Chapter 13

Planning Life Tests for Estimation

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Chapter 13
Planning Life Tests for Estimation

Topics discussed in this chapter are:

- The basic ideas behind planning a life test.
- A simple method to choose a sample size as a function of estimation precision.
- How to use simulation to anticipate life test results, visualize estimation precision, and assess tradeoffs between sample size and length of a study.
- How to obtain large-sample approximate variance factors for a general quantity of interest.
- How to obtain large-sample approximate variance factors for a function of the parameters of a log-location-scale distribution.

Chapter 13

Segment 1

Basic Ideas Behind Life-Test Planning, Planning Values, and the Sample-Size Tool

Basic Ideas in Test Planning

- The enormous cost of reliability studies makes it essential to do careful planning. Frequently asked **questions** include:
  - ▶ How many units do I need to test in order to estimate the 0.1 quantile of life?
  - ▶ How long do I need to run the life test?

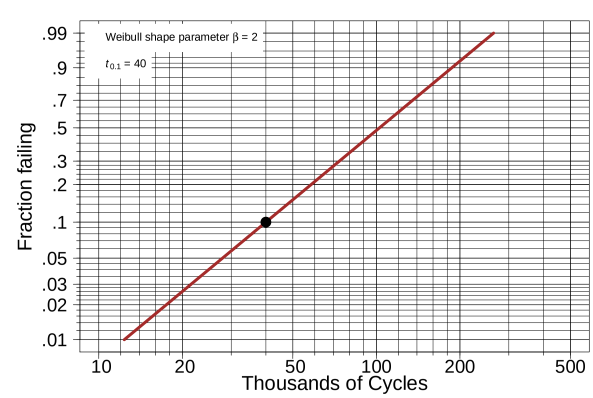
  More test units and more time will provide more information and thus more precision in estimation (e.g., narrower confidence intervals).
- To anticipate the results from a test plan and to respond to the questions above, it is necessary to have some **planning** information about the life distribution to be estimated.

Engineering Planning Values and Assumed Distribution for Planning a Life Test

Want to estimate $t_{0.1}$ of the life distribution of a metal spring. Tests are run at higher than usual cycling rate to cause failures to occur more quickly.

- Information from engineering:
  - ▶ The Weibull distribution will be used to describe the failure-time distribution.
  - ▶ The Weibull shape parameter $\beta^{\square} = 2$ will be used.
  - ▶ Expect about 10% failures by 40 thousand cycles of operation ($t^{\square}_{0.10} = 40$).
- Start by using a simple analytical method to suggest a sample size.
- Use simulation to get insight and fine-tune the test plan.

Weibull Probability Paper Showing the Metal Spring cdf Corresponding to the Test Planning Values $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$



Motivation for Use of Large-Sample Approximate Test Plan Properties

Large-sample approximate test plan properties provide:

- Simple expressions giving **precision** of a specified estimator as a **function of sample size**.
- Simple expressions giving needed **sample size** as a **function of precision** of a specified estimator.
- Simple tables, graphs, and **software** that will allow easy assessments of tradeoffs in test planning decisions like sample size and test length.
- Can be fine tuned with simulation evaluation.

Sample Size Formula for the Mean of a Normal Distribution

- A Wald approximate $100(1-\alpha)\%$ confidence interval for the normal distribution mean μ is

$$[\underset{\sim}{\mu},\ \tilde{\mu}] = \hat{\mu} \mp z_{(1-\alpha/2)}\frac{\hat{\sigma}}{\sqrt{n}} = \left[\hat{\mu}-\widehat{D},\ \hat{\mu}+\widehat{D}\right]$$

where the half-width $\widehat{D} = z_{(1-\alpha/2)}\hat{\sigma}/\sqrt{n}$ can be used to describe the precision for estimating μ as a function of n.

- Substituting the planning value $(\sigma^{\square})^2$ for $\hat{\sigma}$ and the target precision value D_T for $\widehat{D}$ and solving for n gives the needed sample size to estimate μ with **complete data** as

$$n = \frac{z^2_{(1-\alpha/2)}(\sigma^{\square})^2}{D_T^2},$$

- This formula appears in most elmentary textbooks.
- This chapter generalizes this formula to allow for estimation of and desired quantile of a specified (log-)location-scale distributions and allowing for censoring.

Confidence Interval for an Unrestricted Quantile (e.g., $-\infty < y_p < \infty$)

- For an unrestricted quantile y_p a Wald approximate $100(1-\alpha)\%$ confidence interval is given by

$$\begin{aligned}[\underset{\sim}{y_p},\ \tilde{y}_p] &= \hat{y}_p \mp z_{(1-\alpha/2)}\sqrt{\widehat{\mathrm{Var}}(\hat{y}_p)} \\ &= \left[\hat{y}_p-\widehat{D},\ \hat{y}_p+\widehat{D}\right]\end{aligned}$$

where

$$\widehat{D} = z_{(1-\alpha/2)}\sqrt{\widehat{\mathrm{Var}}(\hat{y}_p)} = z_{(1-\alpha/2)}\sqrt{\frac{\hat{\sigma}^2}{n}V_{\hat{y}_p}},$$

where $\mathrm{V}_{\hat{y}_p}$ is a variance factor **depending on** p, the **amount of censoring**, and the underlying distribution $\Phi(z)$.

- The half-width $\widehat{D}$ of the interval and can be used to assess estimation precision for y_p as a function of n and amount of censoring (related to the length of the test).

Sample Size Formulas for an Unrestricted Quantile (e.g., $-\infty < y_p < \infty$)

- Recall the confidence interval half width

$$\widehat{D} = z_{(1-\alpha/2)}\sqrt{\widehat{\mathrm{Var}}(\hat{y}_p)} = z_{(1-\alpha/2)}\sqrt{\frac{\hat{\sigma}^2}{n}V_{\hat{y}_p}},$$

- Substituting the planning value $\sigma^{\square}$ for $\hat{\sigma}$ and the target half-width D_T for $\widehat{D}$ and solving for n gives;

$$n = \frac{z^2_{(1-\alpha/2)}(\sigma^{\square})^2\mathrm{V}_{\hat{y}_p}}{D_T^2}$$

as the sample size needed to estimate y_p with target precision D_T.

- The variance factor $\mathrm{V}_{\hat{y}_p}$ can be obtained from tables, plots or computer algorithms.

Sample Size For Estimating the 0.50 Quantile of Lightbulb Life

- The needed sample size to estimate t_p, a **log-location-scale** distribution p **quantile** with **censored data** and precision R_T is:

$$n = \frac{z^2_{(1-\alpha/2)}(\sigma^{\square})^2 V_{\hat{y}_p}}{[\log(R_T)]^2}$$

where $V_{\hat{y}_p}$ is a variance factor depends on the **quantile of interest** p, the **amount of censoring**, p_c and the underlying distribution $\Phi(z)$.

Confidence Interval for a Positive Quantile (e.g., $0 < t_p < \infty$)

- For a positive quantile t_p a Wald approximate $100(1-\alpha)\%$ confidence interval for $\log(t_p)$ is given by

$$\left[\underset{\sim}{\log(t_p)},\ \widetilde{\log(t_p)}\right] = \log(\hat{t}_p) \pm z_{(1-\alpha/2)}\sqrt{\widehat{\mathrm{Var}}\left[\log(\hat{t}_p)\right]}.$$

Taking antilogs yields a confidence interval for t_p

$$[\underset{\sim}{t_p},\ \tilde{t}_p] = [\hat{t}_p/\hat{R},\ \hat{t}_p\hat{R}]$$

where

$$\hat{R} = \exp\left\{z_{(1-\alpha/2)}\sqrt{\widehat{\mathrm{Var}}\left[\log(\hat{t}_p)\right]}\right\} = \exp\left\{z_{(1-\alpha/2)}\sqrt{\frac{\hat{\sigma}^2}{n}V_{\hat{y}_p}}\right\}.$$

- The unitless $\hat{R} > 1$ **precision factor** is directly related to the width of the confidence interval and can be used to assess estimation precision for t_p as a function of sample size n and the length of the test.

Sample Size Formulas for a Positive Quantile (e.g., $0 < t_p < \infty$)

- The needed sample size to estimate t_p, a **log-location-scale** distribution p **quantile** with **censored data** and precision R_T is:

$$n = \frac{z^2_{(1-\alpha/2)} (\sigma^{\square})^2 V_{\widehat{y}_p}}{[\log(R_T)]^2}$$

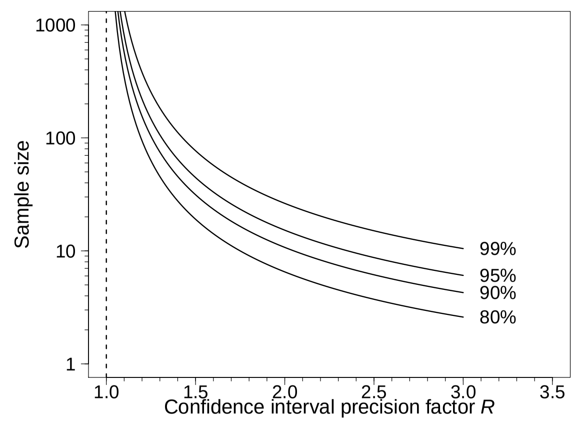
where $V_{\widehat{y}_p}$ is a variance factor depends on the **quantile of interest** p, the **amount of censoring**, p_c and the underlying distribution $\Phi(z)$.

- The variance factor is defined as

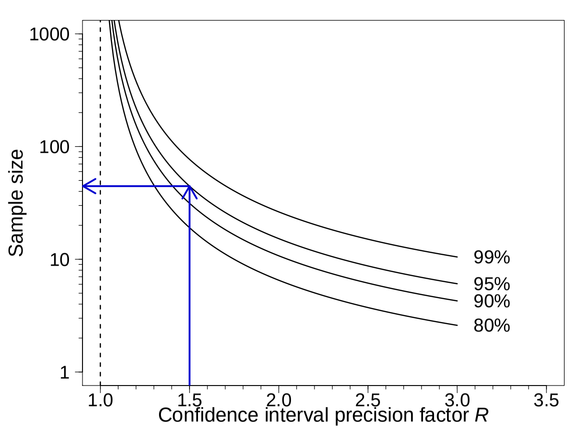
$$\mathrm{V}_{\widehat{y}_p} = \frac{n}{\sigma^2}\mathrm{Avar}\big[\log(\widehat{t}_p)\big] = \frac{n}{\sigma^2}\mathrm{Avar}[\widehat{y}_p]$$

where $\mathrm{Avar}\big[\log(\widehat{t}_p)\big]$ is the large-sample approximate variance of $\log(\widehat{t}_p)$.

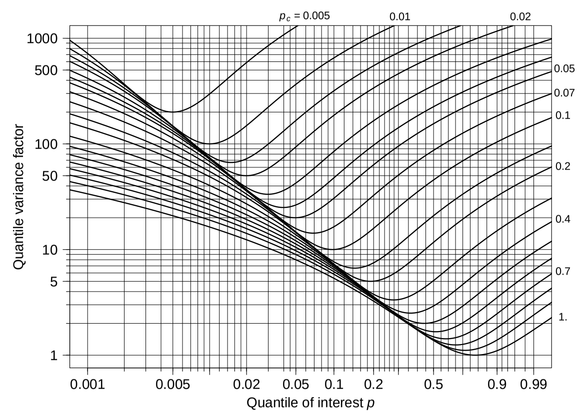
Sample Size Tool Weibull Distribution Test Planning Values $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$ Censoring Time $t_c = 50$ Thousand Cycles



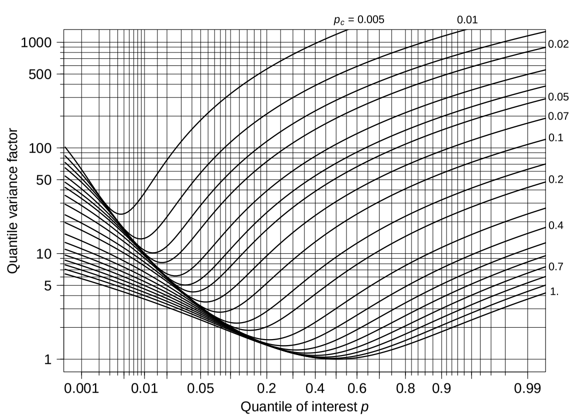
Sample Size Tool Weibull Distribution Test Planning Values $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$ Censoring Time $t_c = 50$ Thousand Cycles



Variance Factor $\mathrm{V}_{\log(\widehat{t}_p)}$ for ML Estimation of Weibull Distribution Quantiles as a Function of p_c, the Population Proportion Failing by Time t_c, and p, the Quantile of Interest



Variance Factor $\mathrm{V}_{\log(\widehat{t}_p)}$ for ML Estimation of Lognormal Distribution Quantiles as a Function of p_c, the Population Proportion Failing by Time t_c, and p, the Quantile of Interest



Figures for Sample Sizes to Estimate Weibull and Lognormal Quantiles

Figures give plots of the factor $\mathrm{V}_{\log(\widehat{t}_p)}$ for the quantile of interest p as a function of $p_c = \Pr(Z \le \zeta_c)$ for the Weibull, lognormal, and loglogistic distributions. The plots show:

- Increasing the length of a life test (increasing the expected proportion of failures) will always reduce the asymptotic variance. After a point, however, the returns are diminishing.

- Estimating quantiles with large or small p generally results in larger variance factors than quantiles somewhat larger than the expected proportion failing p_c.

- When possible, it is better practice to run a life test long enough to avoid extrapolation (i.e., so that $p_c > p$).

Sample Size Formulas Estimating the 0.10 Quantile of Spring Life

- The needed sample size to estimate t_p, a **log-location-scale** distribution p **quantile** with **censored data** and precision R_T is:

$$n = \frac{z^2_{(1-\alpha/2)}(\sigma^{\square})^2 V_{\widehat{y}_p}}{[\log(R_T)]^2}$$

where $V_{\widehat{y}_p}$ is a variance factor depends on the **quantile of interest** p, the **amount of censoring**, p_c and the underlying distribution $\Phi(z)$.

- The variance

Meeting the Precision Criterion

- By the definition of a confidence interval, in repeated samples approximately $100(1-\alpha)\%$ of the intervals for t_p will actually contain the true t_p.
- In repeated samples, $\widehat{\text{Var}}\left[\log(\hat{t}_p)\right]$ is random because $\hat{\sigma}$ and the proportion failing in the test are random.
- If

$$\widehat{\text{Var}}\left[\log(\hat{t}_p)\right] > \text{Avar}\left[\log(\hat{t}_p)\right]$$

then

$$\widehat{R} > R_T.$$

- Generally, $\Pr(\widehat{R} > R_T) \approx 0.50$.
- Thus there is about a 50% chance that the width of the interval will be greater than (or less than) the target.

Chapter 13

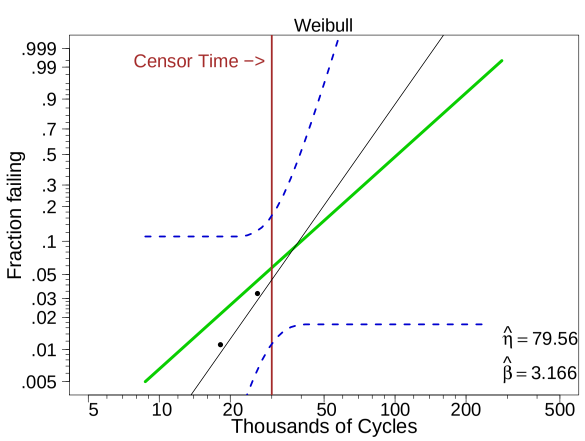
Segment 2

Using Simulation in Life-Test Planning

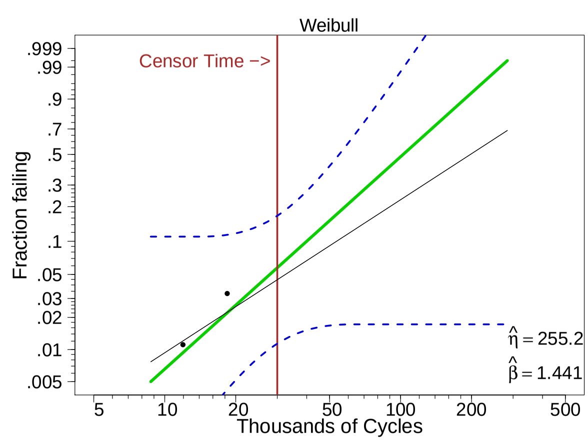
Simulation as a Tool for Test Planning

- Use assumed model and planning values of model parameters to simulate data from the proposed study.
- Analyze the data perhaps under different assumed models.
- Assess precision provided.
- Simulate many times to assess actual sample-to-sample differences.
- Summarize the results.
- Repeat with different test plans to assess tradeoffs.
- Repeat with different input planning values to assess sensitivity to these inputs.

Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 30$ **Thousand Cycles**



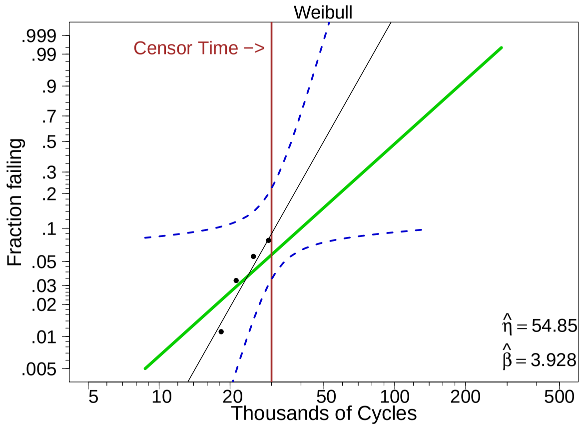
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 30$ **Thousand Cycles**



Simulated Weibull Life Test

Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$

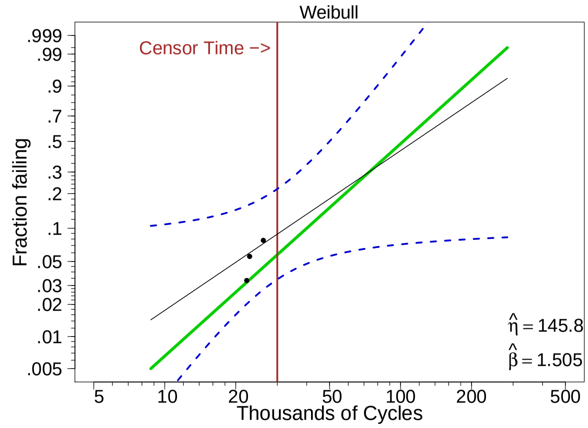
Test Plan: $n = 45$, $t_c = 30$ Thousand Cycles



Simulated Weibull Life Test

Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$

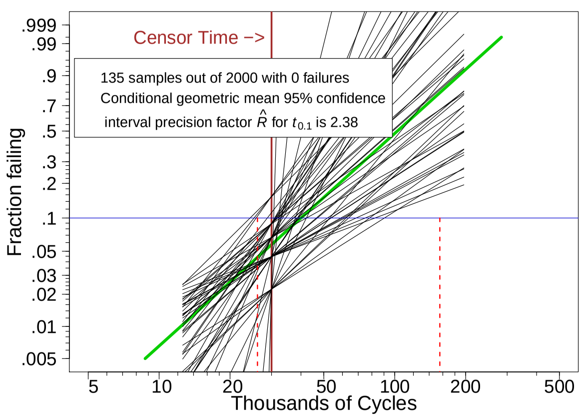
Test Plan: $n = 45$, $t_c = 30$ Thousand Cycles



Summary of Simulated Weibull Life Tests

Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$

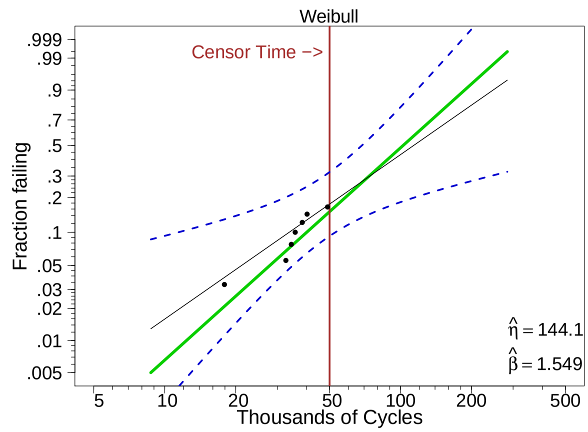
Test Plan: $n = 45$, $t_c = 30$ Thousand Cycles



Simulated Weibull Life Test

Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$

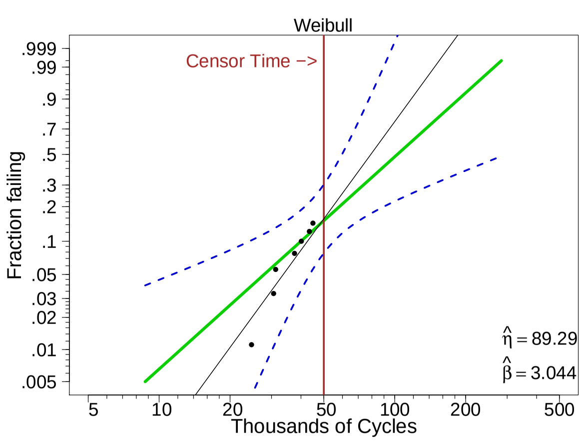
Test Plan: $n = 45$, $t_c = 50$ Thousand Cycles



Simulated Weibull Life Test

Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$

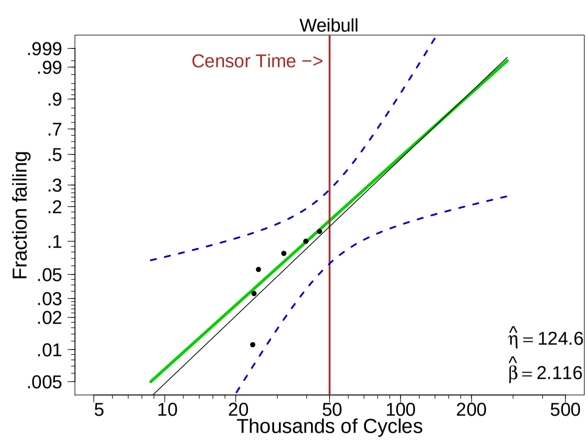
Test Plan: $n = 45$, $t_c = 50$ Thousand Cycles



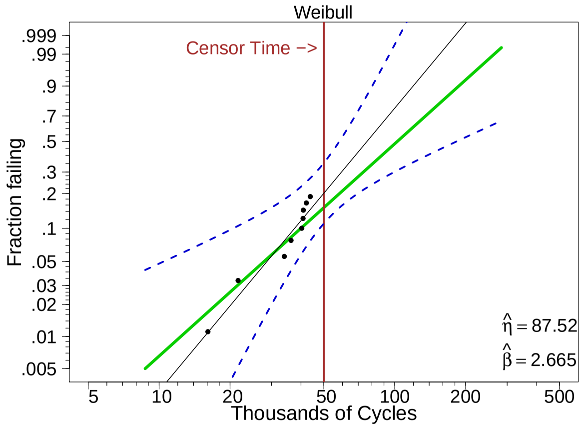
Simulated Weibull Life Test

Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$

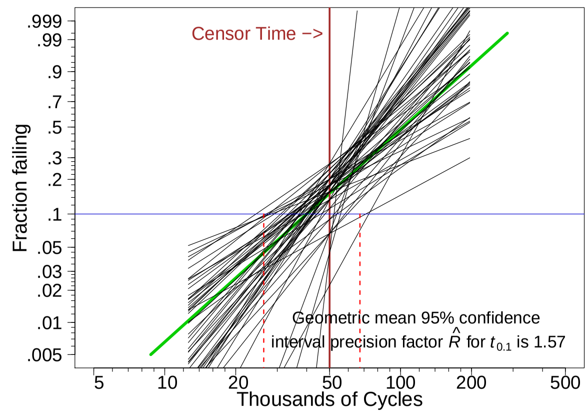
Test Plan: $n = 45$, $t_c = 50$ Thousand Cycles



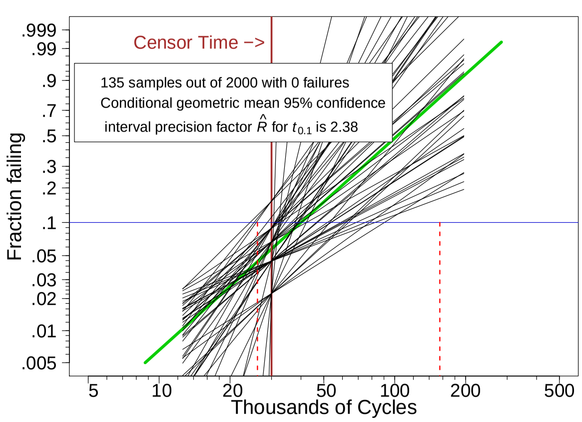
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 50$ Thousand Cycles



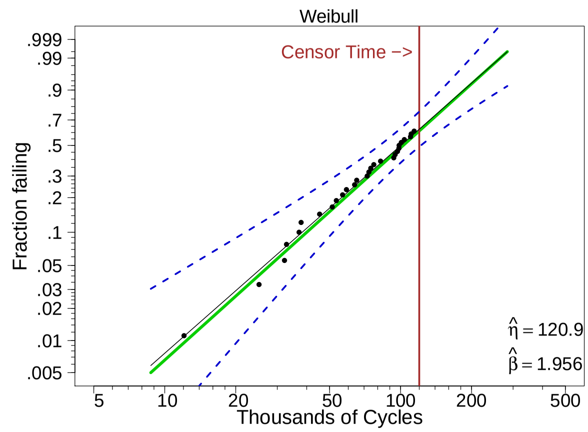
Summary of Simulated Weibull Life Tests
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 50$ Thousand Cycles



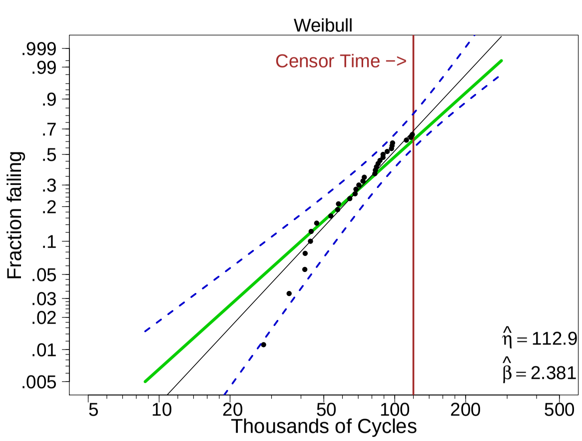
Summary of Simulated Weibull Life Tests
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 30$ Thousand Cycles



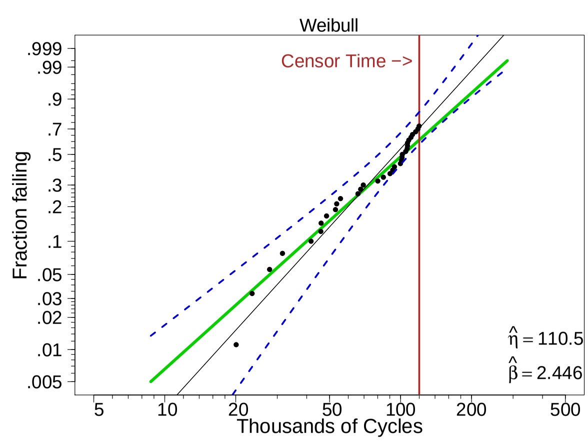
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 120$ Thousand Cycles



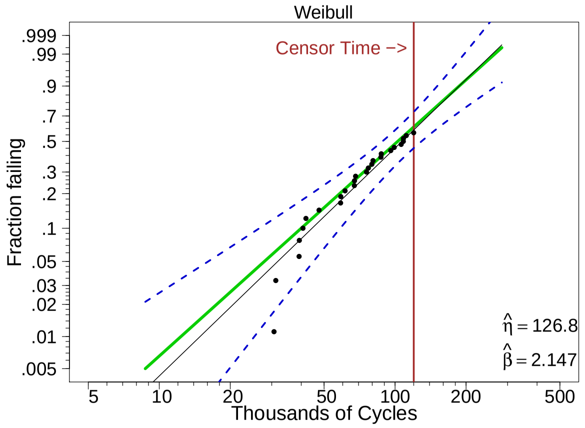
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 120$ Thousand Cycles



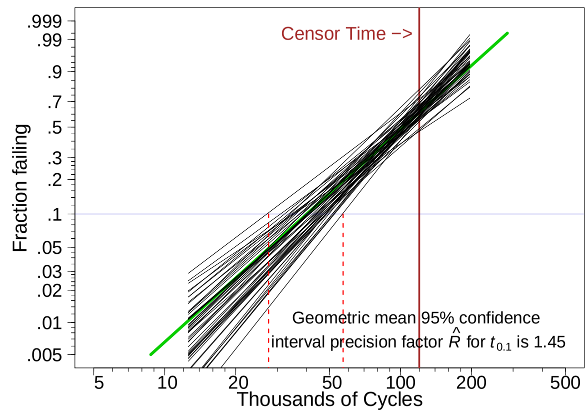
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 120$ Thousand Cycles



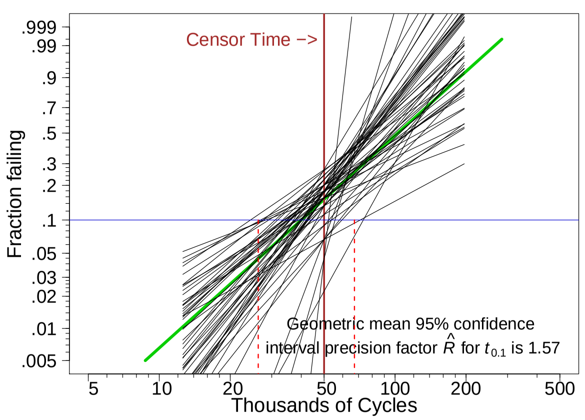
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 120$ **Thousand Cycles**



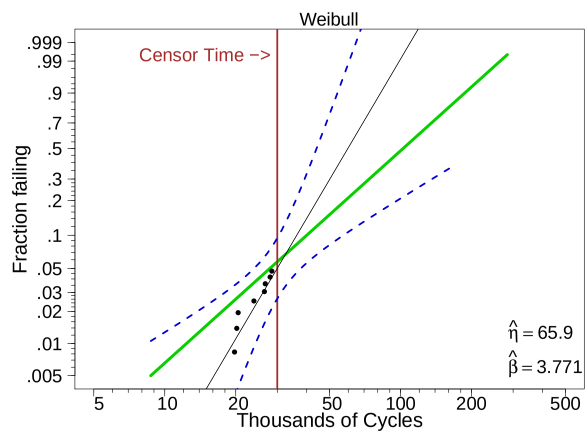
Summary of Simulated Weibull Life Tests
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 120$ **Thousand Cycles**



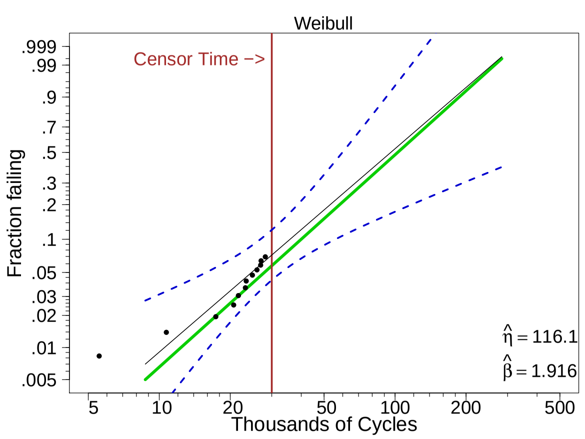
Summary of Simulated Weibull Life Tests
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 50$ **Thousand Cycles**



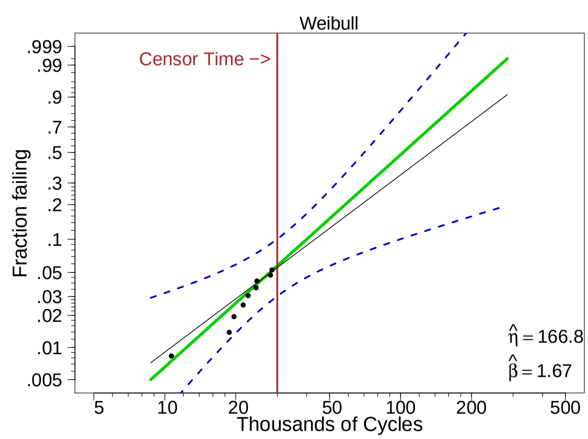
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 180$, $t_c = 30$ **Thousand Cycles**



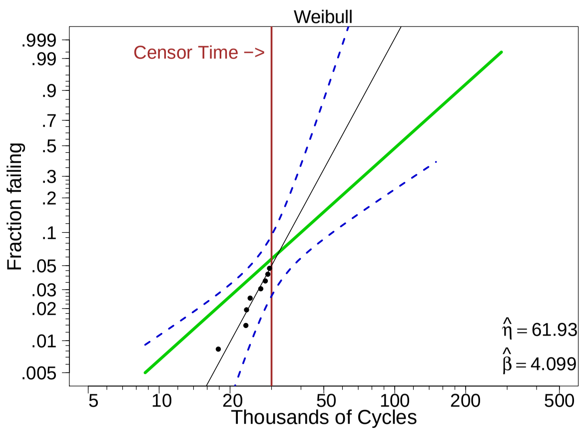
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 180$, $t_c = 30$ **Thousand Cycles**



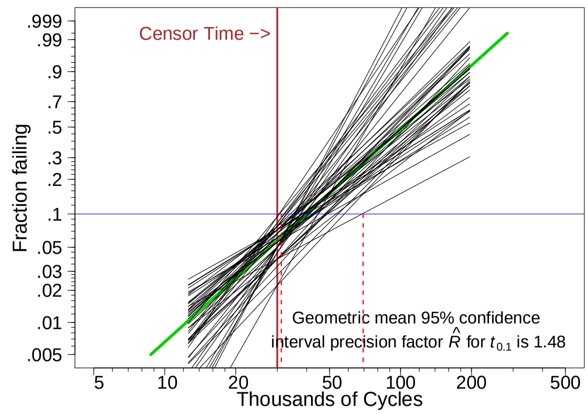
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ **and** $\beta^{\square} = 2$
Test Plan: $n = 180$, $t_c = 30$ **Thousand Cycles**



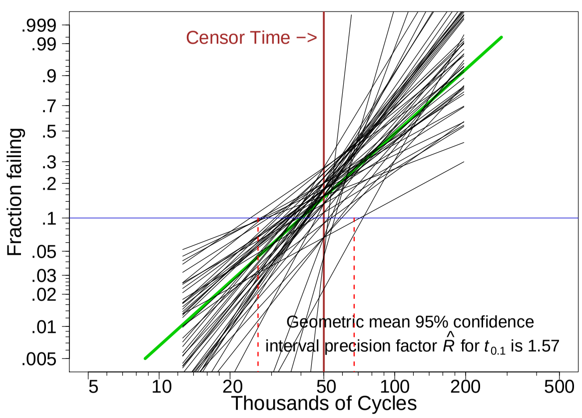
Simulated Weibull Life Test
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 180$, $t_c = 30$ Thousand Cycles



Summary of Simulated Weibull Life Tests
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 180$, $t_c = 30$ Thousand Cycles



Summary of Simulated Weibull Life Tests
Test Planning Values: $t^{\square}_{0.10} = 40$ and $\beta^{\square} = 2$
Test Plan: $n = 45$, $t_c = 50$ Thousand Cycles



Metal Spring Life Tests
Trade-offs Between Test Length and Sample Size

The geometric mean of the precision factors $\widehat{R}$ from 2000 simulated life tests for combinations of sample sizes n and test lengths t_c (conditional on $r \geq 1$ failures).

Test Length t_c	Sample Size n 45	180
30	2.45 (2.6)	1.49 (10.4)
50	1.56 (6.8)	—
120	1.46 (27.6)	—

Numbers in parenthesis are the expected number of failures for the test plans.

Summary of Simulations of the Proposed Metal Spring Life Tests to Estimate $t_{0.10}$

- For the $t_c = 30$ and $n = 45$ life test:
  - Enormous amount of variability in the ML estimates.
  - For many of the simulated data sets, no ML estimates exist because all units were censored.
- For the $t_c = 50$ and $n = 45$ life test:
  - A much more stable estimation process.
  - A substantial improvement in precision.
- For the $t_c = 120$ and $n = 45$ life test:
  - Only a small improvement in estimation of $t_{0.10}$, relative to the $t_c = 50$ and $n = 45$ test.
  - A big improvement for estimation of larger quantiles.
- For the $t_c = 30$ and $n = 180$ life test:
  - Stable estimation and good precision, but
  - Some extrapolation is required.

Chapter 13

Segment 3

Large-Sample Approximate Variances, Justification of the Sample-Size Formula, and Exponential Distribution Example

Large-Sample Approximate Variances

Under certain regularity conditions, the following results hold asymptotically (large sample)

- $\widehat{\boldsymbol{\theta}} \dot{\sim} \text{MVN}(\boldsymbol{\theta}, \Sigma_{\widehat{\boldsymbol{\theta}}})$, where $\Sigma_{\widehat{\boldsymbol{\theta}}} = I_{\boldsymbol{\theta}}^{-1}$, and

$$I_{\boldsymbol{\theta}} = \text{E}\left[-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] = \sum_{i=1}^{n} \text{E}\left[-\frac{\partial^2 \mathcal{L}_i(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right].$$

- Usually, interest centers on a scalar function of $\boldsymbol{\theta}$, such as a quantile or a failure probability.

Large-Sample Approximate Variances of Scalar Functions of the ML Estimators

- For a scalar quantity of interest (or a one-to-one function of a quantity of interest) $g = g(\widehat{\boldsymbol{\theta}}) \dot{\sim} \text{NORM}[g(\boldsymbol{\theta}), \text{Avar}(\widehat{g})]$, where

$$\text{Avar}(\widehat{g}) = \left[\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]' \Sigma_{\widehat{\boldsymbol{\theta}}} \left[\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right].$$

- A one-to-one function of a quantity of interest is often used to set the Wald confidence interal on a scale that is unrestricted (e.g., using $y_p = \log(t_p)$ instead of t_p).
- Generally, a variance factor that does not depend on σ or n can be obtained from

$$V_{\widehat{g}} = \frac{n}{\sigma^2}\text{Avar}(\widehat{g})$$

Sample Size Needed to Estimate the Mean of an Exponential Distribution Used to Describe Insulation Life

- Need a test plan that will estimate the mean life of insulation specimens at highly-accelerated (i.e., higher than usual voltage to get failure information quickly) conditions.
- Desire a 95% confidence interval with endpoints that are approximately 50% away from the estimated mean (so $R_T = 1.5$).
- Can assume an exponential distribution with a mean $\theta^\square = 1000$ hours.
- Simultaneous testing of all units; must terminate the test at 500 hours.

Sample Size Needed to Estimate the Mean of an Exponential Distribution Used to Describe Insulation Life-Continued

- ML estimate of the exponential mean is $\widehat{\theta} = TTT/r$, where TTT is the total time on test and r is the number of failures. It follows that

$$\text{V}_{\widehat{\theta}} = n\text{Avar}(\widehat{\theta}) = \frac{n}{\text{E}\left[-\frac{\partial^2\mathcal{L}(\theta)}{\partial\theta^2}\right]} = \frac{\theta^2}{1-\exp\left(-\frac{t_c}{\theta}\right)}$$

from which, using the delta method,

$$\text{V}^\square_{\log(\widehat{\theta})} = \frac{\text{V}^\square_{\widehat{\theta}}}{(\theta^\square)^2} = \frac{1}{1-\exp\left(-\frac{500}{1000}\right)} = 2.5415.$$

Thus the number of needed specimens (note that implicitly $\sigma^\square = 1$) is

$$n = \frac{z^2_{(1-\alpha/2)}\text{V}^\square_{\log(\widehat{\theta})}}{[\log(R_T)]^2} = \frac{(1.96)^2\, 2.5415}{[\log(1.5)]^2} \approx 60.$$

Chapter 13

Segment 4

Computation of Approximate Variance Factors for Log-Locations-Scale Distributions and an Example

Location-Scale Distributions and Single Right Censoring Asymptotic Variance-Covariance

Here we specialize the computation of sample sizes to situations in which

- $\log(T)$ is location-scale Φ with parameters (μ, σ).
- When the data are Type I singly right censored at t_c,

$$\frac{n}{\sigma^2}\Sigma_{(\widehat{\mu},\widehat{\sigma})} = \begin{bmatrix} \text{V}_{\widehat{\mu}} & \text{V}_{(\widehat{\mu},\widehat{\sigma})} \\ \text{V}_{(\widehat{\mu},\widehat{\sigma})} & \text{V}_{\widehat{\sigma}} \end{bmatrix} = \left[\frac{\sigma^2}{n} I_{(\mu,\sigma)}\right]^{-1} = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}^{-1}$$
$$= \left(\frac{1}{f_{11}f_{22} - f_{12}^2}\right)\begin{bmatrix} f_{22} & -f_{12} \\ -f_{12} & f_{11} \end{bmatrix}$$

where the f_{ij} values depend only on Φ and the standardized censoring time $\zeta_c = [\log(t_c) - \mu]/\sigma$ [or equivalently, the proportion failing by t_c, $p_c = \Phi(\zeta_c)$].

Location-Scale Distributions and Single Right Censoring Fisher Information Elements

The f_{ij} values are defined as:

$$f_{11} = f_{11}(\zeta_c) = \frac{\sigma^2}{n}\mathrm{E}\left[-\frac{\partial^2 \mathcal{L}_i(\mu,\sigma)}{\partial \mu^2}\right]$$
$$f_{22} = f_{22}(\zeta_c) = \frac{\sigma^2}{n}\mathrm{E}\left[-\frac{\partial^2 \mathcal{L}_i(\mu,\sigma)}{\partial \sigma^2}\right]$$
$$f_{12} = f_{12}(\zeta_c) = \frac{\sigma^2}{n}\mathrm{E}\left[-\frac{\partial^2 \mathcal{L}_i(\mu,\sigma)}{\partial \mu \partial \sigma}\right]$$

The f_{ij} values are available from tables or algorithm LSINF for the SEV (Weibull), normal (lognormal), largest extreme value (Fréchet), and logistic (loglogistic) distributions.

For a single fixed censoring time, the asymptotic variance-covariance factors $\mathrm{V}_{\widehat{\mu}}$, $\mathrm{V}_{\widehat{\sigma}}$, and $\mathrm{V}_{(\widehat{\mu},\widehat{\sigma})}$ are easily tabulated as a function of ζ_c.

Table of Information Matrix Elements and Variance Factors

Table C.20 provides for the normal/lognormal distributions, as functions of the standardized censoring time $\zeta_c = [\log(t_c) - \mu]/\sigma$:

- $100\Phi(\zeta_c)$, the percentage in the population failing by the standardized censoring time.
- Fisher information matrix elements f_{11}, f_{22}, and f_{12}.
- The asymptotic variance-covariance factors $\mathrm{V}_{\widehat{\mu}}$, $\mathrm{V}_{\widehat{\sigma}}$, and $\mathrm{V}_{(\widehat{\mu},\widehat{\sigma})}$.
- Asymptotic correlation $\rho_{(\widehat{\mu},\widehat{\sigma})}$ between $\widehat{\mu}$ and $\widehat{\sigma}$.
- The σ-known asymptotic variance factor $\mathrm{V}_{\widehat{\mu}|\sigma} = (n/\sigma^2)\mathrm{Avar}(\widehat{\mu}|\sigma)$, and the μ-known factor $\mathrm{V}_{\widehat{\sigma}|\mu} = (n/\sigma^2)\mathrm{Avar}(\widehat{\sigma}|\mu)$.

Sample Size to Estimate a Quantile of T when $\log(T)$ is Location-Scale (μ, σ)

- Let $g(\boldsymbol{\theta}) = t_p$ be the p quantile of T. Then $y_p = \log(t_p) = \mu + \Phi^{-1}(p)\sigma$, where $\Phi^{-1}(p)$ is the p quantile of the standardized random variable $Z = [\log(T) - \mu]/\sigma$. Suppose that the censoring time is t_c.
- The needed sample size, for a given target precision R_T factor is n is
$$n = \frac{z^2_{(1-\alpha/2)}(\sigma^{\square})^2\,\mathrm{V}_{y_p}}{[\log(R_T)]^2}$$
where
$$\mathrm{V}_{y_p} = \mathrm{V}_{\widehat{\mu}} + \left[\Phi^{-1}(p)\right]^2\mathrm{V}_{\widehat{\sigma}} + 2\left[\Phi^{-1}(p)\right]\mathrm{V}_{(\widehat{\mu},\widehat{\sigma})}$$
is obtained a function of the quantile of interest p and and the proportion failing at the end of the test $p_c = \Pr(T \le t_c)$.
- Figure 10.5 gives V_{y_p} as a function of p and p_c for the Weibull distribution.

Generalization: Location-Scale Parameters and Multiple Censoring

In some applications, a life test may run in parts, each part having a different censoring time (e.g., testing at two different locations or beginning as lots of units to be tested are received). In this case we need to generalize the single-censoring formula. Assume that a proportion δ_i ($\sum_{i=1}^{k} \delta_i = 1$) of data are to be run until right censoring time t_{c_i} or failure (which ever comes first). In this case,

$$\frac{n}{\sigma^2}\Sigma_{(\widehat{\mu},\widehat{\sigma})} = \begin{bmatrix} \mathrm{V}_{\widehat{\mu}} & \mathrm{V}_{(\widehat{\mu},\widehat{\sigma})} \\ \mathrm{V}_{(\widehat{\mu},\widehat{\sigma})} & \mathrm{V}_{\widehat{\sigma}} \end{bmatrix} = \left[\frac{\sigma^2}{n} I_{(\mu,\sigma)}\right]^{-1}$$
$$= \left(\frac{1}{J_{11}J_{22} - J_{12}^2}\right)\begin{bmatrix} J_{22} & -J_{12} \\ -J_{12} & J_{11} \end{bmatrix}$$

where $J_{11} = \sum_{i=1}^{k} \delta_i f_{11}(z_{c_i})$, $J_{22} = \sum_{i=1}^{k} \delta_i f_{22}(z_{c_i})$, and $J_{12} = \sum_{i=1}^{k} \delta_i f_{12}(z_{c_i})$ where $z_{c_i} = (\log(t_{c_i}) - \mu)/\sigma$.

In this case, the asymptotic variance-covariance factors $\mathrm{V}_{\widehat{\mu}}$, $\mathrm{V}_{\widehat{\sigma}}$, and $\mathrm{V}_{(\widehat{\mu},\widehat{\sigma})}$ depend on Φ, the standardized censoring times z_{c_i}, and the proportions δ_i, $i = 1, \ldots k$.

References

Meeker, W. Q., L. A. Escobar, and F. G. Pascual (2021). *Statistical Methods for Reliability Data* (Second Edition). Wiley. [1]