Chapter 5

System Reliability Concepts and Methods

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Chapter 5
System Reliability Concepts and Methods

Topics discussed in this chapter are:

- Important system reliability concepts like system structure, redundancy, nonrepairable, and repairable systems, maintainability and availability.
- Basic concepts of system reliability modeling.
- Expressions for the distribution of system failure time as a function of individual component failure time distributions.
- Multistate and Markov system reliability models

Chapter 5

Segment 1

Introduction to System Reliability and Series Systems

Definitions

- **System**: a collection of components needed to realize a given task.

- **System structure:** a logic diagram illustrating the function of the components within the system and how they relate to system operational state (usually **operational** or **not operational**).

System Structures and System Failure Probability

System failure probability $F_T(t; \boldsymbol{\theta})$ is the probability that a system fails before t (time to **first failure** for a repairable system).

The failure probability of the system **depends on**:

- Time in operation (or another measure of use) denoted by t
- System structure.
- Reliability of system components, including interconnections, interfaces, and human operators.
- Environmental conditions.

Time Dependency of System Reliability

For the time to first failure of a new system with m components (all components starting a time 0)

- The cdf for component i is $F_i = F_i(t; \boldsymbol{\theta}_i)$. The corresponding survival probability is $S_i = S_i(t; \boldsymbol{\theta}_i) = 1 - F_i(t; \boldsymbol{\theta}_i)$. The $\boldsymbol{\theta}_i$s may have some elements in common. Here, $\boldsymbol{\theta}$ denotes the unique elements in $(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m)$.

- The cdf for the system is denoted by $F_T = F_T(t; \boldsymbol{\theta})$. This cdf is determined by the F_i's and the system structure. Then

$$F_T(t; \boldsymbol{\theta}) = g[F_1(t; \boldsymbol{\theta}_1), \ldots, F_m(t; \boldsymbol{\theta}_m)]$$

or one of the simpler forms

$$F_T(\boldsymbol{\theta}) = g[F_1(\boldsymbol{\theta}_1), \ldots, F_m(\boldsymbol{\theta}_m)]$$
$$F_T = g(F_1, \ldots, F_m).$$

To simplify the presentation, time-(and parameter)-dependency will usually be suppressed in this chapter.

A Series System with Two Components

Examples of Series Systems

- Chain
- Multi-cell battery
- Inexpensive personal computer
- Cell phone

Series System cdf

A **series** structure with m components works iff all the components work. Then

- For two independent components,

$$\begin{aligned} F_T(t) &= \Pr(T \le t) = 1 - \Pr(T > t) \\ &= 1 - \Pr(T_1 > t \cap T_2 > t) \\ &= 1 - \Pr(T_1 > t) \Pr(T_2 > t) \\ &= 1 - (1 - F_1)(1 - F_2) \end{aligned}$$

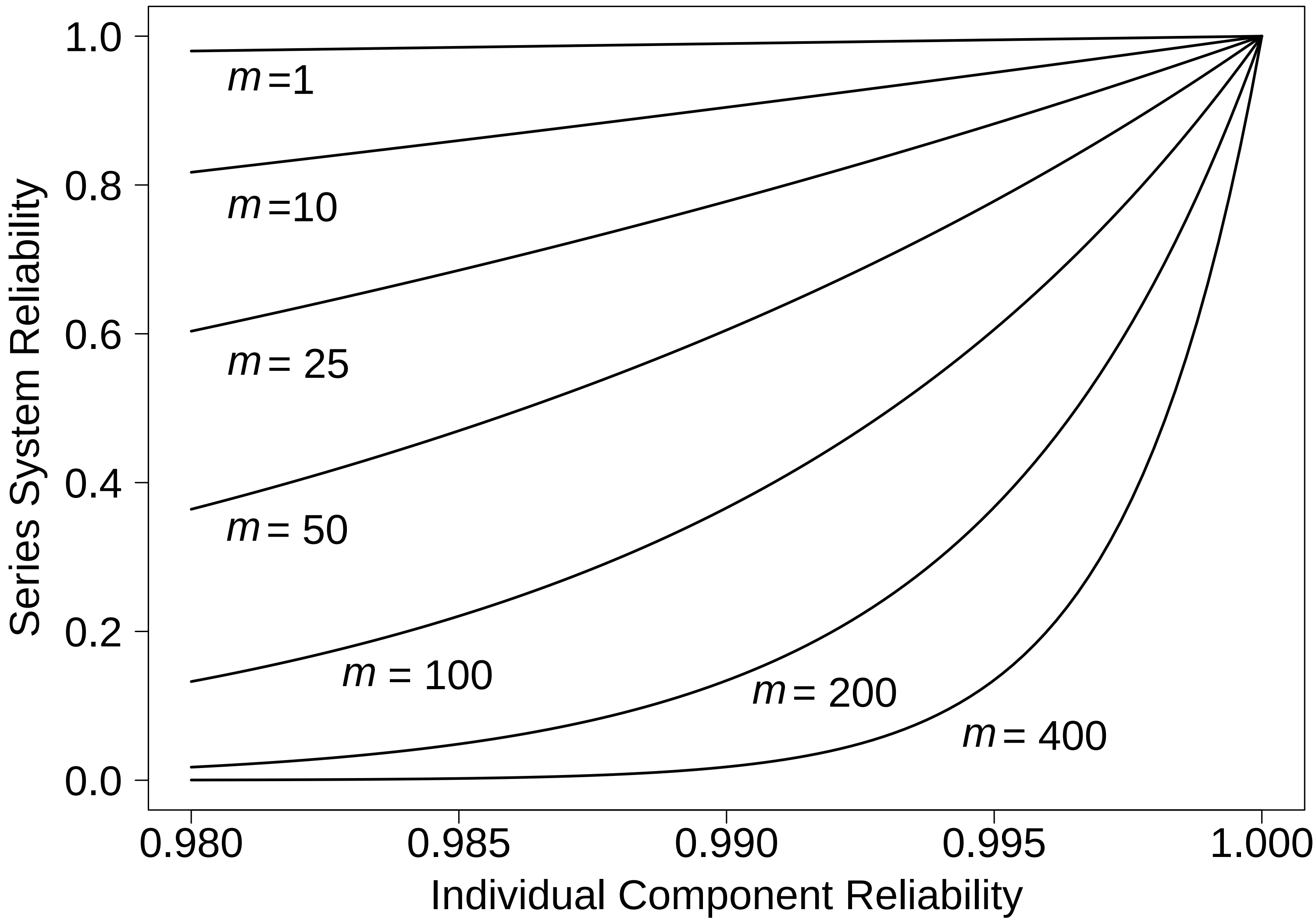
- For m independent components,

$$F_T(t) = 1 - \prod_{i=1}^{m} (1 - F_i)$$

- For m iid components (so $F = F_i$, $i = 1, \ldots, m$),

$$F_T(t) = 1 - (1 - F)^m.$$

Reliability of a System with s Identical Independent Components in Series



Effect of Positive Dependency in a Two-Component Series System

- For a series system with two components and dependent failure times,

  $$F_T(t) = \Pr(T \leq t) = 1 - \Pr(T > t) = 1 - \Pr(T_1 > t \cap T_2 > t).$$

  In this case, the evaluation has to be done with respect to the bivariate distribution of T_1 and T_2.

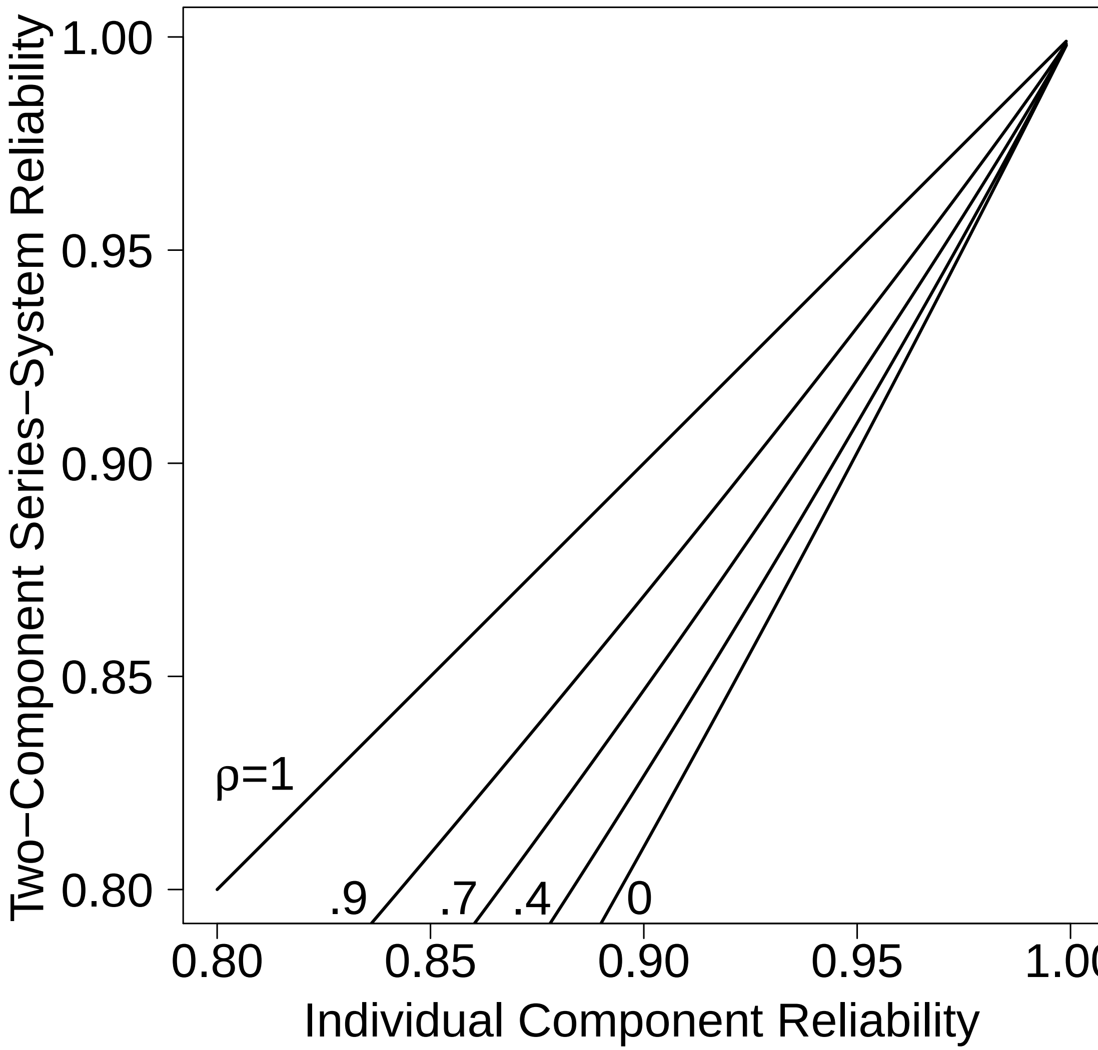
- If the correlation between the two components is positive, then the assumption of independence is conservative in the sense that the actual $F_T(t)$ is **smaller** than that predicted by the independent-component model.

- These results extend to the m components in series, the system $F_T(t)$ would have to be computed with respect to the underlying m-variate distribution. Such computations are, in general, difficult.

Effect of Positive Dependency in a Series System with Two Identical Components Having Lognormal Failure Times

- The distributions of log failure times for the individual components is bivariate normal with the same (arbitrary) mean and standard deviation for both components and correlation ρ.
- The reliability $1 - F_T(t)$ of the system can be expressed as a function of the individual reliability components $1 - F(t)$ and ρ.
- When $\rho = 1$ (so the two components are perfectly dependent and will fail at exactly the same time), the system reliability $1 - F_T(t)$ is the same as the reliability for a single component.
- When $\rho = 0$ (so the two components are independent), $1 - F_T(t)$ corresponds to the system reliability for an $s = 2$ series system with independent components.

Reliability of a Series System with 2 Identical Dependent Components



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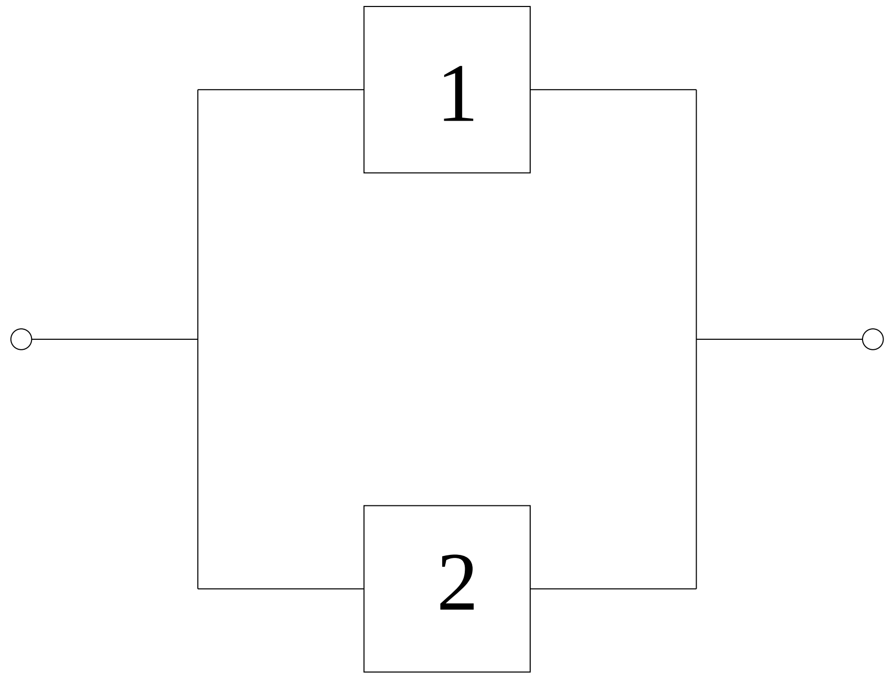
Segment 2

Redundancy and Parallel Systems

A Parallel System with Two Components



Examples of Systems with Components in Parallel

- Automobile headlights
- RAID array computer disk systems
- Stairwells with emergency lighting
- Multiple light banks in an office

Parallel System cdf

A **parallel** structure with m components works if at least one of the components works. Then

- For two independent components,

$$\begin{aligned} F_T(t) &= \Pr(T \le t) \\ &= \Pr(T_1 \le t \cap T_2 \le t) \\ &= \Pr(T_1 \le t)\Pr(T_2 \le t) \\ &= F_1 F_2 \end{aligned}$$

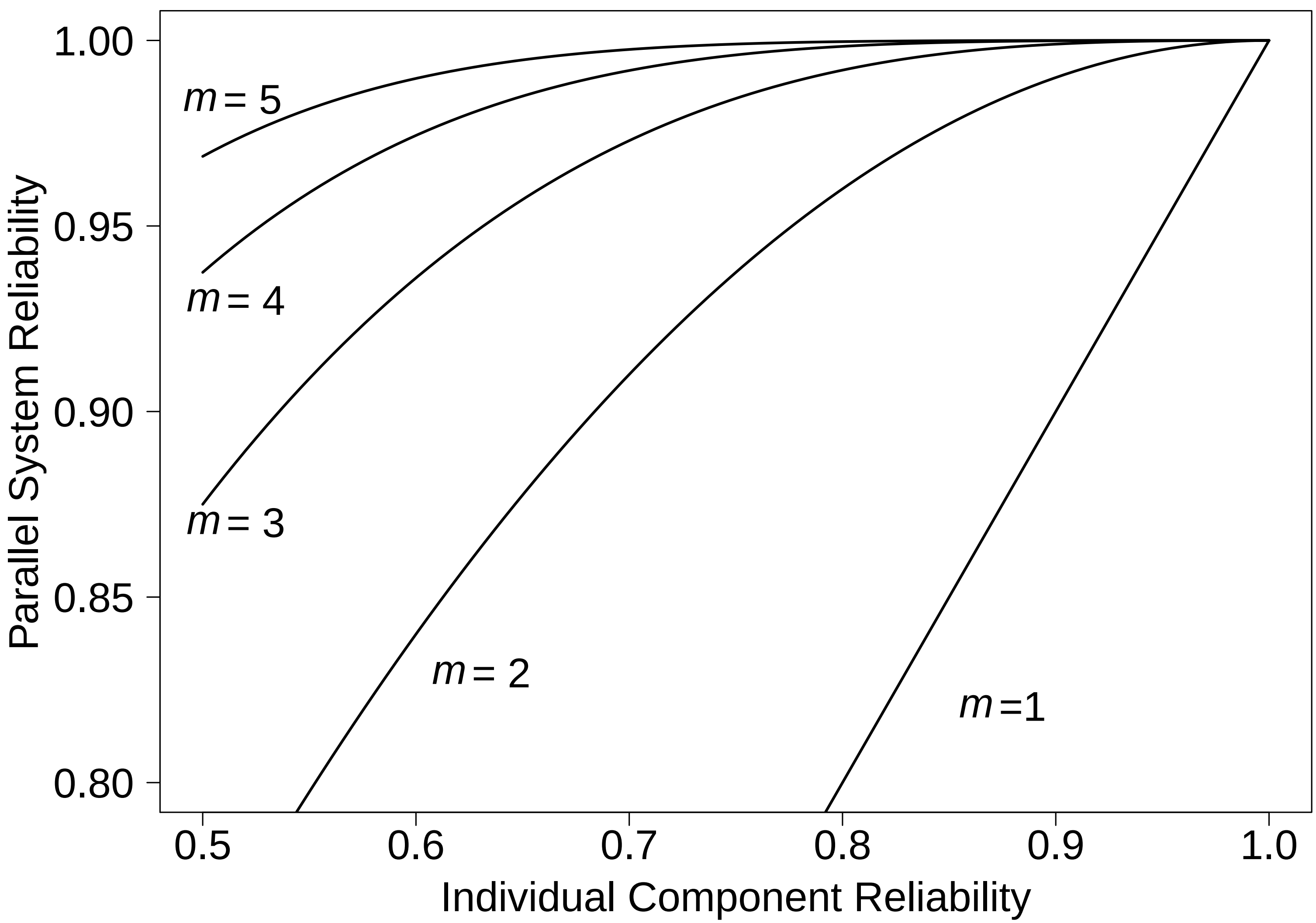
- For m independent components,

$$F_T(t) = \prod_{i=1}^{m} F_i$$

- For m iid components ($F_i = F$, $i = 1, \ldots, m$),

$$F_T(t) = F^m.$$

Reliability of a System with s Identical Independent Components in Parallel



Effect of Positive Dependency in a Two-Component Parallel System

- For a parallel system with two components and dependent failure times,

$$F_T(t) = \Pr(T \le t) = \Pr(T_1 \le t \cap T_2 \le t).$$

  In this case, the evaluation has to be done with respect to the bivariate distribution of T_1 and T_2.

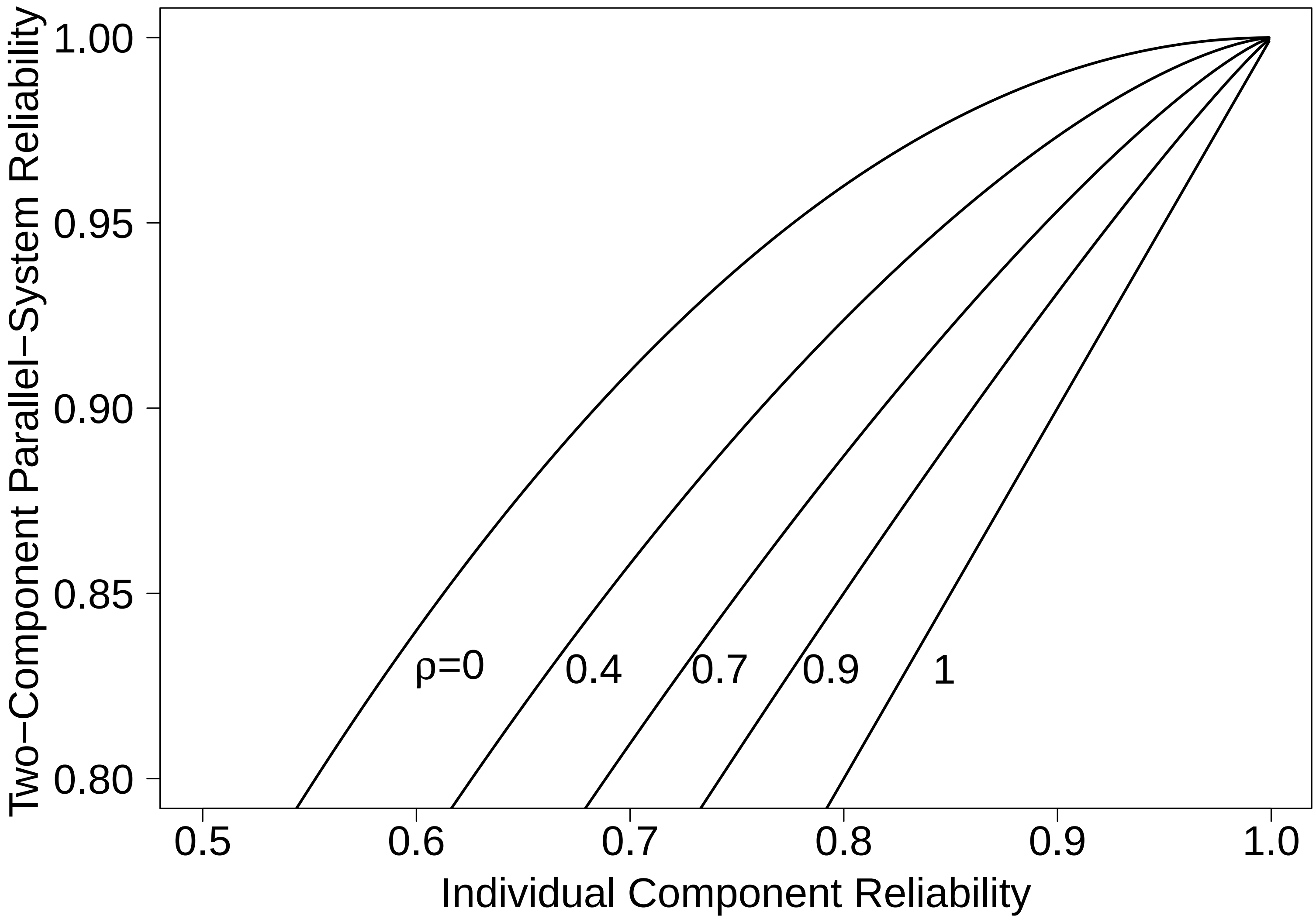
- If the dependency between the two components is positive, then the assumption of independence is anti-conservative in the sense that the actual $F_T(t)$ is larger than that predicted by the independent-component model.

- These results extend to the m components in parallel, the system $F_T(t)$ would have to be computed with respect to the underlying m-variate distribution. Such computations are, in general, difficult.

Effect of Positive Dependency in a Parallel System with Two Identical Components Having Lognormal Failure Times

- The distributions of log failure times for the individual components is bivariate normal with the same (arbitrary) mean and standard deviation for both components and correlation ρ.
- The reliability $1 - F_T(t)$ of the system can be expressed as a function of the individual reliability components $1 - F(t)$ and ρ.
- When $\rho = 1$ (so the two components are perfectly dependent and will fail at exactly the same time), the system reliability $1 - F_T(t)$ is the same as the reliability for a single component.
- When $\rho = 0$ (so the two components are independent), $1 - F_T(t)$ corresponds to the system reliability for an $s = 2$ independent parallel system.
- The advantages of redundancy can be seriously degraded when the failure times of the individual components have positive dependence.

Reliability of a Parallel System with Two Identical Dependent Components Having Lognormal Failure Times



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Segment 3

Series-Parallel Systems

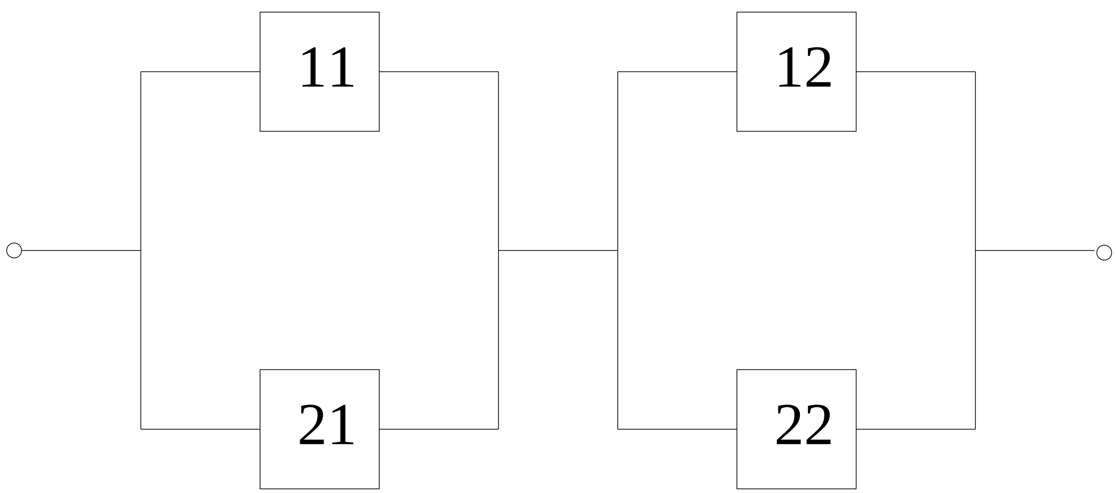
Systems Using a Combination of Series and Parallel Structures

Series and parallel structures are the basis for building models for more complicated systems which use redundancy to increase reliability.

Some examples are:

- Series-parallel systems with component-level redundancy.
- Series-parallel systems with system-level redundancy.
- Series system with parallel redundancy in critical components.

A Series-Parallel System Structure with Component-Level Redundancy



Examples Series-Parallel System Structure with Component-Level Parallel Redundancy

- Spare lasers in each repeater in an under-sea fiber-optic data transmission system

- Automobile with two headlights. Separate front and rear hydraulic brake systems

- Human body (hands, eyes, lungs, kidneys)

Systems with Component-Level Redundancy

A $k \times r$ component-level redundant structure has k series structures each one made of r units in parallel.

- For 2×2 series-parallel with independent components,

$$\begin{aligned} F_T(t) &= 1 - \Pr(T > t) \\ &= 1 - \Pr(\textbf{parallel subsystem 1 OK} \cap \textbf{parallel subsystem 2 OK}) \\ &= 1 - (1 - F_{11}F_{21})(1 - F_{12}F_{22}) \end{aligned}$$

where F_{ij}, $i = 1, 2$ are the cdfs for the parallel subsystem j.

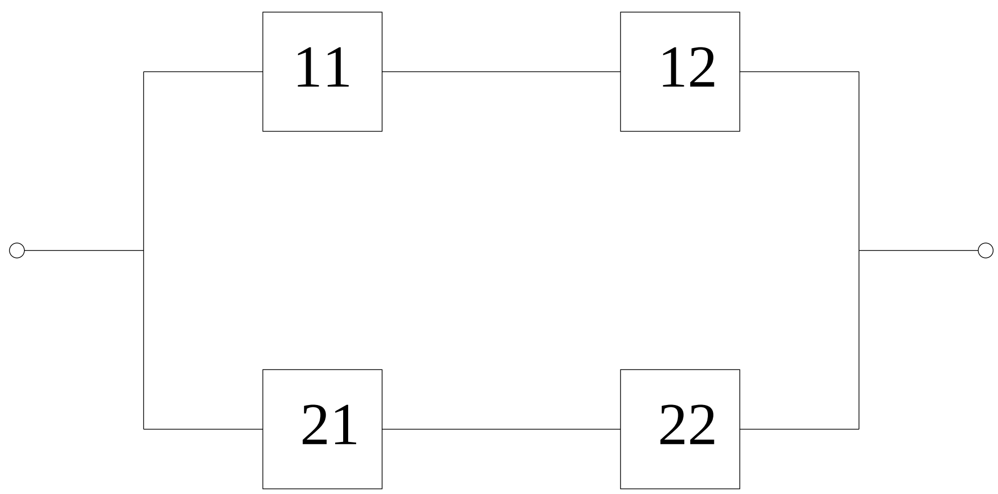
- For a $k \times r$ series-parallel with independent components,

$$F_T(t) = 1 - \prod_{j=1}^{k} \left(1 - \prod_{i=1}^{r} F_{ij} \right)$$

- When all of the components are iid,

$$F_T(t) = 1 - (1 - F^r)^k$$

A Series-Parallel System Structure with System-Level Redundancy



Examples Series-Parallel System Structure with System-Level Parallel Redundancy

- Dual central processors for a system-critical communications switching system

- Multiple computers, working in parallel on the space shuttle

- Multiple trans-Atlantic transmission cables

- Fiber bundle or stranded wire

Series-Parallel Structure with System-Level Redundancy

A $r \times k$ series-parallel system-level redundancy structure has r parallel sets each of k units in series.

- For 2×2 structure with independent components,

$$\begin{aligned} F_T(t) &= \Pr(T \leq t) \\ &= \Pr(\textbf{series subsystem 1 failed} \cap \textbf{series subsystem 2 failed}) \\ &= [1-(1-F_{11})(1-F_{12})][1-(1-F_{21})(1-F_{22})] \end{aligned}$$

where F_{ij}, $j=1,2$ are the cdfs for the series system i.

- For a $r \times k$ structure with independent components,

$$F_T(t) = \prod_{i=1}^{r} \left[1 - \prod_{j=1}^{k} \left(1 - F_{ij}\right) \right]$$

- For a $r \times k$ parallel-series structure with iid components,

$$F_T(t) = \left[1 - (1-F)^k\right]^r$$

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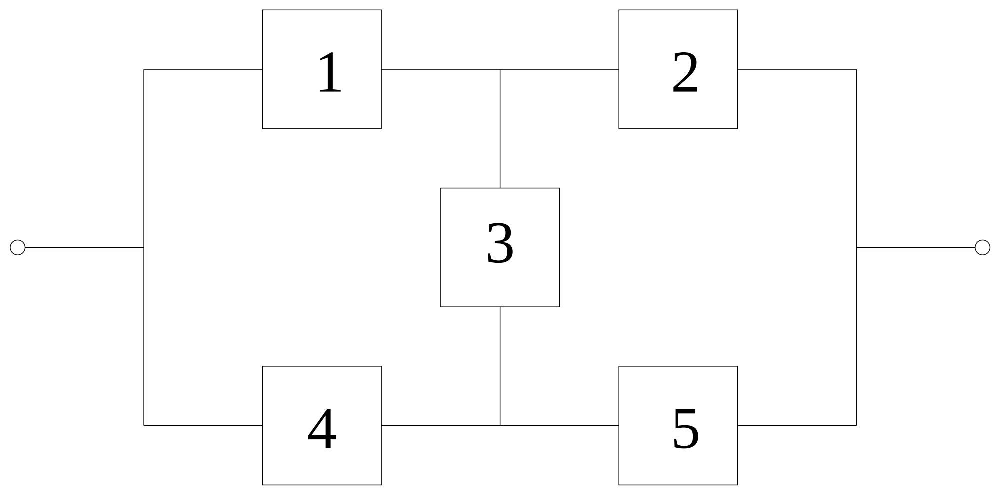
Segment 4

More Complicated System Structures

A Bridge System Structure

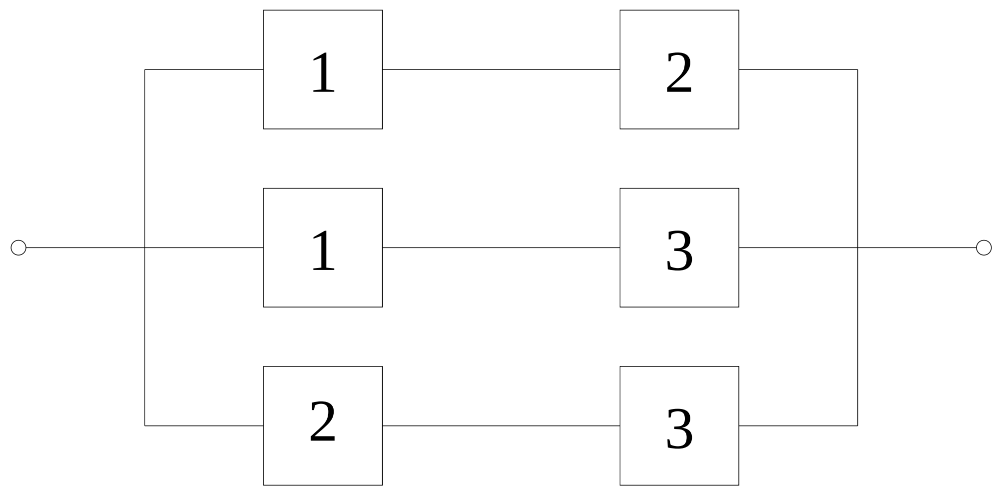


Bridge System Structure cdf

- Let A_i be the event that the i unit is working.
- Conditioning on the event A_3 and using the law of total probability gives

$$
\begin{aligned}
F_T(t) &= \Pr(T \le t \cap A_3) + \Pr(T \le t \cap A_3^c) \\
&= \Pr(T \le t | A_3) \Pr(A_3) + \Pr(T \le t | A_3^c) \Pr(A_3^c) \\
&= \Pr[(A_1^c \cap A_4^c) \cup (A_2^c \cap A_5^c) | A_3] \Pr(A_3) \\
&\quad + \Pr[(A_1^c \cup A_2^c) \cap (A_4^c \cup A_5^c) | A_3^c] \Pr(A_3^c) \\
&= [F_1 F_4 + F_2 F_5 - F_1 F_2 F_4 F_5](1 - F_3) \\
&\quad + [F_1 + F_2 - F_1 F_2][F_4 + F_5 - F_4 F_5] F_3
\end{aligned}
$$

A 2-out-of-3 System Structure



k-out-of-m System Structures

A k out of m system remains operable as long as at least k of the system's m components are operable. Examples include

- A satellite battery system which will continue to operate as long as 6 of 10 batteries to operate correctly.

- Solid-state drives that continue to provide error-free service by having redundant memory locations.

- A web-hosting service that uses a system with ten servers so that the system operates satisfactorily if at least seven of those servers are operating.

2-**out-of**-3 **System Structure cdf**

For a 2-out-of-3 independent components,

$$
\begin{aligned}
F_T(t) &= \Pr(T \le t) \\
&= \Pr(\textbf{exactly two fail}) + \Pr(\textbf{exactly three fail}) \\
&= F_1F_2(1-F_3) + F_1F_3(1-F_2) + F_2F_3(1-F_1) + F_1F_2F_3 \\
&= F_1F_2 + F_1F_3 + F_2F_3 - 2F_1F_2F_3
\end{aligned}
$$

k-out-of-m System Structures

- For k-out-of-m independent components,

$$F_T(t) = \sum_{j=m-k+1}^{m} \left\{ \sum_{\underline{\delta} \in A_j} \left[\prod_{i=1}^{m} F_i^{\delta_i} (1 - F_i)^{(1-\delta_i)} \right] \right\}$$

where $\underline{\delta}' = (\delta_1, \ldots, \delta_m)$ with $\delta_i = 1$ indicating failure of unit i by time t and $\delta_i = 0$ otherwise and A_j is the set of all $\underline{\delta}$ such that $\underline{\delta}'\underline{\delta} = j$.

- $F_T(t)$ can also be viewed as the distribution of the sum of m independent non-identically distributed Bernoulli random variables, and is known as the Poisson-binomial distribution.

- For identically distributed components ($F = F_i, i = 1, \ldots, m$),

$$F_T(t) = \sum_{j=m-k+1}^{m} \binom{m}{j} F^j (1 - F)^{m-j}.$$

Other System Structures

Standby or passive redundancy: a redundant unit is activated only when another unit fails and the redundant unit is needed to keep the system working.

There are many variations of this:

- Cold standby. Component is not turned on until it is needed.
- Partially-loaded redundancy.

Need to consider the reliability of the switching mechanism that activates the standby units.

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Multistate and Markov System Reliability Models

Multistate Systems

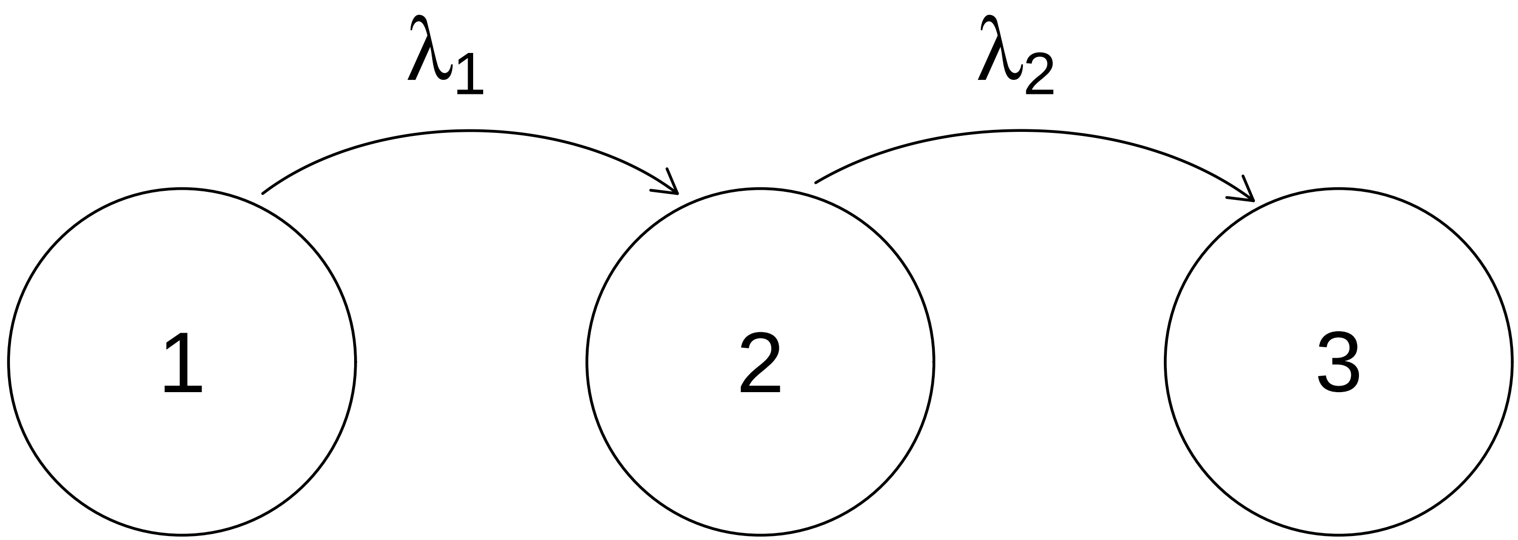
- Multistate reliability models are useful for certain applications.

- Multistate reliability models allow additional flexibility to decribe dependency between system components.

- When transition times in a multistate system are described by an exponential distribution, simple results for common reliability metrics like the the cdf and quantiles of the failure-time distribution are available.

- A simple example of a three-state system is a two-component parallel system and the state would indicate the number of operating components.

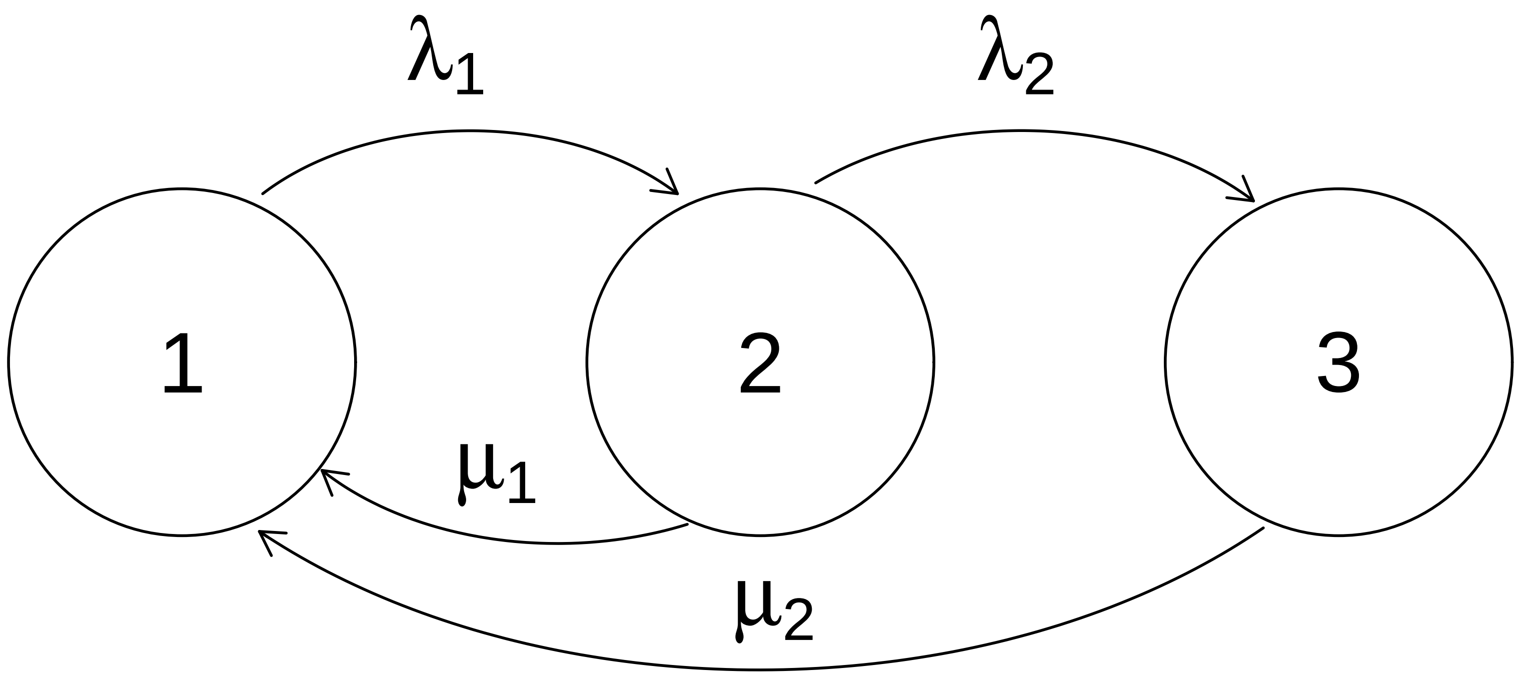
A Three-State Non-Repairable System



Repairable Systems

- Multistate reliability models are also useful for modeling repairable systems.
- There are additional metrics for repairable systems including system availability amd mean time between failures (MTBF).
- When transition times in a multistate system are described by an exponential distribution, simple close-form results for common repairable systems reliability metrics (like availablity and MTBF) are available.

A Three-State Repairable System



Markov and Other More General Models

Markov models allow the modeling of repairable and non-repairable systems, allowing for dependence among components and common-cause failures.

- Markov models are, however, only suitable for relatively small systems.

- The Markov models are also limited by the life and repair distributions that can be employed.

- Non-Markovian generalizations are possible, but lead to computational difficulties. Analysis of non-Markovian models is generally done with simulation.

References

Meeker, W. Q., L. A. Escobar, and F. G. Pascual (2021). *Statistical Methods for Reliability Data* (Second Edition). Wiley. [1]