Chapter 22

Analysis of Repairable System and Other Recurrent Events Data

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Chapter 22
Analysis of Repairable System and Other Recurrent Events Data

Topics discussed in this chapter are:

- An introduction to recurrent events data and recurrent events data analysis.
- Estimation of the mean cumulative function (MCF).
- The variance of the MCF estimator, confidence intervals for the MCF, and a simple example.
- Comparison of two MCFs.
- Analysis of recurrent-event data with multiple event types.

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Segment 1

An Introduction to Recurrent Events Data

Valve Seat Replacement Times

Introduction

Recurrent events data are a sequence of recurrences $T_1, T_2, \ldots$ in time (a point-process). Data may be from one or more than one observational units.

In general, the interest is on:

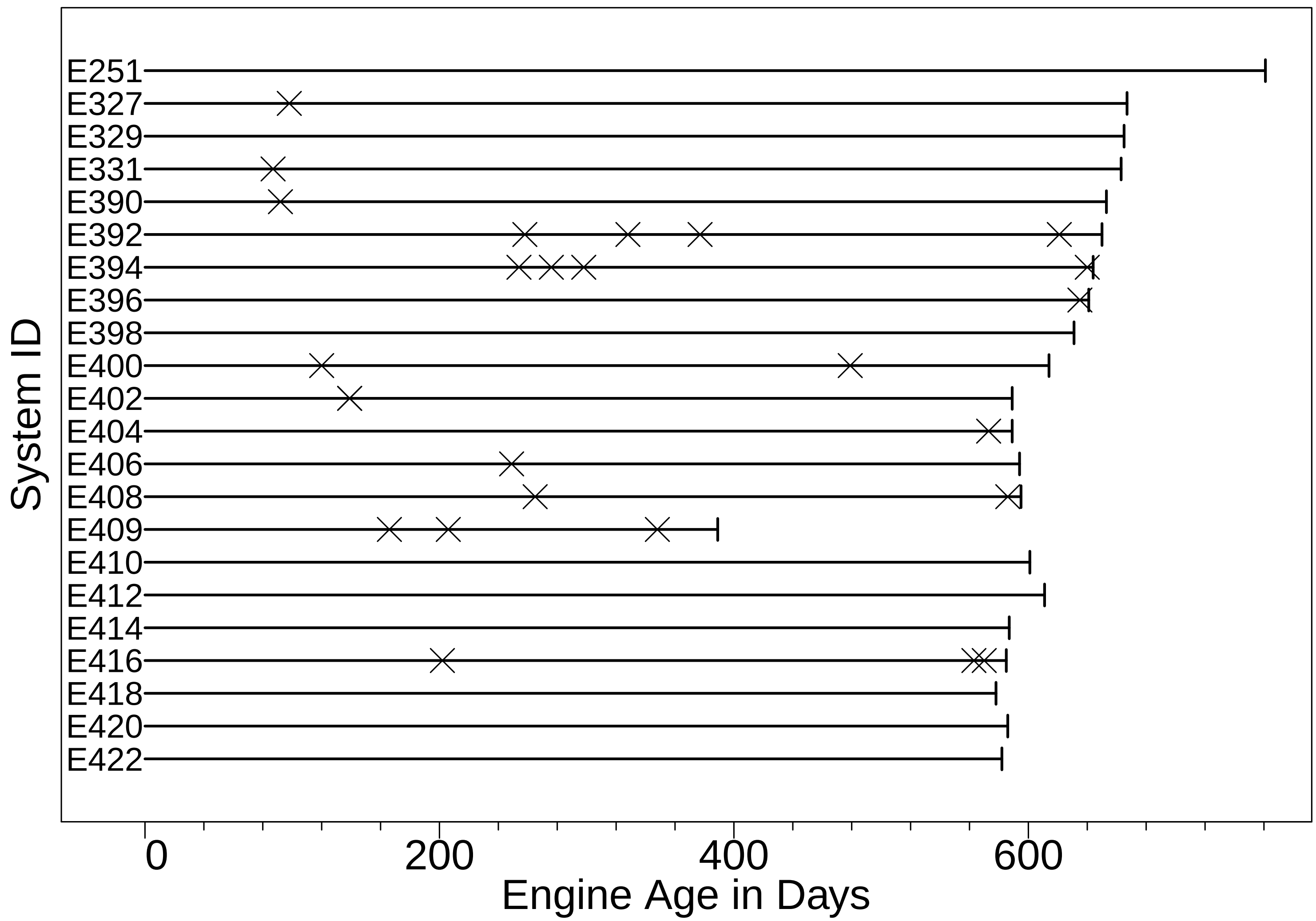
- The number of recurrences in the interval $(0, t]$ as a function of t.
- The expected number of recurrences in the interval $(0, t]$ as a function of t.
- The recurrence rate $\lambda(t)$ as a function of time t.
- The distribution of the times between recurrences, $\tau_j = T_j - T_{j-1}$ $(j = 1, 2, \ldots)$ where $T_0 = 0$.

Valve Seat Replacement Times

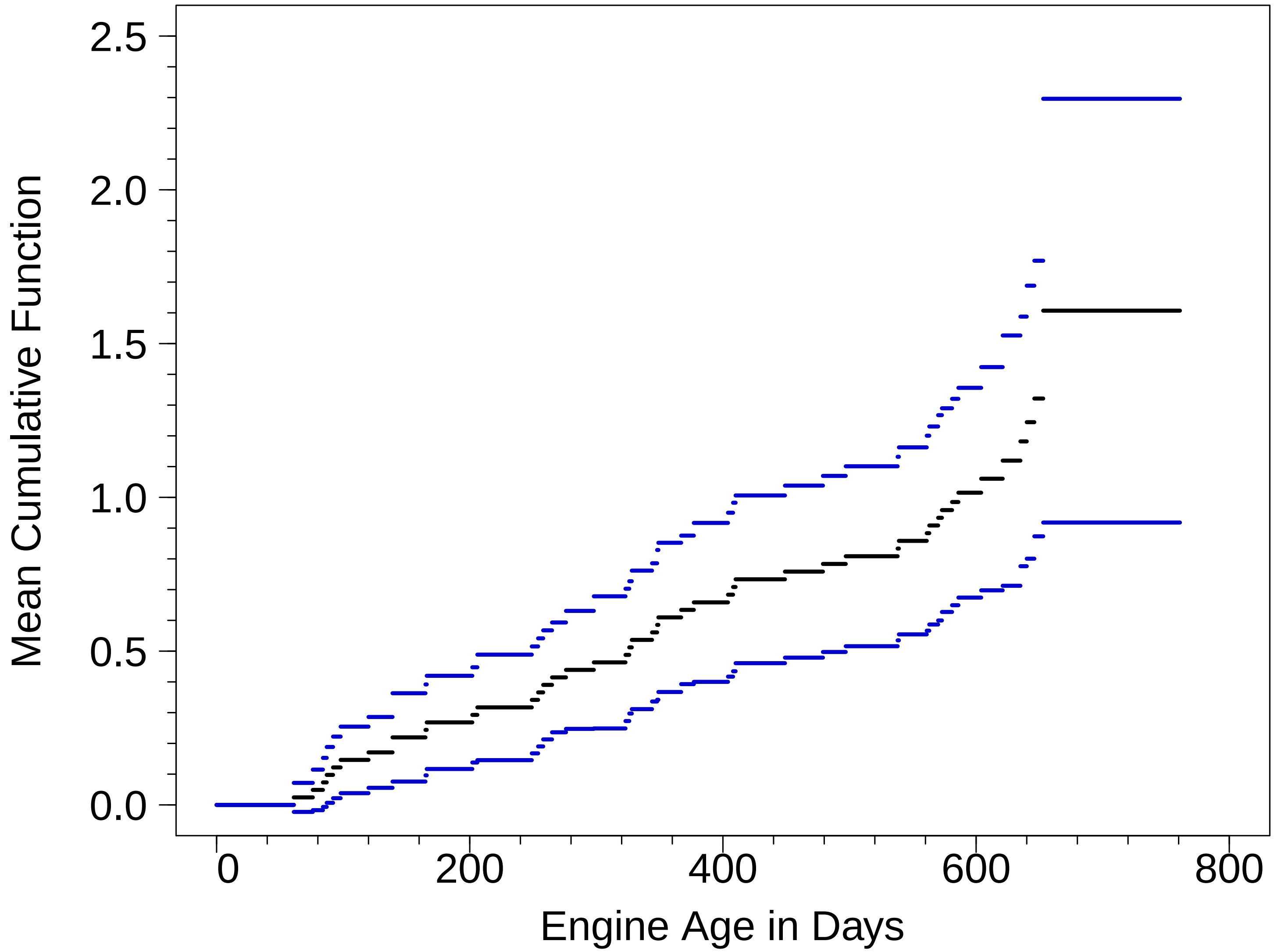
Data collected from valve seats from a fleet of 41 diesel engines operated in and around Beijing, China (days of operation).

- Each engine has 16 valves.
- Most failures caused by operating in a dusty environment.
- Does the replacement rate increase with age?
- How many replacement valves will be needed in the future?
- Can valve life in these systems be modeled as a renewal process?
- Data from Nelson (1995).

Valve Seat Replacement Times Event Plot



Estimate of the Mean Cumulative Replacement Function for the Valve Seat Data



Recurrent Events Data

- Recurrences (e.g., failures, returns, or replacements) are observed in a fixed observation interval $(0, t_a]$.

- The data may be reported on several different ways.

  - Single system or multiple systems.

  - Exact recurrence times $t_1 < \ldots < t_r$ $(t_r \leq t_a)$ resulting from continuous inspection in $(0, t_a]$.

  - Number of interval censored recurrences $d_1, \ldots, d_m$ in the intervals $(0, t_1], (t_1, t_2], \ldots (t_{m-1}, t_m]$, $(t_m = t_a)$ resulting from inspections on $(0, t_a]$.

  - Window-observation data when events are recorded only in certain windows of time.

Multiple Systems - Data and Model

- **Data:** For a single system, $N(s,t)$ denotes the cumulative number of recurrences in the interval $(s,t]$. And $N(t) = N(0,t)$.

- **Model:** The mean cumulative function (MCF) at time t is defined as $\Lambda(t) = \mathrm{E}[N(t)]$, where the expectation is over the variability of each system and the unit to unit variability in the population.

- When $\Lambda(t)$ is differentiable,
$$\lambda(t) = \frac{d\mathrm{E}[N(t)]}{dt} = \frac{d\Lambda(t)}{dt}$$
defines the recurrence rate per system (or **mean** recurrence rate for a collection of systems).

- Some times the interest is on cost over time and $\Lambda(t) = \mathrm{E}[C(t)]$ is the mean cumulative cost per unit in $(0,t]$.

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Segment 2

Estimation of the MCF

Earth-Moving Machine Maintenance Actions Cylinder Replacement Data

Nonparametric Methods for Recurrent Events Data

Under the general cumulative recurrent events model the nonparametric analysis provides:

- Nonparametric estimate of the MCF $\Lambda(t)$.
- Variance of the nonparametric estimator of the MCF $\Lambda(t)$.
- Nonparametric confidence interval for $\Lambda(t)$.
- Nonparametric confidence interval for the difference between two cumulative occurrence models.

Nonparametric Estimate of a Population MCF Definition and Assumptions

Here we present a nonparametric estimate of an MCF $\Lambda(t)$. The estimator is nonparametric in the sense that the method does not require specification of a model for the point process recurrence rate.

- Suppose that there is a fleet of n units generating recurrent events.

- Suppose also that the time at which observation on a unit is terminated is not systematically related to any factor related to the recurrence time distribution.

Nonparametric Estimate of MCF
Input Data Notation Conventions

- Let t_{ij} be recurrence time j for system i, $j = 1, \ldots, m_i$ and $i = 1, \ldots, n$.
- Order the unique recurrence times from smallest to largest and collect the distinct recurrences times say $t_1 < \ldots < t_m$. Thus m is the number of unique event times.
- Some applications focus on the number of recurrent events and others focus on values (e.g., cost) associated with the recurrrent events.
- Let $d_i(t_j)$ the total number of recurrences or other quantitative value (such as cost) for unit i **at** time t_j.
- Let

$$\delta_i(t_j) = \begin{cases} 1 & \text{if system } i \text{ is being observed } \textbf{at} \text{ time } t_j \\ 0 & \text{otherwise.} \end{cases}$$

Estimation of the MCF $\Lambda(t)$ with Multiple Systems Notation for Computational Elements

- The total number of system recurrences **at** time t_k is

$$d.(t_k) = \sum_{i=1}^{n} \delta_i(t_k) d_i(t_k),$$

- The size of the risk set **at** time t_k is

$$\delta.(t_k) = \sum_{i=1}^{n} \delta_i(t_k),$$

- The mean number of system recurrences **at** time t_k (or proportion of recurrences if a system can have only one recurrence at a time) is

$$\bar{d}(t_k) = \frac{d.(t_k)}{\delta.(t_k)}$$

Estimation of the MCF $\Lambda(t)$ with Multiple Systems

The nonparametric estimate of the MCF $\widehat{\Lambda}(t)$ is constant between events (which occur at the t_j's) and at time t_j, the estimate jumps to

$$\widehat{\Lambda}(t_j) = \sum_{k=1}^{j} \frac{\sum_{i=1}^{n} \delta_i(t_k) d_i(t_k)}{\sum_{i=1}^{n} \delta_i(t_k)}$$

$$= \sum_{k=1}^{j} \frac{d.(t_k)}{\delta.(t_k)}$$

$$= \sum_{k=1}^{j} \bar{d}(t_k), \qquad j = 1, \ldots, m.$$

Earth-Moving Machine Maintenance Actions

- Fleet of 23 earth-moving machines put into service over time

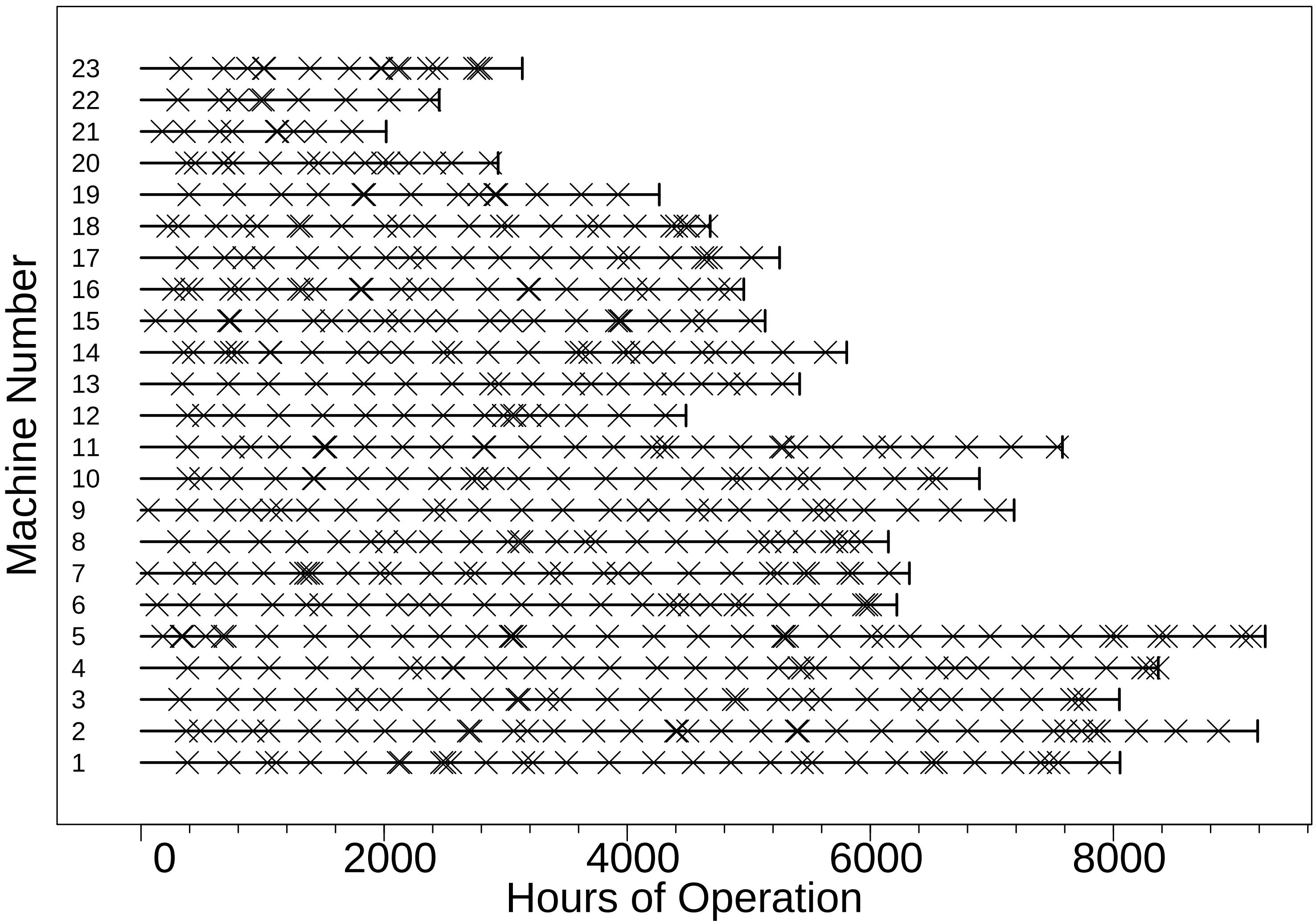
- Preventive maintenance every 300-400 hours of operation

- Major overhaul every 2000-3000 hours of operation

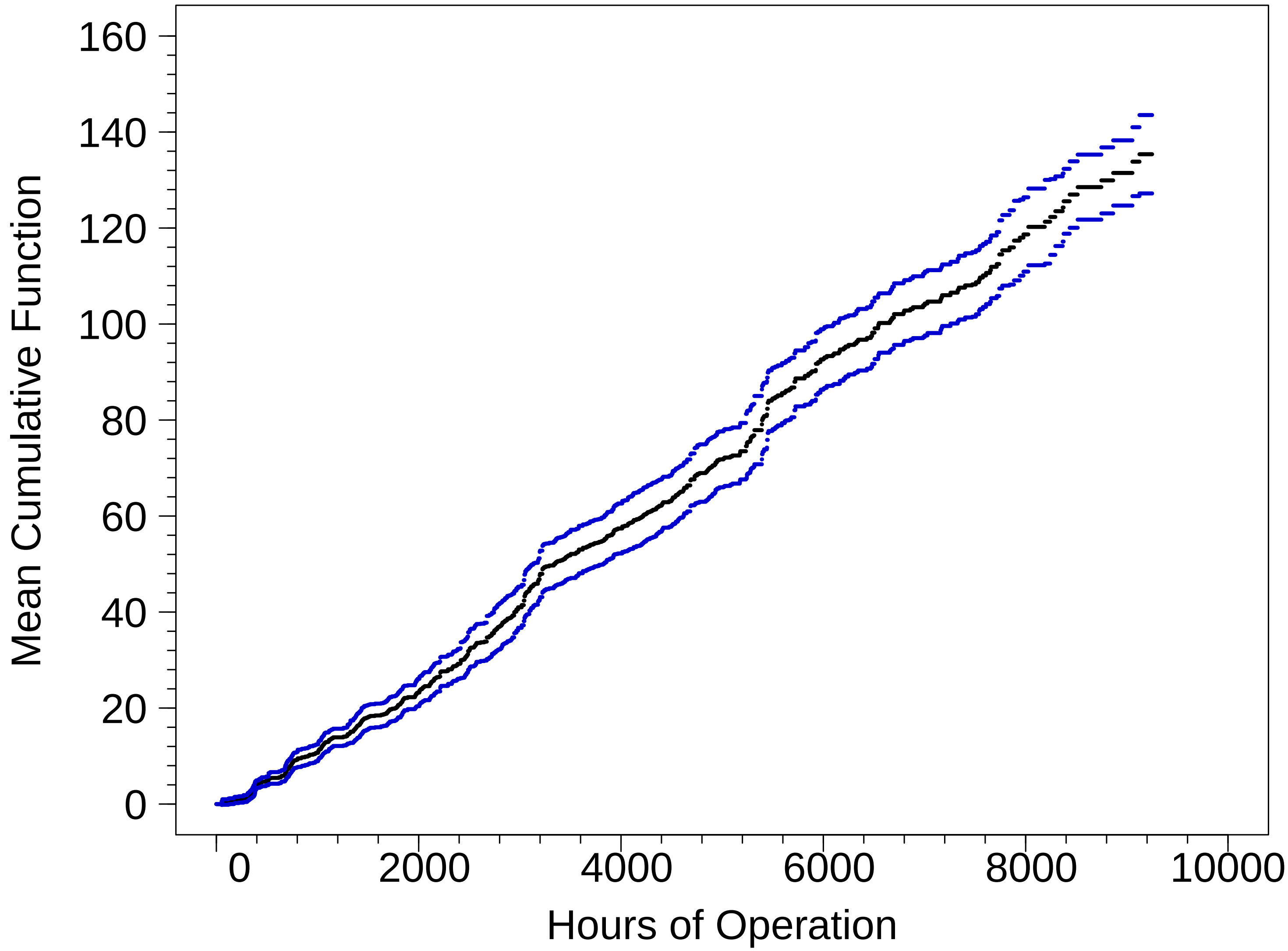
- Many unscheduled maintenance actions

- The response is the number of labor hours for each maintenance action

Event Plot of Earth-Moving Machine Maintenance Actions



MCF plot of Earth-Moving Machine Maintenance Actions



Cylinder Replacement Data

- Cylinder replacement times on 120 locomotive diesel engines.

- Cylinders can develop leaks or have low compression for some other reason.

- Such cylinders are replaced by a rebuilt cylinder.

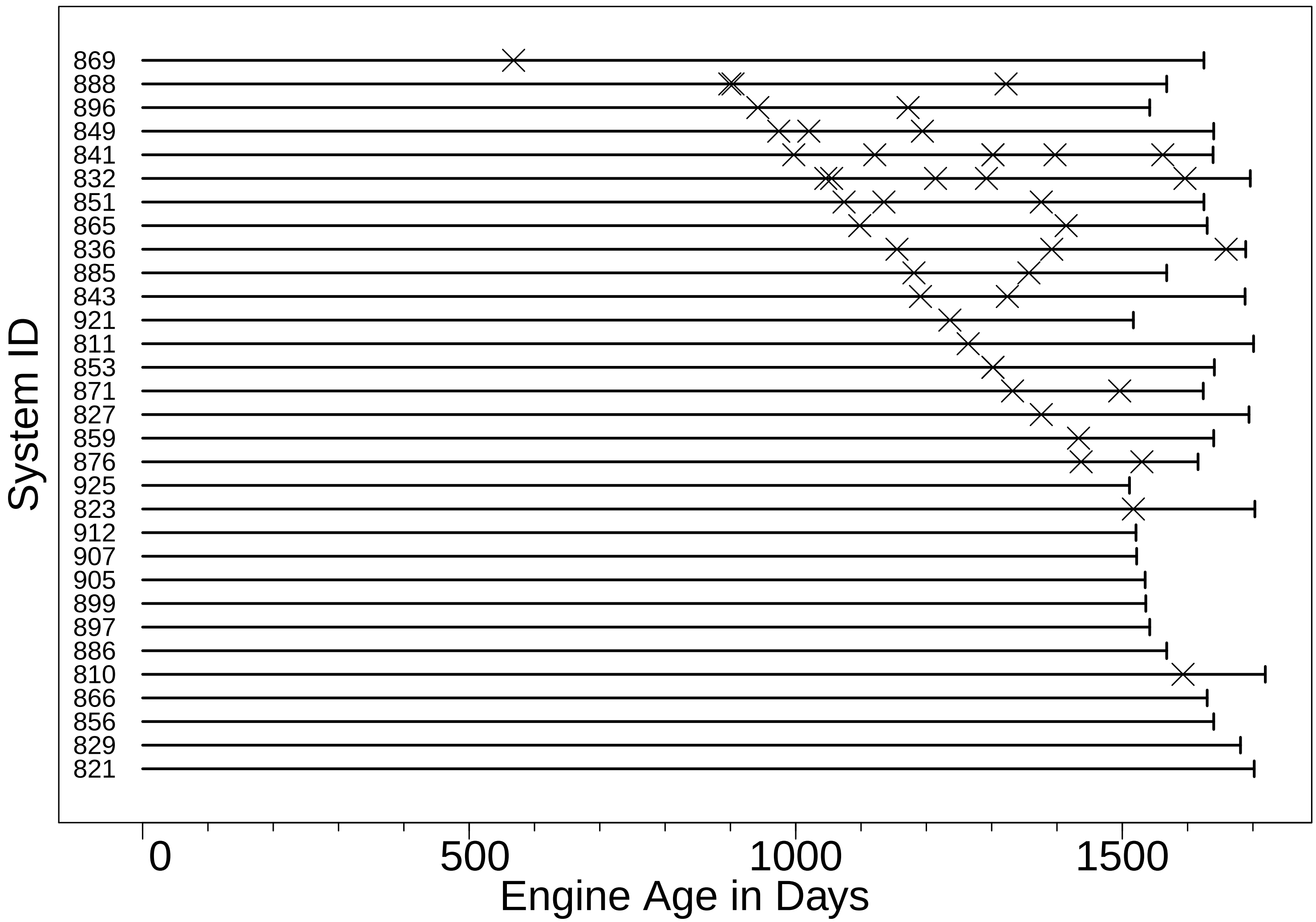
- Each engine has 16 cylinders.

- More than one cylinder may be replaced at an inspection.

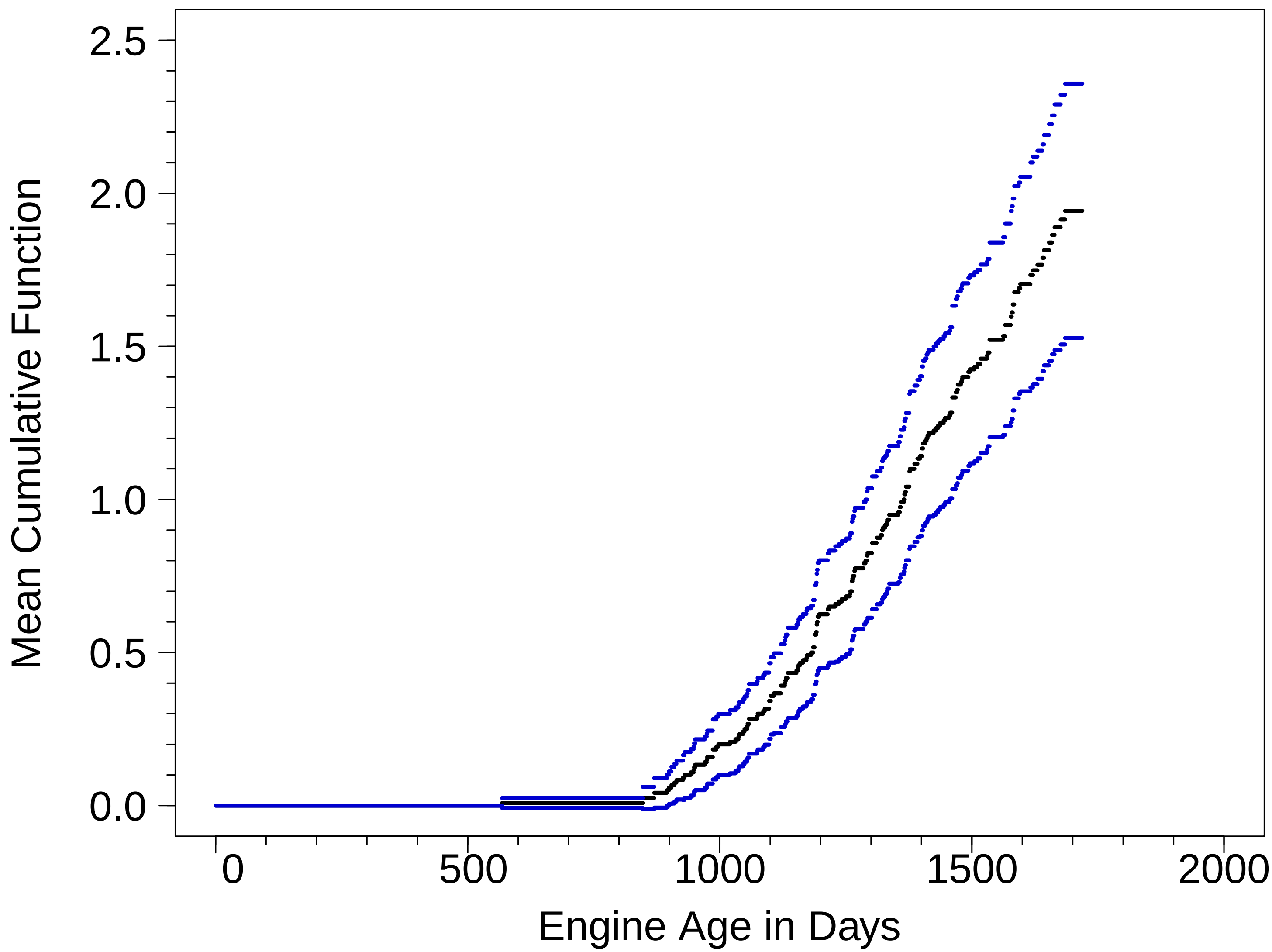
- Is preventive replacement of cylinders appropriate?

- Data from Nelson and Doganaksoy (1989).

Cylinder Replacement Time Event Plot
(Subset of Systems)



Estimate of Mean Cumulative Replacement Function for the Diesel Cylinders



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Analysis of Repairable System and Other Recurrent Events Data

Segment 3

Variance of the MCF Estimator Confidence Intervals for the MCF and a Simple Example

Variance of $\widehat{\Lambda}(t)$

- Suppose that the observation times are fixed. Then the number of recurrent events is random.
- Suppose that the systems are independent.
- Define $d(t_k)$ as the random variable that describes the number of system recurrences or the amount of the variable (such as cost) of interest **at** t_k for a system sampled at random from the population of systems.
- Recall that

$$\widehat{\Lambda}(t_j) = \sum_{k=1}^{j} \bar{d}(t_k), \qquad j = 1, \ldots, m.$$

- Then, direct computations give

$$\begin{aligned}
\text{Var}[\widehat{\Lambda}(t_j)] &= \sum_{k=1}^{j} \text{Var}[\bar{d}(t_k)] + 2 \sum_{k=1}^{j-1} \sum_{v=k+1}^{j} \text{Cov}[\bar{d}(t_k), \bar{d}(t_v)] \\
&= \sum_{k=1}^{j} \frac{\text{Var}[d(t_k)]}{\delta.(t_k)} + 2 \sum_{k=1}^{j-1} \sum_{v=k+1}^{j} \frac{\text{Cov}[d(t_k), d(t_v)]}{\delta.(t_k)}.
\end{aligned}$$

Estimate of Var$[\widehat{\Lambda}(t)]$

- To estimate Var$[d(t_k)]$, we use the assumption that $d_i(t_k)$, $i = 1, \ldots, n$ is a random sample from $d(t_k)$.

  The moment estimators are

$$\widehat{\mathrm{Var}}[d(t_k)] = \sum_{i=1}^{n} \frac{\delta_i(t_k)}{\delta.(t_k)}[d_i(t_k) - \bar{d}(t_k)]^2$$

$$\widehat{\mathrm{Cov}}[d(t_k), d(t_v)] = \sum_{i=1}^{n} \frac{\delta_i(t_v)}{\delta.(t_v)}[d_i(t_k) - \bar{d}(t_k)]d_i(t_v).$$

- Substituting these into the variance formula, and after simplifications, one gets

$$\widehat{\mathrm{Var}}[\widehat{\mu}(t_j)] = \sum_{k=1}^{j} \frac{\widehat{\mathrm{Var}}[d(t_k)]}{\delta.(t_k)} + 2\sum_{k=1}^{j-1} \sum_{v=k+1}^{j} \frac{\widehat{\mathrm{Cov}}[d(t_k), d(t_v)]}{\delta.(t_k)}$$

$$= \sum_{i=1}^{n} \left\{ \sum_{k=1}^{j} \frac{\delta_i(t_k)}{\delta.(t_k)} \Big[d_i(t_k) - \bar{d}.(t_k) \Big] \right\}^2.$$

Comment on Other Estimates of Var$[\widehat{\Lambda}(t)]$

- An alternative to the moment estimators of variances and covariances, one can use Nelson's (slightly different) unbiased estimators given by

$$\widehat{\text{Var}}[d(t_k)] = \sum_{i=1}^{n} \frac{\delta_i(t_k)}{\delta.(t_k) - 1}[d_i(t_k) - \bar{d}(t_k)]^2$$

$$\widehat{\text{Cov}}[d(t_k), d(t_v)] = \sum_{i=1}^{n} \frac{\delta_i(t_v)}{\delta.(t_v) - 1}[d_i(t_k) - \bar{d}(t_k)]d_i(t_v).$$

- Using the unbiased estimates can result in a *negative estimate* for Var$[\widehat{\Lambda}(t)]$. The probability of this event is small unless the number of units under observation is small (e.g., fewer than 20).

Simple Example for 3 Systems Data

Consider 3 systems with the following system failures and censoring times

System	System Failures	Censoring Time
1	5, 8	12
2		16
3	1, 8, 16	20

Then the collection of all system failures is

$$t_1 = 1, t_2 = 5, t_3 = 8, t_4 = 16$$

Simple Example Estimation of $\mu(t)$

- Point estimation:

j	t_j	δ_1	δ_2	δ_3	d_1	d_2	d_3	$\delta.$	$d.$	$\bar{d}$	$\widehat{\mu}(t_j)$
1	1	1	1	1	0	0	1	3	1	1/3	1/3
2	5	1	1	1	1	0	0	3	1	1/3	2/3
3	8	1	1	1	1	0	1	3	2	2/3	4/3
4	16	0	1	1	0	0	1	2	1	1/2	11/6

- Estimation of variances:

$$\widehat{\text{Var}}[\widehat{\mu}(t_1)] = [(1/3) \times (0-1/3)]^2 + [(1/3) \times (0-1/3)]^2 \\ + [(1/3) \times (1-1/3)]^2 = \frac{6}{81}.$$

Similar computations yield:

$$\widehat{\text{Var}}[\widehat{\mu}(t_2)] = 6/81 = 0.0741$$
$$\widehat{\text{Var}}[\widehat{\mu}(t_3)] = 24/81 = 0.296$$
$$\widehat{\text{Var}}[\widehat{\mu}(t_4)] = 163/216 = 0.755$$

Estimation of the MCF $\Lambda(t)$ with Finite Populations

Sometimes with field data, the number of systems is small and the inference of interest is on the number of recurrences and cost of *those* units.

- In this case, finite population methods are appropriate.

- The point estimator for the MCF $\Lambda(t)$ is the same. But to take in consideration sampling from a finite population the following estimates are used in computing $\widehat{\text{Var}}[\widehat{\Lambda}(t)]$:

$$\widehat{\text{Var}}[d(t_k)] = \left[1 - \frac{\delta.(t_k)}{N}\right] \sum_{i=1}^{n} \frac{\delta_i(t_k)}{\delta.(t_k)}[d_i(t_k) - \bar{d}(t_k)]^2$$

$$\widehat{\text{Cov}}[d(t_k), d(t_v)] = \left[1 - \frac{\delta.(t_v)}{N}\right] \sum_{i=1}^{n} \frac{\delta_i(t_v)}{\delta.(t_v)}[d_i(t_k) - \bar{d}(t_k)]d_i(t_v)$$

where N is the total number of systems in the population of interest.

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Segment 4

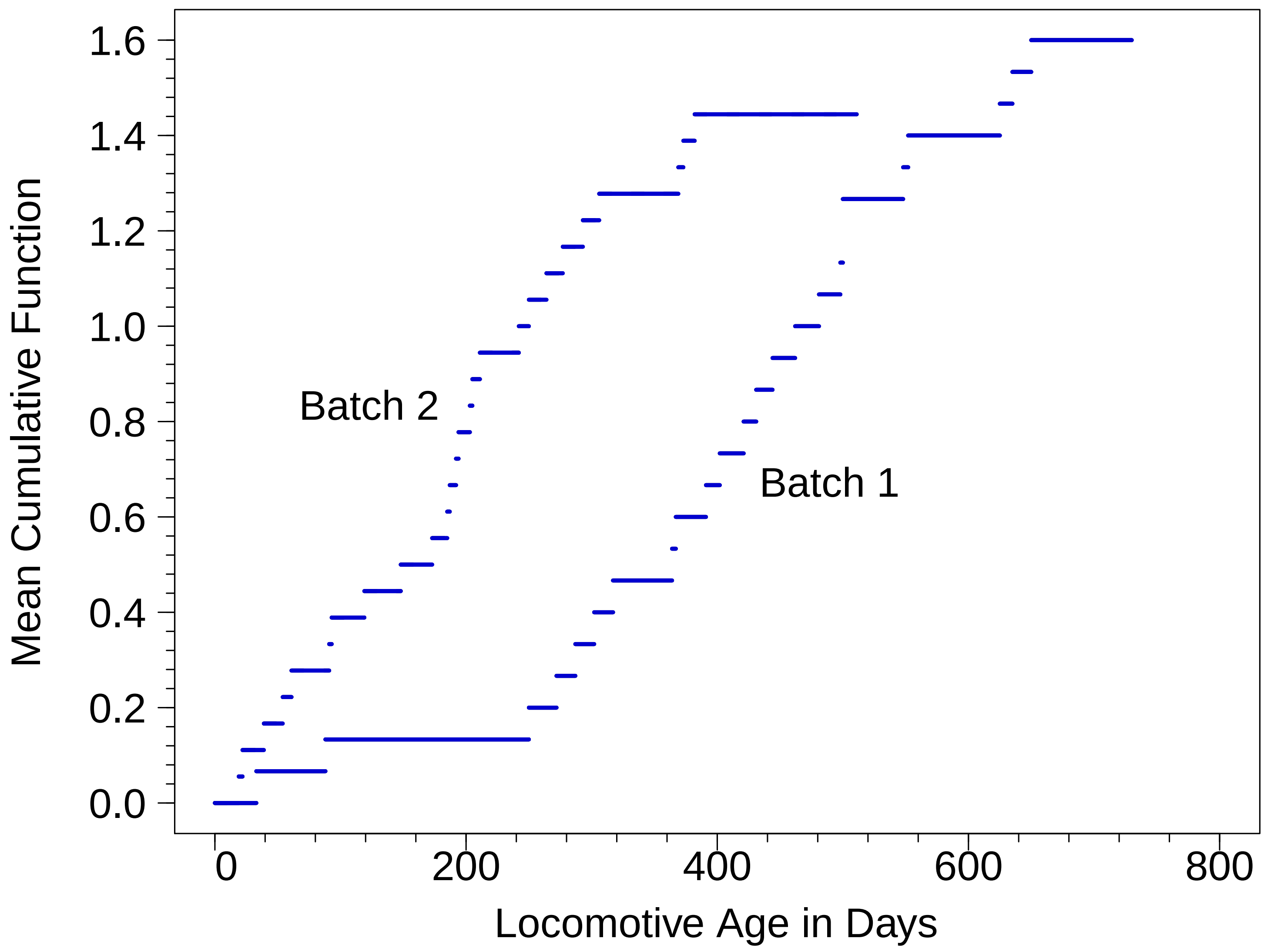
Comparison of Two MCFs

Braking Grid Replacement Frequency Comparison

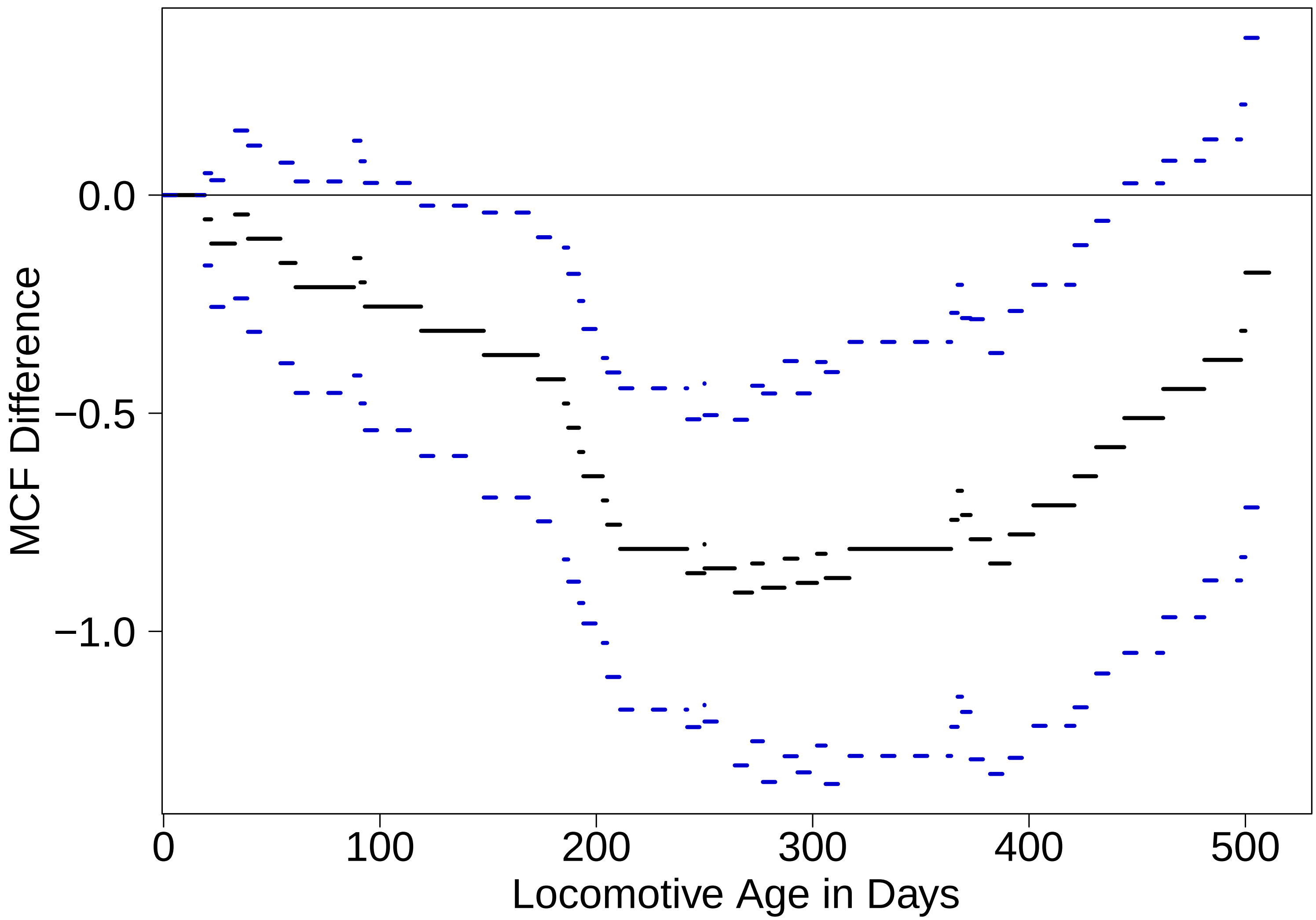
Braking Grid Replacement Frequency Comparison

- A particular type of locomotive has six braking grids.
- Data available on locomotive age when a braking grid is replaced and the age at the end of the observation period.
- A comparison of two different braking grids production batches is desired.
- The data are from Doganaksoy and Nelson (1998).

Comparison of MCFs for the Braking Grids from Production Batches 1 and 2



Difference $\widehat{\Lambda}_1 - \widehat{\Lambda}_2$ Between Sample MCFs for Batches 1 and 2 and Pointwise Approximate 95% Confidence Intervals for the Population Difference



Nonparametric Comparison of Two Samples of Recurrent Events Data

- Suppose that there are two independent samples of recurrent events data with MCF estimates given by $\widehat{\Lambda}_1(t)$ and $\widehat{\Lambda}_2(t)$, respectively.
- Let $\Delta(t) = \Lambda_1(t) - \Lambda_2(t)$ represent the MCF difference at t.
- A nonparametric estimate of $\Delta(t)$ is

$$\widehat{\Delta}(t) = \widehat{\Lambda}_1(t) - \widehat{\Lambda}_2(t)$$

  with estimated variance given by

$$\widehat{\mathrm{Var}}[\widehat{\Delta}(t)] = \widehat{\mathrm{Var}}[\widehat{\Lambda}_1(t)] + \widehat{\mathrm{Var}}[\widehat{\Lambda}_2(t)].$$

- The standard error for $\widehat{\Delta}(t)$ is

$$\mathrm{se}_{\widehat{\Delta}(t)} = \sqrt{\widehat{\mathrm{Var}}[\widehat{\Delta}(t)]}.$$

- An approximate $100(1-\alpha)\%$ confidence interval for $\Delta(t)$ is

$$\left[\underset{\sim}{\Delta}_{\mu}, \quad \widetilde{\Delta}_{\mu}\right] = \left[\widehat{\Delta} - z_{(1-\alpha/2)}\mathrm{se}_{\widehat{\Delta}}, \quad \widehat{\Delta} + z_{(1-\alpha/2)}\mathrm{se}_{\widehat{\Delta}}\right].$$

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Segment 5

Analysis of Recurrent Events Data with Multiple Event Types

Analyzing the System E Recurrent Events Data

System E Recurrent Events Data

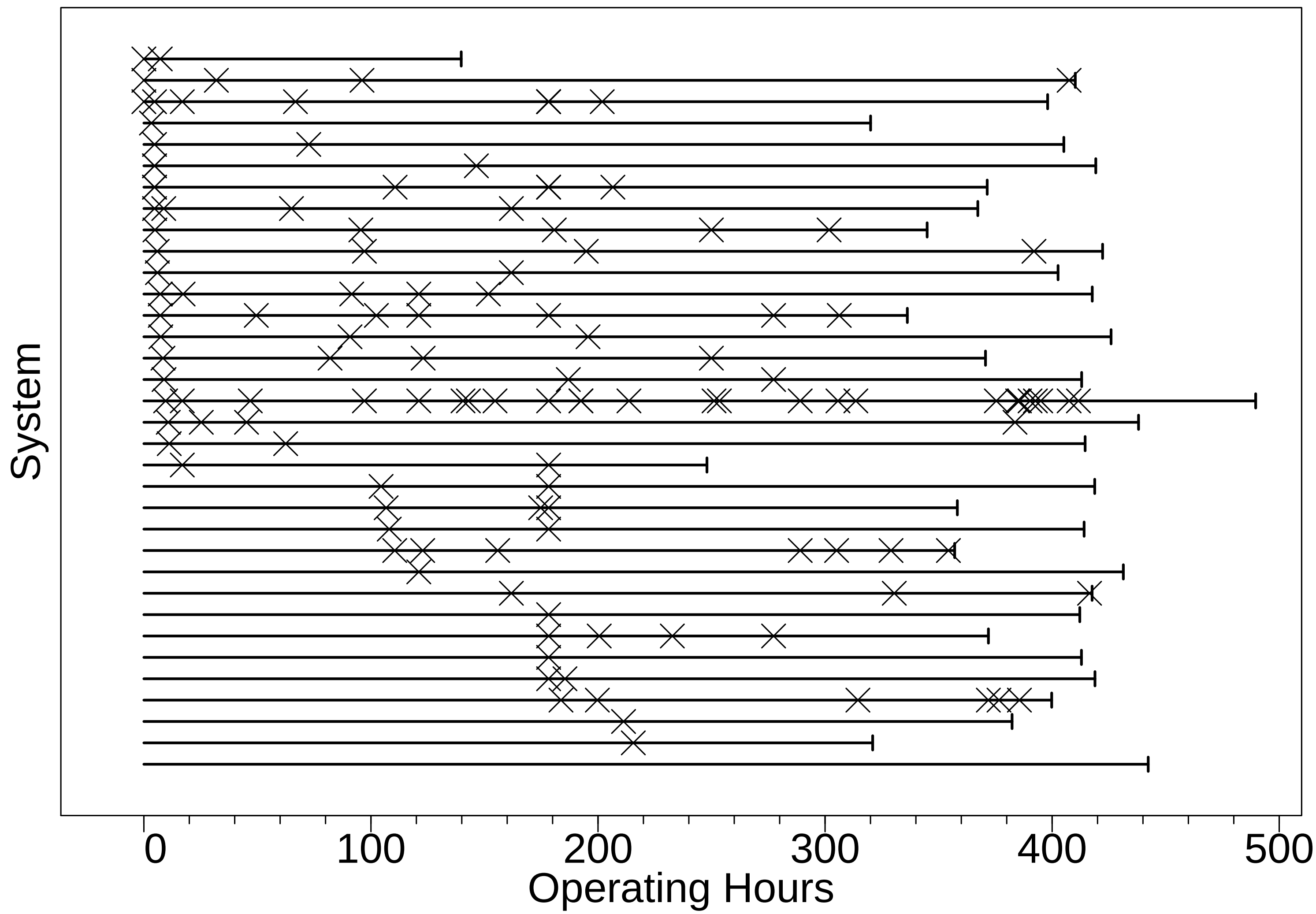
- System operators and others by technicians who periodically perform tests on the system to detect, identify, and mitigate potential failure-causing problems.

- 34 systems were monitored in the field for the occurrence of event types A, B, and C.

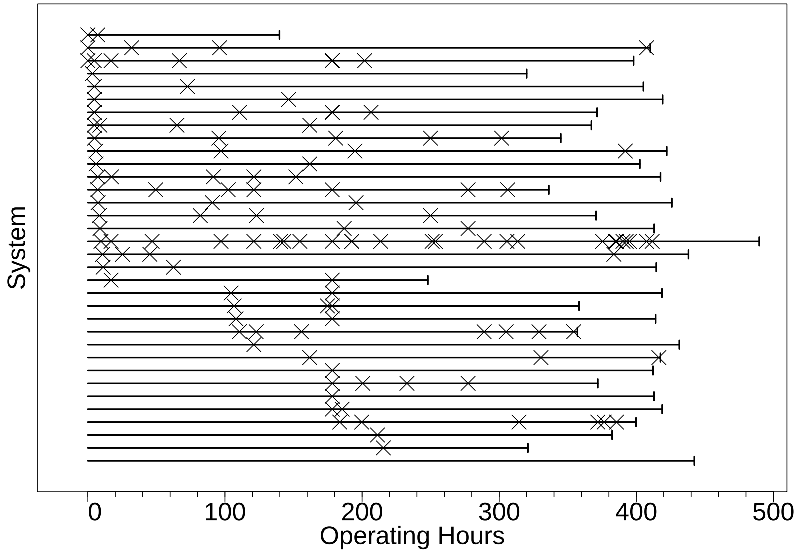
- There was interest in how much each event type contributes to the total MCF.

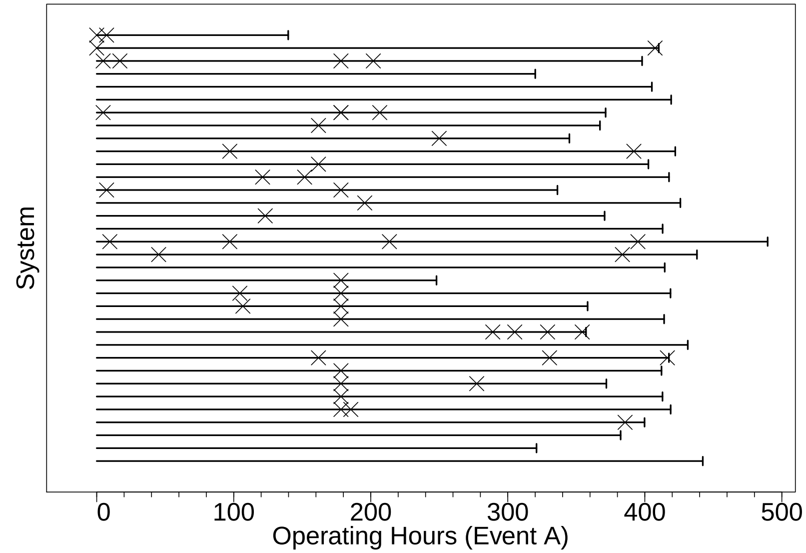
- Information would be fed back to improve system design for better reliability and availability.

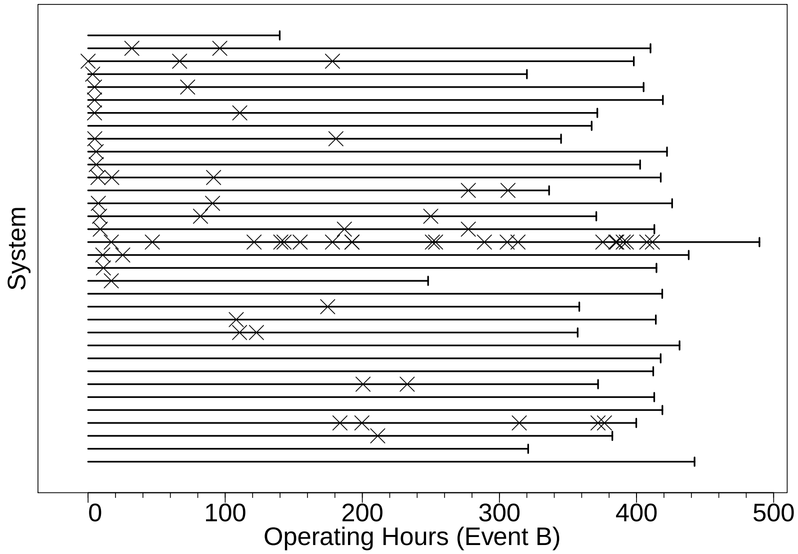
System E Event Plot Showing All Event Types

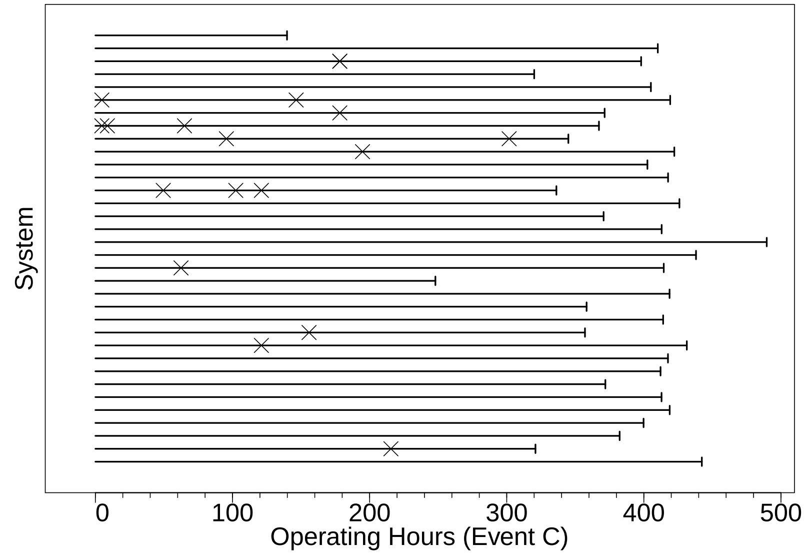


System E Event Plots

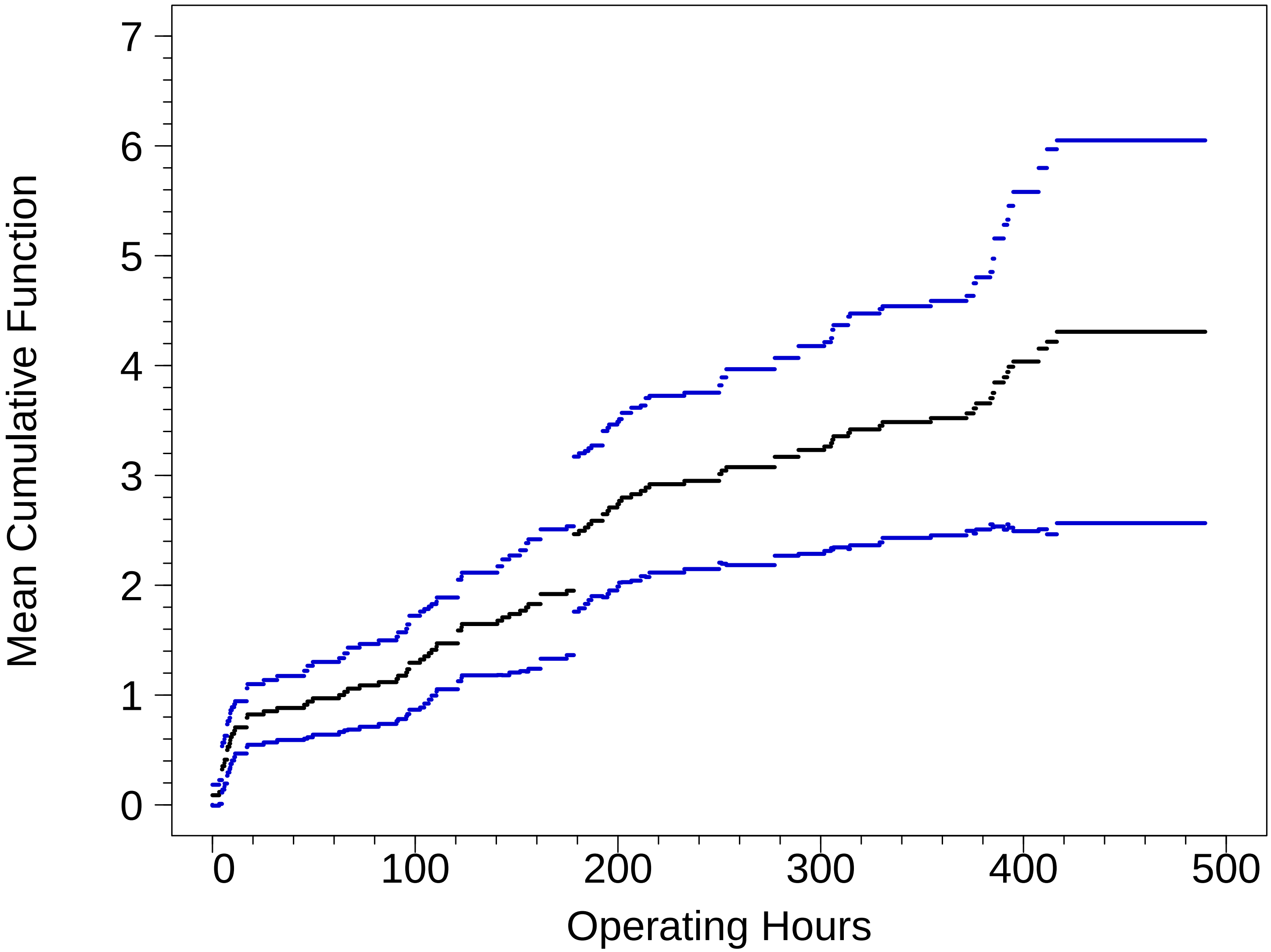








System E MCF Estimate
Ignoring Event-Type Information



MCF Estimation Using Event-Type Information

- Separate MCF estimates can be computed for each event type.
- Suppose that there are L event types.
- Let $d_{\ell i}(t_j)$ be the total number of recurrences of event type ℓ for system i at time t_j for $\ell = 1, \dots, L$, $i = 1, \dots, n$, and $j = 1, \dots, m$.
- Then the MCF estimate for event type ℓ at time t_j is

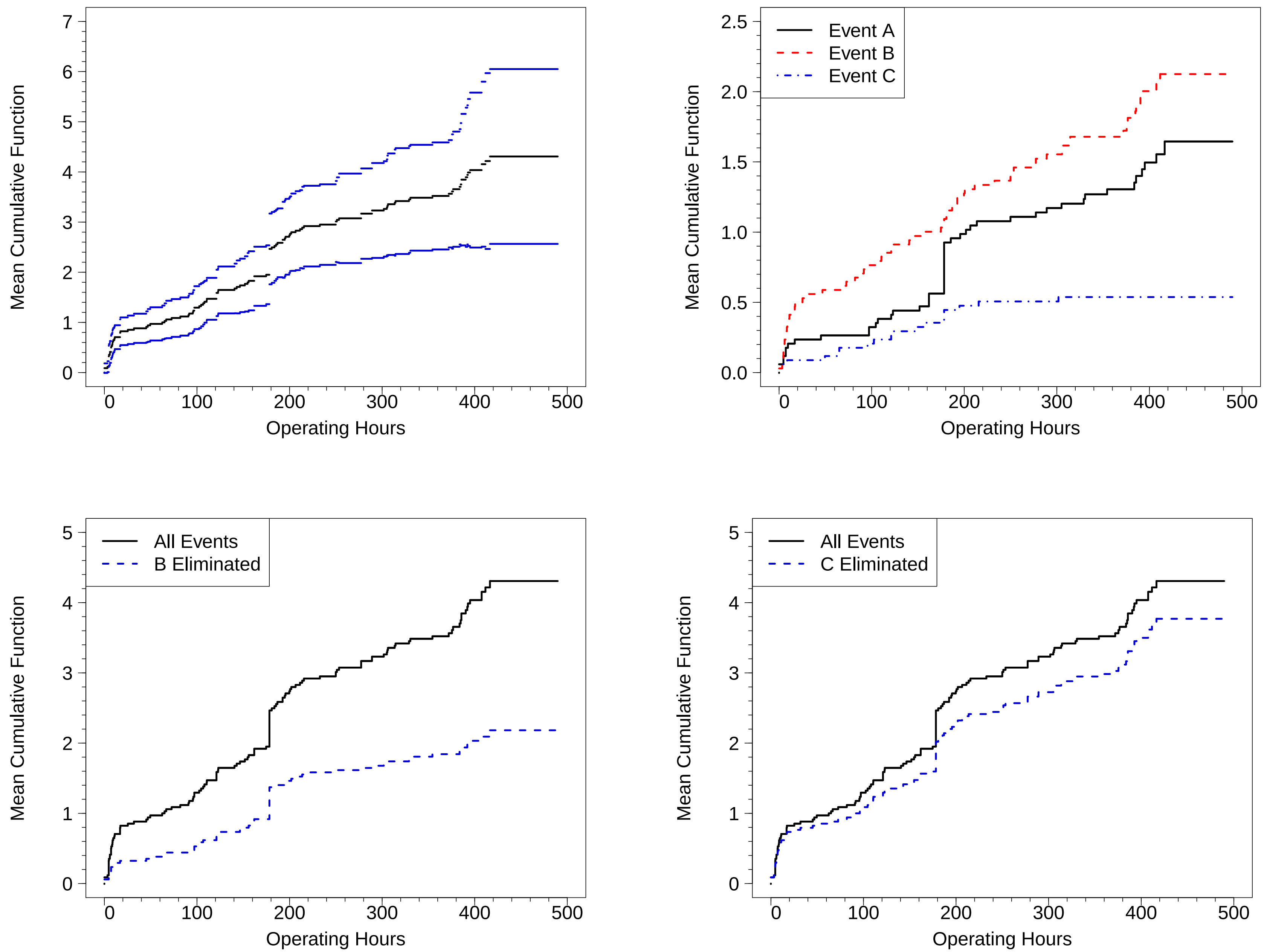
$$\widehat{\mu}_{\ell}(t_j) = \sum_{k=1}^{j} \frac{\sum_{i=1}^{n} \delta_i(t_k) d_{\ell i}(t_k)}{\sum_{i=1}^{n} \delta_i(t_k)}$$

- Then

$$\widehat{\mu}(t_j) = \sum_{\ell=1}^{L} \widehat{\mu}_{\ell}(t_j)$$

- The effect of eliminating one or more event types can be estimated by summing over the remaining event types.

System E Event MCF Estimate



Other Topics In Recurrent Events Data Analysis

- Window-observation data.
- Parametric models for recurrent event data (e.g., for prediction).
  - Nonhomogeneous Poisson process.
  - Renewal process.
  - Trend renewal process.
- Covariate adjustment/regression models for recurrent event data.
  - Fixed covariates.
  - Dynamic (time-varying) covariates.
  - Random effects (frailities) to describe unit-to-unit variability.

References

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