Expressions for $\text{se}_{\widehat{F}}$ and $\text{se}_{\widehat{z}_e}$ are obtained by using the delta method.

Wald Confidence Interval for $F(t_e; \mu, \sigma)$—Continued

Comments:

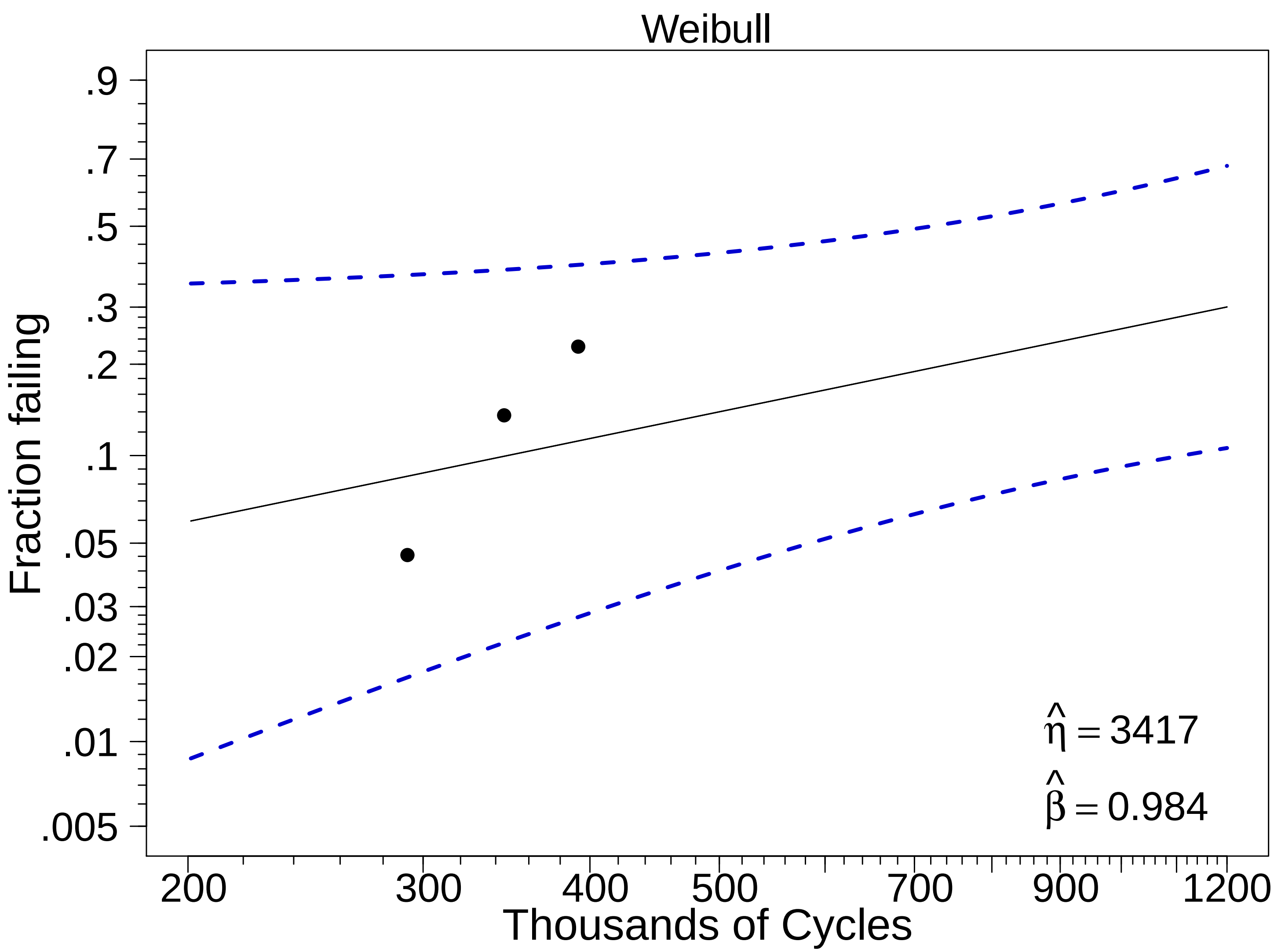
- The confidence interval procedure based on the asymptotic normality of $Z_{\widehat{F}}$ has poor statistical properties because $Z_{\widehat{F}}$ converges slowly toward normality.

- The confidence interval procedure based on $\widehat{z}_e$ has better statistical properties because $\widehat{z}_e$ converges to normality faster than $Z_{\widehat{F}}$.

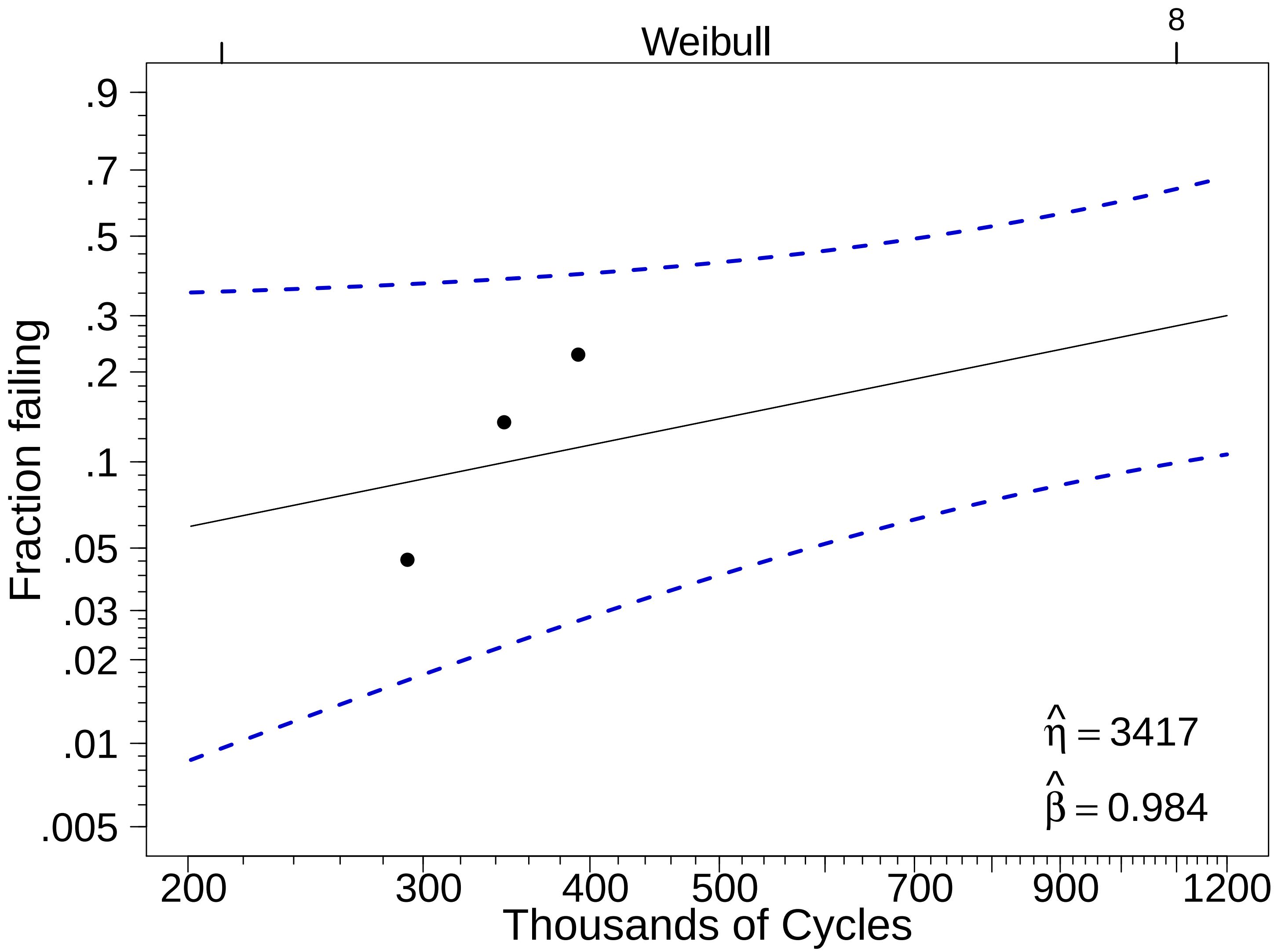
Example: Bearing-A Life Test Results

- Continuous-run test for a newly-designed bearing.
- Sample of 12 units put on test; one early removal; 3 failures.
- Test terminated at 1100 thousand cycles.
- What is the failure-time distribution of the bearing?

Bearing-A Weibull Probability Plot and ML Estimate



Bearing-A Weibull Probability Plot and ML Estimate



Bearing-A Life Test Example Conclusions

- Sometimes the ML estimate does not go through the points on a Weibull (or other) probability plot.

- When the ML estimate does not go through the points, it is an indication that the Weibull distribution does not agree with the data.

- It is important to find the reason that the line does not fit.

Chapter 8

Segment 5

Weibull Distribution Inference with Few Failures

Weibull Inference with Few Failures

- Suppose that β is given. Knowledge of the failure mechanism will often provide information about β.

- Simplifies problem. Only one parameter with r failures and $t_1, \ldots, t_n$ failures and censor times

$$\widehat{\eta} = \left(\frac{\sum_{i=1}^{n} t_i^{\beta}}{r} \right)^{1/\beta}, \quad \text{se}_{\widehat{\eta}} = \frac{\widehat{\eta}}{\beta} \sqrt{\frac{1}{r}}.$$

- Provides much more precision, especially with small r.

- Requires **sensitivity analysis** because β is unknown.

Bearing-Cage Fracture Field Data

- Data from the Weibull Handbook Abernethy et al. (1983).

- $n = 1703$ units had been introduced into service over time; oldest unit had 2220 hours of operation.

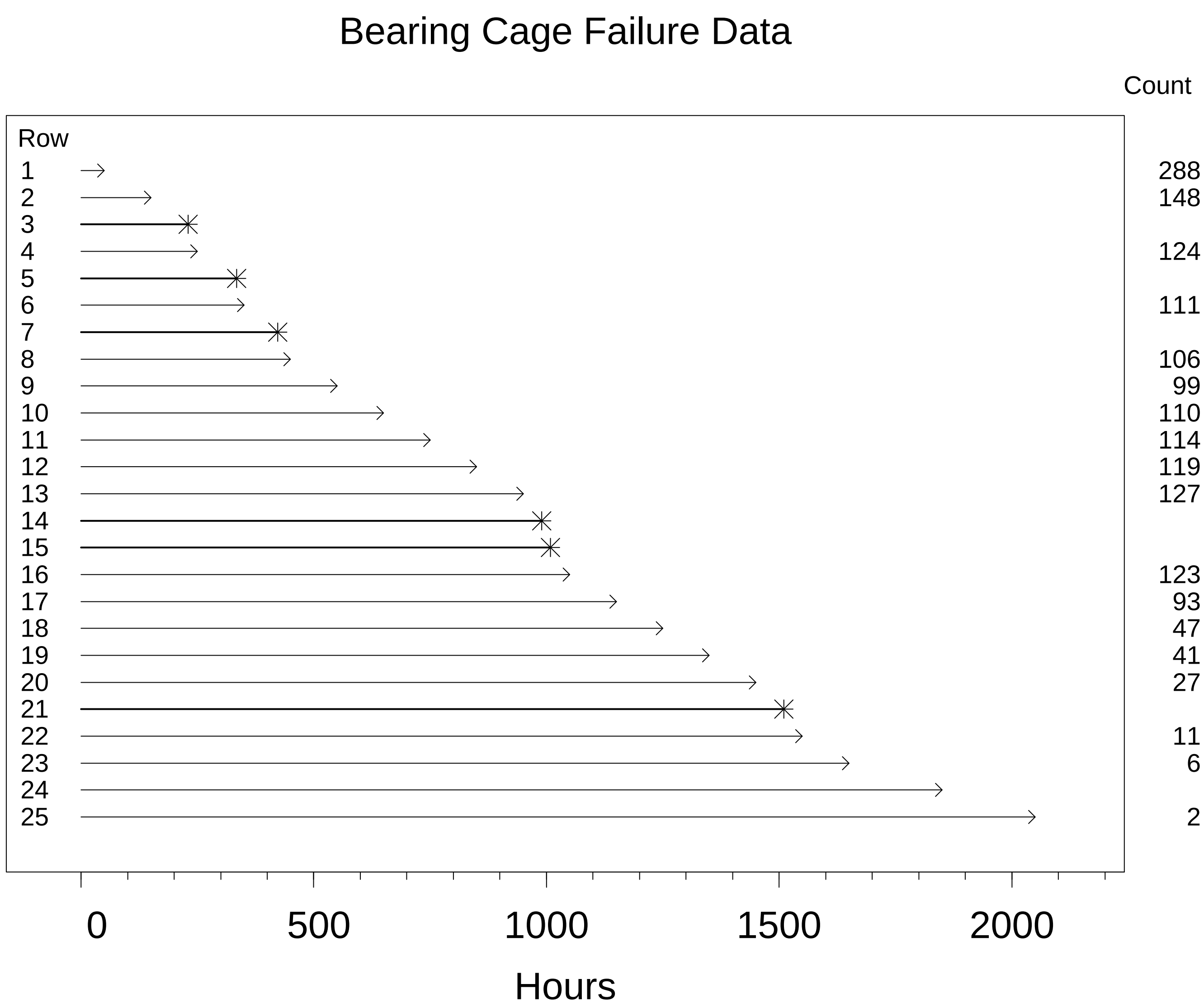
- 6 units had failed.

- Design life specification was B10 $= t_{0.1} = 8000$ hours.

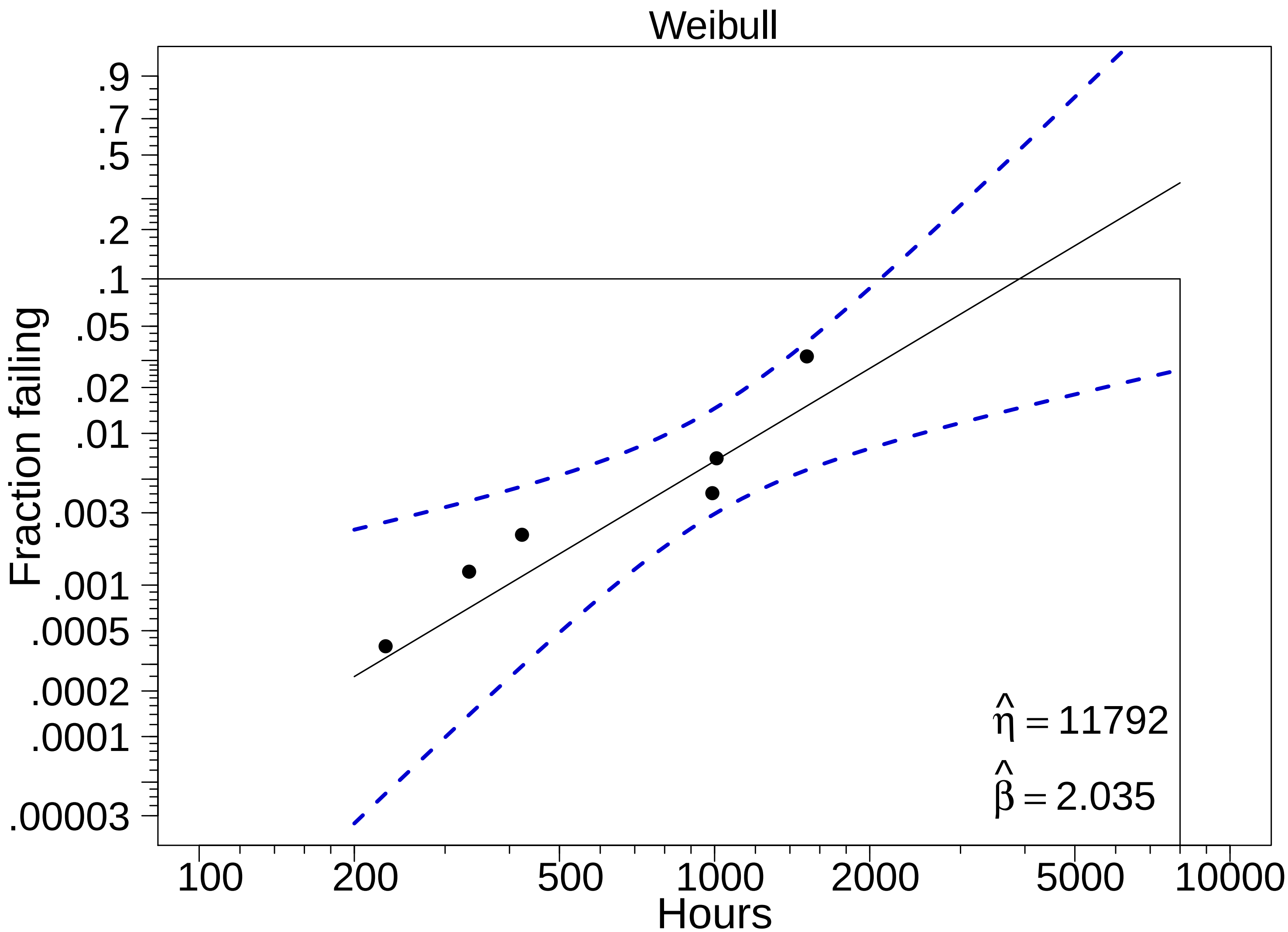
- ML estimate is $\widehat{t}_{0.1} = 3.903$ thousand hours. Does this indicate a problem?

- How many replacement parts will be needed?

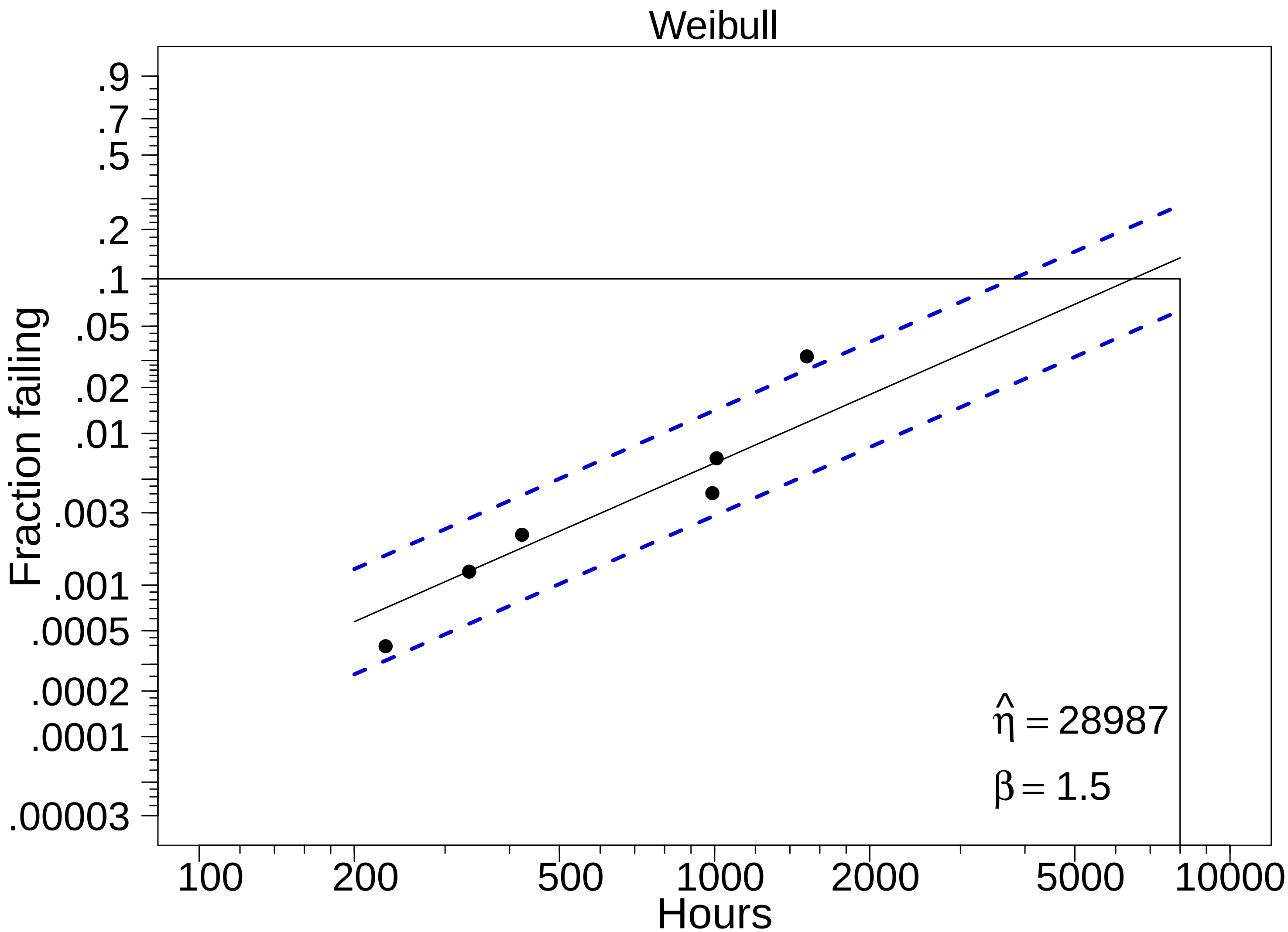
Bearing-Cage Fracture Data Event Plot



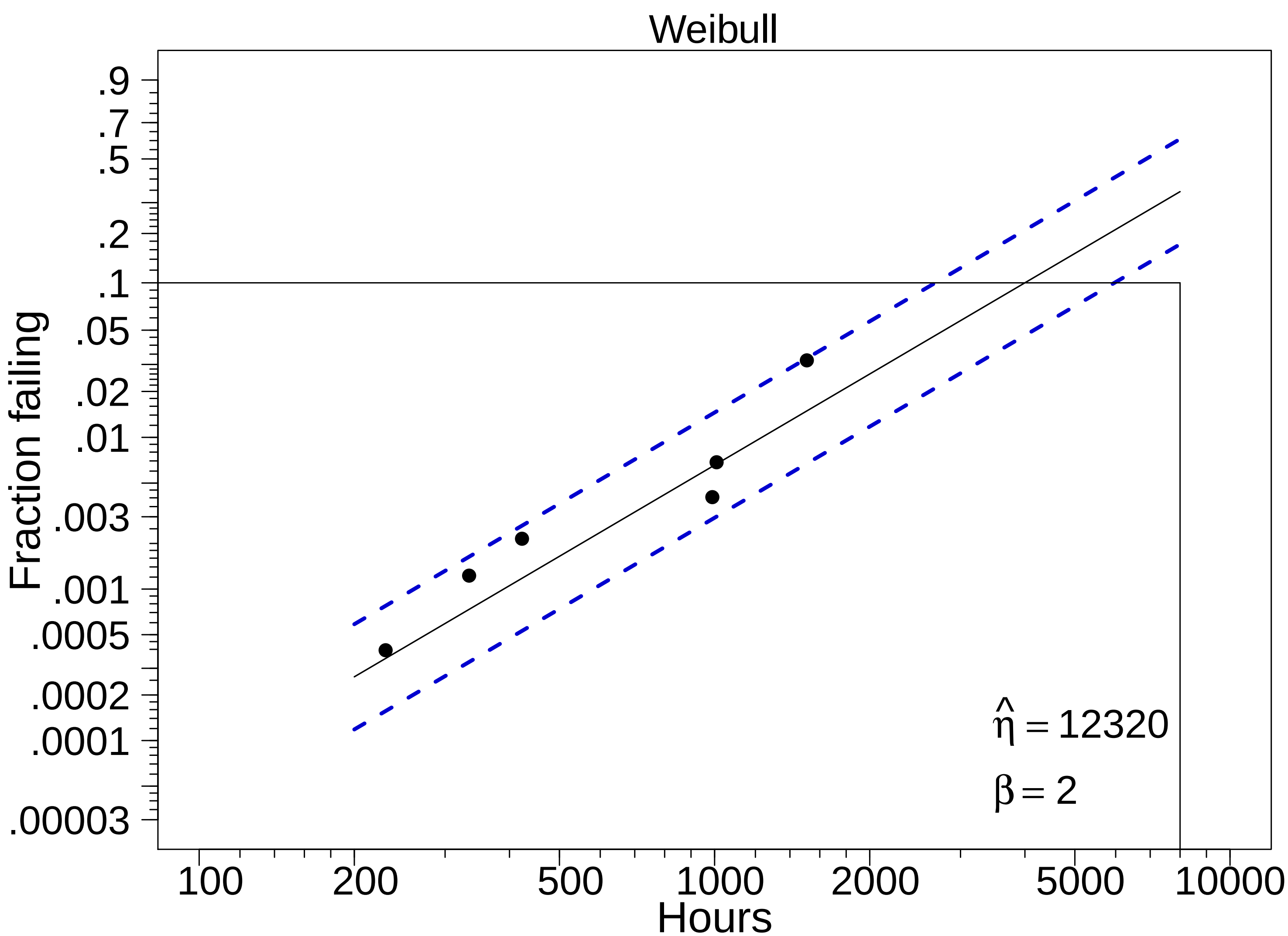
Weibull Probability Plots Bearing-Cage Fracture Data with Weibull ML Estimates and Sets of 95% Pointwise Confidence Intervals for $F(t)$ with β Estimated



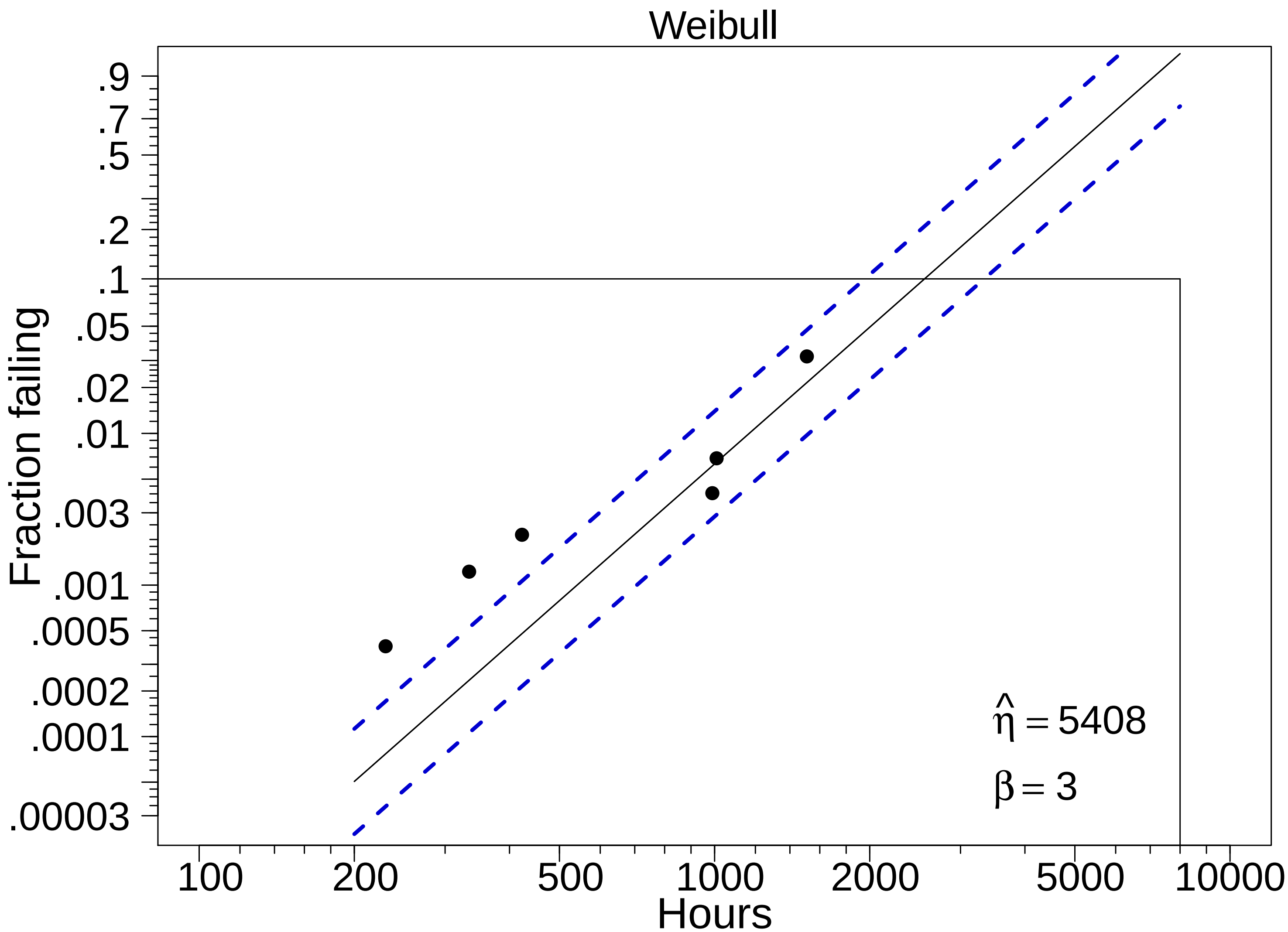
Weibull Probability Plots Bearing-Cage Fracture Data with Weibull ML Estimates and Sets of 95% Pointwise Confidence Intervals for $F(t)$ with Given $\beta = 1.5$



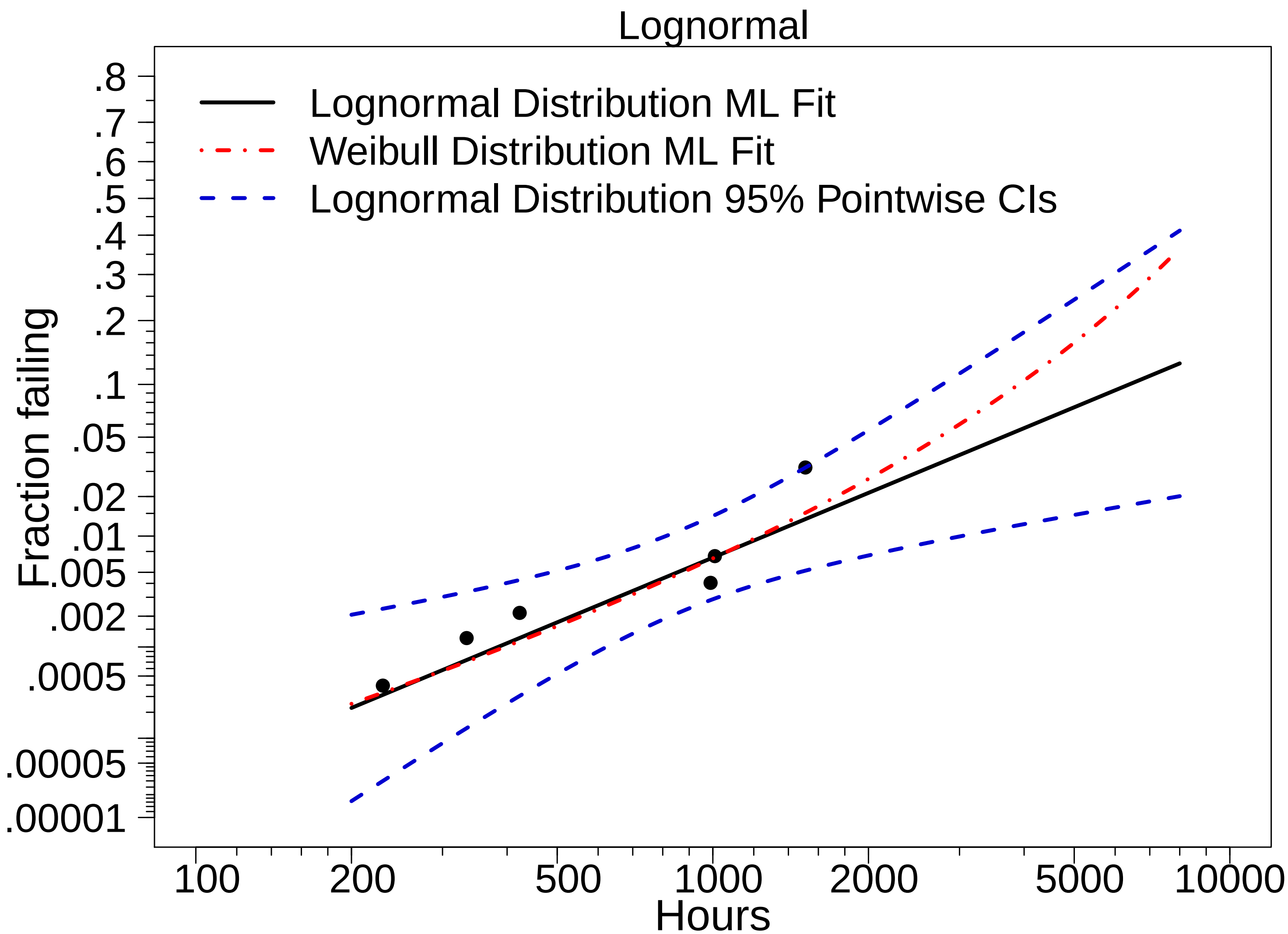
Weibull Probability Plots Bearing-Cage Fracture Data with Weibull ML Estimates and Sets of 95% Pointwise Confidence Intervals for $F(t)$ with Given $\beta = 2$



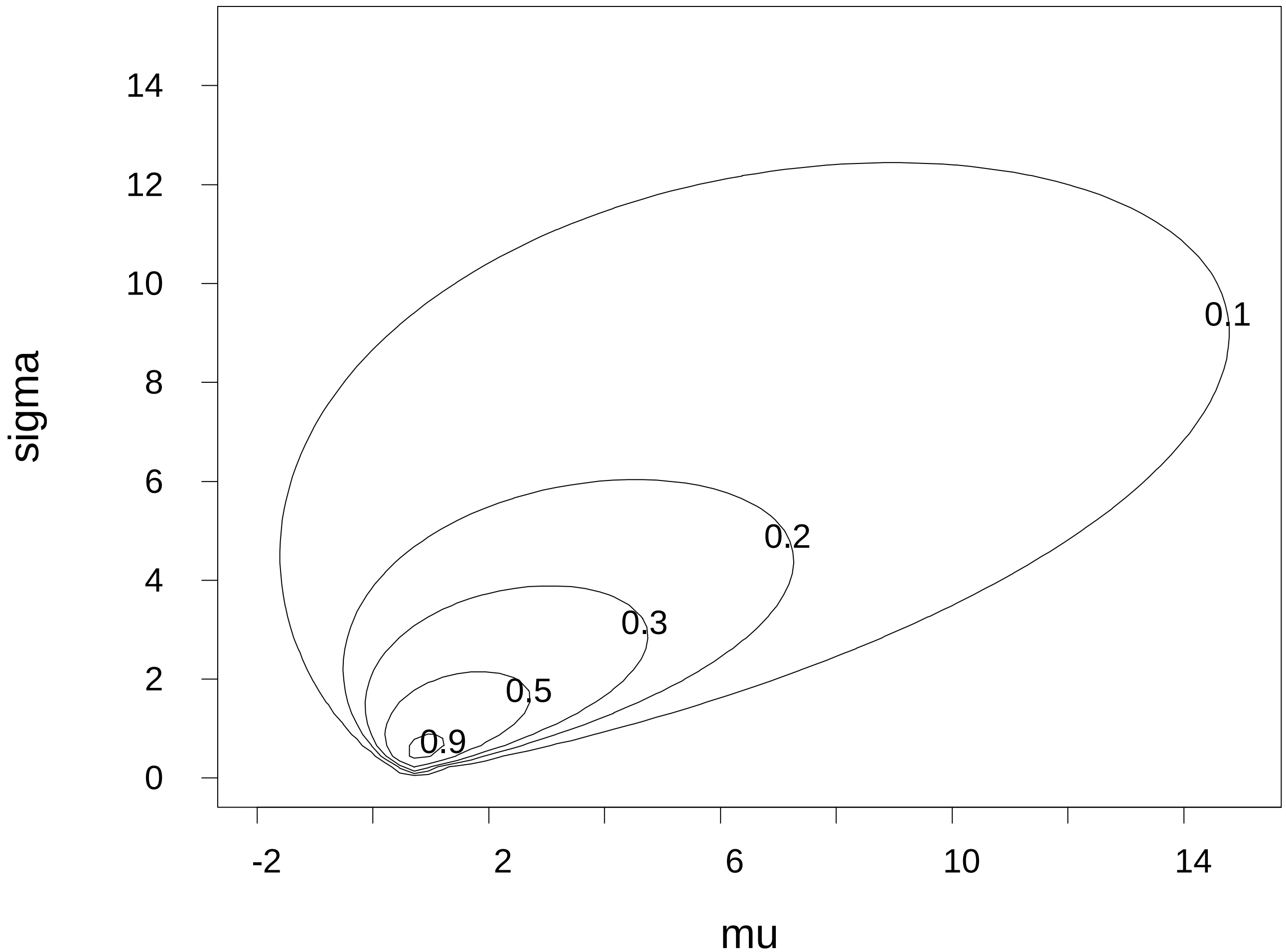
Weibull Probability Plots Bearing-Cage Fracture Data with Weibull ML Estimates and Sets of 95% Pointwise Confidence Intervals for $F(t)$ with Given $\beta = 3$



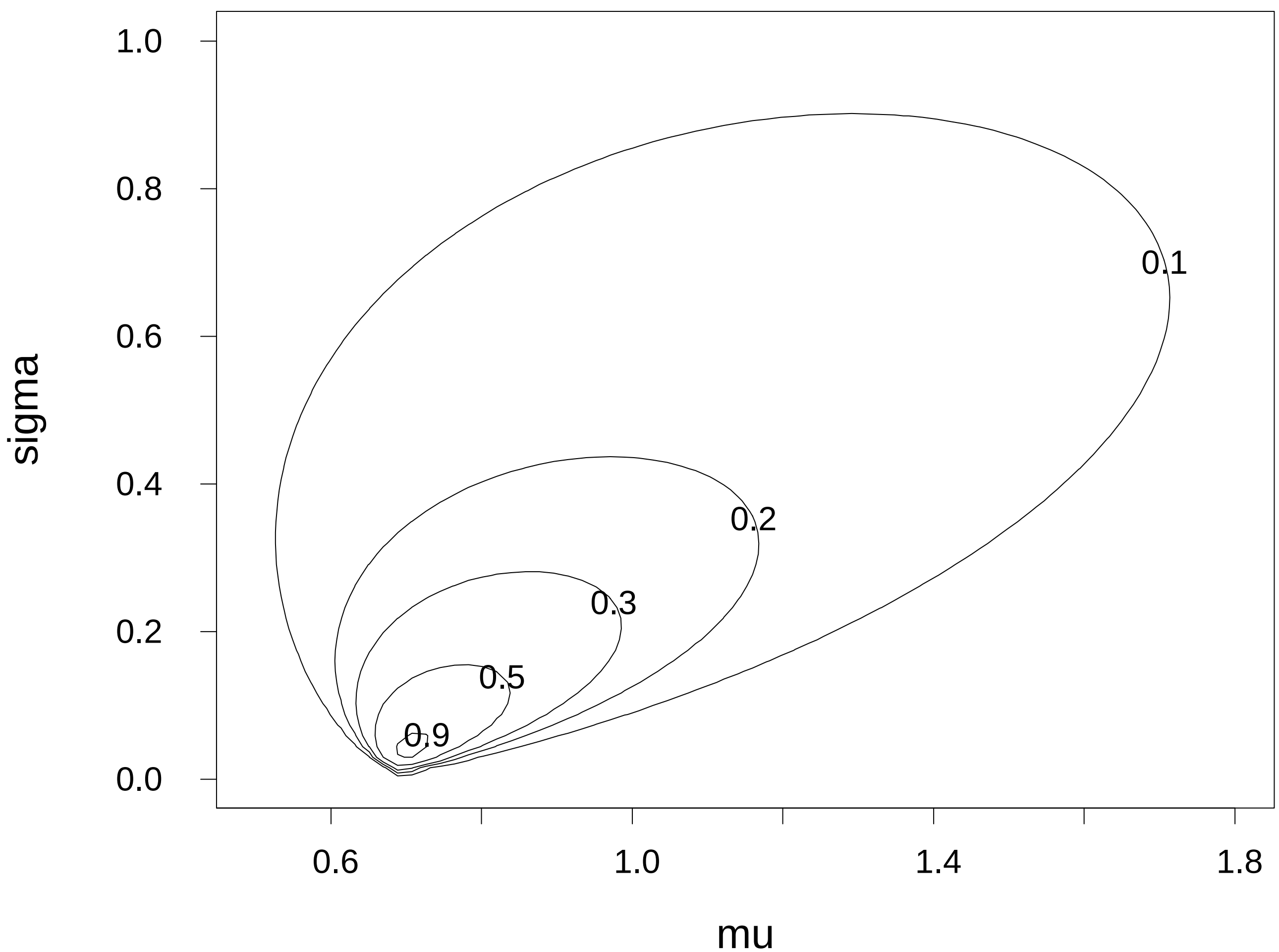
Lognormal and Weibull Comparison
Bearing-Cage Fracture Field Data
Lognormal Probability Paper



Relative Weibull Likelihood
with One Failure at 1 and One Survivor at 2



Relative Weibull Likelihood with One Failure at 1.9 and One Survivor at 2



Chapter 8

Segment 6

Weibull Distribution Inference with Zero Failures

Weibull Distribution with Given β and Zero Failures

- ML estimate for the Weibull scale parameter η cannot be computed unless the available data contains one or more failures.

- For a sample of n units with running times $t_1, \ldots, t_n$ and no failures, a conservative $100(1-\alpha)\%$ lower confidence bound for η is

$$\underset{\sim}{\eta} = \left(\frac{2 \sum_{i=1}^{n} t_i^{\beta}}{\chi^2_{(1-\alpha;2)}} \right)^{\frac{1}{\beta}}.$$

- The lower bound $\underset{\sim}{\eta}$ can be translated into a lower confidence bound for functions like t_p for specified p or an upper confidence bound for $F(t_e)$ for a specified t_e.

Component A Safe Data

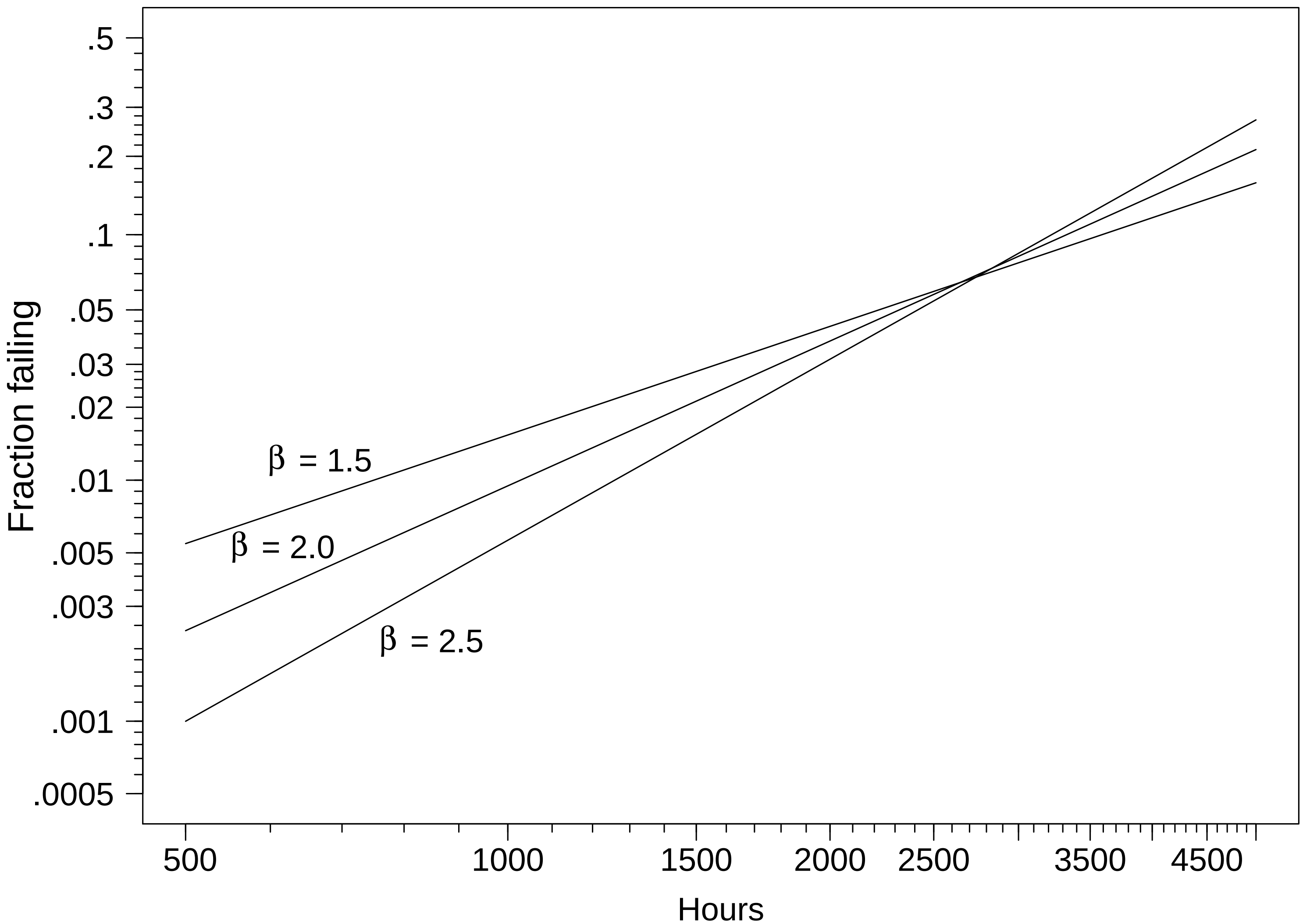
- A metal component in a ship's propulsion system fails from fatigue-caused fracture.
- Because of persistent reliability problems, the component was redesigned to have a longer service life.
- Previous experience suggests that the Weibull shape parameter is near $\beta = 2$, and almost certainly between 1.5 and 2.5.
- Many copies of a newly designed component were put into service during the past year and no failures have been reported.

Hours:	500	1000	1500	2000	2500	3000	3500	4000
Number of Units:	10	12	8	9	7	9	6	3

Staggered entry data, with no reported failures.

- Can the replacement time be safely increased from 2000 hours to 4000 hours?

Weibull Model 95% Upper Confidence Bounds on $F(t)$ for Component-A with Different Fixed Values for the Weibull Shape Parameter



Chapter 8

Segment 7

Regularity Conditions and Other Topics

Regularity Conditions

- Each technical result (e.g., the asymptotic distribution of an estimator) has its own set of conditions on the model (see Lehmann 1983, Rao 1973).
- Frequent reference to Regularity Conditions which give rise to simple results.
- For special cases the regularity conditions are easy to state and check. For example, for some location-scale distributions the needed conditions are:

$$\lim_{z \to -\infty} \frac{z^2 \phi^2(z)}{\Phi(z)} = 0$$
$$\lim_{z \to +\infty} \frac{z^2 \phi^2(z)}{1 - \Phi(z)} = 0.$$

- In **non-regular** models, asymptotic behavior is more complicated (e.g., behavior depends on $\boldsymbol{\theta}$), but there are still useful asymptotic results.

Regularity Conditions − Continued

Some **typical** regularity conditions include:

- Support does not depend on unknown parameters.
- Number of parameters does not grow too fast with n.
- Continuous derivatives of log likelihood (w.r.t. $\boldsymbol{\theta}$).
- Bounded derivatives of likelihood.
- Can exchange the order of differentiation of log likelihood w.r.t. $\boldsymbol{\theta}$ and integration w.r.t. data.
- Identifiability.

Other Topics Related to Parametric Likelihood Covered in the Book

- Bayesian methods (Chapter 9).
- Truncated data (Chapter 11).
- Threshold parameters (Chapter 11).
- Other distributions (e.g., generalized gamma) (Chapters 4, 11).
- Comparison of failure-time distributions (Chapter 12).
- Prediction (Chapter 15).
- Multiple failure modes (Chapter 16).
- Regression anaysis and accelerated testing (Chapters 17-19).

References

Abernethy, R. B., J. E. Breneman, C. H. Medlin, and G. L. Reinman (1983). *Weibull Analysis Handbook*. Air Force Wright Aeronautical Laboratories Technical Report AFWAL-TR-83-2079. Available from: http://apps.dtic.mil/dtic/tr/fulltext/u2/a143100.pdf. []

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