Weighted LISA (Local indicator of spatial autocorrelation): sketch, v1

F.Bavaud, UNIL

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1 Introduction

To perform local (or global) weighted auto-correlation, two ingredients are needed :

- a $n \times n$ exchange matrix $\boldsymbol{E} = (e_{ij})$ between "positions" i and j, which is symmetric, non-negative, and normalized to $e_{\bullet\bullet} = 1$. In addition, $\boldsymbol{E}$ has to be *weight-compatible* in the sense that the weights $\boldsymbol{f} = (f_i)$ defined by $f_i := e_{i\bullet} > 0$ are the relevant *strictly positive* weights under consideration.
- a $n \times n$ matrix of squared Euclidean distances $\boldsymbol{D} = (d_{ij})$.

1.1 Markov chains

Let $\boldsymbol{\Pi} := \text{diag}(\boldsymbol{f})$. Define $\boldsymbol{W} := c^{-1}\boldsymbol{E}$, that is $w_{ij} := \frac{e_{ij}}{f_i}$. By construction, $\boldsymbol{W}$ is the $n \times n$ transition matrix of a reversible Markov chain (we assume to be *regular*, that is irreducible and aperiodic, with stationary distribution $\boldsymbol{f}$. It obeys $\boldsymbol{\Pi W} = \boldsymbol{W}^\top \boldsymbol{\Pi} = \boldsymbol{E}$.

1.2 Relative autocorrelation index δ

Define the global and local inertia by

$$\Delta := \frac{1}{2}\sum_{ij} f_i f_j d_{ij} \qquad\qquad \Delta_{\text{loc}} := \frac{1}{2}\sum_{ij} e_{ij} d_{ij} \tag{1}$$

The *relative autocorrelation index* δ (a weighted, multivariate generalization of Moran's I) is

$$\delta := \frac{\Delta - \Delta_{\text{loc}}}{\Delta} \in [-1, 1] \tag{2}$$

1.3 Local autocorrelation index δ_i (LISA)

There are many ways to define a local autocorrelation index δ_i such that $\delta = \sum_{i=1}^{n} f_i \delta_i$. Presumably the most elegant (unpublished, but cited in "Flow autocorrelation : a dyadic approach" by F. Bavaud, M. Kordi, C. Kaiser, The Annals of Regional Science (2018) Vol. 61, Issue 1, pp 95–111, https ://doi.org/10.1007/s00168-018-0860-y) is

$$\delta_i := \frac{(\boldsymbol{W}\boldsymbol{B})_{ii}}{\Delta} \tag{3}$$

where $\boldsymbol{B} := -\frac{1}{2}\boldsymbol{H}\boldsymbol{D}\boldsymbol{H}^\top$ is the matrix of (unweighted) *scalar products* and $\boldsymbol{H} := \boldsymbol{I}_n - \boldsymbol{1}_n \boldsymbol{f}^\top$ is the (idempotent, generally non symmetric) weighted *centration matrix*.

By construction, $\boldsymbol{H}\boldsymbol{1}_n = \boldsymbol{0}_n$, and $\boldsymbol{H}^\top \boldsymbol{f} = \boldsymbol{0}_n$.

Note that $\boldsymbol{E}\boldsymbol{H} = \boldsymbol{E} - \boldsymbol{f}\boldsymbol{f}^\top$ and $\boldsymbol{H}^T\boldsymbol{E} = \boldsymbol{E} - \boldsymbol{f}\boldsymbol{f}^\top$ and finally
$\boldsymbol{H}^T\boldsymbol{E}\boldsymbol{H} = \boldsymbol{E} - \boldsymbol{f}\boldsymbol{f}^\top$

By construction,

$$\begin{aligned}\sum_i f_i(\boldsymbol{W}\boldsymbol{B})_{ii} &= \mathrm{trace}(\boldsymbol{\Pi}\boldsymbol{W}\boldsymbol{B}) = \mathrm{trace}(\boldsymbol{E}\boldsymbol{B}) = -\frac{1}{2}\mathrm{trace}(\boldsymbol{E}\boldsymbol{H}\boldsymbol{D}\boldsymbol{H}^\top) \\ &= -\frac{1}{2}\mathrm{trace}(\boldsymbol{H}^\top\boldsymbol{E}\boldsymbol{H}\boldsymbol{D}) = \frac{1}{2}\mathrm{trace}((\boldsymbol{f}\boldsymbol{f}^\top - \boldsymbol{E})\boldsymbol{D}) = \Delta - \Delta_{\text{loc}}\end{aligned}$$

which proves $\delta = \sum_{i=1}^{n} f_i \delta_i$.