

# Transportation network with multiple lines

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## 1 Formalism

### 1.1 The transportation network with multiple lines

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a simple, oriented, and connected graph representing a transportation network between  $|\mathcal{V}| = n$  nodes, having  $|\mathcal{E}| = m$  edges, and possessing  $p$  different transportation lines. Each node belongs to only one line, i.e.  $\mathcal{V} = \bigcup_{k=1}^p \mathcal{V}_k$  and  $\bigcap_{k=1}^p \mathcal{V}_k = \emptyset$ , where  $\mathcal{V}_k$  represents the set of nodes in line  $k$ . The edge set  $\mathcal{E}$ , can also be decomposed with

$$\mathcal{E} = \mathcal{E}_W \cup \mathcal{E}_B, \quad \mathcal{E}_W := \bigcup_{k=1}^p \mathcal{E}_k, \quad \mathcal{E}_W \cap \mathcal{E}_B = \emptyset, \quad \mathcal{E}_k \cap \mathcal{E}_l = \emptyset, \quad \forall k, l. \quad (1)$$

where  $\mathcal{E}_k$  is the set of edges composing line  $k$ ,  $\mathcal{E}_W$  the set containing all edges inside lines, and  $\mathcal{E}_B$  the set of transfer edges, connecting the different lines. The graph  $\mathcal{G}$  can be represented by its adjacency matrix  $\mathbf{A} = (a_{ij})$ , which can also be decomposed with

$$\mathbf{A} = \mathbf{A}_W + \mathbf{A}_B, \quad \mathbf{A}_W = \sum_{k=1}^p \mathbf{A}_k, \quad (2)$$

with  $\mathbf{A}_k = (a_{ij}^k)$  are edges of line  $k$ ,  $\mathbf{A}_W = (a_{ij}^W)$  edges of inside all lines, and  $\mathbf{A}_B = (a_{ij}^B)$  transfer edges. We suppose that there is an uniquely define route inside lines, i.e.

$$a_{i\bullet}^k \leq 1 \text{ and } a_{\bullet i}^k \leq 1, \quad \forall i, k. \quad (3)$$

where  $\bullet$  designates a summation over the replaced index.

### 1.2 The origin-destination matrix

The  $(n \times n)$  *origin-destination matrix*, denoted by  $\mathbf{N} = (n_{st})$ ,  $n_{st} \geq 0$ ,  $\forall s, t$ , contains the flow (e.g. the number of passengers) entering the network in source node  $s$  and leaving it in target node  $t$ . We can denote its margins with

$$\boldsymbol{\sigma}_{\text{in}} := \mathbf{N} \mathbf{e}_n \quad (4)$$

$$\boldsymbol{\sigma}_{\text{out}} := \mathbf{N}^\top \mathbf{e}_n \quad (5)$$

where  $\mathbf{e}_n$  is the vector of ones of size  $n$ . The vector  $\boldsymbol{\sigma}_{\text{in}} = (\sigma_i^{\text{in}})$  is the *vector of flow entering the network* and  $\boldsymbol{\sigma}_{\text{out}} = (\sigma_i^{\text{out}})$  is the *vector of flow leaving the network*. We have

$$\sigma_{\bullet}^{\text{in}} = \sigma_{\bullet}^{\text{out}}. \quad (6)$$

### 1.3 The flow matrix

A flow on edges is represented by the  $(n \times n)$  *flow matrix*  $\mathbf{X} = (x_{ij})$ , verifying

$$x_{ij} \geq 0, \quad \forall i, j, \quad (7)$$

$$a_{ij} = 0 \Rightarrow x_{ij} = 0, \quad \forall i, j, \quad (8)$$

$$x_{i\bullet} + \sigma_i^{\text{out}} = x_{\bullet i} + \sigma_i^{\text{in}}, \quad \forall i. \quad (9)$$

Again, we can decompose the flow matrix with

$$\mathbf{X} = \mathbf{X}_W + \mathbf{X}_B \quad \mathbf{X}_W := \sum_{k=1}^l \mathbf{X}_k \quad (10)$$

where  $\mathbf{X}_k$  represent the flow inside line  $k$ ,  $\mathbf{X}_W$  is the flow inside all lines, and  $\mathbf{X}_B$  the flow between lines. This decomposition allows us to define the *vector of flow entering lines*  $\boldsymbol{\rho}_{\text{in}} = (\rho_i^{\text{in}})$  and the *vector of flow leaving lines*  $\boldsymbol{\rho}_{\text{out}} = (\rho_i^{\text{out}})$ , with

$$\boldsymbol{\rho}_{\text{in}} := \boldsymbol{\sigma}_{\text{in}} + \mathbf{X}_B^{\top} \mathbf{e}_n, \quad (11)$$

$$\boldsymbol{\rho}_{\text{out}} := \boldsymbol{\sigma}_{\text{out}} + \mathbf{X}_B \mathbf{e}_n, \quad (12)$$

where  $\mathbf{e}_n$  is the vector of ones of size  $n$ . It is easy to see that we still have  $\rho_{\bullet}^{\text{in}} = \rho_{\bullet}^{\text{out}}$ .

### 1.4 Shortest-paths flow

Let  $\mathcal{P}_{st}$  be the set of *admissible* shortest-paths between  $s$  and  $t$  on  $\mathcal{G}$ . We can denote by  $P_{st}(i, j)$  the probability of having edge  $(i, j) \in \wp$  when drawing a path  $\wp$  from  $\mathcal{P}_{st}$ . We have

$$P_{st}(i, j) := \frac{1}{|\mathcal{P}_{st}|} \sum_{\wp \in \mathcal{P}_{st}} \delta((i, j) \in \wp), \quad (13)$$

where  $\delta(\cdot)$  designate the indicator function. If we are given an origin-destination matrix  $\mathbf{N} = (n_{st})$ , we can compute the *shortest-path flow matrix*, noted  $\mathbf{X}_{\text{sp}} = (x_{ij}^{\text{sp}})$ , with:

$$x_{ij}^{\text{sp}} = \sum_{st} P_{st}(i, j) n_{st} \quad (14)$$

This matrix contains the flow on each edge if we suppose that the flow follows shortest-paths.

We can rewrite equation (14) by defining the  $(n^2 \times n^2)$  *shortest-path - edge matrix*  $\mathbf{S} = (s_{\alpha\beta})$  with

$$s_{\alpha\beta} = \begin{cases} P_{st}(i, j) & \text{if } \alpha = t + n(s - 1) \text{ and } \beta = j + n(i - 1), \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

Then (14) writes

$$\text{vec}(\mathbf{X}) = \mathbf{S}^\top \text{vec}(\mathbf{N}), \quad (16)$$

where  $\text{vec}(\cdot)$  denotes the vectorization function of a matrix, obtained by stacking matrix columns on top of one another.

### 1.5 Problem definition

We suppose that we know the flow entering and leaving each line, i.e.  $\rho_{\text{in}}$  and  $\rho_{\text{out}}$  and we are interested in finding trajectories  $n_{st}$  of the flow in the network knowing the set  $\mathcal{A}$  of admissible pair  $(s, t)$ , i.e. pairs of nodes where there is a possible use of the network for traveling. This set can be given by the matrix  $\mathbf{T} = (t_{st})$  defined by

$$t_{st} = \begin{cases} 1 & \text{if } (s, t) \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

### 1.6 Algorithm

Set  $\sigma_{\text{in}}^{(0)} = \rho_{\text{in}} + \epsilon$  and  $\sigma_{\text{out}}^{(0)} = \rho_{\text{out}} + \epsilon$ , with  $\epsilon$  a small positive scalar. Until convergence, do:

1. Compute  $\mathbf{N}^{(i)} = \mathbf{Diag}(\mathbf{a}^{(i)})(\mathbf{T} + \epsilon)\mathbf{Diag}(\mathbf{b}^{(i)})$  with iterative fitting, such that  $\mathbf{N}^{(i)}\mathbf{e}_n = \sigma_{\text{in}}^{(i)}$  and  $(\mathbf{N}^{(i)})^\top \mathbf{e}_n = \sigma_{\text{out}}^{(i)}$ .
2. Compute the associated shortest-path flow matrix  $\mathbf{X}^{(i)}$  with (14).
3. Compute  $\sigma_{\text{in}}^{(i+1)} = \rho_{\text{in}} - (\mathbf{X}_{\text{B}}^{(i)})^\top \mathbf{e}_n$  and  $\sigma_{\text{out}}^{(i+1)} = \rho_{\text{out}} - \mathbf{X}_{\text{B}}^{(i)} \mathbf{e}_n$ .