인공지능프로그래밍

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https://github.com/ggorr/Machine-Learning/tree/master/Python

Function

- Surface

$$z = f(x, y)$$

- Level curve

$$f(x, y) = c$$

Example

(1) $z = x^2 + y^2$

level curve is a circle

$$x^2 + y^2 = c$$

(2) $z = 3x^2 + 2y^2$

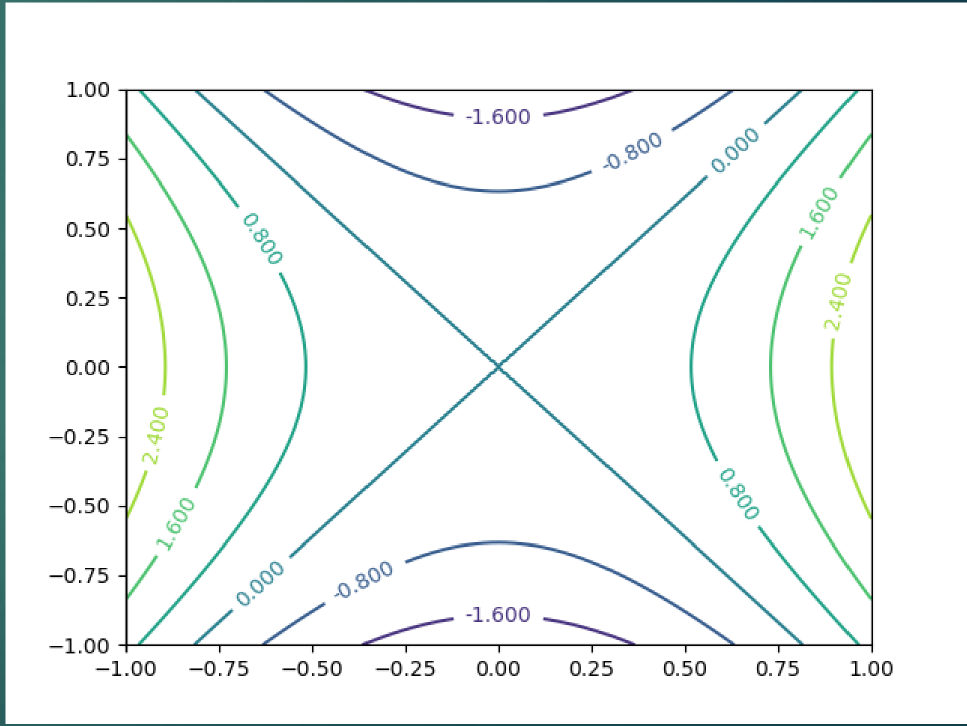
level curve is an ellipse

$$3x^2 + 2y^2 = c$$

(3) $z = 3x^2 - 2y^2$

level curve is an hyperbola

$$3x^2 - 2y^2 = c$$



Gradient

- 함수: $z = f(x, y)$
- 그래디언트(gradient): $\nabla f(x, y) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial x}\right) = \left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)\right)$
- Example

$$z = x^2 + y^2$$

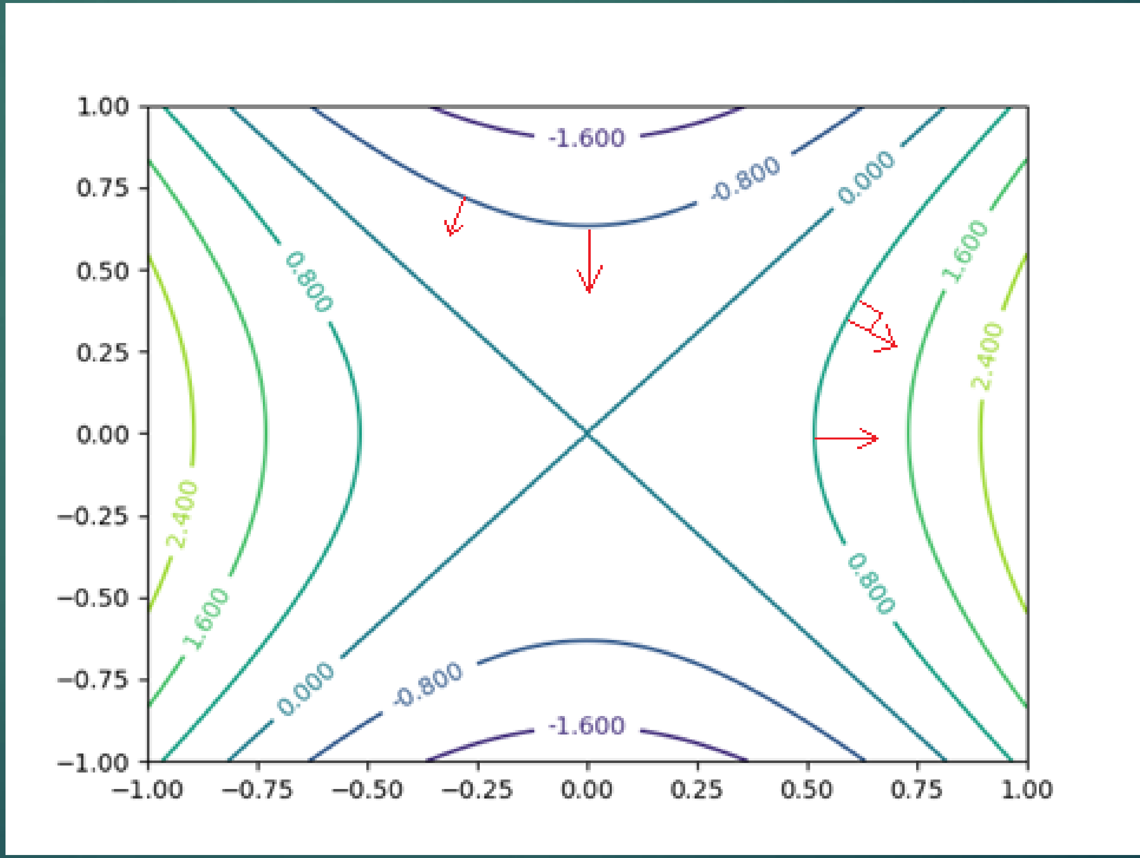
$$\nabla f(x, y) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = (2x, 2y)$$

$$\nabla f(-2,3) = (-4,6)$$

- vector or matrix

$$\nabla f(x, y) = \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) \\ \frac{\partial f}{\partial y}(x, y) \end{bmatrix}$$

- Gradient is the direction in which the function increases fastest (渐变是功能增加最快的方向)
- The gradient is perpendicular to the level curve (梯度垂直于水平曲线)



Finding Maximum

- maximum of $z = f(x, y)$
- Algorithm

  $\lambda \leftarrow$ small number

  start at any $p \leftarrow (x_0, y_0)$

  repeat forever

  $$p \leftarrow p + \lambda \nabla f(p)$$

- 연습문제

  Implement the algorithm for $z = -3x^2 + 6x - 2y^2 - y$

Finding Minimum

- $p \leftarrow p - \lambda \nabla f(p)$

Function composition

- Function composition

  $z = f(x, y)$

  $x = g(s, t),\ y = h(s, t)$

  $z = f(g(s, t), h(s, t))$

- Example

  $z = x^2 + y^2$

  $x = s + 2t,\ y = s^2 - t + 1$

Chain rule

- $z = f(x, y)$
  $x = g(s, t),\ y = h(s, t)$
  Then
  $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
  $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
- Example
  $$z = x^2 + y^2$$
  $$x = s + 2t,\ y = s^2 - t + 1$$
  Then
  $$\frac{\partial z}{\partial s} = 2x \cdot 1 + 2y \cdot 2s = 2(s + 2t) + 2(s^2 - t + 1) \cdot 2s$$
  $$\frac{\partial z}{\partial t} = 2x \cdot 2 + 2y \cdot (-1) = 2(s + 2t) \cdot 2 + 2(s^2 - t + 1) \cdot (-1)$$

연습 문제

- 연습문제 1. $z = xy,\ x = s - t,\ y = s + t,$
  $\frac{\partial z}{\partial s} = ?,\ \frac{\partial z}{\partial t} = ?$
- 연습문제 2. $z = xy + 2w,\ x = s - t,\ y = s + t,\ w = st$
  $\frac{\partial z}{\partial s} = ?,\ \frac{\partial z}{\partial t} = ?$
- 연습문제 3. $z = xy + w^2,\ x = s^2,\ y = e^s,\ w = s + 1$
  $\frac{dz}{ds} = ?$

General Form

- $z = g(y_1, \dots, y_m)$ with

$$\begin{gathered} y_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ y_m = f_m(x_1, \dots, x_n) \end{gathered}$$

- $\frac{\partial z}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_i}, \; i = 1, \dots, n$
- matrix form

$$\begin{bmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial y_1} \\ \vdots \\ \frac{\partial z}{\partial y_m} \end{bmatrix}$$

Vector Form

- $z = g(y_1, \dots, y_m)$ with
$$\begin{gathered} y_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ y_m = f_m(x_1, \dots, x_n) \end{gathered}$$
- $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_m)$, $F = (f_1, \dots, f_m)$
$$z = g(\boldsymbol{y}),\ \boldsymbol{y} = F(\boldsymbol{x})$$
$$z = g\big(F(\boldsymbol{x})\big)$$
- Gradient
$$\nabla_y z = \begin{bmatrix} \frac{\partial z}{\partial y_1} \\ \vdots \\ \frac{\partial z}{\partial y_m} \end{bmatrix},\ \nabla_x z = \begin{bmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_n} \end{bmatrix}$$

Jacobian

- Jacobian: $m \times n$ matrix

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = JF(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\boldsymbol{x}) \end{bmatrix}$$

- Chain rule

$$\nabla_x z = JF(\boldsymbol{x})^T \nabla_y z$$

연습문제

- 연습문제 4. $z = xy$, $(x, y) = (\log s - t, s + \log t)$
  $$\nabla_{s,t} z = ?$$
- 연습문제 5. $z = xe^y + 2w$, $(x, y, z) = (s - t, s + t, st)$
  $$\nabla_{s,t} z = ?$$
- 연습문제 6. $z = \cos x - \sin y + w^2$, $(x, y, z) = (s - 1, s + 1, e^s)$
  $$\nabla_s z = ?$$