Received: 14 May 2018 | Revised: 18 June 2018 | Accepted: 27 June 2018
DOI: 10.1111/rsp3.12141

ORIGINAL ARTICLE

An urban metric system based on space-economy: Foundations and implementation

Luc-Normand Tellier[1] | Jérémy Gelb[2]

[1] Département d'études urbaines et touristiques, Université du Québec à Montréal, Montreal, Quebec, Canada
[2] Centre Urbanisation Culture Société (UCS), Institut National de la Recherche Scientifique (INRS), Montreal, Quebec, Canada

Correspondence
Luc-Normand Tellier, Département d'études urbaines et touristiques, Université du Québec à Montréal, Case postale 8888, succursale Centre-ville, Montreal, Quebec, Canada H3C 3P8.
Email: tellier.luc-n@uqam.ca

JEL Classification: R1; R2; O21; O18

Abstract
An urban metric system (UMS) is here implemented in order to henceforth solve the numerous problems stemming from the absence of a mathematical definition of the boundaries of the various concepts of urban areas. It is based on space-economy, especially, land-rent theory, the concepts of attractive and repulsive forces, as well as vector fields. The system is applied to the central cities, agglomerations and metropolitan areas of Montreal, Toronto and Quebec City. Based on the proposed UMS, a synthetic index of urban sprawl is defined and computed for the studied metropolitan areas.

KEYWORDS
attractive and repulsive forces, urban areas, urban limits, urban metric system, urban sprawl, vector field analysis

1 | INTRODUCTION

"The delimitation of urban areas has been a challenging problem for a long time, and there are no standard methods to solve it to date" (Goerlich Gisbert, Cantarino Martí, & Gielen, 2017, p. 690). Liang, Li, and Mao (2010) have reviewed the methods of delimitation for the spatial scope of urban settlements, and they came to that same conclusion. Presently, the study of the evolution of urban sprawl resorts to a multiplicity of indicators including the following: (i) median single family dwellerial lot size; (ii) housing density; (iii) median number of rooms; (iv) population density; (v) average household size; (vi) mean distance to commercial zone; (vii) mean distance to public park; (viii) mean distance to K-12 schools; (ix) mean distance to transit stops; (x) ratio streets to intersections; (xi) median perimeter of dwellerial blocks; (xii) ratio cul-de-sac to streets; (xiii) median length of cul-de-sacs; (xiv) land use contiguity; (xv) land use richness; (xvi) land use diversity; (xvii) population working outside city of residence; (xviii) renter-owner balance, etc. (see Lowry & Lowry, 2014). Measuring urban sprawl by means of a single mathematical tool still represents an unreachable dream for most regional scientists.

© 2018 The Author(s). Regional Science Policy and Practice © 2018 RSAI

This paper presents a method to delimitate urban areas, and proposes a hierarchy of such areas in the framework of an urban metric system (UMS). That proposal stems from Tellier (2014, forthcoming), but various tests led to improve the initial proposal. This article presents some results of those tests, and the most recent version of UMS. The method is applied to three morphologically different Canadian cities: Montreal, Quebec City and Toronto. Montreal is an island surrounded by rivers. Quebec City is a river town located on the right bank of the St. Lawrence with extensions on the left bank. Toronto is a coastal city located on Lake Ontario. Quebec City, Montreal, and Toronto have successively dominated the Canadian economy, but their present populations and urban morphologies significantly differ. They have been chosen because they were better known by the authors, which mattered in order for them to assess the realism of the obtained boundaries.

The main objective of the UMS is to propose a method that:

- has theoretical foundations in space-economy and location theory;
- delimits urban areas in a standard clear-cut non-fractal way;
- offers operational and hierarchical definitions of cities, agglomerations, metropolitan areas, megacities, megalopolises, urban systems, urban macro-systems, continental systems, intercontinental systems and global systems;
- can be applied to past as well as present and future urban areas;
- allows someone to compare urban areas of very dissimilar cities as well as cities of different countries and continents;
- uses easily available data about few variables (like population and employment) for which historical statistics exist (this excludes data stemming from recent sophisticated technologies based on satellite photographs); and
- enables to measure mathematically the progress of urban sprawl through time, even in contexts where the built environment remains unchanged while the number of inhabitants per household varies through time and space (the urbanization of rural lands is just one aspect of urban sprawl).

To reach such an ambitious objective, the proposed approach resorts to space-economy, location theory, land-rent theory, urban systems theory, vector field analysis, and the concepts of attractive and repulsive locational forces. A presentation of its theoretical foundations will be followed by a description of the urban metric system, and its application to the cases of Montreal, Toronto, and Quebec City.

2 | THE THEORETICAL FOUNDATIONS OF THE URBAN METRIC SYSTEM

2.1 | From von Thünen to urban densities

Tellier (1985) introduced the concept of repulsive force in location theory by formulating the attraction-repulsion problem, which generalizes both the Fermat and Weber problems.[1] The introduction of repulsive forces led to reinterpreting most of the space-economic theory. For instance, the whole theory of land rent can be summed up by the rule according to which each economic agent, whether consumer or producer, locates at the equilibrium point between the "attractive force that is independent from land values" and the "repulsive force linked to land values"

[1]The Fermat problem, formulated by Pierre de Fermat before 1640, looks for the minimum of the sum of the distances between a given point and various attraction points exerting the same attractive force. The problem studied by Weber (1909/1929) consists in finding the optimal location of an activity with respect to various points exerting different attractive forces; it is generally presented as a problem of minimization of the total transportation cost between an activity to be located and its clients and providers located at various attraction points. The attraction-repulsion problem refers to the finding of the optimal location of an activity on which are exerted both attractive and repulsive forces stemming from various reference points.

exerted by a single reference point that is the market in von Thünen's (1826) approach or the central business district in the model of Alonso (1964).

The von Thünen (1826) model can be applied to the form of the city if one considers various competing land-uses corresponding to different “population + workers” densities under the assumption that transportation costs are proportional to the Euclidean distances to the center of the city, which is at the origin. Let us take a simple example involving four types of land-uses:

1. office buildings areas with a density of 400 up to 500 dwellers plus workers per hectare;
2. apartment buildings areas with a density of 200 up to 400 dwellers plus workers per hectare;
3. townhouses areas with a density of 100 up to 200 dwellers plus workers per hectare; and
4. single-family homes areas with a density of 1 up to 100 dwellers plus workers per hectare.

Figure 1 illustrates the bid-functions corresponding to this example. The value of the intercept P of the office building-dominated areas bid-function PP′ for buying one hectare is the highest because there are 400–500 persons/workers to pay, whereas the second highest value of the intercept is the one of the apartment buildings-dominated areas bid-function TT′ since that land-use involves just 200–400 persons/workers per hectare, the third and fourth values of the intercept being those of the townhouse-dominated areas and single-family home-dominated areas for similar reasons. In the von Thünen model, the slope of the four bid-functions reflects the transportation costs of the people living or working on one hectare in each case. The higher the total cost, the steeper the slope of the bid-function. Of course, the total transportation cost of moving the 400–500 persons living/working on one hectare in an office building-dominated area is higher than that of moving the 200–400 persons of an hectare in an apartment building-dominated area, which is higher than that of moving the 100–200 persons of an hectare in a townhouse-dominated area that is higher than that of moving the 1–100 persons of an hectare in a single-family home-dominated area. Every hectare being bought by the highest bidder, in the end, if there is a fierce competition between a large number of bidders for all four land-uses, from the origin to distance A′, office building-dominated land-uses and a density of 400–500 dwellers/workers per hectare will prevail; from distance A′ to distance B′, apartment buildings-

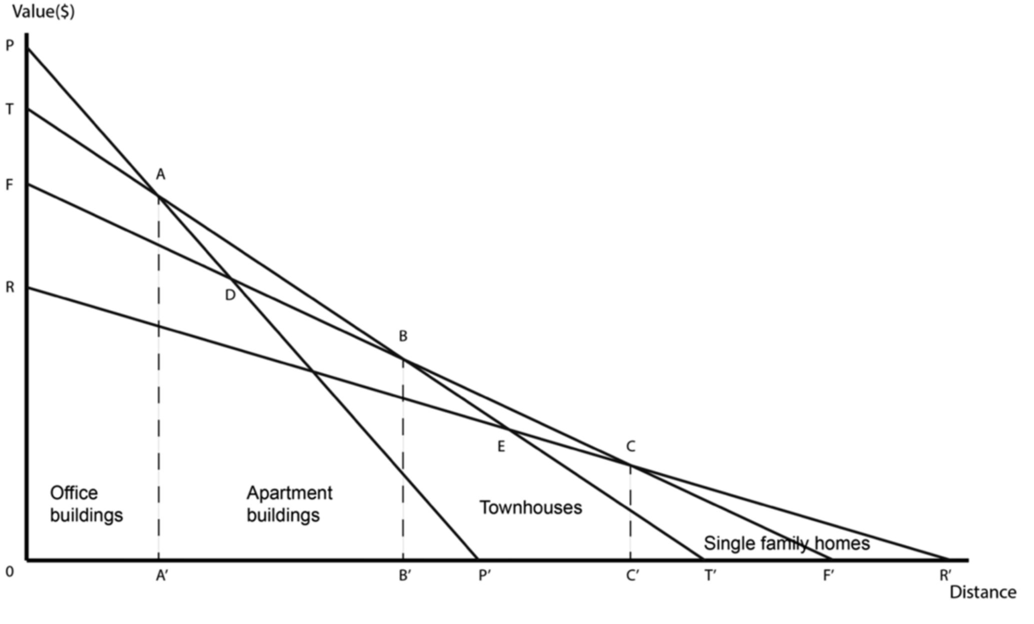

FIGURE 1 Von Thünen's bid-curves applied to the urban land-uses

dominated land-uses win with a density of 200–400 dwellers/workers per hectare; from distance B′ to distance C′, townhouse-dominated land-uses supersede with a density of 100–200 dwellers/workers per hectare, while single-family home-dominated land-uses with a density of 1–100 dwellers/workers per hectare are left with the territory lying between distances C′ and R′.

In the end, the land price curve corresponds to the envelop curve PABCR′, and the dwellers/workers density declines with the distance to the central business district. In summary, each land-use type ends up being at a distance from the central business district where the attractive force exerted on it by the central business district (which corresponds to the slope of its bid-curve multiplied by −1) is cancelled by the repulsive force exerted on it by the central business district through the land prices (that force being equal to the slope of the envelop curve PABCR′).

Each individual living in each land-use zone has the same desire to locate at the central business district, but, since everybody cannot locate there, each dweller is forced to consider other locations at which he will incur different transportation costs in order to travel to the central business district. It is his marginal transportation cost as well as that of his co-dwellers in the apartment building (taking that example) that determine the slope of the apartment building bid-curve, which influences both the land prices and the location of apartment buildings. Ultimately, the resulting land prices exert a repulsive force on all dwellers since they are decreasing as the distance from the central business district increases. In other words, from the land-price point of view, the central business district exerts a repulsion force on the individuals. The urban areas are then the product of the interaction of both attractive and repulsive forces, and, in the end, everybody settles at a location where the resultant of the attractive and repulsive forces exerted on him by the center is equal to zero.

2.2 | From urban densities to attractive and repulsive forces

There is an alternative way to look at that same story. It consists in translating Figure 1 into Figure 2 where the vertical axis refers to dwellers/workers densities instead of land prices. In Figure 2, densities are assumed to diminish in each land-use type. Office building-dominated land-uses vary from a density of 500 down to 400 at distance A′; apartment buildings-dominated land-uses prevail from a density of 400 down to 200 at distance B′; townhouse-dominated land-uses, from 200 down to 100 at distance C′; and single-family home-dominated land-uses, from 100 down to 0.

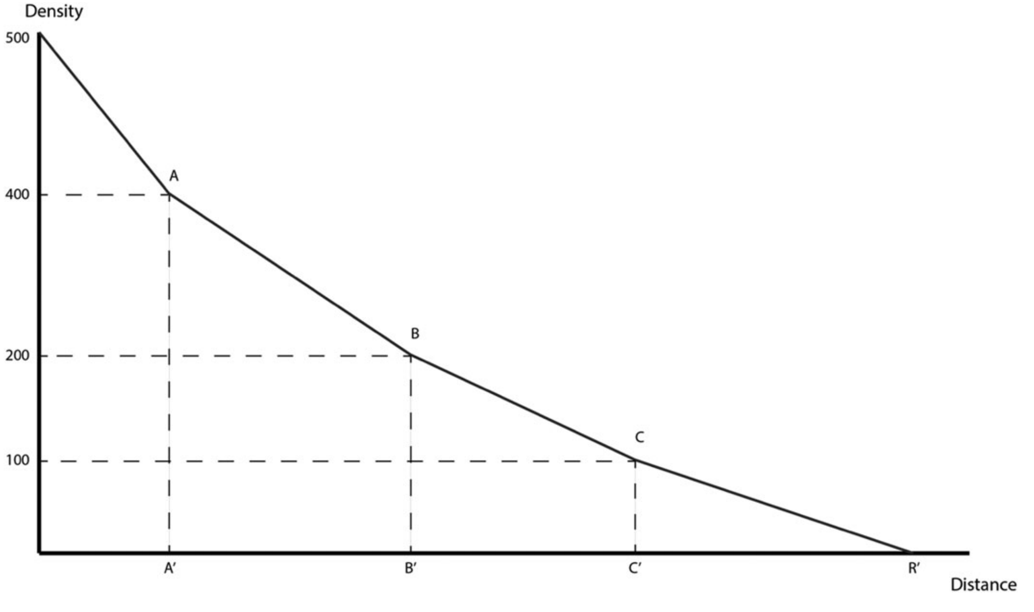

FIGURE 2 The distribution of densities and urban land-uses from the central business district

In Figure 2, each land-use type ends up being at a distance from the central business district at which the attractive force exerted on it by the central business district (which corresponds to the slope of its bid-curve multiplied by −1) is cancelled by the repulsive force exerted on it by the central business district through the competition between land-uses of various densities (that force being equal to the slope of the envelop curve PABCR′). That repulsive force stems from the fact that higher densities exert an outward pressure on lower densities that are pushed away to the periphery due to the fact that higher densities compete with them.

The limits A′, B′, and C′ between the four density types correspond to the distances where the attractive force exerted by the central business district on the higher-density land-uses is cancelled by the repulsive force exerted, through competition, by the centre on the lower-density land-uses. Conversely, any given distance x can be seen as separating the zone where the attractive force exerted by the centre on agglomerating activities looking for densities that are higher than the one that prevails at distance x is cancelled by the repulsive force exerted by competition on the disseminating activities looking for densities that are lower than the one that prevails at distance x. The method for delimiting urban areas presented in this paper stems directly from that observation.

2.3 | From an exclusive centre to reality

Now let us ask ourselves what gives the central business district those attractive and repulsive characteristics we just considered as being given. The usual assumption according to which everybody works, buys and sells at the same single point located at the origin is a vision of the mind. Actually, production and consumption activities are spread all over urban areas. They are more numerous in certain districts than in others. Assuming that this is the case makes everything complicated in location theory, which explains why the usual assumption prevails. However, attempting to propose a mathematical definition of the various types of urban territorial concepts (central city, inner suburbs, agglomeration, outer suburbs, metropolitan area, megacity, etc.) forces us to face reality. This is why, ideally, our method uses both dwellers and workers data (as will be the case here for Montreal and Toronto).

The solution we adopted was to consider that each dweller or worker exerts a minimal attractive influence on "agglomerating activities" looking for higher densities, as well as a minimal repulsive influence on "disseminating

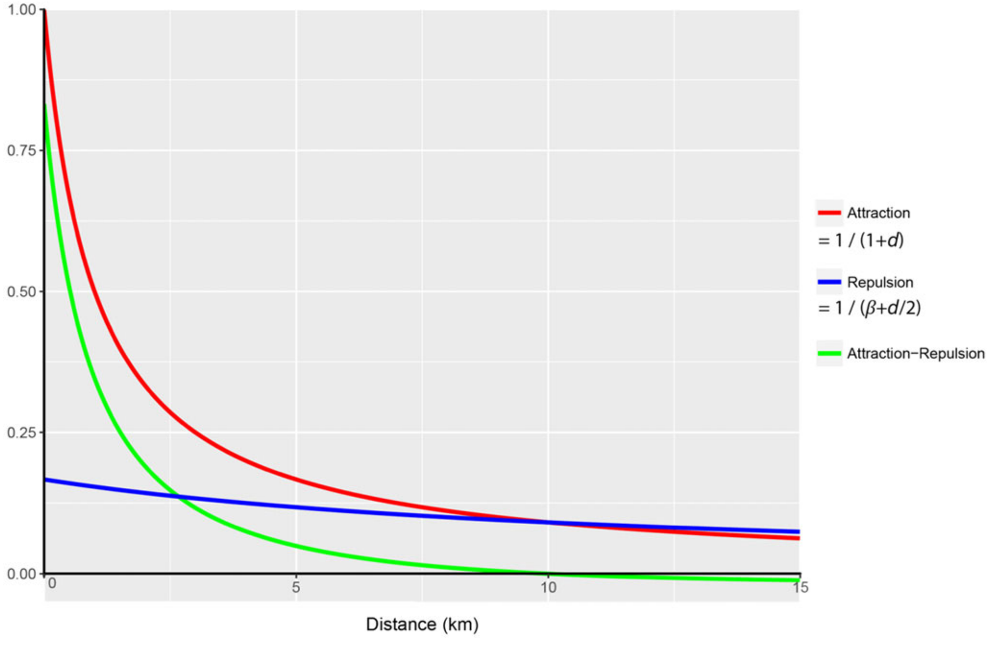

FIGURE 3 The inverse-distance attraction and repulsion functions

activities" looking for lower densities, and, consequently, any group of dwellers and workers exert both a higher attractive influence on "agglomerating activities," and a higher repulsive influence on "disseminating activities" than any individual does. This being, the shape of the attraction and repulsion functions defined with respect to distance will be determined on an individual basis, even if all individuals will be assumed to have the same attraction and repulsion functions depending on the "zero resultant radius," that is the distance at which the repulsive effect starts exceeding the attractive one (for example, distance 10 km in Figure 3).

2.4 | From land-rent theory to vector field analysis

The world of locations is a world of interaction between attractive and repulsive locational forces. The best tools to study it stem from vector field analysis. The UMS resorts to "vectors-resultants of attractive and repulsive forces" (see Tellier, forthcoming) that are a generalization of the "vectors-resultants of gravity attraction" used and analysed by Tobler (1981, 1991), McQuistan (1965) and Dorigo and Tobler (1983).

A vector-resultant of vector attraction G indicates the most likely direction of the movement of the population located at a given point that is attracted by a large number of attractors distributed over the considered space. The vector-resultant gives the direction of the movement, but also its magnitude. That magnitude is proportional to the norm of the vector. The attraction exerted by each attractor on the population of generators is computed by means of an attraction function usually based on a gravity model.

Gravity models estimate probabilities of interaction between pairs of points in the space. Is a gravity model any model that is such that the probability of interaction between two points increases when the number of "attractors" and "generators" increases at each point, and decreases when the distance between the two points increases. The impact of the number of "attractors" and "generators" on the interaction probability is called the "mass effect," whereas the impact of distance on the interaction probability is called the "deterrence effect" of distance. The interaction probability takes the form of a scalar attached to a pair of points. The interaction probability I_{ij} between points p_i and p_j is given by:

$$I_{ij} = \left(\frac{m_i m_j / d_{ij}{}^{\alpha}}{1 + m_i m_j / d_{ij}{}^{\alpha}} \right),$$

where m_i is the "mass" of the ith point, d_{ij} is the distance between points $\{p_i, p_j\}$ and where α is a positive real number representing the "deterrence effect" of distance. In the case where the deterrence effect can clearly be distinguished from the "mass effect," we can define:

$$I_{ij} = \left(\frac{m_i m_j}{1 + m_i m_j} \right) \times \left(\frac{d_{ij}{}^{\alpha}}{1 + d_{ij}{}^{\alpha}} \right).$$

The concept of vector-resultant of vector attraction is static since it takes into account the spatial distribution of attractors at a given time. Its mathematical expression is the following when the attraction function takes the form of an inverse-distance gravity model:

$$G_i = \sum_{j \neq i} \frac{m_i m_j}{d_{ij}{}^{\alpha}} \frac{(p_j - p_i)}{d_{ij}}.$$

And the probability of the most likely movement is proportional to:

$$P(G_i) = \frac{|G_i|}{1 + |G_i|}.$$

McQuistan (1965), and the other users of the concept of vector-resultant of gravity attraction did not consider the existence of repulsive forces. The concept of "vector-resultant of vector attraction and repulsion" is similar to the previous one except for the introduction of repulsive forces pointing into the opposite direction of their corresponding repulsors.[2] In the UMS, each dweller or worker is assumed to be, at the same time, an attractor and a repulsor.

A vector field is defined as a portion of the space where the force exerted on a point depends only on its position. Such a force can be computed on every points of the vector field. Convergence points are such that all the vectors that directly surround them point in their direction. A sector of the vector field is defined as the portion of the space of the vector field where all the forces are oriented towards a convergence point. Divergence points are directly surrounded with forces that point in outward directions as if they originated from the divergence points. Since, normally, divergence points lie on the limits of the various sectors of a vector field, they are quite useful for demarcating those sectors. One of the first steps in demarcating such limits precisely consists in spotting the divergence points. Those concepts are used in conceiving the Urban Metric System presented further. Actually, in that system, urban areas correspond to vector field sectors.

2.5 | The urban metric system, and its attraction and repulsion functions

The UMS is based on the fact that urban zones can be identified and delimited by identifying sectors in vector fields defined with respect to observed "dwellers + workers" distributions and predetermined functions attributed to each unit of "dwellers + workers" exerting both an attractive force on agglomerating activities and a repulsive force on disseminating activities. These functions bear the following characteristics:

1. The attractive force is assumed to be equal to $1/ (1 + d)$, where d = distance from the unit of "dwellers + workers"; the attractive force varies between 1 and 0, and it diminishes as distance increases; the value 0 is reached at infinity;
2. The repulsive force is assumed to be equal to $1/ (\beta + d/2)$, where β = a parameter characteristic of each level of the hierarchy ($\beta > 1$); the repulsive force varies between $1/\beta$ and 0, and it diminishes as distance increases; the value 0 is reached at infinity;
3. The boundary of each type of urban zone depends on the distance at which, for each dweller or worker, the attractive force equals the repulsive force, this distance being called "zero resultant radius"; for example, in the case of a "city" boundary, that radius is equal to 10 kilometres, whereas it is equal to 20 kilometres in the case of the "agglomeration" boundary.

The "inverse-distance" form of the attraction and repulsion functions illustrated in Figure 3, is a compromise between the traditional gravity models where the exponent of distance is closer to 2 than to 1 (as it is here), and the linear functions corresponding to constant marginal transportation costs illustrated in Figure 1.

It is understood that, between the unit of "dwellers + workers" and the zero resultant radius, the attractive force of the unit of "dwellers + workers" is greater than its repulsive force, whereas, for distances greater than the zero resultant radius, the attractive force of the unit of "dwellers + workers" is smaller than its repulsive force. Both types of forces are such that their intensity diminishes as distance increases. The value of β varies from one level of the UMS to the other (see Table 1).

It is possible to compute a boundary for each possible value of "zero resultant radius" (and its associated β value). For a given settlement, the generated boundaries delimit concentric "rings" that never intersect each other. Various observed densities characterize those rings. We arbitrarily fix at 100 inhabitants per square kilometre the threshold that separates rural and urban areas. We call the limit between these areas, the "rural/urban limit". The territory lying inside that limit is defined as urban, whereas the one located outside is rural. However, as we shall see, that limit varies from one level of the hierarchy to the other. We shall distinguish between level 1, level 2, level 3, etc, "rural/urban limits".

Tellier (2014) has proposed a UMS aimed at defining operationally various urban concepts. It has two versions: the discrete one based on rectangular or triangular standard cells having the same surface, where each zone is made

[2]Repulsive forces are less likely to be defined with respect to classical gravity models, but they do generally decrease with distance.

TABLE 1 The structure of the urban metric system

Level	Name of the areas at each level	Zero resultant radius	Value of β
0	Districts	10×2^{-1} = 1 kilometre	$1 + (10 \times 2^{-1}/2) = 3{,}5$
1	Cities (or rural regions)	10×2^{0} = 10 kilometres	$1 + (10 \times 2^{0}/2) = 6$
2	Agglomerations	10×2^{1} = 20 kilometres	$1 + (10 \times 2^{1}/2) = 11$
3	Metropolises	10×2^{2} = 40 kilometres	$1 + (10 \times 2^{2}/2) = 21$
4	Megacities	10×2^{3} = 80 kilometres	$1 + (10 \times 2^{3}/2) = 41$
5	Megalopolises	10×2^{4} = 160 kilometres	$1 + (10 \times 2^{4}/2) = 81$
6	Urban systems	10×2^{5} = 320 kilometres	$1 + (10 \times 2^{5}/2) = 161$
7	Urban macro-systems	10×2^{6} = 640 kilometres	$1 + (10 \times 2^{6}/2) = 321$
8	Continental systems	10×2^{7} = 1 280 kilometres	$1 + (10 \times 2^{7}/2) = 641$
9	Intercontinental systems	10×2^{8} = 2 560 kilometres	$1 + (10 \times 2^{8}/2) = 1\,281$
10	Global systems	10×2^{9} = 5 120 kilometres	$1 + (10 \times 2^{9}/2) = 2\,561$

of a set of complete cells, and the continuous one, where the border of a zone does not necessarily coincide with cells' limits. The system includes 10 levels, each level being determined by a "radius of integration" in the case of the discrete version, and a "zero resultant radius" in the continuous case presented here. Table 1 illustrates the structure of the continuous system, which is recent and much more adapted to real situations than the discrete one described in Tellier (2014).

A settlement is said to be a city, if its level 1 "rural/urban limit" lies outside or coincides with the $\beta = 6$ boundary. It is considered an urban agglomeration, if its level 2 "rural/urban limit" lies outside or coincides with the $\beta = 11$ boundary. It is a metropolitan area, if its level 3 "rural/urban limit" lies outside or coincides with the $\beta = 21$ boundary, and so on. Each level of the urban metric system has its own "rural/urban limit," but, for all levels, the "rural/urban limit corresponds to the threshold of 100 inhabitants per square kilometre. For example, a secondary city (different from the central city) that is included in an agglomeration and a metropolitan area has its own "100-inhabitants-per-square-kilometre" limit, while the agglomeration and the metropolitan area it belongs to have their own "100-inhabitants-per-square-kilometre" limits. This paper focuses on levels 1 (central city), 2 (agglomeration) and 3 (metropolitan area) of UMS. The applications of the method to Montreal, Quebec City and Toronto are presented and commented.

3 | THE MONTREAL APPLICATION

Montreal is located on a 50-kilometre-long island having an area of 499 square kilometres. It neighbours the 247-square-kilometre Laval Island. Together, the two islands, lying in the middle of the St. Lawrence River, are a little larger than Singapore (746 km^2 versus 720 km^2). Except for the Mount Royal located northwest of the central business district, the two islands and their surroundings are flat, and could be urbanized rather easily, the main obstacle being the main branch of the St. Lawrence, which runs south of the islands. Montreal has dominated the Canadian economy for about 130 years, from the 1830s, when she superseded Quebec City, till the 1960s, when Toronto replaced her as the dominant pole of Canada.

According to the 2016 Census, the metropolitan region of Montreal has a population of 4.099 million, which corresponds almost exactly to the 4.1 million population of the metropolitan area we obtain with the 40-kilometre "zero resultant radius" (see Table 2). The boundary of the central city we draw with our method is very different from that of the City of Montreal, but their respective populations are similar (1.633 million compared with 1.705 million in 2016).[3] Figure 4 illustrates the layout of the computed Montreal agglomeration

[3]It must be noticed that the administrative boundaries of the City of Montreal have changed a lot in the last 20 years. On 1 January 2002, all the municipalities of the Island of Montreal were merged: that is the City of Montreal, plus 27 other municipalities of the Island. In 2006, 15 of the 27 "other municipalities" were "demerged," and recovered their autonomy.

TABLE 2 Measuring urban sprawl

Urban area	Statistics	Quebec City	Montreal	Toronto
Central city	Population 2006	530,000	1,542,600	1,104,800
	Population 2016	573,500	1,633,500	1,222,500
	Pop. change 2006–2016	8.21%	5.89%	10.65%
	Area 2006 (km^2)	327.67	292.69	197.49
	Area 2016 (km^2)	333.87	290.92	194.62
	Area change 2006–2016	1.89%	−0.60%	−1.45%
	Density 2006 (inh/km^2)	1,617.48	5,270.40	5,594.23
	Density 2016 (inh/km^2)	1,717.71	5,614.87	6,281.37
	Density change 2006–2016	6.20%	6.54%	12.28%
	Part metro. Area 2006	69.10%	41.11%	21.64%
	Part metro. Area 2016	66.55%	39.76%	21.63%
Inner Suburbs	Pop. change 2006–2016	18.15%	7.63%	8.82%
	Area change 2006–2016	0.57%	−0.14%	−2.32%
	Density change 2006–2016	17.48%	7.77%	11.41%
	Part metro. Area 2006	19.97%	36.49%	42.14%
	Part metro. Area 2016	21.00%	35.86%	41.42%
Agglomeration	Population 2006	683,200	2,911,700	3,255,900
	Population 2016	754,500	3,107,000	3,563,300
	Pop. change 2006–2016	10.44%	6.71%	9.44%
	Area 2006 (km^2)	1,302.52	1,361.82	1,211.19
	Area 2016 (km^2)	1,314.24	1,358.57	1,184.77
	Area change 2006–2016	0.90%	−0.24%	−2.18%
	Density 2006 (inh/km^2)	524.52	2,138.09	2,688.18
	Density 2016 (inh/km^2)	574.10	2,286.96	3,007.58
	Density change 2006–2016	9.45%	6.96%	11.88%
	Part metro. Area 2006	89.07%	77.60%	63.78%
	Part metro. Area 2016	87.55%	75.62%	63.06%
Outer Suburbs	Pop. change 2006–2016	28.04%	19.21%	12.93%
	Area change 2006–2016	1.26%	0.80%	3.59%
	Density change 2006–2016	26.45%	18.26%	9.02%
	Part metro. Area 2006	10.93%	22.40%	36.22%
	Part metro. Area 2016	12.45%	24.38%	36.94%
Metropolitan area	Population 2006	767,000	3,752,100	5,104,500
	Population 2016	861,800	4,108,800	5,651,000
	Pop. change 2006–2016	12.36%	9.51%	10.71%
	Area 2006 (km^2)	5,016.78	5,480.77	4,190.48
	Area 2016 (km^2)	5,075.15	5,510.57	4,271.09
	Area change 2006–2016	1.16%	0.54%	1.92%
	Density 2006 (inh/km^2)	152.89	684.59	1,218.12
	Density 2016 (inh/km^2)	169.81	745.62	1,323.08
	Density change 2006–2016	11.07%	8.92%	8.62%

centre and boundary. Out of the boundary, the vectors are outward oriented, while they are inward oriented inside, and the vectors that surround the centre are converging towards it. Figure 5 presents the estimated borderlines of the Montreal central city, agglomeration and metropolitan area for 2016, as well as the subway system.

The Metropolitan region of Montreal is experiencing a strong urban sprawl.[4] Even though the 2006–2016 period was marked by the Great recession of 2007–2012, the outward movement of the Montreal population went on. During that period, the population of the outer suburbs grew by 19.21% while the population growth rate of the central city was just 5.89% and that of the inner suburbs, 7.63%. Consequently, the central city's part of the metropolitan area declined from 41.11% in 2006 to 39.76% in 2016, and that of the agglomeration, from 77.60% down to 75.62%. However, it must be noted that, according to our method, the surface of the Montreal metropolitan area

[4]See Jaeger (2016, 2017), as well as Nazarnia, Schwick, and Jaeger (2016).

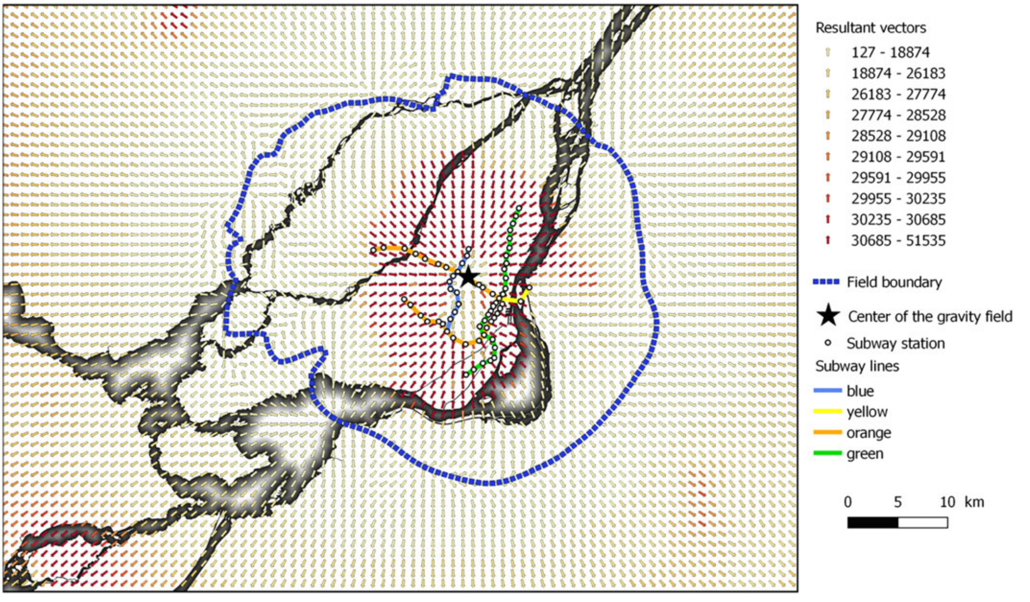

FIGURE 4 Vector fields and estimated boundary of the Montreal agglomeration, 2006

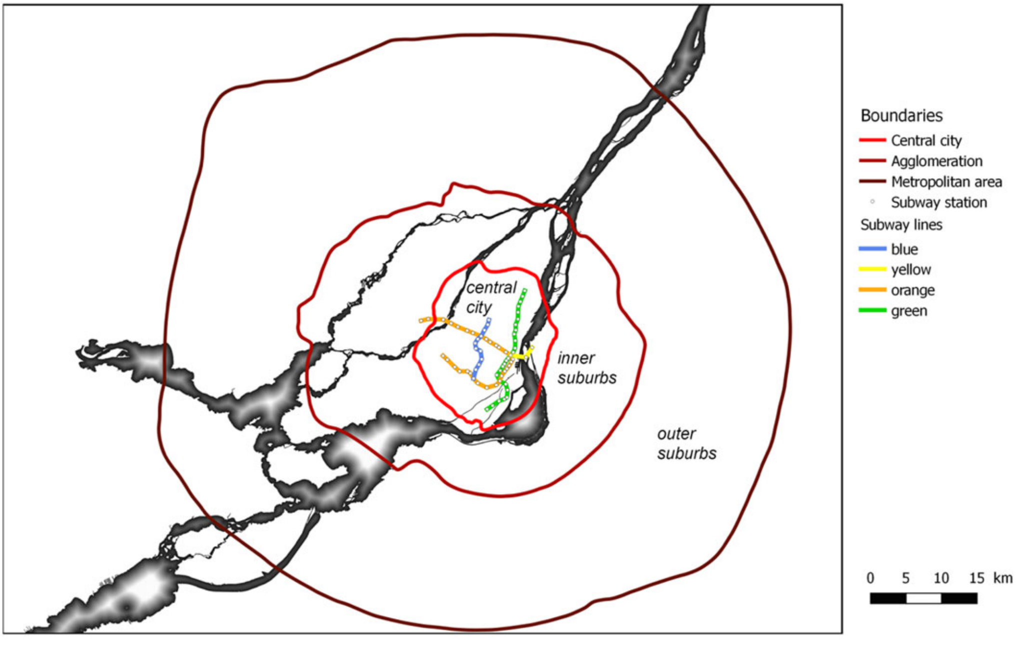

FIGURE 5 The estimated boundaries of the central city, inner suburbs, agglomeration, outer suburbs and metropolitan area of Montreal, 2016

increased by only 0.54% between 2006 and 2016, while that of the Quebec City and Toronto metropolitan areas grew by 1.16% and 1.92%, respectively. It must be noted that the extension of the rural/urban boundary differs from the boundary of the metropolitan area, as we have seen, and as we shall illustrate later.

4 | THE TORONTO APPLICATION

The City of Toronto is located on the shore of Lake Ontario in a region that is called the Golden Horseshoe, which stretches south from Lake Erie and north to Lake Scugog. Toronto's site differs completely from that of Montreal. While Montreal is an island located in the centre of the St. Lawrence River, and an urban region characterized by the possibility to expand in all directions, Toronto can only spread towards the southwest, the north and the northeast, its central business area being located close to the lakeshore. In fact, Toronto's expansion mainly consists in progressively integrating the numerous urban areas distributed along Lake Ontario, from St. Catharines to Oshawa and beyond, as well as their hinterland. The Golden Horseshoe is the most industrialized region of Canada, and accounts for over 21% of the Canadian population.

In 2016, the Census Metropolitan Area of Toronto had a population of 5.928 million, which differs a little from the 5.651 million population of the metropolitan area we obtain with the 40-kilometre "zero resultant radius" (see Table 2). The boundary of the central city we draw with our method is very different from those of both the new and old Cities of Toronto.[5] Their respective 2016 populations also differ: 1.222 million for our estimated central city, 2.731 million for the New City of Toronto, and 0.798 million for the Old one. This is a good illustration of the disparities administrative definitions of cities can generate, and of the need for a standard statistical definition of urban areas. Figure 6 presents the estimated borderlines of the Toronto central city, agglomeration and metropolitan area, as well as the subway system.

The present metropolitan area of Toronto is already characterized by an important urban sprawl (see Furberg & Ban, 2012). However, during the 2006–2016 period, population densities grew faster in the central city (12.28%) than in the inner suburbs (11.41%), and even the outer suburbs (9.02%), which indicates that urban sprawl may not be a fatality. However, due to the shift of the boundaries and the differences in surface between the various urban areas, the central city's part of the metropolitan area stayed about the same during that period (it went from 21.64% to 21.63%); the part of the inner suburbs declined a little from 42.14% to 41.42%, and the part of the outer suburbs went from 36.22% to 36.94%. The density and relative weight evolutions do not contradict each other since, according to our computations, the territories of the central city, and the inner suburbs have shrunk a little between 2006 and 2016, whereas that of the outer suburbs has increased by 3.39%.

Overall, urban sprawl appears under control inside the estimated metropolitan area of Toronto, which does not mean that it is not going on in the areas located outside of our estimated metropolitan area boundary or that the Toronto rural/urban boundary is not expanding anymore. It must be noticed that the evolution of the metropolitan region of Toronto is very much influenced by the existence, north of the region, of the 7,200 square kilometres Greenbelt established by the Ontario government in 2005. That green belt includes the Niagara Escarpment Biosphere Reserve and the Oak Ridges Moraine.

5 | THE QUEBEC CITY APPLICATION

Quebec City is the old capital of Canada, and it dominated its economy for many years till around 1830. It has long been a military and administrative city, but, over the last thirty years, it has developed a lot, especially in the high technologies sector, and it has one of the lowest unemployment rates in Canada. It is located on the left bank of the St. Lawrence River at a very strategic site where the river suddenly narrows upstream. Once the largest city of Canada, Quebec City is now the eleventh largest municipality of Canada in terms of population.

[5] In 1998, the Old City of Toronto was merged with the other cities of Metropolitan Toronto (York, North York, Etobicoke, and Scarborough), and the Borough of East York. And the new Toronto metropolitan area was called the Greater Toronto Area, which consists of the New City of Toronto and four surrounding regional municipalities: Durham, Halton, Peel, and York. The territory of the Greater Toronto Area is 20% larger than the Census Metropolitan Area of Toronto.

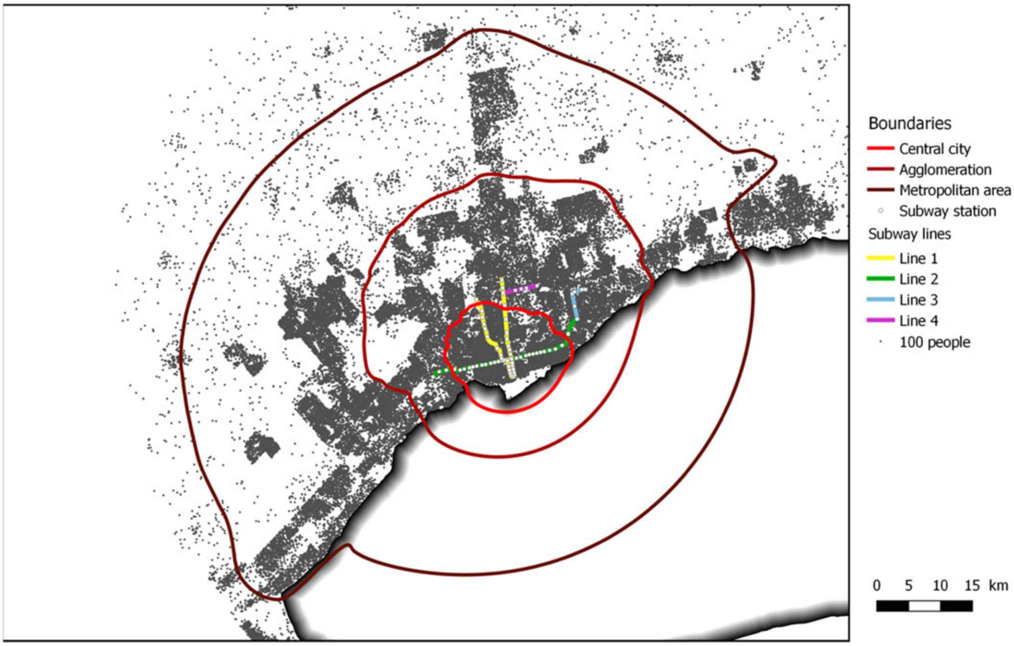

FIGURE 6 The estimated boundaries of the central city, inner suburbs, agglomeration, outer suburbs and metropolitan area of Toronto, 2016

In 2016, the Metropolitan Community of Quebec had a population of 0.798 million, which is close to the 0.862 million population of the metropolitan area we obtain with the 40-kilometre "zero resultant radius" (see Table 2). The official Quebec agglomeration had a population of 0.575 million in 2016, while the boundary corresponding to the 20-kilometre "zero resultant radius" encompassed a population of 0.755 million. The boundary of the central city we draw with a 10-kilometre "zero resultant radius" includes a part of the right bank of the St. Lawrence River, which is not the case of the municipality of Quebec. The population of the estimated central city is 0.574 million, whereas the official population of Quebec City for 2016 is 0.539 million.[6] Figure 7 presents the estimated borderlines of the Quebec central city, agglomeration and metropolitan area.

Quebec City has a very special topography that may have been an obstacle to urban sprawl. By contrast, between 2006 and 2016, the part of the central city in the metropolitan area decreased from 69.10% to 66.55%, while the part of the outer suburbs has climbed from 10.93% to 12.45%. This reflects the fact that the population of the central city grew by 8.21% during that period, while the population of the metropolitan area increased by 12.36%.

6 | A SIMPLE SYNTHETIC INDEX OF URBAN SPRAWL

A systematic and universal method for delimiting urban areas having been elaborated, it becomes possible to define a synthetic index *S* of urban area. According to our research, the most reliable one is simple in the case of the urban sprawl of a metropolitan area. It takes the following form:

$$S = (\text{density of outer suburbs}) \ / \ (\text{density of the central city}).$$

Table 3 gives the values of *S* obtained for Montreal, Quebec City and Toronto in 2006 and 2016. We see that, in 2016, Toronto was 2.5 times more sprawled than Montreal, which was 2.6 times more sprawled than Quebec City.

[6]The territory of Quebec City has been redefined on 1 January 2006. The Quebec Metropolitan Community was created on 1 January 2002, and the Quebec Agglomeration, on 1 January 2006.

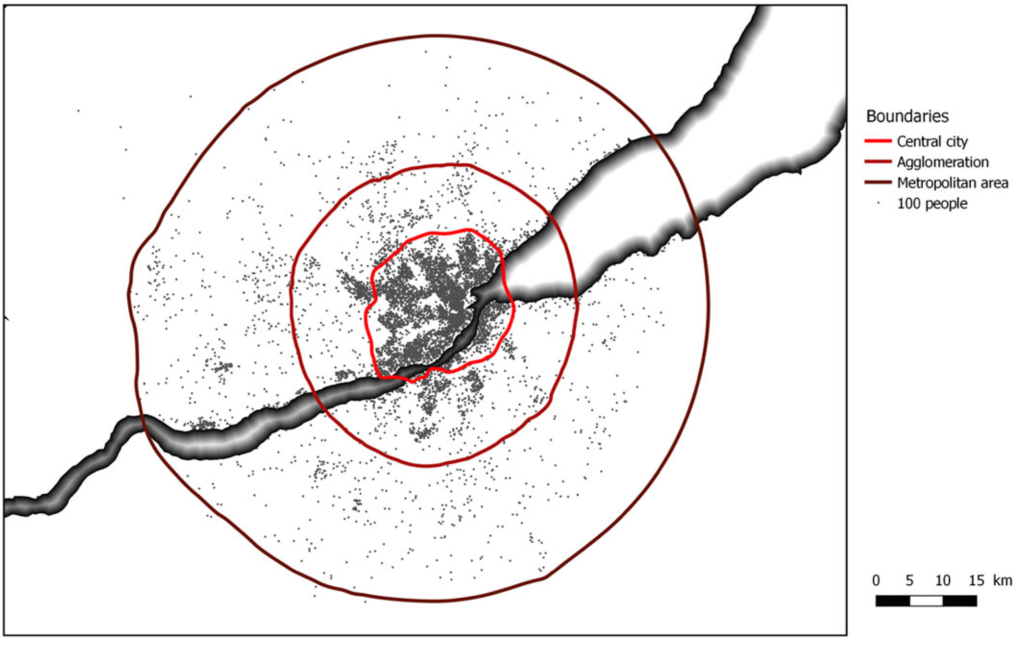

FIGURE 7 The estimated boundaries of the central city, inner suburbs, agglomeration, outer suburbs and metropolitan area of Quebec City, 2016

TABLE 3 Urban sprawl indices of Montreal, Quebec City and Toronto in 2006 and 2016

	Quebec City		Montreal		Toronto	
	2006	2016	2006	2016	2006	2016
Urban sprawl index	1.39%	1.66%	3.87%	4.30%	11.09%	10.77%
Variation 2006–2016	19.08%		11.00%		−2.91%	

However, between 2006 and 2016, while urban sprawl has decreased by 2.91% in Toronto, it has increased by 11.00% in Montreal, and by 19.08% in Quebec City.

This index can also be computed at other levels of the Urban Metric System. For instance, at the level of a simple city, it would be applied with the following values of the "zero resultant radius": 2.25 km, 5 km, and 10 km. In the case of a simple agglomeration, the corresponding values would be: 5 km, 10 km, and 20 km.

The presented method permits us to draw a map illustrating the progression of urban sprawl by finding the 2006 and 2016 values of the "zero resultant radius" corresponding to the limit between densities higher and lower than 100 inhabitants per square kilometre. Figure 8 presents the expansion of the level-3 rural/urban limit in the Montreal metropolitan region between 2006 and 2016. The 2006 limit corresponds to the case where the "zero resultant radius" is equal to 39.30 km (β = 20.65), while the 2016 limit corresponds to the case where the "zero resultant radius" is equal to 39.86 km (β = 20.93). According to our computations, the terrestrial area within the level 3 rural/urban limit went from,5289 km^2 in 2006 to 5,477 km^2 in 2016. The 188 additional square kilometres represent an increase of 3.554% in ten years, and equal 64,62% of the estimated surface of the 2016 central city of Montreal (188 km^2/290.92 km^2), which is far from being negligible. It is clear that the progression of the Montreal metropolitan urban sprawl between 2006 and 2016 was due to both the "densification" of the outer suburbs, and the enlarging of the level 3 rural/urban limit.[7]

[7]About the relation that exists between density and sprawl, see Ewing (1997).

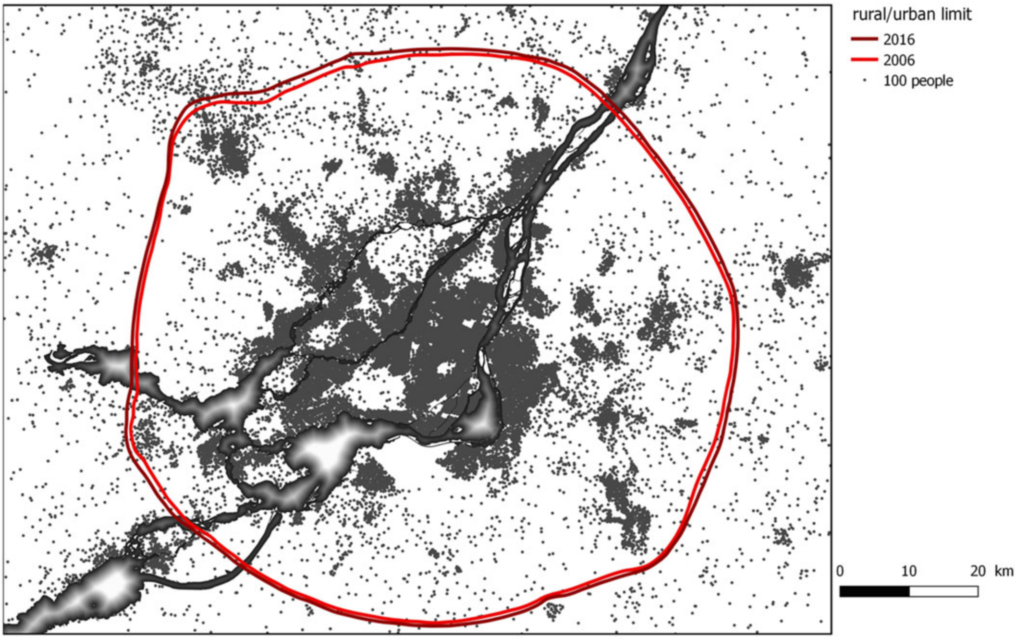

FIGURE 8 Expansion of the Montreal level-3 rural/urban limit between 2006 and 2016

It must be noticed that this level 3 limit, and the level 1, level 2 or other similar limits must be distinguished from the very micro notion of "urbanized lots," which has a fractal nature, and precludes, to some extent, drawing the type of smooth curve illustrated in Figure 7, and which makes very hard, if not impossible, measuring the surface of the progression of urban sprawl, as is done here. That being said, satellite photographs remain useful for specifying at the micro level where the urban sprawl takes place, despite that they hardly measure all aspects of suburbs densification, which is an important aspect of urban sprawl.

7 | CONCLUSION

This research has applied Tellier's method for delimiting three types of urban areas (central city, agglomeration, and metropolitan area) by resorting all together to the concepts of attractive force, repulsive force, and vector field. It led to the following conclusions:

1. The method is "universal" in the sense that it can be applied throughout the world, and in whatever historical period.
2. It is based on the concepts of traditional land-rent theory, as well as on a Weberian approach, and the physics notion of vector fields.
3. It uses spatially disaggregated population and employment data. Data relating to the location of workers are optional, but they are particularly useful in the cases where the central business district is sparsely populated, and located close to an empty space (as in the case of Toronto, where the central business district lies very close to the shore of Lake Ontario). In most developed countries, disaggregated population and employment data are easily available in the case of major cities.
4. The population statistics resulting of the application of the method to Montreal, Toronto, and Quebec City central cities, agglomerations, and metropolitan areas are somehow congruent with the Canadian Census estimations.

5. The method allows analysts to delimit central cities, agglomerations, and metropolitan areas, and to estimate their populations without referring to the administrative boundaries. This is especially worthwhile in cases like that of the City of Toronto whose official population went from 0.798 million to 2.731 million overnight in 1998.
6. The method enables researchers to compare the level, and evolution of urban sprawl in a systematic way in various contexts and cases. It also leads to follow the shifting of the rural/urban limits around separate cities, agglomerations, and metropolitan areas.
7. The urban sprawl index we have developed makes it possible to measure the progression of the phenomenon, and to compare different cities in a systematic way.

In our opinion, the main contributions of the Urban Metric System, and the urban sprawl index are scientific, and urbanistic. For instance, they can systematize international urban statistical data, and permit checking in a more scientific way the famous rank-size rule. They are likely to mark a turning point in the vast literature on urban sprawl, where the very idea of attempting to delimit urban areas was generally considered far-fetched. Finally, they can provide city planners with precise instruments to convince local authorities to better fight urban sprawl.

This last remark is important. Right now, so many indices of urban sprawl are proposed that each local authority can find some according to which its sector performs rather well in terms of urban sprawl control. This leads to finding good excuses to do nothing, or, at least, nothing more than what is already being done. Having a synthetic index of urban sprawl almost unanimously accepted, limits the possibilities to get away with it by means of such excuses.

Similarly, since controlling urban sprawl generally requires resorting to regional planning, and convincing local governments that benefit from urban sprawl to curb their ambitions, often the compromise that ends up being accepted at the regional level is to encourage all local authorities, whether central or suburban, to "densify" their urban fabric. Such a policy allows suburban authorities to declare that they are not "sprawling" anymore, and that they are just "densifying" their territory, which, actually, most of the time, results in increasing the metropolitan urban sprawl. The best way to demonstrate this is to resort to a convincing synthetic index of urban sprawl that proves that densifying peripheral sectors results in aggravating the urban sprawl if the central sectors are not densifying at the same pace as the suburbs do.

It remains to apply the presented method to more urban areas, and to have the urban metric system accepted by researchers. It must be stressed that the UMS is basically a system of definitions, and, as such, its use will become common if it happens to clarify urban statistics, to facilitate inter-city comparisons, and to convince local authorities to better deal with urban sprawl.

REFERENCES

Alonso, W. (1964). *Location and land use: Toward a general theory of land rent*. Cambridge, MA: Harvard University Press.

Dorigo, G., & Tobler, W. (1983). Push-pull migration laws. *Annals of American Geographers*, *73*(1), 2–17.

Ewing, R. (1997). Is Los Angeles-style sprawl desirable? *Journal of the American Planning Association*, *63*(1), 107–126.

Furberg, D., & Ban, Y. (2012). Satellite monitoring of urban sprawl and assessment of its potential environmental impact in the Greater Toronto Area between 1985 and 2005. *Environmental Management*, *50*(6), 1068–1088.

Goerlich Gisbert, F. J., Cantarino Martí, I., & Gielen, E. (2017). Clustering cities through urban metrics analysis. *Journal of Urban Design*, *22*(5), 689–708.

Jaeger, J. A. G. (2016). Social and ecological impacts of the exponential increase of urban sprawl in Montréal. *Canada Communicable Disease Report*, *42*(10), 207–208.

Jaeger, J. A. G. (2017). Exponential increase of urban sprawl in Montreal in the last 60 years should be taken into consideration in decisions about future urban development. Document presented to the borough of Pierrefonds (West), Montreal, May.

Liang, J., Li, F., & Mao, L. (2010). Review of the methods of delimitation for the spatial scope of urban agglomeration. In *The 18th International Conference on Geoinformatics: GIScience in Change, Geoinformatics 2010*, 18–20. Beijing, June: Peking University.

Lowry, J. H., & Lowry, M. B. (2014). Comparing spatial metrics that quantify urban form. *Computers, Environment and Urban Systems*, *44*, 59–67.

McQuistan, R. B. (1965). *Scalar and vector fields*. New York: Wiley.

Nazarnia, N., Schwick, C., & Jaeger, J. A. G. (2016). Accelerated urban sprawl in Montreal, Quebec City, and Zurich: Investigating the differences using time series 1951–2011. *Ecological Indicators*, *60*, 1229–1251.

Tellier, L.-N. (1985). *Économie spatiale: Rationalité économique de l'espace habité*. Chicoutimi: Gaëtan Morin éditeur.

Tellier, L.-N. (2014). Un système métrique urbain. In M.-U. Proulx (Ed.), *Sciences du territoire II: Methodologies* (pp. 72–74). Québec: Presses de l'Université du Québec.

Tellier, L.-N. (forthcoming). Approaching the *New Urban World* in the light of the topodynamic theory, and by means of the topodynamic model. *Land Use Policy* (under publication).

Tobler, W. (1981). A model of geographical movement. *Geographical Analysis*, *13*(1), 1–20.

Tobler, W. (1991). Visual evidence for urban potential fields. *Mappemonde*, *1*, 46–47.

von Thünen, J. H. (1826). *Der isolierte staat in beziehung auf landwirtschaft und nationalökonomie*. Hamburg: Puthes.

Weber, A. (1909/1929). *Über den standort der industrien*. Tübingen, J.C.B. Mohr—English translation: *The theory of the location of industries*. Chicago, IL: Chicago University Press.

How to cite this article: Tellier L-N, Gelb J. An urban metric system based on space-economy: Foundations and implementation. *Reg Sci Policy Pract*. 2018;10:145–160. https://doi.org/10.1111/rsp3.12141

DOI: 10.1111/rsp3.12141

Resumen. En este estudio se implementa un sistema de métricas urbanas (SMU) para resolver en el futuro los numerosos problemas derivados de la ausencia de una definición matemática de los límites de los diversos conceptos de áreas urbanas. Se basa en la economía del espacio, especialmente en la teoría tierra-renta, los conceptos de fuerzas atractivas y repulsivas, y en los campos vectoriales. El sistema se aplica a las ciudades centrales, las aglomeraciones y las áreas metropolitanas de Montreal, Toronto y la ciudad de Quebec. A partir de la SMU propuesta, se define y calcula un índice sintético de dispersión urbana para las áreas metropolitanas estudiadas.

抄録: 都市部の様々なコンセプトの境界が数学的に定義されていないことから生じる多数の問題が、今後解消されるように都市計量システム(urban metric system : UMS)を使用する。このシステムは、空間経済の理論、特に引力と斥力のコンセプトである地代の理論、そしてベクトル場に基づいている。このシステムを、モントリオール、トロント、ケベックの中心都市の集積地域と大都市圏に適用する。提示したUMSに基づいて、都市スプロール現象の総合指数を定義して、対象の大都市圏の指数を算出する。

© 2018 The Author(s). Regional Science Policy and Practice © 2018 RSAI