Notes from MAT296 - Calculus II

Notetaker: Grant Griffiths

Semester: Spring 2012 - Syracuse University

Contents

Chapter 6: Applications of Integration

Lesson 6.1: Areas Between Curves

- **Area between two curves:** $A = \int_a^b |f(x) - g(x)| dx$

Lesson 6.2: Volumes

- **Definition of Volume** Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plan P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_a^b A(x) dx$$

Lesson 6.3: Volumes by Cylindrical Shells

- The **volume** of a solid, obtained by **rotating about the y-axis** under the curve $f(x)$ is
  - $V = \int circumference * height * thickness$
  - $V = \int_a^b 2\pi x f(x) dx$ where $0 \leq a < b$
- The **volume** of a solid, obtained by **rotating about the x-axis** under the curve $f(y)$ is
  - $V = \int circumference * height * thickness$
  - $V = \int_a^b 2\pi y f(y) dy$ where $0 \leq a < b$

Lesson 6.4: Work

- **Work:** $W = Fd$
- **Hooke's Law (Force on spring):** $F = kx$
- **Work done pulling rope up:**
  - $W = \int weight\ density * displacement * thickness$
  - $W = \int_a^b kx\ dx$ where k is the weight density
- **Work done in moving an object from a to b:** $W = \int_a^b f(x) dx$

Chapter 7: Techniques of Integration

Lesson 7.1: Integration by Parts

- **Formula:** $\int u\ dv = uv - \int v\ du$

Lesson 7.2: Trigonometric Integrals

- **General Advice:** Memorize Trigonometric Identities and set yourself up for an easy substitution.
- **Strategy for Evaluating** $\int sin^m x\ cos^n x\ dx$
  - If the **power of cosine is odd**, save one cosine factor and use $cos^2 x = 1 - sin^2 x$ to express the remaining factors in terms of sine.
  - If the **power of sine is odd**, save one sine factor and use $sin^2 x = 1 - cos^2 x$ to express the remaining factors in terms of cosine.
  - If the **powers of both sine and cosine are even**, use half angle identities

    $$sin^2 x = \frac{1}{2}(1 - cos(2x)) \qquad cos^2 x = \frac{1}{2}(1 + cos(2x))$$

    It is sometimes helpful to use the identity

    $$sin(x)\ cos(x) = \frac{1}{2} sin(2x)$$

- **Strategy for Evaluating** $\int tan^m x\ sec^n x\ dx$
  - If the **power of secant is even**, save a factor of $sec^2 x$ and use $sec^2 x = 1 + tan^2 x$ to express the remaining factors in terms of $tan\ x$ Then substitute $u = tan\ x$
  - If the **power of tangent is odd**, save a factor of $sec(x)\ tan(x)$ and use $tan^2 x = sec^2 x - 1$ to express the remaining factors in terms of $sec\ x$ Then substitute $u = sec\ x$
- $\int tan\, x\ dx = ln|sec\, x| + C$
- $\int sec\, x\ dx = ln|sec\, x + tan\, x| + C$

Lesson 7.3: Trigonometric Substitution

- Use the following table to match substitutions and identities depending on what expression you're trying to integrate.

Expression	**Substitution**	**Identity**
$\sqrt{a^2 - x^2}$	$x = a\, sin\, \theta$	$1 - sin^2\theta = cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\, tan\, \theta$	$1 + tan^2\theta = sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\, sec\, \theta$	$sec^2\theta - 1 = tan^2\theta$

Lesson 7.4: Integration of Rational Functions by Partial Fractions

- Fractions can be broken down using coefficients.

Lesson 7.5: Strategy for Integration

- Use all integration techniques we have learned so far. Use your judgement to decide which technique to use depending on what problem it is.
- **Integration Formulas**
  1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
  2. $\int \frac{1}{x} = ln|x|$
  3. $\int e^x \, dx = e^x$
  4. $\int a^x \, dx = \frac{a^x}{ln\, a}$
  5. $\int sin\, x \, dx = -cos\, x$
  6. $\int cos\, x \, dx = sin\, x$
  7. $\int sec^2 x \, dx = tan\, x$
  8. $\int sec\, x \, dx = ln|sec\, x + tan\, x|$
  9. $\int tan\, x \, dx = ln|tan\, x|$

     Note: Integrals for hyperbolic trigonometric function, cosecant, and cotangent also exist

Lesson 7.8: Improper Integrals

- **Formulas for finding infinite improper integrals:**
  1. If $\int_a^b f(x)\, dx$ exists for every number $t \geq a$, then
  $$\int_a^\infty f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx$$
  2. If $\int_a^b f(x)\, dx$ exists for every number $t \geq a$, then
  $$\int_{-\infty}^b f(x)\, dx = \lim_{t \to -\infty} \int_t^b f(x)\, dx$$
- **Formulas for finding improper integrals with discontinuous points:**
  1. If f is continuous on $[a, b)$, and is discontinuous at b, then
  $$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx$$
  2. If f is continuous on $(a, b]$, and is discontinuous at a, then
  $$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^a f(x)\, dx$$
- **Convergent Improper Integral:** if the limit inside the improper exists
- **Divergent Improper Integral:** if the limit inside the improper does not exist
- If **both of the bounds are infinite**, we can break it into the sum of two improper integrals

Chapter 8: Further Applications of Integration

Lesson 8.1: Arc Length

- **The Arc Length Formula:** If f' is continuous on [a,b], then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Same idea but different variables for functions in terms of y

Lesson 8.2: Area of a Surface of Revolution

- **Surface Area of Revolution:** $S = \int 2\pi x\, ds$ where ds is $\sqrt{1 + f'(x)^2} dx$
- Together, the **Formula For a Surface area of Revolution** is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx \qquad \text{or, in Leibniz notation,} \qquad S = \int_a^b 2\pi y \sqrt{1 + (\tfrac{dy}{dx})^2} dx$$

Chapter 10: Parametric Equations and Polar Coords

Lesson 10.3: Polar Coordinates

- **Converting from Polar coordinates to Cartesian coordinates and vice versa:**
  1. $x = r\,cos\,\theta$
  2. $y = r\,sin\,\theta$
  3. $r^2 = x^2 + y^2$
  4. $tan\,\theta = \frac{y}{x}$

  Finding slopes of parametric curves:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} sin\,\theta + r\,cos\,\theta}{\frac{dr}{d\theta} cos\,\theta - r\,sin\,\theta}$$

Lesson 10.4: Areas and Lengths in Polar Coordinates

- **Formula for a polar area:** $A = \int\limits_a^b \frac{1}{2} r^2\,d\theta$
- **Formula for length of a polar curve:** $L = \int\limits_a^b \sqrt{r^2 + (\frac{dr}{d\theta})}\,d\theta$

Chapter 11: Infinite Sequences and Series

Lesson 11.1: Sequences

- **Sequence** - a list of numbers written in definite order.
- Sequences can have limits, both convergent and divergent.
- The limit laws from section 2.3 hold for sequences.

Lesson 11.2: Series

- **Infinite Series** (or just a **series**): $\sum_{n=1}^{\infty}$
- Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$$

- If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n$ exists, then $\sum a_n$ is a **convergent series**.
- If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.
- The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ...$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

if $|r| \geq 1$, the geometric series is divergent.

- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
- **Test for Divergence:** If $\lim_{n\to\infty} a_n$ does not exists or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- A series can be broken down into the sum or difference of two series. Constants can also be pulled out of the series.

Lesson 11.3: The Integral Test and Estimates of Sums

- **The Integral Test:** Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int\limits_1^{\infty} f(X)dx$ is convergent. In other words:

$$\text{If } \int\limits_1^{\infty} f(x)dx \text{ is convergent, then } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

$$\text{If } \int\limits_1^{\infty} f(x)dx \text{ is divergent, then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

- The p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

Lesson 11.4: The Comparison Tests

- **The Comparison Test:** Suppose that $\sum A_n$ and $\sum B_n$ are series with positive terms.

  $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
  If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.

Lesson 11.5: Alternating Series

- **Alternating Series Test:** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + ... \qquad b_n > 0$$

satisfies

1. $b_{n+1} \leq b_n$ for all n
2. $\lim\limits_{n\to\infty} b_n = 0$

then the series is convergent.

Lesson 11.6: Absolute Convergence and the Ratios and Roots Tests

- **Definition:** A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- **Definition:** A series $\sum a_n$ is called **conditionally convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- **Theorem:** If a series $\sum a_n$ is absolutely convergent, then it is convergent.
- **The Ratio Test**:
  - If $\lim\limits_{n\leftarrow\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.
  - If $\lim\limits_{n\leftarrow\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ or $\lim\limits_{n\leftarrow\infty} |\frac{a_{n+1}}{a_n}| = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.
  - If $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the ratio test is inconclusive. No conclusion can be drawn.
- **The Root Test**:
  - If $\lim\limits_{n\leftarrow\infty} \sqrt[n]{|\frac{a_{n+1}}{a_n}|} = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.
  - If $\lim\limits_{n\leftarrow\infty} \sqrt[n]{|\frac{a_{n+1}}{a_n}|} = L > 1$ or $\lim\limits_{n\leftarrow\infty} \sqrt[n]{|\frac{a_{n+1}}{a_n}|} = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.
  - If $\lim\limits_{n\to\infty} \sqrt[n]{|\frac{a_{n+1}}{a_n}|} = 1$, the root test is inconclusive. No conclusion can be drawn.

Lesson 11.7: Strategy for Testing Series

- Series can be tested in more than one way. However, there is usually an easiest way to test a given series.
- This is one way to prioritize the tests:
  1. If the function inside the series appears to not go to zero, then try the **Test for Divergence**.
  2. If it is a **geometric series**, you can easily find out if it converges or not.
  3. If the series has $(-1)^n$ in it, try the **Alternating Series Test**.
  4. If the series has a form similar to a p-series or geometric series, then try a **Comparison Test**.
  5. If the series involves factorials, try the **Ratio Test**.
  6. If the series is of the form $(b_n)^n$, then the **Root Test** may be useful.
  7. If the function inside the series can be integrated, try the **Integral Test**.
- **Warning:** Always remember to check if the series satisfies the pre-requirements of each test.

Lesson 11.8: Power Series

- A **power series** is a series in the form $\sum_{n=0}^{\infty} c_n x^n$
- The c_n terms are called the coefficient terms.
- The a term is called the center
- **Theorem** For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:
  1. The series converges only when $x = a$
  2. The series converges for all x.
  3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$
- **Radius of convergence:** The number "R" from above is the radius of converge
- **Interval of Convergence:** The interval that consists all values of x for which the series converges.

Lesson 11.9: Representations of Functions as Power Series

- **Theorem** If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

  is differentiable (and therefore) continuous on the interval (a-R,a+R) and

  - $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
  - $\int f(x)\,dx = C + \sum_{0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

  The radii of convergence of the power series in the two equations above are both R

- **Representing:** In order to represent a function as a power series, you have to get it into the general form $\frac{1}{1-x}$. Once in this form, the function can be represented as a power series:

$$\sum_{n=0}^{\infty} (x)^n \qquad \text{where x is any function.}$$

Lesson 11.10: Taylor and Maclaurin Series

- **Taylor Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
- **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$
- $\lim_{n \to \infty} \frac{x^n}{n!} = 0$ for every real number x
- **Taylor Series expansions to remember:**
  1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
  2. $sin\ x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
  3. $cos\ x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
  4. $tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
  5. $ln|1+x| = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

Bibliography

Book used: Calculus Early Transcendentals 7th Edition
Professor: Notes from Dr. Graham Leuschke's Spring 2012 Calculus II course