Chapter 5
Monte Carlo Methods

- Monte Carlo method
  - Learning from experience
    - Actual experience
    - Simulated experience
  - Learning from sample episodes
    - Sequences of states, actions, and rewards from experience

5.1 Monte Carlo Prediction

- Monte Carlo(MC) methods
  - Learning the state-value function for a given policy
    - Policy π
    - State-value function v_π
  - Estimating v_π from experience
    - Sampling episodes
    - Estimation

$$\begin{aligned} v_\pi(s) &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots | S_t = s] \\ &= \mathbb{E}[G_t | S_t = s] \end{aligned}$$

    - Nonstationary estimation

$$v_\pi(S_t) \leftarrow v_\pi(S_t) + \alpha\big(G_t - v_\pi(S_t)\big)$$

- The first-visit MC method
  - $v_\pi(s)$ is estimated as the average of the returns following first visits to s
- The every-visit MC method
  - $v_\pi(s)$ is estimated as the average of the returns following all visits to s

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

  $V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

  $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

  Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

  $G \leftarrow 0$

  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

    $G \leftarrow \gamma G + R_{t+1}$

    Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

      Append G to $Returns(S_t)$

      $V(S_t) \leftarrow \text{average}(Returns(S_t))$

Example 5.1: Blackjack

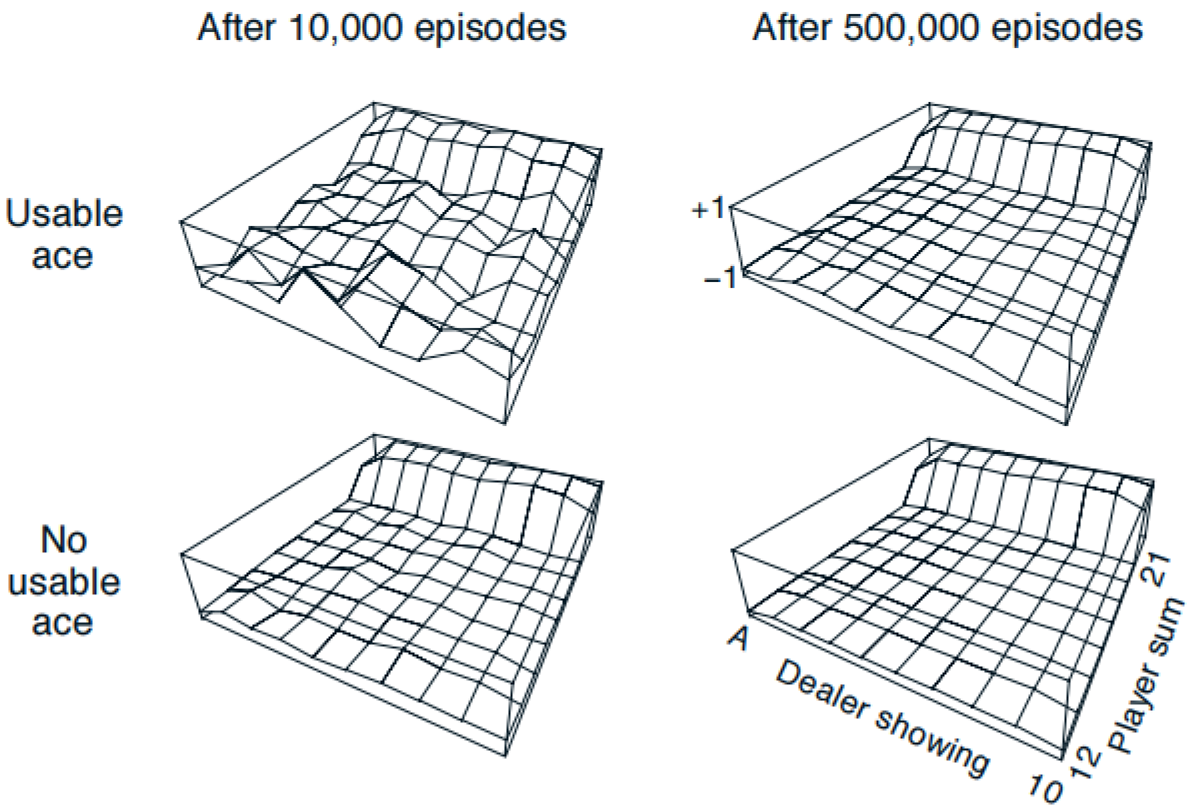


Figure 5.1: Approximate state-value functions for the blackjack policy that sticks only on 20 or 21, computed by Monte Carlo policy evaluation. ■

Usable ace: ace that can be counted as 11

State: player sum and dealer showing
Action: hit(one more card), stick(stop)
Reward: +1 for winning
−1 for losing
0 for drawing

$\gamma = 1$

5.2 Monte Carlo Estimation of Action Values

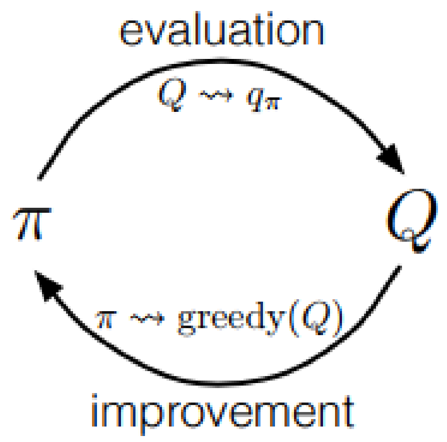
- If a model(of environment) is not available
  - One cannot determine the next state
    - Example) Alpha-go, black jack
  - One may estimate action values rather than state values

- Estimating action values $q_\pi(s, a)$
    - All state-action pairs will be visited an infinite number of times in the limit of an infinite number of episodes
- Exploring starts
    - Every state–action pair has a nonzero probability of being selected as the start
    - All state–action pairs will be visited an infinite number of times

5.3 Monte Carlo Control

- Update action values through episodes
- Update policies by action values

$$\pi(s) = \underset{a}{\operatorname{argmax}}\, q(s,a)$$



Control: finding optimal policy

Monte Carlo ES (Exploring Starts), for estimating $\pi \approx \pi_*$

Initialize:

$\pi(s) \in \mathcal{A}(s)$ (arbitrarily), for all $s \in \mathcal{S}$

$Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

$Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Loop forever (for each episode):

Choose $S_0 \in \mathcal{S}$, $A_0 \in \mathcal{A}(S_0)$ randomly such that all pairs have probability > 0

Generate an episode from S_0, A_0, following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$

Example 5.3: Solving Blackjack

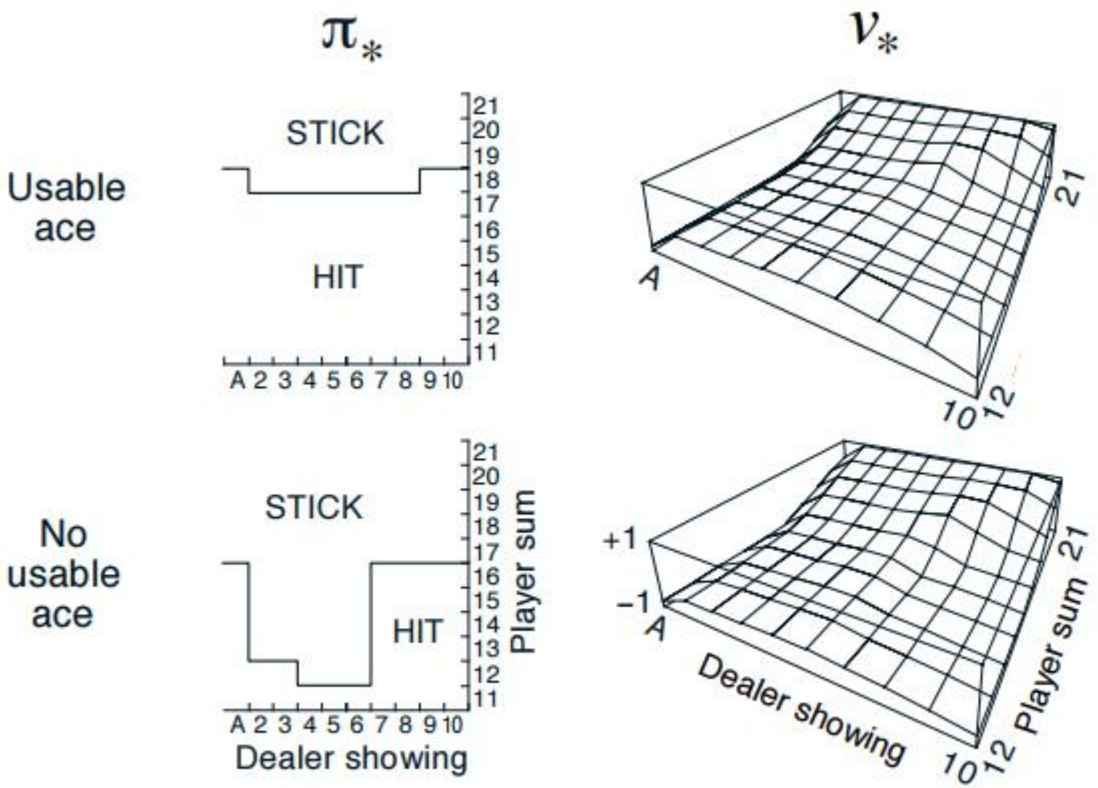
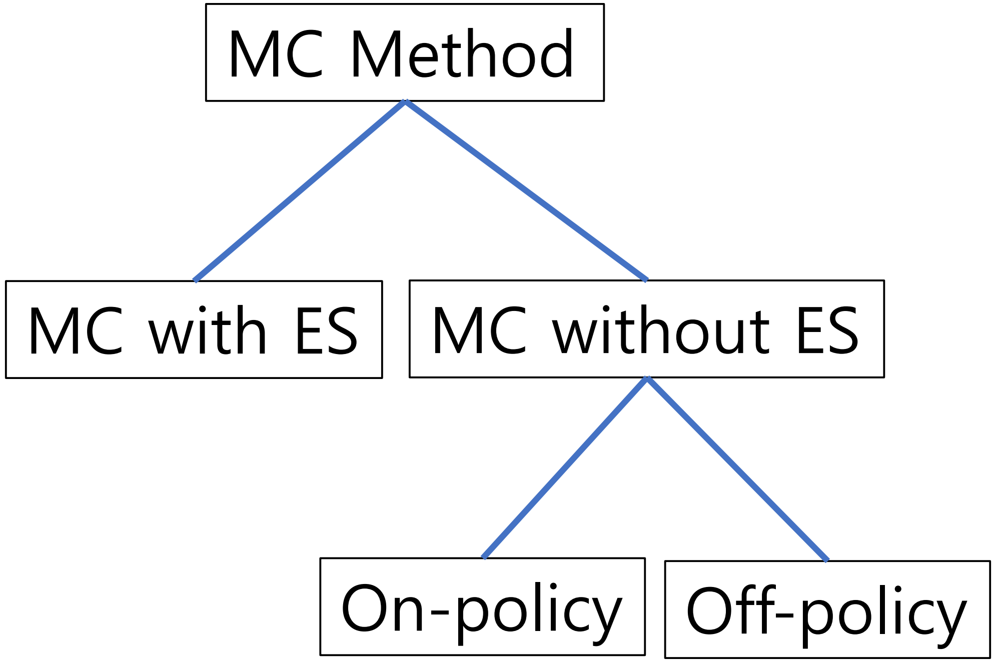


Figure 5.2: The optimal policy and state-value function for blackjack, found by Monte Carlo ES. The state-value function shown was computed from the action-value function found by Monte Carlo ES. ■

5.4 Monte Carlo Control without Exploring Starts

- On-policy methods
  - To improve a policy, action values are evaluated using the policy
- Off-policy methods
  - To improve a policy, action values are evaluated using the other policy



- On-policy method
  - The policy is soft, in general
  - Meaning that $\pi(a|s) > 0$ for all $s \in \mathcal{S}$ and all $a \in \mathcal{A}(s)$
- Example) ϵ-greedy method
  - Most of the time we choose an action that has maximal estimated action value as

$$\Pr(a) = \begin{cases} \dfrac{\epsilon}{|\mathcal{A}(s)|}, & \text{if } a \text{ is not maximal} \\ 1 - \epsilon + \dfrac{\epsilon}{|\mathcal{A}(s)|}, & \text{otherwise} \end{cases}$$

for some $\epsilon > 0$

On-policy first-visit MC control (for ε-soft policies), estimates $\pi \approx \pi_*$

Algorithm parameter: small $\varepsilon > 0$

Initialize:

    $\pi \leftarrow$ an arbitrary ε-soft policy
    $Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$
    $Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

    Generate an episode following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$
    $G \leftarrow 0$
    Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
        $G \leftarrow \gamma G + R_{t+1}$
        Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:
            Append G to $Returns(S_t, A_t)$
            $Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$
            $A^* \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken arbitrarily)
            For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

5.5 Off-policy Prediction via Importance Sampling

- Dilemma
  - To calculate an accurate action value, we trace maximal action values
  - To find a maximal action, we explorer all actions
- Off-policy learning
  - Introducing another policy for explorering
  - Example
    - To use two policies, the target policy and the behavior policy
    - The target policy is learned about, and that becomes the optimal policy
    - The behavior policy is more exploratory, and is used to generate behavior

- The assumption of coverage
  - $q_\pi(a|s)$ must be evaluated if $\pi(a|s) > 0$
  - Hence $\pi(a|s) > 0$ implies $b(a|s) > 0$
- Importance sampling
  - An off-policy prediction
  - A general technique for estimating expected values under one distribution given samples from another

- The probability of the state–action trajectory under π

$$\Pr\{A_t, S_{t+1}, A_{t+1}, \ldots, S_T | S_t\} = \pi(A_t|S_t)p(S_{t+1}|S_t, A_t)\pi(A_{t+1}|S_{t+1}) \cdots p(S_T|S_{T-1}, A_{T-1})$$
$$= \prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k)$$

- Importance-sampling ratio

$$\rho_{t:T-1} = \frac{\prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k)}{\prod_{k=t}^{T-1} b(A_k|S_k)\, p(S_{k+1}|S_k, A_k)} = \prod_{k=t}^{T-1} \frac{\pi(A_k|S_k)}{b(A_k|S_k)}$$

- Value of a state

$$v_\pi(s) = \mathbb{E}_\pi[G_t|S_t = s] = \mathbb{E}_b[\rho_{t:T-1}G_t|S_t = s] = \mathbb{E}[\rho_{t:T-1}G_t|S_t = s]$$

where $\mathbb{E}$ is the empirical mean

- Estimation

$$V(s) = \frac{1}{|\mathcal{T}(s)|} \sum_{t \in \mathcal{T}(s)} \rho_{t:T(t)-1} G_t$$

where $\mathcal{T}(s)$ denotes the set of all time steps in which state s is visited

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- Ordinary importance sampling

$$V(s) = \frac{1}{|\mathcal{T}(s)|} \sum_{t \in \mathcal{T}(s)} \rho_{t:T(t)-1} G_t$$

- Weighted importance sampling

$$V(s) = \frac{\sum_{t \in \mathcal{T}(s)} \rho_{t:T(t)-1} G_t}{\sum_{t \in \mathcal{T}(s)} \rho_{t:T(t)-1}}$$

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