Notes from MAT275 - Abstract Mathematics

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Contents

Chapter 1: Logic and Proof

Lesson 1.1: Proofs, What and Why?

- **Proof:** logically sound argument or explanation that takes in account all generalities of the situation and reaches the desired conclusion.
- **Prime Number:** positive integer with exactly two divisors
- **Composite number:** an integer 2 that is not a prime number.
- **Factorial Function**

Lesson 1.2: Statements and Non-statements

- **Statement:** any sentence that has exactly one truth value.
- **Paradox:** a sentence with proper grammatical structure, yet one that cannot have a truth value.
- **Propositional Function:** Can be true depending on the input
- **Truth Set:** the set of objects for which a propositional function has value True

Lesson 1.3: Logical Operations and Logical Equivalence

- **Conjunction** of P with Q, written $P \wedge Q$, is given by the following **truth table**:

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

- **Disjunction** of P with Q, written $P \vee Q$, is given by the following **truth table**:

P	Q	$P \vee Q$
F	F	T
F	T	T
T	F	T
T	T	F

- **Negation** of P, written $\neg P$, is given by the following **truth table**:

P	$\neg P$
T	F
F	T

- Two expressions are **logically equivalent**, written $E_1 \Leftrightarrow E_2$, if their truth tables match.
- **Proposition:** Let P, Q, and R be statements. Then
    1. $\neg(\neg P) \Longleftrightarrow P$
    2. $\neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q$
    3. $\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$

4. $P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$
5. $P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$

- Rules 4 and 5 are the **distributive laws**.
- The two logical operations $\wedge$ and $\vee$ satisfy the **commutative** and associative laws.
- **Definition:** Let P and Q be statements. We define the **exclusive or** operation, written $P \oplus Q$, by the following table.

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Lesson 1.4: Conditionals, Tautologies, and Contradictions

- Let P and Q be statements. The **conditional** "if P, then Q," written $P \Rightarrow Q$, has truth value according to the truth table below:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- For the above statement, P is the **hypothesis**, and Q is the **conclusion**.
- For the conditional $P \Rightarrow Q$, the statement $Q \Rightarrow P$ is its **converse**, and the statement $\neg Q \Rightarrow \neg P$ is its **contrapositive**.
- **Theorem:** Let P and Q be any two statements. Then $P \Rightarrow Q \Longleftrightarrow \neg Q \Rightarrow \neg P$.
- Let P and Q be statements "P if and only if Q" is the **biconditional** of P with Q, written $P \Leftrightarrow Q$. The truth value of the bicondtional is given by the following truth table:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

- **Tautology:** An expression is whose truth value is **T** for all combinations of truth values
- **Contradiction:** An expression is whose truth value is **F** for all combinations of truth values

Lesson 1.5: Methods of Proof

- **Direct Proof:** explaining the reasoning behind your idea in words
- **Proof by the contrapositive:** proving $\neg Q \Rightarrow \neg P$ (contrapositive)
- **Proof by contradiction:** Considering the opposite of what you want to prove and proving that the opposite creates a contradiction.

Lesson 1.6: Quantifiers

- **Definition:** Let $P(x)$ is a propositional function with universal set X. The sentence

$$\text{For all } x \in X,\ P(x)$$

is a **universally quantified statement** whose truth value is **T** if the truth set of $P(x)$ is the universal set X and F otherwise. We write:

$$(\forall x \in X)P(x) \qquad \text{where } \forall \text{ is the } \textbf{universal quantifier}.$$

- **Definition:** Let $P(x)$ be a propositional function with universal set X. The sentence

$$\text{There exists } x \in X \text{ such that } P(x)$$

is an **existentially quantified statement** whose truth value **F** if the truth set of $P(x)$ has no elements and **T** otherwise. We write

$$(\exists x \in X)P(x) \qquad \text{where } \exists \text{ is the } \textbf{existential quantifier}.$$

- **Definition:** Let $P(x)$ be a propositional function with universal set X. The sentence

$$\text{There exists a unique } x \in X \text{ such that } P(x)$$

is an **uniquely existentially quantified statement** whose truth value is **T** if the truth set of $P(x)$ has exactly one element and **F** otherwise. We Write

$$(\exists x! \in X)P(x) \qquad \text{where } \exists! \text{ is the } \textbf{unique existential quantifier}.$$

Theorem: Let $P(x)$ be a propositional function with universal set X. Then the following hold:

1. $\neg[(\exists x)P(x)] \iff (\forall x)[\neg P(x)]$
2. $\neg[(\forall x)P(x)] \iff (\exists x)[\neg P(x)]$
3. **Definition:** Let X be the universal set for $P(x)$. An element x_0 is a **counterexample** to the statement $(\forall x)P(x)$ provided that $P(x_0)$ is false.

Chapter 2: Numbers

Lesson 2.1: Basic Ideas of Sets

- **Set:** collection of objects
- **Elements:** the objects in a set
- **Set-builder notation:** the way mathematicians formulate sets

  Colon (:) - stands for such that

Lesson 2.2: Sets of Numbers

- **Natural Numbers:** $\mathbb{N} = 1, 2, 3, 4, 5, 6, 7, 8, ...$
- **Positive Even Numbers:** $\mathbb{E} = 2, 4, 6, 8, 10, ...$
- **Rational Numbers:** $\mathbb{Q} = a/b : a, b \in Z \wedge b \neq 0$
- **Real Numbers:** All points on a number line.
- **Complex Numbers:** $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, where $i^2 = -1$

Lesson 2.3: Some Properties of $\mathbb{N}$ and $\mathbb{Z}$

- **Even:** Let $n \in Z$ Then n is even whenever there exists some $k \in \mathbb{Z}$ such that $n = 2k$
- **Odd:** Let $n \in Z$ Then n is odd whenever there exists some $k \in \mathbb{Z}$ such that $n = 2k + 1$
- **Definition:** Let $a, b \in \mathbb{Z}$ with $a \neq 0$. Then a **divides** b, written $a \,|\, b$, when there exists an integer k such that $b = ak$. Equivalently, we may say that b is **divisible** by a, or that b is a **multiple** of a, or that a is a **divisor** of b. If $a \,|\, b$ and $1 < a < |b|$, then a is a **proper divisor** of b.

Lesson 2.4: Prime Numbers

- **Definition:** The number $p \in \mathbb{N}$ is **prime** if p has no proper divisor.
- An integer greater than 1 that is not prime is **composite** . A **prime factorization** of any integer n is a representation of n as a product $n = (\pm)p_1p_2...p_k$ whose factors are prime numbers.
- **The Fundamental Theorem of Arithmetic:** Every integer greater than 1 has a prime factorization that is unique up to the order in which the factors occur.
- **Theorem:** There are infinitely prime numbers.

Lesson 2.5: gcd's and lcm's

- **Definition:** Let $a, b \in \mathbb{Z}$. Then $c \in \mathbb{N}$ is a **common divisor** of a and b whenever $c \,|\, a$ and $c \,|\, b$.
- **Definition:** Let $a, b \in \mathbb{Z}$ with a and b not both 0. Let $D(a, b)$ be the set of common divisors of a and b; that is,

$$D(a, b) = \{c \in \mathbb{N} : c \,|\, a \wedge c \,|\, b\}$$

- The **greatest common divisor of a and b**, denoted $gcd(a,b)$, is the largest element $D(a,b)$. We denote this element by $gcd(a,b)$. Thus,

$$(\forall c \in D(a,b))[c \leq gcd(a,b)]$$

- When $gcd(a,b) = 1$, we say that a and b are **relatively prime**.
- **Definition:** Let a, b be nonzero integers. Let $M(a,b)$ be the set of common multiples of a and b; that is

$$M(a,b) = \{m \in \mathbb{N} : a \mid m \wedge b \mid m\}$$

- The **least common multiple of a and b**, denoted $lcm(a,b)$ is the smallest element of $M(a,b)$. Thus

$$(\forall m \in M(a,b))[lcm(a,b) \leq m]$$

- **Proposition:** For all $a, b \in \mathbb{N}$

$$ab = lcm(a,b) \cdot gcd(a,b)$$

Lesson 2.6: Euclid's Algorithm

- **Lemma**: Let $a, b, x \in \mathbb{Z}$ with a and b not both 0. Then

$$gcd(a,b) = gcd(a, b + ax)$$

- **Theorem (Euclid's Algorithm):** Let $a, b \in \mathbb{N}$. By applying the Division Algorithm repeatedly, let

$$a = bq_1 + r_1 \qquad \text{with} \quad 0 < r_1 < b;$$

$$b = r_1 q_2 + r_2 \qquad \text{with} \quad 0 < r_2 < r_1;$$

$$r_1 = r_2 q_3 + r_3 \qquad \text{with} \quad 0 < r_3 < r_2;$$

$$\ldots$$

$$r_{j-2} = r_{j-1} q_j + r_j \qquad \text{with} \quad 0 < r_j < r_{j-1};$$

$$r_{j-1} = r_j q_{j+1}.$$

Lesson 2.7: Rational Numbers and Algebraic Numbers

- A **rational number** q is written **in lowest terms** when $q = \frac{a}{b}$ and a, b are integer such that $gcd(|a|, |b|) = 1$
- We defined the set $\mathbb{I}$ of **irrational numbers** by $\mathbb{I} = \{x \in \mathbb{R} : x \ni \mathbb{Q}\}$

Chapter 3: Sets

Lesson 3.1: Subsets

- **Definition:** Let A and B be sets. Then A is a **subset** of B, written $A \subseteq B$, when the statement $(\forall x)[x \in A \Rightarrow x \in B]$ is true.
- For $B \supseteq A$, we say that B is a superset of A.
- **Definition:** When sets are given in context of a subject, they have an assumed **universal set**.
- **Proposition:** Let A, B, and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- **Definition:** A set with no elements is an **empty set**.
- Let A and B be sets. Then A **equals** B, written $A = B$, when both $A \subseteq B$ and $B \subseteq A$. Thus the symbols A and B denote the same set.
- **Definition:** A set A is a **proper subset** of a set B, written $A \subset B$, when A is a subset of B but $A \neq B$.
- A statement of the form $(\forall x \in \emptyset)P(x)$ is a **vacuous statement**.
- **Definition:** Let A be a set. The set whose elements are all of the subsets of A is the **power set** of A, denoted $\mathcal{P}(A)$, and defined by :

$$\mathcal{P}(A) = \{S : S \subseteq A\}$$

Lesson 3.2: Operations with Sets

- **Definition:** Let A and B be sets.
  - The **intersection** of A and B, written $A \cap B$, is the set
    $A \cap B = \{x : x \in A \wedge x \in B\}$
  - The **union** of A and B, written $A \cup B$, is the set
    $A \cup B = \{x : x \in A \vee x \in B\}$
- **Proposition:** Let A,B,C be sets. Then all of the following hold:
  1. $A \cap A = A$ and $A \cup A = A$
  2. $\emptyset \cap A = \emptyset$ and $\emptyset \cup A = A$
  3. $(A \cap B) \subseteq A$
  4. $A \subseteq (A \cup B)$
  5. $A \cap (B \cap C) = (A \cap B) \cap C$
  6. $A \cup (B \cup C) = (A \cup B) \cup C$
  7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Lesson 3.3: The Complement of a Set

- **Definition:** Let A and B be sets. The **complement** of B relative to A, written $A \backslash B$, is the set

$$A \backslash B = \{x : x \in A \wedge x \notin B\}$$

- **Definition:** Let U be a universal set, then $U' = \emptyset$ and $\emptyset = U$. In the universe $\mathbb{N}$, the set of all odd natural numbers is

$$A' = U \backslash A = \{x \in U : x \notin A\}$$

- **Proposition:** Let A and B be subsets of a universal set U. Then
  1. $A \backslash B = A \cap B'$
  2. $(B')' = B$
  3. $(A \cup B)' = A' \cap B'$
  4. $(A \cap B)' = A' \cup B'$
  5. $A \subseteq B$ if and only if $B' \subseteq A$
  6. $A \cup A' = U$
  7. $A \cap A' = \emptyset$
  8. $A \cap B = \emptyset$ if and only if $A \subseteq B'$
  9. $A \subseteq B$ if and only if $A \backslash B = \emptyset$

- Parts 3 and 4 are called **De Morgan's Laws**

Lesson 3.4: The Cartesian Product

- **Definition:** Let A and B be sets. The **Cartesian product of A by B**, written $A \times B$, is the set

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

- **Proposition:** Let A,B,C and D be sets. Then
  1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
  2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
  3. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
  4. $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$

Chapter 4: Induction

Lesson 4.1: An Inductive Example

- **Definition:** Define a set of lines in the plane to be in **general position** when

  1. No two of the lines are parallel, and
  2. No three lines meet a common point

  In these terms, your quest is now to find the number of regions created by 100 lines in the plane in general position.

Lesson 4.2: The Principle of Mathematical Induction

- **Theorem: (The Principle of Mathematical Induction):** Let $n_0 \in \mathbb{Z}$. For each integer $n \geq n_0$, let $\mathbf{P}(n)$ be a statement about n. Suppose that the following two statements are true:

  1. $\mathbf{P}(n_0)$
  2. $(\forall n \geq n_0)[\mathbf{P}(n) \Rightarrow \mathbf{P}(n+1)]$

  Then, for all integer $n \geq n_0$, the statement $\mathbf{P}(n)$ is true.

- **The Well ordering Principle:** Let $n_0 \in \mathbb{Z}$. Every nonempty subset of the set $\{n \in \mathbb{Z} : n \geq n_0\}$

- **Proposition:** For all $n \in \mathbb{N}, \displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

- **Induction Hypthesis:** When you assume $\mathbf{P}(n)$

Lesson 4.3: The Principle of Strong Induction

- **Theorem (The Principle of Strong Induction):** Let $n_0 \in \mathbb{Z}$. For each integer $n \geq n_0$, let $\mathbf{P}(n)$ be a statement about n. Suppose that the following two statements are true:

  1. $\mathbf{P}(n_0)$
  2. $(\forall n \geq n_0)[(\wedge_{k=n_0}^{n} \mathbf{P}(k)) \Rightarrow P(n+1)]$
     Then, for all integers $n \geq n_0$, the statement $\mathbf{P}(n)$ is true.

Chapter 5: Functions

Lesson 5.1: Functional Notation

- **Definition:** Let X and Y be sets. A **function** f from X to Y, written $f : X \to Y$, is a rule that pairs an element $x \in X$ with an element $y \in Y$, written $f(x) = y$, such that the following property holds.

$$(\forall x \in X)(\exists! y \in Y)[f(x) = y]$$

  The set X is the **domain** of f and the set Y is the **codomain** of f. If $f(x) = y$, then y is the **image** of x and x is a preimage of y.

- **Definition:** Two functions are **equal** when
  1. They have the same domain and the same codomain, and
  2. They agree at every element of their domain.

- **Definition:** Let $f : X \to Y$. The **range** of f is the set

$$\{y \in Y : (\exists x \in X)[f(x) = y]\}$$

- **Definition:** Let $f : X \to Y$. The **inverse** of f, denoted f^{-1}, is the pairing defined by the rule that, if $f(x) = y$, then $f^{-1}(y) = x$

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Lesson 5.2: Operations with Functions

- **Definition:** Let X and Y be any sets. A function $f : X \to Y$ is a **constant function** when the following property holds.

$$(\exists a \in Y)(\forall x \in Y)[f(x) = a]$$

- The function $(-1)g$ is written as $-g$ and is called the **negative** of g.

- **Definition:** Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. Then f is **increasing on** S if

$$(\forall x_1, x_2 \in S)[x_1 < x_2 \Rightarrow f(x_1) < f(x_2)]$$

  f is **decreasing on** S if

$$(\forall x_1, x_2 \in S)[x_1 < x_2 \Rightarrow f(x_1) > f(x_2)]$$

  f is **nondecreasing on** S if

$$(\forall x_1, x_2 \in S)[x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)]$$

  and f is **nonincreasing on** S if

$$(\forall x_1, x_2 \in S)[x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)]$$

- **Definition:** Let X,Y, and Z be sets. Let functions $f : X \to Y$ and $g : Y \to Z$ be given.
  Then the **composition of g with f**, written $g \circ f$, is defined by

$$(\forall x \in X)[(g \circ f)(x) = g(f(x))]$$

Lesson 5.3: Induced Set Functions

- **Definition:** Let $f : X \to Y$. The **set function induced by f** is the function $\bar{f} : \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by the rule that, for all $A \in \mathcal{P}(X)$,

$$\bar{f}(A) = \{y \in Y : (\exists x \in A)[f(x) = y]\} = \{f(x) : x \in A\}$$

- **Theorem:** Let $f : X \to Y$ and let $A, B \in \mathcal{P}(X)$. Then the following hold.
  - $A \subseteq B \Rightarrow \bar{f}(A) \subseteq \bar{f}(B)$
  - $\bar{f}(A \cap B) \subseteq \bar{f}(A) \cap \bar{f}(B)$
  - $\bar{f}(A \cup B) = \bar{f}(A) \cup \bar{f}(B)$
- **Proposition:** Let A,B $\subseteq X$ and $f : X \to Y$. Then

$$\bar{f}(A) \backslash \bar{f}(B) \subseteq \bar{f}(A \backslash B)$$

- **Definition:** Let $f : X \to Y$. For each set $B \in \mathcal{P}(Y)$, define the function $f^{\bar{-}1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ by

$$f^{\bar{-}1}(B) = \{x \in X : f(x) \in B\}$$

Lesson 5.4: Surjections, Injections, and Bijections

- **Definition:** A function $f : X \to Y$ with the property

$$(\forall y \in Y)(\exists x \in X)[f(x) = y]$$

  is a **surjection** of X onto Y
- **Proposition:** The composition of two surjections is a surjection.
- **Definition:** A function $f : X \to Y$ with the property

$$(\forall x_1, x_2 \in X)[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$$

  is an **injection** of X into Y
- **Proposition:** The composition of two injections is an injection.
- **Theorem:** If the function $f : X \to Y$ is an injection, then so is its induced set function $\bar{f} : \mathcal{P}(X) \to \mathcal{P}(Y)$
- **Definition:** A function that is both an injection and surjection is a bijection.
- **Corollary:** The composition of two bijections is a bijection.

Bibliography

Book used: Passage to Abstract Mathematics
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