A note on a simplified formula for the free energy

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0.1 Image segmentation by generalized cut minimization

Minimizing the corresponding free energy

$$\mathcal{F}[Z] = \beta\Delta_W[Z] + \frac{\gamma}{2}\mathcal{C}^\kappa[Z] + \mathcal{K}[Z] \qquad\qquad \beta,\gamma \geq 0 \tag{1}$$

where

$$\mathcal{C}^\kappa[Z] = \sum_{g=1}^m \frac{\rho_g^2 - e(g,g)}{\rho_g^\kappa} \quad \text{where} \quad e(g,g) = \sum_{i,j=1}^n e_{ij} z_{ig} z_{jg} \quad \text{and} \quad \kappa \in [0,1] \ . \tag{2}$$

yields the necessary first-order condition

$$z_{ig} = \frac{\rho_g \exp(-H_{ig})}{\sum_h \rho_h \exp(-H_{ih})} \tag{3}$$

with

$$\rho_g[Z] = \sum_{i=1}^n f_i z_{ig} \qquad\qquad H_{ig}[Z] = \beta D_i^g + \gamma \rho_g^{-\kappa}[\rho_g - (Wz^g)_i] - \frac{\gamma\kappa}{2}\rho_g^{-\kappa-1}[\rho_g^2 - e(g,g)] \ . \tag{4}$$

Here $D_i^g = \sum_j f_i^g D_{ij} - \Delta_g$ is the squared Euclidean dissimilarity from i to the centroid of group g, and $(\mathcal{L}z^g)_i = z_{ig} - \sum_j w_{ij} z_{jg} = z_{ig} - (Wz^g)_i$ is the *Laplacian* of membership z^g at pixel i, comparing its value to the average value of its neighbors, and adjusting the former to the latter.

Define $\zeta_i := \sum_h \rho_h \exp(-H_{ih})$. Then

$$\mathcal{K}[Z] = \sum_{ig} f_i z_{ig} \ln \frac{z_{ig}}{\rho_g} = -\sum_{ig} f_i z_{ig} H_{ih} - \sum_i f_i \ln \zeta_i \tag{5}$$

and

$$\begin{aligned} \sum_{ig} f_i z_{ig} H_{ih} &= \beta \sum_{ig} f_i z_{ig} D_i^g + \gamma \sum_{ig} f_i z_{ig} \rho_g^{-\kappa}[\rho_g - (Wz^g)_i] \\ &- \sum_{ig} f_i z_{ig} \frac{\gamma\kappa}{2} \rho_g^{-\kappa-1}[\rho_g^2 - e(g,g)] = \beta \sum_g \sum_i \frac{f_i z_{ig}}{\rho_g} D_i^g + \gamma \sum_g \rho_g^{-\kappa}[\rho_g^2 - e(g,g)] \\ &- \frac{\gamma\kappa}{2} \sum_g \rho_g^{-\kappa}[\rho_g^2 - e(g,g)] = \beta\Delta_W[Z] + \gamma(1 - \frac{\kappa}{2})\mathcal{C}^\kappa[Z] \end{aligned} \tag{6}$$

Hence

$$\begin{aligned}\mathcal{F}[Z] &= -\ln\zeta_i - \beta\Delta_W[Z] - \gamma(1-\frac{\kappa}{2})\mathcal{C}^\kappa[Z] + \beta\Delta_W[Z] + \frac{\gamma}{2}\mathcal{C}^\kappa[Z] \\ &= -\sum_i f_i \ln\zeta_i + \gamma\frac{\kappa-1}{2}\mathcal{C}^\kappa[Z] \end{aligned} \tag{7}$$