# Chapter 11

## Special Parametric Models

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# Special Parametric Models
# Chapter 11 Objectives

Topics discussed in this chapter are:

- ML estimation for the generalized gamma (GENG) distribution and other special distributions.
- ML estimation for the Birnbaum-Saunders distribution.
- ML estimation for the limited failure population model.
- ML estimation for truncated data (or data observed from from truncated distributions).
- ML estimation for threshold-parameter distributions like the 3-parameter lognormal and the 3-parameter Weibull distributions.

# Chapter 11

## Segment 1

## Fitting Other Distributions and Models
## Generalized Gamma Distribution
## Birnbaum-Saunders Distribution

# Fitting Other Distributions and Models

- Likelihood principles similar to location-scale distributions.

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} L_i(\boldsymbol{\theta}; \text{data}_i) = \prod_{i=1}^{n} [f(t_i; \boldsymbol{\theta})]^{\delta_i} [1 - F(t_i; \boldsymbol{\theta})]^{1-\delta_i}$$

  where $\text{data}_i = (t_i, \delta_i)$,

$$\delta_i = \begin{cases} 1 & \text{if } t_i \text{ is an exact failure} \\ 0 & \text{if } t_i \text{ is a right-censored observation} \end{cases}$$

  and $F(t_i; \boldsymbol{\theta})$ and $f(t_i; \boldsymbol{\theta})$ are the specified distribution's cdf and pdf, respectively.

- For some non-location-scale distributions (e.g., threshold distributions) the density approximation breaks down and one should use the actual interval probability instead.

- Left-censored and interval-censored observations also could be included, as described in Chapter 2.

# Confidence Intervals for Other Distributions and Models

Confidence intervals and regions similar to location-scale distributions.

- Wald confidence intervals (using the delta method and appropriate transformations) are simple and are adequate in large samples or for rough approximations.
- Profile likelihood and corresponding likelihood-based intervals provide useful insight into the information available about a particular parameter or functions of parameters.
- Bootstrap and simulation-based intervals will generally provide confidence intervals procedures with good coverage properties.

## Fitting the Generalized Gamma Distribution

- $T \sim \text{GENG}(\mu, \sigma, \lambda)$.

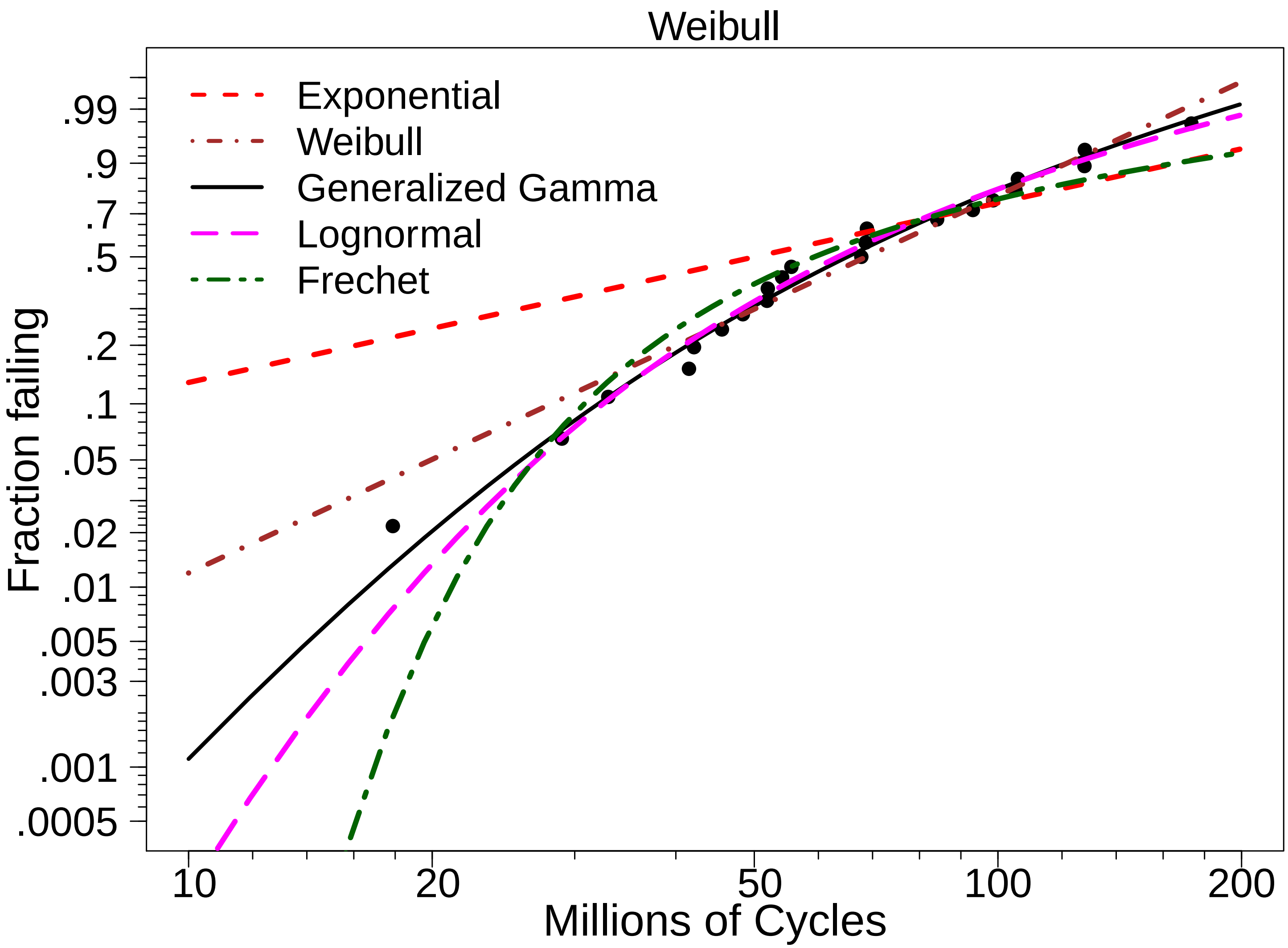
- Special cases: Fréchet ($\lambda = -1$), Lognormal ($\lambda = 0$), and Weibull ($\lambda = 1$).

- A more flexible curve for the data.

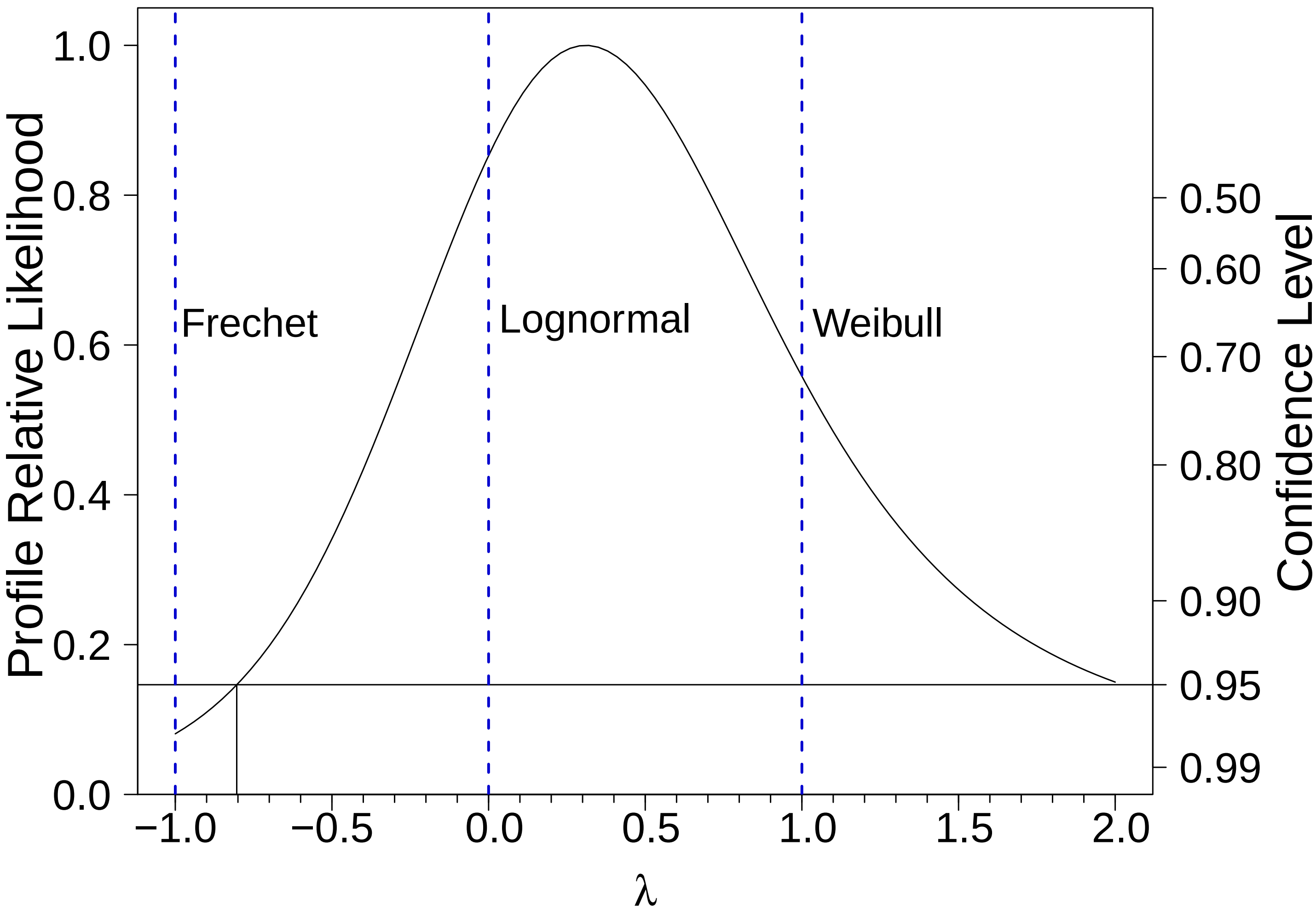
- Can use GENG to see if there is evidence for one distribution over the others.

- Generally not suitable for small sample sizes or when extrapolation is required (because of extremely wide confidence intervals).

## Weibull Probability Plot of the Ball Bearing Failure Data Showing Exponential, Weibull, Lognormal, Fréchet, Generalized Gamma ML Estimates of $F(t)$



# Profile Likelihood Plot for GENG $\lambda$ for the Bearing Failure Data Showing Weibull, Lognormal, and Fréchet, Distributions as Special Cases



# Fitting the GENG Distribution to the Bearing Data-Conclusions

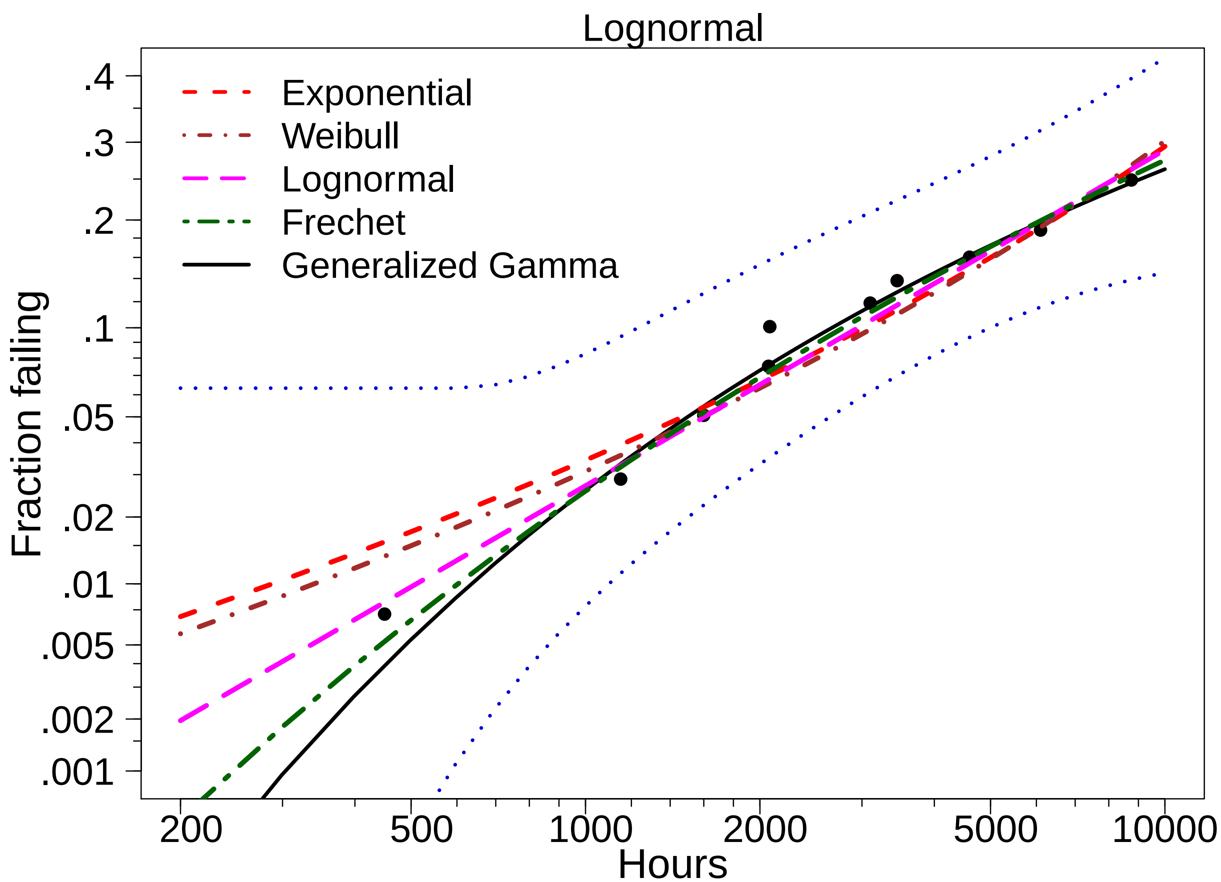
- For the bearing data, the GENG distribution provides a compromise between lognormal and Weibull.

- The GENG, lognormal, Fréchet, and Weibull agree reasonably well within the range of the data. Important deviations in the lower tail of the distribution illustrate the danger of extrapolation.

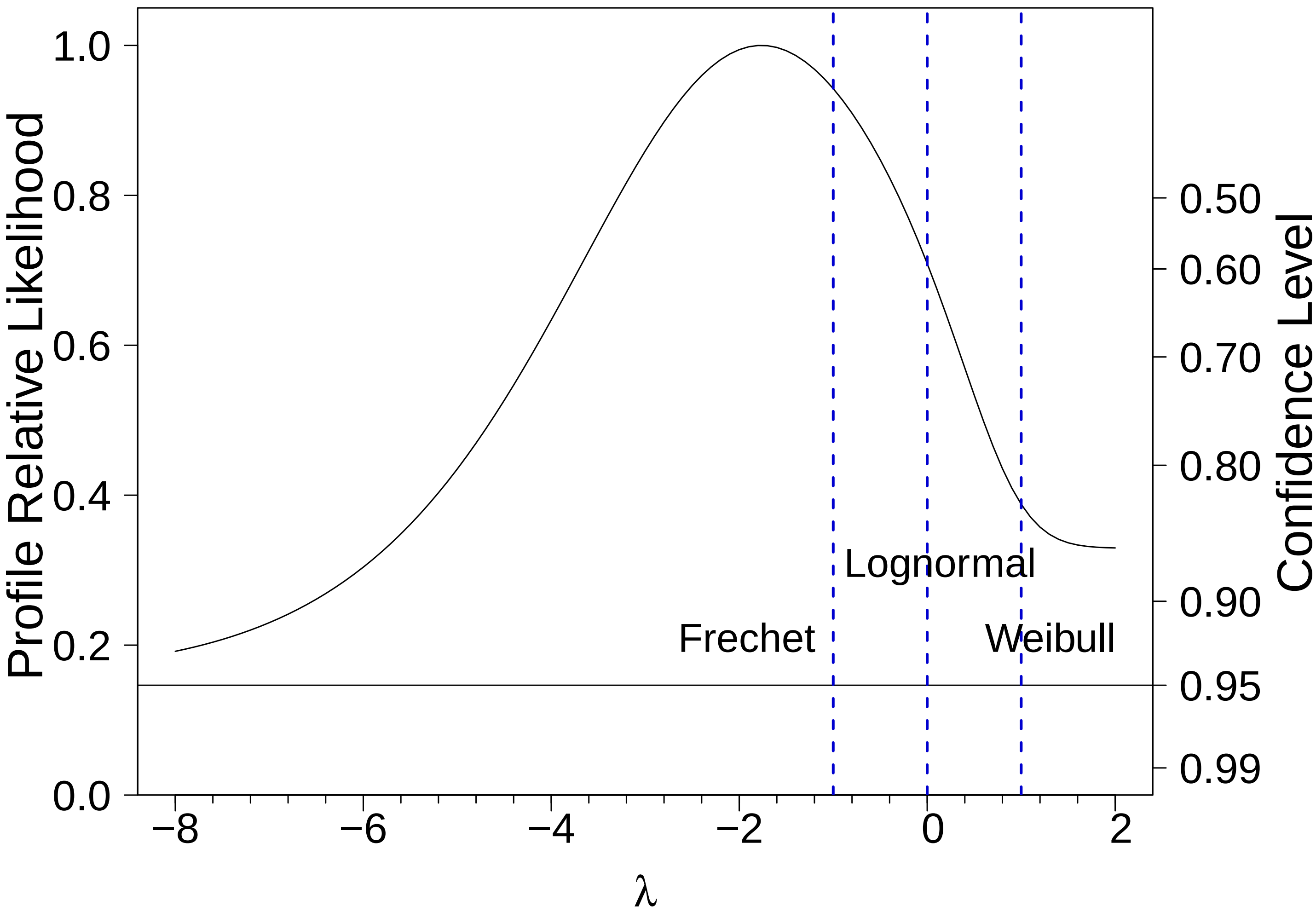
- The profile likelihood shows that the data do not, in this case, provide strong evidence for one distribution over the other.

- Physics of failure suggests the lognormal distribution.

# Lognormal Probability Plot of the <u>Fan Failure</u> Data Showing GNG ML Estimates and Corresponding Approximate 95% Pointwise Confidence Intervals for $F(t)$ along with Exponential, Weibull, Lognormal, and Fréchet ML Estimates of $F(t)$



## Profile Likelihood Plot for GENG $\lambda$ for the Fan Failure Data Showing Weibull, Lognormal, and Fréchet Distributions as Special Cases



## Fitting the GENG Distribution to the Fan Data

- Only 12 failures out of 70 units (multiple censoring).
- Lognormal fits the data well. Weibull and exponential also fit the data reasonably well. Can GENG do better?
- The GENG has a larger likelihood than the other distributions, but the difference is statistically unimportant.
- Comparison shows that the position of the smallest observation does not have much influence on the fit (small order statistics have a large amount of variability).
- Fitting a 3-parameter distribution to 12 failures is **overfitting**.

# Fitting the Birnbaum-Saunders Distribution

- The Birnbaum-Saunders (BISA) distribution is based on a probability describing **discrete-time** growth of fatigue cracks until fracture.

- For a variable $T$ with Birnbaum–Saunders distribution, BISA$(\theta, \beta)$,

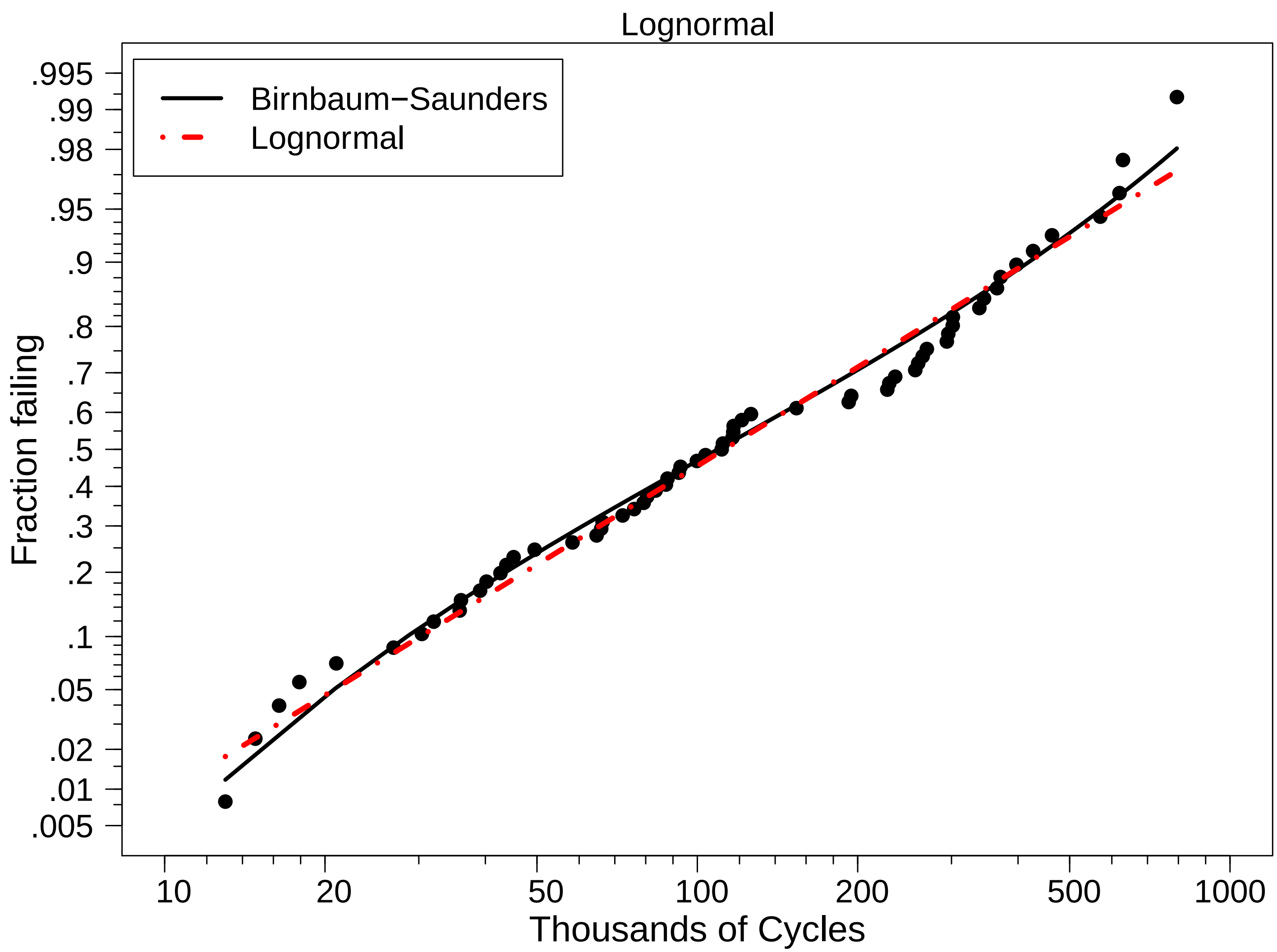
$$\Pr(T \leq t) = F(t;\beta,\theta) = \Phi_{\text{norm}}(\zeta)$$
$$f(t;\beta,\theta) = \frac{\sqrt{\frac{t}{\theta}} + \sqrt{\frac{\theta}{t}}}{2\beta t}\,\phi_{\text{norm}}(\zeta)$$

where $t \geq 0$, $\theta > 0$ is a scale parameter, $\beta > 0$ is a shape parameter, and

$$\zeta = \frac{1}{\beta}\left(\sqrt{\frac{t}{\theta}} - \sqrt{\frac{\theta}{t}}\right).$$

- For some values of their parameters, the Birnbaum-Saunders and lognormal distributions are close to each other.

# Lognormal Probability Plot of Yokobori's Fatigue Failure Data on Cylindrical Specimens at 52.658 ksi Showing Lognormal and BISA Distribution ML Estimates



**Chapter 11**

**Segment 2**

**Fitting the Limited Failure Population Model**

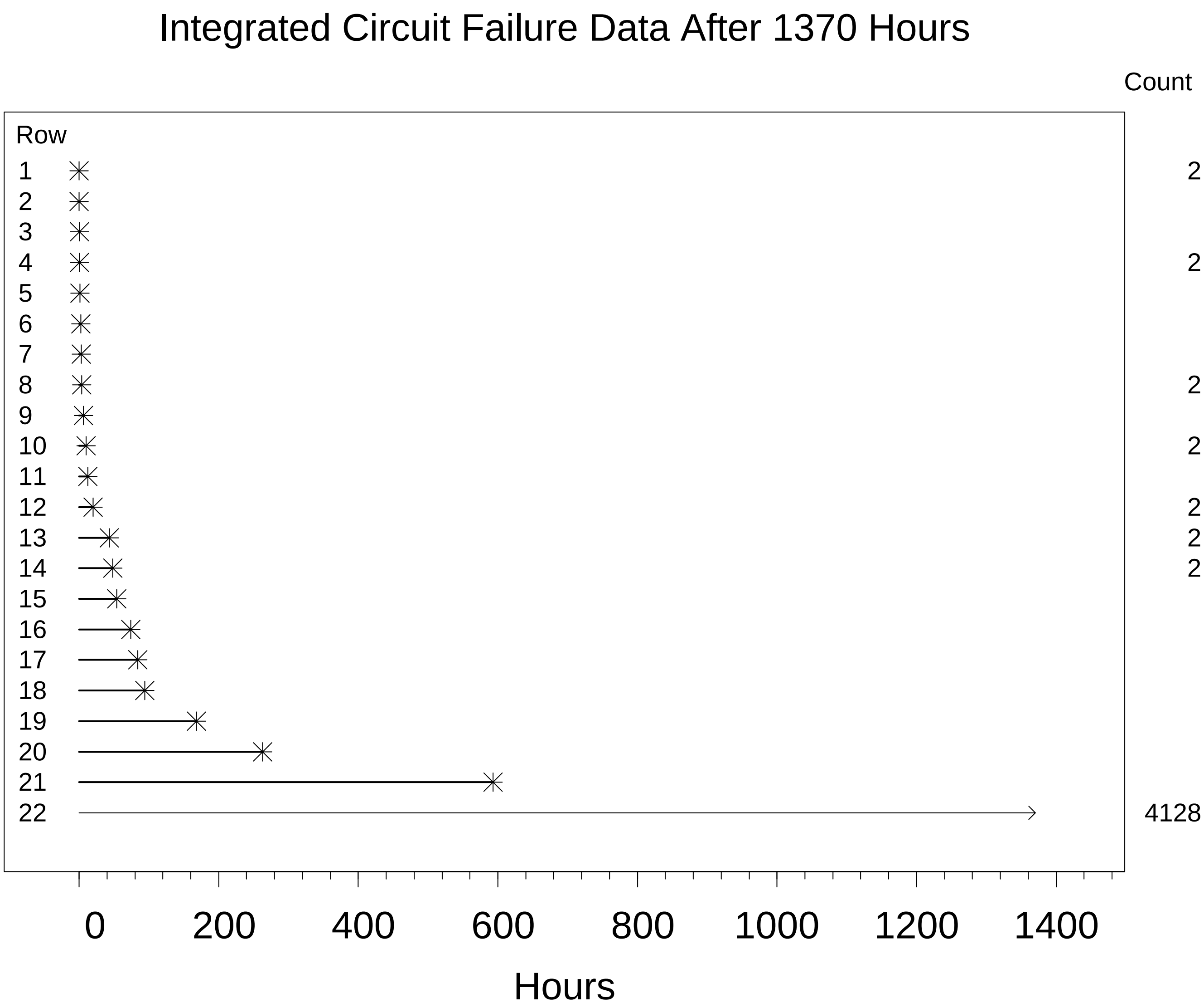
## IC Data

$n = 4{,}156$ IC's tested for 1,370 hours at 80°C and 80% relative humidity; there were 28 failures (Meeker 1987).

| | | | | | |
|---|---|---|---|---|---|
| 0.10 | 0.10 | 0.15 | 0.60 | 0.80 | 0.80 |
| 1.20 | 2.5 | 3.0 | 4.0 | 4.0 | 6.0 |
| 10.0 | 10.0 | 12.5 | 20. | 20. | 43. |
| 43. | 48. | 48. | 54. | 74. | 84. |
| 94. | 168. | 263. | 593. | | |

Want to estimate the proportion of defective units.

# Event Plot
# Integrated Circuit Life Test Data



# Limited Failure Population (LFP) Model (aka Defective Subpopulation Model)

- Only a small proportion $p$ of the population is susceptible to failure.

- The Weibull/LFP model is

$$\Pr(T \le t) = pF(t;\mu,\sigma) = p\Phi_{\text{sev}}\left[\frac{\log(t)-\mu}{\sigma}\right].$$

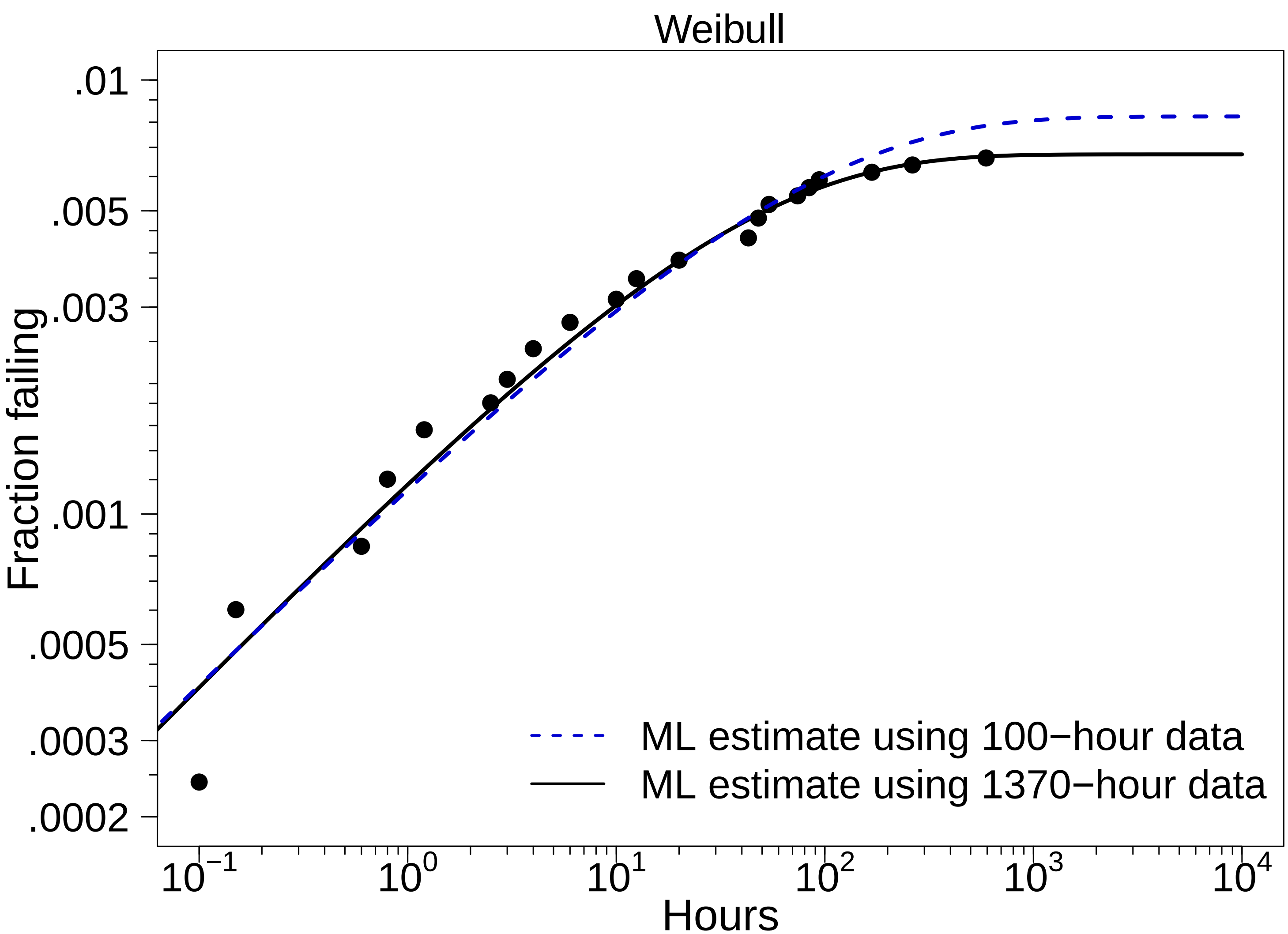
  Similar for lognormal or other distributions.

- ML methods work well. The likelihood has the form

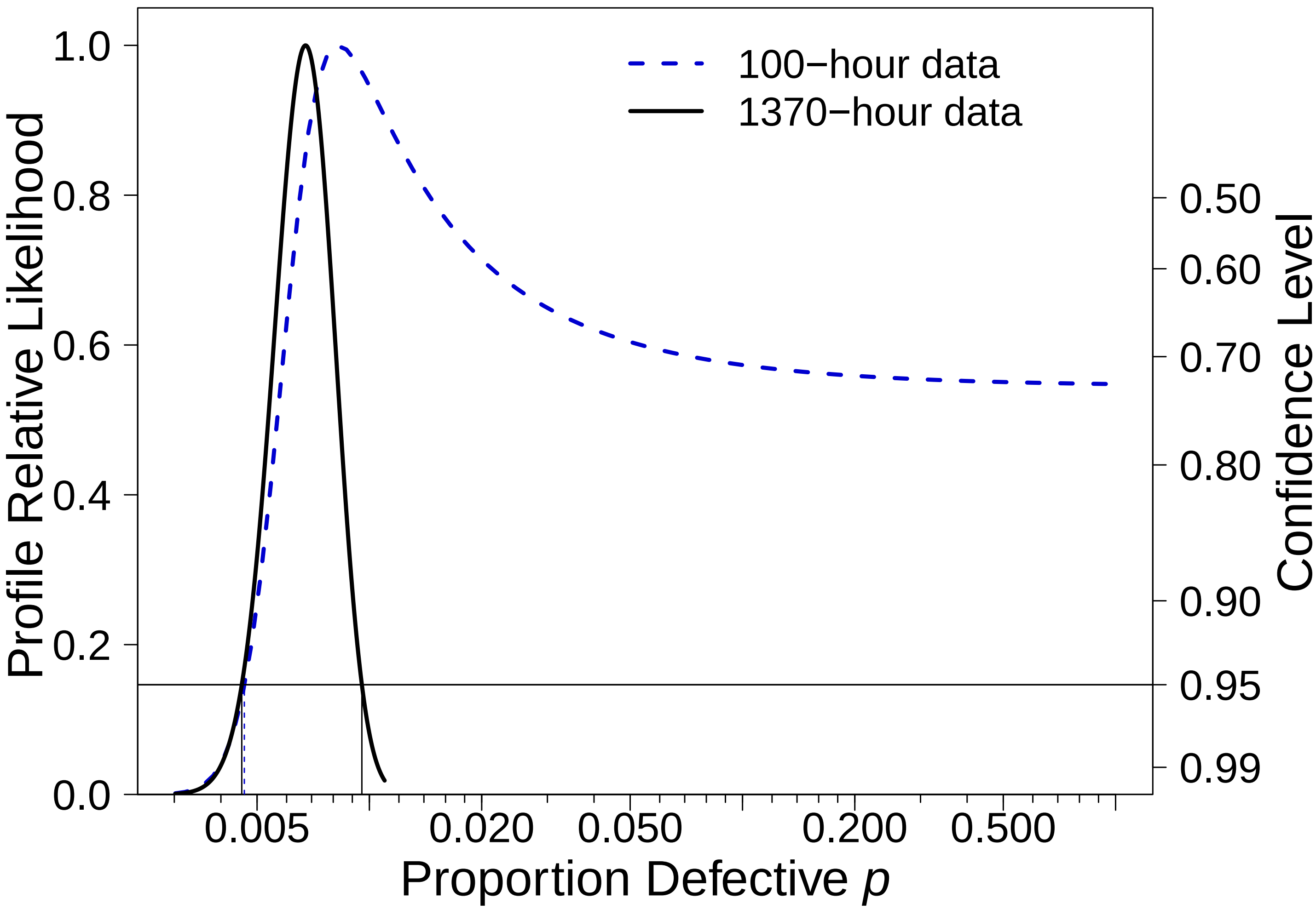
$$L(\mu,\sigma,p) = \prod_{i=1}^{n}\left\{\frac{p}{\sigma}\,\phi_{\text{sev}}\left[\frac{\log(t_i)-\mu}{\sigma}\right]\right\}^{\delta_i} \times \left\{1 - p\Phi_{\text{sev}}\left[\frac{\log(t_i)-\mu}{\sigma}\right]\right\}^{1-\delta_i}.$$

  Need to have had a high proportion (e.g., greater than 0.90) of the defective units fail in order to identify $p$.

# Weibull Probability Plot of Integrated Circuit Failure-Time Data with ML Estimates of the Weibull/LFP Model After 1370 Hours and 100 Hours of Testing



# Comparison of Profile Likelihoods for $p$, the LFP Proportion Defective, After 1370 and 100 Hours of Testing



## Relationship Between Wald and Profile Likelihood-Based Confidence Regions/Intervals

**Result:** Using the Wald (normal-theory) based interval is equivalent to using a quadratic approximation to the log-likelihood profile.

- See Meeker and Escobar (1995) for a proof.
- Likelihood-based intervals are invariant to transformation.
- Simulation and some theory suggest that the likelihood-based interval provides a better asymptotic approximation.
- Differences between the Wald and likelihood intervals for the LFP parameter $p$ provides an extreme example of where the quadratic approximation breaks down.

# Chapter 11

# Segment 3

# Analysis of Truncated Data

## The Difference Between Censoring and Truncation

- Suppose that there is an inspection process to screen out manufactured parts that do not meet specifications.
- In order to meet specifications the part must have a diameter that is between 2.999 and 3.001. Rejected parts are reprocessed.
- It is desired to use a sample of **accepted parts** to estimate the distribution of the diameters.
- The measured sizes are $2.999 \le x_1, x_2, \ldots, x_n \le 3.001$ and that $r_{\text{lower}}$ parts were rejected for being too small and that $r_{\text{upper}}$ were rejected for being too large. Then the the data are a censored sample with $r_{\text{lower}}$ left-censored at 2.999 observations and $r_{\text{upper}}$ right-censored observations at 3.001.
- If $r_{\text{lower}}$ and $r_{\text{upper}}$ are unknown, then the sample is truncated on the left at 2.999 and on the right at 3.001.
- There can be substantially less information in the truncated sample.

## Truncated Data In Reliability Applications

Truncated data arise in certain kinds of reliability applications

- In a study to predict the failure times of power transformers, there was no information about power transformers that had been installed and removed before 1980. There were, however, a large number of transformers that had been installed before 1980 and that were still in service. These transformers are a sample from left-truncated distributions.

- Starting in 2013, Backblaze (a company that provides cloud backup services) has been posting quarterly reports giving information about all of the thousands of disk drives that they are using to store data. Because many of the drives were installed before 2013 and the reported data contain no information about drives installed and removed before 2013, the drives that were installed before 2013 have been sampled from a left-truncated distribution.

## More Truncated Data Applications

- In the manufacturing aircraft engine components, billets of a titanium alloy are inspected with ultrasound to detect material flaws that could, if not detected, lead to component failure. Data from the “finds” (size of the flaw and strength of the ultrasound reflected signal) are used to estimate probability of detection as a function of flaw size). The data are a sample from a left-truncated distribution because the existence of small flaws with signals below the noise floor are not detected.

## Likelihood for Left-Truncated Data

- Likelihood for an interval-censored left-truncated observation.

$$L_i(\boldsymbol{\theta}) = \Pr(t_i^L < T_i \le t_i | T_i > \tau_i^L) = \frac{F(t_i;\boldsymbol{\theta}) - F(t_i^L;\boldsymbol{\theta})}{1 - F(\tau_i^L;\boldsymbol{\theta})}, \quad t_i > t_i^L \ge \tau_i^L.$$

- Likelihood for an exact-failure left-truncated observation.

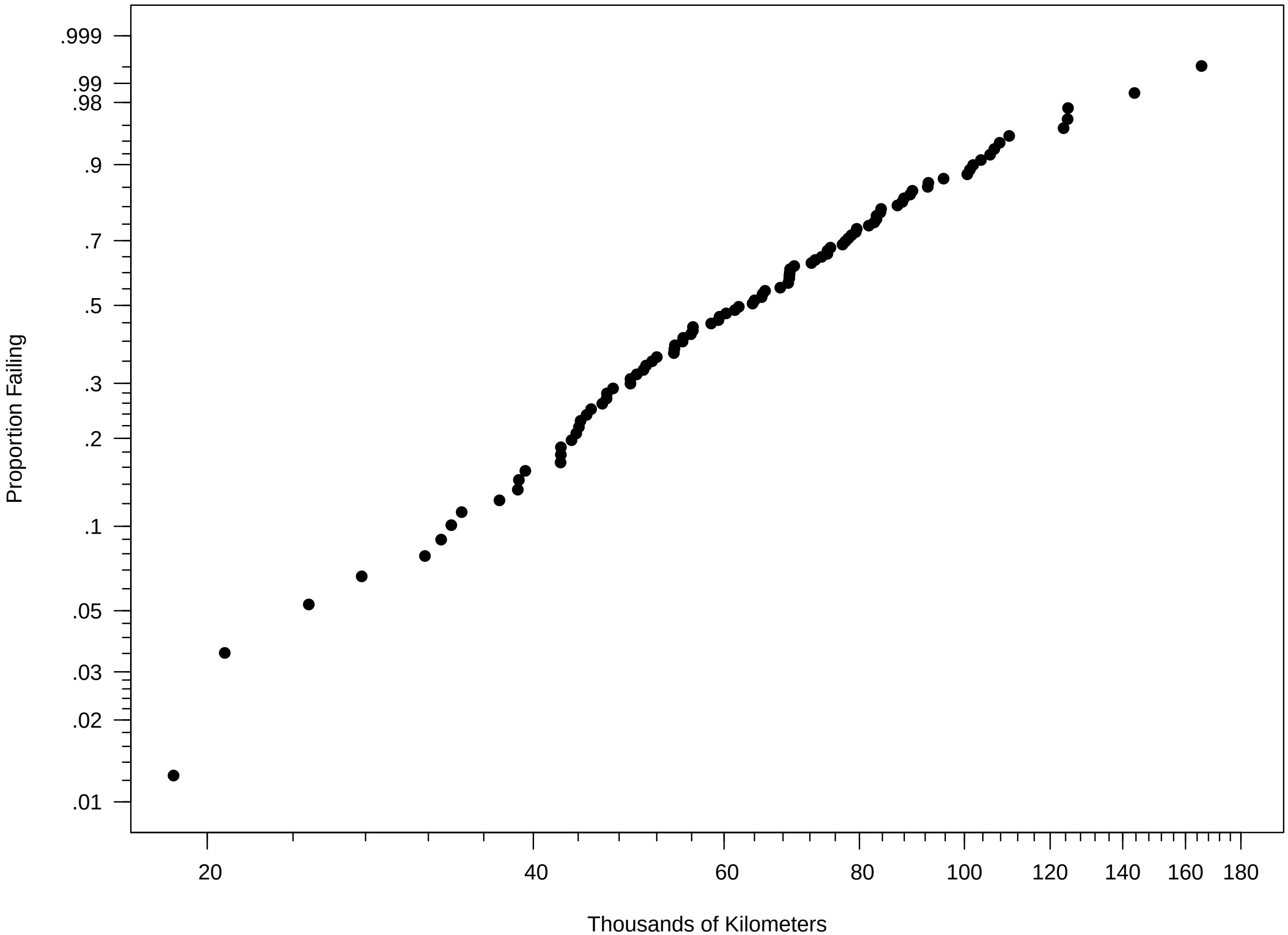
$$L_i(\boldsymbol{\theta}) = \frac{f(t_i;\boldsymbol{\theta})}{1 - F(\tau_i^L;\boldsymbol{\theta})}, \quad t_i > \tau_i^L.$$

- Note that software without the explicit ability to handle truncated data can be tricked into handling an observation at $t_i$ (censored or not) that is left-truncated at $\tau_i^L$ by including a dummy left-censored observation at $\tau_i^L$ with a weight of $-1$. A similar trick can be used for right-truncated observations.

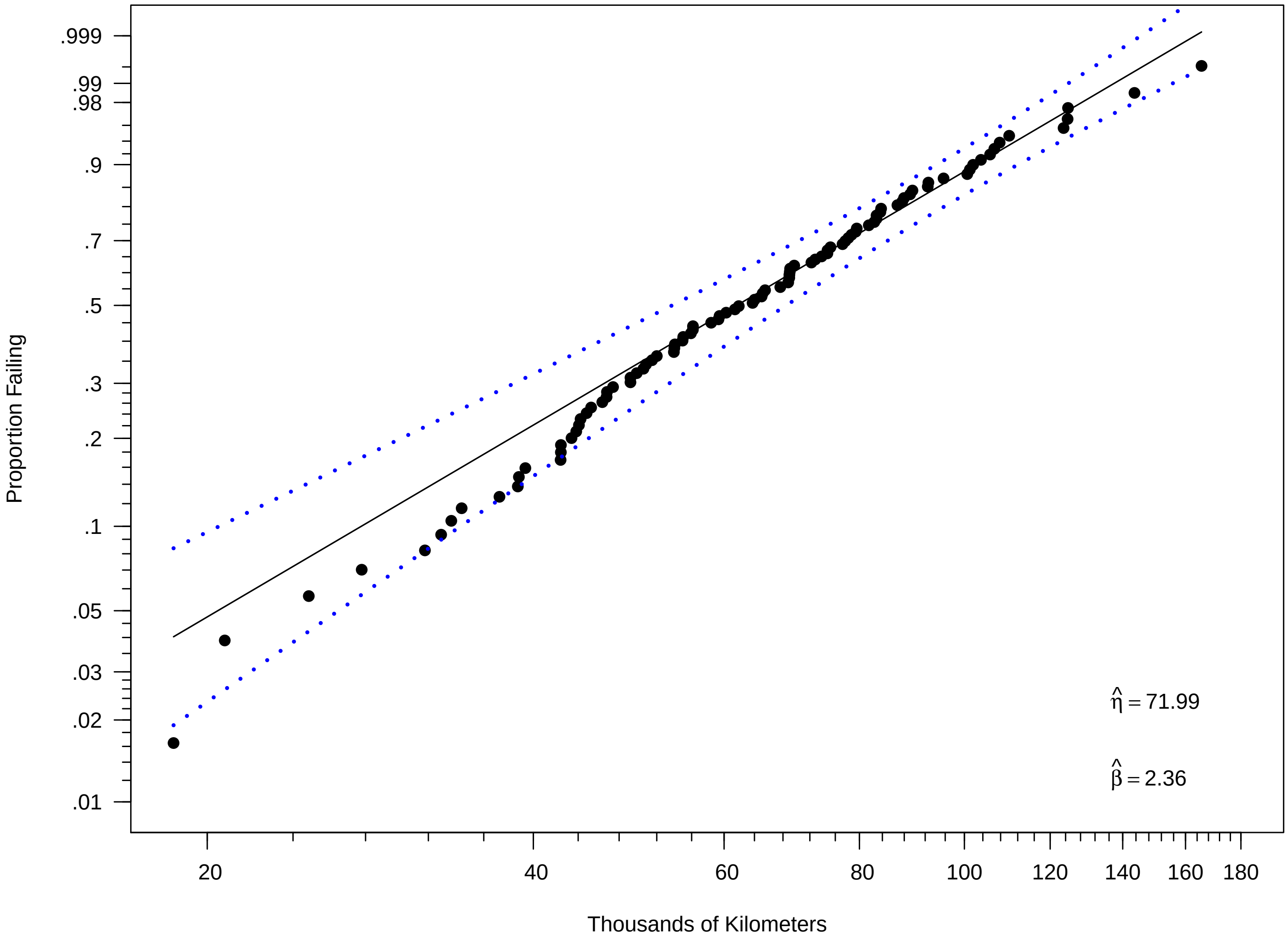
# Distribution of Brake Pad Life from Observational Data

- Pad wear ($W$ as a proportion of wear at the end of life) was measured and distance driven ($V$ in thousands of km) was recorded on automobiles that came in for service. Data from Kalbfleisch and Lawless (1992).
- Time of failure for each pad was imputed from the observed wear rate as $Y = V/W$.
- Units that already had a pad replacement were omitted from the data. Thus, high-rate units are under-represented in the sample.
- To analyze the data, each unit can be viewed as having been left-truncated at its observation time (if it had failed before its observation time, we would not know of the unit's existence because it would have been omitted from the sample).

# Weibull Probability Plot
# of the Nonparametric Estimate of Brake Pad Life, Conditional on Failure After 6.951 Thousand km



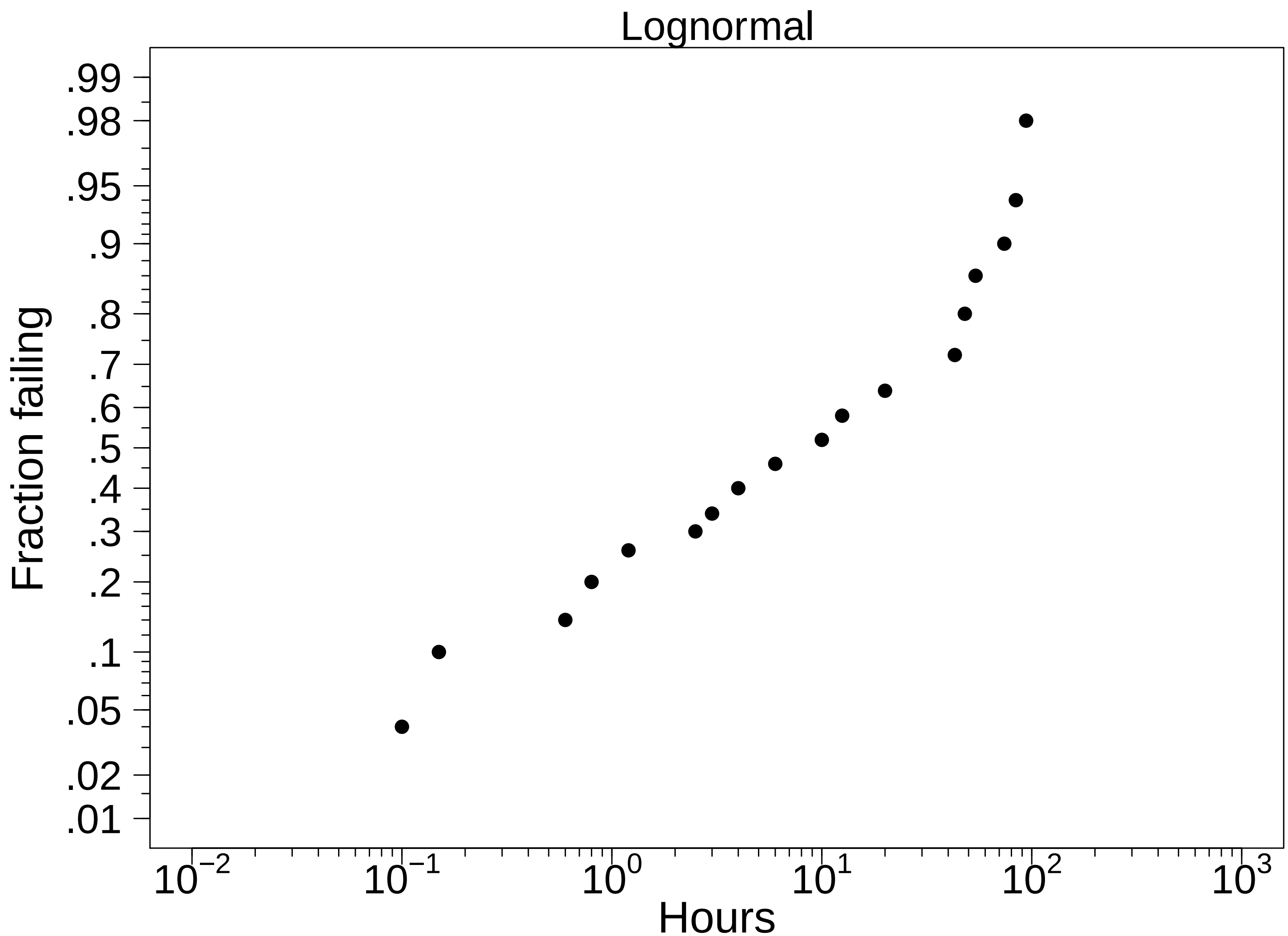
# Weibull Probability Plot of the Weibull-Adjusted Nonparametric Estimate of the Brake Pad Life Distribution



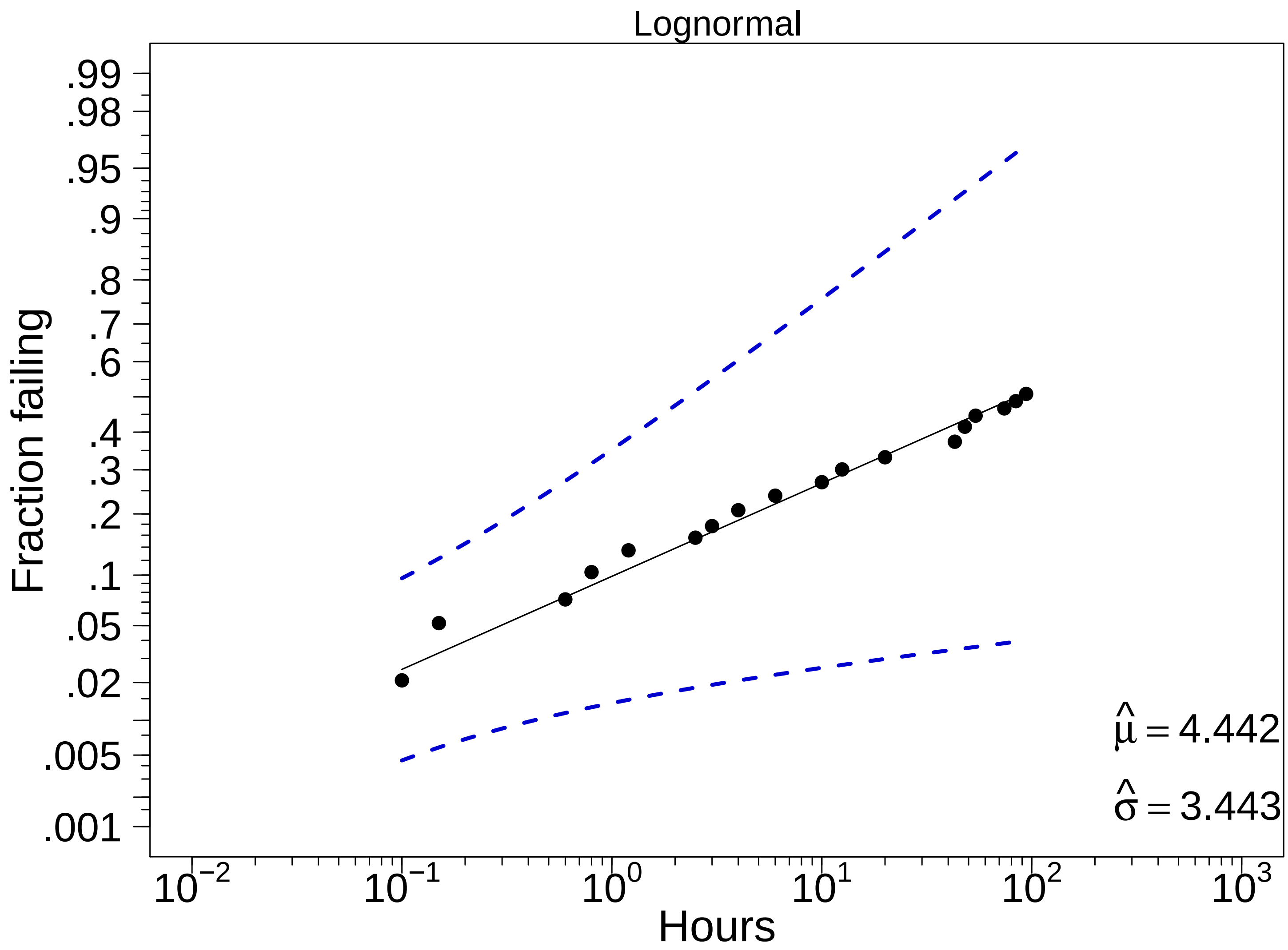
## IC Failure Data from a Limited Failure Population

- Of the $n$ =4156 integrated circuits tested, there were 25 failures in the first 100 hours of testing.
- The number of susceptible units in the sample is unknown.
- The 25 failures can be viewed as a sample from a distribution truncated on the right at 100 hours.

# Lognormal Probability Plot of the Nonparametric Estimate of the IC Failure-Time Distribution Conditional on Failure Before 100 Hours



# Lognormal Probability Plot of the Lognormal-Adjusted (Unconditional) Nonparametric Estimate of the IC Failure-Time Distribution

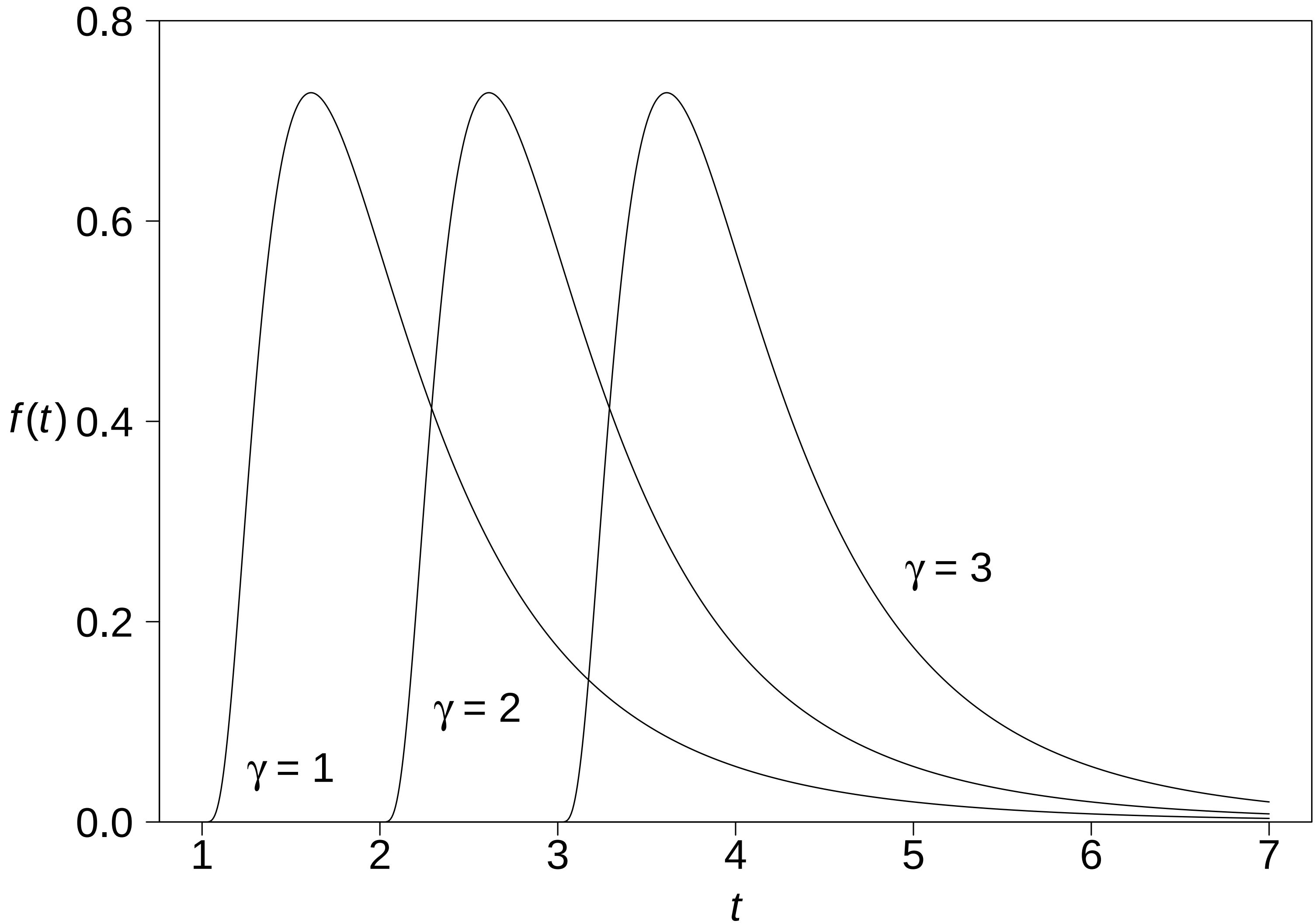


# Chapter 11

## Segment 3

## Fitting Distributions with a Threshold Parameter
## Three-Parameter Weibull Distribution
## Three-Parameter Lognormal Distribution

## pdfs for Threshold (Three-Parameter) Lognormal Distributions for $\mu = 0$ and $\sigma = 0.5$ with $\gamma =$ 1,2,3.



## Threshold Parameter Log-Location-Scale Distributions

- Let $\gamma$ be the threshold parameter. Then

$$F(t;\mu,\sigma,\gamma) = \Phi\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right]$$
$$f(t;\mu,\sigma,\gamma) = \frac{1}{\sigma(t-\gamma)}\,\phi\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right]$$

  for $t > \gamma$.

- The three-parameter Weibull (use $\Phi_{\text{sev}}$ and $\phi_{\text{sev}}$) and the three-parameter lognormal (use $\Phi_{\text{norm}}$ and $\phi_{\text{norm}}$) distributions are special cases.

## Inferences Threshold Parameter Log-Location-Scale Distributions Assuming that the Threshold Parameter $\gamma$ is Given

- If $\gamma$ can be assumed to be known, we can subtract $\gamma$ from all times and fit the two-parameter Weibull distribution to estimate $\mu$ and $\sigma$.

- Need to adjust inferences accordingly (e.g., add $\gamma$ back into estimates of quantiles or subtract $\gamma$ from times before computing probabilities).

- Similar methods can be used for other distributions for positive random variables.

- ML works if used correctly.

# The Three-Parameter Weibull Distribution Likelihood for Right-Censored Data

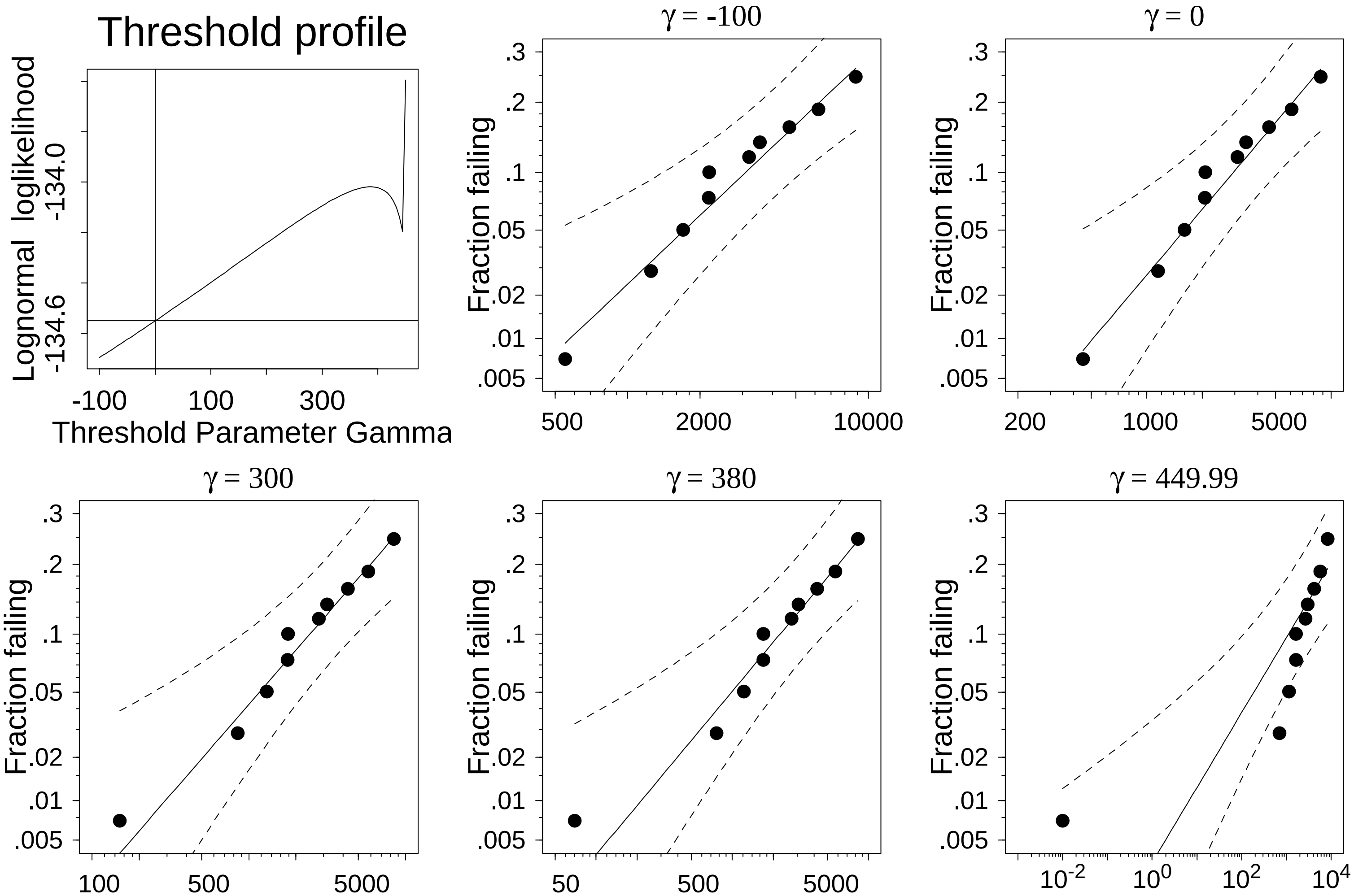
- The likelihood has the form

$$
\begin{aligned}
L(\mu, \sigma, \gamma) &= \prod_{i=1}^{n} L_i(\mu, \sigma, \gamma; \mathsf{data}_i) \\
&= \prod_{i=1}^{n} \{f(t_i; \mu, \sigma, \gamma)\}^{\delta_i} \{1 - F(t_i; \mu, \sigma, \gamma)\}^{1-\delta_i} \\
&= \prod_{i=1}^{n} \left\{ \frac{1}{\sigma(t_i - \gamma)} \phi_{\mathsf{sev}} \left[ \frac{\log(t_i - \gamma) - \mu}{\sigma} \right] \right\}^{\delta_i} \\
&\qquad \times \left\{ 1 - \Phi_{\mathsf{sev}} \left[ \frac{\log(t_i - \gamma) - \mu}{\sigma} \right] \right\}^{1-\delta_i}.
\end{aligned}
$$

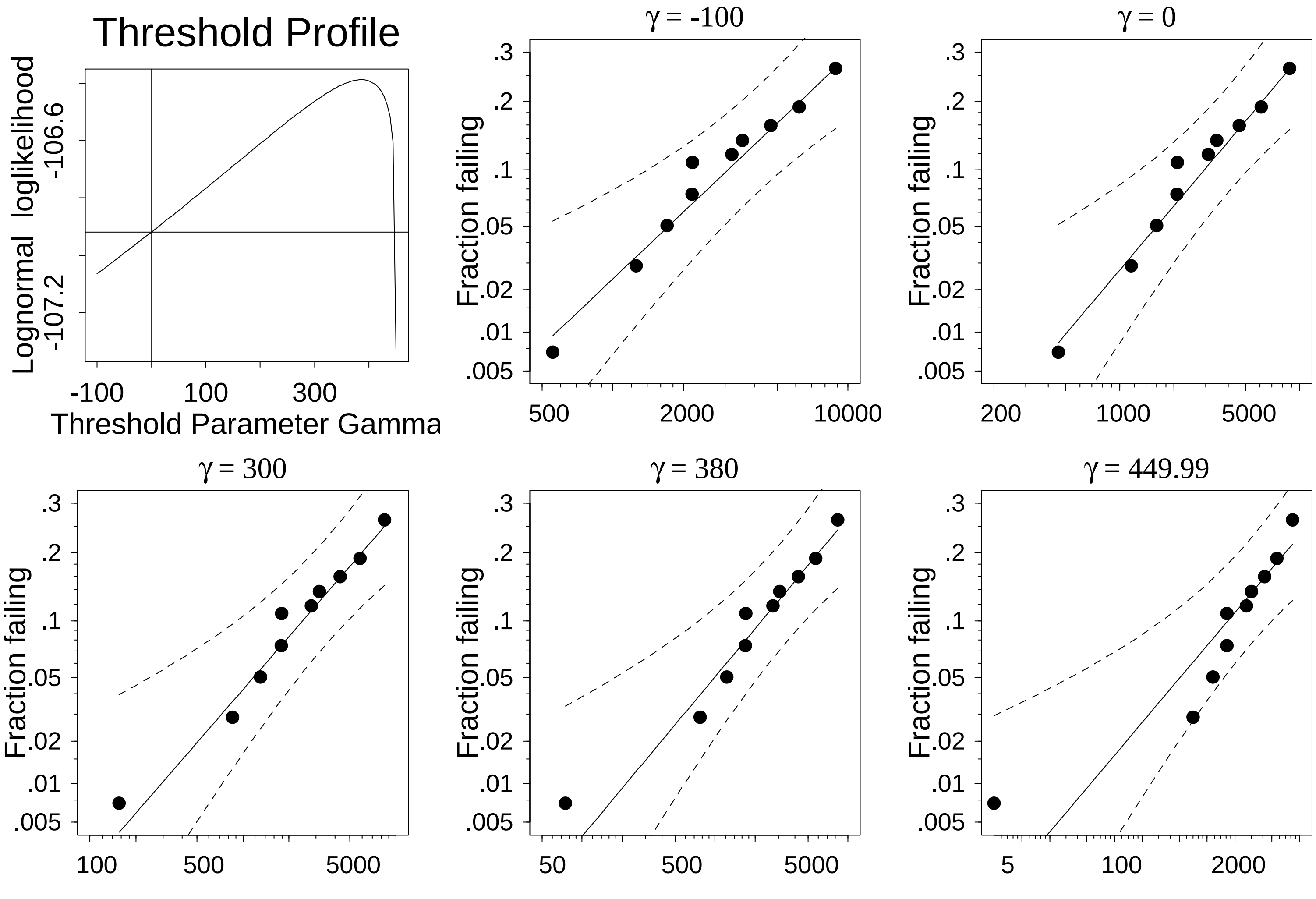
- Problem: when $\gamma \to t_{(1)}$ and $\sigma \to 0$, $L(\mu, \sigma, \gamma) \to \infty$.

- **Solution:** Do not use the density approximation; use the correct likelihood (based on small intervals).

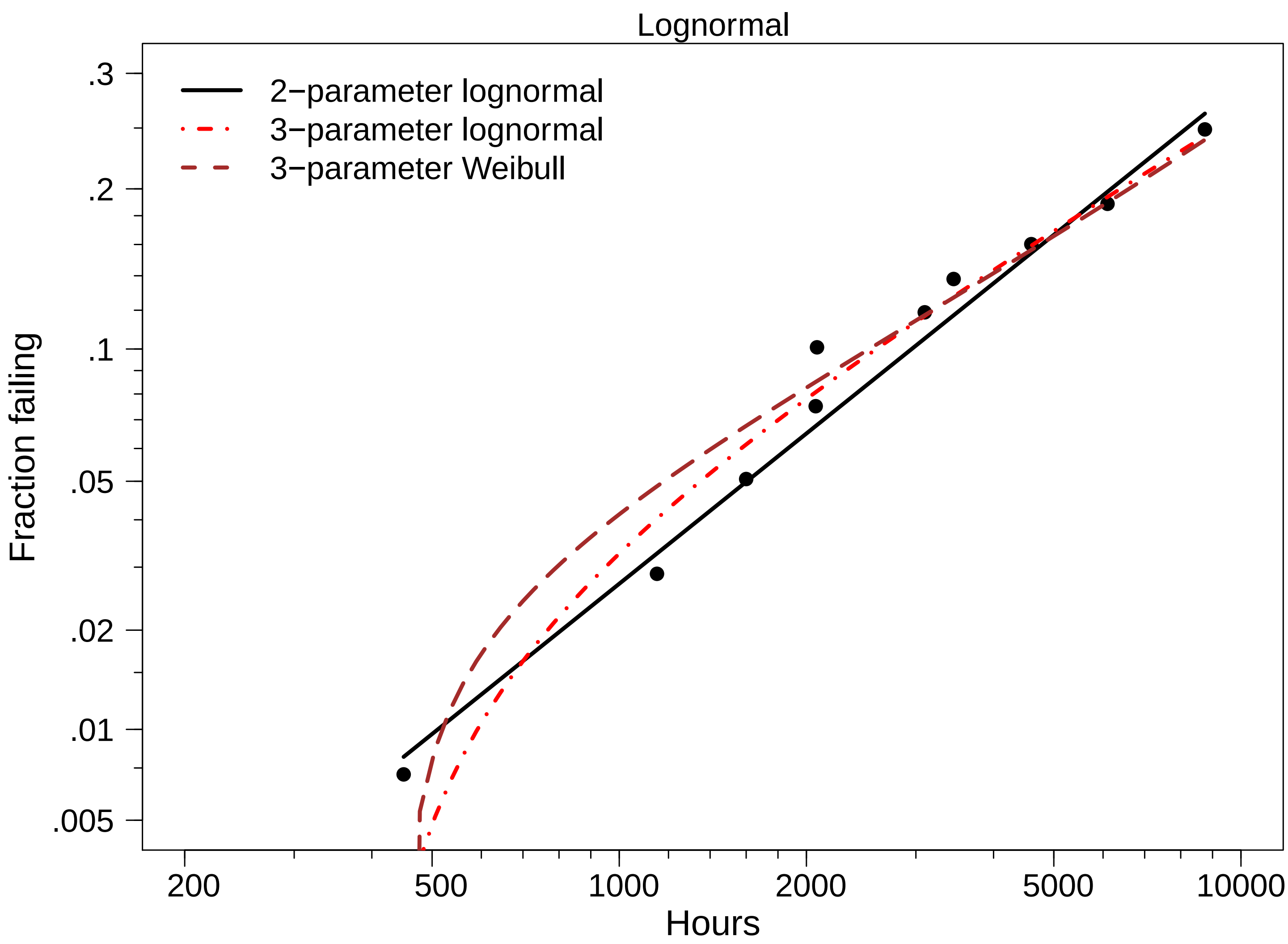
# Density Approximation Profile Likelihood for $\gamma$ and 3-Parameter Lognormal Probability Plots of the Fan Data with $\gamma$ Varying Between $-100$ and $449.999$



# Correct Likelihood ($\Delta = 0.01$) Profile Likelihood for $\gamma$ and 3-Parameter Lognormal Probability Plots of the Fan Data with $\gamma$ Varying Between $-100$ and $449.999$



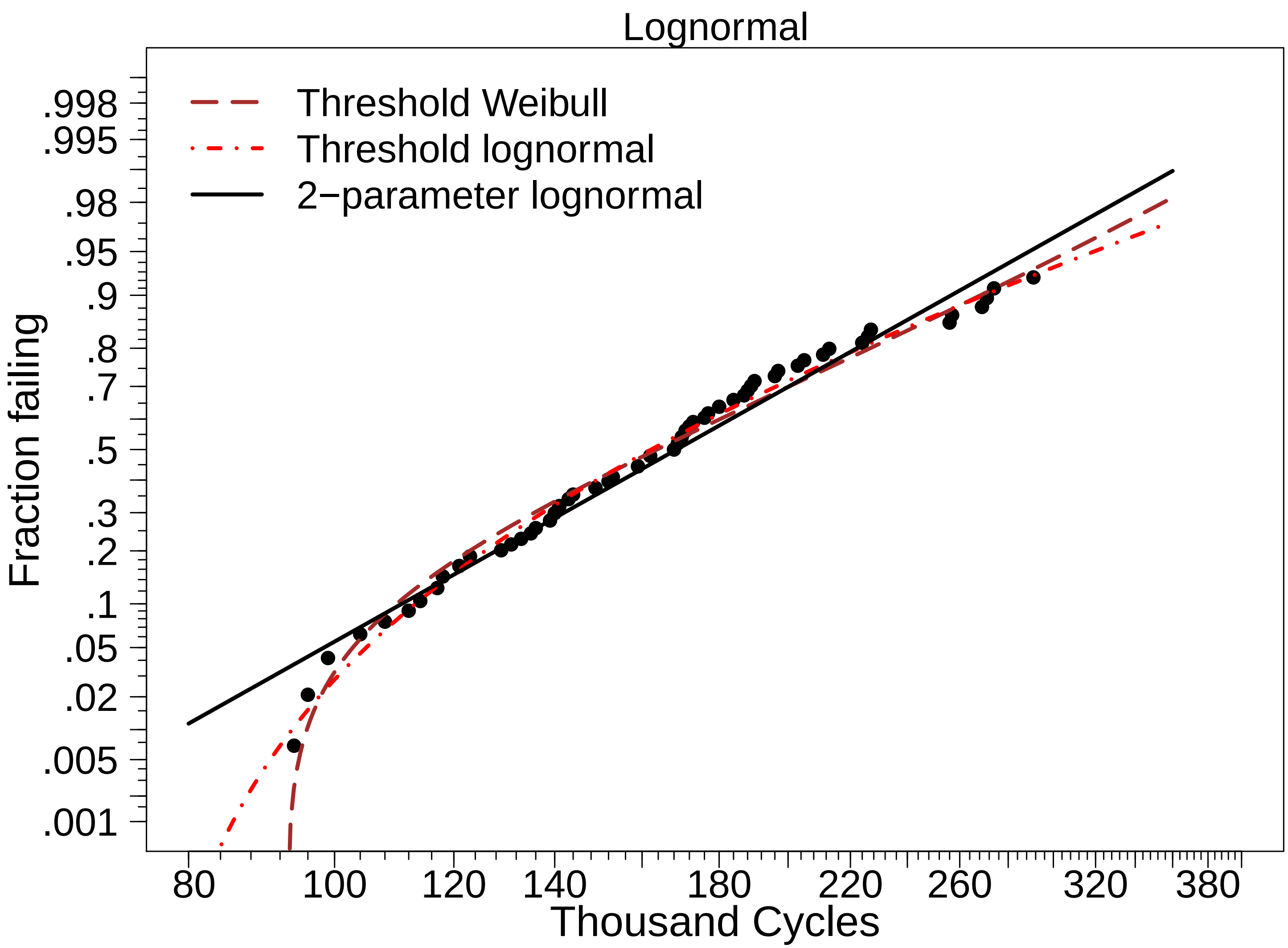
# Lognormal Probability Plot Comparing ML Estimates Three-Parameter Lognormal and Three-Parameter Weibull Distributions for the Turbine Fan Data



# Fitting the 3-Parameter Lognormal and 3-Parameter Weibull Distributions to the Fan Data

- Only 12 failures out of 70 units (multiple censoring).
- 2-parameter Lognormal fits the data well. Weibull and exponential also fit the data reasonably well. Can 3-parameter distributions do better?
- For the 3-parameter distributions, the density approximation breaks down. One should use the **correct** likelihood.
- ML suggests that there is a positive threshold, but the level of improvement is statistically unimportant.
- Fitting a 3-parameter distribution to 12 failures is **overfitting**.

## Lognormal Probability Plot Comparing Three-Parameter Lognormal and Three-Parameter Weibull Distributions for the Alloy T7987 Data



**References**


Kalbfleisch, J. D. and J. F. Lawless (1992). Some useful statistical methods for truncated data. *Journal of Quality Technology 24*, 145–152. []

Meeker, W. Q. and L. A. Escobar (1995). Teaching about approximate confidence regions based on maximum likelihood estimation. *The American Statistician 49*, 48–53. []

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