5.2 The Master Theorem

Master Theorem

In the last section, we saw three different kinds of behavior for recurrences of the form

$$T(n) = \begin{cases} aT(n/2) + n & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

These behaviors depended upon whether $a < 2$, $a = 2$, and $a > 2$. Remember that a was the number of subproblems into which our problem was divided. Dividing by 2 cut our problem size in half each time, and the n term said that after we completed our recursive work, we had n additional units of work to do for a problem of size n. There is no reason that the amount of additional work required by each subproblem needs to be the size of the subproblem. In many applications it will be something else, and so in Theorem 5.1 we consider a more general case. Similarly, the sizes of the subproblems don't have to be 1/2 the size of the parent problem. We then get the following theorem, our first version of a theorem called the *Master Theorem.* (Later on we will develop some stronger forms of this theorem.)

Theorem 5.1 *Let a be an integer greater than or equal to 1 and b be a real number greater than 1. Let c be a positive real number and d a nonnegative real number. Given a recurrence of the form*

$$T(n) = \begin{cases} aT(n/b) + n^c & \text{if } n > 1 \\ d & \text{if } n = 1 \end{cases}$$

then for n a power of b,

1. *if $\log_b a < c$, $T(n) = \Theta(n^c)$,*
2. *if $\log_b a = c$, $T(n) = \Theta(n^c \log n)$,*
3. *if $\log_b a > c$, $T(n) = \Theta(n^{\log_b a})$.*

Proof: In this proof, we will set $d = 1$, so that the bottom level of the tree is equally well computed by the recursive step as by the base case. It is straightforward to extend the proof for the case when $d \neq 1$.

Let's think about the recursion tree for this recurrence. There will be $\log_b n$ levels. At each level, the number of subproblems will be multiplied by a, and so the number of subproblems at level i will be a^i. Each subproblem at level i is a problem of size (n/b^i). A subproblem of size n/b^i requires $(n/b^i)^c$ additional work and since there are a^i problems on level i, the total number of units of work on level i is

$$a^i(n/b^i)^c = n^c\left(\frac{a^i}{b^{ci}}\right) = n^c\left(\frac{a}{b^c}\right)^i.$$

Recall from above that the different cases for $c = 1$ were when the work per level was decreasing, constant, or increasing. The same analysis applies here. From our formula for work on level i, we see that the work per level is decreasing, constant, or increasing exactly when $(\frac{a}{b^c})^i$

is decreasing, constant, or increasing. These three cases depend on whether $(\frac{a}{b^c})$ is 1, less than 1, or greater than 1. Now observe that

$$\begin{array}{rc} & (\frac{a}{b^c}) = 1 \\ \Leftrightarrow & a = b^c \\ \Leftrightarrow & \log_b a = c \log_b b \\ \Leftrightarrow & \log_b a = c \end{array}$$

Thus we see where our three cases come from. Now we proceed to show the bound on $T(n)$ in the different cases. In the following paragraphs, we will use the facts (whose proof is a straightforward application of the definition of 1ogartihms and rules of exponents) that for any x, y and z, each greater than 1, $x^{\log_y z} = z^{\log_y x}$ and that $\log_x y = \Theta(\log_2 y)$. (See Problem 3 at the end of this section and Problem 4 at the end of the previous section.)

In general, we have that the total work done is

$$\sum_{i=0}^{\log_b n} n^c \left(\frac{a}{b^c}\right)^i = n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i$$

In case 1, (part 1 in the statement of the theorem) this is n^c times a geometric series with a ratio of less than 1. Theorem 4.4 tells us that

$$n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i = \Theta(n^c).$$

Exercise 5.2-1 Prove Case 2 of the Master Theorem.

Exercise 5.2-2 Prove Case 3 of the Master Theorem.

In Case 2 we have that $\frac{a}{b^c} = 1$ and so

$$n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i = n^c \sum_{i=0}^{\log_b n} 1^i = n^c(1 + \log_b n) = \Theta(n^c \log n)\ .$$

In Case 3, we have that $\frac{a}{b^c} > 1$. So in the series

$$\sum_{i=0}^{\log_b n} n^c \left(\frac{a}{b^c}\right)^i = n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i,$$

the largest term is the last one, so by Theorem 4.4,the sum is $\Theta\left(n^c \left(\frac{a}{b^c}\right)^{\log_b n}\right)$. But

$$\begin{aligned} n^c \left(\frac{a}{b^c}\right)^{\log_b n} &= n^c \frac{a^{\log_b n}}{(b^c)^{\log_b n}} \\ &= n^c \frac{n^{\log_b a}}{n^{\log_b b^c}} \\ &= n^c \frac{n^{\log_b a}}{n^c} \\ &= n^{\log_b a}. \end{aligned}$$

Thus the solution is $\Theta(n^{\log_b a})$. ■

We note that we may assume that a is a real number with $a > 1$ and give a somewhat similar proof (replacing the recursion tree with an iteration of the recurrence), but we do not give the details here.

Solving More General Kinds of Recurrences

So far, we have considered divide and conquer recurrences for functions $T(n)$ defined on integers n which are powers of b. In order to consider a more realistic recurrence in the master theorem, namely

$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + n^c & \text{if } n > 1 \\ d & \text{if } n = 1, \end{cases}$$

or

$$T(n) = \begin{cases} aT(\lfloor n/b \rfloor) + n^c & \text{if } n > 1 \\ d & \text{if } n = 1, \end{cases}$$

or even

$$T(n) = \begin{cases} a'T(\lceil n/b \rceil) + (a - a')T(\lfloor n/b \rfloor) + n^c & \text{if } n > 1 \\ d & \text{if } n = 1, \end{cases}$$

it turns out to be easiest to first extend the domain for our recurrences to a much bigger set than the nonnegative integers, either the real or rational numbers, and then to work backwards.

For example, we can write a recurrence of the form

$$t(x) = \begin{cases} f(x)t(x/b) + g(x) & \text{if } x \geq b \\ k(x) & \text{if } 1 \leq x < b \end{cases}$$

for two (known) functions f and g defined on the real [or rational] numbers greater than 1 and one (known) function k defined on the real [or rational] numbers x with $1 \leq x < b$. Then so long as $b > 1$ it is possible to prove that there is a unique function t defined on the real [or rational] numbers greater than or equal to 1 that satisfies the recurrence. We use the lower case t in this situation as a signal that we are considering a recurrence whose domain is the real or rational numbers greater than or equal to 1.

Exercise 5.2-3 How would we compute $t(x)$ in the recurrence

$$t(x) = \begin{cases} 3t(x/2) + x^2 & \text{if } x \geq 2 \\ 5x & \text{if } 1 \leq x < 2 \end{cases}$$

if x were 7? How would we show that there is one and only one function t that satisfies the recurrence?

Exercise 5.2-4 Is it the case that there is one and only one solution to the recurrence

$$T(n) = \begin{cases} f(n)T(\lceil n/b \rceil) + g(n) & \text{if } n > 1 \\ k & \text{if } n = 1 \end{cases}$$

when f and g are (known) functions defined on the positive integers, and k and b are (known) constants with b an integer larger than or equal to 2? (Note that $\lceil n/b \rceil$ denotes the *ceiling* of n/b, the smallest integer greater than or equal to n/b. Similarly $\lfloor x \rfloor$ denotes the *floor* of x, the largest integer less than or equal to x.)

To compute $t(7)$ in Exercise 5.2-3 we need to know $t(7/2)$. To compute $t(7/2)$, we need to know $t(7/4)$. Since $1 < 7/4 < 2$, we know that t(7/4) = 35/4. Then we may write

$$t(7/2) = 3 \cdot \frac{35}{4} + \frac{49}{4} = \frac{154}{4} = \frac{77}{2}.$$

Next we may write

$$\begin{aligned} t(7) &= 3t(7/2) + 7^2 \\ &= 3 \cdot \frac{77}{2} + 49 \\ &= \frac{329}{2}. \end{aligned}$$

Clearly we can compute $t(x)$ in this way for any x, though we are unlikely to enjoy the arithmetic. On the other hand suppose all we need to do is to show that there is a unique value of $t(x)$ determined by the recurrence, for all real numbers $x \geq 1$. If $1 \leq x < 2$, then $t(x) = 5x$, which uniquely determines $t(x)$. Given a number $x \geq 2$, there is a smallest integer i such that $x/2^i < 2$, and for this i, we have $1 < x/2^i$. We can now prove by induction on i that $t(x)$ is uniquely determined by the recurrence relation.

In Exercise 5.2-4 there is one and only one solution. Why? Clearly $T(1)$ is determined by the recurrence. Now assume inductively that $n > 1$ and that $T(m)$ is uniquely determined for positive integers $m < n$. We know that $n \geq 2$, so that $n/2 \leq n - 1$. Since $b \geq 2$, we know that $n/2 \geq n/b$, so that $n/b \leq n - 1$. Therefore $\lceil n/b \rceil < n$, so that we know by the inductive hypothesis that $T(\lceil n/b \rceil)$ is uniquely determined by the recurrence. Then by the recurrence,

$$T(n) = f(n)T\left(\left\lceil \frac{n}{b} \right\rceil\right) + g(n),$$

which uniquely determines $T(n)$. Thus by the principle of mathematical induction, $T(n)$ is determined for all positive integers n.

For every kind of recurrence we have dealt with, there is similarly one and only one solution. Because we know solutions exist, we don't find formulas for solutions to demonstrate that solutions exist, but rather to help us understand properties of the solutions. In the last section, for example, we were interested in how fast the solutions grew as n grew large. This is why we were finding Big-O and Big-Θ bounds for our solutions.

Recurrences for general n

We will now show how recurrences for arbitrary real numbers relate to recurrences involving floors and ceilings. We begin by showing that the conclusions of the Master Theorem apply to recurrences for arbitrary real numbers when we replace the real numbers by "nearby" powers of b.

Theorem 5.2 *Let a and b be positive real numbers with $b > 1$ and c and d be real numbers. Let $t(x)$ be the solution to the recurrence*

$$t(x) = \begin{cases} at(x/b) + x^c & \text{if } x \geq b \\ d & \text{if } 1 \leq x < b. \end{cases}$$

Let $T(n)$ be the solution to the recurrence

$$T(n) = \begin{cases} aT(n/b) + n^c & \text{if } n \geq 0 \\ d & \text{if } n = 1. \end{cases}$$

where n is a nonnegative integer power of b. Let $m(x)$ be the smallest integer power of b greater than or equal to x. Then $t(x) = \Theta(T(m(x)))$

Proof: If iterate (or, in the case that a is an integer, draw recursion trees for) the two recurrences, we can see that the results of the iterations are nearly identical. This means the solutions to the recurrences have the same big-Θ behavior. See the Appendix to this Section for details. ■

Removing Floors and Ceilings

We have also pointed out that a more realistic Master Theorem would apply to recurrences of the form $T(n) = aT(\lfloor n/b \rfloor) + n^c$, or $T(n) = aT(\lceil n/b \rceil) + n^c$, or even $T(n) = a'T(\lceil n/b \rceil) + (a - a')T(\lfloor n/b \rfloor) + n^c$. For example, if we are applying mergesort to an array of size 101, we really break it into pieces, one of size 50 and one of size 51. Thus the recurrence we want is not really $T(n) = 2T(n/2) + n$, but rather $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$.

We can show, however, that one can essentially "ignore" the floors and ceilings in typical divide-and-conquer recurrences. If we remove the floors and ceilings from a recurrence relation, we convert it from a recurrence relation defined on the integers to one defined on the rational numbers. However we have already seen that such recurrences are not difficult to handle.

The theorem below says that in recurrences covered by the master theorem, if we remove ceilings, our recurrences still have the same Big-Θ bounds on their solutions. A similar proof shows that we may remove floors and still get the same Big-Θ bounds. The condition that $b > 2$ can be replaced by $b > 1$, but the base case for the recurrence will depend on b. Since we may remove either floors or ceilings, that means that we may deal with recurrences of the form $T(n) = a'T(\lceil n/b \rceil) + (a - a')T(\lfloor n/b \rfloor) + n^c$

Theorem 5.3 *Let a and b be positive real numbers with $b \geq 2$ and let c and d be real numbers. Let $T(n)$ be the function defined on the integers by the recurrence*

$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + n^c & \text{if } n > 1 \\ d & n = 1 \end{cases},$$

and let $t(x)$ be the function on the real numbers defined by the recurrence

$$t(x) = \begin{cases} at(x/b) + x^c & \text{if } x \geq b \\ d & \text{if } 1 \leq x < b \end{cases}.$$

Then $T(n) = \Theta(t(n))$. The same statement applies with ceilings replaced by floors.

Proof: As in the previous theorem, we can consider iterating the two recurrences. It is straightforward (though dealing with the notation is difficult) to show that for a given value of n, the iteration for computing $T(n)$ has at most two more levels than the iteration for computing

$t(n)$. The work per level also has the same Big-Θ bounds at each level, and the work for the two additional levels of the iteration for $T(n)$ has the same Big-Θ bounds as the work at the bottom level of the recursion tree for $t(n)$. We give the details in the appendix at the end of this section. ■

Theorem 5.2 and Theorem 5.3 tell us that the Big-Θ behavior of solutions to our more realistic recurrences

$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + n^c & \text{if } n > 1 \\ d & \text{n=1} \end{cases}$$

is determined by their Big-Θ behavior on powers of the base b.

A version of the Master Theorem for more general recurrences.

We showed that in our version of the master theorem, we could ignore ceilings and assume our variables were powers of b. In fact we can ignore them in circumstances where the function telling us the "work" done at each level of our recursion tree is $\Theta(x^c)$ for some positive real number c. This lets us apply the master theorem to a much wider variety of functions.

Theorem 5.4 *Theorems 5.3 and 5.2 apply to recurrences in which the x^c term is replaced by a function f in which $f(x) = \Theta(x^c)$.*

Proof: We iterate the recurrences or construct recursion trees in the same way as in the proofs of the original theorems, and find that the condition $f(x) = \Theta(x^c)$ gives us enough information to again bound one of the solutions above and below with a multiple of the other solution. The details are similar to those in the original proofs. ■

Exercise 5.2-5 If $f(x) = x\sqrt{x+1}$, what can you say about the Big-Θ behavior of solutions to

$$T(n) = \begin{cases} 2T(\lceil n/3 \rceil) + f(n) & \text{if } n > 1 \\ d & \text{if } n = 1, \end{cases}$$

where n can be any positive integer, and the solutions to

$$T(n) = \begin{cases} 2T(n/3) + f(n) & \text{if } n > 1 \\ d & \text{if } n = 1, \end{cases}$$

where n is restricted to be a power of 3?

Since $f(x) = x\sqrt{x+1} \geq x\sqrt{x} = x^{3/2}$, we have that $x^{3/2} = O(f(x))$. Since

$$\sqrt{x+1} \leq \sqrt{x+x} = \sqrt{2x} = \sqrt{2}\sqrt{x}$$

for $x > 1$, we have $f(x) = x\sqrt{x+1} \leq \sqrt{2}x\sqrt{x} = \sqrt{2}x^{3/2} = O(x^{3/2})$. Thus the big-$\Theta$ behavior of the solutions to the two recurrences will be the same.

Extending the Master Theorem

As Exercise 5.2-5 suggests, Theorem 5.4 opens up a whole range of interesting recurrences to analyze. These recurrences have the same kind of behavior predicted by our original version of the Master Theorem, but the original version of the Master Theorem does not apply to them, just as it does not apply to the recurrences of Exercise 5.2-5.

We now state a second version of the Master Theorem. A still stronger version of the theorem may be found in CLR, but the version here captures much of the interesting behavior of recurrences that arise from the analysis of algorithms. The condition that $b \geq 2$ in this theorem can be replaced by $b > 1$, but then the base case depends on b and is not the case with $n = 1$.

Theorem 5.5 *Let a and b be positive real numbers with $a \geq 1$ and $b \geq 2$. Let $T(n)$ be defined by*

$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + f(n) & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

Then

1. *if $f(n) = \Theta(x^c)$ where $\log_b a < c$, then $T(n) = \Theta(n^c) = \Theta(f(n))$.*
2. *if $f(n) = \Theta(n^c)$, where $\log_b a = c$, then $T(n) = \Theta(n^{\log_b a} \log_b n)$*
3. *if $f(n) = \Theta(n^c)$, where $\log_b a > c$, then $T(n) = \Theta(n^{\log_b a})$*

The same results apply with ceilings replaced by floors.

Proof: Since we have assumed that $f(n) = \Theta(n^c)$, we know by Theorem 5.4 that we may restrict our domain to exact powers of b. We mimic the original proof of the Master theorem, substituting the appropriate $\Theta(n^c)$ for $f(n)$ in computing the work done at each level. But this means there are constants c_1 and c_2, independent of the level, so that the work at each level is between $c_1 n^c \left(\frac{a}{b^c}\right)^i$ and $c_2 n^c \left(\frac{a}{b^c}\right)^i$ so from this point on the proof is largely a translation of the original proof. ■

Exercise 5.2-6 What does the Master Theorem tell us about the solutions to the recurrence

$$T(n) = \begin{cases} 3T(n/2) + n\sqrt{n+1} & \text{if } n > 1 \\ 1 & \text{if } n = 1? \end{cases}$$

As we saw in our solution to Exercise 5.2-5 $x\sqrt{x+1} = \Theta(x^{3/2})$. Since $2^{3/2} = \sqrt{2^3} = \sqrt{8} < 3$, we have that $\log_2 3 > 3/2$. Then by conclusion 3 of version 2 of the Master Theorem, $T(n) = \Theta(n^{\log_2 3})$.

Appendix: Proofs of Theorems

For convenience, we repeat the statements of the earlier theorems whose proofs we merely outlined.

Theorem 5.6 *Let a and b be positive real numbers with $b > 1$ and c and d be real numbers. Let $t(x)$ be the solution to the recurrence*

$$t(x) = \begin{cases} at(x/b) + x^c & \text{if } x \geq b \\ d & \text{if } 1 \leq x < b. \end{cases}$$

Let $T(n)$ be the solution to the recurrence

$$T(n) = \begin{cases} aT(n/b) + n^c & \text{if } n \geq 0 \\ d & \text{if } n = 1, \end{cases}$$

where n is a nonnegative integer power of b. Let $m(x)$ be the smallest integer power of b greater than or equal to x. Then $t(x) = \Theta(T(m(x)))$

Proof: By iterating each recursion 4 times (or using a four level recursion tree in the case that a is an integer), we see that

$$t(x) = a^4 t\Big(\frac{x}{b^4}\Big) + \Big(\frac{a}{b^c}\Big)^3 x^c + \Big(\frac{a}{b^c}\Big)^2 x^c + \frac{a}{b^c} x^c$$

and

$$T(n) = a^4 T\Big(\frac{n}{b^4}\Big) + \Big(\frac{a}{b^c}\Big)^3 n^c + \Big(\frac{a}{b^c}\Big)^2 n^c + \frac{a}{b^c} n^c$$

Thus continuing until we have a solution, in both cases we get a solution that starts with a raised to an exponent that we will denote as either $e(x)$ or $e(n)$ when we want to distinguish between them and e when it is unnecessary to distinguish. The solution will be a^e times $T(x/b^e)$ plus x^c or n^c times a geometric series $G(x) = \sum_{i=0}^{e} \Big(\frac{a}{b^c}\Big)^i$. In both cases $T(x/b^e)$ (or $T(n/b^e)$) will be d. In both cases the geometric series will be $\Theta(1)$, $\Theta(e)$ or $\Theta\Big(\frac{a}{b^c}\Big)^e$, depending on whether $\frac{a}{b^c}$ is less than 1, equal to 1, or greater than one. Clearly $e(n) = \log_b n$. Since we must divide x by b an integer number greater than $\log_b x - 1$ times in order to get a value in the range from 1 to b, $e(x) = \lfloor \log_b x \rfloor$. Thus if m is the smallest integer power of b greater than or equal to x, then $0 \leq e(m) - e(x) < 1$. Then for any real number r we have $r^0 \leq r^{e(m)-e(x)} < r$, or $r^{e(x)} \leq r^{e(m)} \leq r \cdot r^{e(x)}$. Thus we have $r^{e(x)} = \Theta(r^{e(m)})$ for every real number r, including $r = a$ and $r = \frac{a}{b^c}$. Finally, $x^c \leq m^c \leq b^c x^c$, and so $x^c = \Theta(m^c)$. Therefore, every term of $t(x)$ is Θ of the corresponding term of $T(m)$. Further, there are only a finite number of different constants involved in our Big-Θ bounds. Therefore since $t(x)$ is composed of sums and products of these terms, $t(x) = \Theta(T(m))$. ■

Theorem 5.7 *Let a and b be positive real numbers with $b \geq 2$ and let c and d be real numbers. Let $T(n)$ be the function defined on the integers by the recurrence*

$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + n^c & \text{if } n \geq b \\ d & n = 1, \end{cases}$$

and let $t(x)$ be the function on the real numbers defined by the recurrence

$$t(x) = \begin{cases} at(x/b) + x^c & \text{if } x \geq b \\ d & \text{if } 1 \leq x < b. \end{cases}$$

Then $T(n) = \Theta(t(n))$.

Proof: As in the previous proof, we can iterate both recurrences. Let us compare what the results will be of iterating the recurrence for $t(n)$ and the recurrence for $T(n)$ the same number of times. Note that

$$\begin{aligned} \lceil n/b \rceil &< n/b + 1 \\ \lceil \lceil n/b \rceil / b \rceil < \lceil n/b^2 + 1/b \rceil &< n/b^2 + 1/b + 1 \\ \lceil \lceil \lceil n/b \rceil / b \rceil / b \rceil < \lceil n/b^3 + 1/b^2 + 1/b \rceil &< n/b^3 + 1/b^2 + 1/b + 1 \end{aligned}$$

This suggests that if we define $n_0 = n$, and $n_i = \lceil n_{i-1}/b \rceil$, then it is straightforward to prove by induction that $n_i < n/b^i + 2$. The number n_i is the argument of T in the ith iteration of the recurrence for T. We have just seen it differs from the argument of t in the ith iteration of t by at most 2. In particular, we might have to iterate the recurrence for T twice more than we iterate the recurrence for t to reach the base case. When we iterate the recurrence for t, we get the same solution we got in the previous theorem, with n substituted for x. When we iterate the recurrence for T, we get for some integer j that

$$T(n) = a^j d + \sum_{i=0}^{j-1} a^i n_i^c,$$

with $\frac{n}{b^i} \leq n_i \leq \frac{n}{b^i} + 2$. But, so long as $n/b^i \geq 2$, we have $n/b^i + 2 \leq n/b^{i-1}$. Since the number of iterations of T is at most two more than the number of iterations of t, and since the number of iterations of t is $\lfloor \log_b n \rfloor$, we have that j is at most $\lfloor \log_b n \rfloor + 2$. Therefore all but perhaps the last three values of n_i are less than or equal to n/b^{i-1}, and these last three values are at most b^2, b, and 1. Putting all these bounds together and using $n_0 = n$ gives us

$$\begin{aligned} \sum_{i=0}^{j-1} a^i \left(\frac{n}{b^i}\right)^c &\leq \sum_{i=0}^{j-1} a^i n_i^c \\ &\leq n^c + \sum_{i=1}^{j-4} a^i \left(\frac{n}{b^{i-1}}\right)^c + a^{j-2}(b^2)^c + a^{j-1}b^c + a^j 1^c\ , \end{aligned}$$

or

$$\begin{aligned} \sum_{i=0}^{j-1} a^i \left(\frac{n}{b^i}\right)^c &\leq \sum_{i=0}^{j-1} a^i n_i^c \\ &\leq n^c + b\sum_{i=1}^{j-4} a^i \left(\frac{n}{b^i}\right)^c + a^{j-2}\left(\frac{b^j}{b^{j-2}}\right)^c + a^{j-1}\left(\frac{b^j}{b^{j-1}}\right)^c + a^j\left(\frac{b^j}{b^j}\right)^c\ . \end{aligned}$$

As we shall see momentarily these last three "extra" terms and the b in front of the summation sign do not change the Big-Θ behavior of the right-hand side.

As in the proof of the master theorem, the Big-Θ behavior of the left hand side depends on whether a/b^c is less than 1, in which case it is $\Theta(n^c)$, equal to 1, in which case it is $\Theta(n^c \log_b n)$, or greater than one in which case it is $\Theta(n^{\log_b a})$. But this is exactly the Big-Θ behavior of the right-hand side, because $n < b^j < nb^2$, so $b^j = \Theta(n)$, which means that $\left(\frac{b^j}{b^i}\right)^c = \Theta\left(\left(\frac{n}{b^i}\right)^c\right)$, and the b in front of the summation sign does not change its Big-Θ behavior. Adding $a^j d$ to the middle term of the inequality to get $T(n)$ does not change this behavior. But this modified middle term is exactly $t(n)$. ■

Important Concepts, Formulas, and Theorems

1. *Master Theorem, simplified version.* The simplified version of the *Master Theorem* states: Let a be an integer greater than or equal to 1 and b be a real number greater than 1. Let c be a positive real number and d a nonnegative real number. Given a recurrence of the form

$$T(n) = \begin{cases} aT(n/b) + n^c & \text{if } n > 1 \\ d & \text{if } n = 1 \end{cases}$$

then for n a power of b,

   (a) if $\log_b a < c$, $T(n) = \Theta(n^c)$,

   (b) if $\log_b a = c$, $T(n) = \Theta(n^c \log n)$,

   (c) if $\log_b a > c$, $T(n) = \Theta(n^{\log_b a})$.

2. *Properties of Logarithms.* For any x, y and z, each greater than 1, $x^{\log_y z} = z^{\log_y x}$. Also, $\log_x y = \Theta(\log_2 y)$.

3. *Important Recurrences have Unique Solutions.* The recurrence

$$T(n) = \begin{cases} f(n)T(\lceil n/b \rceil) + g(n) & \text{if } n > 1 \\ k & \text{if } n = 1 \end{cases}$$

when f and g are (known) functions defined on the positive integers, and k and b are (known) constants with b an integer larger than 2 has a unique solution.

4. *Recurrences Defined on the Positive Real Numbers and Recurrences Defined on the Positive Integers.* Let a and b be positive real numbers with $b > 1$ and c and d be real numbers. Let $t(x)$ be the solution to the recurrence

$$t(x) = \begin{cases} at(x/b) + x^c & \text{if } x \geq b \\ d & \text{if } 1 \leq x < b. \end{cases}$$

Let $T(n)$ be the solution to the recurrence

$$T(n) = \begin{cases} aT(n/b) + n^c & \text{if } n \geq 0 \\ d & \text{if } n = 1, \end{cases}$$

where n is a nonnegative integer power of b. Let $m(x)$ be the smallest integer power of b greater than or equal to x. Then $t(x) = \Theta(T(m(x)))$

5. *Removing Floors and Ceilings from Recurrences.* Let a and b be positive real numbers with $b \geq 2$ and let c and d be real numbers. Let $T(n)$ be the function defined on the integers by the recurrence
$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + n^c & \text{if } n > 1 \\ d & n = 1 \end{cases},$$
and let $t(x)$ be the function on the real numbers defined by the recurrence
$$t(x) = \begin{cases} at(x/b) + x^c & \text{if } x \geq b \\ d & \text{if } 1 \leq x < b \end{cases}.$$
Then $T(n) = \Theta(t(n))$. The same statement applies with ceilings replaced by floors.

6. *Solutions to Realistic Recurrences.* The theorems summarized in 4 and 5 tell us that the Big-Θ behavior of solutions to our more realistic recurrences
$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + n^c & \text{if } n > 1 \\ d & \text{n=1} \end{cases}$$
is determined by their Big-Θ behavior on powers of the base b.

7. *Master Theorem, More General Version.* Let a and b be positive real numbers with $a \geq 1$ and $b \geq 2$. Let $T(n)$ be defined by
$$T(n) = \begin{cases} aT(\lceil n/b \rceil) + f(n) & \text{if } n > 1 \\ d & \text{if } n = 1 \end{cases}$$
Then
   (a) if $f(n) = \Theta(x^c)$ where $\log_b a < c$, then $T(n) = \Theta(n^c) = \Theta(f(n))$.
   (b) if $f(n) = \Theta(n^c)$, where $\log_b a = c$, then $T(n) = \Theta(n^{\log_b a} \log_b n)$
   (c) if $f(n) = \Theta(n^c)$, where $\log_b a > c$, then $T(n) = \Theta(n^{\log_b a})$.

   The same results apply with ceilings replaced by floors. A similar result with a base case that depends on b holds when $1 < b < 2$.

Problems

1. Use the master theorem to give Big-Θ bounds on the solutions to the following recurrences. For all of these, assume that $T(1) = 1$ and n is a power of the appropriate integer.
   (a) $T(n) = 8T(n/2) + n$
   (b) $T(n) = 8T(n/2) + n^3$
   (c) $T(n) = 3T(n/2) + n$
   (d) $T(n) = T(n/4) + 1$
   (e) $T(n) = 3T(n/3) + n^2$
2. Extend the proof of the Master Theorem, Theorem 5.1 to the case $T(1) = d$.
3. Show that for any x, y and z, each greater than 1, $x^{\log_y z} = z^{\log_y x}$.

4. Show that for each real number $x \geq 0$ there is one and only one value of $T(x)$ given by the recurrence
$$T(x) = \begin{cases} 7xT(x-1)+1 & \text{if } x \geq 1 \\ 1 & \text{if } 0 \leq x < 1. \end{cases}$$

5. Show that for each real number $x \geq 1$ there is one and only one value of $T(x)$ given by the recurrence
$$T(x) = \begin{cases} 3xT(x/2)+x^2 & \text{if } x \geq 2 \\ 1 & \text{if } 1 \leq x < 2 \end{cases}.$$

6. How many solutions are there to the recurrence
$$T(n) = \begin{cases} f(n)T(\lceil n/b \rceil)+g(n) & \text{if } n > 1 \\ k & \text{if } n = 1 \end{cases}$$
if $b < 2$? If $b = 10/9$, what would we have to replace the condition that $T(n) = k$ if $n = 1$ by in order to get a unique solution?

7. Give a big-Θ bound on the solution to the recurrence
$$T(n) = \begin{cases} 3T(\lceil n/2 \rceil)+\sqrt{n+3} & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

8. Give a big-Θ bound on the solution to the recurrence
$$T(n) = \begin{cases} 3T(\lceil n/2 \rceil)+\sqrt{n^3+3} & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

9. Give a big-Θ bound on the solution to the recurrence
$$T(n) = \begin{cases} 3T(\lceil n/2 \rceil)+\sqrt{n^4+3} & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

10. Give a big-Θ bound on the solution to the recurrence
$$T(n) = \begin{cases} 2T(\lceil n/2 \rceil)+\sqrt{n^2+3} & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$