Notes from MAT331 - First Course in Linear Algebra

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Chapter 1: Linear Equations in Linear Algebra

Lesson 1.1: Systems of Linear Equations

- **Linear Equation** - An equation that can be written in the form $a_1x_1 + a_2x_2 + ... + a_nx_n = b$
- **Systems of Linear Equations** - a collection of one or more linear equations involving the same variables
- **Elementary Row Operations**
    1. **Replacement** - Replace one row by the sum of itself and a multiple of another row
    2. **Interchange** - Interchange two rows
    3. **Scaling** - Multiply all entries in a row by a nonzero constant
- **Row Equivalent** - when a matrix can be transformed into a another one using the row operations.
- **Fact:** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Lesson 1.2: Row Reduction and Echelon Forms

- **Leading Entry** - the leftmost nonzero entry in a nonzero row
- **Echelon form** requirements:
    1. All non-zero rows are above any rows of all zeros
    2. Each leading entry of a row is in a column to the right of the leading entry of the row above it
    3. All entries in a column below a leading entry are zeros
- **Reduced Echelon form** additional requirements
    1. The leading entry in each nonzero row is 1
    2. Each leading 1 is the only nonzero entry in its column
- **Echelon Matrix** - matrix that is in echelon form
- **Fact** - Any nonzero matrix may be **row reduced**
- **Pivot Position** - location in matrix A that corresponds to a leading 1 in the reduced echelon form of A
- **Pivot Column** - is a column of A that contains a pivot position
- **Using Row Reduction to solve a Linear System**
    1. Write the augmented matrix of the system
    2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
    3. Continue row reduction to obtain the reduced echelon form. Write the system of equations corresponding to the matrix obtained in step 3.
    4. Write the system of equations corresponding to the matrix obtained in step 3.
    5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing the equation.

Lesson 1.3: Vector Equations

- **Column Vector** - a matrix with only one column
- **Equal** - two vectors are equal iff their corresponding entries are equal.
- **Vector Addition** - Given two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$, their sum is the vector u + v obtained by adding corresponding entries of u and v.
- **Scalar Multiple** - Given a vector u and a real number c, the scalar multiple of u by c is the vector cu
- **Parallelogram Rule for Addition** - If u and v in $\mathbb{R}^2$ are represented as points in the plane, then u + v corresponds to the fourth vertex of the parallelogram whose other vertices are u, 0, and v.
- If $v_1, ..., v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, ..., v_p$ is denoted by $Span\{v_1, ..., v_p\}$, and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $v_1, ..., v_p$. That is, $Span\{v1, ..., vp\}$ is the collection of vectors that can be written in form $c_1v_1 + c_2v_2 + ... + c_pv_p$ with $c_1...c_p$ scalars.

Lesson 1.4: The Matrix Equation Ax = b

- If A is an m×n matrix, with columns $a_1...a_n$, and $\vec{x}$ is in $\mathbb{R}^2$, then the **product of A and** $\vec{x}$, denoted by A$\vec{x}$, is the **linear combination of the columns of A using the corresponding entries in** $\vec{x}$ **as weights**; that is,

$$A\vec{x} = [a_1a_2...a_n] \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\vec{a_n}$$

- If A is an m × n matrix, with columns $a_1, ..., a_n$, and if b is in $\mathbb{R}^m$, the matrix equation

$$A\vec{x} = \vec{b}$$

  has the same solution set as the vector equation $x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\vec{a_n} = \vec{b}$ which, in turn, has the same solution set as the system of linear equations whose augmented matrix is $[\vec{a_1}\ \vec{a_2}\ ...\ \vec{a_n}]$
- The equation A$\vec{x} = b$ has a solution iff $\vec{b}$ is a linear combination of the columns of A.
- Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.
  1. For each $\vec{b}$ in $\mathbb{R}^m$, the equation A$\vec{x} = b$ has a solution.
  2. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
  3. The columns of A span $\mathbb{R}^m$
  4. A has a pivot position in every row.
- **Row-Vector Rule for computing** A$\vec{x}$
  If the product A$\vec{x}$ is defined, then the ith entry in A$\vec{x}$ is the sum of products of corresponding entries from row i of A and from the vector $\vec{x}$
- **Identity Matrix** - when there are a set of diagonal 1's from the top left to bottom right in a square matrix.
- If A is an m×n matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:
  1. A($\vec{u}$+$\vec{v}$) = A$\vec{u}$+A$\vec{v}$
  2. A(c$\vec{u}$) = c(A$\vec{u}$)

Lesson 1.5: Solution Sets of Linear System

- A system of linear equations is said to be **homogeneous** if it can be written in the form A$\vec{x}$=0, where A is in an m×n matrix and 0 is the zero factor in $\mathbb{R}^m$
- The homogeneous equation A$\vec{x}$=0 has a nontrivial solution iff the equation has at least one free variable.
  Such a system A$\vec{x}$=0 always has at least one solution, namely, x=0. This zero solution is usually called the **trivial solution**.
- **Nontrivial Solution** - a nonzero vector $\vec{x}$ that satisfies A$\vec{x}$=0
- The homogeneous equation A$\vec{x}$=0 has a nontrivial solution iff the equation has at least one free variable.
- Whenever a solution set is described explicitly with vectors, we say that the solution is in **parametric vector form.**
- Equation of a line through p parallel to v: x = **p** + t**v**
- Suppose the equation A$\vec{x}$=b is consistent for some given b, and let p be a solution. Then the solution set of A$\vec{x}$=b is the set of all vectors of the form **w**=**p**+$\mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation A$\vec{x}$=0.
- **Writing a solution set in parametric vector form**
  1. Row reduce the augmented matrix to reduced echelon form
  2. Express each basic variable in terms of any free variables appearing in an equation.
  3. Write a typical solution $\vec{x}$ as a vector whose entries depend on the free variables, if any.
  4. Decompose $\vec{x}$ into a linear combination of vectors(with numeric entries) using the free variables as parameters.

Lesson 1.7: Linear Independence

- **Linearly independent** - There are no free variables in a vector equation
- **Linearly dependent** - There is at least one free variable in a vector equation
- The columns of a matrix A are linearly indepedent iff the equation A$\vec{x}$=0 has only the trivial solution.
- A set of two vectors $\vec{v}_1, \vec{v}_2$ is linearly depednet if at least one of the vectors is a multiple of te other. The set is linearly indepdent iff neither of the cvectors is a multiple of the other.
- An indexed set S = $v_1, ..., v_p$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others.
- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
- if a set S = $v_1, ..., v_p$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Lesson 1.8: Introduction to Linear Transformations

- A **transformation (or function or mapping)** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\vec{x}$ in $\mathbb{R}^n$ a vector T($\vec{x}$) in $\mathbb{R}^m$. The set $\mathbb{R}^n$is called the **domain** of T, and $\mathbb{R}^m$ is called the codomain of T.
- **Image -** the vector T($\vec{x}$)
- **Range -** the set of all images T($\vec{x}$)
- Let A = $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation T:$\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by T($\mathbf{x}$)=A$\mathbf{x}$ is called the **shear transformation**.
- A transformation (or mapping) T is **linear** if:
    1. T($\vec{u}$+Vv)=T($\vec{u}$)+T($\vec{v}$) for all $\vec{u}$,$\vec{v}$ in the domain of T
    2. T(c$\vec{u}$)=cT($\vec{u}$) for all scalars c and all $\vec{u}$ in the domain of T
- If T is a linear transformation, then

$$\mathrm{T}(\mathbf{0})=\mathbf{0}$$

and

$$\mathrm{T}(\mathrm{c}\vec{u}+\mathrm{d}\vec{v})=\mathrm{cT}(\vec{u})+\mathrm{dT}(\vec{v})$$

for all vectors $\vec{u}$, $\vec{v}$, in the domain of T and all scalars c,d.

- Given a scalar r, define T:$\mathbb{R}^2 \rightarrow \mathbb{R}^2$ by T($\vec{x}$)=r$\vec{x}$. T is called **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$.

Lesson 1.9: The Matrix of a Linear Transformation

- Let T:$\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation Then there exists a unique matrix A such that

  $$T(\vec{x})=A\vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n$$

  In fact, A is the m×n matrix whose jth column is the vector T($\vec{e}_j$) where $\vec{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

  $$A=\begin{bmatrix} T(\vec{e}_1) & ... & T(\vec{e}_n) \end{bmatrix}$$

- **Reflections**
  - Reflection through the x_1-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - Reflection through the x_2-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Reflection through line x_2=x_1: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - Reflection through line x_2=-x_1: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
  - Reflection through the origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
- **Contractions and Expansions**
  - Horizontal contraction and expansion
  - Vertical contraction and expansion
- **Shears**
  - Horizontal Shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
  - Vertical Shear: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$
- **Projections**
  - Projection onto the x_1-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
  - Projection onto the x_2-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- A mapping T:$\mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$.
- A mapping T:$\mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$.
- Let T:$\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one iff the equation T($\vec{x}$)=0 has only the trivial solution.
- Let T:$\mathbb{R}^n \to \mathbb{R}^m$ and Let A be the standard matrix for T. Then:
  1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A spans $\mathbb{R}^m$
  2. T is one-to-one iff the columns of A are linearly independent.

Chapter 2: Matrix Algebra

Lesson 2.1: Matrix Operations

- The **diagonal entries** in an m×n matrix A = $[a_{ij}]$ are a_{11},a_{22},a_{33},..., and they form the **main diagonal** of A.
- A **diagonal matrix** is a square $n \times n$ matrix whose non-diagonal entries are zero.
- A matrix whose entries are all 0 is a **zero matrix** and is written as 0.
- Two matrices are **equal** if they have the same size and their corresponding columns are equal.
- The sum of two matrices A and B is the matrix whose columns are the sums of the columns of A and B.
- **Scalar Multiple** - matrix rA whose columns are r times the corresponding columns A.
- Matrices follow standard algebraic laws such as commutativity, associativity, distributivity, A + 0 = A, and IA = A.
- AB = A[b_1 b_2 ... b_p]=[Ab_1 Ab_2 ... Ab_p]
- Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.
- **Row-columns rule for computing AB**
  If the product AB is defined, then the entry in row i and colun j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(\mathrm{AB})_{ij}$ denotes the (i,j)-entry in AB, and if A is an m×n matrix, then

$$(\mathrm{AB})_{ij}{=}a_{i1}b_{1j} + a_{i2}b_{2j} + ..a_{in}b_{nj}$$

- **WARNINGS:**
  1. In general, AB≠BA
  2. The cancellation laws do not hold for matrix multiplication. That is, if AB = AC, then it is *not* true in general that B = C.
  3. If a product AB is the zero matrix, you *cannot* conclude in general that either A = 0 or B = 0.
- $\mathrm{A}^k = \underbrace{A \cdots A}$ for k times.
- Given an m×n matrix A, the **transpose** of A is the n×m matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathrm{B} = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

  Then

$$A^T{=} \begin{bmatrix} a & c \\ b & d \end{bmatrix}, B^T{=} \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, C^T{=} \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

- Let A and B denote matrices whose sizes are appropriate for the following sums and products.
  1. $(A^T)^T = A$
  2. $(A+B)^T = A^T + B^T$
  3. For any scalar r, $(rA)^T = rA^T$
  4. $(AB)^T = B^T A^T$

Lesson 2.2: The Inverse of a Matrix

- **Singular Matrix** - matrix that cannot be inverted
- **Nonsingular Matrix** - matrix that can be inverted
- $A^{-1}A = I$ and $AA^{-1} = I$
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \tfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  If $ad - bc = 0$, then A is not invertible.

- The quantity $det\,A = ad - bc$ is called the **determinant** of A.
- If A is an invertible matrix, then for each b in $\mathbb{R}^n$, the equation A$\vec{x}$=b has the unique solution $\vec{x} = A^{-1}b$.
- If A and B are invertible matrices then,
  1. $(A^{-1})^{-1} = A$
  2. $(AB)^{-1} = B^{-1}A^{-1}$
  3. $(A^T)^{-1} = (A^{-1})^T$
- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.
- An n×n matrix A is invertible iff A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n, also transforms I_n into A^{-1}.
- How to find A^{-1}
  1. Row reduce the augmented matrix [A I]
  2. If A is row equivalent to I, then [A I] is row equivalent to [I A^{-1}]
  3. Else, A does not have an inverse

Lesson 2.3: Characterizations of invertible Matrices

- Let A be a square n×n matrix. Then, the following statements must be all true or all false:
  1. A is an invertible matrix
  2. A is row equivalent to the n×n identity matrix
  3. A has n pivot position
  4. The equation A$\vec{x}$=0 has only the trivial solution
  5. The columns of A form a linearly independent set
  6. The Linear Transformation x $\rightarrow$ A$\vec{x}$ is one-to-one
  7. The equation A$\vec{x}$=b has at least one solution for each b in $\mathbb{R}^n$
  8. The linear transformation x $\rightarrow$ A$\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
  9. There is an n×n matrix C such that CA = I
  10. There is an n×n matrix D such that AD = I
  11. A^T is an invertible matrix
- Let A and B be square matrices. If $AB = I$, then A and B are both invertible with $B = A^{-1}$ and $A = B^{-1}$

Lesson 2.4 Partitioned Matrices

- You can split a matrix into parts called **Partitions**

Lesson 2.8 Subspaces of R

- A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:
  1. The Zero vector is in H
  2. For each $\vec{u}$ and $\vec{v}$ in H, the sum $\vec{u} + \vec{v}$ is in H.
  3. For each $\vec{u}$ in H and each scalar c, the vector c$\vec{u}$ is in H.
- We now refer to $Span\{v_1, ..., v_p\}$ as the **subspace spanned** (or **generated**) by $v_1, ..., v_p$
- **Zero subspace** - subspace containing only the zero vector
- **Column Space** of a matrix A is the set Col A of all linear combinations of the columns of A.
- If $A = [a_1 ... a_n]$, with the columns in $\mathbb{R}^n$, then Col A is the same as $Span\{a_1, ..., a_n\}$.
- Col A equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise, Col A is only part of $\mathbb{R}^m$
- The **null space** of a matrix A is the set Nul A of all solutions of the homogenous equation A$\vec{x}$=0
- The null space of an m×n matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system A$\vec{x}$=0 of m homogenous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.
- A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.
- The pivot columns of a matrix A form a basis for the columns of A.
- **Warning:** Be careful to use *pivot columns* of A *itself* for the basis of Col A. The columns of an echelon form B are often not in the column space of A.

Lesson 2.9 Dimension and Rank

- Suppose the set $\beta = b_1, ..., b_p$ is a basis for a subspace H. For each x in H, the **coordinates of x relative to the basis** β are the weights $c_1, ..., c_p$ such that $x = c_1 b_1 + ... + c_p b_p$, and the vector in $\mathbb{R}^p$

$$[X]_\beta = \begin{bmatrix} c_1 \\ ... \\ c_p \end{bmatrix}$$

  is called the **coordinate vector of x (relative to β)** or the **β-Coordinate vector of x**

- The **dimension** of a nonzero subspace H, denoted by dim H, is t he number of vectors in any basis for H. The dimension of the zero subspace 0 is defined to be zero.
- The **rank** of matrix A, denoted by rank A, is the dimension of the column space of A.
- **The Rank Theorem:** If a matrix A has n columns, then rank A + dim Nul A = n
- **The Basis Theorem:** Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.
- **The Invertible Matrix Theorem (continued):**
  Let A be an n×n matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

    1. The columns of A form a basis of $\mathbb{R}^n$
    2. Col A = $\mathbb{R}^n$
    3. dim Col A = n
    4. rank A = n
    5. Nul A = 0
    6. dim Nul A = 0

Chapter 3: Determinants

Lesson 3.1 Introduction to Determinants

- If A is an 2×2 matrix, then the determinant of A, written det A, $= a_{11}a_{22} - a_{12}a_{21}$
- For $n \geq 2$, the **determinant** of an n×n matrix $A = [a_{ij}]$ is

$$\sum_{j=1}^{n} (-1)^{1+j} a_{1j} det A_{1j}$$

- The determinant of an n×n matrix A can be computed by a cofactor expansion across any column. The expansion across the ith row using the cofactors in $C_{ij} = (-1)^{i+j}\text{det } A_{ij}$ is

$$\text{det A} = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$

  The cofactor expansion down the jth column is

$$\text{det A} = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{2j}C_{2j}$$

- If A is a triangular matrix, then det A is the product of the entries in the main diagonal of A.

Lesson 3.2 Properties of Determinants

- Let A be a square matrix
  1. If a multiple of one row of A is added to another row to produce a matrix B, then det B = det A
  2. If two rows of A are interchanged to produce B, then det B = -det A
  3. If one row of A is multiplied by k to produce B, then det B = k det A
- A square matrix A is invertible iff det $A \neq 0$
- If A is an n×n matrix, then det A^T=det A
- **Multiplicative Property** - If A and B are n×n matrices, then det AB = $(detA)(detB)$
- **Warning:** det(A+B)≠det(A)+det(B) in general.

Lesson 3.3 Cramer's Rule, Volume, and Linear Transformations

- **Cramer's Rule** - Let A be an invertible n×n matrix. For any $\vec{b}$ in $\mathbb{R}^n$, the unique solution $\vec{b}$ of A$\vec{x}$=$\vec{b}$ has entries given by

$$x_i = \frac{det(A_i)(b)}{det(A)},\ i = 1, 2, ..., n$$

- A matrix of cofactors is called a **adjugate** (or **classical adjoint**) of A, denoted by adj A
- **An Inverse Formula** - Let A be an invertible n×n matrix. Then,

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|detA|$
- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|detA|$
- Let T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of T(S)}\}{=}|detA| \cdot \{area\ of\ S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^2$, then

$$\{\text{volume of T(S)}\}{=}|detA| \cdot \{volume\ of\ S\}$$

Chapter 5: Eigenvalues and Eigenvectors

Lesson 5.1 Eigenvectors and Eigenvalues

- An **eigenvector** of an n×n matrix A is a non zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$; such an $\vec{x}$ is called an *eigenvector corresponding to* λ.
- **Eigenspace**: The subspace defined as the set of all solutions to $(A - \lambda I)\vec{x} = 0$
- The eigenvalues of a triangular matrix are the entries on its main diagonal
- If $v_1, ..., v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, ..., \lambda_r$ of an n×n matrix A, then the set $\{v_1, ..., v_r\}$ is linearly independent.

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