Technical Report
Composites

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Comparison

FEA and Classical Lamination Theory

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Contents

Introduction

The purpose of this report is to compare the analytical models with FEA. As usual, analytical models have to be simplified due to mathematical difficulties or simply lack of the highly specific test data. The work begins with a micro model which represents the packing of the fibers inside the matrix. In this section the influence of the anisotropy of the carbon fiber is discussed. Next, a lamina composed of the orthoropic 2D plies is concerned. This step is based on the classical lamination theory (CLT), which will be later modified to model a thin-walled, closed-cross section, beam-like profiles.

Materials data

For sample calculation data for Torayca T700S fiber and Biresin CR84 are used. The full, original data-sheet may be found on the manufacturer website. They can be also found in the appendix A.
The most important information are listed below:

Torayca T700S:

- Tensile Strength - $\sigma_{1ult} = 4\ 900$ MPa
- Tensile Modulus - Ef1 = 230 GPa
- Density - ρ 1800 kg/m^3

Biresin CR84 - approx. values after 8 h / 70řC (source: Sika internal)

- Tensile Strength - $\sigma_{1ult} = 89$ MPa
- Tensile Modulus - Em = 3.55 GPa
- Flexural Strength - $\sigma_{1ult} = 124$ MPa
- Flexural Modulus - Em = 3.25 GPa
- Density - ρ 1150 kg/m^3

Effective moduli of a continuous fiber-reinforcement lamina

Analytical models

Fiber packing

The fiber packing may be realized by different patterns, which are square array or triangular array.[1]

[1] Ronald F. Gibson. *Principles Of Composite Materials Mechanics*. McGraw Hill, Inc., 1994, p. 65.

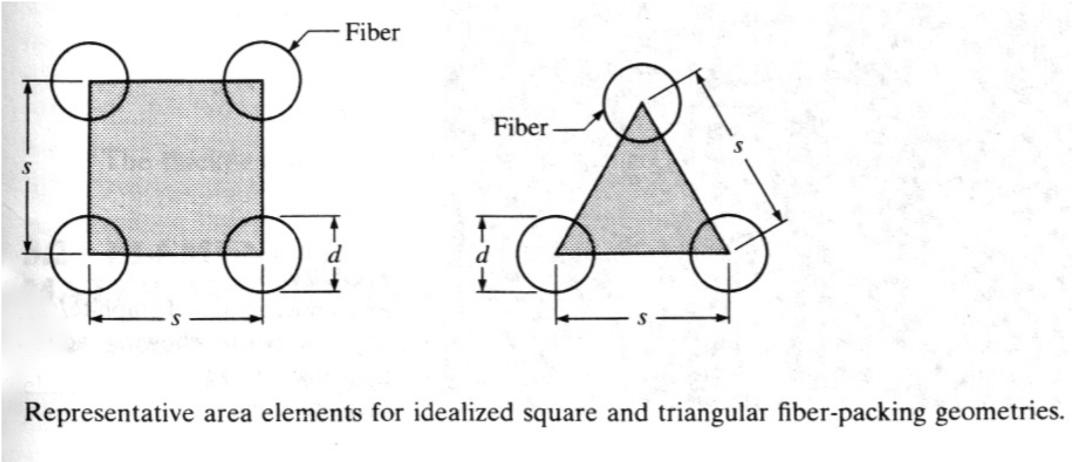
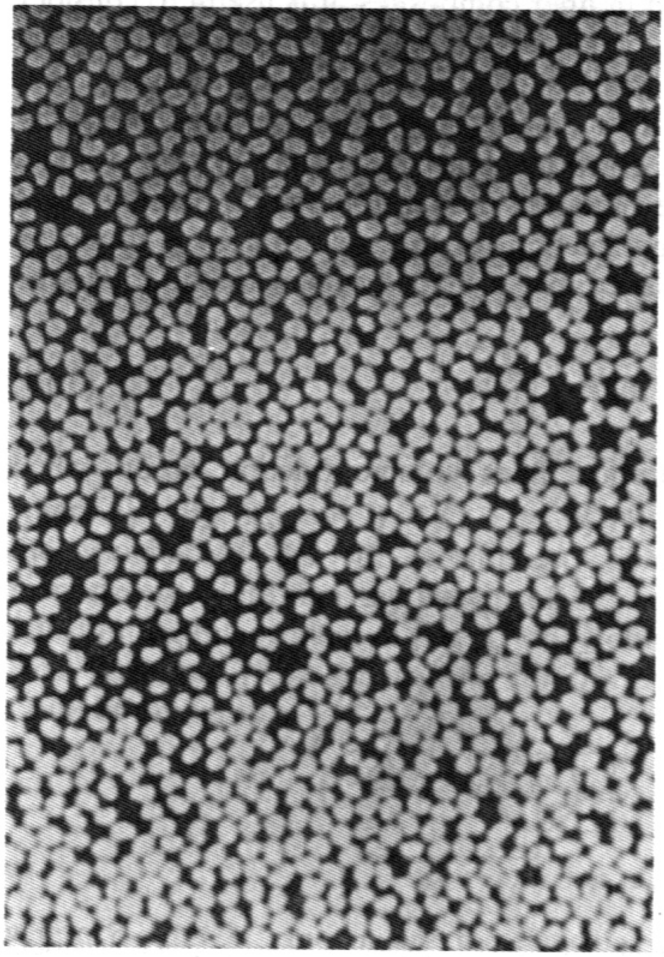


Figure 1: Theoretical fiber packing pattern

The real fiber packing geometry is of a random nature.[2]

[2]Ibid., p. 66.



Photomicrograph of graphite/epoxy composite showing actual fiber-packing geometry at 400× magnification.

Figure 2: Microphotograph fiber packing

The fiber volume fraction for the square array is found by dividing the area of fiber enclosed in the shaded square by the total area of the square:

$$v_f = \frac{\pi}{4}(\frac{d}{s})^2 \tag{1}$$

The geometrical constrains enforce an upper limit for the maximal fiber volume ration, which is 0.785 for the square array and 0.907 for the circular array.

The rule of mixtures

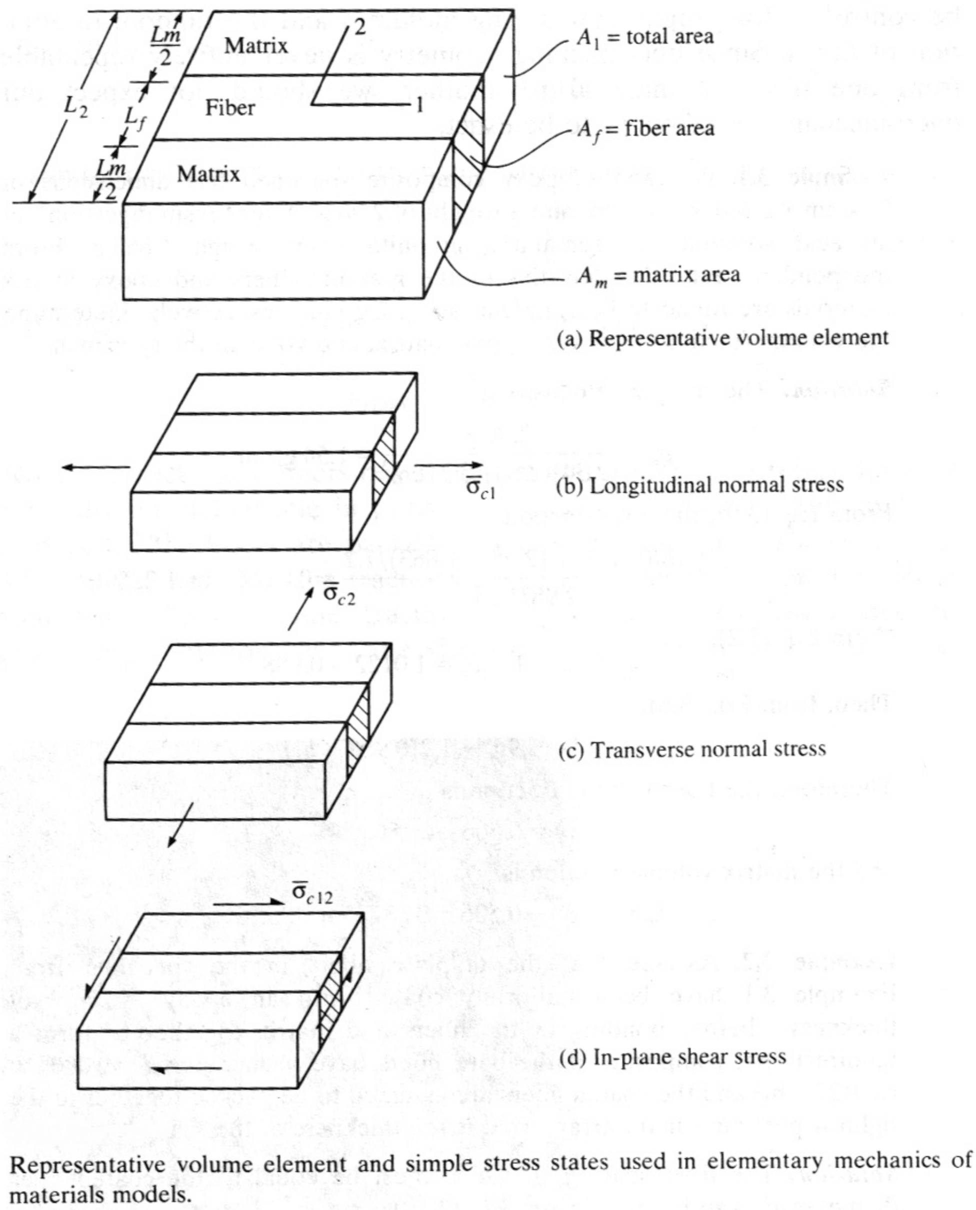


Figure 3: RVE - Representative Volume Element

To find the transverse modulus, E_2 it is assumed that the total transverse composite displacement δ_{c2} is equal to the sum of δ_{f2} (fiber displacement) and δ_{m2} (matrix displacement):

$$\delta_{c2} = \delta_{f2} + \delta_{m2} \tag{2}$$

Next, from the definition of normal strain ($\epsilon = \delta L$) :

$$\epsilon_{c2} L_2 = \epsilon_{f2} L_f + \epsilon_{m2} L_m \tag{3}$$

Since the longitudinal dimension of the RVE does not change, the length fractions must be equal to the volume fractions:

$$\epsilon_{c2} = \epsilon_{f2} v_f + \epsilon_{m2} v_m \tag{4}$$

Applying the Hooke's law ($\sigma = E\epsilon$) and the fact that $1 = v_f + v_m$:

$$\frac{\sigma_{c2}}{E_2} = \frac{\sigma_{f2} v_f}{E_{f2}} + \frac{\sigma_{c2}(1 - v_f)}{E_m} \tag{5}$$

Finally, it is assumed that the stresses in the composite matrix and fiber are all equall, which yields to the 'inverse rule of mixtures' expressing the transverse modulus, E_2 as:

$$\frac{1}{E_2} = \frac{v_f}{E_{f2}} + \frac{1 - v_f}{E_m} \tag{6}$$

Where:

E_2 - transverse Young modulus of the lamina

E_{f2} - transverse Young modulus of the fiber

E_m - Young modulus of the matrix

v_f - volume fraction of fibers

This simple model does not fit experiments, thus more advanced analytical approaches has been introduced. However, it turns out that the semi-empirical formulas are even better. They are beyond the scope of this work.

Micromechanical model

The figure 4 shows that the structure of the carbon fiber is like a paper harmonica. This means that its properties are considerably lower in the transverse direction. It is inconvenient (or even impossible) to find the E_{f2} in the experimental manner. However, some back-calculation can be done by substitution of measured composite properties and matrix properties in SME (Simplified Micromechanics Equations).

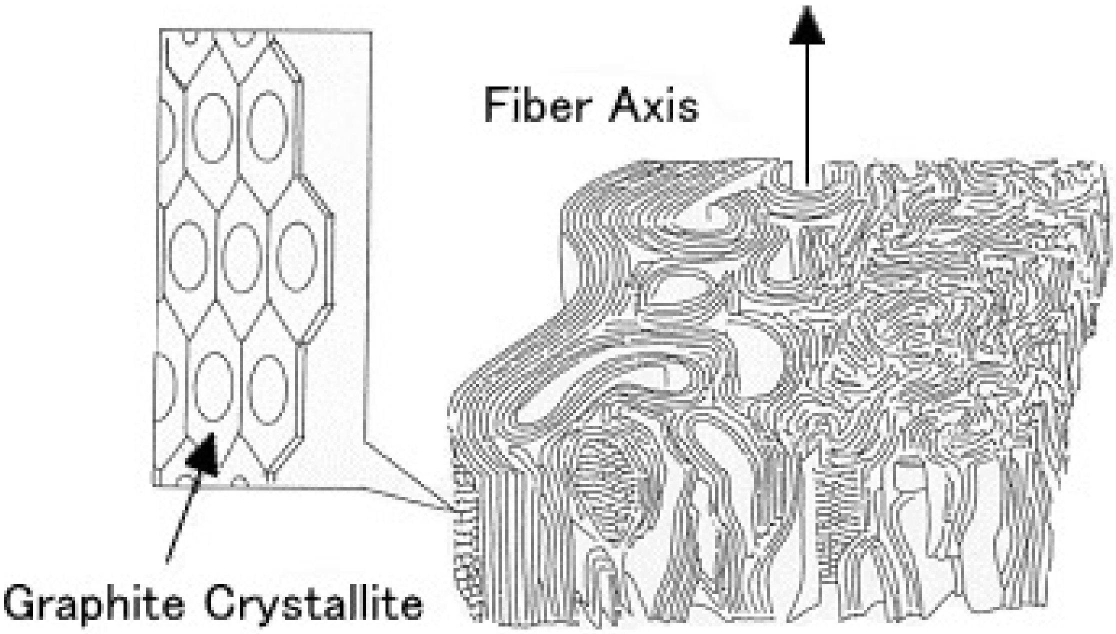
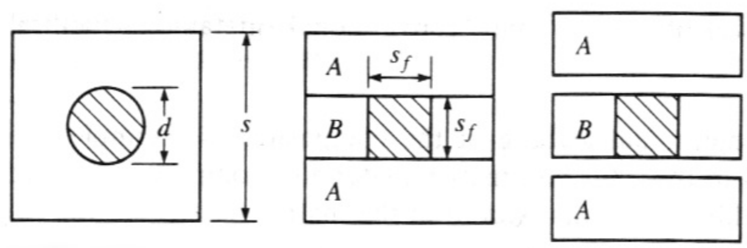


Figure 4: Structure of the carbon fiber



Division of representative volume element into subregions based on square fiber having equivalent fiber volume fraction.

Figure 5: RVE - subregions

In the micromechamical approach the equation for E2 is derived in the same ways as in the 'inverse rule of mixtures'. The only difference is that the RVE is split and additional 'A' elements appears next to 'B'. In the 'inverse rule of mixtures' the transverse behaviour was modelled only by 'B' element.

$$E_2 = E_m \left[(1 - \sqrt{v_f}) + \frac{\sqrt{v_f}}{1 - \sqrt{v_f}(1 - E_m/E_{f2})} \right] \tag{7}$$

The detailed derivation may be found in literature.[3]

Comparison of the analytical models

Plots below illustrate the influence of the transverse fibers' modulus in two cases Ef2 = Ef1 and Ef2 ≠ Ef1

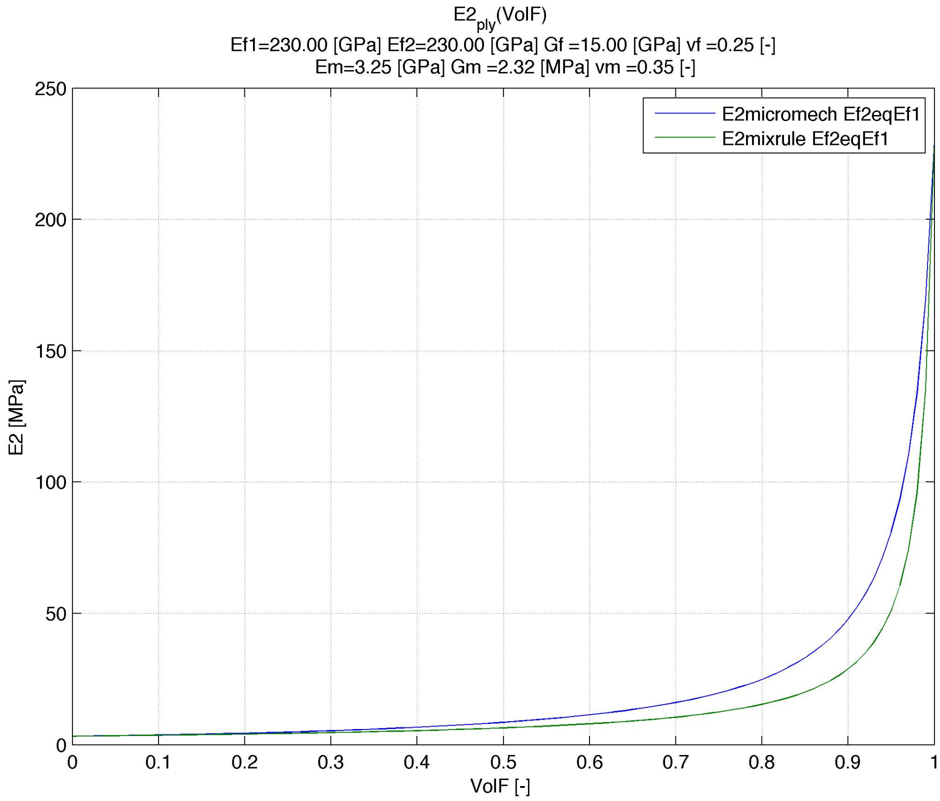


Figure 6: Ef2 = Ef1

[3] Ibid., p. 77-79.

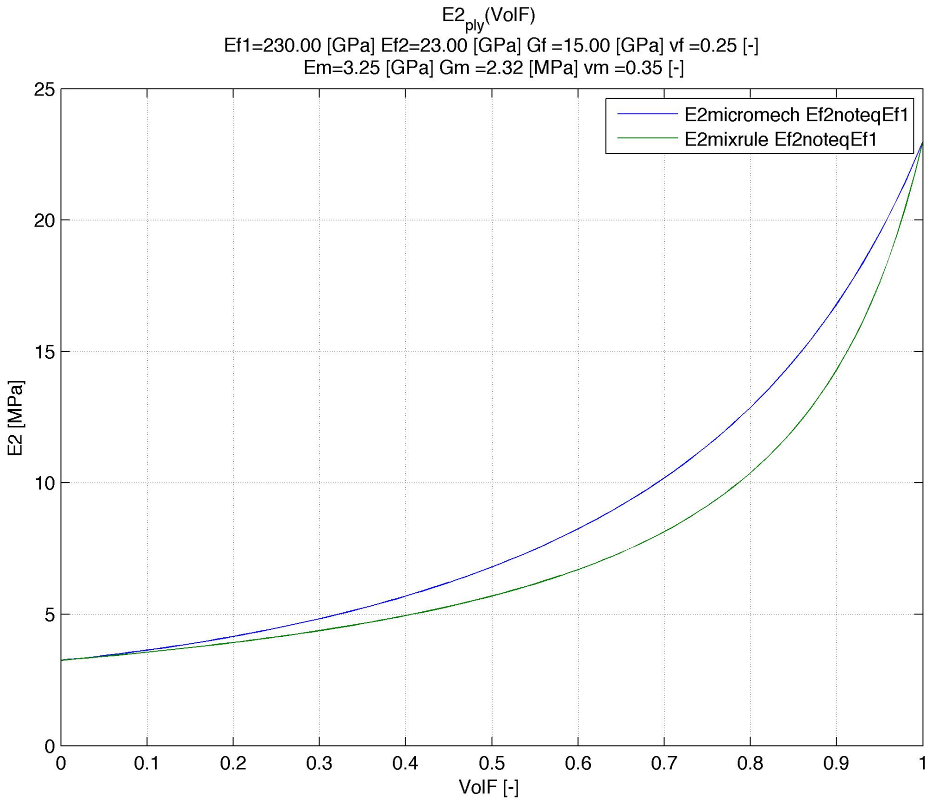


Figure 7: Ef2 $\neq$ Ef1

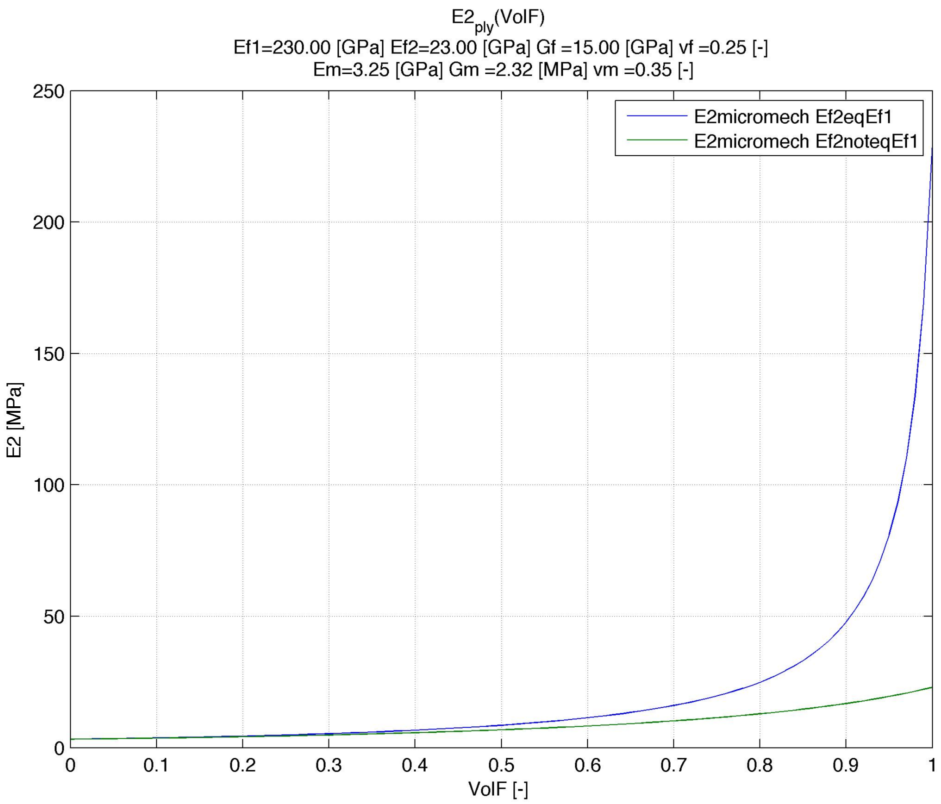


Figure 8: Micromechanical approach, influence of Ef2

It is worth noticing that the influence of anistropy of carbon fibres is not so dramatic as it may appear at the beginning. The disproportion becomes large above VolF 0.7 which is rather difficult to achieve from technical point of view.

FEA - transverse to fibers

Figures below show sample FEA for fibers in a square array. The fiber volume ratio is 0.5. To create a valid material model in FEA additional properties (not provided by the manufacturer) have been assumed:

- $v_m = 0.35$ poisson's ratio of the matrix
- $v_m = 0.25$ poisson's ratio of the fibers
- fibers are assumed to be isothropic (Ef = 230 GPa) or transversely orthotropic (Ef1 = 230 GPa, Ef2 = 23 GPa) - depending on the case

FEA Boundary conditions:

- load: imposed displacement in the X direction $\sim 2[\%]$
- support: symmetry X, symmetry Y, symmetry Z
- support: constrained DOF - Z (wall opposite to the symmetry BC), constrained DOF - Y (wall opposite to the symmetry BC).
  The constrained DOF BC is necessary to ensure that the whole wall of cube deforms by the same amount despite the fact that it consist of two different materials.

FEA - square array

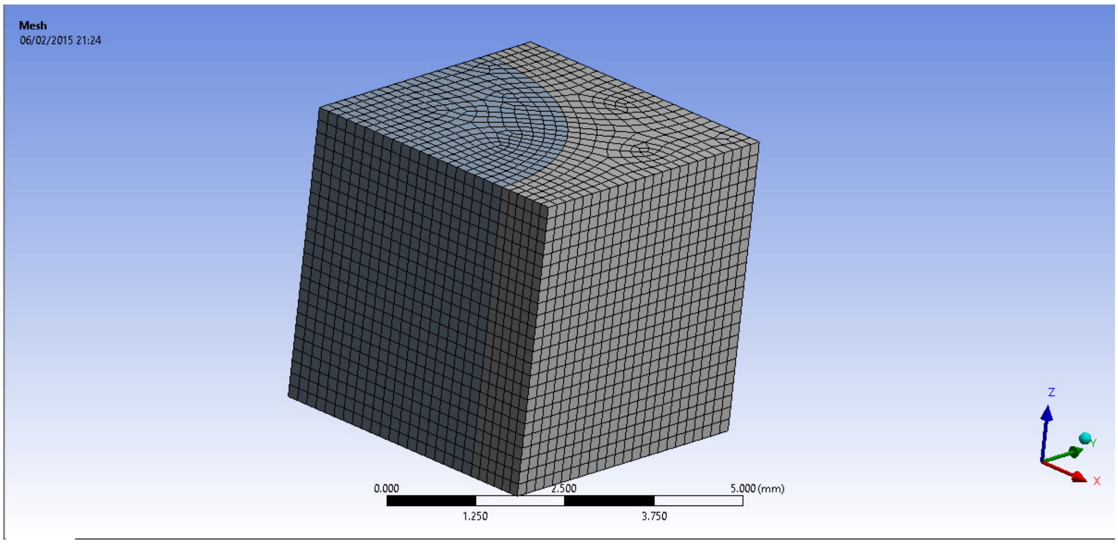


Figure 9: Mesh

Assumption: Isotropic fibers

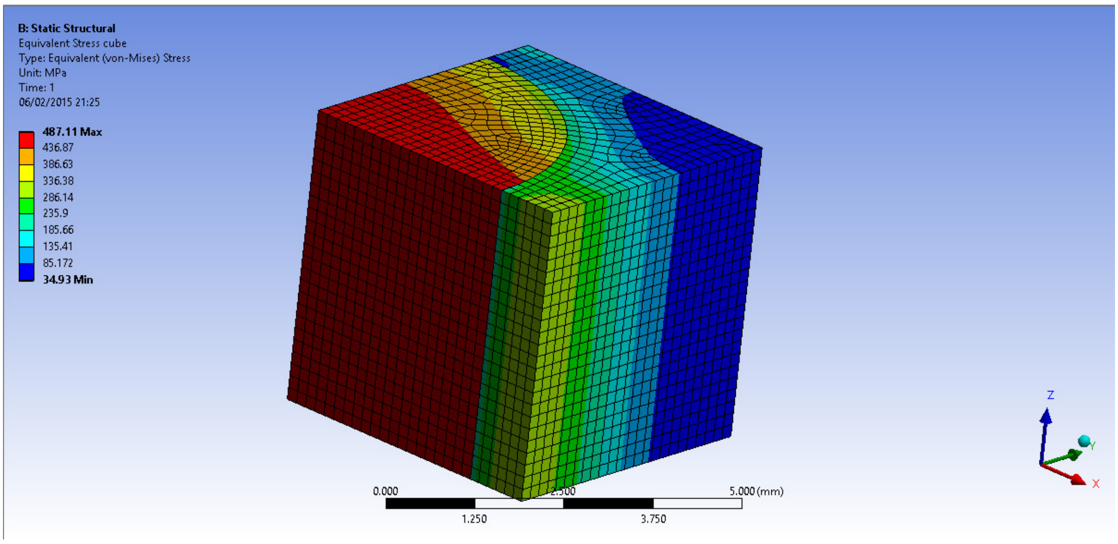


Figure 10: von Mises equivalent stress, Ef2 = Ef1

The figure 10 shows that the assumption of equal stresses in matrix and fiber used for derivation of the 'inverse rule of mixtures' 3.1.2 is not valid. This invalidty can be also shown with the strain energy approach[4]

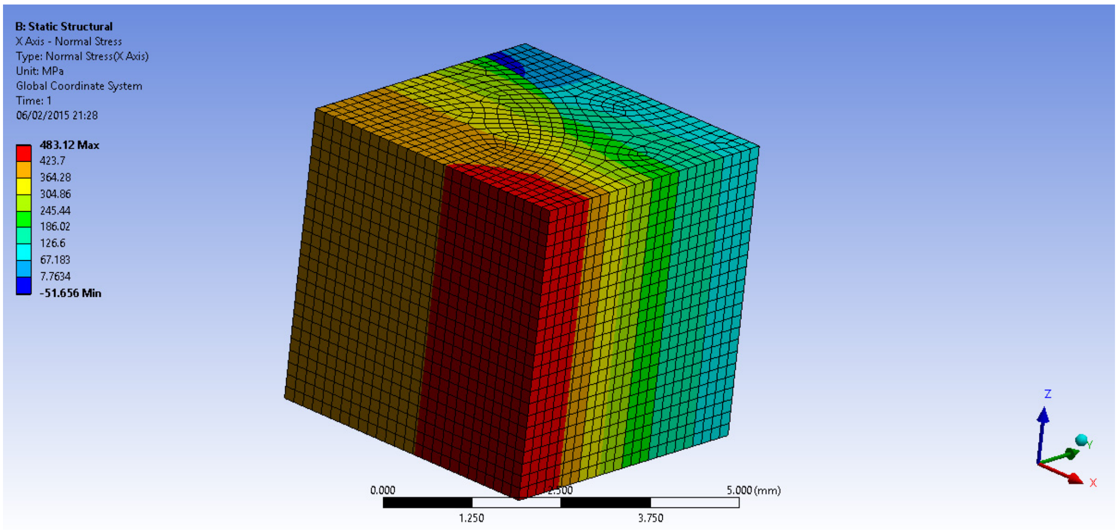


Figure 11: Stress - X direction, Ef2 = Ef1

[4] Ibid., p. 74.

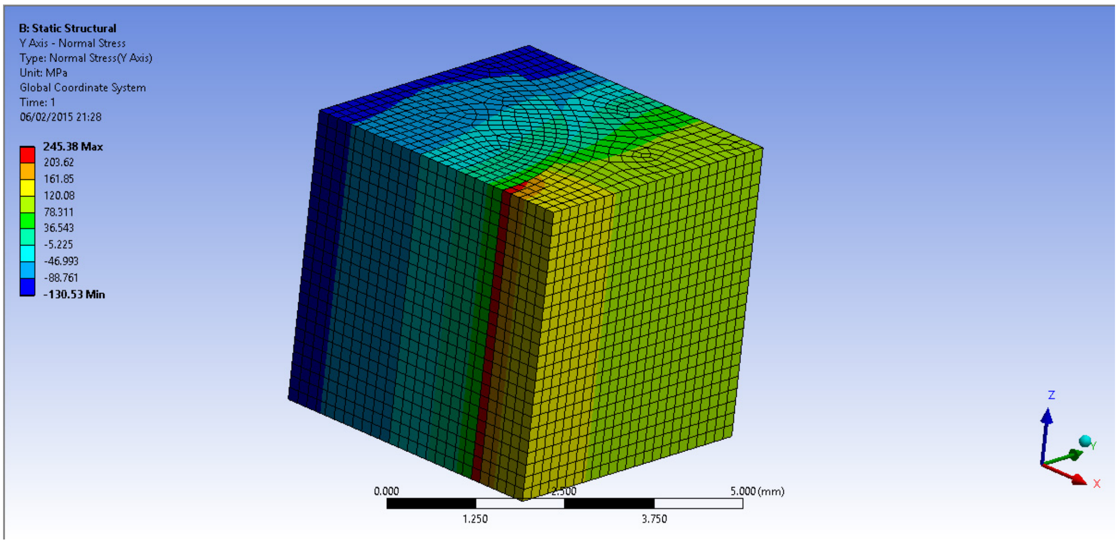


Figure 12: Stress - Y direction, Ef2 = Ef1

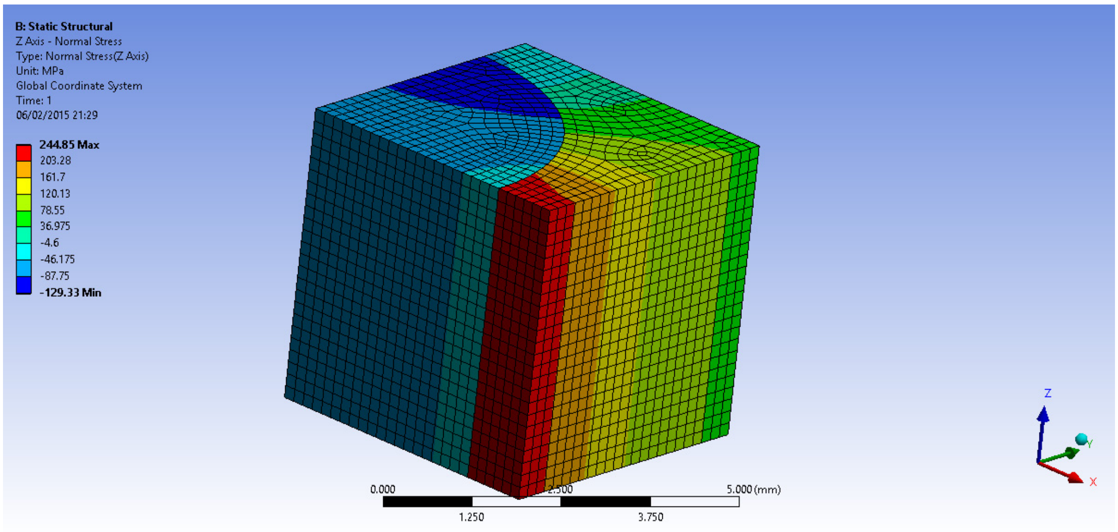


Figure 13: Stress - Z direction, Ef2 = Ef1

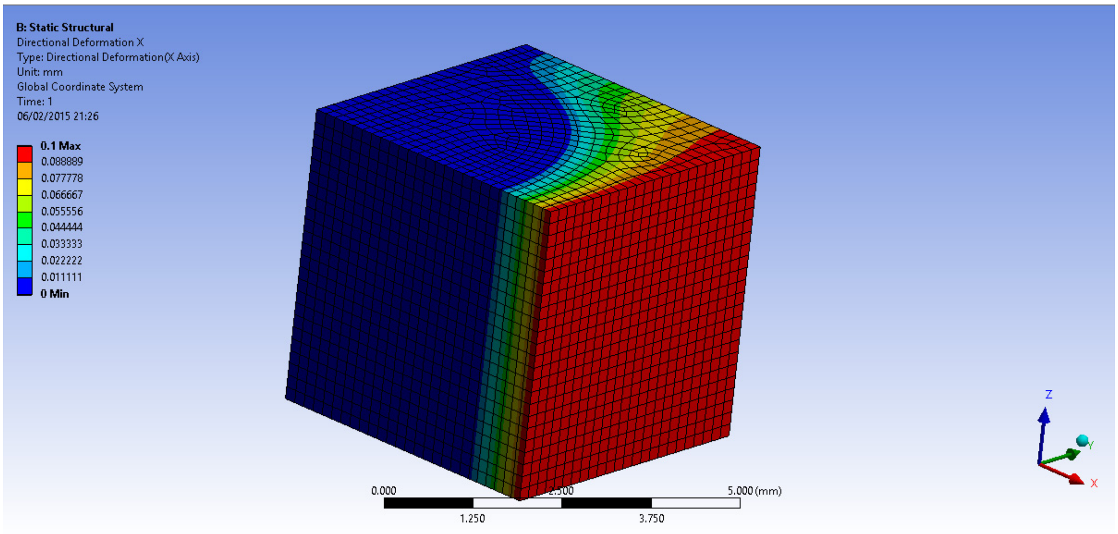


Figure 14: Displacement - X direction, Ef2 = Ef1
It is clearly visible that the fibers are much stiffer than matrix.

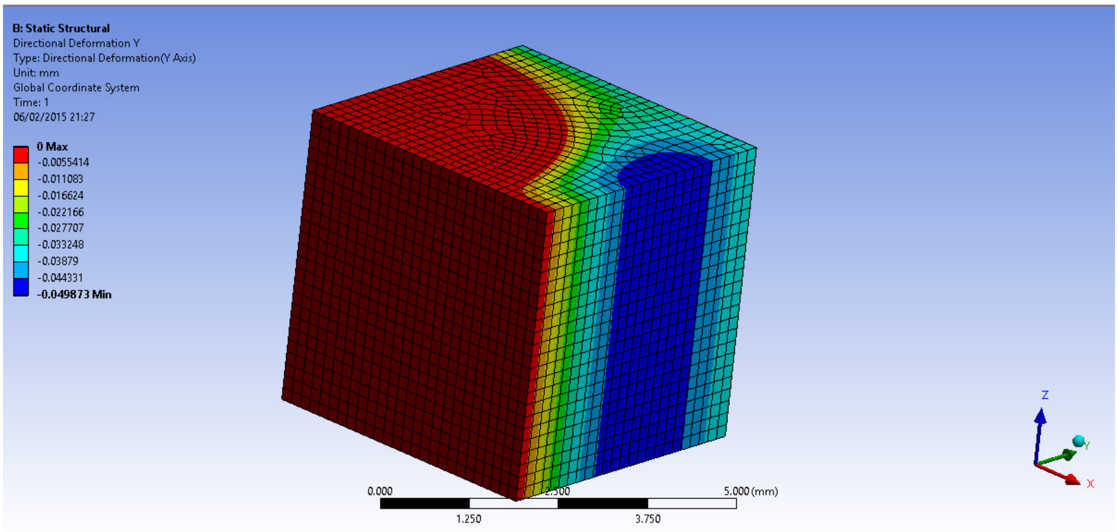


Figure 15: Displacement - Y direction, Ef2 = Ef1

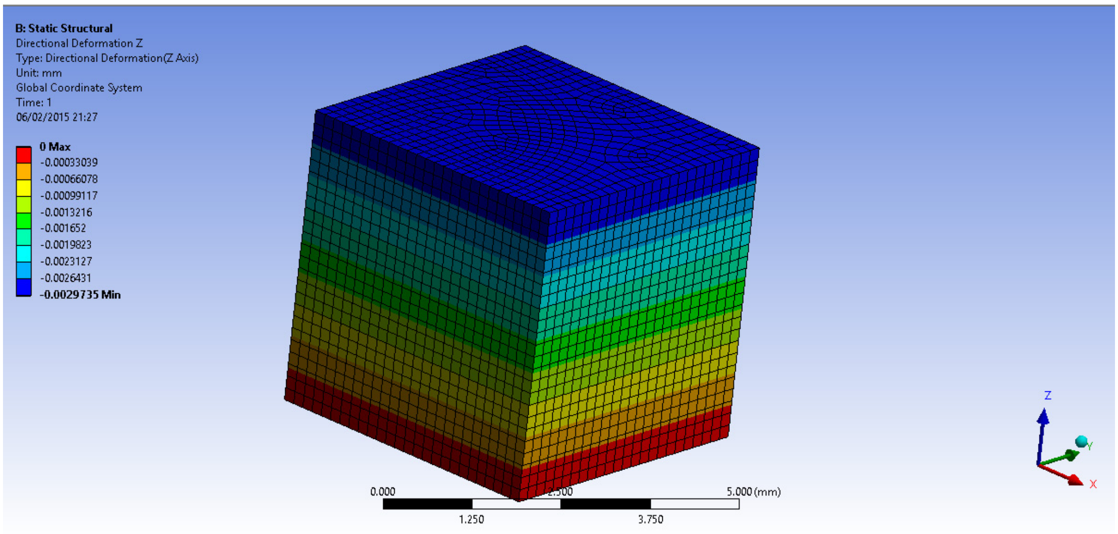


Figure 16: Displacement - Z direction, Ef2 = Ef1

The figure 16 illustrates linear compression of the model due to Poisson effect. Of course the Poisson's ratio of fibers and matrix is different but the model-brick has to preserve its shape since it is virtually adjacent to the others.

Assumption: Anisotropic fibers

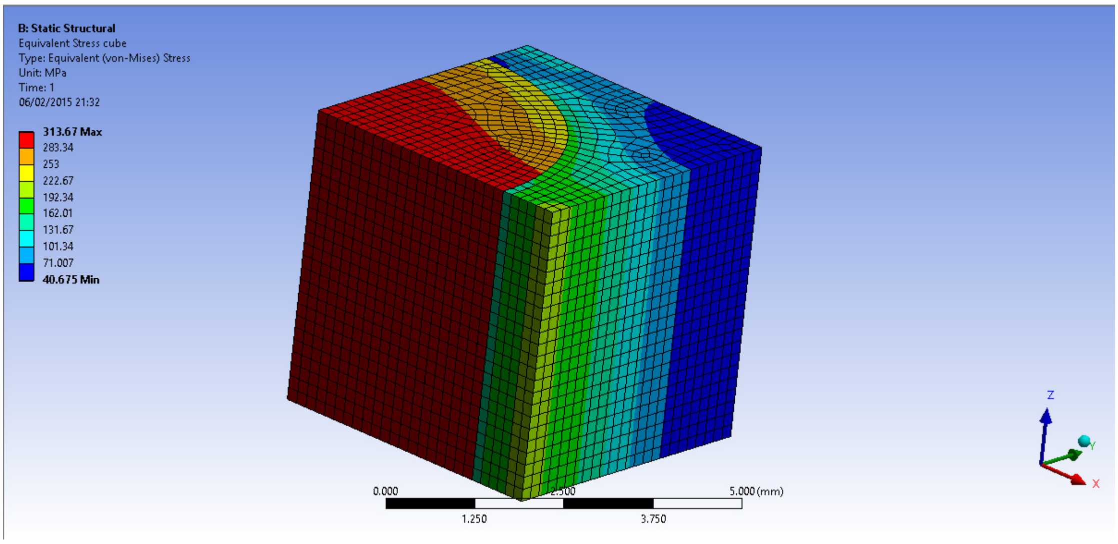


Figure 17: von Mises equivalent stress, Ef2 $\neq$ Ef1
Again, the stresses in matrix and fibers are different

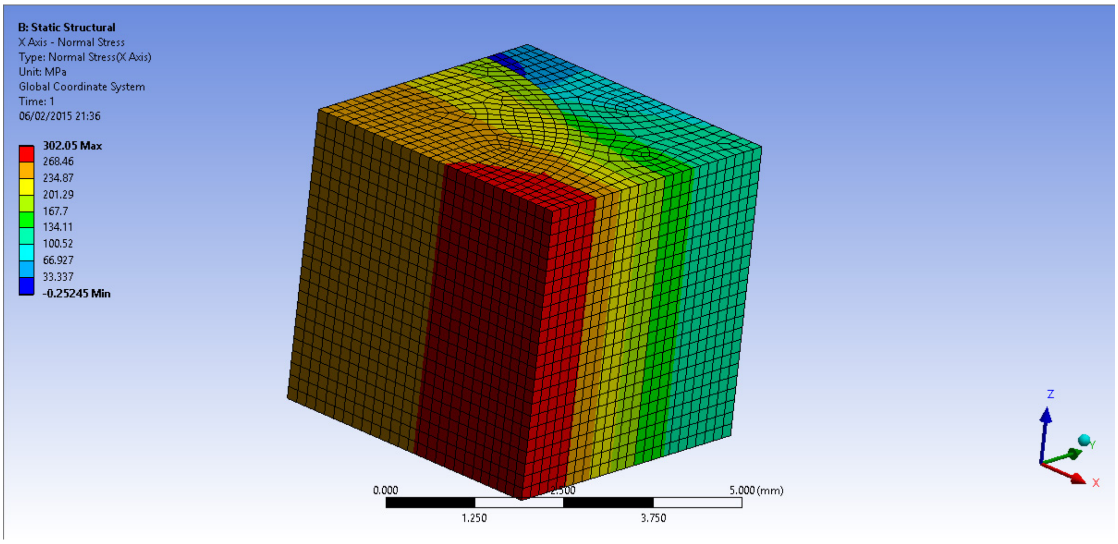


Figure 18: Stress - X direction, Ef2 $\neq$ Ef1

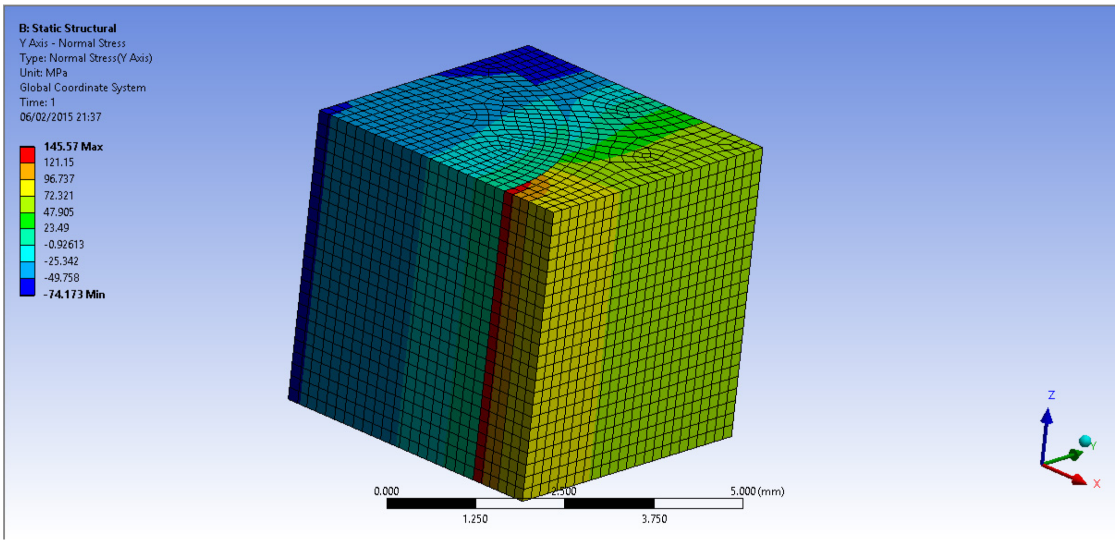


Figure 19: Stress - Y direction, Ef2 $\neq$ Ef1

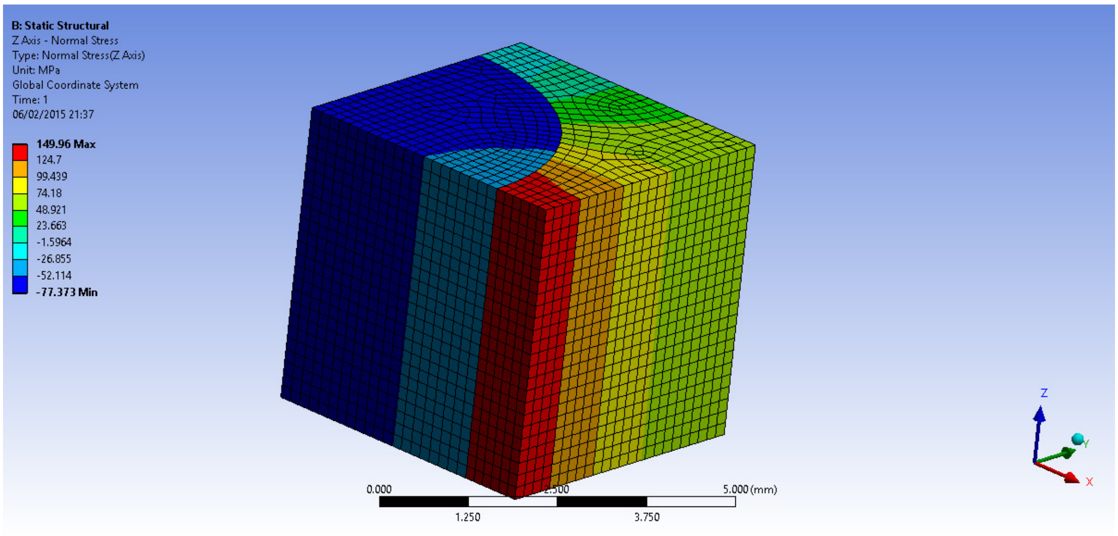


Figure 20: Stress - Z direction, Ef2 $\neq$ Ef1

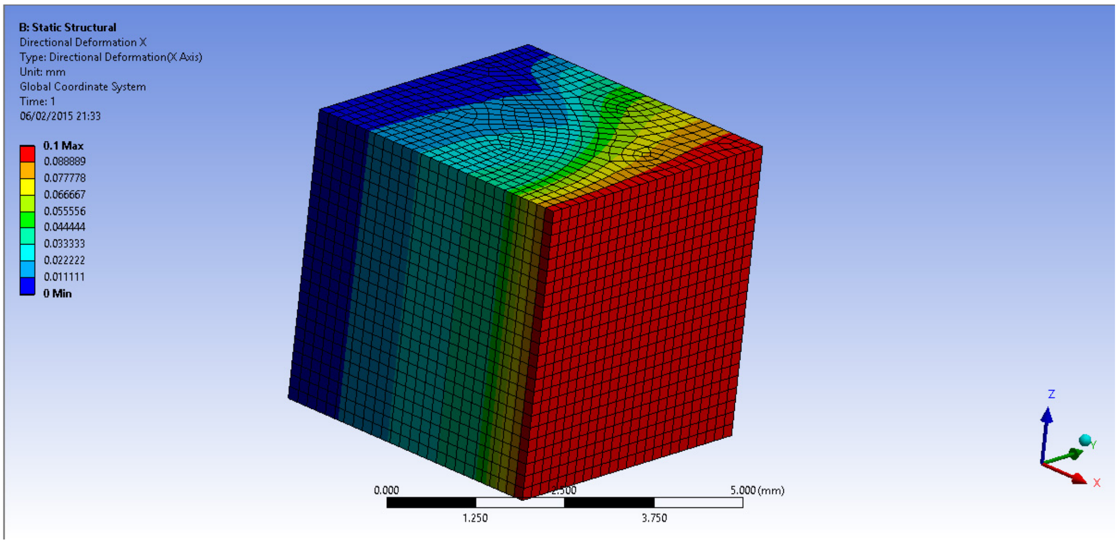


Figure 21: Displacement - X direction, Ef2 $\neq$ Ef1

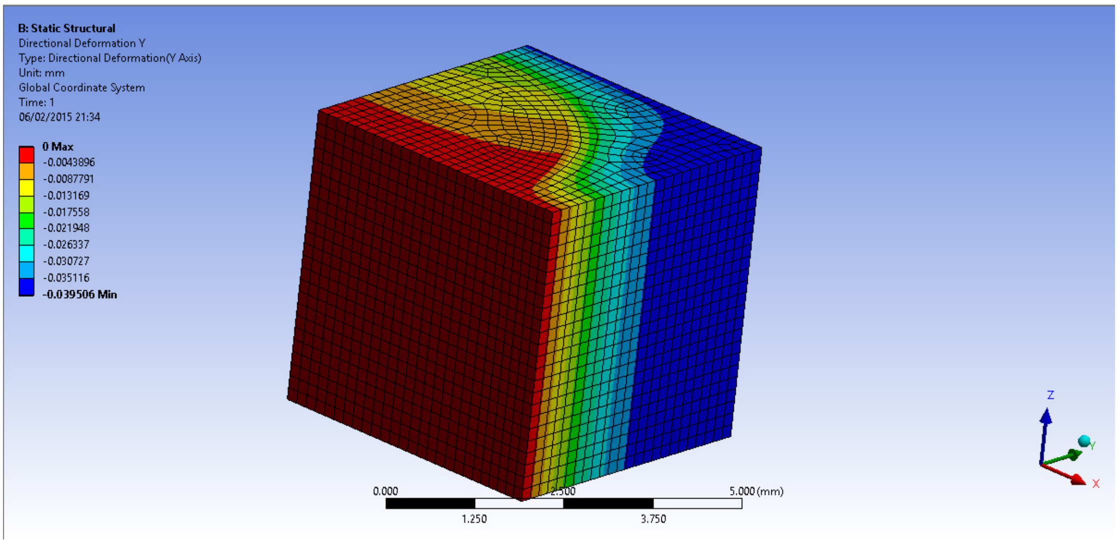


Figure 22: Displacement - Y direction, Ef2 $\neq$ Ef1
The stiffness of both the matrix and fibers is comparable in the transverse direction, which results in similar displacements

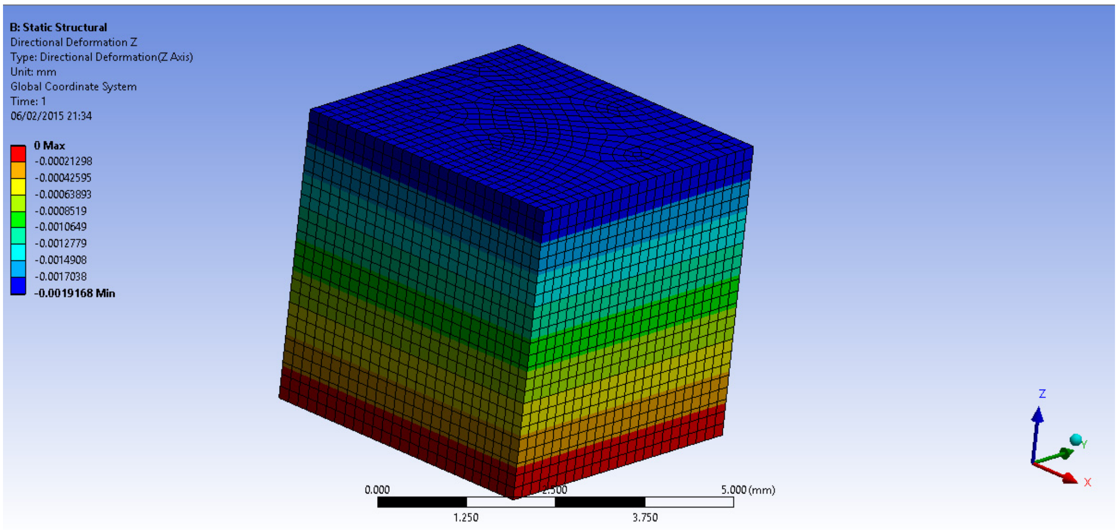


Figure 23: Displacement - Z direction, Ef2 $\neq$ Ef1

FEA - triangular array

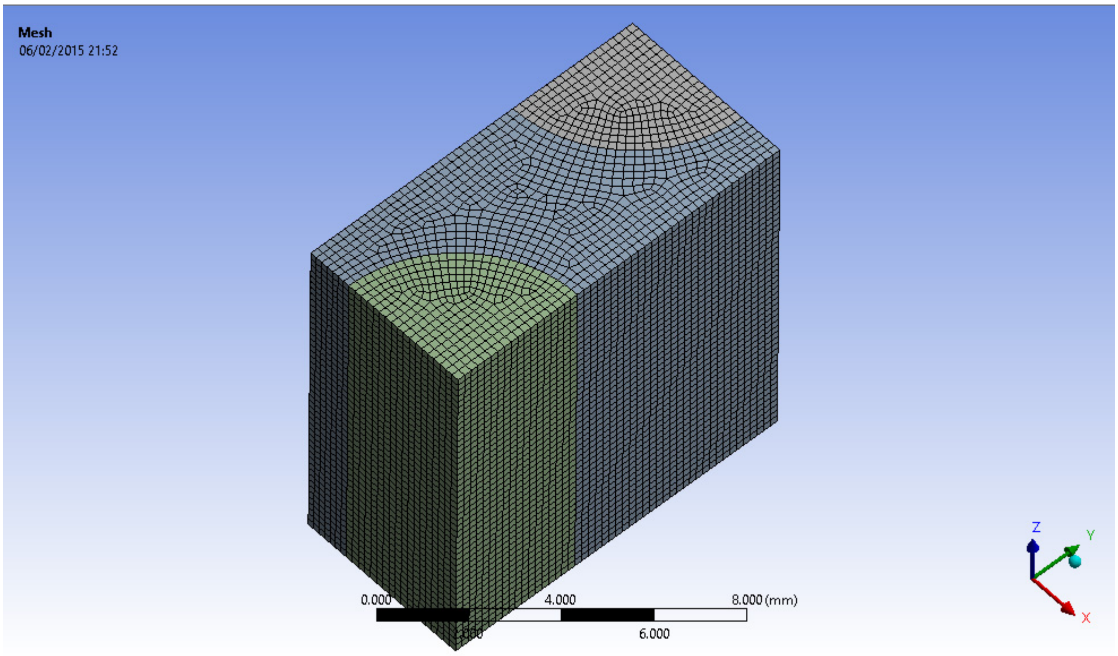


Figure 24: Mesh

Assumption: Isotropic fibers

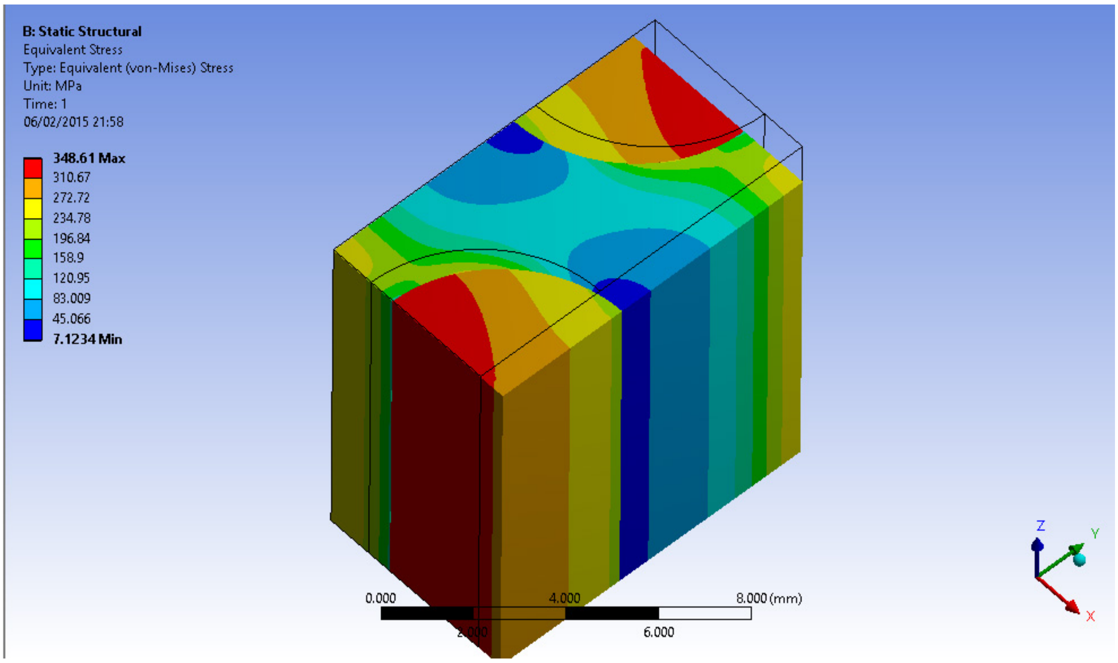


Figure 25: von Mises equivalent stress, Ef2 = Ef1

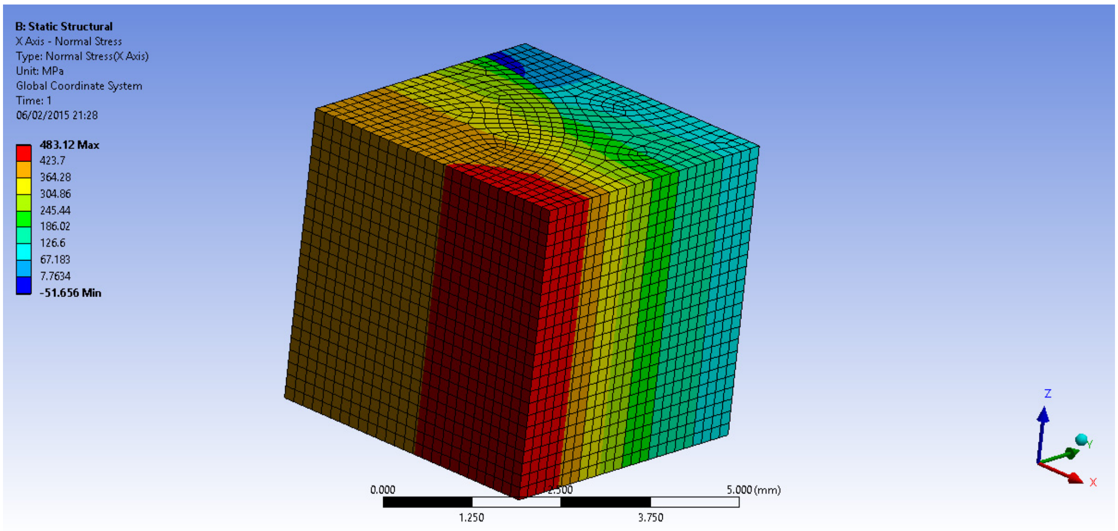


Figure 26: Stress - X direction, Ef2 = Ef1

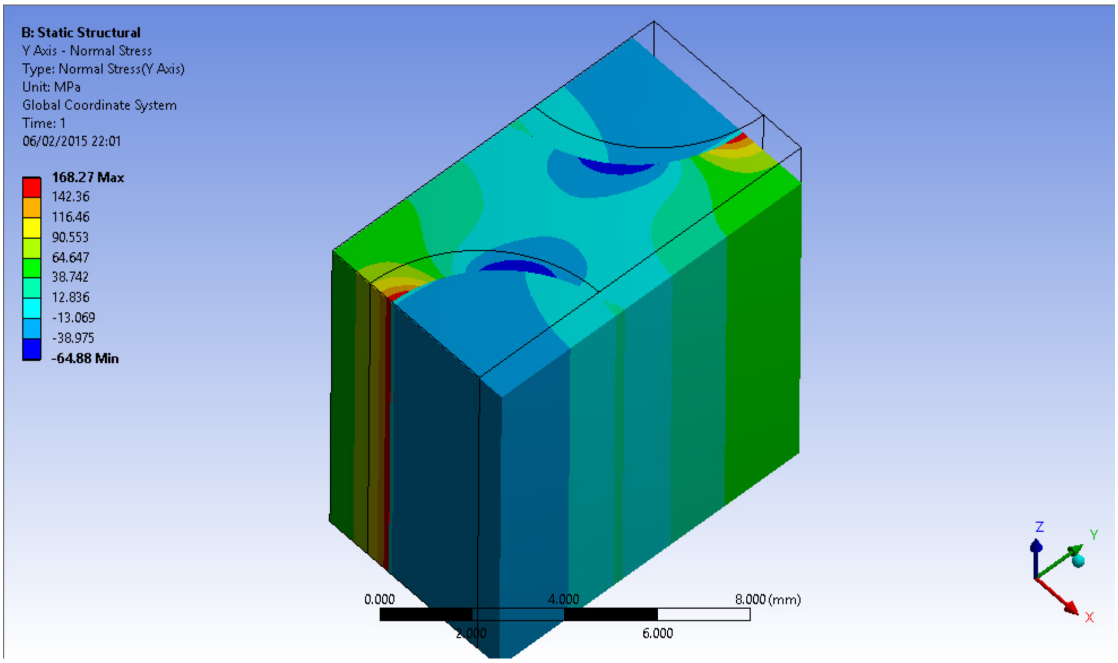


Figure 27: Stress - Y direction, Ef2 = Ef1

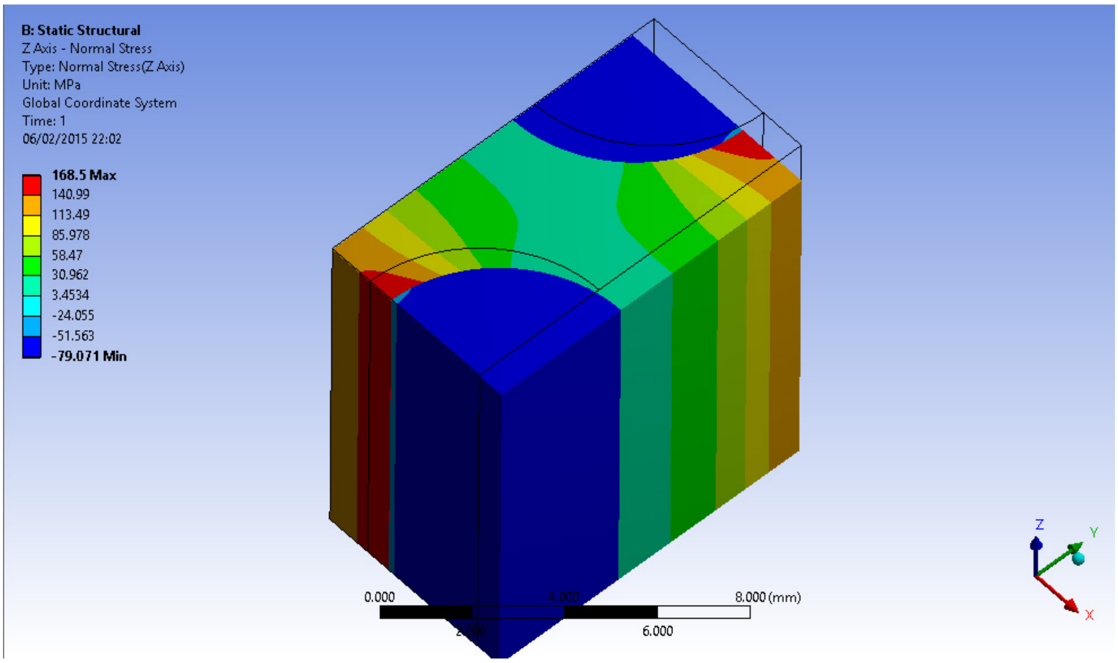


Figure 28: Stress - Z direction, Ef2 = Ef1

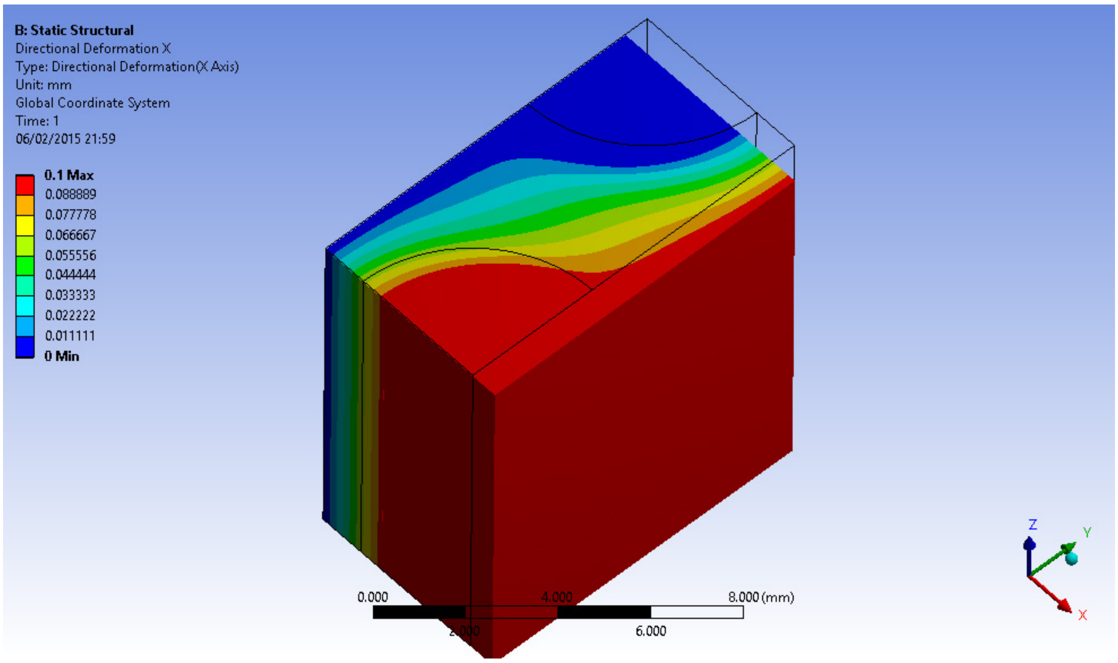


Figure 29: Displacement - X direction, Ef2 = Ef1

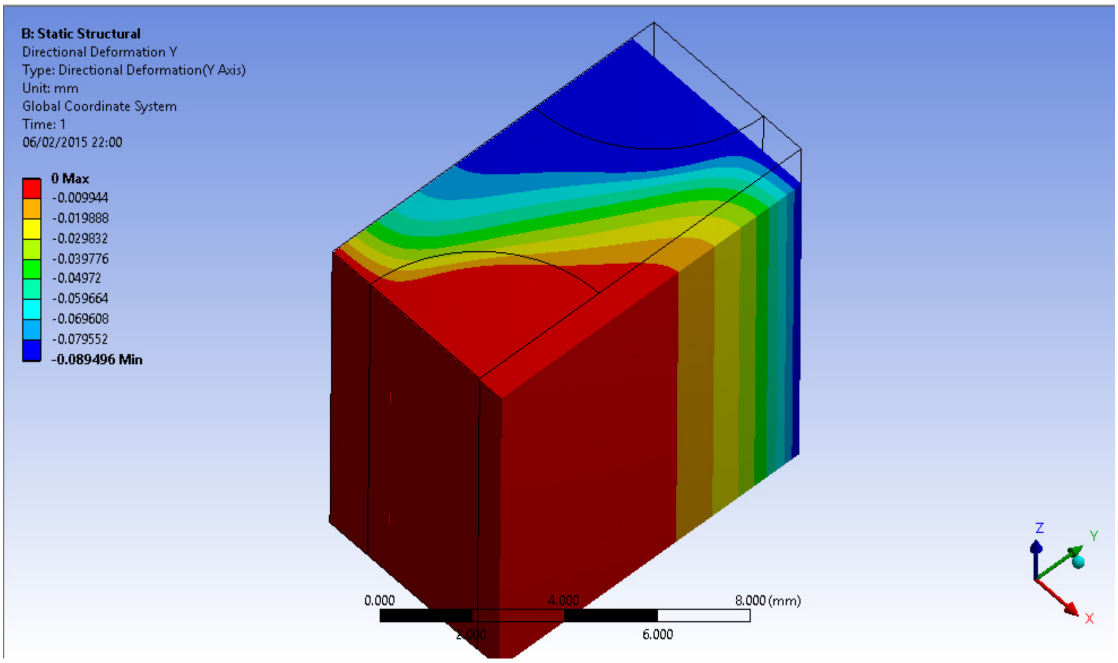


Figure 30: Displacement - Y direction, Ef2 = Ef1

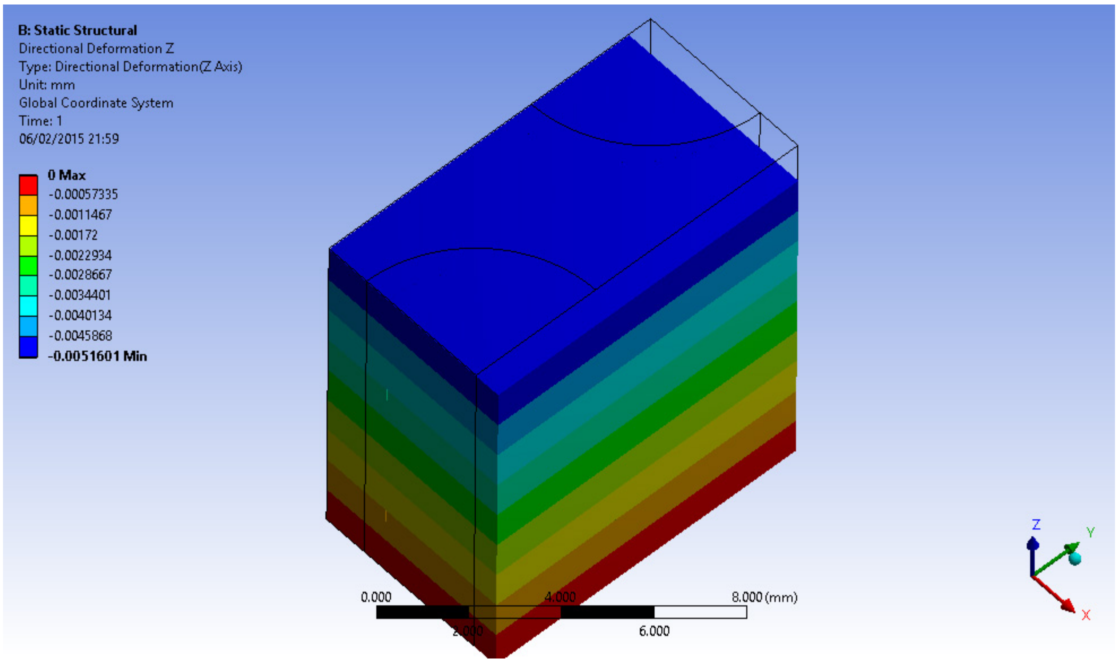


Figure 31: Displacement - Z direction, Ef2 = Ef1

Assumption: Anisotropic fibers

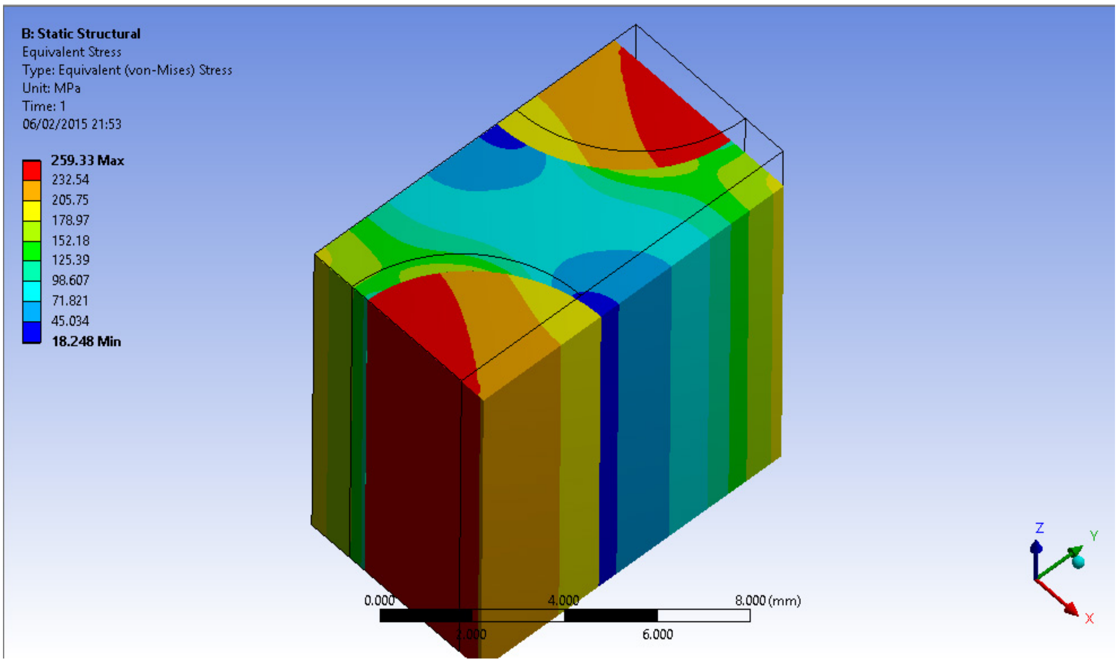


Figure 32: von Mises equivalent stress, Ef2 $\neq$ Ef1

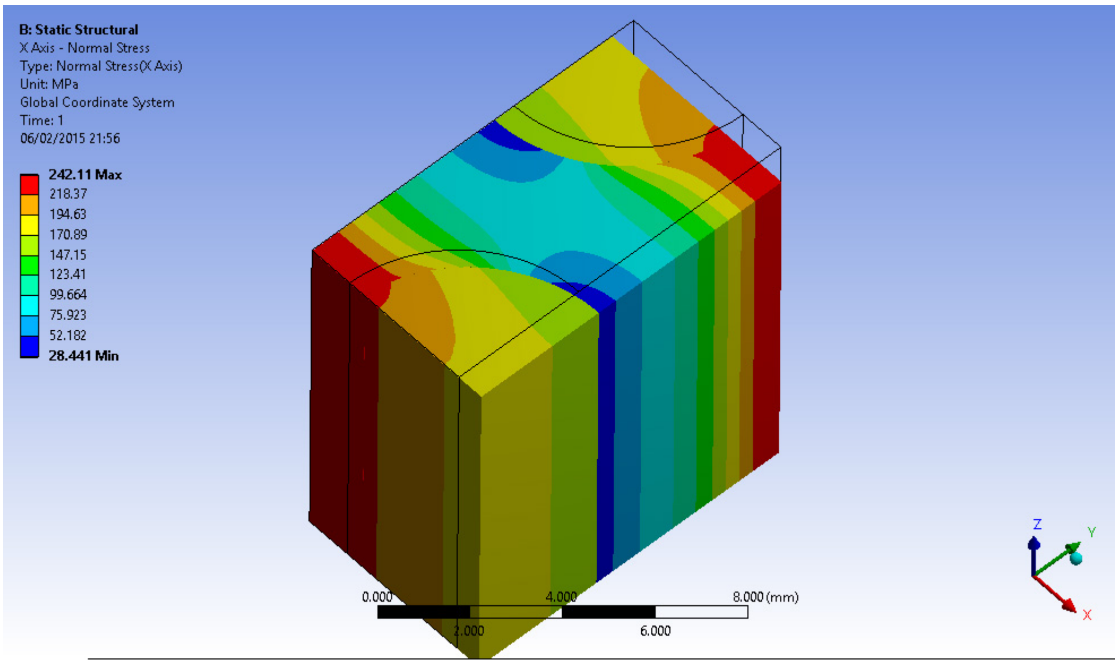


Figure 33: Stress - X direction, Ef2 $\neq$ Ef1

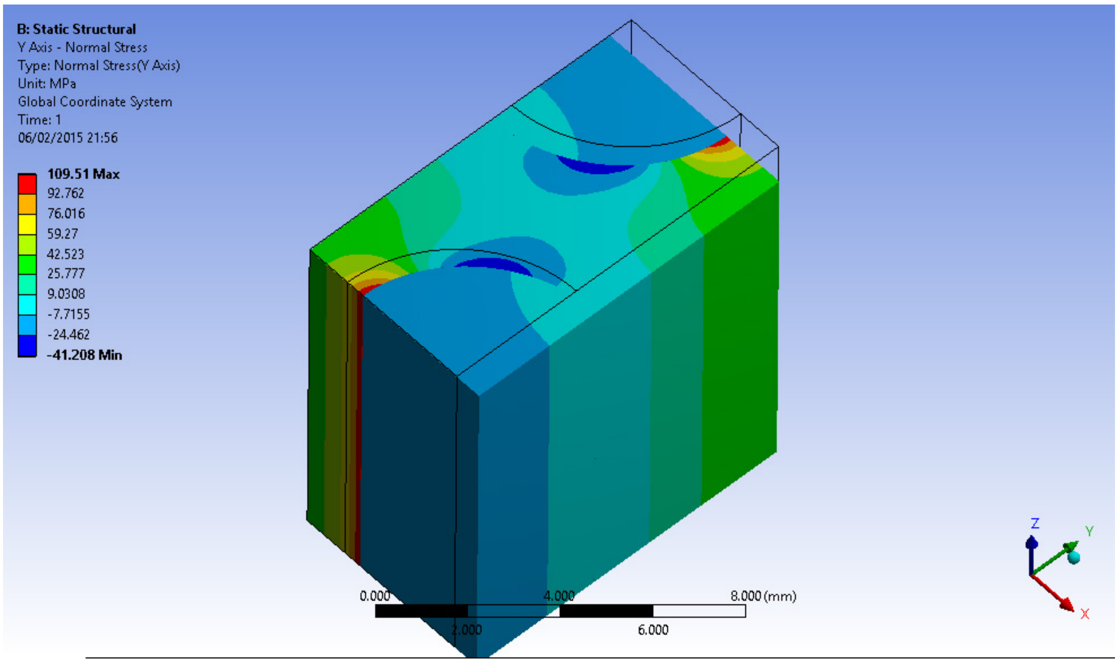


Figure 34: Stress - Y direction, Ef2 $\neq$ Ef1

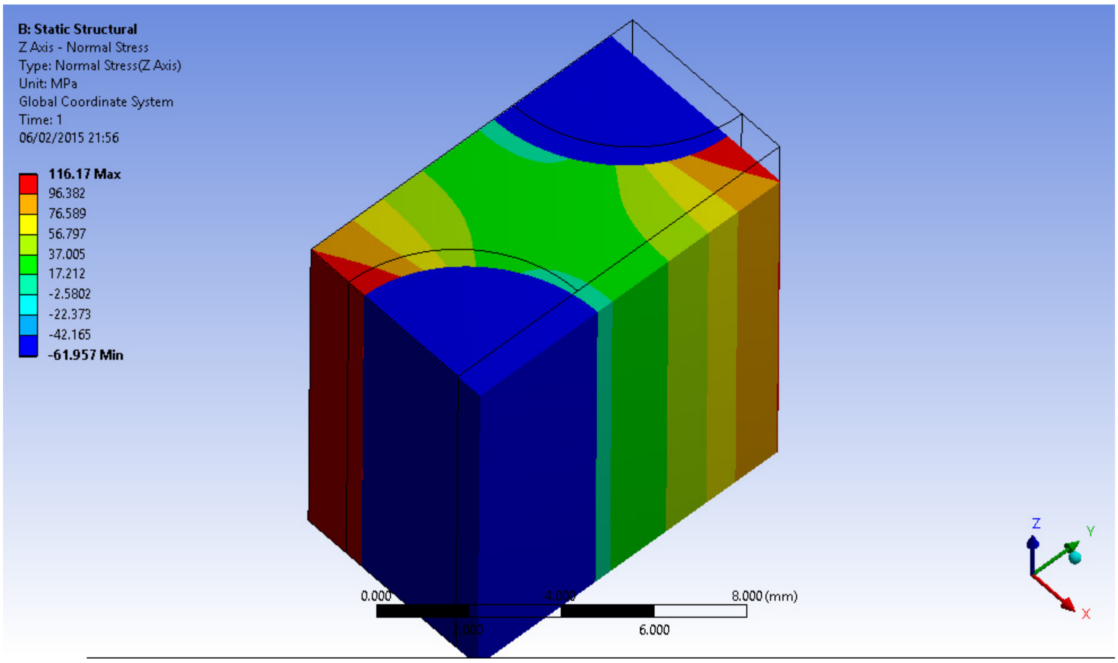


Figure 35: Stress - Z direction, Ef2 $\neq$ Ef1

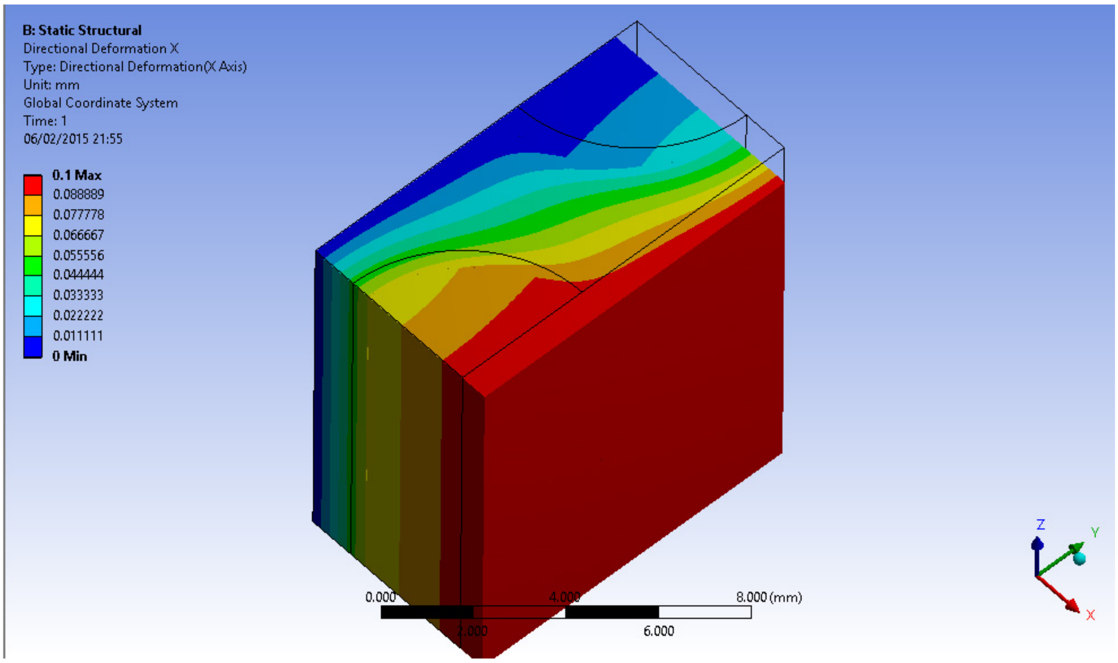


Figure 36: Displacement - X direction, Ef2 $\neq$ Ef1

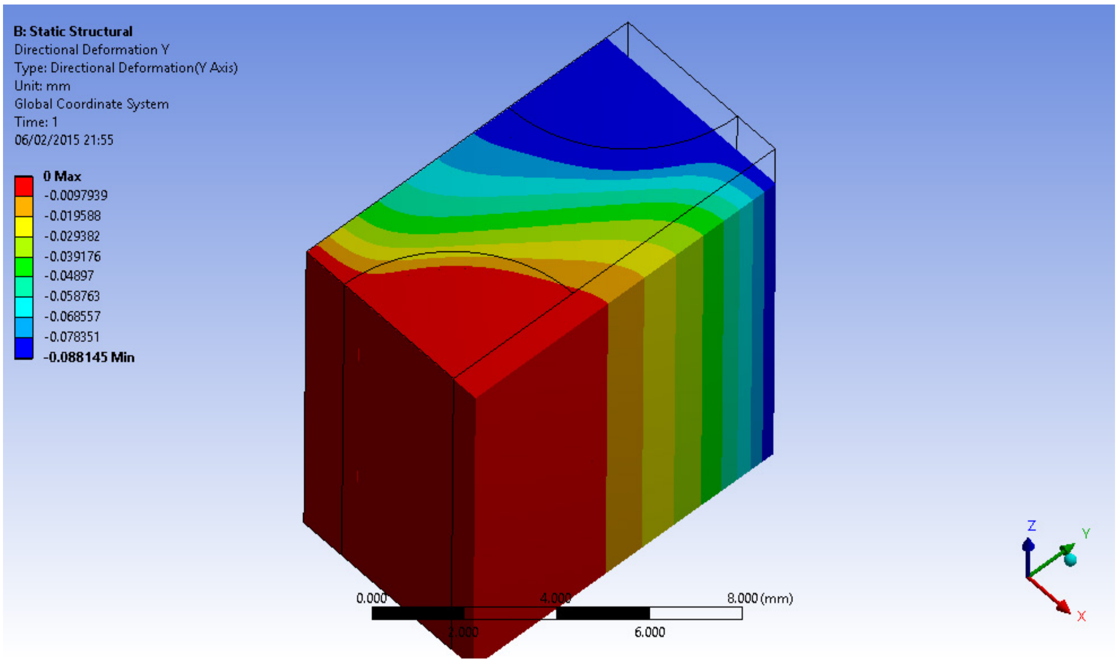


Figure 37: Displacement - Y direction, Ef2 $\neq$ Ef1

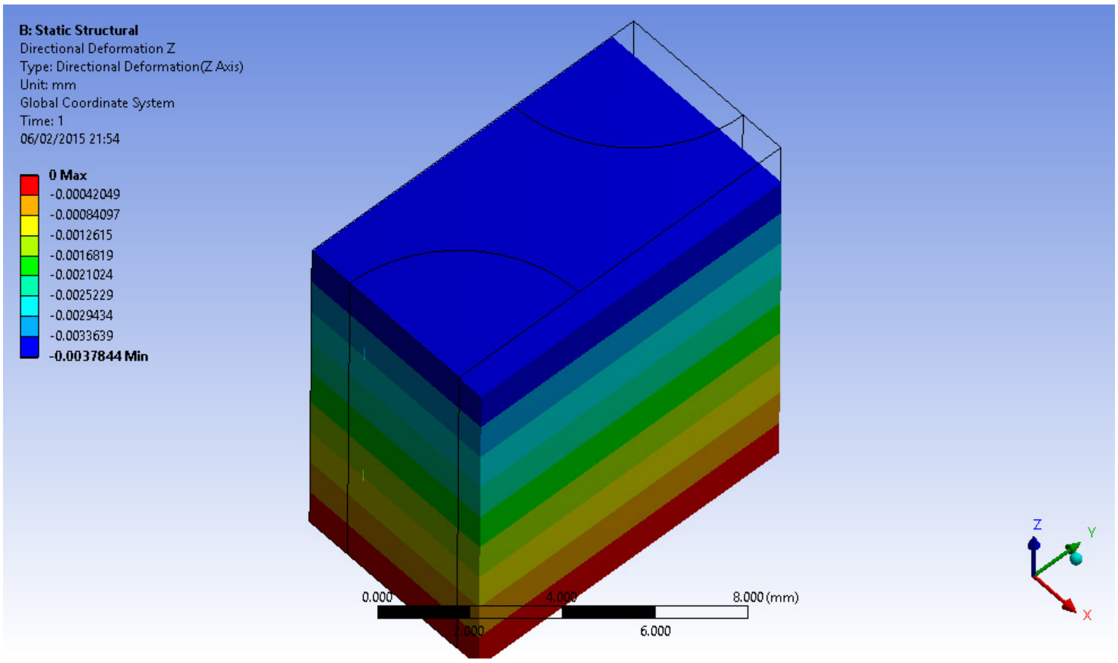


Figure 38: Displacement - Z direction, Ef2 $\neq$ Ef1

FEA - results

Values of obtained Youngs' modules are presented in a table. It is clear that in 1st direction (along fibers) the results are almost identical. However, in the 2nd direction the agreement is not so good. The micromechanical model is closer to the FEM results. Moreover, the literature says that it fits better to the experimental data than the mixing rule.

	FEM - square array	FEM - triangular array	mix rule
E1 [Pa]	1.17E+011	1.01E+011	1.17E+011

Case I: Ef1 = Ef2

	FEM - square array	FEM - triangular array	micromechanics	mix rule
E2 [Pa]	1.18E+010	8.80E+009	8.54E+009	6.40E+009
v12 [-]	0.2931	0.3028	0.3	
v21 [-]	0.0297	0.0258	0.0210	

Case II: Ef1 $\neq$ Ef2

	FEM - square array	FEM - triangular array	micromechanics	mix rule
E2 [Pa]	8.36E+009	6.98E+009	6.80E+009	5.70E+009
v12 [-]	0.2931	0.2810	0.3	
v21 [-]	0.0298	0.0189	0.0210	

In real word fibers are random packed, thus the FEM results should be concerned as an average between square and triangular array. To figure out a general trend between FEM and analytical models a study through the range of different volume ratios should be performed, which is beyond the scope of this study.

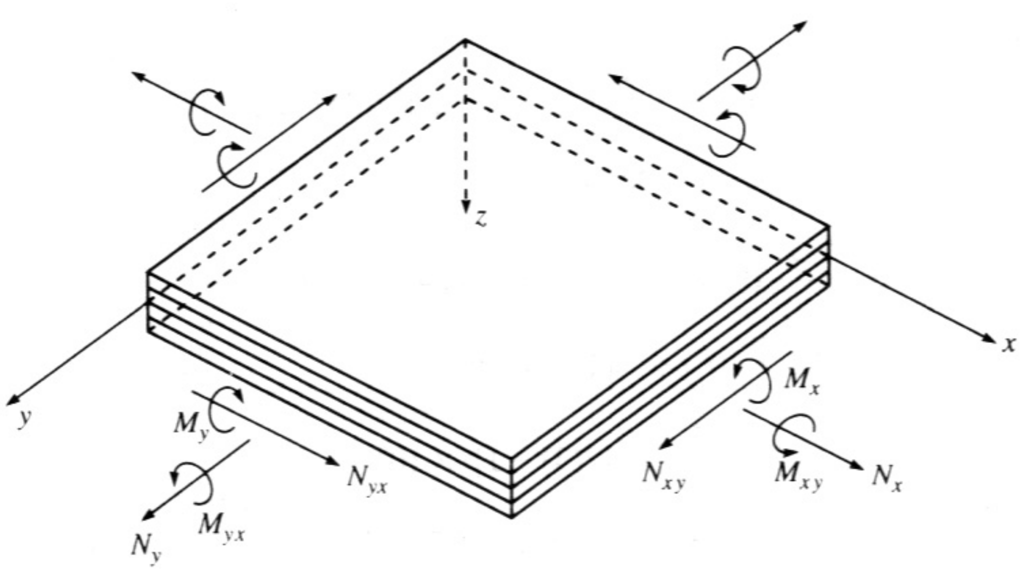
Microscale - conclusions

Different approaches used to find the effective moduli of a continuous fiber-reinforcement lamina shows excellent agreement in the 1st direction.
The result should be taken with care since the transverse fibre properties are rather a rough estimates. Moreover, in real composite additional factors like voids, matrix-fiber interface or other imperfections can influence the result.

Composite plate - Effective moduli

Classical Lamination Theory (CLT)

A quick remind of the CLT is given below:[5]



Coordinate system and stress resultants for laminated plate.

Figure 39: Laminated plate - coordinate system

The assumptions of the CLT:

[5] Ibid., p. 202.

1. The plate consists of orthotropic laminae bonded together, with the principal material axes of the orthotropic laminae oriented along arbitrary directions with respect to the xy axes.
2. The thickness of the plate, t, is much smaller than the lengths along the plate edges, a and b.
3. The displacements u, v, and w are small compared with the plate thickness
4. The in-plane strains $\varepsilon_x, \varepsilon_y$, and γ_{xy} are small compared with unity.
5. Transverse shear strains γ_{xz}, γ_{yz} are negligible.
6. Tangential displacements u and v are linear functions of the z coordinate.
7. The transverse normal strain ε_z is negligible
8. Each ply obeys Hooke's law.
9. The plate thickness is constant
10. Transverse shear stresses τ_{xz} and τ_{yz} vanish on the plate surfaces by $z = \pm t/2$

The general laminate force - deformation equations are expressed as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} & B_{21} & B_{22} & B_{23} \\ A_{31} & A_{32} & A_{33} & B_{31} & B_{32} & B_{33} \\ B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\ B_{21} & B_{22} & B_{23} & D_{21} & D_{22} & D_{23} \\ B_{31} & B_{32} & B_{33} & D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{8}$$

where:

N - force per unit length

M - moment per unit length

κ - curvature

γ - engineering shear strain: $\gamma_{xy}/2 = \varepsilon_{xy}$

The laminate extensional stifnesses are given by:

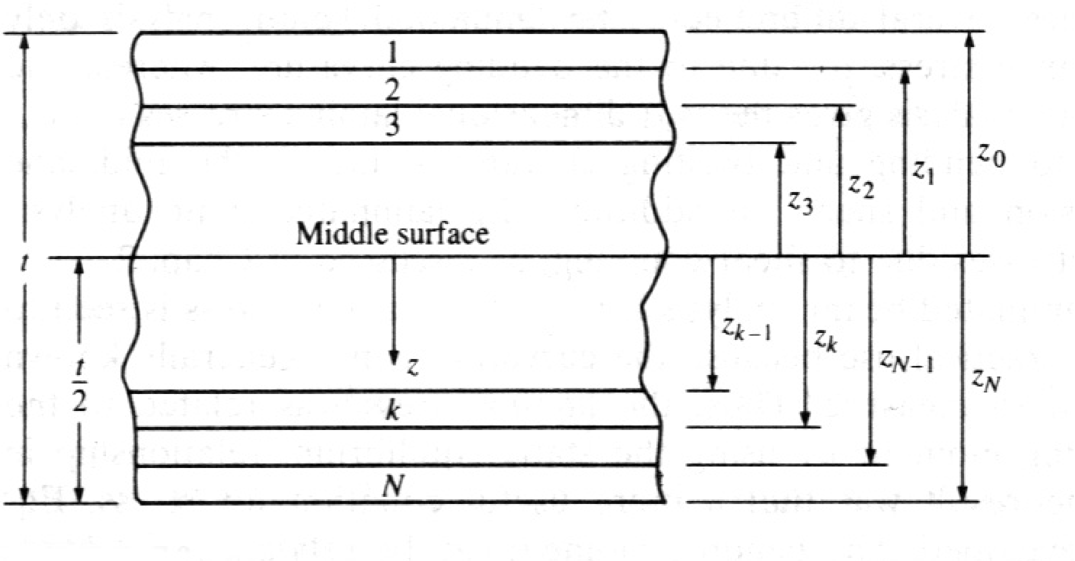
$$A_{ij} = \iint\limits_{-t/2}^{t/2} (\overline{Q}_{ij})_k \, \mathrm{d}z = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k - z_{k-1}) \tag{9}$$

The laminate coupling stifnesses are given by:

$$B_{ij} = \iint_{-t/2}^{t/2} (\overline{Q}_{ij})_k z \, \mathrm{d}z = \frac{1}{2} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \tag{10}$$

The laminate bending stifnesses are given by:

$$D_{ij} = \iint_{-t/2}^{t/2} (\overline{Q}_{ij})_k z^2 \, \mathrm{d}z = \frac{1}{3} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \tag{11}$$



Laminated plate geometry and ply numbering system.

Figure 40: Ply numbering system

The $\overline{Q}$ matrix is the transformed lamina stiffness matrix:

$$\overline{Q} = [T]^{-1}[Q][T] \tag{12}$$

Transformation matrix is defined as:

$$[T] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \tag{13}$$

where $c = cos\theta$ and $s = sin\theta$

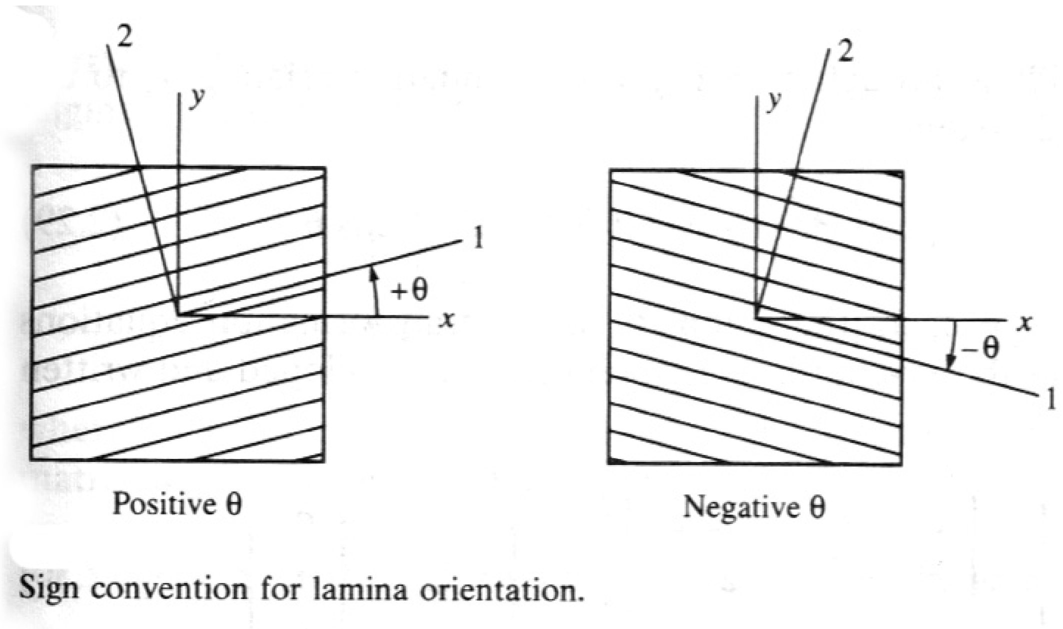


Figure 41: Sign convention

The stiffness matrix [Q] is an inverse of the compliance matrix [S]: $[Q] = [S]^{-1}$
The strain - stress relationship is expressed as $[\epsilon] = [S][\sigma]$:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{11} & S_{22} & 0 \\ 0 & 0 & S_{33} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \tag{14}$$

The compliance matrix [S] is defined as:

$$S_{11} = \frac{1}{E_1} \tag{15}$$

$$S_{22} = \frac{1}{E_2} \tag{16}$$

$$S_{12} = S_{21} = -\frac{v_{21}}{E_2} = -\frac{v_{12}}{E_1} \tag{17}$$

$$S_{33} = \frac{1}{G_{12}} \tag{18}$$

$$\tag{19}$$

In partitioned form as:

$$\left[\begin{array}{c} N \\ \hdashline M \end{array}\right] = \left[\begin{array}{c:c} A & B \\ \hdashline B & D \end{array}\right] \left[\begin{array}{c} \varepsilon^0 \\ \hdashline \kappa \end{array}\right] \tag{20}$$

The compliance matrix is an inverse of the stiffness matrix. It is used in a deformation - force equations:

$$\left[\begin{array}{c}\varepsilon^0 \\ \hdashline \kappa\end{array}\right] = \left[\begin{array}{c:c}A' & B' \\ \hdashline C' & D'\end{array}\right]^{-1} \left[\begin{array}{c}N \\ \hdashline M\end{array}\right] \tag{21}$$

The A'_{ij} is defined as a term from the compliance matrix which place corresponds to the one in the stiffness matrix A_{ij}
The detailed derivation may be found in any textbook concerning composites.[6]

Laminate Engineering Constants

The effective longitudinal Young's modulus of the laminate, E_x, governs the response of the laminate under the single axial load per unit length N_x with $N_y = N_{xy} = 0$ and is defined as:

$$E_x = \frac{\sigma^x}{\epsilon_x^0} = \frac{N_x/t}{A'_{11}N_x} = \frac{1}{tA'_{11}} \tag{22}$$

where:
t - thickness
E_y and G_{xy} can be calculated similarly

Case study

For the purpose of the study a ply with following parameters is analysed:

- $E1 = 138[GPa]$
- $E2 = 9[GPa]$
- $G12 = 6.9[GPa]$
- $v12 = 0.3[-]$
- $v21 = 0.0196[-]$
- $thickness = 0.00025[m]$
- $\varrho = 1500[kg/m^3]$
- $length = 1[m]$
- $width = 0.05[m]$

[6] Ibid., p. 206.

FEM Boundary conditions:

- load: $\sigma = 1.44[MPa]$
- support - line fixed DisplacementY and fixed RotationX
- support: vertex at CSYS - fixed DisplacementXYZ
- support: edge vertex - fixed DisplacementZ

Symmetrical layup: $[45/-45/-45/45]$

Classical Lamination Theory

$$[A] = \begin{bmatrix} 45.22 & 31.42 & 0 \\ 31.42 & 45.22 & 0 \\ 0 & 0 & 35.6 \end{bmatrix} GPa - mm$$

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} GPa - mm^2$$

$$[D] = \begin{bmatrix} 3.77 & 2.62 & 2.03 \\ 2.62 & 3.77 & 2.03 \\ 2.03 & 2.03 & 2.97 \end{bmatrix} GPa - mm^3$$

The symmetry condition causes $[B] = 0$. A_{13} and A_{23} do not have to vanish. According to CLT (Classical Lamination Theory), the Laminate Engineering Constants are:

$$E_x = 23.4[GPa]$$
$$E_y = 23.4[GPa]$$
$$G_{xy} = 35.6[GPa]$$

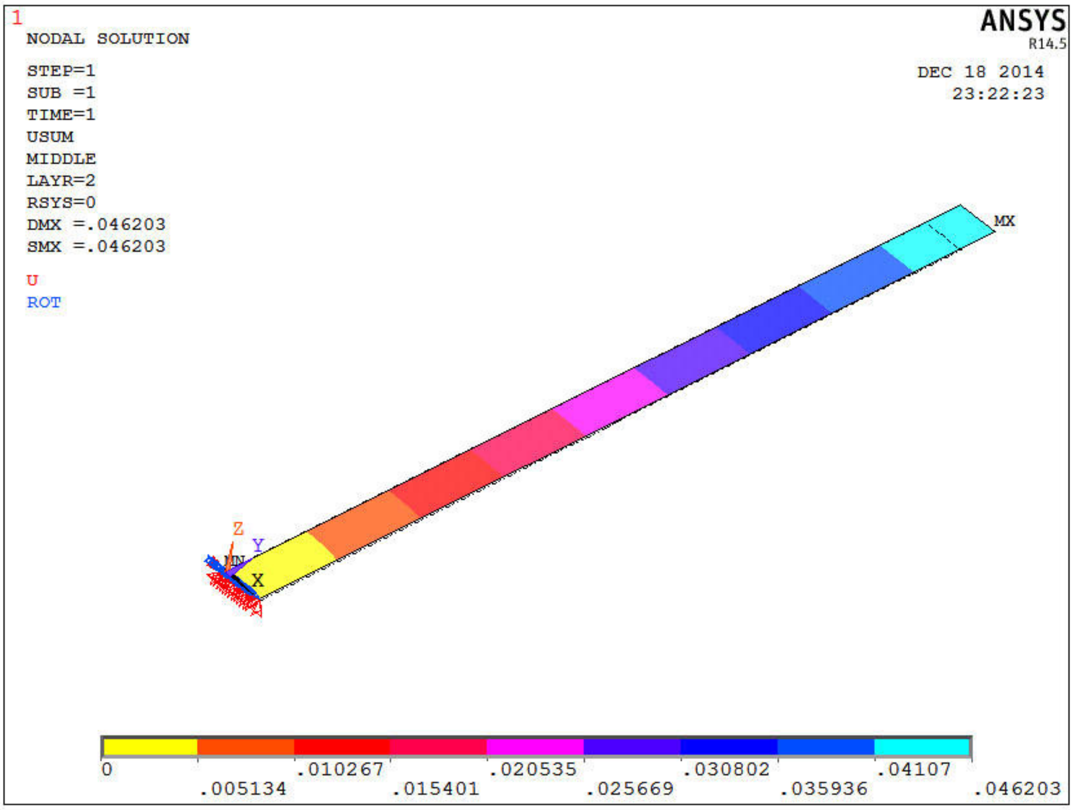


Figure 42: Symmetrical layup: $[45/-45/-45/45]$ - total displacement

Substituting FEM results to the formula for Young's modulus gives:

$$E = \frac{\sigma}{\epsilon} = 23.4[GPa]$$

Quasi-Isotropic layup: $[60/0/-60]$

In quasi - isotropic laminate the angle between adjacent plies must be π/N, where N is the total number of plies.

$$[A] = \begin{bmatrix} 44.68 & 12.80 & 0 \\ 12.80 & 44.68 & 0 \\ 0 & 0 & 15.94 \end{bmatrix} GPa - mm$$

$$[B] = \begin{bmatrix} 0 & 0 & -1.96 \\ 0 & 0 & -5.06 \\ -1.96 & -5.06 & 0 \end{bmatrix} GPa - mm^2$$

$$[D] = \begin{bmatrix} 0.856 & 0.824 & 0 \\ 2.62 & 3.77 & 0 \\ 0 & 0 & 0.972 \end{bmatrix} GPa - mm^3$$

In the quasi-isotropic laminas only the A_{ij} remain unchanged when the overall lamina orientation is altered (eg. [75/15/-45] or [30/-30/-90] - the angle between plies have to be maintained). The [B] and [D] matrices may change. In fact, the laminate is isotropic with respect to the in plane behaviour only.

The Laminate Engineering Constants are:

$$E_x = 53.8[GPa]$$
$$E_y = 24.1[GPa]$$
$$G_{xy} = 8.83[GPa]$$

To impose the quasi-isotropic behaviour the [B] matrix must be set to 0, then:

$$\overline{E} = 54.6[GPa]$$
$$\overline{G} = 2.13[GPa]$$
$$\overline{v} = 0.287$$

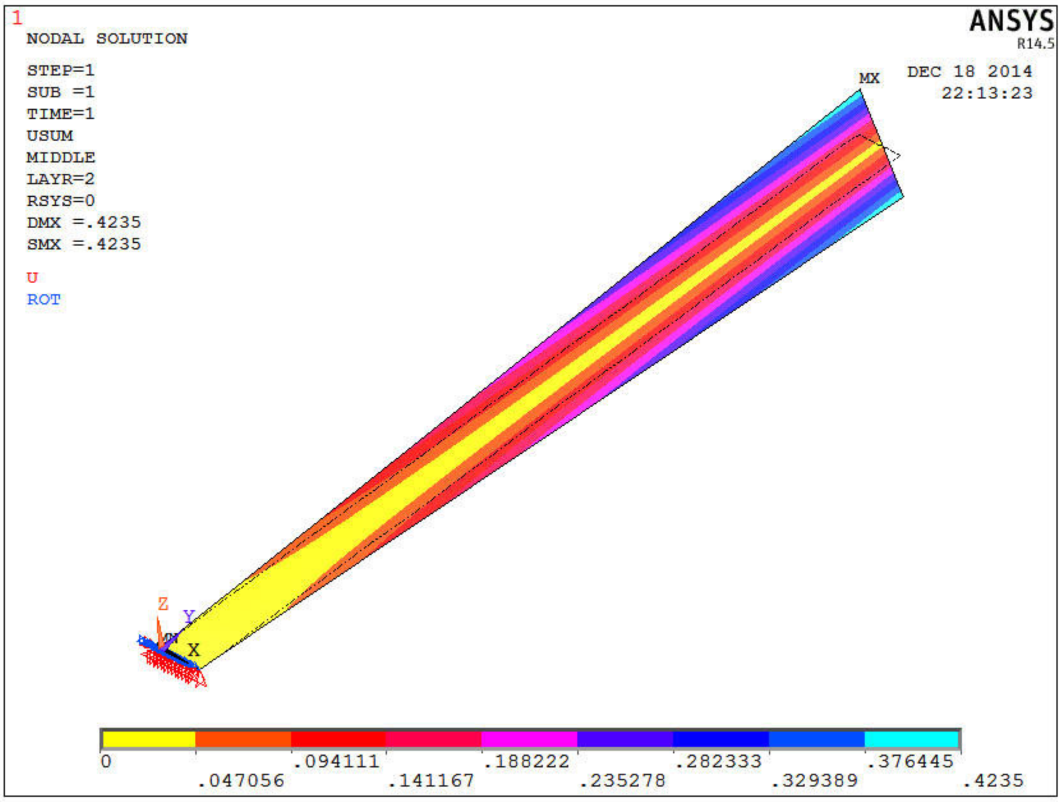


Figure 43: Quasi-Isotropic layup: $[60/0/-60]$ - total displacement

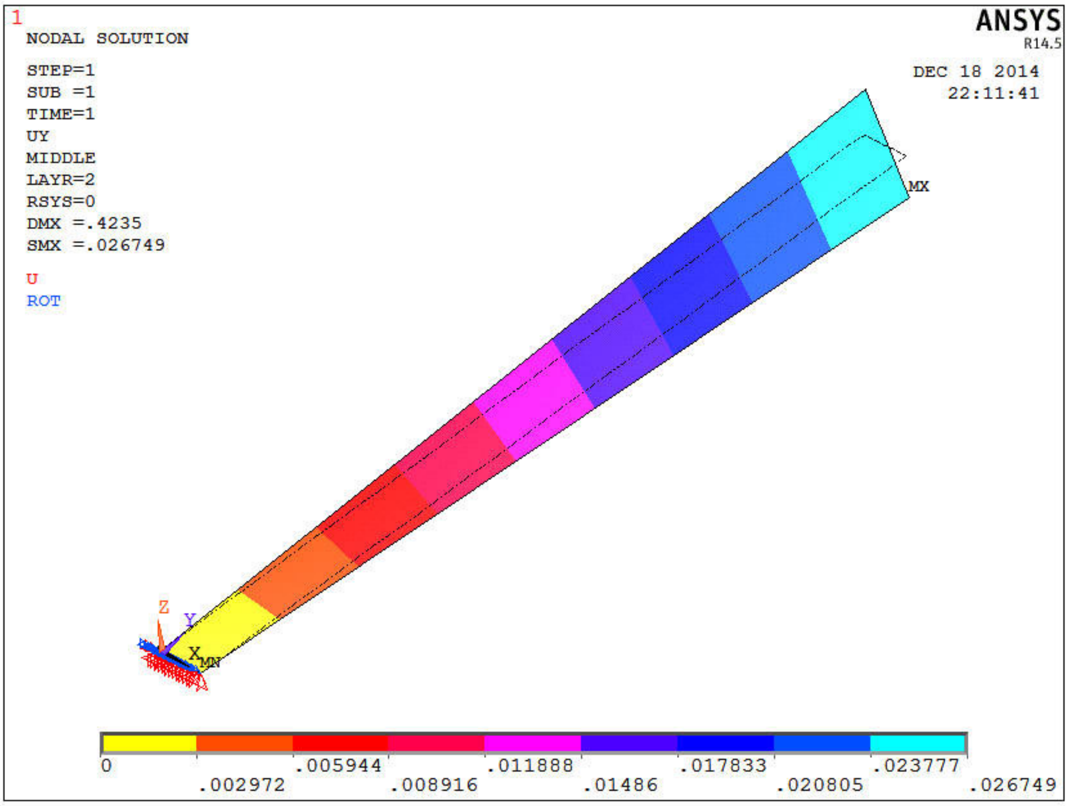


Figure 44: Quasi-Isotropic layup: $[60/0/-60]$ - Y displacement

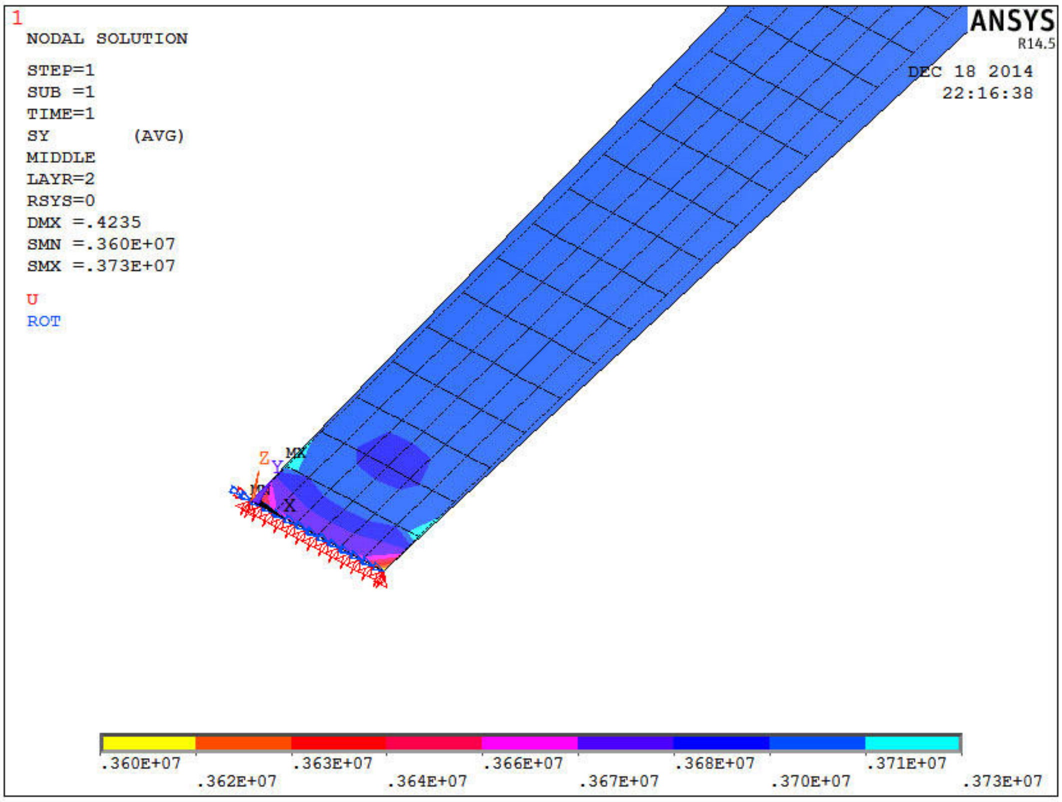


Figure 45: Quasi-Isotropic layup: $[60/0/-60]$ - Y Stress, middle layer: 0deg

Substituting FEM results to the formula for Young's modulus gives:

$$E = \frac{\sigma}{\epsilon} = 53.8[GPa]$$

For rotated layup $+90°$, which is $[30/90/-30]$

$$E = \frac{\sigma}{\epsilon} = 24.3[GPa]$$

Composite plate - conclusions

The results agrees with the Laminate Engineering Constants E_x and E_y . The influence of the [B] can not be neglected, which means that the quasi-isotropic $\overline{E}$ is not valid.

Composite with closed cross-section - Effective moduli

Software dedicated to the analytical analysis of composites (Autodesk Simulation Composite Design) says that the Laminate Engineering Constants can be used in closed cross-section parts if they are under simple (uniaxial) load. Such parts should behave as a symmetric laminates even if the layup is non symmetric. It can be mathematically reached be assigning zeros to selected terms from the stiffness matrix:

$$A_{13} = A_{23} = B_{ij} = 0 \tag{23}$$

Condition imposed in 23 is refereed as zero-coupling. This condition is very strong and it is to be compared with FEM. Now, the general laminate force - deformation equations looks as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{1,2} & 0 & & & \\ A_{21} & A_{22} & 0 & & 0 & \\ 0 & 0 & A_{33} & & & \\ & & & D_{11} & D_{12} & D_{13} \\ & 0 & & D_{21} & D_{22} & D_{23} \\ & & & D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{24}$$

Next, the modified stiffness matrix is inverted and used as before to get the E_x, E_y, G_{xy}

Effective Flexural Modulus

Knowing the properties of the composite with a closed cross-section it is tempting to estimate stiffness the effective flexural/torsional modulus of a whole beam. To do so it is useful to introduce E_f - effective flexural modulus (which is same as Young's modulus of beam material for a homogeneous isotropic beam).
The simplest approach uses the effective moduli of the composite with closed cross-section, which was derived in the previous chapter. Then, for example, the differential equation for the transverse deflection, w, of a laminated beam would be of the form:

$$\begin{gathered} E_f = E_x \\ M = E_f I \frac{d^2 w}{dx^2} \end{gathered} \tag{25}$$

where:

M - bending moment

I - moment of inertia of cross section about the neutral axis

More sophisticated approach includes the effect of layup order in the analysis. The assumption used in developing the analysis are as follows:

1. Plane sections which are initially normal to the longitudinal axis of the beam remain plane and normal during flexure. $\varepsilon_x = \varepsilon_0 + \kappa z$
2. The plies are perfectly bonded together, so that no slip occurs at ply interfaces

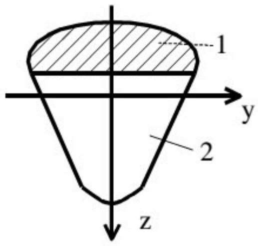


Figure 46: Beam composed from different materials

The equilibrium equation of a cross-section[7]

$$N = \iint_{A_i} \sigma_i \, \mathrm{d}A = \sum_i (E_i A_i \varepsilon_0 + E_i S_i \kappa) = \sum_i (E_i A_i)\varepsilon_0 + \sum_i (E_i S_i)\kappa \quad (26)$$

$$M = \iint_{A_i} \sigma_i z \, \mathrm{d}A = \iint_{A_i} E_i(\varepsilon_0 + z\kappa) z \, \mathrm{d}A = \sum_i (E_i \varepsilon_0 S_i + E_i I_i \kappa) = \sum_i (E_i S_i)\varepsilon_0 + \sum_i (E_i I_i)\kappa \quad (27)$$

The matter of interest are thin walled beams wrapped with laminate. For simplicity assume that the beam is symmetrical with respect to horizontal and vertical axis.
Term $\sum_i (E_i S_i) = 0$ with respect to Neutral Axis. In the coordinate system related to the NA, the tension and bending can be separated:

$$N = \sum_i (E_i A_i)\varepsilon_0 \quad (28)$$

$$M = \sum_i (E_i I_i)\kappa \quad (29)$$

Focus on 29

[7] Adam Zaborski. *Belki Zlozone i Zespolone*.

Assume $\kappa \simeq w''$

$$w'' = \frac{M}{\sum_i E_i I_i} = \frac{M}{E_f I_f} \tag{30}$$

where

$$\begin{aligned} I_f &= \sum_i I_i \\ E_f &= \frac{\sum_i E_i I_i}{I_f} \end{aligned} \tag{31}$$

Stress in the j-th layer:

$$(\sigma_x)_j = \frac{Mz}{I_f} \frac{(E_x)_j}{E_f} \tag{32}$$

Case study

The properties of the ply remain unchanged. The pipe is 2m length and has diamater d = 0.05 m.

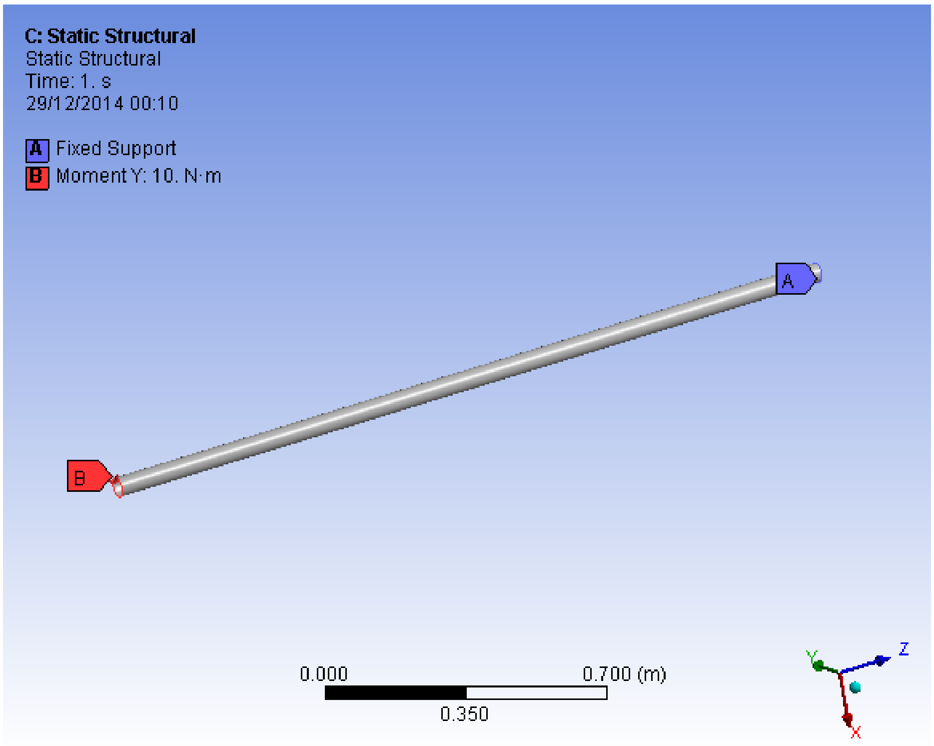


Figure 47: Pipe - boundary conditions

FEM Boundary conditions:

- support - fixed
- load:
  tensile force (along axis) Ft = 1000N or
  bending force (perpendicular to axis) Fb = 10N or
  bebning moment Mb = 10 Nm

Symmetrical layup: $[45/-45]_s$

Quasi-Isotropic layup: $[60/0/-60]$

FEA - results

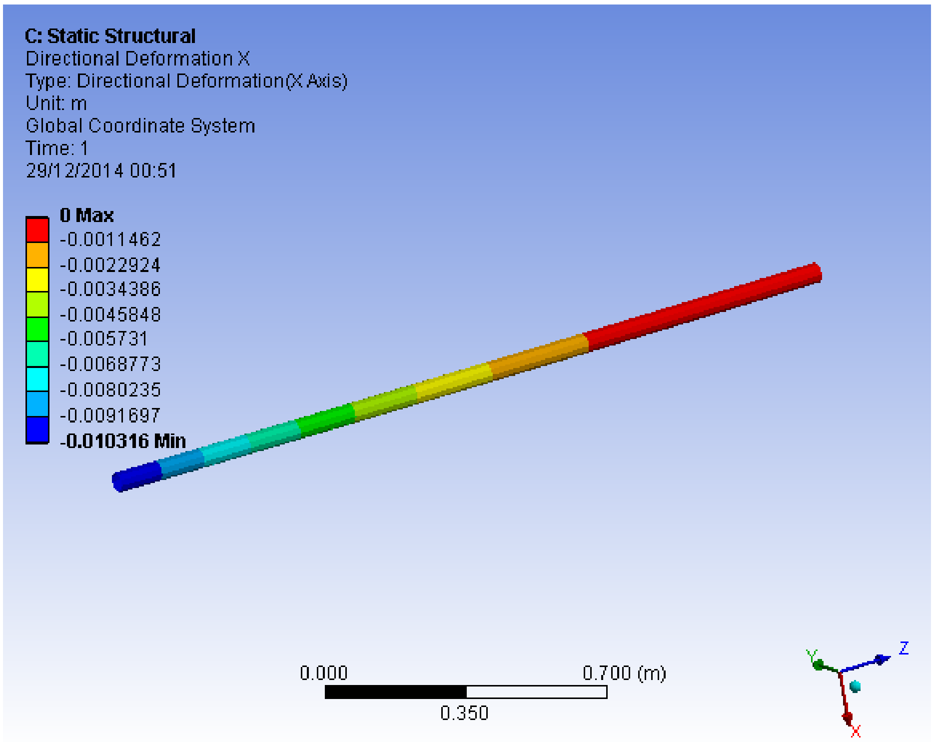


Figure 48: Pipe - X deflection

To calculate the Young's modulus displacement of the pipe is compared with well know handbook formulas for loading of a cantilever beam. Values of obtained Youngs' moduli are presented in tables.

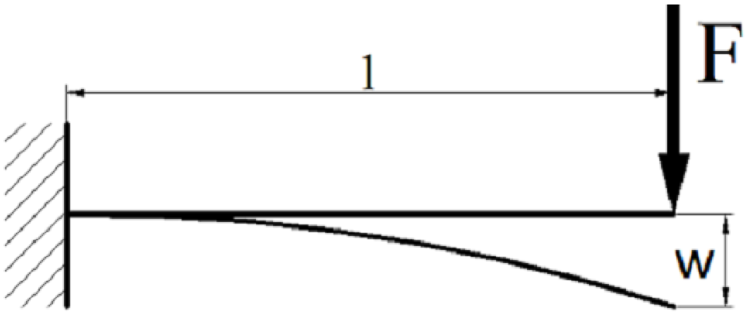


Figure 49: Cantilever beam - handbook's example: $w = \dfrac{Fl^3}{3EI}$

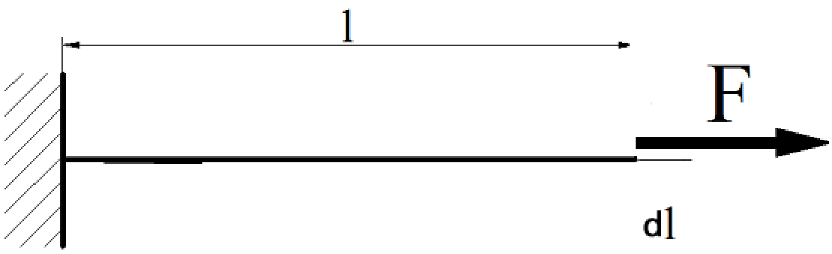


Figure 50: Cantilever beam - handbook's example: $dl = \dfrac{Fl}{AE}$

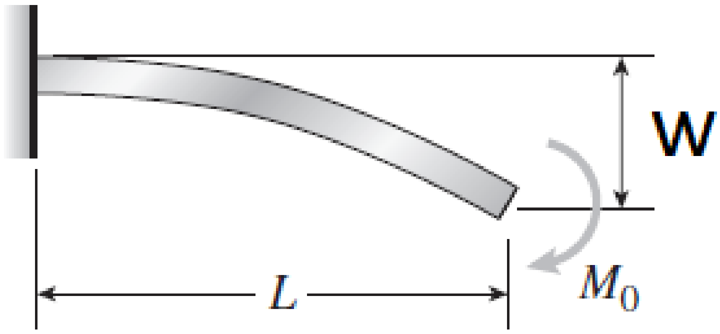


Figure 51: Cantilever beam - handbook's example: $w = \frac{Ml^2}{2EI}$

where:

- Ft - beam loaded with a tensile force as in figure 49
- Ft - beam loaded with a shearing force as in figure 50
- Mb - beam loaded with a bending moment as in figure 51
- E - Effective longitudinal Young's modulus of the laminate in the subscripted direction (in pipe's csys: x - axially or y - radially) as defined in section Laminate Engineering Constants 4.2
- $\overline{E}$ - Effective longitudinal Young's modulus with imposed 'zero-coupling' condition
- E_f - Effective Flexural Modulus

Steel - benchmark

	FEM Ft	FEM Fb	FEM Mb	Analytical
Ex [Pa]	1.95E+011			2.00E+011
Ey [Pa]				2.00E+011
$\overline{Ex}$ [Pa]				2.00E+011
$\overline{Ey}$ [Pa]				2.00E+011
E_f [Pa]		1.95E+011	1.95E+011	2.00E+011

Layup: [0/0/0/0]

	FEM Ft	FEM Fb	FEM Mb	Analytical
Ex [Pa]	1.34E+011			1.38E+011
Ey [Pa]				9.00E+009
$\overline{Ex}$ [Pa]				1.38E+011
$\overline{Ey}$ [Pa]				9.00E+009
E_f [Pa]		1.33E+011	1.35E+011	1.38E+011

Layup: $[45/-45]_s$ **- symmetric laminate**

	FEM Ft	FEM Fb	FEM Mb	Analytical
Ex [Pa]	2.29E+010			2.34E+010
Ey [Pa]				2.34E+010
$\overline{Ex}$ [Pa]				2.34E+010
$\overline{Ey}$ [Pa]				2.34E+010
E_f [Pa]		2.20E+010	2.29E+010	2.30E+010

Layup: $[60/0/-60]$ **- quasi isotropic laminate**

	FEM Ft	FEM Fb	FEM Mb	Analytical
Ex [Pa]	5.32E+010			5.38E+010
Ey [Pa]				2.41E+010
$\overline{Ex}$ [Pa]				5.46E+010
$\overline{Ey}$ [Pa]				5.46E+010
E_f [Pa]		5.36E+010	5.37E+010	5.47E+010

Layup: $[30/90/-30]$ **- quasi isotropic laminate**

	FEM Ft	FEM Fb	FEM Mb	Analytical
Ex [Pa]	4.48+010			2.41E+010
Ey [Pa]				5.38E+010
$\overline{Ex}$ [Pa]				5.46E+010
$\overline{Ey}$ [Pa]				5.46E+010
E_f [Pa]		4.96E+010	4.97E+010	3.99E+010

Layup: $[60/0/30]$ **- 'random' laminate (without any special properties)**

	FEM Ft	FEM Fb	FEM Mb	Analytical
Ex [Pa]	6.27E+010			5.86E+010
Ey [Pa]				1.53E+010
$\overline{Ex}$ [Pa]				7.35E+010
$\overline{Ey}$ [Pa]				3.44E+010
E_f [Pa]		6.35E+010	6.38E+010	6.86E+010

Composite with closed cross-section - conclusions

There is a good agreement between the analytical data and FEM. However, the E obtained from FEM can be slightly different than expected.

Summary

The work shows that the analytical theories are comparable with FEM.
The effective moduli based on the analytical micro models is a good guide for the estimation of the ply Young's modulus in the longitudinal and transverse direction. There also exist a bunch of semi-empirical models which are not covered in this report. In engineering applications the load should be carried along fibers, thus the transverse properties are not of major interest. Since the transverse Young's modulus of the lamina is of the order of magnitude lower than the longitudinal modulus the influence of the carbon fiber anisothropy can be neglected. In other words, the assumption of the isotrophy of the carbon fiber can be justified from the engineering point of view.
Section covering the Classical Lamination Theory indicates that the laminae coupling stiffness terms B_{ij} play a significant role in the mechanics of a plane plate. Its role can not be neglected, the best proof is provided by the 'quasi isotropic' laminates like [60/ 0 / -60] which would be equivalent to the [30/90/-30] under the assumption of zero-coupling. The last part describes the closed section profiles where the zero-coupling is imposed regardless to the ply orientation. The closed shape enforces the laminae to behave in a more rigid way. In engineering application, when the layup is designed to carry the load along the fibers, i.e., bending in case of a beam/mast or torsion in a shaft such assumption is reasonable and gives accurate results. However, there exists 'random' layups where the effect of coupling is more exposed.
To sum up the above, the analytical models can provide a quick estimation of the engineering properties of the laminate structures.

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Manufacturer data-sheet

TECHNICAL DATA SHEET
No. CFA-005

TORAYCA®

T700S DATA SHEET

Highest strength, standard modulus fiber available with excellent processing characteristics for filament winding and prepreg. This never twisted fiber is used in high tensile applications like pressure vessels, recreational, and industrial.

FIBER PROPERTIES

		English	Metric	Test Method
Tensile Strength		711 ksi	4,900 MPa	TY-030B-01
Tensile Modulus		33.4 Msi	230 GPa	TY-030B-01
Strain		2.1 %	2.1 %	TY-030B-01
Density		0.065 lbs/in^3	1.80 g/cm^3	TY-030B-02
Filament Diameter		2.8E-04 in.	7 µm	
Yield	6K	3,724 ft/lbs	400 g/1000m	TY-030B-03
	12K	1,862 ft/lbs	800 g/1000m	TY-030B-03
	24K	903 ft/lbs	1,650 g/1000m	TY-030B-03

Sizing Type	50C	1.0 %	TY-030B-05
& Amount	60E	0.3 %	TY-030B-05
	F0E	0.7 %	TY-030B-05
	Twist	Never twisted	

FUNCTIONAL PROPERTIES

CTE	-0.38 $\alpha \cdot 10^{-6}$/°C
Specific Heat	0.18 Cal/g·°C
Thermal Conductivity	0.0224 Cal/cm·s·°C
Electric Resistivity	1.6 x 10^{-3} Ω·cm
Chemical Composition: Carbon	93 %
Na + K	<50 ppm

COMPOSITE PROPERTIES*

	English	Metric	Test Method
Tensile Strength	370 ksi	2,550 MPa	ASTM D-3039
Tensile Modulus	20.0 Msi	135 GPa	ASTM D-3039
Tensile Strain	1.7 %	1.7 %	ASTM D-3039
Compressive Strength	215 ksi	1,470 MPa	ASTM D-695
Flexural Strength	245 ksi	1,670 MPa	ASTM D-790
Flexural Modulus	17.5 Msi	120 GPa	ASTM D-790
ILSS	13 ksi	9 kgf/mm^2	ASTM D-2344
90° Tensile Strength	10.0 ksi	69 MPa	ASTM D-3039

* Toray 250°F Epoxy Resin. Normalized to 60% fiber volume.

T700S

C O M P O S I T E   P R O P E R T I E S **

Tensile Strength	355 ksi	2,450 MPa	ASTM D-3039
Tensile Modulus	18.0 Msi	125 GPa	ASTM D-3039
Tensile Strain	1.7 %	1.7 %	ASTM D-3039
Compressive Strength	230 ksi	1,570 MPa	ASTM D-695
Compressive Modulus	--- Msi	--- GPa	ASTM D-695
In-Plane Shear Strength	14 ksi	98 MPa	ASTM D-3518
ILSS	15.5 ksi	11 kgf/mm^2	ASTM D-2344
90° Tensile Strength	10.0 ksi	70 MPa	ASTM D-3039

** Toray Semi-Toughened 350°F Epoxy Resin. Normalized to 60% fiber volume.

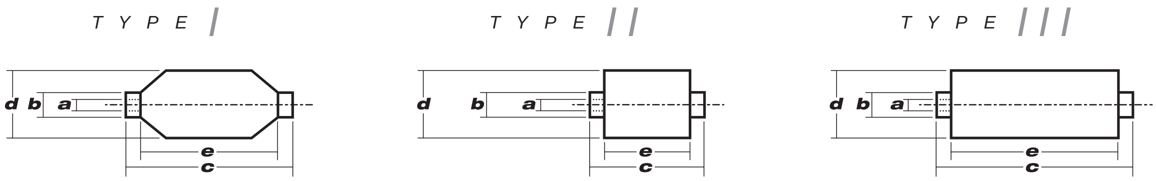
See Section 4 for Safety & Handling information. The above properties do not constitute any warranty or guarantee of values. These values are for material selection purposes only. For applications requiring guaranteed values, contact our sales and technical team to establish a material specification document.

P A C K A G I N G

The table below summarizes the tow sizes, twists, sizing types, and packaging available for standard material. Other bobbin sizes may be available on a limited basis.

Tow Sizes	Twist[1]	Sizing	Bobbin Net Weight (kg)	Bobbin Type[2]	Bobbin Size (mm) a	b	c	d	e	Spools per Case	Case Net Weight (kg)
6K	C	50C	2.0	III	76.5	82.5	280	140	252	12	24
12K	C	50C	6.0	III	76.5	82.5	280	200	252	4	24
	C	60E	6.0	III	76.5	82.5	280	200	252	4	24
	C	F0E	6.0	III	76.5	82.5	280	200	252	4	24
24K	C	50C	6.0	III	76.5	82.5	280	200	252	4	24
	C	60E	6.0	III	76.5	82.5	280	200	252	4	24
	C	F0E	6.0	III	76.5	82.5	280	200	252	4	24

1 **Twist** A: Twisted yarn B: Untwisted yarn made from a twisted yarn through an untwisting process C: Never twisted yarn
2 **Bobbin Type** See Diagram below



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