# Estimation of flow trajectories in a multiple lines network

Experiments with *transports publics de la région lausannoise* (tl) data

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# Table of contents

# Introduction

# Context

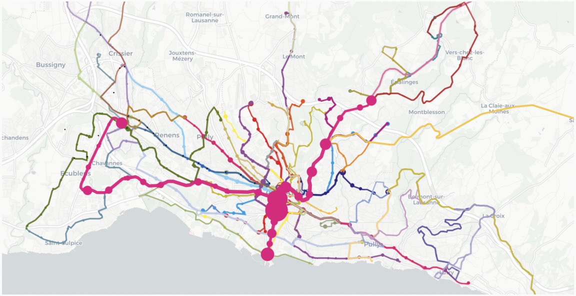
The **tl dataset**, used by Romain Loup for his PhD:

- 1 year of data (2019).
- 115 millions of passengers.
- 42 bus and subway lines.
- 1361 stops and 497 “superstops”.
- Every journey data: traveling time, waiting time, embarking and disembarking passengers at each stops, etc.

# Context

```
##   stop_id  stop_name line_id direction order embarkment disembarkment
## 1 MALAD_N  Maladière       1         A     1     164558             0
## 2 MTOIE_E    Montoie       1         A     2     136236         12705
## 3 BATEL_E  Batelière       1         A     3     203045         13409
## 4 RTCOU_E Riant-Cour       1         A     4     156015         24909
```

```
##    stop_id  stop_name line_id direction order embarkment disembarkment
## 42 RTCOU_O Riant-Cour       1         R    19      23634        132201
## 43 BATEL_O  Batelière       1         R    20      13707        168884
## 44 MTOIE_O    Montoie       1         R    21       4259        128255
## 45 MALAD_N  Maladière       1         R    22          0        146798
```



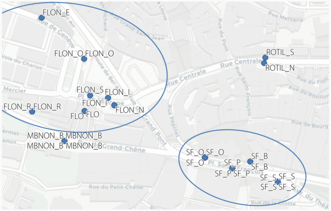
# The multiple lines network

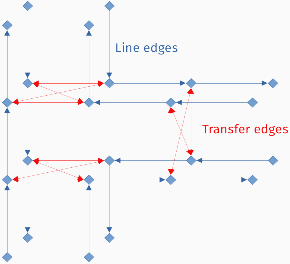
Having only lines data, the structure is a **disconnected oriented graph**.

In addition to **line edges**, it is possible to construct **transfer edges** to make the graph connected, by using, e.g.,

- Superstops names,
- Pedestrian time,
- Distance.

With transfer edges, we have a **unilaterally connected graph**.





# Problematic

This dataset offers multiple axes of research. In this presentation, we will focus on one question:

> *Knowing* **(1) the network structure** *and* **(2) the number of passengers embarking and disembarking at each stop**, *can we deduce* **trajectories of the passengers** *in the network ?*

Short answer: **No**.

Thank you for your attention !

Questions ?

(just kidding)

# The problematic

Exact trajectories are impossible to know, but, with additional assumptions, we can **estimate** them.

We will divide this problematic into two parts:

- The estimation of trajectories on a **single line**.
- The estimation of trajectories on the **multiple lines network**.

# The single line problem

# Formal problem definition

Let a line (in one direction), which have $n$ stops, indexed by line order. Let $\boldsymbol{\rho}_{\text{in}} = (\rho_s^{\text{in}})$ and $\boldsymbol{\rho}_{\text{out}} = (\rho_t^{\text{out}})$ be two vectors representing, respectively, the **passengers entering and leaving lines at each stop**.

We search a $(n \times n)$ **origin-destination matrix** $\mathbf{N} = (n_{st})$ where components represents

$$n_{st} = \text{“the number of passengers entering line at } s \text{ and leaving at } t\text{''}.$$

These components must verify

1. $n_{st} \geq 0$,
2. $n_{s\bullet} = \rho_s^{\text{in}}$,
3. $n_{\bullet t} = \rho_t^{\text{out}}$.

($\bullet$ indicates a sum on the replaced index)

# Formal problem definition

It reads:

$$
\mathbf{N} = \begin{array}{cc}
 & \begin{array}{ccccc} \rho_1^{\text{out}} & \rho_2^{\text{out}} & \cdots & \rho_{n-1}^{\text{out}} & \rho_n^{\text{out}} \end{array} \\
\begin{array}{c} \rho_1^{\text{in}} \\ \rho_2^{\text{in}} \\ \vdots \\ \rho_{n-1}^{\text{in}} \\ \rho_n^{\text{in}} \end{array} &
\begin{pmatrix}
n_{11} & n_{12} & \cdots & n_{1,n-1} & n_{1n} \\
n_{21} & n_{22} & \cdots & n_{2,n-1} & n_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
n_{n-1,1} & n_{n-1,2} & \cdots & n_{n-1,n-1} & n_{n-1,n} \\
n_{n,1} & n_{n,2} & \cdots & n_{n,n-1} & n_{n,n}
\end{pmatrix}
\end{array}
$$

In fact, we already know that some components are null:

$$
\mathbf{N} = \begin{array}{cc}
 & \begin{array}{ccccc} \mathbf{0} & \rho_2^{\text{out}} & \cdots & \rho_{n-1}^{\text{out}} & \rho_n^{\text{out}} \end{array} \\
\begin{array}{c} \rho_1^{\text{in}} \\ \rho_2^{\text{in}} \\ \vdots \\ \rho_{n-1}^{\text{in}} \\ \mathbf{0} \end{array} &
\begin{pmatrix}
\mathbf{0} & n_{12} & \cdots & n_{1,n-1} & n_{1n} \\
\mathbf{0} & \mathbf{0} & \ddots & \cdot & n_{2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & n_{n-1,n} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}
\end{pmatrix}
\end{array}
$$

# Formal problem definition

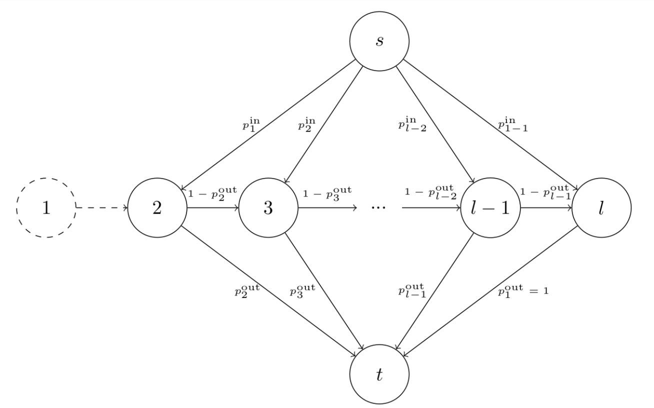
In this form, the problem is **ill posed**, because it has multiple solutions. An exemple of solution is to make passengers follow a **first in, first out (FIFO)** scheme.

A principle of mathematical modeling is to find the solution which makes the **least assumptions about passenger behavior**, in other words the **maximum entropy solution**.

In this case, it translates by supposing that there is the **same probability of leaving the line for every passenger which have traveled at least one stop**.

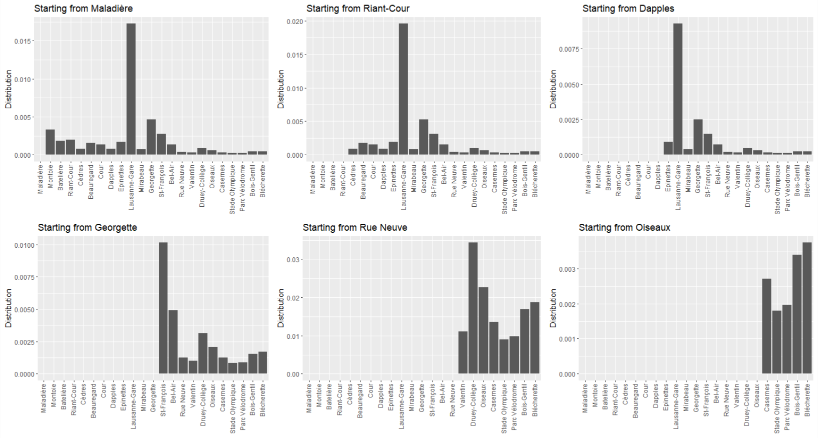
# Solution with Markov chain modeling

We can then model passenger flow with a **Markov chain**:

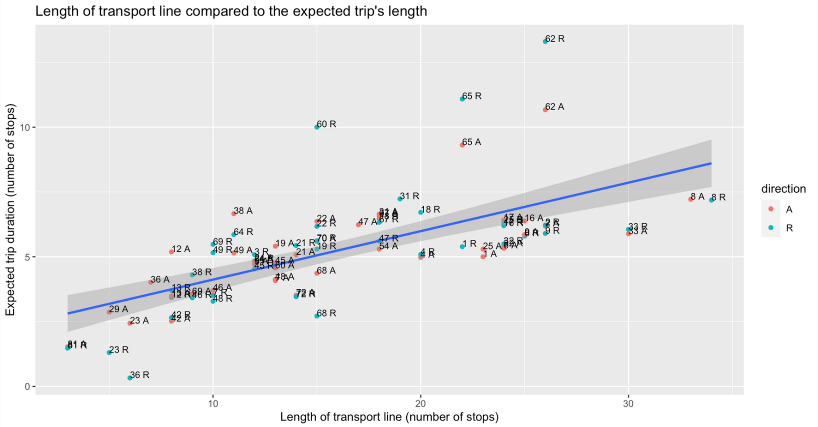



with $p_i^{\text{in}} = \frac{\rho_i^{\text{in}}}{\rho_\bullet^{\text{in}}}$ and $p_i^{\text{out}} = \frac{\rho_i^{\text{out}}}{\sum_{1\leq k\leq(i-1)}(\rho_k^{\text{in}}-\rho_k^{\text{out}})}$.

# Solution with Markov chain modeling



# Solution with Markov chain modeling



# Iterative proportional fitting

# Iterative proportional fitting

The same solution can be obtained with the **iterative proportional fitting (IPF)** algorithm [Bishop et al., 1975]. Let

1. $\mathbf{P} = (p_{ij})$ a $(n \times m)$ matrix,
2. $\mathbf{u} = (u_i)$ a n-length vector, and
3. $\mathbf{v} = (v_i)$ a m-length vector,

all of them with strictly positive components. We can find two vectors $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$ such that the matrix $\mathbf{Q} = (q_{ij})$, defined with

$$q_{ij} = a_i b_j p_{ij},$$

verifies

- $q_{i\bullet} = u_i$,
- $q_{\bullet j} = v_j$,
- $K(\mathbf{Q}|\mathbf{P}) := \sum_{ij} \frac{q_{ij}}{q_{\bullet\bullet}} \log\left(\frac{q_{ij}/q_{\bullet\bullet}}{p_{ij}/p_{\bullet\bullet}}\right)$ is minimum.

# Solution with iterative proportional fitting

In our context, it means that if we define an **origin-destination affinity matrix $\mathbf{S} = (s_{st})$** with

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

we can find the **maximum entropy solution with iterative proportional fitting**, i.e., we can find $\mathbf{a} = (a_s)$ and $\mathbf{b} = (b_t)$, such that $n_{ij} = a_s b_t s_{st}$ verifies

1. $n_{s\bullet} = \rho_s^{\text{in}}$,
2. $n_{\bullet t} = \rho_t^{\text{out}}$,
3. $K(\mathbf{N}|\mathbf{S})$ is minimum.

(a small number $\epsilon$ has to be added on null components).

# Solution with iterative proportional fitting

By **decreasing (resp. increasing)** $s_{st}$, we **reduce (resp. expand)** the resulting number of passengers going from $s$ to $t$ obtained with IPF.

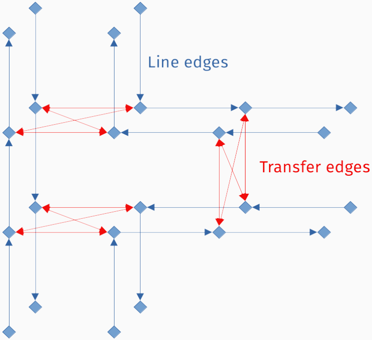
Thus, this approach is more **flexible**, because we could give a **specific affinity matrix** $\mathbf{S} = (s_{st})$, based on other data (additional assumptions).

The relationship $n_{st} = a_s b_t s_{st}$ can be seen as a **gravity model** (still to investigate).

# The multiple lines problem

# The multiple lines problem

In this problem, we will have to use the whole multiple lines network, which is composed of **line edges** and **transfer edges**.




It is not possible to use a Markov chain modeling in this case, but the **iterative proportional fitting** approach is still valid. However, there are **additional difficulties**.

# The flow in/out lines/network

To begin with, we have to distinguish between

- The passengers **entering and leaving lines** at each stops, represented by vectors $\boldsymbol{\rho}_{\mathbf{in}}$ and $\boldsymbol{\rho}_{\mathbf{out}}$ and
- The passengers **entering and leaving the network** at each stops, represented by vectors $\boldsymbol{\sigma}_{\mathbf{in}}$ and $\boldsymbol{\sigma}_{\mathbf{out}}$, which are **unknown**.

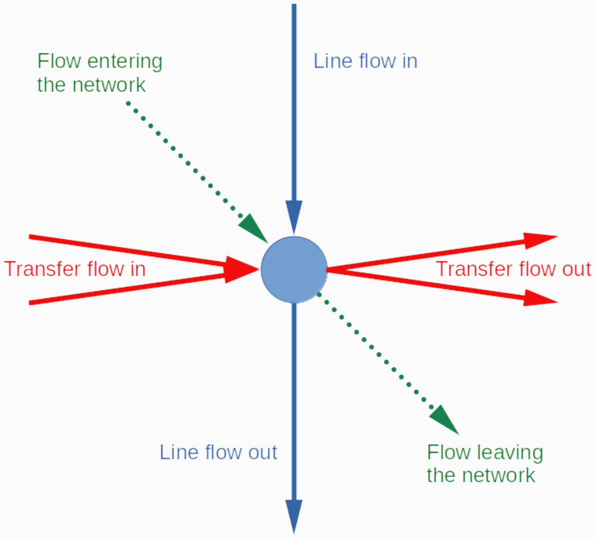
At each stop $i$, we have

$$\rho_i^{\text{in}} = \sigma_i^{\text{in}} + x_{\bullet i}^{\text{B}},$$
$$\rho_i^{\text{out}} = \sigma_i^{\text{out}} + x_{i\bullet}^{\text{B}},$$

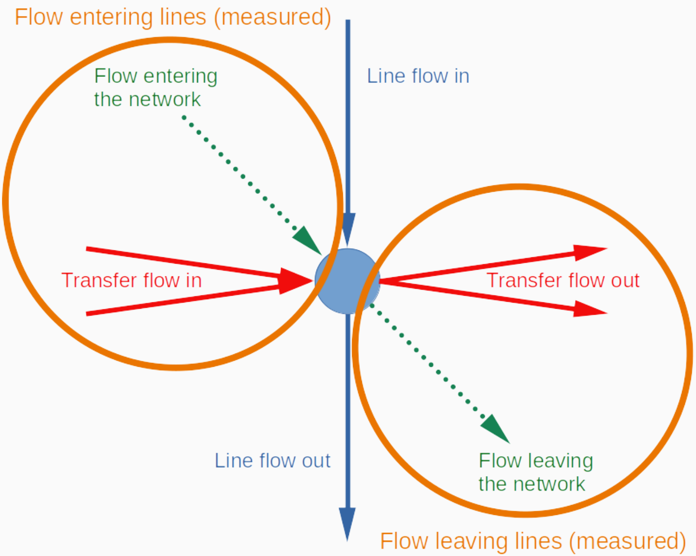
where

- $x_{\bullet i}^{\text{B}}$ is the **transfer flow entering the node** $i$ and
- $x_{i\bullet}^{\text{B}}$ is the **transfer flow leaving the node** $i$.

# The flow in/out lines/network



# The flow in/out lines/network



# The flow in/out lines/network

The flow entering/leaving the lines, $\boldsymbol{\rho}_{\text{in}}$ and $\boldsymbol{\rho}_{\text{out}}$, are **known**.

If we know **transfer flow on edges**, i.e. $\mathbf{X}_{\text{B}} = (x_{ij}^{\text{B}})$, we can **compute the flow entering/leaving the network**, $\boldsymbol{\sigma}_{\text{in}}$ and $\boldsymbol{\sigma}_{\text{in}}$.

The flow entering/leaving the lines, $\rho_i^{\text{in}}$ and $\rho_i^{\text{out}}$, acts as **constraints on the in/out transfer flow**, $x_{\bullet i}^{\text{B}}$ and $x_{i\bullet}^{\text{B}}$.

When there are no transfers on $i$, we have $\rho_i^{\text{in}} = \sigma_i^{\text{in}}$ and $\rho_i^{\text{out}} = \sigma_i^{\text{out}}$.

# Flow behavior

The second difficulty is that there are generally **multiple routes** to reach node $t$ from node $s$. This can be solved by making a new assumption.

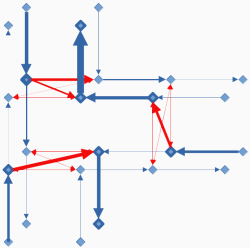
In the multiple lines network, we will suppose that passengers take **shortest-paths** in order to reach node $t$ from node $s$.

If multiple shortest-paths exists between $s$ and $t$, the passenger flow is **divided equally** among them.

# Flow behavior

This assumption unlocks a very useful property. If we have an **origin-destination matrix** $\mathbf{N} = (n_{st})$, we can compute the $(n \times n)$ **flow matrix on edges** $\mathbf{X} = (x_{ij})$.

$$\mathbf{N} = \begin{pmatrix} 0 & 13055 & 243 & \cdots & 144 \\ 3498 & 0 & 24429 & \cdots & 7523 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1508 & & \cdots & 0 & 5093 \\ 8903 & 6343 & \cdots & 53 & 0 \end{pmatrix} \rightarrow$$



# Algorithm outline

Let us make an outline for the **iterative Algorithm**. There is **4 steps** at each iteration:

$$\left.\begin{array}{r}\text{OD affinity matrix } \mathbf{S}\\ \text{Flow in the network } \boldsymbol{\sigma}_{\text{in}}\\ \text{Flow out the network } \boldsymbol{\sigma}_{\text{out}}\end{array}\right\} \xrightarrow{IPF} \text{OD matrix } \mathbf{N} \tag{1}$$

$$\text{OD matrix } \mathbf{N} \xrightarrow{SP} \text{Transfer flow } \mathbf{X}^{\text{B}} \tag{2}$$

$$\left.\begin{array}{r}\text{Transfer flow } \mathbf{X}^{\text{B}}\\ \text{Flow in lines } \boldsymbol{\rho}_{\text{in}}\\ \text{Flow out lines } \boldsymbol{\rho}_{\text{out}}\end{array}\right\} \xrightarrow{\text{Constraints}} \left\{\begin{array}{l}\text{Allowed transfer flow } \widetilde{\mathbf{X}}^{\text{B}}\\ \text{Flow in the network } \boldsymbol{\sigma}_{\text{in}}\\ \text{Flow out the network } \boldsymbol{\sigma}_{\text{out}}\end{array}\right. \tag{3}$$

$$\left.\begin{array}{r}\text{Transfer flow } \mathbf{X}^{\text{B}}\\ \text{Allowed transfer flow } \widetilde{\mathbf{X}}^{\text{B}}\end{array}\right\} \xrightarrow{\text{Affinity update}} \text{OD affinity matrix } \mathbf{S} \tag{4}$$

# Step 1: iterative proportional fitting

At the beginning of the algorithm, we have to set

- An **initial flow in the network** . We can set it to $\boldsymbol{\sigma}_{\text{in}}^{\text{init}} = \boldsymbol{\rho}_{\text{in}}$,
- An **initial flow out the network** . We can set it to $\boldsymbol{\sigma}_{\text{out}}^{\text{init}} = \boldsymbol{\rho}_{\text{out}}$,
- An **initial affinity matrix between origin-destination**, $\mathbf{S}^{\text{init}}$.

The initial affinity matrix $\mathbf{S}^{\text{init}} = (s_{st}^{\text{init}})$ is crafted in order to have:

- $s_{st}^{\text{init}} = 1$ if $s$ is a **valid trajectory** for using the network.
- $s_{st}^{\text{init}} = 0$ otherwise.

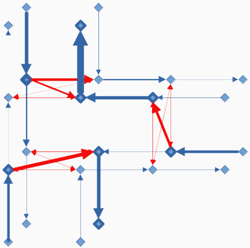
It now possible to obtain a first origin-destination matrix $\mathbf{N}$ with **iterative proportional fitting**.

# Step 2: shortest-paths flow

In this step, we use the assumption that passengers use **shortest-paths** in the network to obtain flow on edges, and in particular, **flow on transfer edges**:

$$\mathbf{N} \longrightarrow \mathbf{X}^{\mathrm{B}}$$

$$\mathbf{N} = \begin{pmatrix} 0 & 13055 & 243 & \cdots & 144 \\ 3498 & 0 & 24429 & \cdots & 7523 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1508 & & \cdots & 0 & 5093 \\ 8903 & 6343 & \cdots & 53 & 0 \end{pmatrix} \rightarrow$$

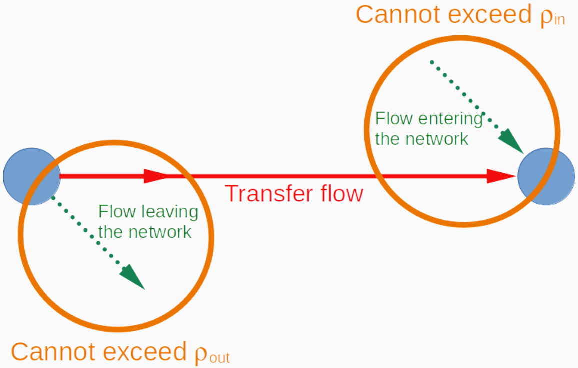


# Step 3: corrected transfer flow

This transfer flow $\mathbf{X}^{\mathrm{B}}$ could be used to update $\boldsymbol{\sigma}_{\text{in}}$ and $\boldsymbol{\sigma}_{\text{out}}$, with

- $\sigma_i^{\text{in}} = \rho_i^{\text{in}} - x_{\bullet i}^{\mathrm{B}}$,
- $\sigma_i^{\text{out}} = \rho_i^{\text{out}} - x_{i \bullet}^{\mathrm{B}}$,

However, there is **no guarantee that the flow will not exceed limits given by $\rho_{\text{in}}$ and $\rho_{\text{in}}$**.



# Step 3: allowed transfer flow

For each $x_{ij}^{\mathsf{B}}$, we use $\rho_i^{\text{out}}$ and $\rho_j^{\text{in}}$ in order to compute a **allowed transfer flow** $\widetilde{x}_{ij}^{\mathsf{B}} \leq x_{ij}^{\mathsf{B}}$. There are multiple choices:

1. **Constraint thresholds** could be reachable.
2. We can set a **percentage limit** for the transfer flow among the flow in/out of the lines.
3. We can set a **soft limit** to the transfer flow, with, e.g., an exponential law.

When this corrected flow is computed, $\boldsymbol{\sigma}_{\text{in}}$ and $\boldsymbol{\sigma}_{\text{out}}$ can be **updated** with

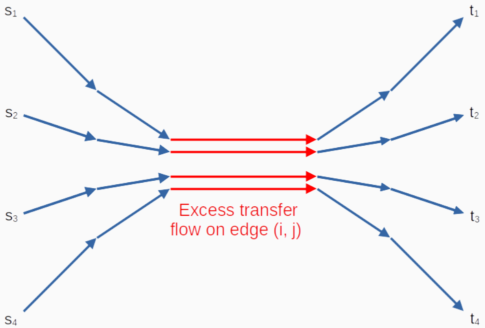
$$\sigma_i^{\text{in}} = \rho_i^{\text{in}} - \widetilde{x}_{\bullet i}^{\mathsf{B}},$$
$$\sigma_i^{\text{out}} = \rho_i^{\text{out}} - \widetilde{x}_{i\bullet}^{\mathsf{B}}.$$

# Step 4: Affinity update

What about the **excess flow on transfer edges**, i.e., $x_{ij}^{B} - \widetilde{x}_{ij}^{B}$ ?

Note that we only updated **margins distribution**, but we need also need a way to reduce **dependencies** between particular $s$ and $t$.

Having flow following **shortest-paths**, we know which couples $s, t$ are "responsible" for the excess flow on edge $(i, j)$.



# Step 4: Affinity update

On each transfer edge, we can compute the **proportion of allowed flow**:

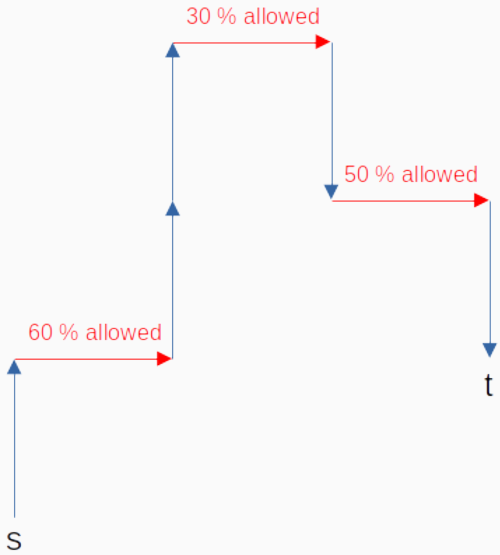
$$p_{ij}^{\text{allowed}} = \frac{\widetilde{x}_{ij}^{\text{B}}}{x_{ij}^{\text{B}}}$$

Each transfer edge will then "send a signal" to all couples $s, t$ using this edge on their shortest-paths, in order to **reduce their affinities $s_{st}$**.

Each couple $s, t$, will recieve a **list of proportion of allowed flow**, from all transfer edges on its shortest-path. The **update factor for the affinity $s_{st}$** is constructed by using the minimum allowed flow received.

$$s_{st}^{\text{new}} = \min\{p_{i_1,j_1}^{\text{allowed}}, \ldots, p_{i_k,j_k}^{\text{allowed}}\} \cdot s_{st}^{\text{old}}$$

# Step 4: Affinity update




In this example, $s_{st}^{\text{new}} = 0.3 \cdot s_{st}^{\text{old}}$

# Results (demo)

# Conclusion

# Conclusion

As it is still a work in progress, the conclusion will take the form of a to-do list:

- **Optimizing** speed/memory (almost done).
- Running the algorithm on **all lines**.
- Using carefully crafted **OD affinities**.
- Finding **applications**.
- Finding a **proof of convergence**.
- Finding if the solution is **unique** (doubt it).
- Making links with **gravity model**, **transportation problem**, ...
- **Adapting** the algorithm to closely related problems.

Thank you for your attention !

Questions ?

(for real this time)