Some notes on plasma lensing: Construction of the field

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1 Derivation of the lens equation

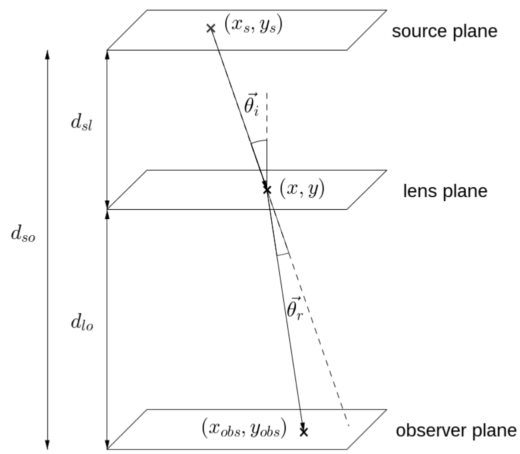

Figure 1: Lensing geometry.

The geometrical optics approximation treats the radio pulses emitted from a pulsar as rays, and in the case of lensing we are concerned with mapping the outgoing rays from the source and lens planes to the observer plane. From the geometry in Figure 1, we see that the 2D angle of incidence $\vec{\theta}_i$ and the deviation angle $\vec{\theta}_r$ are given (in the paraxial approximation) by

$$\vec{\theta}_i = \frac{\vec{x}_s - \vec{x}}{d_{sl}} \tag{1}$$

$$\vec{\theta}_r = \frac{\vec{x}_{obs} - \vec{x}}{d_{lo}} - \vec{\theta}_i \tag{2}$$

Eliminating $\vec{\theta}_i$ and using $d_{so} = d_{lo} + d_{sl}$ gives the lens equation in terms of $\vec{\theta}_r$

$$\vec{x}_s \left(\frac{d_{lo}}{d_{so}}\right) + \vec{x}_{obs} \left(\frac{d_{sl}}{d_{so}}\right) = \vec{x} + \vec{\theta}_r \left(\frac{d_{sl} d_{lo}}{d_{so}}\right) \tag{3}$$

We now define a new set of coordinates $\vec{x}'$ as a combination of the source and observer coordinates scaled by the distances, namely

$$\vec{x}' \equiv \vec{x}_s \left(\frac{d_{lo}}{d_{so}}\right) + \vec{x}_{obs} \left(\frac{d_{sl}}{d_{so}}\right) \tag{4}$$

and write the lens equation in the simpler form

$$\vec{x}' = \vec{x} + \vec{\theta}_r \left(\frac{d_{sl} d_{lo}}{d_{so}} \right) \tag{5}$$

Note that this is a general expression that not only applies to plasma lensing, but to other types of lensing as well, like gravitational lensing. The expression for $\vec{\theta}_r$ in the case of plasma lensing is

$$\vec{\theta}_r = -\frac{c^2 r_e DM_l}{2\pi\nu^2} \nabla\psi(\vec{x}) \tag{6}$$

where r_e corresponds to the classical radius of the electron, DM_l is the maximum value of the electron density perturbation due to the lens, and can be positive (overdensity, diverging lens) or negative (underdensity, converging lens), and is given in units of pc cm^{-3}. $\psi(\vec{x})$ is an arbitrary 2D function with unit maximum that describes the shape of the electron density variation in the lens plane. We define the Fresnel scale as $r_F = \left(\frac{c d_{sl} d_{lo}}{2\pi\nu d_{so}} \right)^{1/2}$, the strength of the scattering screen as $\phi_0 = -\frac{c r_e DM_l}{\nu}$, and a new parameter $\alpha = r_F^2 \phi_0$, which enables us to write (5) as

$$\vec{x}' = \vec{x} + \alpha\nabla\psi(\vec{x}) \tag{7}$$

Finally, we adimensionalize in terms of the characteristic lens scales a_x and a_y, defining $u'_x = \frac{x}{a_x}$ and $u'_y = \frac{y}{a_y}$, and explicitly write (7) in its adimensionalized component form. Using the notation $\psi_{ij} = \frac{\partial^{i+j}\psi}{\partial u_x^i \partial u_y^j}$, we get

$$\begin{aligned} \begin{bmatrix} u'_x \\ u'_y \end{bmatrix} &= \begin{bmatrix} u_x + \frac{\alpha}{a_x^2}\psi_{10}(u_x, u_y) \\ u_y + \frac{\alpha}{a_y^2}\psi_{01}(u_x, u_y) \end{bmatrix} \\ &= \begin{bmatrix} u_x + \alpha_x \psi_{10} \\ u_y + \alpha_y \psi_{01} \end{bmatrix} \end{aligned} \tag{8}$$

Depending on the nature of ψ, the mapping between the u' plane and the u plane can be solved for explicitly; in most cases, however, this is not possible, and (8) must be solved numerically using a root finding algorithm. In this context, solving the lens equation implies finding the u_x and u_y that satisfy (8) given a set of coordinates in the u' plane.

2 Amplitude and phase

The solutions to the lens equation $\vec{u} = \vec{u}_j^0$ can be interpreted as the number of rays that reach the observer at a certain position, or equivalently as the number of images of the source that the observer can perceive. Each of these rays is said to have an amplitude A_j and a phase Φ_j. The normalized (ie. it is equal to unity in the absence of a lens) amplitude is a function of the determinant of the Jacobian matrix $\mathcal{J}$ of the mapping between the planes evaluated at $\vec{u}_j^0$, namely

$$A_j(\vec{u}') = |\mathcal{J}|^{-1/2} = \left[|(1 + \alpha_x\psi_{20})(1 + \alpha_y\psi_{02}) - \alpha_x\alpha_y\psi_{11}|^{-1/2} \right]_{\vec{u}=\vec{u}_j^0} \tag{9}$$

whereas the phase is given by

$$\Phi_j(\vec{u}') = \left\{ \frac{1}{2r_F^2} \left[a_x^2(u_x - u'_x)^2 + a_y^2(u_y - u'_y)^2 \right] + \phi_0\psi(\vec{u}) \right\}_{\vec{u}=\vec{u}_j^0} \tag{10}$$

which can be written as

$$\Phi_j(\vec{u}') = \left[\frac{r_F^2 \phi_0^2}{2} \left(\frac{\psi_{10}^2}{a_x^2} + \frac{\psi_{01}^2}{a_y^2} \right) + \phi_0 \psi(\vec{u}) \right]_{\vec{u}=\vec{u}_j^0} \tag{11}$$

It is easy to see how the expression for the amplitude can be problematic in certain regions where the value of $\mathcal{J}$ approaches zero. Näive geometrical optics fails at these regions as it predicts an infinite amplitude. The curves at which $\mathcal{J} = 0$ are known as caustics, and are the envelopes of families of rays. As one crosses a caustic, the number of solutions to the lens equation changes by at least two, and the observer sees either the merging or the appearance of two or more images. The number of images involved determines the type of so-called catastrophe that occurs at the caustic. The study of catastrophes and their classification is a part of mathematics known as catastrophe theory.

It is important to note that given these definitions, we can write A in terms of the derivatives of the phase with respect to $\vec{u}$,

$$A_j(\vec{u}') = \left[\frac{a_x a_y}{r_F^2} \left| \Phi_{20}\Phi_{02} - \Phi_{11}^2 \right|^{-1/2} \right]_{\vec{u}=\vec{u}_j^0} \tag{12}$$

and thus the geometrical optics approximation is equivalent to solving the 2D Kirchhoff diffraction integral by the method of stationary phase. This integral gives the normalized wave optics field ε as a function of the observer coordinates $\vec{u}'$ by integrating over an angular spectrum of plane waves,

$$\varepsilon(\vec{u}') = \frac{a_x a_y}{2\pi r_F^2} \iint d^2u \exp(\imath\Phi) \tag{13}$$

Solving (13) by the method of stationary phase in two dimensions yields (12) as the absolute value of the contribution of a stationary point to the integral.

3 Field and gain

3.1 Field far from a caustic region

Using the concepts of the previous section, it is straightforward to construct what we call the first order geometrical optics field, which is a good approximation to the wave optics field far from the caustic regions, in cases where all of the images are well separated, ie. where the difference between the phases of each of the images is greater than π rad. Under these circumstances, the field ε can be written as

$$\varepsilon(\vec{u}') = \sum_{j=1}^{n} A_j \exp\left\{ \imath \left[\Phi_j + \frac{\pi}{4}(\delta_j + 1)\sigma_j \right] \right\} \tag{14}$$

where $\delta_j = \mathrm{sgn}(\Phi_{20}\Phi_{02} - \Phi_{11}^2)$, $\sigma_j = \mathrm{sgn}(\Phi_{02})$ evaluated at $\vec{u} = \vec{u}_j^0$, and the summation is over all of the n images. The parameters δ_j and σ_j account for the phase shift that occurs when a ray crosses a caustic. The normalized intensity, or gain, can then be found by taking the square of the absolute value of (14), that is, $G = |\varepsilon|^2$. The presence of the phasor term implies that the gain will oscillate in the presence of more than one image, as these images interfere with each other.

This expression for the geometrical optics field is well known in the context of the geometrical theory of diffraction, and emerges naturally from the stationary phase solution of the Kirchhoff diffraction integral; however, it is not often employed in an astronomical context. Melrose & Watson 2006 use this kind of expression in their study of 1D plasma lenses, but other papers such as Clegg et al. 1998, Cordes et al. 2017, and Er & Rogers 2017 disregard the phasor term, and simply add up the squares of the amplitudes in order to obtain the total gain at a particular observer position. We call this the zeroth order geometrical optics approximation. The relationship between the approaches is simply that in the limit of large ϕ_0 and across certain paths described by the observer in the u' plane, the oscillations resulting from the first order approximation can be very fast, and the zeroth order gain then describes the envelope of these oscillations. For weaker screens, and also for certain kinds of slices across the u' plane, however, the zeroth order approximation can be very misleading.

Both the zeroth and first order approximations fail to describe the field close to caustic regions. It is possible to approximate the gain exactly at the caustic location by employing an expression that involves the third derivative of the phase in 1D (as in Cordes et al. 2017) and an equivalent process in 2D (see Cooke 1982), but this procedure is not very useful in the context of relatively weak screens, when the maximum value of the gain is not located extremely close to the caustic, because the geometrical optics gain begins to diverge some distance away from the singularity, and the value obtained by using the third derivative will then be less than the value of the diverging field around it.

Note that unless we are very far away from a caustic region, (14) fails to accurately describe the field when there is only one image present unless we allow ourselves to look for complex solutions to the lens equation. Mathematically, the points at which two images merge correspond to points at which two real solutions to the lens equation become a complex conjugate pair of solutions. The field of one of these solutions then decreases exponentially, and the other increases exponentially, since the phase is now complex and the formerly purely oscillatory exp function becomes both oscillatory and exponential. It is possible to reproduce the wave optics field that we obtain from a full solution of the Kirchoff diffraction integral if we include the exponentially decreasing field of one of the complex solutions, and we can thus observe oscillations in the first order geometrical optics gain even in the presence of a single real image. This scheme, however, is only useful starting some distance away from the caustic, because close to the caustic the intensity of the field due to the complex solution also diverges.

3.2 Fields close to caustic regions and uniform asymptotics

When two or more images approach each other, the above approximations fail. There are three ways of working around this. The first and most obvious one is to fully solve the Kirchhoff diffraction integral. Unfortunately, this is only possible analytically for a restricted number of lens shapes ψ, and although it is possible to solve the integral numerically using the FFT, this can be done only for relatively small lenses because the quadratic phase factor oscillates extremely rapidly at the edges of larger lenses and thus cannot be appropriately sampled using a grid of reasonable size. Large values of ϕ_0 also lead to sampling issues. Furthermore, solving the integral directly gives us no information on the number of images seen by the observer, and this information can be useful in the radio astronomical context, where we are also interested in the time of arrival and dispersion measure perturbations due to the lens, and in the pulse shape as perceived by the telescope.

A second way of overcoming the field divergences close to the caustics involves the use of transitional approximations, as Melrose & Watson do for the 1D case. Depending on the number of images that merge at a caustic, different transitional forms need to be used. A caustic in which two images merge corresponds to a fold catastrophe, and close to a fold the field follows an Airy function pattern. If three images merge, the catastrophe is known as a cusp, and close to the cusp the field is described by the Pearcey integral. Higher order catastrophes are described by different classes of so-called generalized Airy functions. The problem with using transitional approximations is that one must choose a precise boundary on which to employ the transitional approximation, and another in which to employ the expression given by geometrical optics. A better approach is to use an expression that is valid both close to and far from the caustic, ie. a uniform asymptotic. It is possible to do this on the basis of the geometrical optics amplitudes and phases, although an explicit expression is only possible for the case of the fold caustic: for higher order catastrophes, construction of the asymptotic requires numerically solving sets of nonlinear systems of equations. In the present context, we are overwhelmingly likely to observe fold catastrophes instead of any of the higher order ones, because fold catastrophes show up as curves in the u' plane, whereas cusps show up as points and higher order catastrophes such as umbilics occur only at very specific values of distances and lens parameters.

The uniform asymptotic for the fold caustic is given by two expressions, one for the bright side of the caustic and another one for the dark side. In the bright side of the caustic, where the interacting solutions to the lens equation are real, the uniform asymptotic is given by

$$\varepsilon_{bright}(\vec{u}') = \sqrt{\pi}\left[(A_1 + A_2)(-\xi)^{1/4}\,\mathrm{Ai}(\xi) - \imath(A_1 - A_2)(-\xi)^{-1/4}\,\mathrm{Ai}'(\xi)\right]\exp\left[\imath\left(\chi + \frac{\sigma\pi}{4}\right)\right] \tag{15}$$

where $\sigma = \mathrm{sgn}(\Phi_{20})$, the sign of the second derivative of the phase with respect to u'_x evaluated at the caustic, $\chi = \frac{1}{2}(\Phi_1 + \Phi_2)$, $\xi = -\left[\frac{3}{4}(\Phi_2 - \Phi_1)\right]^{2/3}$, and Ai is the aforementioned Airy function. The condition $\Phi_2 - \Phi_1 > 0$ determines the labeling of the pair of interacting solutions.

In the caustic's dark side, the solutions become complex and form a conjugate pair of solutions. Here, the uniform asymptotic simplifies to

$$\varepsilon_{dark}(\vec{u}') = 2\sqrt{\pi}A\xi^{1/4}\,\mathrm{Ai}(\xi)\exp\left[\imath\left(\chi + \frac{\sigma\pi}{4}\right)\right] \tag{16}$$

where $\chi = Re(\Phi)$, $\xi = \left[\frac{3}{2}|Im(\Phi)|\right]^{2/3}$, and A and Φ are the geometrical optics amplitude and phase of either of the solutions.[1] Even though $A_1$and A_2 in (15) diverge as we approach the caustic, the products $(A_1 + A_2)(-\xi)^{1/4}$ and $(A_1 - A_2)(-\xi)^{-1/4}$ tend to a finite limit. The same occurs with the product $A\xi^{1/4}$ in (16). As one moves away from the caustic and the solutions become more separated, it can be shown that these expressions become equivalent to the first order geometrical optics approximation given by (14). It is important to note that in the present context, the field will contain at least one other image apart from the two that merge at the folds, and this image must be taken into account when constructing the overall field. In the bright side of the caustic, the total field is then given by

$$\varepsilon_{total} = \varepsilon_{bright} + \sum_{j=1}^{n} A_j \exp\left\{\imath\left[\Phi_j + \frac{\pi}{4}(\delta + 1)\sigma\right]\right\} \tag{17}$$

where the summation is over all of the images that are not involved in the creation of the catastrophe. The same expression works for the dark side of the caustic, with the substitution $\varepsilon_{bright} \to \varepsilon_{dark}$.

4 Bibliographical notes

I've decided to keep the mentions of the literature to the bare minimum above in order to make these notes easier to read, but there is of course a lot more to be said on this topic, and I didn't come up with any of the expressions given on my own. The starting point of this project is the 2017 paper by Cordes et al., which builds on Clegg et al. 1998 and applies a 1D model of a Gaussian lens to FRBs. The pair of papers by Watson & Melrose 2006 deal with the construction of the wave optics intensity on the basis of geometrical optics amplitudes and phases for 1D lenses of different shapes, including the behavior of the field at fold caustics. Some other relevant papers in the area of astrophysical plasma lensing, its relationship to scintillations, and the role of caustics that I've found useful are Goodman et al. 1987, Pen & Levin 2014, Simard & Pen 2017, Tuntsov et al. 2016, and Er & Rogers 2017, among others. Some numerical simulations of 2D plasma lenses were carried out by Stinebring et al. 2007.

[1]The amplitude of both solutions is the same, because the expression for the amplitude (9) does not include the oscillatory exponential that is used to construct the field in (14), and we take the absolute value of $\mathcal{J}$ before taking its square root. This is a subtle point that can be confusing, and it is important to remember that the geometrical optics field is not the same as the geometrical optics amplitude.

The foundations of geometrical optics and the derivation of the expression of the amplitude and field in relation to the Jacobian of the mapping from the u' plane to the u plane and the geometrical phase involve the transport and eikonal equations, and are not often referenced directly in papers in astrophysics. More details can be found in the classic textbook by Born & Wolf 1999, and other texts on geometrical optics such as Kravstov & Orlov 1999, Borovikov & Kinber 1994, and Berry & Upstill 1980. The last three works also deal at length with caustics and catastrophe theory, and the paper by Berry & Upstill in particular is regularly referenced in the context of catastrophe theory and its applications to optics. These texts also include discussions concerning the relationship between geometrical optics and wave optics, in particular regarding the asymptotic expansion of the Kirchhoff diffraction integral by the method of stationary phase and its relationship to the lens equation. Important works that deal explicitly with the asymptotic expansions to multiple integrals include Dingle 1973, Bleistein & Handelsman 1975, Cooke 1982, and Stamnes 1986.

The topic of uniform asymptotic expansions at caustics is discussed at length by Kravstov & Orlov 1999, Borovikov & Kinber 1994, Chester et al. 1957, Ludwig 1966, Stamnes 1986, and Kryukovskii et al. 2006. The 2000 paper by Qiu & Wong is also very useful in this context, as it contains a formal derivation of the asymptotic in the general context of multidimensional highly oscillatory integrals. The concept of complex rays is discussed by Kravstov et al. 1999, and their application to the simulation of propagation of radio waves is exemplified by Budden & Terry 1971, and Budden 1976. Complex stationary phase points also arise in the develoment of hyperasymptotic expansions of integrals, as in Kaminsky 1994 and Howls 1997.

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