Notes from MAT397 - Calculus III

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Chapter 12: Vectors and The Geometry of Space

Lesson 12.1: Three Dimensional Coordinate Systems

- **Coordinate Planes**

  XY-Plane: Z is always 0 and x,y can be any real numbers

  XZ-Plane: Y is always 0 and x,z can be any real numbers

  YZ-Plane: X is always 0 and y,z can be any real numbers

- These three coordinate planes divide space into eight parts, call **octants**. The first octant is where x,y,z are all positive.

- A point P in $\mathbb{R}^3$ with coordinates (x,y,z) is in the **three-dimensional rectangular coordinate system.**

- **Distance Formula in Three Dimensions**: The distance $|P_1P_2|$ between the points P_1 and P_2 is

  $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- **Equation of a Sphere:** An equation of a sphere with center C(h,h,l) and radius r is

  $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

  In particular, if the center is the origin O, then the equation for the sphere is $x^2 + y^2 + z^2 = r^2$

Lesson 12.2: Vectors

- **Definition of Vector Addition:** If **u** and **v** are vector positioned so the initial point of **v** is at the terminal pointer of **u** , then the **sum v +u** is the vector from the initial point of **u** to the terminal point of **v** .

- **Definition of Scalar Multiplication:** If c is a scalar and **v** is a vector, then the **scalar multiple** c**v** is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and opposite to **v** if $c < 0$. if $c = 0$ or $v = 0$, then c**v** =0.

- Vectors are **parallel** if they are scalar multiples of one another.

- If a vector has the same magnitude but opposite direction, we call it the **negative** of **v**

- **Vector subtraction**: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

- **Components of a Vector:** $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ where a_1, a_2, and a_3 are the **components**

- **Position Vector:** Vector from the origin to a point.

  Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

  $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

- **The Magnitude of a vector:** The square root of the sum of all the components squared.

  $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$ or $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

- **Vector Operations:** If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

  $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$

  $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

  $c\mathbf{a} = \langle ca_1, ca_2 \rangle$

- **Properties of Vectors:** If **a**, **b**, **c** are vectors in V_n, and c and d are scalars, then

    1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
    2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
    3. $\mathbf{a} + 0 = \mathbf{a}$
    4. $\mathbf{a} + -(\mathbf{a}) = 0$
    5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
    6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
    7. $(cd)\mathbf{a} = c(d\mathbf{a})$
    8. $1\mathbf{a} = \mathbf{a}$

- **Standard Basis Vectors:**
  $\hat{i} = \langle 1, 0, 0\rangle$
  $\hat{j} = \langle 0, 1, 0\rangle$
  $\hat{k} = \langle 0, 0, 1\rangle$

- **Unit Vectors:**

  Vector divided by it's magnitude. $\dfrac{\mathbf{a}}{|\mathbf{a}|}$

Lesson 12.3: The Dot Product

- **Definition:** If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **dot product** is t he number $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$
- **Properties of Dot Products:** **a** , **b** , and **c** are vectors in V_3 and c is a scalar, then

    1. $a \cdot a = |a|^2$
    2. $a \cdot b = b \cdot a$
    3. $a \cdot (b + c) = a \cdot b + a \cdot c$
    4. $(ca) \cdot b = c(a \cdot b) = a \cdot (cb)$
    5. $0 \cdot a = 0$

- **Theorem:** If θ is the angle between the vectors **a** and **b** , then
  $a \cdot b = |a||b|cos\theta$
- **Corollary** If θ is the angle between the nonzero vectors **a** and **b**, then
  $cos\,\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|a|\,|b|}$
- The vectors **a** and **b** are orthogonal if and only if $a \cdot b = 0$
- **Scalar projection** of **b** onto **a**:
  $comp_a\mathbf{b} = \frac{a\cdot b}{|a|}$
- **Vector projection** of **b** onto **a**:
  $proj_a\mathbf{b} = (\frac{a\cdot b}{|a|})\frac{a}{|a|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

Lesson 12.4: The Cross Product

- **Definition:** If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **cross product** of **a** and **b** is the vector

  $$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

- **Theorem:** The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.
- **Theorem:** If θ is the angle between **a** and **b** , then

  $$|\mathbf{a} \times \mathbf{b}| = |a||b|sin\,\theta$$

- **Corollary:** Two nonzero vectors **a** and **b** are parallel if and only if

  $$\mathbf{a} \times \mathbf{b} = 0$$

- The magnitude of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.
- **Theorem:** If **a** , **b** , and **c** are vectors and c is a scalar, then

  1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
  2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
  3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
  4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
  5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
  6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

- The **volume of the parallelpiped** determined by the vectors **a** ,**b** , and **c** is the magnitude of their scalar triple product:

  $$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Lesson 12.5: Equations of Lines and Planes

- **Vector Equation** of a Line

  $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

- **Parametric Equations** of a Line

  $$x = x_0 + at$$
  $$y = y_0 + bt$$
  $$z = z_0 + ct$$

- **Symmetric Equations** of a Line

  $$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

- The line segment from r_0 to r_1 is given by the vector equation $\mathbf{r}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1} \qquad 0 \leq t \leq 1$

- **Vector Equation of the plane**:

  $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$

  $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$

- **Scalar Equation of the Plane** through $P_0(x_0, y_0, z_0$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$:

  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

  At $P_0(0, 0, 0)$, can be rewritten as $ax + by + cz + d = 0$ where $d = -(ax_0 + by_0 + cz_0)$.

- Two planes are **parallel** if their normal vectors are parallel.

- **Distance between two planes:**

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

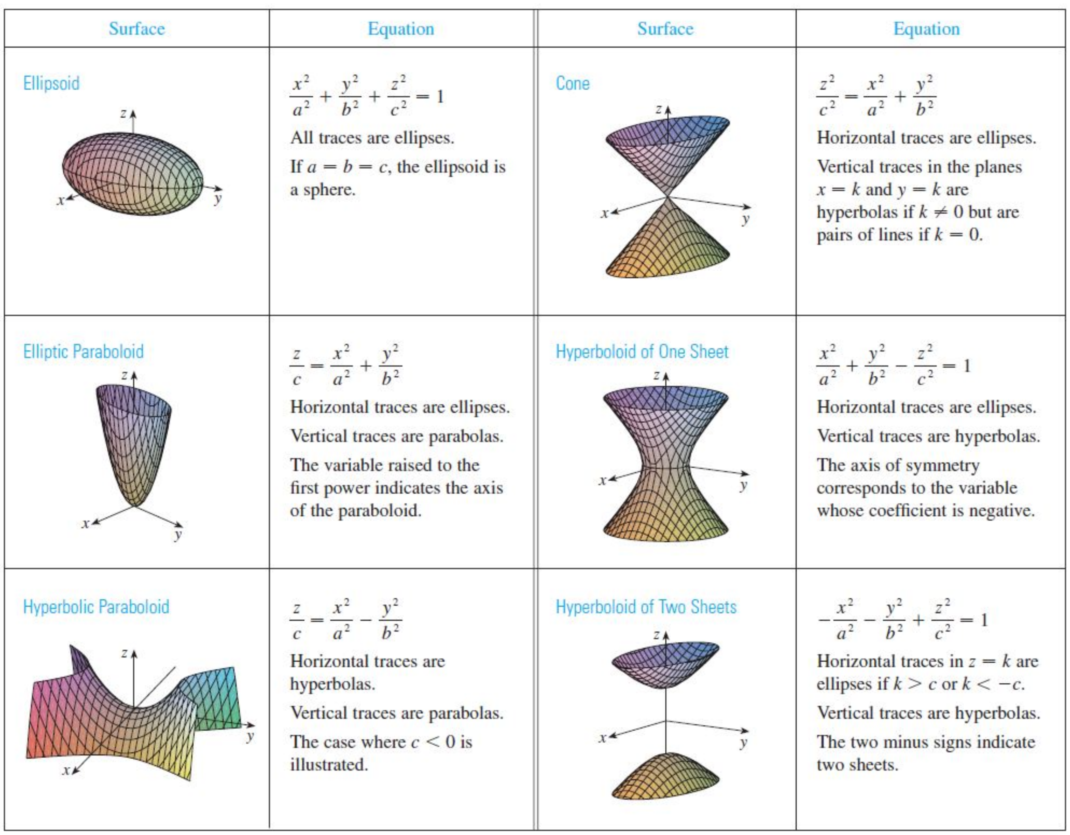
Lesson 12.6: Cylinders and Quadric Surfaces

- **Quadric Surface General Equation**

  $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

- **Cylinder:** A surface that consist of all lines that are parallel to a given line and pass through a given plane curve.

- **Parabolic Cylinder:** Surface made up infinitely many shifted copies of the same parabola.



Surface	Equation	Surface	Equation
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Figure 12.1: Image from Calculus Early Transcendentals 6th Edition

Chapter 13: Vector Functions

Lesson 13.1: Vector Functions and Space Curves

- **Vector Function:** A function whose domain is a set of real numbers and whose range is a set of vectors.

  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

- The **limit** of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions as follows.

  If $\mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$

  provided the limits of the component functions exist.

- A vector function $\mathbf{r}$ is **continuous** at a if

  $$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$$

- **Parametric equations of a curve C** with a **parameter** t:

  $$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Lesson 13.2: Derivatives and Integrals of Vector Functions

- **Derivative of a vector function r**:

  $$\frac{d\mathbf{r}}{dt} = \mathbf{r}\,'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

- **Tangent Vector:** $\mathbf{r}'(t)$

- **Unit Tangent Vector:**

  $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

- **Theorem:** If $\mathbf{r}(t) = \langle f(t), g(t), g(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ where f, g, and h are differentiable functions, then

  $$\mathbf{r}\,'(t) = \langle f'(t), g'(t), g'(t) \rangle = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$$

- **Theorem:** Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

  1. **Sums**: $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}\,'(t) + \mathbf{v}\,'(t)$
  2. **Scalars**: $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}\,'(t)$
  3. **Product Rule** (Multiplicative): $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}\,'(t)$
  4. **Product Rule** (Dot Product): $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}\,'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}\,'(t)$
  5. **Product Rule** (Cross Product): $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}\,'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}\,'(t)$
  6. **Chain Rule**: $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}\,'(f(t))$

- **Definite Integral of a vector function:**

  $$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{i} + \left(\int_a^b g(t)\,dt\right)\hat{j} + \left(\int_a^b h(t)\,dt\right)\hat{k}$$

Lesson 13.3: Arc Length and Curvature

- **Arc Length of a function function:**

  $L = \int\limits_a^b |\mathbf{r}'(t)|\, dt$

- A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if **r**' is continuous and $\mathbf{r}'(t) \neq 0$

- **Definition:** The **curvature** of a curve is

  $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$

- **Theorem:** The curvature of the curve give by the vector function **r** is

  $\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

- **Unit Normal Vector:**

  $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

- **Binormal Vector:**

  $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

Lesson 13.4: Motion in Space: Velocity and Acceleration

- **Velocity Vector** $\mathbf{v}$(t) at time t

  $\mathbf{v}(t) = \lim\limits_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$

- The speed of a particle at a time is the magnitude of the velocity $|\mathbf{v}(t)|$

- **Acceleration** of a particle: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

- **Parametric Equations of Trajectory**:

$$x = (v_0 \, cos \, \alpha) \, t \qquad\qquad y = (v_0 \, sin \, \alpha) \, t - \frac{1}{2} g t^2$$

- Acceleration of a particle(2): $\mathbf{a} = v' \, \mathbf{T} + \kappa v^2 \mathbf{N}$

Chapter 14: Partial Derivatives

Lesson 14.1: Functions of Several Variables

- **Function f of two variables**: $f(x, y)$
- **Level Curves** of a function f are the curves with equation $f(x, y) = k$ where k is a constant
- **Function f of three variables**: $f(x, y, z)$

Lesson 14.2: Limits and Continuity

- **Definition:** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of f(x,y)** as **x,y approaches (a,b)** is L and we write:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

  if for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that
  $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$

- If the limit as $f(x, y)$ approaches a point P is different with different paths, then the limit does not exist at P.
- **Definition:** A function f of two variables is called **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

  We say f is **continuous on D** if f is continuous at every point (a, b) in D

Lesson 14.3: Partial Derivatives

- **Partial Derivative** of f with respect to x at (a, b):

$$f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)$$

- If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h,y)-(fx,y)}{h}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x,y+h)-(fx,y)}{h}$$

- **Notations for Partial Derivatives:** If $z = f(x, y)$ we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

- **Rule for Finding Partial Derivative of** $z = f(x, y)$
    1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x
    2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y
- **Second Partial Derivatives** of f:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

- **Clairaut's Theorem:** Suppose f is defined on a disk D that contains the point (a, b). If the function f_{xy} and f_{yx} are both continuous on D. Then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Lesson 14.4: Tangent Planes and Linear Approximations

- Suppose f has continuous partial derivatives. An equation of the **tangent plane** to the surface is
  $$z = z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
- **Linearization** of f at (a, b):
  $$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
- If $z = f(x, y)$ then f is **differentiable** at (a, b) if Δz can be expressed in the form

  $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where ϵ_1 and $\epsilon_2 \to 0 as (\Delta x, \Delta y) \to (0, 0)$.
- **Theorem:** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).
- The **differential** of $y = f(x)$ is defined as $dy = f'(x)\,dx$
- For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy. The **total differential** dz can be defined by:

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Lesson 14.5: The Chain Rule

- **The Chain Rule (Case 1)**: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable function of t. Then z is a differentiable function of t Then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

- **The Chain Rule (Case 2)**: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable function of t. Then z is a differentiable function of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

- **The Chain Rule (General Version)**: Suppose that u is a differentiable function of n variables $x_1, x_2, ..., x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ ... \ + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

- **Implicit Function Theorem:**

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Lesson 14.6: Directional Derivatives and the Gradient Vector

- The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h} \qquad \text{if this limit exists}$$

- **Theorem:** If f is differentiable function of x and y, then f has a directional derivative in the direction of any vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

- **Definition:** If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

- **Directional derivative** of a differentiable function:

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

- **Definition:** The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0, y_0, z_0)}{h} \qquad \text{if this limit exists}$$

- **Gradient Vector** of a function $f(x, y, z)$ can be defined

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- **Directional derivative** of a differentiable function of three variables:

  $D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$

- **Theorem:** Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivatives $D_U f(x)$ is $|\nabla f(x)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x)$.

- **Tangent Plane to the level surface:**

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

Lesson 14.7: Maximum and Minimum Values

- **Definition:** A function of two variables has a **local maximum** at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b). The number $f(a,b)$ is called a **local maximum value**. If $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b), then f has a **local maximum value** at (a,b) and $f(a,b)$ is a **local minimum value**.

- If the inequalities above hold for *all* points (x,y) in the domain of f, then f has an **absolute maximum or minimum** at (a,b). **Theorem:** If f has a local maximum or minimum at (a,b) and the first-order partial derivative of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

- **Critical Point:** Point of f where $f_x = 0$, $f_y = 0$, or one of the partial derivatives does not exist.

- **Second Derivative Test:** Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

  - If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
  - If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
  - If $D < 0$, then $f(a,b)$ is not a local maximum or minimum. (**Saddle Point**)

- **Extreme Value Theorem for Functions of Two Variables:** If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points (x_1,y_1) and (x_2,y_2) in D.

- **To find the absolute max and min values** of a continuous function f on a closed, bounded set D:

  1. Find the values of f at the critical points of f in D.
  2. Find the extreme values of f on the boundary of D.
  3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Lesson 14.8: Lagrange Multipliers

- **Definition:** $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$ where λ is called a **Lagrange multiplier**.

- **Method of Lagrange Multipliers:** To find the max and min values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ [assuming that these extreme values exist and $\nabla g \ne 0$ on the surface $g(x,y,z) = k$]:

  1. Find all values of x, y, z and λ such that

     $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \qquad g(x,y,z) = k$
  2. Evaluate f at all the points (x,y,z) that result from step (1). The largest of these values is the **maximum** value of f; the smallest is the **minimum** value of f

Chapter 15: Multiple Integrals

Lesson 15.1: Double integrals over Rectangles

- **Definition:** The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA$$

- The **volume** V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

Lesson 15.2: Iterated Integrals

- **Fubini's Theorem:** If f is continuous on the rectangle $R = (x,y)\,|\,a \leq x \leq b,\ c \leq y \leq d$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

  More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exit.

Lesson 15.3: Double Integrals over General Regions

- If F is integrable over R, then we define the **double integral of f over D** by

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

  where $F(x,y) = f(x,y)$ if (x,y) is in D and $F(x,y) = 00$ if (x,y) is in R but not in D

- If f is continuous on a type I region D such that $D = (x,y)\,|\,a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$

  then $\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

Lesson 15.4: Double Integrals in Polar Coordinates

- **Polar Coordinates:**
  - Functions of the form $f(r,\theta)$
  - $r^2 = x^2 + y^2$
  - $x = r\,cos\,\theta$
  - $y = r\,sin\,\theta$

- **Change to Polar Coordinates in a Double Integrals**: If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\,cos\,\theta, r\,sin\,\theta)\,r\,dr\,d\theta$$

- If f is continuous on a polar region of the form $D = (r,\theta) \,|\, \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)$

$$\iint\limits_D dA = \int\limits_{\alpha}^{\beta} \int\limits_{h_1(\theta)}^{h_2(\theta)} f(r\,cos\,\theta, r\,sin\,\theta)\, r\, dr\, d\theta$$

Lesson 15.5: Applications of Double Integrals

- **Density** at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. This means that $\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$
- **Mass** with density ρ: $m = \iint\limits_D \rho(x, y)\, dA$
- **Moment** about the $x - axis$: $M_x = \iint\limits_D y\, \rho(x, y)\, dA$
- **Moment** about the $y - axis$: $M_y = \iint\limits_D x\, \rho(x, y)\, dA$
- The coordinates $(\bar{x}, \bar{y})$ of the **center of mass** of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint\limits_D x\rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint\limits_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint\limits_D \rho(x, y)\, dA$$

- **Moment of Inertia** of the lamina **about the x-axis**: $I_x = \iint\limits_D y^2 \rho(x, y)\, dA$
- **Moment of Inertia** of the lamina **about the y-axis**: $I_y = \iint\limits_D x^2 \rho(x, y)\, dA$
- **Moment of inertia** of the lamina **about the origin**: $I_0 = \iint\limits_D (x^2 + y^2)\rho(x, y)\, dA$

Lesson 15.6: Surface Area

- The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous is,

$$A(S) = \iint\limits_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

or using partial derivative notation...

$$A(s) = \iint\limits_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial x})^2}\; dA$$

Lesson 15.7: Triple Integrals

- **Definition**: The **triple integral** of f over the box B is

$$\iiint_B f(x,y,z)\,dV$$

- **Fubini's Theorem for Triple Integrals:** If f is continuous on the rectangular box $B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

- **Triple Integral over a General Region E**.
If $E=(x,y,z)\,|\,a\le x\le b, g_1(x)\le y\le g_2(x), u_1(x,y)\le z\le u_2(x,y)$, then

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Lesson 15.8: Triple Integrals in Cylindrical Coordinates

- **Function Format:** $f(r,\theta,z)$.
- **Cylindrical to Rectangular Coordinates:**

$$x = r\,cos\,\theta \qquad y = r\,sin\,\theta \qquad z = z$$

- **Rectangular to Cylindrical Coordinates:**

$$r^2 = x^2+y^2 \qquad tan\,\theta = \tfrac{y}{x} \qquad z = z$$

- **Triple Integral in Cylindrical Coordinates:**
If $E=(x,y,z)\,|\,(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)$, then

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1}^{u_2} f(r\,cos\,\theta, r\,sin\,\theta, z)\,r\,dz\,dr\,d\theta$$

Notice the single r inserted near the end of the equation.
This is the *jacobian* of our transformation into polar coordinates. (covered in 15.10)

Lesson 15.9: Triple Integrals in Spherical Coordinates

- **Function Format:** $f(\rho, \theta, \phi)$
- **Spherical to Rectangular Coordinates:**

$$x = \rho\, sin\,\phi\, cos\,\theta \qquad y = \rho\, sin\,\phi\, sin\,\theta \qquad z = \rho\, cos\,\theta$$

- **Rectangular to Spherical:**

$$\rho = x^2 + y^2 + z^2$$

- **Triple Integration in Spherical Coordinates:**
  If E is the spherical wedge given by $E = (\rho, \theta, \phi)\,|\,a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d$, then

$$\iiint\limits_E f(x,y,z)\,dV = \int\limits_c^d \int\limits_\alpha^\beta \int\limits_a^b f(\rho\, sin\,\phi\, cos\,\theta, \rho\, sin\,\phi\, sin\,\theta, \rho\, cos\,\phi)\, \rho^2 sin\,\phi\, d\rho\, d\theta\, d\phi$$

Lesson 15.10: Change of Variables in Multiple Integrals

- **Definition:** The **Jacobian** of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- **Charge of Variables in a Double Integral:** Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f(x(u,v), y(u,v))\left|\tfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

- **Jacobian** for a transformation in space, where $x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$

is the following 3×3 determinant: $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- Similar to how we do double integrals with their Jacobians, we can use the above Jacobian when finding a triple integral after a transformation T.

Chapter 16: Vector Calculus

Lesson 16.1: Vector Fields

- **Definition:** Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field** on $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in D a two-dimensional vector **F**(x,y)
- **Definition:** Let E be a subset $\mathbb{R}^3$. A **vector field** on $\mathbb{R}^3$ is a function **F** that assigns to each point (x, y, z) in E three-dimensional vector $\mathbf{F}(x, y, z)$

Lesson 16.2: Line Integrals

- **Definition:** If f is defined on a smooth curve C given by parametric equations, then the **line integral of f along C** is

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\,dt$$

- **Line Integral with respect to Arc Length:**

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

- **Definition:** Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

Lesson 16.3: The Fundamental Theorem for line Integrals

- **Theorem:** Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- **Theorem:** $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.
- **Theorem:** Suppose **F** is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then **F** is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$

Lesson 16.4: Green's Theorem

- **Green's Theorem:** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be a region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Bibliography

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