Differential Equations

and

Linear Algebra

A Course for
Science and Engineering

Part I: Chapters 1-7
Part II: Chapters 8-12

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Chapter 1

Fundamentals

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1.2 Exercises

Growth-Decay Model Solve the given initial value problem using the growth-decay formula; see page **??** and Example 1.1, page 9.

1. $y' = -3y,\ y(0) = 20$
Solution: $y(x) = 20\,e^{-3x}$ by the growth-decay formula page **??**.

2. $y' = 3y,\ y(0) = 1$

3. $3A' = A,\ A(0) = 1$
Solution: $A(t) = e^{t/3}$

4. $4A' + A = 0,\ A(0) = 3$

5. $3P' - P = 0,\ P(0) = 10$
Solution: $P(t) = 10\, e^{t/3}$

6. $4P' + 3P = 0,\ P(0) = 11$

7. $I' = 0.005I,\ I(t_0) = I_0$
Solution: $I(t) = I_0\, e^{(t-t_0)/200}$

8. $I' = -0.015I,\ I(t_0) = I_0$

9. $y' = \alpha y,\ y(t_0) = 1$
Solution: $y(t) = e^{\alpha(t-t_0)}$

10. $y' = -\alpha y,\ y(t_0) = y_0$

Growth-decay Theory

11. Graph without a computer $y = 10(2^x)$ on $-3 \le x \le 3$.

12. Graph without a computer $y = 10(2^{-x})$ on $-3 \le x \le 3$.

13. Find the doubling time for the growth model $P = 100e^{0.015t}$.

14. Find the doubling time for the growth model $P = 1000e^{0.0195t}$.

15. Find the elapsed time for the decay model $A = 1000e^{-0.11237t}$ until $|A(t)| < 0.00001$.

16. Find the elapsed time for the decay model $A = 5000e^{-0.01247t}$ until $|A(t)| < 0.00005$.

Newton Cooling Recipe Solve the given cooling model. Follow Example 1.2 on page 10.

17. $u' = -10(u-4),\ u(0) = 5$

18. $y' = -5(y-2),\ y(0) = 10$

19. $u' = 1 + u,\ u(0) = 100$

20. $y' = -1 - 2y,\ y(0) = 4$

21. $u' = -10 + 4u,\ u(0) = 10$

22. $y' = 10 + 3y,\ y(0) = 1$

23. $2u' + 3 = 6u$, $u(0) = 8$

24. $4y' + y = 10$, $y(0) = 5$

25. $u' + 3(u + 1) = 0$, $u(0) = -2$

26. $u' + 5(u + 2) = 0$, $u(0) = -1$

27. $\alpha' = -2(\alpha - 3)$, $\alpha(0) = 10$

28. $\alpha' = -3(\alpha - 4)$, $\alpha(0) = 12$

Newton Cooling Model The cooling model $u(t) = u_0 + A_0 e^{-ht}$ is applied; see page **??**. Methods parallel those in the flask cooling example, page **??**, and the baking example, page **??**.

29. **(Ingot Cooling)** A metal ingot cools in the air at temperature 20C from 130C to 75C in one hour. Predict the cooling time to 23C.

30. **(Rod Cooling)** A plastic rod cools in a large vat of 12-degree Celsius water from 75C to 20C in 4 minutes. Predict the cooling time to 15C.

31. **(Murder Mystery)** A body discovered at 1:00 in the afternoon, March 1, 1929, had temperature 80F. Over the next hour the body's temperature dropped to 76F. Estimate the date and time of the murder.

32. **(Time of Death)** A dead person found in a 40F river had body temperature 70F. The coroner requested that the body be left in the river for 45 minutes, whereupon the body's temperature was 63F. Estimate the time of death, relative to the discovery of the body.

Verhulst Model Solve the given Verhulst logistic equation using formula (**??**). Follow Example 1.3 on page 10.

33. $P' = P(2 - P)$, $P(0) = 1$

34. $P' = P(4 - P)$, $P(0) = 5$

35. $y' = y(y - 1)$, $y(0) = 2$

36. $y' = y(y - 2)$, $y(0) = 1$

37. $A' = A - 2A^2$, $A(0) = 3$

38. $A' = 2A - 5A^2$, $A(0) = 1$

39. $F' = 2F(3 - F)$, $F(0) = 2$

40. $F' = 3F(2-F)$, $F(0) = 1$

Inverse Modeling Given the model, find the differential equation and initial condition.

41. $A = A_0 e^{4t}$

42. $A = A_0 e^{-3t}$

43. $P = 1000e^{-0.115t}$

44. $P = 2000e^{-7t/5}$

45. $u = 1 + e^{-3t}$

46. $u = 10 - 2e^{-2t}$

47. $P = \dfrac{10}{10 - 8e^{-2t}}$

48. $P = \dfrac{5}{15 - 14e^{-t}}$

49. $P = \dfrac{1}{5 - 4e^{-t}}$

50. $P = \dfrac{2}{4 - 3e^{-t}}$

Populations The following exercises use Malthusian population theory, page **??**, and the Malthusian model $P(t) = P_0 e^{kt}$. Methods appear in Examples **??** and 1.5; see page **??**.

51. **(World Population)** In June of 1993, the world population of $5,500,000,000$ people was increasing at a rate of $250,000$ people per day. Predict the date when the population reaches 10 billion.

52. **(World Population)** Suppose the world population at time $t = 0$ is 5 billion. How many years before that was the population one billion?

53. **(Population Doubling)** A population of rabbits increases by 10% per year. In how many years does the population double?

54. **(Population Tripling)** A population of bacteria increases by 15% per day. In how many days does the population triple?

55. **(Population Growth)** Trout in a river are increasing by 15% in 5 years. To what population size does 500 trout grow in 15 years?

56. **(Population Growth)** A region of 400 acres contains 1000 forest mushrooms per acre. The population is decreasing by 150 mushrooms per acre every 2 years. Find the population size for the 400-acre region in 15 years.

Verhulst Equation Write out the solution to the given differential equation and, when it makes sense, report the carrying capacity

$$M = \lim_{t\to\infty} P(t).$$

57. $P' = (1 - P)P$

58. $P' = (2 - P)P$

59. $P' = 0.1(3 - 2P)P$

60. $P' = 0.1(4 - 3P)P$

61. $P' = 0.1(3 + 2P)P$

62. $P' = 0.1(4 + 3P)P$

63. $P' = 0.2(5 - 4P)P$

64. $P' = 0.2(6 - 5P)P$

65. $P' = 11P - 17P^2$

66. $P' = 51P - 13P^2$

Logistic Equation The following exercises use the Verhulst logistic equation $P' = (a - bP)P$, page **??**. Some methods appear on page **??**.

67. **(Protozoa)** Experiments on the protozoa *Paramecium* determined growth rate $a = 2.309$ and carrying capacity $a/b = 375$ using initial population $P(0) = 5$. Establish the formula $P(t) = \dfrac{375}{1 + 74e^{-2.309t}}$.

68. **(World Population)** Demographers projected the world population in the year 2000 as 6.5 billion, which was corrected by census to 6.1 billion. Use $P(1965) = 3.358 \times 10^9$, $a = 0.029$ and carrying capacity $a/b = 1.0760668 \times 10^{10}$ to compute the logistic equation projection for year 2000.

69. **(Harvesting)** A fish population satisfying $P' = (a - bP)P$ is subjected to harvesting, the new model being $P' = (a - bP)P - H$. Assume $a = 0.04$, $a/b = 5000$ and $H = 1000$. Using algebra, rewrite it as $P' = b(\alpha - P)(P - \beta)$ in terms of the roots α, β of $ay - by^2 - H = 0$. Apply the change of variables $u = (\alpha - P)/(P - \beta)$ to solve it.

70. (Extinction) Let an endangered species satisfy $P' = bP^2 - aP$ for $a > 0$, $b > 0$. The term bP^2 represents births due to chance encounters of males and females, while the term aP represents deaths. Use the change of variable $u = P/(bP - a)$ to solve it. Show from the answer that population sizes below a/b become extinct.

71. (Logistic Answer Check) Let $P = au/(1 + bu)$, $u = u_0 e^{at}$, $u_0 = P_0/(a - bP_0)$. Verify that $P(t)$ is a solution the differential equation $P' = (a - bP)P$ and $P(0) = P_0$.

72. (Logistic Equation) Let k, α, β be positive constants, $\alpha < \beta$. Solve $w' = k(\alpha - w)(\beta - w)$, $w(0) = w_0$ by the substitution $u = (\alpha - w)/(\beta - w)$, showing that $w = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha-\beta)kt}$, $u_0 = (\alpha - w_0)/(\beta - w_0)$. This equation is a special case of the harvesting equation $P' = (a-bP)P+H$.

Growth-Decay Uniqueness Proof

73. State precisely and give a calculus text reference for *Rolle's Theorem*, which says that a function vanishing at $x = a$ and $x = b$ must have slope zero at some point in $a < x < b$.

74. Apply Rolle's Theorem to prove that a differentiable function $v(x)$ with $v'(x) = 0$ on $a < x < b$ must be constant.

1.3 Exercises

Light Intensity The following exercises apply the theory of light intensity on page **??**, using the model $I(t) = I_0 e^{-kx}$ with x in meters. Methods parallel Example 1.8 on page 27.

1. The light intensity is $I(t) = I_0 e^{-1.4x}$ in a certain swimming pool. At what depth does the light intensity fall off by 50%?

2. The light intensity in a swimming pool falls off by 50% at a depth of 2.5 meters. Find the depletion constant k in the exponential model.

3. Plastic film is used to cover window glass, which reduces the interior light intensity by 10%. By what percentage is the intensity reduced, if two layers are used?

4. Double-thickness colored window glass is supposed to reduce the interior light intensity by 20%. What is the reduction for single-thickness colored glass?

RC-Electric Circuits In the exercises below, solve for $Q(t)$ when $Q_0 = 10$ and graph $Q(t)$ on $0 \le t \le 5$.

5. $R = 1$, $C = 0.01$.

6. $R = 0.05$, $C = 0.001$.

7. $R = 0.05$, $C = 0.01$.

8. $R = 5$, $C = 0.1$.

9. $R = 2$, $C = 0.01$.

10. $R = 4$, $C = 0.15$.

11. $R = 4$, $C = 0.02$.

12. $R = 50$, $C = 0.001$.

LR-Electric Circuits In the exercises below, solve for $I(t)$ when $I_0 = 5$ and graph $I(t)$ on $0 \le t \le 5$.

13. $L = 1$, $R = 0.5$.

14. $L = 0.1$, $R = 0.5$.

15. $L = 0.1$, $R = 0.05$.

16. $L = 0.01$, $R = 0.05$.

17. $L = 0.2$, $R = 0.01$.

18. $L = 0.03$, $R = 0.01$.

19. $L = 0.05$, $R = 0.005$.

20. $L = 0.04$, $R = 0.005$.

Interest and Continuous Interest Financial formulas which appear on page **??** are applied below, following the ideas in Examples 1.11, 1.12 and 1.13, pages 28–29.

21. (Total Interest) Compute the total daily interest and also the total continuous interest for a 10-year loan of $5,000$ dollars at 5% per annum.

22. (Total Interest) Compute the total daily interest and also the total continuous interest for a 15-year loan of $7,000$ dollars at $5\frac{1}{4}\%$ per annum.

23. (Monthly Payment) Find the monthly payment for a 3-year loan of $8,000$ dollars at 7% per annum compounded continuously.

24. (Monthly Payment) Find the monthly payment for a 4-year loan of $7,000$ dollars at $6\frac{1}{3}\%$ per annum compounded continuously.

25. (Effective Yield) Determine the effective annual yield for a certificate of deposit at $7\frac{1}{4}\%$ interest per annum, compounded continuously.

26. (Effective Yield) Determine the effective annual yield for a certificate of deposit at $5\frac{3}{4}\%$ interest per annum, compounded continuously.

27. (Retirement Funds) Assume a starting salary of $35,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $10\frac{1}{2}\%$ of salary, deposited monthly, growing at $5\frac{1}{2}\%$ continuous interest per annum. Find the retirement amount after 30 years.

28. (Retirement Funds) Assume a starting salary of $45,000$ dollars per year, which is expected to increase 3% per year. Retirement contributions are $9\frac{1}{2}\%$ of salary, deposited monthly, growing at $6\frac{1}{4}\%$ continuous interest per annum. Find the retirement amount after 30 years.

29. (Actual Cost) A van is purchased for $18,000$ dollars with no money down. Monthly payments are spread over 8 years at $12\frac{1}{2}\%$ interest per annum, compounded continuously. What is the actual cost of the van?

30. **(Actual Cost)** Furniture is purchased for $15,000$ dollars with no money down. Monthly payments are spread over 5 years at $11\frac{1}{8}\%$ interest per annum, compounded continuously. What is the actual cost of the furniture?

Radioactive Decay Assume the decay model $A' = -kA$ from page **??**. Below, $A(T) = 0.5A(0)$ defines the *half-life* T. Methods parallel Examples 1.14– 1.17 on pages 31– 32.

31. **31.**(Half-Life) Determine the half-life of a radium sample which decays by 5.5% in 13 years.

32. **(Half-Life)** Determine the half-life of a radium sample which decays by 4.5% in 10 years.

33. **(Half-Life)** Assume a radioactive isotope has half-life 1800 years. Determine the percentage decayed after 150 years.

34. **(Half-Life)** Assume a radioactive isotope has half-life 1650 years. Determine the percentage decayed after 99 years.

35. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 5.5% in 13 years.

36. **(Disintegration Constant)** Determine the constant k in the model $A' = -kA$ for radioactive material that disintegrates by 4.5% in 10 years.

37. **(Radiocarbon Dating)** A fossil found near the town of Dinosaur, Utah contains carbon-14 at a ratio of 6.21% to the atmospheric value. Determine its approximate age according to Libby's method.

38. **(Radiocarbon Dating)** A fossil found in Colorado contains carbon-14 at a ratio of 5.73% to the atmospheric value. Determine its approximate age according to Libby's method.

39. **(Radiocarbon Dating)** In 1950, the Lascaux Cave in France contained charcoal with 14.52% of the carbon-14 present in living wood samples nearby. Estimate by Libby's method the age of the charcoal sample.

40. **(Radiocarbon Dating)** At an excavation in 1960, charcoal from building material had 61% of the carbon-14 present in living wood nearby. Estimate the age of the building.

41. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 5% in 12 years. By what percentage is it reduced in 99 years?

42. **(Percentage of an Isotope)** A radioactive isotope disintegrates by 6.5% in $1,000$ years. By what percentage is it reduced in $5,000$ years?

Chemical Reactions Assume below the model $A' = kA$ for a first-order reaction. See page ?? and Example 1.18, page 33.

43. **(First-Order $A+B \longrightarrow C$)** A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $N\%$ of A remains after t_0 minutes.

44. **(First-Order $A+B \longrightarrow C$)** A first order reaction produces product C from chemical A and catalyst B. Model the production of C, given $M\%$ of A is depleted after t_0 minutes.

45. **(Law of Mass-Action)** Consider a second-order chemical reaction $X(t)$ with $k = 0.14$, $\alpha = 1$, $\beta = 1.75$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 2$.

46. **(Law of Mass-Action)** Consider a second-order chemical reaction $X(t)$ with $k = 0.015$, $\alpha = 1$, $\beta = 1.35$, $X(0) = 0$. Find an explicit formula for $X(t)$ and graph it on $t = 0$ to $t = 10$.

47. **(Mass-Action Derivation)** Let k, α, β be positive constants, $\alpha < \beta$. Solve $X' = k(\alpha - X)(\beta - X)$, $X(0) = X_0$ by the substitution $u = (\alpha - X)/(\beta - X)$, showing that $X = (\alpha - \beta u)/(1 - u)$, $u = u_0 e^{(\alpha-\beta)kt}$, $u_0 = (\alpha - X_0)/(\beta - X_0)$.

48. **(Mass-Action Derivation)** Let k, α, β be positive constants, $\alpha < \beta$. Define $X = (\alpha - \beta u)/(1 - u)$, where $u = u_0 e^{(\alpha-\beta)kt}$ and $u_0 = (\alpha - X_0)/(\beta - X_0)$. Verify by calculus computation that (1) $X' = k(\alpha - X)(\beta - X)$ and (2) $X(0) = X_0$.

Drug Dosage Employ the drug dosage model $D(t) = D_0 e^{-ht}$ given on page ??. Let h be determined by a half-life of three hours. Apply the techniques of Example 1.19, page 33.

49. **(Injection Dosage)** Bloodstream injection of a drug into an animal requires a minimum of 20 milligrams per pound of body weight. Predict the dosage for a 12-pound animal which will maintain a drug level 3% higher than the minimum for two hours.

50. **(Injection Dosage)** Bloodstream injection of an antihistamine into an animal requires a minimum of 4 milligrams per pound of body weight. Predict the dosage for a 40-pound animal which will maintain an antihistamine level 5% higher than the minimum for twelve hours.

51. **(Oral Dosage)** An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Predict the number of hours after the first dose at which to take a second dose, in order to maintain a blood level of at least 180 milligrams for three hours.

52. **(Oral Dosage)** An oral drug with first dose 250 milligrams is absorbed into the bloodstream after 45 minutes. Determine three (small) dosage amounts, and their administration time, which keep the blood level above 180 milligrams but below 280 milligrams over three hours.

1.4 Exercises

Solution Verification Given the differential equation, initial condition and proposed solution y, verify that y is a solution. Don't try to *solve* the equation!

1. $\dfrac{dy}{dx} = y$, $y(0) = 2$, $y = 2e^x$

2. $y' = 2y$, $y(0) = 1$, $y = e^{2x}$

3. $y' = y^2$, $y(0) = 1$, $y = (1 - x)^{-1}$

4. $\dfrac{dy}{dx} = y^3$, $y(0) = 1$,
$y = (1 - 2x)^{-1/2}$

5. $D^2 y(x) = y(x)$, $y(0) = 2$,
$Dy(0) = 2$, $y = 2e^x$

6. $D^2 y(x) = -y(x)$, $y(0) = 0$,
$Dy(0) = 1$, $y = \sin x$

7. $y' = \sec^2 x$, $y(0) = 0$, $y = \tan x$

8. $y' = -\csc^2 x$, $y(\pi/2) = 0$,
$y = \cot x$

9. $y' = e^{-x}$, $y(0) = -1$, $y = -e^{-x}$

10. $y' = 1/x$, $y(1) = 1$, $y = \ln x$

Explicit and Implicit Solutions Identify the given solution as *implicit* or *explicit*. If *implicit*, then solve for y in terms of x by college algebra methods.

11. $y = x + \sin x$

12. $y = x + \sin x$

13. $2y + x^2 + x + 1 = 0$

14. $x - 2y + \sin x + \cos x = 0$

15. $y = e^{\pi}$

16. $e^y = \pi$

17. $e^{2y} = \ln(1 + x)$

18. $\ln|1+y^2| = e^x$

19. $\tan y = 1 + x$

20. $\sin y = (x-1)^2$

Tables and Explicit Equations For the given explicit equation, make a table of values $x = 0$ to $x = 1$ in steps of 0.2.

21. $y = x^2 - 2x$

22. $y = x^2 - 3x + 1$

23. $y = \sin \pi x$

24. $y = \cos \pi x$

25. $y = e^{2x}$

26. $y = e^{-x}$

27. $y = \ln(1+x)$

28. $y = x\ln(1+x)$

Tables and Approximate Equations Make a table of values $x = 0$ to $x = 1$ in steps of 0.2 for the given approximate equation. Identify precisely the *recursion* formulas applied to obtain the next table pair from the previous table pair.

29. $y(x+0.2) \approx y(x) + 0.2(1 - y(x))$, $y(0) = 1$

30. $y(x+0.2) \approx y(x) + 0.2(1 + y(x))$, $y(0) = 1$

31. $y(x+0.2) \approx y(x) + 0.2(x - y(x))$, $y(0) = 0$

32. $y(x+0.2) \approx y(x) + 0.2(2x + y(x))$, $y(0) = 0$

33. $y(x+0.2) \approx y(x) + 0.2(\sin x + xy(x))$, $y(0) = 2$

34. $y(x+0.2) \approx y(x) + 0.2(\sin x - x^2 y(x))$, $y(0) = 2$

35. $y(x+0.2) \approx y(x) + 0.2(e^x - 7y(x))$, $y(0) = -1$

36. $y(x+0.2) \approx y(x) + 0.2(e^{-x} - 5y(x))$, $y(0) = -1$

37. $y(x+0.2) \approx y(x) + 0.1(e^{-2x} - 3y(x))$, $y(0) = 2$

38. $y(x+0.2) \approx y(x) + 0.2(\sin 2x - 2y(x))$, $y(0) = 2$

1.4 Exercises

Hand Graphing Make a graphic by hand on engineering paper, using 6 data points. Cite the divisions assigned horizontally and vertically. Label the axes and the center of coordinates. Supply one sample hand computation per graph. Employ a computer program or calculator to obtain the data points.

39. $y = 3x$, $x = 0$ to $x = 1$.

40. $y = 5x^3$, $x = 0$ to $x = 1$.

41. $y = 2x^5$, $x = 0$ to $x = 1$.

42. $y = 3x^7$, $x = 0$ to $x = 1/2$.

43. $y = 2x^4$, $x = 0$ to $x = 1$.

44. $y = 3x^6$, $x = 0$ to $x = 1$.

45. $y = \sin x$, $x = 0$ to $x = \pi/4$.

46. $y = \cos x$, $x = 0$ to $x = \pi/4$.

47. $y = \dfrac{x+1}{x+2}$, $x = 0$ to $x = 1$.

48. $y = \dfrac{x-1}{x+1}$, $x = 0$ to $x = 1$.

49. $y = \ln(1 + x)$, $x = 0$ to $x = 1$.

50. $y = \ln(1 + 2x)$, $x = 0$ to $x = 1$.

1.5 Exercises

Window and Grid Find the equilibrium solutions, then determine a graph window which includes them and construct a 5×5 uniform grid. Follow Example 1.25.

1. $y' = 2y$

2. $y' = 3y$

3. $y' = 2y + 2$

4. $y' = 3y - 2$

5. $y' = y(1 - y)$

6. $y' = 2y(3 - y)$

7. $y' = y(1 - y)(2 - y)$

8. $y' = 2y(1 - y)(1 + y)$

9. $y' = 2(y - 1)(y + 1)^2$

10. $y' = 2y^2(y - 1)^2$

11. $y' = (x + 1)(y + 1)(y - 1)y$

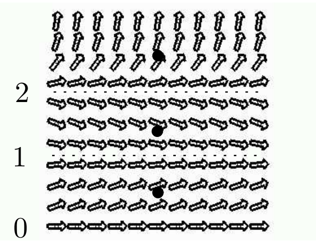
12. $y' = 2(x + 1)y^2(y + 1)(y - 1)^2$

13. $y' = (x + 2)y(y - 3)(y + 2)$

14. $y' = (x + 1)y(y - 2)(y + 3)$

Threading Solutions Each direction field below has window $0 \le x \le 3$, $0 \le y \le 3$. Start each threaded solution at a black dot and continue it left and right across the field. Dotted horizontal lines are equilibrium solutions. See Example 1.26.

15.



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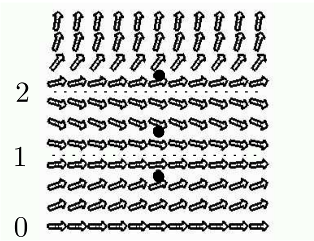

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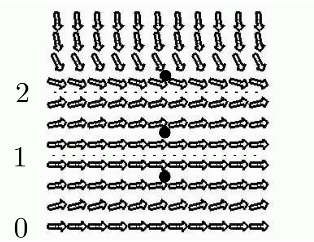

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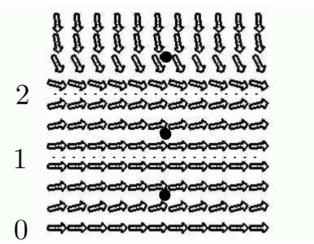

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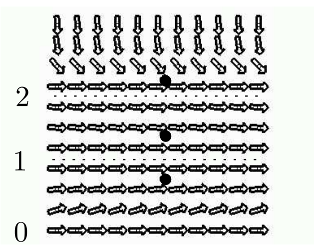

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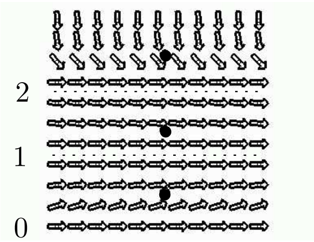

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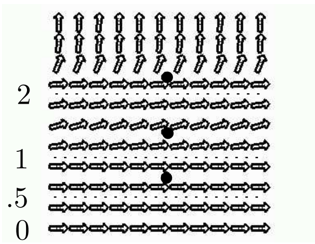

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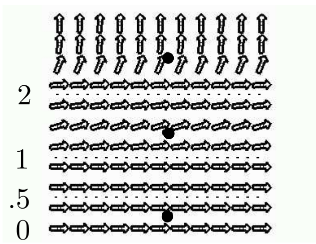
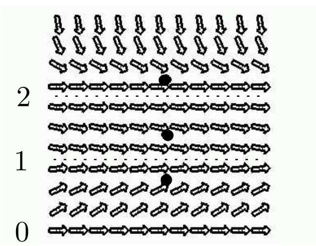
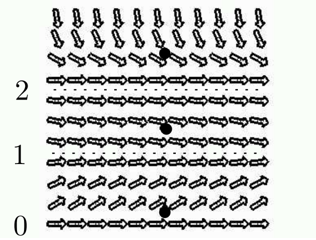


23.



24.



Uniform Grid Method Apply the uniform grid method as in Example 1.27, page 57 to make a direction field of 11×11 grid points for the given differential equation on $-1 \leq x \leq 1$, $-2 \leq y \leq 2$. If using a computer program, then use about 20×20 grid points.

25. $y' = 2y$

26. $y' = 3y$

27. $y' = 1 + y$

28. $y' = 2 + 3y$

29. $y' = x + y(2 - y)$

30. $y' = x + y(1 - 2y)$

31. $y' = 1 + y(2 - y)$

32. $y' = 1 + 2y(2 - y)$

33. $y' = x - y$

34. $y' = x + y$

35. $y' = y - \sin(x)$

36. $y' = y + \sin(x)$

Isocline Method Apply the isocline method as in Example 1.28, page 59 to make a direction field of about 11×11 points for the given differential equation on $0 \leq x \leq 1$, $0 \leq y \leq 2$. Computer programs are used on these kinds of problems to find grid points as intersections of isoclines and lines. Graphics are expected to be done by hand. Large white spaces in the graphic should be filled by choosing a richer set of slopes.

37. $y' = x - y^2$

38. $y' = 2x - y^2$

39. $y' = 2y/(x+1)$

40. $y' = -y^2/(x+1)^2$

41. $y' = \sin(x - y)$

42. $y' = \cos(x - y)$

43. $y' = xy$

44. $y' = x^2 y$

45. $y' = xy + 2x$

46. $y' = x^2 y + 2x^2$

1.6 Exercises

Stability-Instability Test

Find all equilibria for the given differential equation and then apply Theorem 1.3, page 69, to obtain a classification of each equilibrium as a **source**, **sink** or **node**. Do not draw a phase line diagram.

1. $P' = (2 - P)P$

2. $P' = (1 - P)(P - 1)$

3. $y' = y(2 - 3y)$

4. $y' = y(1 - 5y)$

5. $A' = A(A - 1)(A - 2)$

6. $A' = (A - 1)(A - 2)^2$

7. $w' = \dfrac{w(1 - w)}{1 + w^2}$

8. $w' = \dfrac{w(2 - w)}{1 + w^4}$

9. $v' = \dfrac{v(1 + v)}{4 + v^2}$

10. $v' = \dfrac{(1 - v)(1 + v)}{2 + v^2}$

Phase Line Diagram

Draw a phase line diagram, with detail similar to Figure 20.

11. $y' = y(2 - y)$

12. $y' = (y + 1)(1 - y)$

13. $y' = (y - 1)(y - 2)$

14. $y' = (y - 2)(y + 3)$

15. $y' = y(y - 2)(y - 1)$

16. $y' = y(2 - y)(y - 1)$

17. $y' = \dfrac{(y - 2)(y - 1)}{1 + y^2}$

18. $y' = \dfrac{(2-y)(y-1)}{1+y^2}$

19. $y' = \dfrac{(y-2)^2(y-1)}{1+y^2}$

20. $y' = \dfrac{(y-2)(y-1)^2}{1+y^2}$

Phase Portrait

Draw a phase portrait of threaded curves, using the phase line diagram constructed in the previous ten exercises.

21. $y' = y(2-y)$

22. $y' = (y+1)(1-y)$

23. $y' = (y-1)(y-2)$

24. $y' = (y-2)(y+3)$

25. $y' = y(y-2)(y-1)$

26. $y' = y(2-y)(y-1)$

27. $y' = \dfrac{(y-2)(y-1)}{1+y^2}$

28. $y' = \dfrac{(2-y)(y-1)}{1+y^2}$

29. $y' = \dfrac{(y-2)^2(y-1)}{1+y^2}$

30. $y' = \dfrac{(y-2)(y-1)^2}{1+y^2}$

Bifurcation Diagram

Draw a stack of phase line diagrams and construct from it a succinct bifurcation diagram with abscissa k and ordinate $y(0)$. Don't justify details at a bifurcation point.

31. $y' = (2-y)y - k$

32. $y' = (3-y)y - k$

33. $y' = (2-y)(y-1) - k$

34. $y' = (3 - y)(y - 2) - k$

35. $y' = y(2 - y)(y - 1) - k$

36. $y' = y(2 - y)(y - 2) - k$

37. $y' = y(y - 1)^2 - k$

38. $y' = y^2(y - 1) - k$

39. $y' = y(0.5 - 0.001y) - k$

40. $y' = y(0.4 - 0.045y) - k$

Details and Proofs

Supply details for the following statements.

41. (Stability Test)
Verify **(b)** of Theorem 1.3, page 69, by altering the proof given in the text for **(a)**.

42. (Stability Test)
Verify **(b)** of Theorem 1.3, page 69, by means of the change of variable $x \to -x$.

43. (Autonomous Equations)
Let $y' = f(y)$ have solution $y(x)$ on $a < x < b$. Then for any c, $a < c < b$, the function $z(x) = y(x + c)$ is a solution of $z' = f(z)$.

44. (Autonomous Equations)
The method of isoclines can be applied to an autonomous equation $y' = f(y)$ by choosing equally spaced horizontal lines $y = c_i$, $i = 1, \ldots, k$. Along each horizontal line $y = c_i$ the slope is a constant $M_i = f(c_i)$, and this determines the set of invented slopes $\{M_i\}_{i=1}^{k}$ for the method of isoclines.

1.7 Exercises

Multiple Solution Example

Define $f(x,y) = 3(y-1)^{2/3}$. Consider $y' = f(x,y)$, $y(0) = 1$.

1. Do an answer check for $y(x) = 1$. Do a second answer check for $y(x) = 1+x^3$.

2. Let $y(x) = 1$ on $0 \le x \le 1$ and $y(x) = 1 + (x-1)^3$ for $x \ge 1$. Do an answer check for $y(x)$.

3. Does $f_y(x,y)$ exist for all (x,y)?

4. Verify that Picard's theorem does not apply to $y' = f(x,y)$, $y(0) = 1$, due to discontinuity of f_y.

5. Verify that Picard's theorem applies to $y' = f(x,y)$, $y(0) = 2$.

6. Let $y(x) = 1 + (x+1)^3$. Do an answer check for $y' = f(x,y)$, $y(0) = 2$. Does another solution exist?

Discontinuous Equation Example

Consider $y' = \dfrac{2y}{x-1}$, $y(0) = 1$. Define $y_1(x) = (x-1)^2$ and $y_2(x) = c(x-1)^2$. Define $y(x) = y_1(x)$ on $-\infty < x < 1$ and $y(x) = y_2(x)$ on $1 < x < \infty$. Define $y(1) = 0$.

7. Do an answer check for $y_1(x)$ on $-\infty < x < 1$. Do an answer check for $y_2(x)$ on $1 < x < \infty$. Skip condition $y(0) = 1$.

8. Justify one-sided limits $y(1+) = y(1-) = 0$. The functions y_1 and y_2 join continuously at $x = 1$ with common value zero and the formula for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

9. (a) For which values of c does $y'(1)$ exist? (b) For which values of c is $y(x)$ continuously differentiable?

10. Find all values of c such that $y(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Finite Blowup Example

Consider $y' = 1 + y^2$, $y(0) = 0$. Let $y(x) = \tan x$.

11. Do an answer check for $y(x)$.

12. Find the partial derivative f_y for $f(x,y) = 1 + y^2$. Justify that f and f_y are everywhere continuous.

13. Justify that Picard's theorem applies, hence $y(x)$ is the only possible solution to the initial value problem.

14. Justify for $a = -\pi/2$ and $b = \pi/2$ that $y(a+) = -\infty$, $y(b-) = \infty$. Hence $y(x)$ blows up for finite values of x.

Numerical Instability Example

Let $f(x, y) = y - 2e^{-x}$.

15. Do an answer check for $y(x) = e^{-x}$ as a solution of the initial value problem $y' = f(x, y)$, $y(0) = 1$.

16. Do an answer check for $y(x) = ce^x + e^{-x}$ as a solution of $y' = f(x, y)$.

Multiple Solutions

Consider the initial value problem $y' = 5(y-2)^{4/5}$, $y(0) = 2$.

17. Do an answer check for $y(x) = 2$. Do a second answer check for $y(x) = 2 + x^5$.

18. Verify that the hypotheses of Picard's theorem fail to apply.

19. Find a formula which displays infinitely many solutions to $y' = f(x, y)$, $y(0) = 2$.

20. Verify that the hypotheses of Peano's theorem apply.

Discontinuous Equation

Consider $y' = \dfrac{y}{x-1}$, $y(0) = 1$. Define $y(x)$ piecewise by $y(x) = -(x-1)$ on $-\infty < x < 1$ and $y(x) = c(x-1)$ on $1 < x < \infty$. Leave $y(1)$ undefined.

21. Do an answer check for $y(x)$. The initial condition $y(0) = 1$ applies only to the domain $-\infty < x < 1$.

22. Justify one-sided limits $y(1+) = y(1-) = 0$. The piecewise definitions of $y(x)$ join continuously at $x = 1$ with common value zero and the formula for $y(x)$ gives one continuous formal solution for each value of c (∞-many solutions).

23. (a) For which values of c does $y'(1)$ exist? (b) For which values of c is $y(x)$ continuously differentiable?

24. Find all values of c such that $y(x)$ is a continuously differentiable function that satisfies the differential equation and the initial condition.

Picard Iteration

Find the Picard iterates y_0, y_1, y_2, y_3.

25. $y' = y + 1$, $y(0) = 2$

26. $y' = y + 1$, $y(0) = 2$

27. $y' = y + 1$, $y(0) = 0$

28. $y' = 2y + 1$, $y(0) = 0$

29. $y' = y^2$, $y(0) = 1$

30. $y' = y^2$, $y(0) = 2$

31. $y' = y^2 + 1$, $y(0) = 0$

32. $y' = 4y^2 + 4$, $y(0) = 0$

33. $y' = y + x$, $y(0) = 0$

34. $y' = y + 2x$, $y(0) = 0$

Picard Iteration and Taylor Series

Find the Taylor polynomial $P_n(x) = y(0) + y'(0)x + \cdots + y^{(n)}(0)x^n/n!$ and compare with the Picard iterates. Use a computer algebra system, if possible.

35. $y' = y$, $y(0) = 1$, $n = 4$,
$y(x) = e^x$

36. $y' = 2y$, $y(0) = 1$, $n = 4$,
$y(x) = e^{2x}$

37. $y' = x - y$, $y(0) = 1$, $n = 4$,
$y(x) = -1 + x + 2e^{-x}$

38. $y' = 2x - y$, $y(0) = 1$, $n = 4$,
$y(x) = -2 + 2x + 3e^{-x}$

Numerical Instability

Use a computer algebra system or numerical laboratory. Let $f(x, y) = y - 2e^{-x}$.

39. Solve $y' = f(x, y)$, $y(0) = 1$ numerically for $y(30)$.

40. Solve $y' = f(x, y)$, $y(0) = 1 + 0.0000001$ numerically for $y(30)$.

Closed–Form Existence

Solve these initial value problems using a computer algebra system.

41. $y' = y$, $y(0) = 1$

42. $y' = 2y$, $y(0) = 2$

43. $y' = 2y + 1$, $y(0) = 1$

44. $y' = 3y + 2$, $y(0) = 1$

45. $y' = y(y - 1)$, $y(0) = 2$

46. $y' = y(1 - y)$, $y(0) = 2$

47. $y' = (y - 1)(y - 2)$, $y(0) = 3$

48. $y' = (y - 2)(y - 3)$, $y(0) = 1$

49. $y' = -10(1 - y)$, $y(0) = 0$

50. $y' = -10(2 - 3y)$, $y(0) = 0$

Lipschitz Condition

Justify the following results.

51. The function $f(x, y) = x - 10(2 - 3y)$ satisfies a Lipschitz condition on the whole plane.

52. The function $f(x, y) = ax + by + c$ satisfies a Lipschitz condition on the whole plane.

53. The function $f(x, y) = xy(1 - y)$ satisfies a Lipschitz condition on $D = \{(x, y) : |x| \leq 1, \quad |y| \leq 1\}$.

54. The function $f(x, y) = x^2 y(a - by)$ satisfies a Lipschitz condition on $D = \{(x, y) : x^2 + y^2 \leq R^2\}$.

55. If f_y is continuous on D and the line segment from y_1 to y_2 is in D, then $f(x, y_1) - f(x, y_2) = \int_{y_1}^{y_2} f_y(x, u) du$.

56. If f and f_y are continuous on a disk D, then f is Lipschitz with $M = \max_D\{|f_y(x, u)|\}$.

Chapter 2

First Order Differential Equations

Contents

2.2 Exercises

Quadrature

Find a candidate solution for each initial value problem and verify the solution. See Example 2.1 and Example 2.2, page 95.

1. $y' = 4e^{2x}$, $y(0) = 0$.

2. $y' = 2e^{4x}$, $y(0) = 0$.

3. $(1+x)y' = x$, $y(0) = 0$.

4. $(1-x)y' = x$, $y(0) = 0$.

5. $y' = \sin 2x$, $y(0) = 1$.

6. $y' = \cos 2x$, $y(0) = 1$.

7. $y' = xe^x$, $y(0) = 0$.

8. $y' = xe^{-x^2}$, $y(0) = 0$.

9. $y' = \tan x$, $y(0) = 0$.

10. $y' = 1 + \tan^2 x$, $y(0) = 0$.

11. $(1+x^2)y' = 1$, $y(0) = 0$.

12. $(1+4x^2)y' = 1$, $y(0) = 0$.

13. $y' = \sin^3 x$, $y(0) = 0$.

14. $y' = \cos^3 x$, $y(0) = 0$.

15. $(1+x)y' = 1$, $y(0) = 0$.

16. $(2+x)y' = 2$, $y(0) = 0$.

17. $(2+x)(1+x)y' = 2$, $y(0) = 0$.

18. $(2+x)(3+x)y' = 3$, $y(0) = 0$.

19. $y' = \sin x \cos 2x$, $y(0) = 0$.

20. $y' = (1 + \cos 2x)\sin 2x$, $y(0) = 0$.

River Crossing

A boat crosses a river of width w miles at v_b miles per hour with power applied perpendicular to the shoreline. The river's midstream velocity is v_c miles per hour. Find the transit time and the downstream drift to the opposite shore. See Example 2.3, page 97, and the details for (**??**).

21. $w = 1$, $v_b = 4$, $v_c = 12$

22. $w = 1$, $v_b = 5$, $v_c = 15$

23. $w = 1.2$, $v_b = 3$, $v_c = 13$

24. $w = 1.2$, $v_b = 5$, $v_c = 9$

25. $w = 1.5$, $v_b = 7$, $v_c = 16$

26. $w = 2$, $v_b = 7$, $v_c = 10$

27. $w = 1.6$, $v_b = 4.5$, $v_c = 14.7$

28. $w = 1.6$, $v_b = 5.5$, $v_c = 17$

Fundamental Theorem I

Verify the identity. Use the fundamental theorem of calculus part (b), page **??**.

29. $\int_0^x (1+t)^3 dt = \frac{1}{4}\left((1+x)^4 - 1\right)$.

30. $\int_0^x (1+t)^4 dt = \frac{1}{5}\left((1+x)^5 - 1\right)$.

31. $\int_0^x te^{-t} dt = -xe^{-x} - e^{-x} + 1$.

32. $\int_0^x te^{t} dt = xe^{x} - e^{x} + 1$.

Fundamental Theorem II

Differentiate. Use the fundamental theorem of calculus part (b), page **??**.

33. $\int_0^{2x} t^2 \tan(t^3) dt$.

34. $\int_0^{3x} t^3 \tan(t^2) dt$.

35. $\int_0^{\sin x} te^{t+t^2} dt$.

36. $\int_0^{\sin x} \ln(1+t^3) dt$.

Fundamental Theorem III

Integrate $\int_0^1 f(x)dx$. Use the fundamental theorem of calculus part (a), page **??**. Check answers with computer or calculator assist. Some require a clever u-substitution or an integral table.

37. $f(x) = x(x-1)$

38. $f(x) = x^2(x+1)$

39. $f(x) = \cos(3\pi x/4)$

40. $f(x) = \sin(5\pi x/6)$

41. $f(x) = \dfrac{1}{1+x^2}$

42. $f(x) = \dfrac{2x}{1+x^4})$

43. $f(x) = x^2 e^{x^3}$

44. $f(x) = x(\sin(x^2) + e^{x^2})$

45. $f(x) = \dfrac{1}{\sqrt{-1+x^2}}$

46. $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

47. $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

48. $f(x) = \dfrac{1}{\sqrt{1+4x^2}}$

49. $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

50. $f(x) = \dfrac{4x}{\sqrt{1-4x^2}}$

51. $f(x) = \dfrac{\cos x}{\sin x}$

52. $f(x) = \dfrac{\cos x}{\sin^3 x}$

53. $f(x) = \dfrac{e^x}{1+e^x}$

54. $f(x) = \dfrac{\ln |x|}{x}$

55. $f(x) = \sec^2 x$

56. $f(x) = \sec^2 x - \tan^2 x$

57. $f(x) = \csc^2 x$

58. $f(x) = \csc^2 x - \cot^2 x$

59. $f(x) = \csc x \cot x x$

60. $f(x) = \sec x \tan x x$

Integration by Parts

Integrate $\int_0^1 f(x)dx$ by parts, $\int udv = uv - \int vdu$. Check answers with computer or calculator assist.

61. $f(x) = xe^x$

62. $f(x) = xe^{-x}$

63. $f(x) = \ln|x|$

64. $f(x) = x\ln|x|$

65. $f(x) = x^2e^{2x}$

66. $f(x) = (1+2x)e^{2x}$

67. $f(x) = x\cosh x$

68. $f(x) = x\sinh x$

69. $f(x) = x\arctan(x)$

70. $f(x) = x\arcsin(x)$

Partial Fractions

Integrate f by partial fractions. Check answers with computer or calculator assist.

71. $f(x) = \dfrac{x+4}{x+5}$

72. $f(x) = \dfrac{x-2}{x-4}$

73. $f(x) = \dfrac{x^2+4}{(x+1)(x+2)}$

74. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)}$

75. $f(x) = \dfrac{x+4}{(x+1)(x+2)}$

76. $f(x) = \dfrac{x-1}{(x+1)(x+2))}$

77. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x+5)}$

78. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x+3)}$

79. $f(x) = \dfrac{x+4}{(x+1)(x+2)(x-1)}$

80. $f(x) = \dfrac{x(x-1)}{(x+1)(x+2)(x-1)}$

Special Methods

Integrate f by using the suggested u-substitution or method. Check answers with computer or calculator assist.

81. $f(x) = \dfrac{x^2+2}{(x+1)^2}$, $u = x+1$.

82. $f(x) = \dfrac{x^2+2}{(x-1)^2}$, $u = x-1$.

83. $f(x) = \dfrac{2x}{(x^2+1)^3}$, $u = x^2+1$.

84. $f(x) = \dfrac{3x^2}{(x^3+1)^2}$, $u = x^3+1$.

85. $f(x) = \dfrac{x^3+1}{x^2+1}$, use long division.

86. $f(x) = \dfrac{x^4+2}{x^2+1}$, use long division.

2.3 Exercises

Separated Form Test

Test the given equation by the separated form test on page **??**.

Report whether or not the equation *passes* or *fails*, as written. In this test, algebraic operations on the equation are disallowed. See Examples 2.4 and 2.5, page 107.

1. $y' = 2$

2. $y' = x$

3. $y' + y = 2$

4. $y' + 2y = x$

5. $yy' = 2 - x$

6. $2yy' = x + x^2$

7. $yy' + \sin(y') = 2 - x$

8. $2yy' + \cos(y) = x$

9. $2yy' = y' \cos(y) + x$

10. $(2y + \tan(y))y' = x$

Separated Equation

Determine the separated form $y'/G(y) = F(x)$ for the given separable equation. See Example 2.6, page 108.

11. $(1 + x)y' = 2 + y$

12. $(1 + y)y' = xy$

13. $y' = \dfrac{x + xy}{(x + 1)^2 - 1}$

14. $y' = \sin(x)\dfrac{1 + y}{(x + 2)^2 - 4}$

15. $xy' = y\sin(y)\cos(x)$

16. $x^2y' = y\cos(y)\tan(x)$

17. $y^2(x - y)y' = \dfrac{x^2 - y^2}{x + y}$

18. $xy^2(x+y)y' = \dfrac{y^2 - x^2}{x - y}$

19. $xy^2y' = \dfrac{y - x}{x - y}$

20. $xy^2y' = \dfrac{x^2 - xy}{x - y}$

Equilibrium solutions

Determine the equilibria for the given equation. See Examples 2.7 and 2.9.

21. $y' = xy(1+y)$

22. $xy' = y(1-y)$

23. $y' = \dfrac{1+y}{1-y}$

24. $xy' = \dfrac{y(1-y)}{1+y}$

25. $y' = (1+x)\tan(y)$

26. $y' = y(1+\ln y)$

27. $y' = xe^y(1+y)$

28. $xy' = e^y(1-y)$

29. $xy' = e^y(1-y^2)(1+y)^3$

30. $xy' = e^y(1-y^3)(1+y^3)$

Non-Equilibrium Solutions

Find the non-equilibrium solutions for the given separable equation. See Examples 2.8 and 2.10 for details.

31. $y' = (xy)^{1/3}$, $y(0) = y_0$.

32. $y' = (xy)^{1/5}$, $y(0) = y_0$.

33. $y' = 1 + x - y - xy$, $y(0) = y_0$.

34. $y' = 1 + x + 2y + 2xy$, $y(0) = y_0$.

35. $y' = \dfrac{(x+1)y^3}{x^2(y^3 - y)}$, $y(0) = y_0$.

36. $y' = \dfrac{(x-1)y^2}{x^3(y^3+y)}$, $y(0) = y_0$.

37. $2yy' = x(1-y^2)$

38. $2yy' = x(1+y^2)$

39. $(1+x)y' = 1-y$, $y(0) = y_0$.

40. $(1-x)y' = 1+y$, $y(0) = y_0$.

41. $\tan(x)y' = y$, $y(\pi/2) = y_0$.

42. $\tan(x)y' = 1+y$, $y(\pi/2) = y_0$.

43. $\sqrt{x}y' = \cos^2(y)$, $y(1) = y_0$.

44. $\sqrt{1-x}y' = \sin^2(y)$, $y(0) = y_0$.

45. $\sqrt{x^2-16}yy' = x$, $y(5) = y_0$.

46. $\sqrt{x^2-1}yy' = x$, $y(2) = y_0$.

47. $y' = x^2(1+y^2)$, $y(0) = 1$.

48. $(1-x)y' = x(1+y^2)$, $y(0) = 1$.

Independent of x

Solve the given equation, finding all solutions. See Example 2.11.

49. $y' = \sin y$, $y(0) = y_0$.

50. $y' = \cos y$, $y(0) = y_0$.

51. $y' = y(1+\ln y)$, $y(0) = y_0$.

52. $y' = y(2+\ln y)$, $y(0) = y_0$.

53. $y' = y(y-1)(y-2)$, $y(0) = y_0$.

54. $y' = y(y-1)(y+1)$, $y(0) = y_0$.

55. $y' = y^2+2y+5$, $y(0) = y_0$.

56. $y' = y^2+2y+7$, $y(0) = y_0$.

Details in the Examples

Collected here are verifications for details in the examples.

57. **(Example 2.7)** The equation $x(1-y)(1+y) = 0$ was solved in the example, but $x = 0$ was ignored, and only $y = -1$ and $y = 1$ were reported. Why?

58. **(Example 2.8)** An absolute value equation $|u| = w$ was replaced by $u = kw$ where $k = \pm 1$. Justify the replacement using the *definition* $|u| = u$ for $u \geq 0$, $|u| = -u$ for $u < 0$.

59. **(Example 2.8)** Verify directly that $y = (1 + y_0)e^{x^3/3} - 1$ solves the initial value problem $y' = x^2(1 + y)$, $y(0) = y_0$.

60. **(Example 2.9)** The relation $y = 1 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ describes the list $\ldots, 1 - \pi, 1, 1 + \pi, \ldots$. Write the list for the relation $y = -1 + (2n + 1)\frac{\pi}{2}$.

61. **(Example 2.9)** Solve $\sin(u) = 0$ and $\cos(v) = 0$ for u and v. Supply graphs which show why there are infinity many solutions.

62. **(Example 2.10)** Explain why $y_0/2$ does not equal $\text{Arctan}(\tan(y_0/2))$. Give a calculator example.

63. **(Example 2.10)** Establish the identity $\tan(y/2) = \csc y - \cot y$.

64. **(Example 2.11)** Let $y_0 > 0$. Verify that $y = e^{1 - (1 - \ln y_0)e^{-x}}$ solves

$$y' = y(1 - \ln y), \quad y(0) = y_0.$$

2.4 Exercises

Integrating Factor Method

Apply the integrating factor method, page **??**, to solve the given linear equation. See the examples starting on page **??** for details.

1. $y' + y = e^{-x}$

2. $y' + y = e^{-2x}$

3. $2y' + y = e^{-x}$

4. $2y' + y = e^{-2x}$

5. $2y' + y = 1$

6. $3y' + 2y = 2$

7. $2xy' + y = x$

8. $3xy' + y = 3x$

9. $y' + 2y = e^{2x}$

10. $2y' + y = 2e^{x/2}$

11. $y' + 2y = e^{-2x}$

12. $y' + 4y = e^{-4x}$

13. $2y' + y = e^{-x}$

14. $2y' + y = e^{-2x}$

15. $4y' + y = 1$

16. $4y' + 2y = 3$

17. $2xy' + y = 2x$

18. $3xy' + y = 4x$

19. $y' + 2y = e^{-x}$

20. $2y' + y = 2e^{-x}$

Superposition

Find a particular solution with fewest terms. See Example 2.15, page 123.

21. $3y' = x$

22. $3y' = 2x$

23. $y' + y = 1$

24. $y' + 2y = 2$

25. $2y' + y = 1$

26. $3y' + 2y = 1$

27. $y' - y = e^x$

28. $y' - y = xe^x$

29. $xy' + y = \sin x \ (x > 0)$

30. $xy' + y = \cos x \ (x > 0)$

31. $y' + y = x - x^2$

32. $y' + y = x + x^2$

General Solution

Find y_h and a particular solution y_p. Report the general solution $y = y_h + y_p$. See Example 2.17, page 124.

33. $y' + y = 1$

34. $xy' + y = 2$

35. $y' + y = x$

36. $xy' + y = 2x$

37. $y' - y = x + 1$

38. $xy' - y = 2x - 1$

39. $2xy' + y = 2x^2 \ (x > 0)$

40. $xy' + y = 2x^2 \ (x > 0)$

Classification

Classify as linear or non-linear. Use the test $f(x, y) = f(x, 0) + f_y(x, 0)y$ and a computer algebra system, when available, to check the answer. See Example 2.18, page 125.

41. $y' = 1 + 2y^2$

42. $y' = 1 + 2y^3$

43. $yy' = (1 + x) \ln e^y$

44. $yy' = (1 + x) (\ln e^y)^2$

45. $y' \sec^2 y = 1 + \tan^2 y$

46. $y' = \cos^2(xy) + \sin^2(xy)$

47. $y'(1 + y) = xy$

48. $y' = y(1 + y)$

49. $xy' = (x + 1)y - xe^{\ln y}$

50. $2xy' = (2x + 1)y - xye^{-\ln y}$

Shortcuts

Apply theorems for the homogeneous equation $y' + p(x)y = 0$ or for constant coefficient equations $y' + py = r$. Solutions should be done without paper or pencil, then write the answer and check it.

51. $y' - 5y = -1$

52. $3y' - 5y = -1$

53. $2y' + xy = 0$

54. $3y' - x^2y = 0$

55. $y' = 3x^4y$

56. $y' = (1 + x^2)y$

57. $\pi y' - \pi^2 y = -e^2$

58. $e^2y' + e^3y = \pi^2$

59. $xy' = (1 + x^2)y$

60. $e^xy' = (1 + e^{2x})y$

Proofs and Details

61. Prove directly without appeal to Theorem 2.6 that the difference of two solutions of $y' + p(x)y = r(x)$ is a solution of the homogeneous equation $y' + p(x)y = 0$.

62. Prove that y_p^* given by equation (**??**) and $y_p = W^{-1} \int r(x)W(x)dx$ given in the integrating factor method are related by $y_p = y_p^* + y_h$ for some solution y_h of the homogeneous equation.

63. The equation $y' = r$ with r constant can be solved by quadrature, without pencil and paper. Find y.

64. The equation $y' = r(x)$ with $r(x)$ continuous can be solved by quadrature. Find a formula for y.

2.5 Exercises

Variation of Parameters I

Report the shortest particular solution given by the formula

$$y_p(x) = \frac{\int rW}{W}, \quad W = e^{\int p(x)dx}$$

1. $y' = x + 1$

2. $y' = 2x - 1$

3. $y' + y = e^{-x}$

4. $y' + y = e^{-2x}$

5. $y' - 2y = 1$

6. $y' - y = 1$

7. $2y' + y = e^{x}$

8. $2y' + y = e^{-x}$

9. $xy' = x + 1$

10. $xy' = 1 - x^2$

Variation of Parameters II

Compute the particular solution given by

$$y_p^*(x) = \frac{\int_{x_0}^{x} rW}{W(x)}, \quad W(t) = e^{\int_{x_0}^{t} p(x)dx}.$$

11. $y' = x + 1$, $x_0 = 0$

12. $y' = 2x - 1$, $x_0 = 0$

13. $y' + y = e^{-x}$, $x_0 = 0$

14. $y' + y = e^{-2x}$, $x_0 = 0$

15. $y' - 2y = 1$, $x_0 = 0$

16. $y' - y = 1$, $x_0 = 0$

17. $2y' + y = e^{x}$, $x_0 = 1$

18. $2y' + y = e^{-x}$, $x_0 = 1$

19. $xy' = x + 1$, $x_0 = 1$

20. $xy' = 1 - x^2$, $x_0 = 1$

Euler Solution Atoms

Report the list of distinct Euler atoms of the given function $f(x)$. Then $f(x)$ is a sum of constants times the Euler atoms from this list.

21. $x + e^x$

22. $1 + 2x + 5e^x$

23. $x(1 + x + 2e^x)$

24. $x^2(2 + x^2) + x^2e^{-x}$

25. $\sin x \cos x + e^x \sin 2x$

26. $\cos^2 x - \sin^2 x + x^2 e^x \cos 2x$

27. $(1 + 2x + 4x^5)e^x e^{-3x} e^{x/2}$

28. $(1 + 2x + 4x^5 + e^x \sin 2x)e^{-3x/4} e^{x/2}$

29. $\dfrac{x + e^x}{e^{-2x}} \sin 3x + e^{3x} \cos 3x$

30. $\dfrac{x + e^x \sin 2x + x^3}{e^{-2x}} \sin 5x$

Initial Trial Solution

Differentiate repeatedly $f(x)$ and report the list of distinct Euler solution atoms which appear in f and all its derivatives. Then each derivative of $f(x)$ is a sum of constants times the Euler atoms from this list.

31. $12 + 5x^2 + 6x^7$

32. $x^6/x^{-4} + 10x^4/x^{-6}$

33. $x^2 + e^x$

34. $x^3 + 5e^{2x}$

35. $(1 + x + x^3)e^x + \cos 2x$

36. $(x + e^x)\sin x + (x - e^{-x})\cos 2x$

37. $(x + e^x + \sin 3x + \cos 2x)e^{-2x}$

38. $(x^2e^{-x} + 4\cos 3x + 5\sin 2x)e^{-3x}$

39. $(1 + x^2)(\sin x \cos x - \sin 2x)e^{-x}$

40. $(8 - x^3)(\cos^2 x - \sin^2 x)e^{3x}$

Correction Rule

Given the homogeneous solution y_h and an initial trial solution y, determine the final trial solution according to the correction rule.

41. $y_h(x) = ce^{2x}$, $y = d_1 + d_2x + d_3e^{2x}$

42. $y_h(x) = ce^{2x}$, $y = d_1 + d_2e^{2x} + d_3xe^{2x}$

43. $y_h(x) = ce^{0x}$, $y = d_1 + d_2x + d_3x^2$

44. $y_h(x) = ce^{x}$, $y = d_1 + d_2x + d_3x^2$

45. $y_h(x) = ce^{x}$, $y = d_1\cos x + d_2\sin x + d_3e^{x}$

46. $y_h(x) = ce^{2x}$, $y = d_1e^{2x}\cos x + d_2e^{2x}\sin x$

47. $y_h(x) = ce^{2x}$, $y = d_1e^{2x} + d_2xe^{2x} + d_3x^2e^{2x}$

48. $y_h(x) = ce^{-2x}$, $y = d_1e^{-2x} + d_2xe^{-2x} + d_3e^{2x} + d_4xe^{2x}$

49. $y_h(x) = cx^2$, $y = d_1 + d_2x + d_3x^2$

50. $y_h(x) = cx^3$, $y = d_1 + d_2x + d_3x^2$

Trial Solution

Find the form of the **corrected** trial solution y but do not evaluate the undetermined coefficients.

51. $y' = x^3 + 5 + x^2e^x(3 + 2x + \sin 2x)$

52. $y' = x^2 + 5x + 2 + x^3e^x(2 + 3x + 5\cos 4x)$

53. $y' - y = x^3 + 2x + 5 + x^4e^x(2 + 4x + 7\cos 2x)$

54. $y' - y = x^4 + 5x + 2 + x^3e^x(2 + 3x + 5\cos 4x)$

55. $y' - 2y = x^3 + x^2 + x^3e^x(2e^x + 3x + 5\sin 4x)$

56. $y' - 2y = x^3e^{2x} + x^2e^x(3 + 4e^x + 2\cos 2x)$

57. $y' + y = x^2 + 5x + 2 + x^3e^{-x}(6x + 3\sin x + 2\cos x)$

58. $y' - 2y = x^5 + 5x^3 + 14 + x^3e^x(5 + 7xe^{-3x})$

59. $2y' + 4y = x^4 + 5x^5 + 2x^8 + x^3 e^x (7 + 5xe^x + 5\sin 11x)$

60. $5y' + y = x^2 + 5x + 2e^{x/5} + x^3 e^{x/5}(7 + 9x + 2\sin(9x/2))$

Undetermined Coefficients

Compute a particular solution y_p according to the method of undetermined coefficients. Report:
(1) the initial trial solution
(2) the corrected trial solution
(3) undetermined coefficient equations
(4) the formula for y_p.

61. $y' + y = x + 1$

62. $y' + y = 2x - 1$

63. $y' - y = e^x + e^{-x}$

64. $y' - y = xe^x + e^{-x}$

65. $y' - 2y = 1 + x + e^{2x} + \sin x$

66. $y' - 2y = 1 + x + xe^{2x} + \cos x$

67. $y' + 2y = xe^{-2x} + x^3$

68. $y' + 2y = (2 + x)e^{-2x} + xe^x$

69. $y' = x^2 + 4 + xe^x(3 + \cos x)$

70. $y' = x^2 + 5 + xe^x(2 + \sin x)$

2.6 Exercises

Concentration

A lab assistant collects a volume of brine, boils it until only salt crystals remain, then uses a scale to determine the crystal mass or weight.

Find the salt **concentration** of the brine in kilograms per liter.

1. One liter of brine, crystal mass 0.2275 kg

2. Two liters, crystal mass 0.32665 kg

3. Two liters, crystal mass 15.5 grams

4. Five pints, crystals weigh 1/4 lb

5. Eighty cups, crystals weigh 5 lb

6. Five gallons, crystals weigh 200 ounces

One-Tank Mixing

Assume one inlet and one outlet. Determine the amount $x(t)$ of salt in the tank at time t. Use the text notation for equation (**??**).

7. The inlet adds 10 liters per minute with concentration $C_1 = 0.023$ kilograms per liter. The tank contains 110 liters of distilled water. The outlet drains 10 liters per minute.

8. The inlet adds 12 liters per minute with concentration $C_1 = 0.0205$ kilograms per liter. The tank contains 200 liters of distilled water. The outlet drains 12 liters per minute.

9. The inlet adds 10 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 200 liters of brine in which 3 kilograms of salt is dissolved. The outlet drains 10 liters per minute.

10. The inlet adds 12 liters per minute with concentration $C_1 = 0.0375$ kilograms per liter. The tank contains 500 liters of brine in which 7 kilograms of salt is dissolved. The outlet drains 12 liters per minute.

11. The inlet adds 10 liters per minute with concentration $C_1 = 0.1075$ kilograms per liter. The tank contains 1000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 10 liters per minute.

12. The inlet adds 14 liters per minute with concentration $C_1 = 0.1124$ kilograms per liter. The tank contains 2000 liters of brine in which k kilograms of salt is dissolved. The outlet drains 14 liters per minute.

13. The inlet adds 10 liters per minute with concentration $C_1 = 0.104$ kilograms per liter. The tank contains 100 liters of brine in which 0.25 kilograms of salt is dissolved. The outlet drains 11 liters per minute. Determine additionally the time when the tank is empty.

14. The inlet adds 16 liters per minute with concentration $C_1 = 0.01114$ kilograms per liter. The tank contains 1000 liters of brine in which 4 kilograms of salt is dissolved. The outlet drains 20 liters per minute. Determine additionally the time when the tank is empty.

15. The inlet adds 10 liters per minute with concentration $C_1 = 0.1$ kilograms per liter. The tank contains 500 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

16. The inlet adds 11 liters per minute with concentration $C_1 = 0.0156$ kilograms per liter. The tank contains 700 liters of brine in which k kilograms of salt is dissolved. The outlet drains 12 liters per minute. Determine additionally the time when the tank is empty.

Two-Tank Mixing

Assume brine tanks A and B in Figure 4 have volumes 100 and 200 gallons, respectively. Let $A(t)$ and $B(t)$ denote the number of pounds of salt at time t, respectively, in tanks A and B. Distilled water flows into tank A, then brine flows out of tank A and into tank B, then out of tank B. All flows are at r gallons per minute. Given rate r and initial salt amounts $A(0)$ and $B(0)$, find $A(t)$ and $B(t)$.

17. $r = 4$, $A(0) = 40$, $B(0) = 20$.

18. $r = 3$, $A(0) = 10$, $B(0) = 15$.

19. $r = 5$, $A(0) = 20$, $B(0) = 40$.

20. $r = 5$, $A(0) = 40$, $B(0) = 30$.

21. $r = 8$, $A(0) = 10$, $B(0) = 12$.

22. $r = 8$, $A(0) = 30$, $B(0) = 12$.

23. $r = 9$, $A(0) = 16$, $B(0) = 14$.

24. $r = 9$, $A(0) = 22$, $B(0) = 10$.

25. $r = 7$, $A(0) = 6$, $B(0) = 5$.

26. $r = 7$, $A(0) = 13$, $B(0) = 26$

Residential Heating

Assume the Newton cooling model for heating and insulation values $1/4 \le k \le 1/2$. Follow Example 2.23, page 145.

27. The office heat goes off at 7PM. It's 74°F inside and 58°F outside overnight. Estimate the office temperature at 10PM, 1AM and 6AM.

28. The office heat goes off at 6:30PM. It's 73°F inside and 55°F outside overnight. Estimate the office temperature at 9PM, 3AM and 7AM.

29. The radiator goes off at 9PM. It's 74°F inside and 58°F outside overnight. Estimate the room temperature at 11PM, 2AM and 6AM.

30. The radiator goes off at 10PM. It's 72°F inside and 55°F outside overnight. Estimate the room temperature at 2AM, 5AM and 7AM.

31. The office heat goes on in the morning at 6:30AM. It's 57°F inside and 40° to 55°F outside until 11AM. Estimate the office temperature at 8AM, 9AM and 10AM. Assume the furnace provides a five degree temperature rise in 30 minutes and the thermostat is set for 76°F.

32. The office heat goes on at 6AM. It's 55°F inside and 43° to 53°F outside until 10AM. Estimate the office temperature at 7AM, 8AM and 9AM. Assume the furnace provides a seven degree temperature rise in 45 minutes and the thermostat is set for 78°F.

33. The hot water heating goes on at 6AM. It's 55°F inside and 50° to 60°F outside until 10AM. Estimate the room temperature at 7:30AM. Assume the radiator provides a four degree temperature rise in 45 minutes and the thermostat is set for 74°F.

34. The hot water heating goes on at 5:30AM. It's 54°F inside and 48° to 58°F outside until 9AM. Estimate the room temperature at 7AM. Assume the radiator provides a five degree temperature rise in 45 minutes and the thermostat is set for 74°F.

35. A portable heater goes on at 7AM. It's 45°F inside and 40° to 46°F outside until 11AM. Estimate the room temperature at 9AM. Assume the heater provides a two degree temperature rise in 30 minutes and the thermostat is set for 90°F.

36. A portable heater goes on at 8AM. It's 40°F inside and 40° to 45°F outside until 11AM. Estimate the room temperature at 10AM. Assume the heater provides a two degree temperature rise in 20 minutes and the thermostat is set for 90°F.

Evaporative Cooling

Define outside temperature (see Figure 3)

$$a(t) = \begin{cases} 75 - 2\,t & 0 \le t \le 6 \\ 39 + 4\,t & 6 < t \le 9 \\ 30 + 5\,t & 9 < t \le 12 \\ 54 + 3\,t & 12 < t \le 15 \\ 129 - 2\,t & 15 < t \le 21 \\ 170 - 4\,t & 21 < t \le 23 \\ 147 - 3\,t & 23 < t \le 24 \end{cases}.$$

Given k, k_1, $P(t) = wa(t)$ and $u(0) = 69$, then plot $u(t)$, $P(t)$ and $a(t)$ on one graphic.

$$u(t) = u(0)e^{-kt-k_1 t} + \\ (k + wk_1)\int_0^t a(r)e^{(k+k_1)(r-t)}dr.$$

37. $k = 1/4$, $k_1 = 2$, $w = 0.85$

38. $k = 1/4$, $k_1 = 1.8$, $w = 0.85$

39. $k = 3/8$, $k_1 = 2$, $w = 0.85$

40. $k = 3/8$, $k_1 = 2.4$, $w = 0.85$

41. $k = 1/4$, $k_1 = 3$, $w = 0.80$

42. $k = 1/4$, $k_1 = 4$, $w = 0.80$

43. $k = 1/2$, $k_1 = 4$, $w = 0.80$

44. $k = 1/2$, $k_1 = 5$, $w = 0.80$

45. $k = 3/8$, $k_1 = 3$, $w = 0.80$

46. $k = 3/8$, $k_1 = 4$, $w = 0.80$

Radioactive Chain

Let A, B and C be the amounts of three radioactive isotopes. Assume A decays into B at rate a, then B decays into C at rate b. Given a, b, $A(0) = A_0$ and $B(0) = B_0$, find formulas for A and B.

47. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 10$

48. $a = 2$, $b = 3$, $A_0 = 100$, $B_0 = 100$

49. $a = 1$, $b = 4$, $A_0 = 100$, $B_0 = 200$

50. $a = 1$, $b = 4$, $A_0 = 300$, $B_0 = 100$

51. $a = 4$, $b = 3$, $A_0 = 100$, $B_0 = 100$

52. $a = 4,\ b = 3,\ A_0 = 100,\ B_0 = 200$

53. $a = 6,\ b = 1,\ A_0 = 600,\ B_0 = 100$

54. $a = 6,\ b = 1,\ A_0 = 500,\ B_0 = 400$

55. $a = 3,\ b = 1,\ A_0 = 100,\ B_0 = 200$

56. $a = 3,\ b = 1,\ A_0 = 400,\ B_0 = 700$

Electric Circuits

In the LR-circuit of Figure 5, assume $E(t) = A\cos wt$ and $I(0) = 0$. Solve for $I(t)$.

57. $A = 100,\ w = 2\pi,\ R = 1,\ L = 2$

58. $A = 100,\ w = 4\pi,\ R = 1,\ L = 2$

59. $A = 100,\ w = 2\pi,\ R = 10,\ L = 1$

60. $A = 100,\ w = 2\pi,\ R = 10,\ L = 2$

61. $A = 5,\ w = 10,\ R = 2,\ L = 3$

62. $A = 5,\ w = 4,\ R = 3,\ L = 2$

63. $A = 15,\ w = 2,\ R = 1,\ L = 4$

64. $A = 20,\ w = 2,\ R = 1,\ L = 3$

65. $A = 25,\ w = 100,\ R = 5,\ L = 15$

66. $A = 25,\ w = 50,\ R = 5,\ L = 5$

2.7 Exercises

Newton's Laws

Review of units and conversions.

1. An object weighs 100 pounds. Find its mass in slugs and kilograms.

2. An object has mass 50 kilograms. Find its mass in slugs and its weight in pounds.

3. Convert from *fps* to *mks* systems: position 1000, velocity 10, acceleration 2.

4. Derive $g = \dfrac{Gm}{R^2}$, where m is the mass of the earth and R is its radius.

Velocity and Acceleration

Find the velocity x' and acceleration x''.

5. $x(t) = 16t^2 + 100$

6. $x(t) = 16t^2 + 10t + 100$

7. $x(t) = t^3 + t + 1$

8. $x(t) = t(t-1)(t-2)$

Free Fall with Constant Gravity

Solve using the model $x''(t) = -g$, $x(0) = x_0$, $x'(0) = v_0$.

9. A brick falls from a tall building, straight down. Find the distance it fell and its speed at three seconds.

10. An iron ingot falls from a tall building, straight down. Find the distance it fell and its speed at four seconds.

11. A ball is thrown straight up from the ground with initial velocity 66 feet per second. Find its maximum height.

12. A ball is thrown straight up from the ground with initial velocity 88 feet per second. Find its maximum height.

13. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground.

14. An arrow is shot straight up from the ground with initial velocity 44 meters per second. Find the flight time back to the ground.

15. A car travels 140 kilometers per hour. Brakes are applied, with deceleration 10 meters per second per second. Find the distance the car travels before stopping.

16. A car travels 120 kilometers per hour. Brakes are applied, with deceleration 40 feet per second per second. Find the distance the car travels before stopping.

17. An arrow is shot straight down from a height of 500 feet, with initial velocity 44 feet per second. Find the flight time to the ground and its impact speed.

18. An arrow is shot straight down from a height of 200 meters, with initial velocity 13 meters per second. Find the flight time to the ground and its impact speed.

Linear Air Resistance

Solve using the linear air resistance model $mx''(t) = -kx'(t) - mg$. An equivalent model is $x'' = -\rho x - g$, where $\rho = k/m$ the drag coefficient.

19. An arrow is shot straight up from the ground with initial velocity 23 meters per second. Find the flight time back to the ground. Assume $\rho = 0.035$.

20. An arrow is shot straight up from the ground with initial velocity 27 meters per second. Find the maximum height. Assume $\rho = 0.04$.

21. A parcel is dropped from an aircraft at $32,000$ feet. It has a parachute that opens automatically after 25 seconds. Assume drag coefficient $\rho = 0.16$ without the parachute and $\rho = 1.45$ with it. Find the descent time to the ground.

22. A first aid kit is dropped from a helicopter at $12,000$ feet. It has a parachute that opens automatically after 15 seconds. Assume drag coefficient $\rho = 0.12$ without the parachute and $\rho = 1.55$ with it. Find the impact speed with the ground.

23. A motorboat has velocity v satisfying $1100v'(t) = 6000 - 110v$, $v(0) = 0$. Find the maximum speed of the boat.

24. A motorboat has velocity v satisfying $1000v'(t) = 4000 - 90v$, $v(0) = 0$. Find the maximum speed of the boat.

25. A parachutist falls until his speed is 65 miles per hour. He opens the parachute. Assume drag coefficient $\rho = 1.57$. About how many seconds must elapse before his speed is reduced to within 1% of terminal velocity?

26. A parachutist falls until his speed is 120 kilometers per hour. He opens the parachute. Assume drag coefficient $\rho = 1.51$. About how many seconds must elapse before his speed is reduced to within 2% of terminal velocity?

27. A ball is thrown straight up with initial velocity 35 miles per hour. Find the ascent time and the descent time. Assume drag coefficient 0.042

28. A ball is thrown straight up with initial velocity 60 kilometers per hour. Find the ascent time and the descent time. Assume drag coefficient 0.042

Linear Ascent and Descent Times

Find the ascent time t_1 and the descent time t_2 for the linear model $v' = -\rho v - g$, $\rho = k/m$ is the drag coefficient. Use equation (**??**) for t_1. Find t_2 from $x(t_2) = 0$, where $v = x'$ and $v' = -\rho v - g$, $v(0) = 0$, $x(0) = y_0$ and $y_0 = \rho^{-1} \int_0^{t_1} (-g + (v_0\rho + g)e^{-\rho t})dt$.

29. $\rho = 0.01$

30. $\rho = 0.015$

31. $\rho = 0.02$

32. $\rho = 0.018$

33. $\rho = 0.022$

34. $\rho = 0.025$

35. $\rho = 1.5$

36. $\rho = 1.55$

37. $\rho = 1.6$

38. $\rho = 1.65$

39. $\rho = 1.45$

40. $\rho = 1.48$

Nonlinear Air Resistance

Assume ascent velocity v_1 satisfies $v_1' = -\rho v_1^2 - g$. Assume descent velocity v_2 satisfies $v_2' = \rho v_2^2 - g$. Let t_1 and t_2 be the ascent and descent times, so that $t_1 + t_2$ is the flight time. Let $v_1(0) = v_0$ and $v_2(t_1) = v_1(t_1) = 0$. Units are mks. Assume $g = 9.8$. Define $M =$ maximum height and $v_f =$ impact velocity.

41. Let $\rho = 0.0012$, $v_0 = 50$. Find t_1, t_2

42. Let $\rho = 0.0012$, $v_0 = 30$. Find t_1, t_2

43. Let $\rho = 0.0015$, $v_0 = 50$. Find t_1, t_2

44. Let $\rho = 0.0015$, $v_0 = 30$. Find t_1, t_2

45. Let $\rho = 0.001$, $v_0 = 50$. Find M, v_f.

46. Let $\rho = 0.001$, $v_0 = 30$. Find M, v_f.

47. Let $\rho = 0.0014$, $v_0 = 50$. Find M, v_f.

48. Let $\rho = 0.0014$, $v_0 = 30$. Find M, v_f.

49. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 60$.

50. Find t_1, t_2, M and v_f for $\rho = 0.00152$, $v_0 = 40$.

Terminal Velocity

Find the terminal velocity for (a) a linear air resistance $a(t) = \rho v(t)$ and (b) a nonlinear air resistance $a(t) = \rho v^2(t)$. Use the model equation $v' = a(t) - g$ and the given drag coefficient ρ.

51. $\rho = 0.15$

52. $\rho = 0.155$

53. $\rho = 0.015$

54. $\rho = 0.017$

55. $\rho = 1.5$

56. $\rho = 1.55$

57. $\rho = 2.0$

58. $\rho = 1.89$

59. $\rho = 0.001$

60. $\rho = 0.0015$

Parachutes

A parachute opens at timer value $t = 0$ and the body falls at speed v given by (a) linear resistance model $v' = \rho v - g$ or (b) nonlinear resistance model $v' = \rho v^2 - g$. Given the drag coefficient ρ and initial velocity $v(0) = v_0$, compute the elapsed distance and elapsed time until the body reaches 98% of its terminal velocity. Report two values for (a) and two values for (b).

61. $\rho = 1.446$, $v_0 = -66$ ft/sec.

62. $\rho = 1.446$, $v_0 = -44$ ft/sec.

63. $\rho = 1.5$, $v_0 = -66$ ft/sec.

64. $\rho = 1.5$, $v_0 = -44$ ft/sec.

65. $\rho = 1.55$, $v_0 = -21$ ft/sec.

66. $\rho = 1.55$, $v_0 = -11$ ft/sec.

67. $\rho = 1.442$, $v_0 = 0$ ft/sec.

68. $\rho = 1.442$, $v_0 = -5$ ft/sec.

69. $\rho = 1.37$, $v_0 = -44$ ft/sec.

70. $\rho = 1.37$, $v_0 = -22$ ft/sec.

Lunar Lander

A lunar lander falls to the moon's surface at v_0 miles per hour. The retrorockets in free space provide a deceleration effect on the lander of a miles per hour per hour. Estimate the retrorocket activation height above the surface which will give the lander zero touch-down velocity. Follow Example 2.30, page 164.

71. $v_0 = -1000$, $a = 18000$

72. $v_0 = -980$, $a = 18000$

73. $v_0 = -1000$, $a = 20000$

74. $v_0 = -1000$, $a = 19000$

75. $v_0 = -900$, $a = 18000$

76. $v_0 = -900$, $a = 20000$

77. $v_0 = -1100$, $a = 22000$

78. $v_0 = -1100$, $a = 21000$

79. $v_0 = -800$, $a = 18000$

80. $v_0 = -800$, $a = 21000$

Escape velocity

Find the escape velocity of the given planet, given the planet's mass m and radius R.

81. **(Planet A)** $m = 3.1 \times 10^{23}$ kilograms, $R = 2.4 \times 10^{7}$ meters.

82. (Mercury) $m = 3.18 \times 10^{23}$ kilograms, $R = 2.43 \times 10^6$ meters.

83. (Planet B) $m = 5.1 \times 10^{24}$ kilograms, $R = 6.1 \times 10^6$ meters.

84. (Venus) $m = 4.88 \times 10^{24}$ kilograms, $R = 6.06 \times 10^6$ meters.

85. (Mars) $m = 6.42 \times 10^{23}$ kilograms, $R = 3.37 \times 10^6$ meters.

86. (Neptune) $m = 1.03 \times 10^{26}$ kilograms, $R = 2.21 \times 10^7$ meters.

87. (Jupiter) $m = 1.90 \times 10^{27}$ kilograms, $R = 6.99 \times 10^7$ meters.

88. (Uranus) $m = 8.68 \times 10^{25}$ kilograms, $R = 2.33 \times 10^7$ meters.

89. (Saturn) $m = 5.68 \times 10^{26}$ kilograms, $R = 5.85 \times 10^7$ meters.

90. (Pluto) $m = 1.44 \times 10^{22}$ kilograms, $R = 1.5 \times 10^6$ meters.

Lunar Lander Experiments

91. (Lunar Lander) Verify that the variable field model for Example 2.30 gives a soft landing flight model in MKS units

$$\begin{aligned} u''(t) &= 2.2352 - \frac{c_1}{(c_2 + u(t))^2}, \\ u(0) &= 127233.2115, \\ u'(0) &= -429.1584, \end{aligned}$$

where $c_1 = 4910591999000$ and $c_2 = 1740000$.

92. (Lunar Lander: Numerical Experiment) Using a computer, solve the flight model of the previous exercise. Determine the flight time $t_0 \approx 625.5$ seconds by solving $u(t) = 0$ for t.

Details and Proofs

93. (Linear Rise Time) Using the inequality $e^u > 1 + u$ for $u > 0$, show that the ascent time t_1 in equation (**??**) satisfies

$$g(1 + \rho t_1) < g e^{\rho t_1} = v_0 \rho + g.$$

Conclude that $t_1 < v_0/g$, proving Lemma 2.2.

94. (Linear Maximum) Verify that Lemma 2.2 plus the inequality $x(t) < -gt^2/2 + v_0 t$ imply $x(t_1) < g v_0^2/2$. Conclude that the maximum for $\rho > 0$ is less than the maximum for $\rho = 0$.

95. (Linear Rise Time) Consider the ascent time $t_1(\rho, v_0)$ given by equation (**??**). Prove that

$$\frac{dt_1}{d\rho} = \frac{\ln \frac{g}{v0\rho+g}}{\rho^2} + \frac{v0}{\rho(v0\rho+g)}.$$

96. (Linear Rise Time) Consider $dt_1(\rho, v_0)/d\rho$ given in the previous exercise. Let $\rho = gx/v_0$. Show that $dt_1/d\rho < 0$ by considering properties of the function $-(x+1)\ln(x+1)+x$. Then prove Lemma 2.2.

97. (Compare Rise Times) Show that the nonlinear ascent time for the model $v' = -\rho v^2 - g$ is less than the linear ascent time from model $v' = -\rho v - g$.

98. (Compare Fall Times) Show that the nonlinear descent time for the model $v' = \rho v^2 - g$ is less than the linear descent time from model $v' = -\rho v - g$.

2.8 Exercises

Limited Environment

Find the equilibrium solutions and the carrying capacity for each logistic equation.

1. $P' = 0.01(2 - 3P)P$

2. $P' = 0.2P - 3.5P^2$

3. $y' = 0.01(-3 - 2y)y$

4. $y' = -0.3y - 4y^2$

5. $u' = 30u + 4u^2$

6. $u' = 10u + 3u^2$

7. $w' = 2(2 - 5w)w$

8. $w' = -2(3 - 7w)w$

9. $Q' = Q^2 - 3(Q - 2)Q$

10. $Q' = -Q^2 - 2(Q - 3)Q$

Spread of a Disease

In each model, find the number of susceptibles and then the number of infectives at $t = 0.557$. Follow Example 2.34, page 177. A calculator is required for approximations.

11. $y' =' (5 - 3P)y$, $y(0) = 1$.

12. $y' = (13 - 3y)y$, $y(0) = 2$.

13. $y' = (5 - 12y)y$, $y(0) = 2$.

14. $y' = (15 - 4y)y$, $y(0) = 10$.

15. $P' = (2 - 3P)P$, $P(0) = 500$.

16. $P' = (5 - 3P)P$, $P(0) = 200$.

17. $P' = 2P - 5P^2$, $P(0) = 100$.

18. $P' = 3P - 8P^2$, $P(0) = 10$.

Explosion–Extinction

Classify the model as **explosion** or **extinction**.

19. $y' = 2(y - 100)y$, $y(0) = 200$

20. $y' = 2(y - 200)y$, $y(0) = 300$

21. $y' = -100y + 250y^2$, $y(0) = 200$

22. $y' = -50y + 3y^2$, $y(0) = 25$

23. $y' = -60y + 70y^2$, $y(0) = 30$

24. $y' = -540y + 70y^2$, $y(0) = 30$

25. $y' = -16y + 12y^2$, $y(0) = 1$

26. $y' = -8y + 12y^2$, $y(0) = 1/2$

Constant Harvesting

Find the carrying capacity N and the threshold population M.

27. $P' = (3 - 2P)P - 1$

28. $P' = (4 - 3P)P - 1$

29. $P' = (5 - 4P)P - 1$

30. $P' = (6 - 5P)P - 1$

31. $P' = (6 - 3P)P - 1$

32. $P' = (6 - 4P)P - 1$

33. $P' = (8 - 5P)P - 2$

34. $P' = (8 - 3P)P - 2$

35. $P' = (9 - 4P)P - 2$

36. $P' = (10 - P)P - 2$

Variable Harvesting

Re-model the variable harvesting equation as $y' = (a - by)y$ and solve the equation by recipe (**??**), page **??**.

37. $P' = (3 - 2P)P - P$

38. $P' = (4 - 3P)P - P$

39. $P' = (5 - 4P)P - P$

40. $P' = (6 - 5P)P - P$

41. $P' = (6 - 3P)P - P$

42. $P' = (6 - 4P)P - P$

43. $P' = (8 - 5P)P - 2P$

44. $P' = (8 - 3P)P - 2P$

45. $P' = (9 - 4P)P - 2P$

46. $P' = (10 - P)P - 2P$

Restocking

Make a direction field graphic by computer, following Example 2.38. Using the graphic, report (a) an estimate for the carrying capacity C and (b) approximations for the amplitude A and period T of a periodic solution which oscillates about $y = C$.

47. $P' = (1 - P)P - \sin(5\pi t)$

48. $P' = (1 - P)P - 1.5\sin(5\pi t)$

49. $P' = (2 - P)P - 3\sin(7\pi t)$

50. $P' = (2 - P)P - \sin(7\pi t)$

51. $P' = (4 - 3P)P - 2\sin(3\pi t)$

52. $P' = (4 - 2P)P - 3\sin(3\pi t)$

53. $P' = (10 - 9P)P - 3\sin(4\pi t)$

54. $P' = (10 - 9P)P - \sin(4\pi t)$

55. $P' = (5 - 4P)P - 2\sin(8\pi t)$

56. $P' = (5 - 4P)P - 3\sin(8\pi t)$

Richard Function

Ideas of L. von Bertalanffy (1934), A. Pütter (1920) and Verhulst were used by F. J. Richards (1957) to define a sigmoid function $Y(t)$ which generalizes the logistic function. It is suited for data-fitting models, for example forestry, tumor growth and stock-production problems. The Richard function is

$$Y(t) = A + \frac{K - A}{(1 + Qe^{-B(t-M)})^{1/\nu}},$$

where Y = weight, height, size, amount, etc., and t = time.

57. Differentiate for $\alpha > 0$, $\nu > 0$, the specialized Richard function

$$Y(t) = \frac{K}{(1 + Qe^{-\alpha\nu(t-t_0)})^{1/\nu}}$$

to obtain the sigmoid differential equation

$$Y'(t) = \alpha\left(1 - \left(\frac{Y}{K}\right)^{\nu}\right)Y.$$

The relation $Y(t_0) = \frac{K}{(1+Q)^{1/\nu}}$ implies $Q = -1 + \left(\frac{K}{Y(t_0)}\right)^{\nu}$.

58. Solve the differential equation $Y'(t) = \alpha\left(1 - \left(\frac{Y}{K}\right)^{\nu}\right)Y$ by means of the substitution $w = (Y/K)^{\nu}$, which gives the familiar logistic equation $w' = \alpha\nu(1-w)w$.

2.9 Exercises

Tank Draining

1. A cylindrical tank 6 feet high with 6-foot diameter is filled with gasoline. In 15 seconds, 5 gallons drain out. Find the drain times for the next 20 gallons and the half-volume.

2. A cylindrical tank 4 feet high with 5-foot diameter is filled with gasoline. The half-volume drain time is 11 minutes. Find the drain time for the full volume.

3. A conical tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x$, $0 \le x \le 5$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

4. A conical tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x$, $0 \le x \le 5$. The units are in meters. Find the half-volume drain time for the tank, given the first 5 liters drain out in 10 seconds.

5. A spherical tank of diameter 12 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 20 seconds.

6. A spherical tank of diameter 9 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 3 seconds.

7. A hemispherical tank of diameter 16 feet is filled with water. Find the drain time for the tank, given the first 5 gallons drain out in 25 seconds.

8. A hemispherical tank of diameter 10 feet is filled with solvent. Find the half-volume drain time for the tank, given the first gallon drains out in 4 seconds.

9. A parabolic tank is filled with water. The tank geometry is a solid of revolution formed from $y = 2x^2$, $0 \le x \le 2$. The units are in feet. Find the drain time for the tank, given the first 5 gallons drain out in 12 seconds.

10. A parabolic tank is filled with oil. The tank geometry is a solid of revolution formed from $y = 3x^2$, $0 \le x \le 2$. The units are in meters. Find the half-volume drain time for the tank, given the first 4 liters drain out in 16 seconds.

Torricelli's Law and Uniqueness

It it known that Torricelli's law gives a differential equation for which Picard's existence-uniqueness theorem is inapplicable for initial data $y(0) = 0$.

11. Explain why Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ fails to satisfy the hypotheses in Picard's theorem. Cite all failed hypotheses.

12. Consider a typical Torricelli's law equation $y' = k\sqrt{y}$ with initial condition $y(0) = 0$. Argue physically that the depth $y(t)$ of the tank for $t < 0$ can be zero for an arbitrary duration of time t near $t = 0$, even though $y(t)$ is not zero for all t.

13. Display infinitely many solutions $y(t)$ on $-5 \le t \le 5$ of Torricelli's equation $y' = k\sqrt{y}$ such that $y(t)$ is not identically zero but $y(t) = 0$ for $0 \le t \le 1$.

14. Does Torricelli's equation $y' = k\sqrt{y}$ plus initial condition $y(0) = 0$ have a solution $y(t)$ defined for $t \ge 0$? Is it unique? Apply Picard's theorem and Peano's theorem, if possible.

Clepsydra: Water Clock Design

A surface of revolution is used to make a container of height h feet for a water clock. A curve $y = f(x)$ is revolved around the y-axis to make the container shape (e.g., $y = x$ makes a conical shape). Water drains out by gravity at $(0, 0)$. The orifice has diameter d inches. The water level in the tank must fall at a constant rate of r inches per hour. Find d and $f(x)$, given h and r.

15. $h = 5$, $r = 4$

16. $h = 4$, $r = 4$

17. $h = 10$, $r = 7$

18. $h = 10$, $r = 8$

19. $h = 15$, $r = 10$

20. $h = 15$, $r = 8$

Stefan's Law

An unclothed prison inmate is handcuffed to a chair. The inmate's skin temperature is 33° Celsius. Given emissivity $\mathcal{E}$, skin area A square meters and room temperature $T_0(r) = C(r/60) + 273.15$, r in seconds, find the number of Joules of heat lost by the inmate's skin after t_0 minutes. Use equation (**??**), page **??**.

21. $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 10$, $C(t) = 24 + 7t/t_0$

22. $\mathcal{E} = 0.9$, $A = 1.7$, $t_0 = 12$, $C(t) = 21 + 10t/12$

23. $\mathcal{E} = 0.9$, $A = 1.4$, $t_0 = 10$, $C(t) = 15 + 15t/t_0$

24. $\mathcal{E} = 0.9$, $A = 1.5$, $t_0 = 12$, $C(t) = 15 + 14t/t_0$

On the next two exercises, use a computer algebra system (CAS)

25. $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 15$, $C(t) = 15 + 15\sin\pi(t - t_0)/12$

26. $\mathcal{E} = 0.8$, $A = 1.4$, $t_0 = 20$, $C(t) = 15 + 14\sin\pi(t - t_0)/12$

Tsunami Wave Shape

Plot the piecewise solution (**??**). See Figure 12.

27. $x_0 = 2$, $|x - x_0| \le 2$

28. $x_0 = 3$, $|x - x_0| \le 4$.

Tsunami Piecewise Solutions

Display a piecewise solution similar to (**??**). Produce a plot like Figure 12.

29. $x_0 = 2$, $|x - x_0| \le 4$,
$(y')^2 = 16y^2 - 10y^3$.

30. $x_0 = 2$, $|x - x_0| \le 4$,
$(y')^2 = 16y^2 - 12y^3$.

31. $x_0 = 3$, $|x - x_0| \le 4$,
$(y')^2 = 8y^2 - 2y^3$.

32. $x_0 = 4$, $|x - x_0| \le 4$,
$(y')^2 = 16y^2 - 4y^3$.

Tsunami Wavefront

Find non-equilibrium solutions for the given differential equation.

33. $(y')^2 = 16y^2 - 10y^3$.

34. $(y')^2 = 16y^2 - 12y^3$.

35. $(y')^2 = 8y^2 - 2y^3$.

36. $(y')^2 = 16y^2 - 4y^3$.

Gompertz Tumor Equation

Solve the Gompertz tumor equation $y' = (a - b\ln y)y$. Make a phase line diagram.

37. $a = 1$, $b = 1$

38. $a = 1$, $b = 2$

39. $a = -1$, $b = 1$

40. $a = -1$, $b = 2$

41. $a = 4$, $b = 1$

42. $a = 5$, $b = 1$

2.10 Exercises

Exactness Test

Test the equality $M_y = N_x$ for the given equation, as written, and report *exact* when true. Do not try to solve the differential equation. See Example 2.43, page 202.

1. $(y - x)dx + (y + x)dy = 0$

2. $(y + x)dx + (x - y)dy = 0$

3. $(y + \sqrt{xy})dx + (-y)dy = 0$

4. $(y + \sqrt{xy})dx + xydy = 0$

5. $(x^2 + 3y^2)dx + 6xydy = 0$

6. $(y^2 + 3x^2)dx + 2xydy = 0$

7. $(y^3 + x^3)dx + 3xy^2dy = 0$

8. $(y^3 + x^3)dx + 2xy^2dy = 0$

9. $2xydx + (x^2 - y^2)dy = 0$

10. $2xydx + (x^2 + y^2)dy = 0$

Conservation Law Test

Test conservation law $U(x, y) = c$ for a solution to $Mdx + Ndy = 0$. See Example 2.44, page 203.

11. $2xydx + (x^2 + 3y^2)dy = 0$,
$x^2y + y^3 = c$

12. $2xydx + (x^2 - 3y^2)dy = 0$,
$x^2y - y^3 = c$

13. $(3x^2 + 3y^2)dx + 6xydy = 0$,
$x^3 + 3xy^2 = c$

14. $(x^2 + 3y^2)dx + 6xydy = 0$,
$x^3 + 3xy^2 = c$

15. $(y - 2x)dx + (2y + x)dy = 0$,
$xy - x^2 + y^2 = c$

16. $(y + 2x)dx + (-2y + x)dy = 0$,
$xy + x^2 - y^2 = c$

Exactness Theorem

Find an implicit solution $U(x, y) = c$. See Examples ??-2.46, page ??.

17. $(y - 4x)dx + (4y + x)dy = 0$

18. $(y + 4x)dx + (4y + x)dy = 0$

19. $(e^y + e^x)dx + (xe^y)dy = 0$

20. $(e^{2y} + e^x)dx + (2xe^{2y})dy = 0$

21. $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$

22. $(1 + ye^{-xy})dx + (xe^{-xy} - 4y)dy = 0$

23. $(2x + \arctan y)dx + \dfrac{x}{1 + y^2}\,dy = 0$

24. $(2x + \arctan y)dx + \dfrac{x + 2y}{1 + y^2}\,dy = 0$

25. $\dfrac{2x^5 + 3y^3}{x^4 y}dx - \dfrac{2y^3 + x^5}{x^3 y^2}dy = 0$

26. $\dfrac{2x^4 + y^2}{x^3 y}dx - \dfrac{2x^4 + y^2}{2x^2 y^2}dy = 0$

27. $M dx + N dy = 0$, $M = e^x \sin y + \tan y$, $N = e^x \cos y + x \sec^2 y$

28. $M dx + N dy = 0$, $M = e^x \cos y + \tan y$, $N = -e^x \sin y + x \sec^2 y$

29. $\left(x^2 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$

30. $\left(x^3 + \ln y\right) dx + \left(y^3 + x/y\right) dy = 0$

2.11 Exercises

Homogeneous-A Equations

Find f such that the equation can be written in the form $y' = f(y/x)$, then solve for y. Check the answer using a computer algebra system.

1. $xy' = y^2/x$

2. $x^2y' = x^2 + y^2$

3. $yy' = \frac{xy^2}{x^2+y^2}$

4. $yy' = \frac{2xy^2}{4x^2+y^2}$

5. $y' = \frac{y^2}{4x^2+y^2}$

6. $y' = \frac{y^2}{x^2+y^2}$

7. $y' = \frac{y^2}{(x+y)^2}$

8. $y' = \frac{xy}{(x+y)^2}$

9. $y' = \frac{y\left(y^2+4\,yx+5\,x^2\right)}{x(y+2\,x)^2}$

10. $y' = \frac{y^2(y+2\,x)}{x(y+x)^2}$

Homogeneous-C Equations

Decompose $f = G(R(x,y))$ where $R(x,y) = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, then solve $y' = f(x,y)$.

11. $y' = \frac{(y+1)x}{y^2+2\,y+1+x^2}$

12. $y' = 2\,\frac{(y+1)x}{4\,y^2+8\,y+4+x^2}$

13. $y' = \frac{(x+1)^2}{4\,y^2+x^2+2\,x+1}$

14. $y' = \frac{(x+1)^2}{(x+1+y)^2}$

15. $y' = \frac{(y+x)(x+1)}{(2\,x+1+y)^2}$

16. $y' = \frac{x\left(2\,y^2+6\,yx+5\,x^2\right)}{(y+x)(y+2\,x)^2}$

17. $y' = \frac{(y+x)\left(3\,y^2+6\,yx+2\,y+3\,x^2+2\,x\right)}{(x+1+y)(2\,y+2\,x+1)^2}$

18. $y' = \frac{(y+2\,x)^2}{x^2}$

19. $y' = \frac{(2\,y+x)^2}{y^2}$

20. $y' = \frac{x^2}{(y+4\,x)^2}$

Bernoulli's Equation

Identify the exponent n in Bernoulli's equation $y' + p(x)y = q(x)y^n$ and solve for $y(x)$.

21. $y^{-2}y' = 1 + x$

22. $yy' = 1 + x$

23. $y^{-2}y' + y^{-1} = 1 + x$

24. $yy' + y^2 = 1 + x$

25. $y' + y = y^{1/3}$

26. $y' + y = y^{1/5}$

27. $y' - y = y^{-1/2}$

28. $y' - y = y^{-1/3}$

29. $yy' + y^2 = e^x$

30. $y' + y = e^{2x}y^2$

Integrating Factor $x^a y^b$

Report an implicit solution for the given equation $M dx + N dy = 0$, using an integrating factor $Q = x^a y^b$. Follow Example 2.50, page 210.

31. $M = 3xy - 6y^2$, $N = 4x^2 - 15xy$

32. $M = 3xy - 10y^2$, $N = 4x^2 - 25xy$

33. $M = 2\,y - 12\,xy^2$, $N = 4\,x - 20\,x^2y$

34. $M = 2\,y - 21\,xy^2$, $N = 4\,x - 35\,x^2y$

35. $M = 3\,y - 32\,xy^2$, $N = 4\,x - 40\,x^2y$

36. $M = 3\,y - 20\,xy^2$, $N = 4\,x - 25\,x^2y$

37. $M = 12\,y - 30\,x^2y^2$,
$N = 12\,x - 25\,x^3y$

38. $M = 12\,y + 90\,x^2y^2$,
$N = 12\,x + 75\,x^3y$

39. $M = 15\,y + 90\,xy^2$,
$N = 12\,x + 75\,x^2y$

40. $M = 35\,y + 30\,xy^2$,
$N = 28\,x + 25\,x^2y$.

Integrating Factor e^{ax+by}

Report an implicit solution $U(x, y) = c$ for the given equation $Mdx + Ndy = 0$ using an integrating factor $Q = e^{ax+by}$. Follow Example 2.51, page 211.

41. $M = e^x + 2e^{2y}$, $N = e^x + 5e^{5y}$

42. $M = 3e^x + 2e^y$, $N = 4e^x + 5e^y$

43. $M = 12\,e^x + 2$, $N = 20\,e^x + 5$

44. $M = 12\,e^x + 2\,e^{-y}$, $N = 24\,e^x + 5\,e^{-y}$

45. $M = 12\,e^y + 2\,e^{-x}$, $N = 24\,e^y + 5\,e^{-x}$

46. $M = 12\,e^{-2\,y} + 2\,e^{-x}$, $N = 12\,e^{-2\,y} + 5\,e^{-x}$

47. $M = 16\,e^y + 2\,e^{-2\,x+3\,y}$, $N = 12\,e^y + 5\,e^{-2\,x+3\,y}$

48. $M = 16\,e^{-y} + 2\,e^{-2\,x-3\,y}$, $N = -12\,e^{-y} - 5\,e^{-2\,x-3\,y}$

49. $M = -16 - 2\,e^{2\,x+y}$, $N = 12 + 4\,e^{2\,x+y}$

50. $M = -16\,e^{-3\,y} - 2\,e^{2\,x}$, $N = 8\,e^{-3\,y} + 5\,e^{2\,x}$

Integrating Factor $Q(x)$

Report an implicit solution $U(x, y) = c$ for the given equation, using an integrating factor $Q = Q(x)$. Follow Example 2.52, page 212.

51. $(x + 2y)dx + (x - x^2)dy = 0$

52. $(x + 3y)dx + (x - x^2)dy = 0$

53. $(2x + y)dx + (x - x^2)dy = 0$

54. $(2x + y)dx + (x + x^2)dy = 0$

55. $(2x + y)dx + (2x + x^2)dy = 0$

56. $(x + y)dx + (2x + x^2)dy = 0$

57. $(x+y)dx+(3x+x^2)dy=0$

58. $(x+y)dx+(3x+5x^2)dy=0$

59. $(x+y)dx+(3x)dy=0$

60. $(x+y)dx+(7x)dy=0$

Integrating Factor $Q(y)$

61. $(y-y^2)dx+(x+y)dy=0$

62. $(y-y^2)dx+(2x+y)dy=0$

63. $(y-y^2)dx+(2x+3y)dy=0$

64. $(y+y^2)dx+(2x+3y)dy=0$

65. $(y+y^2)dx+(5x+3y)dy=0$

66. $(y+5y^2)dx+(5x+3y)dy=0$

67. $(2y+5y^2)dx+(5x+3y)dy=0$

68. $(2y+5y^2)dx+(7x+11y)dy=0$

69. $(2y+5y^3)dx+(3x+7y)dy=0$

70. $(3y+5y^3)dx+(7x+9y)dy=0$

Chapter 3

Linear Algebraic Equations No Matrices

Contents

3.2 Exercises

Toolkit

Compute the equivalent system of equations.

1. Given $\left|\begin{array}{ccccccc} x & & & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$, find the system that results from `combo(2,1,-1)`.

2. Given $\left|\begin{array}{ccccccc} x & & & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$, find the system that results from `swap(1,2)` followed by `combo(2,1,-1)`.

3. Given $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$, find the system that results from `combo(1,2,-1)`.

4. Given $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$, find the system that results from `swap(1,2)` followed by `combo(1,2,-1)`.

5. Given $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$, find the system that results from `swap(2,3)`, `combo(2,1,-1)`.

6. Given $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$, find the system that results from `mult(2,1/3)`, `combo(1,2,-1)`, `swap(2,3)`, `swap(1,2)`.

Inverse Toolkit

Compute the equivalent system of equations.

7. If $\left|\begin{array}{rcrcrcr} & - & y & & & = & -3 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$ resulted from `combo(2,1,-1)`, then find the original system.

8. If $\left|\begin{array}{rcrcl} & y & & = & 3 \\ x & & + \; 2z & = & 1 \\ & & z & = & 0 \end{array}\right|$ resulted from `swap(1,2)` followed by `combo(2,1,-1)`, then find the original system.

9. If $\left|\begin{array}{rcrcl} x & & + \; 3z & = & 1 \\ & y & - \; 3z & = & 4 \\ & & z & = & 1 \end{array}\right|$ resulted from `combo(1,2,-1)`, then find the original system.

10. If $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$ resulted from `swap(1,2)` followed by `combo(2,1,2)`, then find the original system.

11. If $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$ resulted from `mult(2,-1)`, `swap(2,3)`, `combo(2,1,-1)`, then find the original system.

12. If $\left|\begin{array}{rcrcl} 2y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$ resulted from `mult(2,1/3)`, `combo(1,2,-1)`, `swap(2,3)`, `swap(1,2)`, then find the original system.

Planar System

Solve the xy–system and interpret the solution geometrically as

(a) **parallel lines**

(b) **equal lines**

(c) **intersecting lines**.

13. $\left|\begin{array}{rcrcr} x & + & y & = & 1, \\ & & y & = & 1 \end{array}\right|$

14. $\left|\begin{array}{rcrcr} x & + & y & = & -1 \\ x & & & = & 3 \end{array}\right|$

15. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ x & + & 2y & = & 2 \end{array}\right|$

16. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ x & + & 2y & = & 3 \end{array}\right|$

17. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ 2x & + & 2y & = & 2 \end{array}\right|$

18. $\left|\begin{array}{rcrcr} 2x & + & y & = & 1 \\ 6x & + & 3y & = & 3 \end{array}\right|$

19. $\left|\begin{array}{rcrcr} x & - & y & = & 1 \\ -x & - & y & = & -1 \end{array}\right|$

20. $\left|\begin{array}{rcrcr} 2x & - & y & = & 1 \\ x & - & 0.5y & = & 0.5 \end{array}\right|$

21. $\left|\begin{array}{rcrcr} x & + & y & = & 1 \\ x & + & y & = & 2 \end{array}\right|$

22. $\left|\begin{array}{rcrcr} x & - & y & = & 1 \\ x & - & y & = & 0 \end{array}\right|$

System in Space

For each xyz–system:

(a) If no solution, then report **three identical shelves**, **pup tent**, **two parallel shelves** or **book shelf**.

(b) If infinitely many solutions, then report **one shelf**, **open book** or **saw tooth**.

(c) If a unique intersection point, then report the values of x, y and z.

23. $\left|\begin{array}{rrrrrl} x & - & y & + & z & = 2 \\ x & & & & & = 1 \\ & & y & & & = 0 \end{array}\right|$

24. $\left|\begin{array}{rrrrrl} x & + & y & - & 2z & = 3 \\ x & & & & & = 2 \\ & & & & z & = 1 \end{array}\right|$

25. $\left|\begin{array}{rrrl} x & - & y & = 2 \\ x & - & y & = 1 \\ x & - & y & = 0 \end{array}\right|$

26. $\left|\begin{array}{rrrl} x & + & y & = 3 \\ x & + & y & = 2 \\ x & + & y & = 1 \end{array}\right|$

27. $\left|\begin{array}{rrrrrl} x & + & y & + & z & = 3 \\ x & + & y & + & z & = 2 \\ x & + & y & + & z & = 1 \end{array}\right|$

28. $\left|\begin{array}{rrrrrl} x & + & y & + & 2z & = 2 \\ x & + & y & + & 2z & = 1 \\ x & + & y & + & 2z & = 0 \end{array}\right|$

29. $\left|\begin{array}{rrrrrl} x & - & y & + & z & = 2 \\ 2x & - & 2y & + & 2z & = 4 \\ & & y & & & = 0 \end{array}\right|$

30. $\left|\begin{array}{rrrrrl} x & + & y & - & 2z & = 3 \\ 3x & + & 3y & - & 6z & = 6 \\ & & & & z & = 1 \end{array}\right|$

31. $\left|\begin{array}{rrrrrl} x & - & y & + & z & = 2 \\ & & & & 0 & = 0 \\ & & & & 0 & = 0 \end{array}\right|$

32. $\left|\begin{array}{rrrrrl} x & + & y & - & 2z & = 3 \\ & & & & 0 & = 0 \\ & & & & 1 & = 1 \end{array}\right|$

33. $\left|\begin{array}{rcrcl} x & + & y & = & 2 \\ x & - & y & = & 2 \\ & & y & = & -1 \end{array}\right|$

34. $\left|\begin{array}{rcrcl} x & - & 2z & = & 4 \\ x & + & 2z & = & 0 \\ & & z & = & 2 \end{array}\right|$

35. $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$

36. $\left|\begin{array}{rcrcl} x & + & 2z & = & 1 \\ 4x & + & 8z & = & 4 \\ & & z & = & 0 \end{array}\right|$

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3.3 Exercises

Lead and free variables

For each system assume variable list $x_1, \ldots, x_5$. List the lead and free variables.

1. $$\left|\begin{array}{rcl} x_2 + 3x_3 \quad & = & 0 \\ x_4 \quad & = & 0 \\ 0 & = & 0 \end{array}\right|$$

2. $$\left|\begin{array}{rcl} x_2 \qquad\qquad\quad & = & 0 \\ x_3 \qquad + 3x_5 & = & 0 \\ x_4 + 2x_5 & = & 0 \end{array}\right|$$

3. $$\left|\begin{array}{rcl} x_2 + 3x_3 \quad & = & 0 \\ x_4 \quad & = & 0 \\ 0 & = & 0 \end{array}\right|$$

4. $$\left|\begin{array}{rcl} x_1 + 2x_2 + 3x_3 \quad & = & 0 \\ x_4 \quad & = & 0 \\ 0 & = & 0 \end{array}\right|$$

5. $$\left|\begin{array}{rcl} x_1 + 2x_2 + 3x_3 & = & 0 \\ 0 & = & 0 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}\right|$$

6. $$\left|\begin{array}{rcl} x_1 + x_2 \quad & = & 0 \\ x_3 \quad & = & 0 \\ 0 & = & 0 \end{array}\right|$$

7. $$\left|\begin{array}{rcl} x_1 + x_2 + 3x_3 + 5x_4 \quad & = & 0 \\ x_5 & = & 0 \\ 0 & = & 0 \end{array}\right|$$

8. $$\left|\begin{array}{rcl} x_1 + 2x_2 \qquad + 3x_4 + 4x_5 & = & 0 \\ x_3 + \ x_4 + \ x_5 & = & 0 \\ 0 & = & 0 \end{array}\right|$$

9. $$\left|\begin{array}{rcl} x_3 + 2x_4 \quad & = & 0 \\ x_5 & = & 0 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}\right|$$

10. $\left|\begin{array}{r} x_4 + x_5 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right|$

11. $\left|\begin{array}{lllr} x_2 & & +5x_4 & = 0 \\ & x_3 & +2x_4 & = 0 \\ & & & x_5 = 0 \\ & & & 0 = 0 \end{array}\right|$

12. $\left|\begin{array}{lllr} x_1 & +3x_3 & & = 0 \\ \quad x_2 & & +x_4 & = 0 \\ & & & x_5 = 0 \\ & & & 0 = 0 \end{array}\right|$

Elementary Operations

Consider the 3×3 system

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 2, \\ -2x & + & 3y & + & 4z & = & 0, \\ -3x & + & 5y & + & 7z & = & 3. \end{array}$$

Define symbols **combo**, **swap** and **mult** as in the textbook. Write the 3×3 system which results from each of the following operations.

13. `combo(1,3,-1)`

14. `combo(2,3,-5)`

15. `combo(3,2,4)`

16. `combo(2,1,4)`

17. `combo(1,2,-1)`

18. combo(1,2,-e^2)

19. `mult(1,5)`

20. `mult(1,-3)`

21. `mult(2,5)`

22. `mult(2,-2)`

23. `mult(3,4)`

24. `mult(3,5)`

25. `mult(2,-`π`)`

26. `mult(2,`π`)`

27. `mult(1,`e^2`)`

28. `mult(1,-`e^{-2}`)`

29. `swap(1,3)`

30. `swap(1,2)`

31. `swap(2,3)`

32. `swap(2,1)`

33. `swap(3,2)`

34. `swap(3,1)`

Unique Solution

Create a toolkit sequence for each system, whose final frame displays the unique solution of the system of equations.

35. $\begin{vmatrix} x_1+3x_2= & 0 \\ x_2= & -1 \end{vmatrix}$

36. $\begin{vmatrix} x_1+2x_2= & 0 \\ x_2= & -2 \end{vmatrix}$

37. $\begin{vmatrix} x_1+3x_2=2 \\ x_1-\ \ x_2=1 \end{vmatrix}$

38. $\begin{vmatrix} x_1+\ \ x_2=-1 \\ x_1+2x_2=-2 \end{vmatrix}$

39. $\begin{vmatrix} x_1+3x_2+2x_3=1 \\ x_2+4x_3=3 \\ 4x_3=4 \end{vmatrix}$

40. $\begin{vmatrix} x_1 & & & =1 \\ 3x_1+ & x_2 & & =0 \\ 2x_1+ & 2x_2+ & 3x_3 & =3 \end{vmatrix}$

41. $\begin{vmatrix} x_1+ & x_2+ & 3x_3 & =1 \\ & x_2 & & =2 \\ & & 3x_3 & =0 \end{vmatrix}$

42. $$\left|\begin{aligned} x_1 + 3x_2 + 2x_3 &= 1\\ x_2 &= 3\\ 3x_3 &= 0 \end{aligned}\right|$$

43. $$\left|\begin{aligned} x_1 &= 2\\ x_1 + 2x_2 &= 1\\ 2x_1 + 2x_2 + x_3 &= 0\\ 3x_1 + 6x_2 + x_3 + 2x_4 &= 2 \end{aligned}\right|$$

44. $$\left|\begin{aligned} x_1 &= 3\\ x_1 - 2x_2 &= 1\\ 2x_1 + 2x_2 + x_3 &= 0\\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 2 \end{aligned}\right|$$

45. $$\left|\begin{aligned} x_1 + x_2 &= 2\\ x_1 + 2x_2 &= 1\\ 2x_1 + 2x_2 + x_3 &= 0\\ 3x_1 + 6x_2 + x_3 + 2x_4 &= 2 \end{aligned}\right|$$

46. $$\left|\begin{aligned} x_1 - 2x_2 &= 3\\ x_1 - x_2 &= 1\\ 2x_1 + 2x_2 + x_3 &= 0\\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 1 \end{aligned}\right|$$

47. $$\left|\begin{aligned} x_1 &= 3\\ x_1 - x_2 &= 1\\ 2x_1 + 2x_2 + x_3 &= 0\\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 1\\ 3x_1 \qquad x_3 \qquad + 2x_5 &= 1 \end{aligned}\right|$$

48. $$\left|\begin{aligned} x_1 &= 2\\ x_1 - x_2 &= 0\\ 2x_1 + 2x_2 + x_3 &= 1\\ 3x_1 + 6x_2 + x_3 + 3x_4 &= 1\\ 3x_1 + x_3 + 3x_5 &= 1 \end{aligned}\right|$$

49. $$\left|\begin{aligned} x_1 + x_2 &= 2\\ x_1 - x_2 &= 0\\ 2x_1 + 2x_2 + 2x_3 &= 1\\ 3x_1 + 6x_2 + x_3 + 2x_4 &= 1\\ 3x_1 + x_3 + 3x_5 &= 2 \end{aligned}\right|$$

50. $$\left|\begin{aligned} x_1 - x_2 &= 3\\ x_1 - 2x_2 &= 0\\ 2x_1 + 2x_2 + x_3 &= 1\\ 3x_1 + 6x_2 + x_3 + 3x_4 &= 1\\ 3x_1 + x_3 + x_5 &= 3 \end{aligned}\right|$$

No Solution

Develop a toolkit sequence for each system, whose final frame contains a signal equation (e.g., $0 = 1$), thereby showing that the system has no solution.

51. $\left|\begin{array}{l} x_1+3x_2=0 \\ x_1+3x_2=1 \end{array}\right|$

52. $\left|\begin{array}{r} x_1+2x_2=1 \\ 2x_1+4x_2=2 \end{array}\right|$

53. $\left|\begin{array}{rrrl} x_1 & +3x_2 & +2x_3 & =1 \\ & x_2 & +4x_3 & =3 \\ & x_2 & +4x_3 & =4 \end{array}\right|$

54. $\left|\begin{array}{rrrl} x_1 & & & =0 \\ 3x_1 & +x_2 & +3x_3 & =1 \\ 2x_1 & +2x_2 & +6x_3 & =0 \end{array}\right|$

55. $\left|\begin{array}{rrrl} x_1 & +x_2 & +3x_3 & =1 \\ & x_2 & & =2 \\ x_1 & +2x_2 & +3x_3 & =2 \end{array}\right|$

56. $\left|\begin{array}{rrrl} x_1 & +3x_2 & +2x_3 & =1 \\ & x_2 & +2x_3 & =3 \\ x_1 & & +5x_3 & =5 \end{array}\right|$

57. $\left|\begin{array}{rrrrl} x_1 & & & & =2 \\ x_1 & +2x_2 & & & =2 \\ x_1 & +2x_2 & +x_3 & +2x_4 & =0 \\ x_1 & +6x_2 & +x_3 & +2x_4 & =2 \end{array}\right|$

58. $\left|\begin{array}{rrrrl} x_1 & & & & =3 \\ x_1 & -2x_2 & & & =1 \\ 2x_1 & +2x_2 & +x_3 & +4x_4 & =0 \\ 3x_1 & +6x_2 & +x_3 & +4x_4 & =2 \end{array}\right|$

59. $\left|\begin{array}{rrrrrl} x_1 & & & & & =3 \\ x_1 & -x_2 & & & & =1 \\ 2x_1 & +2x_2 & +x_3 & & & =0 \\ 3x_1 & +6x_2 & +x_3 & +4x_4 & -x_5 & =1 \\ & -6x_2 & -x_3 & -4x_4 & +x_5 & =0 \end{array}\right|$

60. $\left|\begin{array}{rrrrrl} x_1 & & & & & =3 \\ x_1 & -x_2 & & & & =1 \\ 3x_1 & +2x_2 & +x_3 & & & =0 \\ 3x_1 & +6x_2 & +x_3 & +4x_4 & -x_5 & =1 \\ & -6x_2 & -x_3 & -4x_4 & +x_5 & =2 \end{array}\right|$

Infinitely Many Solutions

Display a toolkit sequence for each system, whose final frame has this property: *each nonzero equation has a lead variable.* Then apply the **last frame algorithm** to write out the standard general solution of the system. Assume in each system variable list x_1 to x_5.

61. $$\left|\begin{array}{rrrr} x_1 + x_2 + 3x_3 & & = & 0 \\ x_2 \quad\quad & + x_4 & = & 0 \\ & & 0 = & 0 \end{array}\right|$$

62. $$\left|\begin{array}{rrrr} x_1 \quad\quad + x_3 & & = & 0 \\ x_1 + x_2 + x_3 & + 3x_5 & = & 0 \\ & x_4 + 2x_5 & = & 0 \end{array}\right|$$

63. $$\left|\begin{array}{rrrr} x_2 + 3x_3 & & = & 0 \\ & x_4 & = & 0 \\ & & 0 = & 0 \end{array}\right|$$

64. $$\left|\begin{array}{rrrr} x_1 + 2x_2 + 3x_3 & & = & 0 \\ & x_4 & = & 0 \\ & & 0 = & 0 \end{array}\right|$$

65. $$\left|\begin{array}{rrr} x_1 + 2x_2 + 3x_3 & & = 0 \\ & x_3 + x_4 & 0 = 0 \end{array}\right|$$

66. $$\left|\begin{array}{rrr} x_1 + x_2 & & = 0 \\ x_2 + x_3 & & = 0 \\ x_3 & & 0 = 1 \end{array}\right|$$

67. $$\left|\begin{array}{r} x_1 + x_2 + 3x_3 + 5x_4 + 2x_5 = 0 \\ x_5 = 0 \end{array}\right|$$

68. $$\left|\begin{array}{r} x_1 + 2x_2 + x_3 + 3x_4 + 4x_5 = 0 \\ x_3 + \ \ x_4 + \ \ x_5 = 0 \end{array}\right|$$

69. $$\left|\begin{array}{r} x_3 + 2x_4 + \ \ x_5 = 0 \\ 2x_3 + 2x_4 + 2x_5 = 0 \\ x_5 = 0 \end{array}\right|$$

70. $$\left|\begin{array}{r} x_4 + x_5 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right|$$

71. $$\left|\begin{array}{rr} x_2 + x_3 + 5x_4 & = 0 \\ x_3 + 2x_4 & = 0 \\ & x_5 = 0 \\ & 0 = 0 \end{array}\right|$$

72. $$\left|\begin{array}{l} x_1 \quad + 3x_3 = 0 \\ x_1 + x_2 \quad + x_4 = 0 \\ \hfill x_5 = 0 \\ \hfill 0 = 0 \end{array}\right|$$

Inverses of Elementary Operations

Given the final frame of a toolkit sequence is

$$\left|\begin{array}{rcrcrcr} 3x & + & 2y & + & 4z & = & 2 \\ x & + & 3y & + & 2z & = & -1 \\ 2x & + & y & + & 5z & = & 0 \end{array}\right|$$

and the given operations, find the original system in the first frame.

73. `combo(1,2,-1)`, `combo(2,3,-3)`, `mult(1,-2)`, `swap(2,3)`.

74. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,2)`, `swap(3,2)`.

75. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`.

76. `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

77. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`, `swap(2,3)`.

78. `swap(2,3)`, `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

79. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`, `mult(2,3)`.

80. `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`, `combo(2,3,-3)`.

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3.4 Exercises

Classification

Classify the parametric equations as a point, line or plane, then compute as appropriate the tangent to the line or the normal to the plane.

1. $x = 0,\ y = 1,\ z = -2$

2. $x = 1,\ y = -1,\ z = 2$

3. $x = t_1,\ y = 1 + t_1,\ z = 0$

4. $x = 0,\ y = 0,\ z = 1 + t_1$

5. $x = 1 + t_1,\ y = 0,\ z = t_2$

6. $x = t_2 + t_1,\ y = t_2,\ z = t_1$

7. $x = 1,\ y = 1 + t_1,\ z = 1 + t_2$

8. $x = t_2 + t_1,\ y = t_1 - t_2,\ z = 0$

9. $x = t_2,\ y = 1 + t_1,\ z = t_1 + t_2$

10. $x = 3t_2 + t_1,\ y = t_1 - t_2,\ z = 2t_1$

Reduced Echelon System

Solve the xyz–system and interpret the solution geometrically.

11. $\left|\begin{array}{rcrcl} y & + & z & = & 1 \\ x & + & 2z & = & 2 \end{array}\right|$

12. $\left|\begin{array}{rcrcl} x & + & z & = & 1 \\ y & + & 2z & = & 4 \end{array}\right|$

13. $\left|\begin{array}{rcrcl} y & + & z & = & 1 \\ x & + & 3z & = & 2 \end{array}\right|$

14. $\left|\begin{array}{rcrcl} x & + & z & = & 1 \\ y & + & z & = & 5 \end{array}\right|$

15. $\left|\begin{array}{rcrcl} x & + & z & = & 1 \\ 2x & + & 2z & = & 2 \end{array}\right|$

16. $\left|\begin{array}{rcrcl} x & + & y & = & 1 \\ 3x & + & 3y & = & 3 \end{array}\right|$

17. $\left|\ x + y + z = 1.\ \right|$

18. $\left|\ x + 2y + 4z = 0.\ \right|$

19. $\left|\begin{array}{rcrcrcl} x & + & y & & & = & 2 \\ & & & & z & = & 1 \end{array}\right|$

20. $\left|\begin{array}{rcrcrcl} x & & & + & 4z & = & 0 \\ & & y & & & = & 1 \end{array}\right|$

Homogeneous System

Solve the xyz–system using elimination with variable list order x, y, z.

21. $\left|\begin{array}{rcrcrcl} & & y & + & z & = & 0 \\ 2x & & & + & 2z & = & 0 \end{array}\right|$

22. $\left|\begin{array}{rcrcrcl} x & & & + & z & = & 0 \\ & & 2y & + & 2z & = & 0 \end{array}\right|$

23. $\left|\begin{array}{rcrcl} x & + & z & = & 0 \\ & & 2z & = & 0 \end{array}\right|$

24. $\left|\begin{array}{rcrcl} y & + & z & = & 0 \\ y & + & 3z & = & 0 \end{array}\right|$

25. $\left|\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 0 \\ & & & & 0 & = & 0 \end{array}\right|$

26. $\left|\begin{array}{rcrcl} x & + & 2y & = & 0 \\ & & 0 & = & 0 \end{array}\right|$

27. $\left|\begin{array}{rcrcrcl} & & y & + & z & = & 0 \\ 2x & & & + & 2z & = & 0 \\ x & & & + & z & = & 0 \end{array}\right|$

28. $\left|\begin{array}{rcrcrcl} 2x & + & y & + & z & = & 0 \\ x & & & + & 2z & = & 0 \\ x & + & y & - & z & = & 0 \end{array}\right|$

29. $\left|\begin{array}{rcrcrcl} x & + & y & + & z & = & 0 \\ 2x & & & + & 2z & = & 0 \\ x & & & + & z & = & 0 \end{array}\right|$

30. $\begin{vmatrix} x + y + z = 0 \\ 2x + 2z = 0 \\ 3x + y + 3z = 0 \end{vmatrix}$

Nonhomogeneous 3×3 System

Solve the xyz-system using elimination and variable list order x, y, z.

31. $\begin{vmatrix} y = 1 \\ 2z = 2 \end{vmatrix}$

32. $\begin{vmatrix} x = 1 \\ 2z = 2 \end{vmatrix}$

33. $\begin{vmatrix} y + z = 1 \\ 2x + 2z = 2 \\ x + z = 1 \end{vmatrix}$

34. $\begin{vmatrix} 2x + y + z = 1 \\ x + 2z = 2 \\ x + y - z = -1 \end{vmatrix}$

35. $\begin{vmatrix} x + y + z = 1 \\ 2x + 2z = 2 \\ x + z = 1 \end{vmatrix}$

36. $\begin{vmatrix} x + y + z = 1 \\ 2x + 2z = 2 \\ 3x + y + 3z = 3 \end{vmatrix}$

37. $\begin{vmatrix} 2x + y + z = 3 \\ 2x + 2z = 2 \\ 4x + y + 3z = 5 \end{vmatrix}$

38. $\begin{vmatrix} 2x + y + z = 2 \\ 6x \quad y + 5z = 2 \\ 4x + y + 3z = 2 \end{vmatrix}$

39. $\begin{vmatrix} 6x + 2y + 6z = 10 \\ 6x \quad y + 6z = 11 \\ 4x + y + 4z = 7 \end{vmatrix}$

40. $\begin{vmatrix} 6x + 2y + 4z = 6 \\ 6x \quad y + 5z = 9 \\ 4x + y + 3z = 5 \end{vmatrix}$

Nonhomogeneous 3×4 System

Solve the $yzuv$-system using elimination with variable list order y, z, u, v.

41. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 8v & = 10 \\ & & 2z & - & u & + & v & = 10 \\ 2y & & & - & u & + & 5v & = 10 \end{array}\right|$$

42. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 8v & = 10 \\ & & 2z & - & 2u & + & 2v & = 0 \\ y & + & 3z & + & 2u & + & 5v & = 5 \end{array}\right|$$

43. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 8v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 6v & = 1 \end{array}\right|$$

44. $$\left|\begin{array}{rrrrrrrr} y & + & 3z & + & 4u & + & 8v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 6v & = 1 \end{array}\right|$$

45. $$\left|\begin{array}{rrrrrrrr} y & + & 3z & + & 4u & + & 8v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 4z & + & 2u & + & 7v & = 1 \end{array}\right|$$

46. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 9v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 4z & + & 2u & + & 7v & = 1 \end{array}\right|$$

47. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 9v & = 1 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 4z & + & 2u & + & 7v & = 1 \end{array}\right|$$

48. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 9v & = 10 \\ & & 2z & - & 2u & + & 4v & = 4 \\ y & + & 4z & + & 2u & + & 7v & = 8 \end{array}\right|$$

49. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 9v & = 2 \\ & & 2z & - & 2u & + & 4v & = 4 \\ y & + & 3z & + & 5u & + & 13v & = 0 \end{array}\right|$$

50. $$\left|\begin{array}{rrrrrrrr} y & + & z & + & 4u & + & 3v & = 2 \\ & & 2z & - & 2u & + & 4v & = 4 \\ y & + & 3z & + & 5u & + & 7v & = 0 \end{array}\right|$$

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3.5 Exercises

Rank and Nullity

Compute an abbreviated sequence of `combo`, `swap`, `mult` steps which finds the value of the rank and nullity.

1. $\left|\begin{array}{rrrrrrr} x_1 & + & x_2 & + & 4x_3 & + & 8x_4 = 0 \\ & & 2x_2 & - & x_3 & + & x_4 = 0 \end{array}\right|$

2. $\left|\begin{array}{rrrrr} x_1 & + & x_2 & + & 8x_4 = 0 \\ & & 2x_2 & + & x_4 = 0 \end{array}\right|$

3. $\left|\begin{array}{rrrrrrr} x_1 & + & 2x_2 & + & 4x_3 & + & 9x_4 = 0 \\ x_1 & + & 8x_2 & + & 2x_3 & + & 7x_4 = 0 \end{array}\right|$

4. $\left|\begin{array}{rrrrrrr} x_1 & + & x_2 & + & 4x_3 & + & 11x_4 = 0 \\ & & 2x_2 & - & 2x_3 & + & 4x_4 = 0 \end{array}\right|$

Nullspace

Solve using variable order y, z, u, v. Report the values of the **nullity** and **rank** in the equation **nullity**+**rank**=4.

5. $\left|\begin{array}{rrrrrrr} y & + & z & + & 4u & + & 8v = 0 \\ & & 2z & - & u & + & v = 0 \\ 2y & & & - & u & + & 5v = 0 \end{array}\right|$

6. $\left|\begin{array}{rrrrrrr} y & + & z & + & 4u & + & 8v = 0 \\ & & 2z & - & 2u & + & 2v = 0 \\ y & + & 3z & + & 2u & + & 5v = 0 \end{array}\right|$

7. $\left|\begin{array}{rrrrrrr} y & + & z & + & 4u & + & 8v = 0 \\ & & 2z & - & 2u & + & 4v = 0 \\ y & + & 3z & + & 2u & + & 6v = 0 \end{array}\right|$

8. $\left|\begin{array}{rrrrrrr} y & + & 3z & + & 4u & + & 8v = 0 \\ & & 2z & - & 2u & + & 4v = 0 \\ y & + & 3z & + & 2u & + & 6v = 0 \end{array}\right|$

9. $\left|\begin{array}{rrrrrrr} y & + & 3z & + & 4u & + & 8v = 0 \\ & & 2z & - & 2u & + & 4v = 0 \end{array}\right|$

10. $\left|\begin{array}{rrrrrrr} y & + & z & + & 4u & + & 9v = 0 \\ & & 2z & - & 2u & + & 4v = 0 \end{array}\right|$

11. $\left|\begin{array}{rrrrl} y & + \ z & + \ 4u & + \ 9v & = 0 \\ 3y & + \ 4z & + \ 2u & + \ 5v & = 0 \end{array}\right|$

12. $\left|\begin{array}{rrrrl} y & + \ 2z & + \ 4u & + \ 9v & = 0 \\ y & + \ 8z & + \ 2u & + \ 7v & = 0 \end{array}\right|$

13. $\left|\begin{array}{rrrrl} y & + \ z & + \ 4u & + \ 11v & = 0 \\ & 2z & - \ 2u & + \ 4v & = 0 \end{array}\right|$

14. $\left|\begin{array}{rrrrl} y & + \ z & + \ 5u & + \ 11v & = 0 \\ & 2z & - \ 2u & + \ 6v & = 0 \end{array}\right|$

Dimension of the nullspace

In the homogeneous systems, assume variable order x, y, z, u, v.

(a) Display an equivalent set of equations in reduced echelon form.

(b) Solve for the general solution and check the answer.

(c) Report the dimension of the nullspace.

15. $\left|\begin{array}{rrrrrl} x & + \ y & + \ z & + \ 4u & + \ 8v & = 0 \\ -x & + & 2z & - \ 2u & + \ 2v & = 0 \\ & y & - \ z & + \ 6u & + \ 6v & = 0 \end{array}\right|$

16. $\left|\begin{array}{rrrrrl} x & + \ y & + \ z & + \ 4u & + \ 8v & = 0 \\ & & - \ 2z & - \ u & + \ v & = 0 \\ & 2y & & - \ u & + \ 5v & = 0 \end{array}\right|$

17. $\left|\begin{array}{rrrrrl} & y & + \ z & + \ 4u & + \ 8v & = 0 \\ x & & + \ 2z & - \ 2u & + \ 4v & = 0 \\ 2x & + \ y & + \ 3z & + \ 2u & + \ 6v & = 0 \end{array}\right|$

18. $\left|\begin{array}{rrrrrl} x & + \ y & + \ 3z & + \ 4u & + \ 8v & = 0 \\ 2x & & + \ 2z & - \ 2u & + \ 4v & = 0 \\ x & - \ y & + \ 3z & + \ 2u & + \ 12v & = 0 \end{array}\right|$

19. $\left|\begin{array}{rrrrl} y & + \ 3z & + \ 4u & + \ 20v & = 0 \\ & + \ 2z & - \ 2u & + \ 10v & = 0 \\ - \ y & + \ 3z & + \ 2u & + \ 30v & = 0 \end{array}\right|$

20. $\left|\begin{array}{rrrl} y & + \ 4u & + \ 20v & = 0 \\ & - \ 2u & + \ 10v & = 0 \\ - \ y & + \ 2u & + \ 30v & = 0 \end{array}\right|$

21. $\left|\begin{array}{rrrrl} x & + \ y & + \ z & + \ 4u & = 0 \\ & & - \ 2z & - \ u & = 0 \\ & 2y & & - \ u+ & = 0 \end{array}\right|$

22. $\left|\begin{array}{rrrrrr} & + & z + & 12u & + 8v = 0 \\ x & + & 2z - & 6u & + 4v = 0 \\ 2x & + & 3z + & 6u & + 6v = 0 \end{array}\right|$

23. $\left|\begin{array}{rrrl} y + & z + & 4u & = 0 \\ & 2z - & 2u & = 0 \\ y - & z + & 6u & = 0 \end{array}\right|$

24. $\left|\begin{array}{rrrr} x & + \ z & + & 8v = 0 \\ & - \ 2z & + & v = 0 \\ & & & 5v = 0 \end{array}\right|$

Three possibilities with symbols

Assume variables x, y, z. Determine the values of the constants (a, b, c, k, etc) such that the system has (1) *No solution*, (2) *A unique solution* or (3) *Infinitely many solutions.*

25. $\left|\begin{array}{rcrcl} x & + & ky & = & 0 \\ x & + & 2ky & = & 0 \end{array}\right|$

26. $\left|\begin{array}{rcrcl} kx & + & ky & = & 0 \\ x & + & 2ky & = & 0 \end{array}\right|$

27. $\left|\begin{array}{rcrcl} ax & + & by & = & 0 \\ x & + & 2by & = & 0 \end{array}\right|$

28. $\left|\begin{array}{rcrcl} bx & + & ay & = & 0 \\ x & + & 2y & = & 0 \end{array}\right|$

29. $\left|\begin{array}{rcrcr} bx & + & ay & = & c \\ x & + & 2y & = & b-c \end{array}\right|$

30. $\left|\begin{array}{rcrcr} bx & + & ay & = & 2c \\ x & + & 2y & = & c+a \end{array}\right|$

31. $\left|\begin{array}{rcrcrcl} bx & + & ay & + & z & = & 0 \\ 2bx & + & ay & + & 2z & = & 0 \\ x & + & 2y & + & 2z & = & c \end{array}\right|$

32. $\left|\begin{array}{rcrcrcr} bx & + & ay & + & z & = & 0 \\ 3bx & + & 2ay & + & 2z & = & 2c, \\ x & + & 2y & + & 2z & = & c \end{array}\right|$

33. $\left|\begin{array}{rcrcrcl} 3x & + & ay & + & z & = & b \\ 2bx & + & ay & + & 2z & = & 0 \\ x & + & 2y & + & 2z & = & c \end{array}\right|$

34. $\left|\begin{array}{rcrcrcl} x & + & ay & + & z & = & 2b \\ 3bx & + & 2ay & + & 2z & = & 2c \\ x & + & 2y & + & 2z & = & c \end{array}\right|$

Three Possibilities

The following questions can be answered by using the quantitative expression of the three possibilities in terms of lead and free variables, rank and nullity.

35. Does there exist a homogeneous 3×2 system with a unique solution? Either give an example or else prove that no such system exists.

36. Does there exist a homogeneous 2×3 system with a unique solution? Either give an example or else prove that no such system exists.

37. In a homogeneous 10×10 system, two equations are identical. Prove that the system has a nonzero solution.

38. In a homogeneous 5×5 system, each equation has a leading variable. Prove that the system has only the zero solution.

39. Suppose given two homogeneous systems A and B, with A having a unique solution and B having infinitely many solutions. Explain why B cannot be obtained from A by a sequence of swap, multiply and combination operations on the equations.

40. A 2×3 system cannot have a unique solution. Cite a theorem or explain why.

41. If a 3×3 homogeneous system contains no variables, then what is the general solution?

42. If a 3×3 non-homogeneous solution has a unique solution, then what is the nullity of the homogeneous system?

43. A 7×7 homogeneous system is missing two variables. What is the maximum rank of the system? Give examples for all possible ranks.

44. Suppose an $n \times n$ system of equations (homogeneous or non-homogeneous) has two solutions. Prove that it has infinitely many solutions.

45. What is the nullity and rank of an $n \times n$ system of homogeneous equations if the system has a unique solution?

46. What is the nullity and rank of an $n \times n$ system of non-homogeneous equations if the system has a unique solution?

47. Prove or disprove (by example): A 4×3 nonhomogeneous system cannot have a unique solution.

48. Prove or disprove (by example): A 4×3 homogeneous system always has infinitely many solutions.

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3.6 Exercises

Parametric solutions

1. Is there a 2×3 homogeneous system with general solution having 2 parameters t_1, t_2?

2. Is there a 3×3 homogeneous system with general solution having 3 parameters t_1, t_2, t_3?

3. Give an example of a 4×3 homogeneous system with general solution having zero parameters, that is, $x = y = z = 0$ is the only solution.

4. Give an example of a 4×3 homogeneous system with general solution having exactly one parameter t_1.

5. Give an example of a 4×3 homogeneous system with general solution having exactly two parameters t_1, t_2.

6. Give an example of a 4×3 homogeneous system with general solution having exactly three parameters t_1, t_2, t_3.

7. Consider an $n \times n$ homogeneous system with parametric solution having parameters t_1 to t_k. What are the possible values of k?

8. Consider an $n \times m$ homogeneous system with parametric solution having parameters t_1 to t_k. What are the possible values of k?

Answer Checks

Assume variable list x, y, z and parameter t_1. (a) Display the answer check details. (b) Find the rank. (c) Report whether the given solution is a general solution.

9. $\left|\begin{array}{rcrcrcl} & & y & & & = & 1 \\ & & & & 2z & = & 2 \end{array}\right|$
$x = t_1, y = 1, z = 1.$

10. $\left|\begin{array}{rcrcrcl} x & & & & & = & 1 \\ & & & & 2z & = & 2 \end{array}\right|$
$x = 1, y = t_1, z = 1.$

11. $\left|\begin{array}{rcrcrcl} & & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ x & & & + & z & = & 1 \end{array}\right|$
$x = 0, y = 0, z = 1.$

12. $\left|\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 1 \\ x & & & + & 2z & = & 2 \\ x & + & y & - & z & = & -1 \end{array}\right|$
$x = 2, y = -3, z = 0.$

13. $\left|\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ x & & & + & z & = & 1 \end{array}\right|$
$x = 1 - t_1, y = 0, z = t_1.$

14. $\left|\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ 2x & & & + & 2z & = & 2 \\ 3x & + & y & + & 3z & = & 3 \end{array}\right|$
$x = 1 - t_1, y = 0, z = t_1.$

Failure of Answer Checks

Find the unique solution for $\epsilon > 0$. Discuss how a machine might translate the system to obtain infinitely many solutions.

15. $x + \epsilon y = 1,\ x - \epsilon y = 1$

16. $x + y = 1,\ x + (1+\epsilon)y = 1 + \epsilon$

17. $x + \epsilon y = 10\epsilon,\ x - \epsilon y = 10\epsilon$

18. $x + y = 1 + \epsilon,\ x + (1+\epsilon)y = 1 + 11\epsilon$

Minimal Parametric Solutions

For each given system, determine if the expression is a minimal general solution.

19. $\left|\begin{array}{rcrcrcrl} y & + & z & + & 4u & + & 8v & = 0 \\ & & 2z & - & u & + & v & = 0 \\ 2y & & & - & u & + & 5v & = 0 \end{array}\right|$
$y = -3t_1, z = -t_1,$
$u = -t_1, v = t_1.$

20. $\left|\begin{array}{rcrcrcrl} y & + & z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 2v & = 0 \\ y & - & z & + & 6u & + & 6v & = 0 \end{array}\right|$
$y = -5t_1 - 7t_2, z = t_1 - t_2,$
$u = t_1, v = t_2.$

21. $\left|\begin{array}{rcrcrcrl} y & + & z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 6v & = 0 \end{array}\right|$
$y = -5t_1 + 5t_2, z = t_1 - t_2,$
$u = t_1 - t_2, v = 0.$

22. $\left|\begin{array}{rcrcrcrl} y & + & 3z & + & 4u & + & 8v & = 0 \\ & & 2z & - & 2u & + & 4v & = 0 \\ y & + & 3z & + & 2u & + & 12v & = 0 \end{array}\right|$
$y = 5t_1 + 4t_2, z = -3t_1 - 6t_2,$
$u = -t_1 - 2t_2, v = t_1 + 2t_2.$

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Chapter 4

Numerical Methods with Applications

Contents

4.2 Exercises

Connect-the-Dots

Make a numerical table of 6 rows and a connect-the-dots graphic for the following.

1. $y = 2x + 5$, $x = 0$ to $x = 1$

2. $y = 3x + 5$, $x = 0$ to $x = 2$

3. $y = 2x^2 + 5$, $x = 0$ to $x = 1$

4. $y = 3x^2 + 5$, $x = 0$ to $x = 2$

5. $y = \sin x$, $x = 0$ to $x = \pi/2$

6. $y = \sin 2x$, $x = 0$ to $x = \pi/4$

7. $y = x \ln |1 + x|$, $x = 0$ to $x = 2$

8. $y = x \ln |1 + 2x|$, $x = 0$ to $x = 1$

9. $y = xe^x$, $x = 0$ to $x = 1$

10. $y = x^2 e^x$, $x = 0$ to $x = 1/2$

Rectangular Rule

Apply the rectangular rule to make an xy-table for $y(x)$ with 11 rows and step size $h = 0.1$. Graph the approximate solution and the exact solution. Follow example 4.1.

11. $y' = 2x$, $y(0) = 5$.

12. $y' = 3x^2$, $y(0) = 5$.

13. $y' = 3x^2 + 2x$, $y(0) = 4$.

14. $y' = 3x^2 + 4x^3$, $y(0) = 4$.

15. $y' = \sin x$, $y(0) = 1$.

16. $y' = 2 \sin 2x$, $y(0) = 1$.

17. $y' = \ln(1 + x)$, $y(0) = 1$. Exact $(1 + x) \ln |1 + x| + 1 - x$.

18. $y' = 2 \ln(1 + 2x)$, $y(0) = 1$. Exact $(1 + 2x) \ln |1 + 2x| + 1 - 2x$.

19. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

20. $y' = 2x^2 e^{2x}$, $y(0) = 4$. Exact $2\,x^2 e^x - 4\,xe^x + 4\,e^x$.

Trapezoidal Rule

Apply the trapezoidal rule to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow example 4.2.

21. $y' = 2x$, $y(0) = 1$.

22. $y' = 3x^2$, $y(0) = 1$.

23. $y' = 3x^2 + 2x$, $y(0) = 2$.

24. $y' = 3x^2 + 4x^3$, $y(0) = 2$.

25. $y' = \sin x$, $y(0) = 4$.

26. $y' = 2\sin 2x$, $y(0) = 4$.

27. $y' = \ln(1+x)$, $y(0) = 1$. Exact $(1+x)\ln|1+x| + 1 - x$.

28. $y' = 2\ln(1+2x)$, $y(0) = 1$. Exact $(1+2x)\ln|1+2x| + 1 - 2x$.

29. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

30. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Simpson Rule

Apply Simpson's rule to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow example 4.3.

31. $y' = 2x$, $y(0) = 2$.

32. $y' = 3x^2$, $y(0) = 2$.

33. $y' = 3x^2 + 2x$, $y(0) = 3$.

34. $y' = 3x^2 + 4x^3$, $y(0) = 3$.

35. $y' = \sin x$, $y(0) = 5$.

36. $y' = 2\sin 2x$, $y(0) = 5$.

37. $y' = \ln(1+x)$, $y(0) = 1$. Exact $(1+x)\ln|1+x| + 1 - x$.

38. $y' = 2\ln(1+2x)$, $y(0) = 1$. Exact $(1+2x)\ln|1+2x| + 1 - 2x$.

39. $y' = xe^x$, $y(0) = 1$. Exact $xe^x - e^x + 2$.

40. $y' = 2x^2e^{2x}$, $y(0) = 4$. Exact $2\,x^2e^x - 4\,xe^x + 4\,e^x$.

Simpson's Rule

The following exercises use formulas and techniques found in the proof on page **??** and in Example 4.4, page 288.

41. Verify with Simpson's rule (**??**) for cubic polynomials the equality $\int_1^2 (x^3 + 16x^2 + 4)dx = 541/12$.

42. Verify with Simpson's rule (**??**) for cubic polynomials the equality $\int_1^2 (x^3 + x + 14)dx = 77/4$.

43. Let $f(x)$ satisfy $f(0) = 1$, $f(1/2) = 6/5$, $f(1) = 3/4$. Apply Simpson's rule with one division to verify that $\int_0^1 f(x)dx \approx 131/120$.

44. Let $f(x)$ satisfy $f(0) = -1$, $f(1/2) = 1$, $f(1) = 2$. Apply Simpson's rule with one division to verify that $\int_0^1 f(x)dx \approx 5/6$.

45. Verify Simpson's equality (**??**), assuming $Q(x) = 1$ and $Q(x) = x$.

46. Verify Simpson's equality (**??**), assuming $Q(x) = x^2$.

Quadratic Interpolation

The following exercises use formulas and techniques from the proof on page **??**.

47. Verify directly that the quadratic polynomial $y = x(7 - 4x)$ goes through the points $(0, 0)$, $(1, 3)$, $(2, -2)$.

48. Verify directly that the quadratic polynomial $y = x(8 - 5x)$ goes through the points $(0, 0)$, $(1, 3)$, $(2, -4)$.

49. Compute the quadratic interpolation polynomial $Q(x)$ which goes through the points $(0, 1)$, $(0.5, 1.2)$, $(1, 0.75)$.

50. Compute the quadratic interpolation polynomial $Q(x)$ which goes through the points $(0, -1)$, $(0.5, 1)$, $(1, 2)$.

51. Verify the remaining cases in Lemma 4.1, page 290.

52. Verify the remaining cases in Lemma 4.2, page 290.

4.3 Exercises

Euler's Method

Apply Euler's method to make an xy-table for $y(x)$ with 11 rows and step size $h = 0.1$. Graph the approximate solution and the exact solution. Follow Example 4.5.

1. $y' = 2 + y$, $y(0) = 5$. Exact $y(x) = -2 + 7e^x$.

2. $y' = 3 + y$, $y(0) = 5$. Exact $y(x) = -3 + 8e^x$.

3. $y' = e^{-x} + y$, $y(0) = 4$. Exact $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

4. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact $y(x) = -e^{-2x} + 5e^x$.

5. $y' = y \sin x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos x}$.

6. $y' = 2y \sin 2x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos 2x}$.

7. $y' = y/(1 + x)$, $y(0) = 1$. Exact $y(x) = 1 + x$.

8. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact $y(x) = \sqrt{1 + 2x}$.

9. $y' = yxe^x$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = 1 + (x - 1)e^x$.

10. $y' = 2y(x^2 + x)e^{2x}$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = x^2e^{2x}$.

Heun's Method

Apply Heun's method to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow Example 4.6.

11. $y' = 2 + y$, $y(0) = 5$. Exact $y(x) = -2 + 7e^x$.

12. $y' = 3 + y$, $y(0) = 5$. Exact $y(x) = -3 + 8e^x$.

13. $y' = e^{-x} + y$, $y(0) = 4$. Exact $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

14. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact $y(x) = -e^{-2x} + 5e^x$.

15. $y' = y \sin x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos x}$.

16. $y' = 2y \sin 2x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos 2x}$.

17. $y' = y/(1 + x)$, $y(0) = 1$. Exact $y(x) = 1 + x$.

18. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact $y(x) = \sqrt{1 + 2x}$.

19. $y' = yxe^x$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = 1 + (x-1)e^x$.

20. $y' = 2y(x^2 + x)e^{2x}$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = x^2e^{2x}$.

RK4 Method

Apply the Runge-Kutta method (RK4) to make an xy-table for $y(x)$ with 6 rows and step size $h = 0.2$. Graph the approximate solution and the exact solution. Follow Example 4.7.

21. $y' = 2 + y$, $y(0) = 5$. Exact $y(x) = -2 + 7e^x$.

22. $y' = 3 + y$, $y(0) = 5$. Exact $y(x) = -3 + 8e^x$.

23. $y' = e^{-x} + y$, $y(0) = 4$. Exact $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

24. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact $y(x) = -e^{-2x} + 5e^x$.

25. $y' = y\sin x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos x}$.

26. $y' = 2y\sin 2x$, $y(0) = 1$. Exact $y(x) = e^{1-\cos 2x}$.

27. $y' = y/(1+x)$, $y(0) = 1$. Exact $y(x) = 1 + x$.

28. $y' = y(x)/(1+2x)$, $y(0) = 1$. Exact $y(x) = \sqrt{1+2x}$.

29. $y' = yxe^x$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = 1 + (x-1)e^x$.

30. $y' = 2y(x^2 + x)e^{2x}$, $y(0) = 1$. Exact $y(x) = e^{u(x)}$, $u(x) = x^2e^{2x}$.

Euler and RK4 Methods

Apply the Euler method and the Runge-Kutta method (RK4) to make a table with 6 rows and step size $h = 0.1$. The table columns are x, y_1, y_2, y where y_1 is the Euler approximation, y_2 is the RK4 approximation and y is the exact solution. Graph y_1, y_2, y.

31. $y' = \frac{1}{2}(y-2)^2$, $y(0) = 3$. Exact $y(x) = \dfrac{2x-6}{x-2}$.

32. $y' = \frac{1}{2}(y-3)^2$, $y(0) = 4$. Exact $y(x) = \dfrac{3x-8}{x-2}$.

33. $y' = x^3/y^2$, $y(2) = 3$. Exact $y(x) = \frac{1}{2}\sqrt[3]{6x^4 + 120}$.

34. $y' = x^5/y^2$, $y(2) = 3$. Exact $y(x) = \frac{1}{2}\sqrt[3]{4x^6 - 40}$.

35. $y' = 2x(1+y^2)$, $y(1) = 1$. Exact $y(x) = \tan(x^2 - 1 + \pi/4)$.

36. $y' = 3y^{2/3}$, $y(0) = 1$. Exact $y(x) = (x+1)^3$.

37. $y' = 1 + y^2$, $y(0) = 0$. Exact $y(x) = \tan x$.

38. $y' = 1 + y^2$, $y(0) = 1$. Exact $y(x) = \tan(x + \pi/4)$.

4.4 Exercises

Cumulative Error

Make a table of 6 lines which has four columns x, y_1, y, E. Symbols y_1 and y are the approximate and exact solutions while E is the cumulative error. Find y_1 using Euler's method in steps $h = 0.1$.

1. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^x$.

2. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^x$.

3. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$.

4. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^x$.

Local Error

Make a table of 4 lines which has four columns x, y_1, y, E. Symbols y_1 and y are the approximate and exact solutions while E is the local error. Find y_1 using Euler's method in steps $h = 0.1$. The general solution in each exercise is the solution for $y(0) = c$.

5. $y' = 2 + y$, $y(0) = 5$. General solution $y(x) = -2 + (2 + c)e^x$.

6. $y' = 3 + y$, $y(0) = 5$. General solution $y(x) = -3 + (3 + c)e^x$.

7. $y' = 2e^{-x} + y$, $y(0) = 4$. General solution $y(x) = -e^{-x} + (1 + c)e^x$.

8. $y' = 3e^{-2x} + y$, $y(0) = 4$. General solution $y(x) = -e^{-2x} + (1 + c)e^x$.

Roundoff Error

Compute the roundoff error for $y = 5a + 4b$.

9. Assume 3-digit precision. Let $a = 0.0001$ and $b = 0.0003$.

10. Assume 3-digit precision. Let $a = 0.0002$ and $b = 0.0001$.

11. Assume 5-digit precision. Let $a = 0.000007$ and $b = 0.000003$.

12. Assume 5-digit precision. Let $a = 0.000005$ and $b = 0.000001$.

Truncation Error

Find the truncation error.

13. Truncate $x = 1.123456789$ to 3 digits right of the decimal point.

14. Truncate $x = 1.123456789$ to 4 digits right of the decimal point.

15. Truncate $x = 1.017171717$ to 7 digits right of the decimal point.

16. Truncate $x = 1.03939393939$ to 9 digits right of the decimal point.

Guessing the Step Size

Do a numerical experiment to estimate the step size needed for 7-digit accuracy of the solution. Using the given method, report the step size, which if halved repeatedly, generates a numerical solution with 7-digit accuracy.

17. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^x$. Euler's method.

18. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^x$. Euler's method

19. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$. Euler's method

20. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^x$. Euler's method.

21. $y' = y/(1 + x)$, $y(0) = 1$. Exact solution $y(x) = 1 + x$. Euler's method.

22. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact solution $y(x) = \sqrt{1 + 2x}$. Euler's method.

23. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^x$. Heun's method.

24. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^x$. Heun's method

25. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$. Heun's method

26. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^x$. Heun's method.

27. $y' = y/(1 + x)$, $y(0) = 1$. Exact solution $y(x) = 1 + x$. Heun's method.

28. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact solution $y(x) = \sqrt{1 + 2x}$. Heun's method.

29. $y' = 2 + y$, $y(0) = 5$. Exact solution $y(x) = -2 + 7e^x$. RK4 method.

30. $y' = 3 + y$, $y(0) = 5$. Exact solution $y(x) = -3 + 8e^x$. RK4 method

31. $y' = e^{-x} + y$, $y(0) = 4$. Exact solution $y(x) = -\frac{1}{2}e^{-x} + \frac{9}{2}e^x$. RK4 method

32. $y' = 3e^{-2x} + y$, $y(0) = 4$. Exact solution $y(x) = -e^{-2x} + 5e^x$. RK4 method.

33. $y' = y/(1 + x)$, $y(0) = 1$. Exact solution $y(x) = 1 + x$. RK4 method.

34. $y' = y(x)/(1 + 2x)$, $y(0) = 1$. Exact solution $y(x) = \sqrt{1 + 2x}$. RK4 method.

4.5 Exercises

Computing π

Compute $\pi = y(1)$ from the initial value problem $y' = 4/(1 + x^2)$, $y(0) = 0$, using the given method.

1. Use the Rectangular integration rule. Determine the number of steps for 5-digit precision.

2. Use the Rectangular integration rule. Determine the number of steps for 8-digit precision.

3. Use the Trapezoidal integration rule. Determine the number of steps for 5-digit precision.

4. Use the Trapezoidal integration rule. Determine the number of steps for 8-digit precision.

5. Use classical RK4. Determine the number of steps for 5-digit precision.

6. Use classical RK4. Determine the number of steps for 10-digit precision.

7. Use computer algebra system assist for RK4. Report the number of digits of precision using system defaults.

8. Use numerical workbench assist for RK4. Report the number of digits of precision using system defaults.

Computing $\ln(2)$

Compute $\ln(2) = y(1)$ from the initial value problem $y' = 1/(1 + x)$, $y(0) = 0$, using the given method.

9. Use the Rectangular integration rule. Determine the number of steps for 5-digit precision.

10. Use the Rectangular integration rule. Determine the number of steps for 8-digit precision.

11. Use the Trapezoidal integration rule. Determine the number of steps for 5-digit precision.

12. Use the Trapezoidal integration rule. Determine the number of steps for 8-digit precision.

13. Use classical RK4. Determine the number of steps for 5-digit precision.

14. Use classical RK4. Determine the number of steps for 10-digit precision.

15. Use computer algebra system assist for RK4. Report the number of digits of precision using system defaults.

16. Use numerical workbench assist for RK4. Report the number of digits of precision using system defaults.

Computing e

Compute $e = y(1)$ from the initial value problem $y' = y$, $y(0) = 1$, using the given computer assist. Report the number of digits of precision using system defaults.

17. Improved Euler method, also known as Heun's method.

18. RK4 method.

19. RKF45 method.

20. Adams-Moulton method.

Stiff Differential Equation

The flame propagation equation $y' = y^2(1 - y)$ is known to be **Stiff** for initial conditions $y(0) = y_0$ with $y_0 > 0$ and small. Use classical RK4 and then a stiff solver to compute and plot the solution $y(t)$ in each case. Expect 3000 steps with RK4 versus 100 with a stiff solver.

The exact solution of this equation can be expressed in terms of the **Lambert function** $w(u)$, defined by $u = w(x)$ if and only if $ue^u = x$. For example, $y(0) = 0.01$ gives

$$y(t) = \frac{1}{w\left(99e^{99-t}\right) + 1}.$$

See R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth. "On The Lambert W Function," Advances in Computational Mathematics 5 (1996): 329-359.

21. $y(0) = 0.01$

22. $y(0) = 0.005$

23. $y(0) = 0.001$

24. $y(0) = 0.0001$

4.6 Exercises

Critical Altitude r^*

The symbol r^* is the altitude $r(t)$ at which gravitational effects of the moon take over, causing the projectile to fall to the moon.

1. Justify from the differential equation that $r''(t) = 0$ at $r^* = r(t)$ implies the first relation in (**??**):

$$\frac{Gm_2}{(R_2 - R_1 - r^*)^2} - \frac{Gm_1}{(R_1 + r^*)^2} = 0.$$

2. Solve symbolically the relation of the previous exercise for r^*, to obtain the second equation of (**??**):

$$r^* = \frac{R_2}{1 + \sqrt{m_2/m_1}} - R_1.$$

3. Use the previous exercise and values for the constants R_1, R_2, m_1, m_2 to obtain the approximation

$$r^* = 339,620,820 \text{ meters}.$$

4. Determine the effect on r^* for a one percent error in measurement m_2. Replace m_2 by $0.99m_2$ and $1.01m_2$ in the formula for r^* and report the two estimated critical altitudes.

Escape Velocity v_0^*

The symbol v_0^* is the velocity $r'(0)$ such that $\lim_{t\to\infty} r(t) = \infty$, but smaller launch velocities will cause the projectile to fall back to the earth. Throughout, define

$$F(r) = \frac{Gm_1}{R_1 + r} + \frac{Gm_2}{R_2 - R_1 - r}.$$

5. Let $v_0 = r'(0)$, $r^* = r(t_0)$. Derive the formula

$$\frac{1}{2}\left(r'(t_0)\right)^2 = F(r^*) - F(0) + \frac{1}{2}v_0^2$$

which appears in the proof details.

6. Verify using the previous exercise that $r'(t_0) = 0$ implies

$$v_0^* = \sqrt{2(F(0) - F(r^*))}.$$

7. Verify by hand calculation that $v_0^* \approx 11067.31016$ meters per second.

8. Argue by mathematical proof that $F(r)$ is not minimized at the endpoints of the interval $0 \le r \le R$.

Numerical Experiments

Assume values given in the text for physical constants. Perform the given experiment, using numerical software, on initial value problem (**??**), page **??**. The cases when $v_0 > v_0^*$ escape the earth, while the others fall back to earth.

9. RK4 solver, $v_0 = 11068$, $T = 515000$. Plot the solution on $0 \le t \le T$.

10. Stiff solver, $v_0 = 11068$, $T = 515000$. Plot the solution on $0 \le t \le T$.

11. RK4 solver, $v_0 = 11067.2$, $T = 800000$. Plot the solution on $0 \le t \le T$.

12. Stiff solver, $v_0 = 11067.2$, $T = 800000$. Plot the solution on $0 \le t \le T$.

13. RK4 solver, $v_0 = 11067$, $T = 1000000$. Plot the solution on $0 \le t \le T$.

14. Stiff solver, $v_0 = 11067$, $T = 1000000$. Plot the solution on $0 \le t \le T$.

15. RK4 solver, $v_0 = 11066$, $T = 800000$. Plot the solution on $0 \le t \le T$.

16. Stiff solver, $v_0 = 11066$, $T = 800000$. Plot the solution on $0 \le t \le T$.

17. RK4 solver, $v_0 = 11065$. Find a suitable value T which shows that the projectile falls back to earth, then plot the solution on $0 \le t \le T$.

18. Stiff solver, $v_0 = 11065$. Find a suitable value T which shows that the projectile falls back to earth, then plot the solution on $0 \le t \le T$.

19. RK4 solver, $v_0 = 11070$. Find a suitable value T which shows that the projectile falls to the moon, then plot the solution on $0 \le t \le T$.

20. Stiff solver, $v_0 = 11070$. Find a suitable value T which shows that the projectile falls to the moon, then plot the solution on $0 \le t \le T$.

4.7 Exercises

Terminal Velocity

Assume force $F(v) = av + bv^2 + cv^3$ and $g = 32$, $m = 160/g$. Using computer assist, find the terminal velocity v_∞ from the velocity model $v' = g - \frac{1}{m}F(v)$, $v(0) = 0$.

1. $a = 0$, $b = 0$ and $c = 0.0002$.

2. $a = 0$, $b = 0$ and $c = 0.00015$.

3. $a = 0$, $b = 0.0007$ and $c = 0.00009$.

4. $a = 0$, $b = 0.0007$ and $c = 0.000095$.

5. $a = 0.009$, $b = 0.0008$ and $c = 0.00015$.

6. $a = 0.009$, $b = 0.00075$ and $c = 0.00015$.

7. $a = 0.009$, $b = 0.0007$ and $c = 0.00009$.

8. $a = 0.009$, $b = 0.00077$ and $c = 0.00009$.

9. $a = 0.009$, $b = 0.0007$ and $c = 0$.

10. $a = 0.009$, $b = 0.00077$ and $c = 0$.

Numerical Experiment

Let $F(v) = av + bv^2 + cv^3$ and $g = 32$. Consider the skydiver problem $mv'(t) = mg - F(v)$ and constants m, a, b, c supplied below. Using computer assist, apply a numerical method to produce a table for the elapsed time t, the velocity $v(t)$ and the distance $x(t)$. The table must end at $x(t) \approx 10000$ feet, which determines the flight time.

11. $m = 160/g$, $a = 0$, $b = 0$ and $c = 0.0002$.

12. $m = 160/g$, $a = 0$, $b = 0$ and $c = 0.00015$.

13. $m = 130/g$, $a = 0$, $b = 0.0007$ and $c = 0.00009$.

14. $m = 130/g$, $a = 0$, $b = 0.0007$ and $c = 0.000095$.

15. $m = 180/g$, $a = 0.009$, $b = 0.0008$ and $c = 0.00015$.

16. $m = 180/g$, $a = 0.009$, $b = 0.00075$ and $c = 0.00015$.

17. $m = 170/g$, $a = 0.009$, $b = 0.0007$ and $c = 0.00009$.

18. $m = 170/g$, $a = 0.009$, $b = 0.00077$ and $c = 0.00009$.

19. $m = 200/g$, $a = 0.009$, $b = 0.0007$ and $c = 0$.

20. $m = 200/g$, $a = 0.009$, $b = 0.00077$ and $c = 0$.

Flight Time

Let $F(v) = av + bv^2 + cv^3$ and $g = 32$. Consider the skydiver problem $mv'(t) = mg - F(v)$ with constants m, a, b, c supplied below. Using computer assist, apply a numerical method to find accurate values for the flight time to 10,000 feet and the time required to reach terminal velocity.

21. $mg = 160$, $a = 0.0095$, $b = 0.0007$ and $c = 0.000092$.

22. $mg = 160$, $a = 0.0097$, $b = 0.00075$ and $c = 0.000095$.

23. $mg = 240$, $a = 0.0092$, $b = 0.0007$ and $c = 0$.

24. $mg = 240$, $a = 0.0095$, $b = 0.00075$ and $c = 0$.

Ejected Baggage

Baggage of 45 pounds is dropped from a hovercraft at $15,000$ feet. Assume air resistance force $F(v) = av + bv^2 + cv^3$, $g = 32$ and $mg = 45$. Using computer assist, find accurate values for the flight time to the ground and the terminal velocity. Estimate the time required to reach 99.95% of terminal velocity.

25. $a = 0.0095$, $b = 0.0007$, $c = 0.00009$

26. $a = 0.0097$, $b = 0.00075$, $c = 0.00009$

27. $a = 0.0099$, $b = 0.0007$, $c = 0.00009$

28. $a = 0.0099$, $b = 0.00075$, $c = 0.00009$

4.8 Exercises

Lunar Lander Constant Field

Find the retrorocket activation time T and the activation height $x(T)$. Assume the constant gravitational field model. Units are miles/hour and miles/hour per hour.

1. $v_0 = 1210$, $A = 30020$.

2. $v_0 = 1200$, $A = 30100$.

3. $v_0 = 1300$, $A = 32000$.

4. $v_0 = 1350$, $A = 32000$.

5. $v_0 = 1500$, $A = 45000$.

6. $v_0 = 1550$, $A = 45000$.

7. $v_0 = 1600$, $A = 53000$.

8. $v_0 = 1650$, $A = 53000$.

9. $v_0 = 1400$, $A = 40000$.

10. $v_0 = 1450$, $A = 40000$.

Lunar Lander Variable Field

Find the retrorocket activation time T and the activation height $x(T)$. Assume the variable gravitational field model and mks units.

11. $v_0 = 540.92$, $g_1 = 5.277$.

12. $v_0 = 536.45$, $g_1 = 5.288$.

13. $v_0 = 581.15$, $g_1 = 5.517$.

14. $v_0 = 603.504$, $g_1 = 5.5115$.

15. $v_0 = 625.86$, $g_1 = 5.59$.

16. $v_0 = 603.504$, $g_1 = 5.59$.

17. $v_0 = 581.15$, $g_1 = 5.59$.

18. $v_0 = 670.56$, $g_1 = 6.59$.

19. $v_0 = 670.56$, $g_1 = 6.83$.

20. $v_0 = 715.26$, $g_1 = 7.83$.

Distinguishing Models

The constant field model (**1**) page **??** and the variable field model (**2**) page **??** are verified here, by example, to be distinct. Find the retrorocket activation times T_1, T_2 and the activation heights $x_1(T_1)$, $x_2(T_2)$ for the two models (**1**), (**2**). Relations $A = g_1 - g_0$ and $g_0 = GM/R^2$ apply to compute g_1 for the variable field model.

21. $v_0 = 1200$ mph, $A = 10000$ mph/h. Answer: 72, 66.91 miles.

22. $v_0 = 1200$ mph, $A = 12000$ mph/h. Answer: 60, 56.9 miles.

23. $v_0 = 1300$ mph, $A = 10000$ mph/h. Answer: 84.5, 77.7 miles.

24. $v_0 = 1300$ mph, $A = 12000$ mph/h. Answer: 70.42, 66.26 miles.

4.9 Exercises

Eccentric Anomaly for the Planets

Make a plot of the eccentric anomaly $E(M)$ on $0 \le M \le 2\pi$.

1. Mercury, $e = 0.2056$

2. Venus, $e = 0.0068$

3. Earth, $e = 0.0167$

4. Mars, $e = 0.0934$

5. Jupiter, $e = 0.0483$

6. Saturn, $e = 0.0560$

7. Uranus, $e = 0.0461$

8. Neptune, $e = 0.0097$

Elliptic Path of the Planets

Make a plot of the elliptic path of each planet, using constrained scaling with the given major semi-axis A (in astronomical units AU).

9. Mercury, $e = 0.2056$, $A = 0.39$

10. Venus, $e = 0.0068$, $A = 0.72$

11. Earth, $e = 0.0167$, $A = 1$

12. Mars, $e = 0.0934$, $A = 1.52$

13. Jupiter, $e = 0.0483$, $A = 5.20$

14. Saturn, $e = 0.0560$, $A = 9.54$

15. Uranus, $e = 0.0461$, $A = 19.18$

16. Neptune $e = 0.0097$, $A = 30.06$

Planet Positions

Make a plot with at least 8 planet positions displayed. Use constrained scaling with major semi-axis 1 in the plot. Display the given major semi-axis A and period T in the legend.

17. Mercury, $e = 0.2056$, $A = 0.39$ AU, $T = 0.24$ earth-years

18. Venus, $e = 0.0068$, $A = 0.72$ AU, $T = 0.62$ earth-years

19. Earth, $e = 0.0167$, $A = 1$ AU, $T = 1$ earth-years

20. Mars, $e = 0.0934$, $A = 1.52$ AU, $T = 1.88$ earth-years

21. Jupiter, $e = 0.0483$, $A = 5.20$ AU, $T = 11.86$ earth-years

22. Saturn, $e = 0.0560$, $A = 9.54$ AU, $T = 29.46$ earth-years

23. Uranus, $e = 0.0461$, $A = 19.18$ AU, $T = 84.01$ earth-years

24. Neptune $e = 0.0097$, $A = 30.06$ AU, $T = 164.8$ earth-years

Comet Positions

Make a plot with at least 8 comet positions displayed. Use constrained scaling with major-semiaxis 1 in the plot. Display the given eccentricity e and period T in the legend.

25. Churyumov-Gerasimenko orbits the sun every 6.57 earth-years. Discovered in 1969. Eccentricity $e = 0.632$.

26. Comet Wirtanen was the original target of the Rosetta space mission. This comet was discovered in 1948. The comet orbits the sun once every 5.46 earth-years. Eccentricity $e = 0.652$.

27. Comet Wild 2 was discovered in 1978. The comet orbits the sun once every 6.39 earth-years. Eccentricity $e = 0.540$.

28. Comet Biela was discovered in 1772. It orbits the sun every 6.62 earth-years. Eccentricity $e = 0.756$.

29. Comet Encke was discovered in 1786. It orbits the sun each 3.31 earth-years. Eccentricity $e = 0.846$.

30. Comet Giacobini-Zinner, discovered in 1900, orbits the sun each 6.59 earth-years. Eccentricity $e = 0.708$.

31. Comet Schwassmann-Wachmann, discovered in 1930, orbits the sun every 5.36 earth-years. Eccentricity $e = 0.694$.

32. Comet Swift-Tuttle was discovered in 1862. It orbits the sun each 120 earth-years. Eccentricity $e = 0.960$.

Comet Animations

Make an animation plot of comet positions. Use constrained scaling with major-semiaxis 1 in the plot. Display the given period T and eccentricity e in the legend.

33. Comet Churyumov-Gerasimenko
$T = 6.57$, $e = 0.632$.

34. Comet Wirtanen
$T = 5.46$, $e = 0.652$.

35. Comet Wild 2
$T = 6.39$, $e = 0.540$.

36. Comet Biela
$T = 6.62$, $e = 0.756$.

37. Comet Encke
$T = 3.31$, $e = 0.846$.

38. Comet Giacobini-Zinner
$T = 6.59$, $e = 0.708$.

39. Comet Schwassmann-Wachmann
$T = 5.36$, $e = 0.694$.

40. Comet Swift-Tuttle
$T = 120$, $e = 0.960$.

4.10 Exercises

Constant Logistic Harvesting

The model

$$x'(t) = kx(t)(M - x(t)) - h$$

can be converted to the logistic model

$$y'(t) = (a - by(t))y(t)$$

by a change of variables. Find the change of variables $y = x + c$ for the following pairs of equations.

1. $x' = -3x^2 + 8x - 5$,
$y' = (2 - 3y)y$

2. $x' = -2x^2 + 11x - 14$,
$y' = (3 - 2y)y$

3. $x' = -5x^2 - 19x - 18$,
$y' = (1 - 5y)y$

4. $x' = -x^2 + 3x + 4$,
$y' = (5 - y)y$

Periodic Logistic Harvesting

The periodic harvesting model

$$x'(t) = 0.8x(t)\left(1 - \frac{x(t)}{780500}\right) - H(t)$$

is considered with H defined by

$$H(t) = \begin{cases} 0 & 0 < t < 5, \\ H_0 & 5 < t < 6, \\ 0 & 6 < t < 17, \\ H_0 & 17 < t < 18, \\ 0 & 18 < t < 24. \end{cases}$$

The project is to make a computer graph of the solution on $0 < t < 24$ for various values of H_0 and $x(0)$. See Figures **??** and 18 and the corresponding examples.

5. $H_0 = 156100$, $P(0) = 300000$

6. $H_0 = 156100$, $P(0) = 800000$

7. $H_0 = 800100$, $P(0) = 90000$

8. $H_0 = 800100$, $P(0) = 100000$

von Bertalanffy Equation

Karl Ludwig von Bertalanffy (1901-1972) derived in 1938 the equation

$$\frac{dL}{dt} = r_B(L_\infty - L(t))$$

from simple physiological arguments. It is a widely used growth curve, especially important in fisheries studies. The symbols:

t	time,
$L(t)$	length,
r_B	growth rate,
L_∞	expected length for zero growth.

9. Solve $\frac{dL}{dt} = 2(10 - L)$, $L(0) = 0$. The answer is the length in inches of a fish over time, with final adult size 10 inches.

10. Solve von Bertalanffy's equation to obtain the algebraic model

$$L(t) = L_\infty \left(1 - e^{-r_B(t-t_0)}\right).$$

11. Assume von Bertalanffy's model. Suppose field data $L(0) = 0$, $L(1) = 5$, $L(2) = 7$. Display the nonlinear regression details which determine $t_0 = 0$, $L_\infty = 25/3$ and $r_B = \ln(5/2)$.

12. Assume von Bertalanffy's model with field data $L(0) = 0$, $L(1) = 10$, $L(2) = 13$. Find the expected length L_∞ of the fish.

Chapter 5

Linear Algebra

Contents

5.2 Exercises

Fixed vectors

Perform the indicated operation(s).

1. $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

2. $\begin{pmatrix} 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

3. $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

4. $\begin{pmatrix} 2 \\ -2 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$

5. $2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

6. $3\begin{pmatrix} 2 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

7. $5\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

8. $3\begin{pmatrix} 2 \\ -2 \\ 9 \end{pmatrix} - 5\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$

9. $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

10. $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

Parallelogram Rule

Determine the resultant vector in two ways: (a) the parallelogram rule, and (b) fixed vector addition.

11. $\begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

12. $(2\vec{\imath} - 2\vec{\jmath}) + (\vec{\imath} - 3\vec{\jmath})$

13. $\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$

14. $(2\vec{\imath} - 2\vec{\jmath} + 3\vec{k}) + (\vec{\imath} - 3\vec{\jmath} - \vec{k})$

Toolkit

Let V be the data set of all fixed 2-vectors, $V = \mathcal{R}^2$. Define addition and scalar multiplication componentwise. Verify the following toolkit rules by direct computation.

15. **(Commutative)**
$\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$

16. **(Associative)**
$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$

17. (Zero)
Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$

18. (Negative)
Vector $-\vec{X}$ is defined and
$\vec{X} + (-\vec{X}) = \vec{0}$

19. (Distributive I)
$k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$

20. (Distributive II)
$(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$

21. (Distributive III)
$k_1(k_2\vec{X}) = (k_1k_2)\vec{X}$

22. (Identity)
$1\vec{X} = \vec{X}$

Subspaces

Verify that the given restriction equation defines a subspace S of $V = \mathcal{R}^3$. Use Theorem 5.2, page 366.

23. $z = 0$

24. $y = 0$

25. $x + z = 0$

26. $2x + y + z = 0$

27. $x = 2y + 3z$

28. $x = 0$, $z = x$

29. $z = 0$, $x + y = 0$

30. $x = 3z - y$, $2x = z$

31. $x + y + z = 0$, $x + y = 0$

32. $x + y - z = 0$, $x - z = y$

Test S Not a Subspace

Test the following restriction equations for $V = \mathcal{R}^3$ and show that the corresponding subset S is not a subspace of V. Use Theorem 5.4 page 368.

33. $x = 1$

34. $x + z = 1$

35. $xz = 2$

36. $xz + y = 1$

37. $xz + y = 0$

38. $xyz = 0$

39. $z \geq 0$

40. $x \geq 0$ and $y \geq 0$

41. Octant I

42. The interior of the unit sphere

Dot Product

Find the dot product of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

43. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

44. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

45. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

46. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

47. $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are in $\mathcal{R}^{169}$, $\vec{\mathbf{a}}$ has all 169 components 1 and $\vec{\mathbf{b}}$ has all components -1, except four, which all equal 5.

48. $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are in $\mathcal{R}^{200}$, $\vec{\mathbf{a}}$ has all 200 components -1 and $\vec{\mathbf{b}}$ has all components -1 except three, which are zero.

Length of a Vector

Find the length of the vector $\vec{\mathbf{v}}$.

49. $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

50. $\vec{\mathbf{v}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

51. $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

52. $\vec{\mathbf{v}} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

Shadow Projection

Find the shadow projection $d = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}}/|\vec{\mathbf{b}}|$.

53. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

54. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

55. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

56. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

Projections and Reflections

Let L denote a line through the origin with unit direction $\vec{\mathbf{u}}$.

The **projection** of vector $\vec{\mathbf{x}}$ onto L is $P(\vec{\mathbf{x}}) = d\vec{\mathbf{u}}$, where $d = \vec{\mathbf{x}} \cdot \vec{\mathbf{u}}$ is the shadow projection.

The **reflection** of vector $\vec{\mathbf{x}}$ across L is $R(\vec{\mathbf{x}}) = 2d\vec{\mathbf{u}} - \vec{\mathbf{x}}$ (a generalized complex conjugate).

57. Let $\vec{\mathbf{u}}$ be the direction of the x-axis in the plane. Establish that $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$ are sides of a right triangle and P duplicates the complex conjugate operation $z \to \overline{z}$. Include a figure.

58. Let $\vec{\mathbf{u}}$ be any direction in the plane. Establish that $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$ are sides of a right triangle. Draw a suitable figure, which includes $\vec{\mathbf{x}}$.

59. Let $\vec{\mathbf{u}}$ be the direction of $2\vec{\imath} + \vec{\jmath}$. Define $\vec{\mathbf{x}} = 4\vec{\imath} + 3\vec{\jmath}$. Compute the vectors $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$.

60. Let $\vec{\mathbf{u}}$ be the direction of $\vec{\imath} + 2\vec{\jmath}$. Define $\vec{\mathbf{x}} = 3\vec{\imath} + 5\vec{\jmath}$. Compute the vectors $P(\vec{\mathbf{x}})$ and $R(\vec{\mathbf{x}})$.

Angle

Find the angle θ between the given vectors.

61. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

62. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

63. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

64. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

65. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}$.

66. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$.

67. $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

68. $\vec{\mathbf{a}} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Matrix Multiply

Find the given matrix product or else explain why it does not exist.

69. $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

70. $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

71. $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

72. $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

73. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

74. $\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

75. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

76. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

77. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

78. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

79. $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

80. $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

81. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

82. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Matrix Classification

Classify as square, non-square, upper triangular, lower triangular, scalar, diagonal, symmetric, non-symmetric. Cite as many terms as apply.

83. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

84. $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

85. $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

86. $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$

87. $\begin{pmatrix} 1 & 3 & 4 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

88. $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

89. $\begin{pmatrix} 1 & 3 & 4 \\ 3 & 2 & 0 \\ 4 & 0 & 3 \end{pmatrix}$

90. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

91. $\begin{pmatrix} i & 0 \\ 0 & 2i \end{pmatrix}$

92. $\begin{pmatrix} i & 3 \\ 3 & 2i \end{pmatrix}$

Digital Photographs

Assume integer 24-bit color encoding $x = r + (256)g + (65536)b$, which means r units **red**, g units **green** and b units **blue**. Given matrix $X = R + 256G + 65536B$, find the red, green and blue color separation matrices R, G, B.

93. $X = \begin{pmatrix} 514 & 3 \\ 131843 & 197125 \end{pmatrix}$

94. $X = \begin{pmatrix} 514 & 3 \\ 131331 & 66049 \end{pmatrix}$

95. $X = \begin{pmatrix} 513 & 7 \\ 131333 & 66057 \end{pmatrix}$

96. $X = \begin{pmatrix} 257 & 7 \\ 131101 & 66057 \end{pmatrix}$

97. $X = \begin{pmatrix} 257 & 17 \\ 131101 & 265 \end{pmatrix}$

98. $X = \begin{pmatrix} 65537 & 269 \\ 65829 & 261 \end{pmatrix}$

99. $X = \begin{pmatrix} 65538 & 65803 \\ 65833 & 7 \end{pmatrix}$

100. $X = \begin{pmatrix} 259 & 65805 \\ 299 & 5 \end{pmatrix}$

Matrix Properties

Verify the result.

101. Let C be an $m \times n$ matrix. Let $\vec{X}$ be column i of the $n \times n$ identity I. Define $\vec{Y} = C\vec{X}$. Verify that $\vec{Y}$ is column i of C.

102. Let A and C be an $m \times n$ matrices such that $AC = \mathbf{0}$. Verify that each column $\vec{Y}$ of C satisfies $A\vec{Y} = \vec{\mathbf{0}}$.

103. Let A be a 2×3 matrix and let $\vec{Y}_1$, $\vec{Y}_2$, $\vec{Y}_3$ be column vectors packaged into a 3×3 matrix C. Assume each column vector $\vec{Y}_i$ satisfies the equation $A\vec{Y}_i = \vec{\mathbf{0}}$, $1 \le i \le 3$. Show that $AC = \mathbf{0}$.

104. Let A be an $m \times n$ matrix and let $\vec{Y}_1, \ldots, \vec{Y}_n$ be column vectors packaged into an $n \times n$ matrix C. Assume each column vector $\vec{Y}_i$ satisfies the equation $A\vec{Y}_i = \vec{\mathbf{0}}$, $1 \le i \le n$. Show that $AC = \mathbf{0}$.

Triangular Matrices

Verify the result.

105. The product of two upper triangular 2×2 matrices is upper triangular.

106. The product of two lower triangular 2×2 matrices is lower triangular.

107. The product of two triangular 2×2 matrices is not necessarily triangular.

108. The product of two upper triangular $n \times n$ matrices is upper triangular.

109. The product of two lower triangular $n \times n$ matrices is lower triangular.

110. The only 3×3 matrices which are both upper and lower triangular are the 3×3 diagonal matrices.

Matrix Multiply Properties

Verify the result.

111. The associative law $A(BC) = (AB)C$ holds for matrix multiplication.
Sketch: Expand $L = A(BC)$ entry L_{ij} according to matrix multiply rules. Expand $R = (AB)C$ entry R_{ij} the same way. Show $L_{ij} = R_{ij}$.

112. The distributive law $A(B + C) = AB + AC$ holds for matrices.
Sketch: Expand $L = A(B + C)$ entry L_{ij} according to matrix multiply rules. Expand $R = AB + AC$ entry R_{ij} the same way. Show $L_{ij} = a_{ik}(b_{kj} + c_{kj})$ and $R_{ij} = a_{ik}b_{kj} + a_{ik}c_{kj}$. Then $L_{ij} = R_{ij}$.

113. For any matrix A the transpose formula $(A^T)^T = A$ holds.
Sketch: Expand $L = (A^T)^T$ entry L_{ij} according to matrix transpose rules. Then $L_{ij} = a_{ij}$.

114. For matrices A, B the transpose formula $(A + B)^T = A^T + B^T$ holds.
Sketch: Expand $L = (A + B)^T$ entry L_{ij} according to matrix transpose rules. Repeat for entry R_{ij} of $R = A^T + B^T$. Show $L_{ij} = R_{ij}$.

115. For matrices A, B the transpose formula $(AB)^T = B^T A^T$ holds.
Sketch: Expand $L = (AB)^T$ entry L_{ij} according to matrix multiply and transpose rules. Repeat for entry R_{ij} of $R = B^T A^T$. Show $L_{ij} = R_{ij}$.

116. For a matrix A and constant k, the transpose formula $(kA)^T = kA^T$ holds.

Invertible Matrices

Verify the result.

117. There are infinitely many 2×2 matrices A, B such that $AB = 0$.

118. The zero matrix is not invertible.

119. The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ is not invertible.

120. The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is invertible.

121. The matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ satisfy

$$AB = BA = (ad - bc)I.$$

122. If $AB = 0$, then one of A or B is not invertible.

Symmetric Matrices

Verify the result.

123. The product of two symmetric $n \times n$ matrices A, B such that $AB = BA$ is symmetric.

124. The product of two symmetric 2×2 matrices may not be symmetric.

125. If A is symmetric, then so is A^{-1}.
Sketch: Let $B = A^{-1}$. Compute B^T using transpose rules.

126. If B is an $m \times n$ matrix and $A = B^T B$, then A is $n \times n$ symmetric.
Sketch: Compute A^T using transpose rules.

5.3 Exercises

Identify RREF

Mark the matrices which pass the RREF Test, page **??**. Explain the failures.

1. $\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Lead and Free Variables

For each matrix A, assume a homogeneous system $A\vec{X} = \vec{\mathbf{0}}$ with variable list $x_1, \ldots, x_n$. List the lead and free variables. Then report the rank and nullity of matrix A.

5. $\begin{pmatrix} 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

6. $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

7. $\begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

13. $\begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

14. $\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

15. $\begin{pmatrix} 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

16. $\begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Elementary Matrices

Write the 3×3 elementary matrix E and its inverse E^{-1} for each of the following operations, defined on page **??**.

17. `combo(1,3,-1)`

18. `combo(2,3,-5)`

19. `combo(3,2,4)`

20. `combo(2,1,4)`

21. `combo(1,2,-1)`

22. combo(1,2,-e^2)

23. `mult(1,5)`

24. `mult(1,-3)`

25. `mult(2,5)`

26. `mult(2,-2)`

27. `mult(3,4)`

28. `mult(3,5)`

29. mult(2,-π)

30. mult(1,e^2)

31. `swap(1,3)`

32. `swap(1,2)`

33. `swap(2,3)`

34. `swap(2,1)`

35. `swap(3,2)`

36. `swap(3,1)`

Elementary Matrix Multiply

For each given matrix B_1, perform the toolkit operation (`combo, swap, mult`) to obtain the result B_2. Then compute the elementary matrix E for the identical toolkit operation. Finally, verify the matrix multiply equation $B_2 = EB_1$.

37. $\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$, `mult(2,1/3)`.

38. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, `mult(1,3)`.

39. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, `combo(3,2,-1)`.

40. $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, `combo(2,1,-3)`.

41. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, `swap(2,3)`.

42. $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, `swap(1,2)`.

Inverse Row Operations

Given the final frame B of a sequence starting with matrix A, and the given operations, find matrix A. Do not use matrix multiply.

43. $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,-3)`, `mult(1,-2)`, `swap(2,3)`.

44. $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,2)`, `swap(3,2)`.

45. $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`.

46. $B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

Elementary Matrix Products

Given the first frame B_1 of a sequence and elementary matrix operations E_1, E_2, E_3, find matrices $F = E_3E_2E_1$ and $B_4 = FB_1$. Hint: Compute $\left\langle B_4|F \right\rangle$ from toolkit operations on $\left\langle B_1|I \right\rangle$.

47. $B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,-3)`, `mult(1,-2)`.

48. $B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,3)`, `swap(3,2)`.

49. $B_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `mult(1,4)`, `swap(1,3)`.

50. $B_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, operations `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`.

Miscellany

51. Justify with English sentences why all possible 2×2 matrices in reduced row-echelon form must look like

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $*$ denotes an arbitrary number.

52. Display all possible 3×3 matrices in reduced row-echelon form. Besides the zero matrix and the identity matrix, please report five other forms, most containing symbol $*$ representing an arbitrary number.

53. Determine all possible 4×4 matrices in reduced row-echelon form.

54. Display a 6×6 matrix in reduced row-echelon form with rank 4 and only entries of zero and one.

55. Display a 5×5 matrix in reduced row-echelon form with nullity 2 having entries of zero, one and two, but no other entries.

56. Display the rank and nullity of any $n \times n$ elementary matrix.

57. Let $F = \left\langle C|D \right\rangle$ and let E be a square matrix with row dimension matching F. Display the details for the equality

$$EF = \left\langle EC|ED \right\rangle.$$

58. Let $F = \left\langle C|D \right\rangle$ and let E_1, E_2 be $n \times n$ matrices with n equal to the row dimension of F. Display the details for the equality

$$E_2E_1F = \left\langle E_2E_1C|E_2E_1D \right\rangle.$$

59. Display details explaining why $\mathbf{rref}(\langle A|I\rangle)$ equals the matrix $\langle \mathbf{rref}(A)|B\rangle$, where matrix $B = E_k \cdots E_1$. Symbols E_i are elementary matrices in toolkit steps taking $\langle A|I\rangle$ into reduced row-echelon form. Suggestion: Use the preceding exercises.

60. Assume E_1, E_2 are elementary matrices in toolkit steps taking A into reduced row-echelon form. Prove that $A^{-1} = E_2E_1$. In words, A^{-1} is found by doing the same toolkit steps to the identity matrix.

61. Assume $E_1, \ldots, E_k$ are elementary matrices in toolkit steps taking $\langle A|I\rangle$ into reduced row-echelon form. Prove that $A^{-1} = E_k \cdots E_1$.

62. Assume A, B are 2×2 matrices. Assume $\mathbf{rref}(\langle A|B\rangle) = \langle I|D\rangle$. Explain why the first column $\vec{x}$ of D is the unique solution of $A\vec{x} = \vec{b}$, where $\vec{b}$ is the first column of B.

63. Assume A, B are $n \times n$ matrices. Explain how to solve the matrix equation $AX = B$ for matrix X using the augmented matrix of A, B.

5.4 Exercises

Determinant Notation

Write formulae for x and y as quotients of 2×2 determinants. Do not evaluate the determinants!

1. $\begin{pmatrix} 1 & -1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \end{pmatrix}$

3. $\begin{pmatrix} 0 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \end{pmatrix}$

4. $\begin{pmatrix} 0 & -3 \\ 3 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Sarrus' 2×2 rule

Evaluate $\det(A)$.

5. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

6. $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

7. $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$

8. $A = \begin{pmatrix} 5a & 1 \\ -1 & 2a \end{pmatrix}$

Sarrus' rule 3×3

Evaluate $\det(A)$.

9. $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

10. $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

11. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

12. $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

Inverse of a 2×2 Matrix

Define matrix A and its adjugate C:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

13. Verify $AC = |A| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

14. Display the details of the argument that $|A| \neq 0$ implies A^{-1} exists and $A^{-1} = \dfrac{C}{|A|}$.

15. Show that A^{-1} exists implies $|A| \neq 0$. Suggestion: Assume not, then $AB = BA = I$ for some matrix B and also $|A| = 0$. Find a contradiction using $AC = |A|I$.

16. Calculate the inverse of $\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$ using the formula developed in these exercises.

Unique Solution of a 2×2 System

Solve $A\vec{\mathbf{X}} = \vec{\mathbf{b}}$ for $\vec{\mathbf{X}}$ using Cramer's rule for 2×2 systems.

17. $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

18. $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$

19. $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$

20. $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} -10 \\ 10 \end{pmatrix}$

Definition of Determinant

21. Let A be 3×3 with zero first row. Use the college algebra definition of determinant to show that $\det(A) = 0$.

22. Let A be 3×3 with equal first and second row. Use the college algebra definition of determinant to show that $\det(A) = 0$.

23. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Use the college algebra definition of determinant to verify that $|A| = ad - bc$.

24. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Use the college algebra definition of determinant to verify that the determinant of A equals

$$\begin{array}{l} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} \\ +a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} \\ -a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{array}$$

Four Properties

Evaluate $\det(A)$ using the four properties for determinants, page **??**.

25. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

26. $A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

27. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

28. $A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

29. $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$

30. $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

31. $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

32. $A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$

Elementary Matrices and the Four Rules

Find $\det(A)$.

33. A is 3×3 and obtained from the identity matrix I by three row swaps.

34. A is 7×7, obtained from I by swapping rows 1 and 2, then rows 4 and 1, then rows 1 and 3.

35. A is obtained from the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ by swapping rows 1 and 3, then two row combinations.

36. A is obtained from the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ by two row combinations, then multiply row 2 by -5.

More Determinant Rules

Cite the determinant rule that verifies $\det(A) = 0$. **Never** expand $\det(A)$! See page **??**.

37. $A = \begin{pmatrix} -1 & 5 & 1 \\ 2 & -4 & -4 \\ 1 & 1 & -3 \end{pmatrix}$

38. $A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -4 \\ 1 & 1 & -3 \end{pmatrix}$

39. $A = \begin{pmatrix} 4 & -8 & -8 \\ 2 & -4 & -4 \\ 1 & 1 & -3 \end{pmatrix}$

40. $A = \begin{pmatrix} -1 & 5 & 0 \\ 2 & -4 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

41. $A = \begin{pmatrix} -1 & 5 & 3 \\ 2 & -4 & 0 \\ 1 & 1 & 3 \end{pmatrix}$

42. $A = \begin{pmatrix} -1 & 5 & 4 \\ 2 & -4 & -2 \\ 1 & 1 & 2 \end{pmatrix}$

Cofactor Expansion and College Algebra

Evaluate the determinant with an efficient cofactor expansion.

43. $\begin{vmatrix} 2 & 5 & 1 \\ 2 & 0 & -4 \\ 1 & 0 & 0 \end{vmatrix}$

44. $\begin{vmatrix} 2 & 5 & 1 \\ 2 & 0 & -4 \\ 1 & 0 & 1 \end{vmatrix}$

45. $\begin{vmatrix} 2 & 5 & 0 & 0 \\ 2 & 1 & 4 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}$

46. $\begin{vmatrix} 0 & 2 & 0 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{vmatrix}$

47. $\begin{vmatrix} 2 & 5 & 1 & -1 & 1 \\ 0 & -1 & -4 & 1 & -1 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{vmatrix}$

48. $\begin{vmatrix} 2 & 0 & 1 & -1 & 1 \\ 0 & -1 & -4 & 1 & -1 \\ 1 & 2 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \end{vmatrix}$

Minors and Cofactors

Write out and then evaluate the minor and cofactor of each element cited for the matrix $A = \begin{pmatrix} 2 & 5 & y \\ x & -1 & -4 \\ 1 & 2 & z \end{pmatrix}$

49. Row 1 and column 3.

50. Row 2 and column 1.

51. Row 3 and column 2.

52. Row 3 and column 1.

Cofactor Expansion

Use cofactors to evaluate the determinant.

53. $\begin{vmatrix} 2 & 7 & 1 \\ -1 & 0 & -4 \\ 1 & 0 & 3 \end{vmatrix}$

54. $\begin{vmatrix} 2 & 7 & 7 \\ -1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$

55. $\begin{vmatrix} 0 & 2 & 7 & 7 \\ 0 & -1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{vmatrix}$

56. $\begin{vmatrix} 0 & 2 & 7 & 7 \\ 0 & -1 & y & 0 \\ x & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{vmatrix}$

57. $\begin{vmatrix} 0 & 2 & 7 & 7 & 3 \\ 0 & -1 & 0 & 0 & 1 \\ x & 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{vmatrix}$

58. $\begin{vmatrix} 0 & 2 & 7 & 7 & 3 \\ 0 & -1 & 2 & 0 & 1 \\ x & 1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{vmatrix}$

Adjugate and Inverse Matrix

Find the adjugate of A and the inverse B of A. Check the answers via the formulas $A\,\mathbf{adj}(A) = \det(A)I$ and $AB = I$.

59. $A = \begin{pmatrix} 2 & 7 \\ -1 & 0 \end{pmatrix}$

60. $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

61. $A = \begin{pmatrix} 5 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix}$

62. $A = \begin{pmatrix} 5 & 1 & 2 \\ 2 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

63. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 2 \end{pmatrix}$

64. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

Transpose and Inverse

65. Verify that $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ satisfies $A^T = A^{-1}$.

66. Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\det(A) = 1$ and $A^T = A^{-1}$.

67. Find all 3×3 diagonal matrices A such that $A^T = A^{-1}$.

68. Find all 3×3 upper triangular matrices A such that $A^T = A^{-1}$.

69. Find all $n \times n$ diagonal matrices A such that $A^T = A^{-1}$.

70. Determine the $n \times n$ triangular matrices A such that $\det(A) = 1$ and $A^T = \mathbf{adj}(A)$.

Elementary Matrices

Find the determinant of A from the given equation.

71. Let $A = 5E_2E_1$ be 3×3, where E_1 multiplies row 3 of the identity by -7 and E_2 swaps rows 3 and 1 of the identity. Hint: $A = (5I)E_2E_1$.

72. Let $A = 2E_2E_1$ be 5×5, where E_1 multiplies row 3 of the identity by -2 and E_2 swaps rows 3 and 5 of the identity.

73. Let $A = E_2E_1B$ be 4×4, where E_1 multiplies row 2 of the identity by 3 and E_2 is a combination. Find $|A|$ in terms of $|B|$.

74. Let $A = 3E_2E_1B$ be 3×3, where E_1 multiplies row 2 of the identity by 3 and E_2 is a combination. The answer for $|A|$ is in terms of $|B|$.

75. Let $A = 4E_2E_1B$ be 3×3, where E_1 multiplies row 1 of the identity by 2, E_2 and E_3 are combinations and $|B| = -1$.

76. Let $A = 2E_2E_1B^3$ be 3×3, where E_1 multiplies row 2 of the identity by -1, E_2 and E_3 are swaps and $|B| = -2$.

Determinants and the Toolkit

Display the toolkit steps for $\mathbf{rref}(A)$. Using only the steps, report:

- The determinant of the elementary matrix for each step.
- The determinant of A.

77. $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

78. $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

79. $A = \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

80. $A = \begin{pmatrix} 2 & 3 & 1 & 2 \\ 0 & 3 & 0 & 0 \\ 2 & 6 & 1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

Determinant Product Rule

Apply the product rule $\det(AB) = \det(A)\det(B)$.

81. Let $\det(A) = 5$ and $\det(B) = -2$. Find $\det(A^2B^3)$.

82. Let $\det(A) = 4$ and $A(B - 2A) = 0$. Find $\det(B)$.

83. Let $A = E_1E_2E_3$ where E_1, E_2 are elementary swap matrices and E_3 is an elementary combination matrix. Find $\det(A)$.

84. Assume $\det(AB + A) = 0$ and $\det(A) \neq 0$. Show that $\det(B + I) = 0$.

Cramer's 2×2 Rule

Assume

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}.$$

85. Derive the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} e & b \\ f & d \end{pmatrix}.$$

86. Derive the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} = \begin{pmatrix} a & e \\ c & f \end{pmatrix}.$$

87. Use the determinant product rule to derive the Cramer's Rule formula

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

88. Derive, using the determinant product rule, the Cramer's Rule formula

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

Cramer's 3×3 Rule

Let A be the coefficient matrix in the equation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

89. Derive the formula

$$A \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}$$

90. Derive the formula

$$A \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}$$

91. Derive, using the determinant product rule, the Cramer's Rule formula

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

92. Use the determinant product rule to derive the Cramer's Rule formula

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

Cayley-Hamilton Theorem

93. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$. Expand $|A - rI|$ to compute the characteristic polynomial of A. Answer: $r^2 - 4r + 5$.

94. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$. Apply the Cayley-Hamiltion theorem to justify the equation

$$A^2 - 4A + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

95. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Expand $|A - rI|$ by Sarrus' Rule to obtain $r^2 - (a+b)r + (ad - bc)$.

96. The result of the previous exercise is often written as $(-r)^2 + \textbf{trace}(A)(-r) + |A|$ where $\textbf{trace}(A) = a + d =$ sum of the diagonal elements. Display the details.

97. Let $\lambda^2 - 2\lambda + 1 = 0$ be the characteristic equation of a matrix A. Find a formula for A^2 in terms of A and I.

98. Let A be an $n \times n$ triangular matrix with all diagonal entries zero. Prove that $A^n = 0$.

99. Find all 2×2 matrices A such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, discovered from values of $\textbf{trace}(A)$ and $|A|$.

100. Find four 2×2 matrices A such that $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Applied Definition of Determinant

Miscellany for permutation matrices and the sampled product page **??**

$$\begin{aligned} A.P =& (A_1 \cdot P_1)(A_2 \cdot P_2) \cdots (A_n \cdot P_n) \\ =& a_{1\sigma_1} \cdots a_{n\sigma_n}. \end{aligned}$$

101. Compute the sampled product of $\begin{pmatrix} 5 & 3 & 1 \\ 0 & 5 & 7 \\ 1 & 9 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

102. Compute the sampled product of $\begin{pmatrix} 5 & 3 & 3 \\ 0 & 2 & 7 \\ 1 & 9 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

103. Determine the permutation matrices P required to evaluate $\det(A)$ when A is 2×2.

104. Determine the permutation matrices P required to evaluate $\det(A)$ when A is 4×4.

Three Properties

Reference: Page **??**, three properties that define a determinant

105. Assume $n = 3$. Prove that the three properties imply $D = 0$ when two rows are identical.

106. Assume $n = 3$. Prove that the three properties imply $D = 0$ when a row is zero.

5.5 Exercises

Scalar and Vector General Solution

Given the scalar general solution of $A\vec{x} = \vec{0}$, find the vector general solution

$$\vec{x} = t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots$$

where symbols $t_1,\ t_2,\ \ldots$ denote arbitrary constants and $\vec{u}_1,\ \vec{u}_2,\ \ldots$ are fixed vectors.

1. $x_1 = 2t_1$, $x_2 = t_1 - t_2$, $x_3 = t_2$

2. $x_1 = t_1 + 3t_2$, $x_2 = t_1$, $x_3 = 4t_2$, $x_4 = t_2$

3. $x_1 = t_1$, $x_2 = t_2$, $x_3 = 2t_1 + 3t_2$

4. $x_1 = 2t_1 + 3t_2 + t_3$, $x_2 = t_1$, $x_3 = t_2$, $x_4 = t_3$

Vector General Solution

Find the vector general solution $\vec{x}$ of $A\vec{x} = \vec{0}$.

5. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

7. $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

8. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

9. $A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 \end{pmatrix}$

10. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

Dimension

11. Give four examples in $\mathcal{R}^3$ of $S = \textbf{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ (3 vectors required) which have respectively dimensions $0, 1, 2, 3$.

12. Give an example in $\mathcal{R}^3$ of 2-dimensional subspaces S_1, S_2 with only the zero vector in common.

13. Let $S = \textbf{span}(\vec{v}_1, \vec{v}_2)$ in abstract vector space V. Explain why $\dim(S) \le 2$.

14. Let $S = \textbf{span}(\vec{v}_1, \ldots, \vec{v}_k)$ in abstract vector space V. Explain why $\dim(S) \le k$.

15. Let S be a subspace of $\mathcal{R}^3$ with basis $\vec{v}_1, \vec{v}_2$. Define $\vec{v}_3$ to be the **cross product** of $\vec{v}_1, \vec{v}_2$. What is $\dim(\textbf{span}(\vec{v}_2, \vec{v}_3))$?

16. Let S_1, S_2 be subspaces of $\mathcal{R}^4$ such that $\dim(S_1)\dim(S_2) = 2$. Assume S_1, S_2 have only the zero vector in common. Prove or give a counter-example: the span of the union of S_1, S_2 equals $\mathcal{R}^4$.

Independence in Abstract Spaces

17. Assume linear combinations of vectors $\vec{v}_1$, $\vec{v}_2$ are uniquely determined, that is, $a_1\vec{v}_1 + a_2\vec{v}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$ implies $a_1 = b_1$, $a_2 = b_2$. **Prove** this result: If $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, then $c_1 = c_2 = 0$.

18. Assume the zero linear combination of vectors $\vec{v}_1$, $\vec{v}_2$ is uniquely determined, that is, $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ implies $c_1 = c_2 = 0$. **Prove** this result: If $a_1\vec{v}_1 + a_2\vec{v}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$, then $a_1 = b_1$, $a_2 = b_2$.

19. Prove that two **nonzero** vectors $\vec{v}_1$, $\vec{v}_2$ in an abstract vector space V are independent if and only if $\vec{v}_1$ is not a constant multiple of $\vec{v}_2$.

20. Let $\vec{v}_1$ be a vector in an abstract vector space V. Prove that the one-element set $\vec{v}_1$ is independent if and only if $\vec{v}_1$ is not the zero vector.

21. Let V be an abstract vector space and assume $\vec{v}_1$, $\vec{v}_2$ are independent vectors in V. Define $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 + 2\vec{v}_2$. Prove that $\vec{u}_1$, $\vec{u}_2$ are independent in V.

Advice: Fixed vectors not assumed! Bursting the vector packages is impossible, there are no components.

22. Let V be an abstract vector space and assume $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are independent vectors in V. Define $\vec{u}_1 = \vec{v}_1 + \vec{v}_2$, $\vec{u}_2 = \vec{v}_1 + 4\vec{v}_2$, $\vec{u}_3 = \vec{v}_3 - \vec{v}_1$. Prove that $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ are independent in V.

23. Let S be a finite set of independent vectors in an abstract vector space V. Prove that none of the vectors can be the zero vector.

24. Let S be a finite set of independent vectors in an abstract vector space V. Prove that no vector in the list can be a linear combination of the other vectors.

The Spaces $\mathcal{R}^n$

25. **(Scalar Multiply)** Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ have components measured in centimeters. Report constants c_1, c_2, c_3 for re-scaled data $c_1\vec{x}$, $c_2\vec{x}$, $c_3\vec{x}$ in units of kilometers, meters and millimeters.

26. **(Matrix Multiply)** Let $\vec{u} = \begin{pmatrix} x_1, x_2, x_3, p_1, p_2, p_3 \end{pmatrix}^T$ have position x-units in kilometers and momentum p-units in kilogram-centimeters per millisecond. Determine a matrix M such that the vector $\vec{y} = M\vec{u}$ has SI units of meters and kilogram-meters per second.

27. Let $\vec{v}_1$, $\vec{v}_2$ be two independent vectors in $\mathcal{R}^n$. Assume $c_1\vec{v}_1 + c_2\vec{v}_2$ lies strictly interior to the parallelogram determined by $\vec{v}_1$, $\vec{v}_2$. Give geometric details explaining why $0 < c_1 < 1$ and $0 < c_2 < 1$.

28. Prove the 4 scalar multiply toolkit properties for fixed vectors in $\mathcal{R}^3$.

29. Define

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, -\vec{v} = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix}.$$

Prove the 4 addition toolkit properties for fixed vectors in $\mathcal{R}^3$.

30. Use the 8 property toolkit in $\mathcal{R}^3$ to prove that zero times a vector is the zero vector.

31. Let A be an invertible 3×3 matrix. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be a basis for $\mathcal{R}^3$. Prove that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is a basis for $\mathcal{R}^3$.

32. Let A be an invertible 3×3 matrix. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be dependent in $\mathcal{R}^3$. Prove that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3$ is a dependent set in $\mathcal{R}^3$.

Digital Photographs

Let V be the vector space of all 2×3 matrices. A matrix in V is a 6-pixel digital photo, a sub-section of a larger photo.

Replace one zero in the 2×3 zero matrix with a one. The 6 answers $B_1, \ldots,$ B_6 are numbered by $B_j[n, m] = 1$ when $3(n-1) + m = j$.

33. Prove that $B_1, \ldots, B_6$ are independent and span V: they are a **basis** for V. Each B_i **lights up** one pixel in the 2×3 sub-photo.

34. Define red, green and blue monochrome matrices R, G, B by

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 5 & 8 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 5 \end{pmatrix}.$$

Define base $x = 16$. Compute $A = R + xG + x^2B$.

35. Let $A = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Assume a black and white image and 0 means black. Describe photo A, from the checkerboard analogy.

36. Let c_1, c_2 be integer encodings of RGB intensities. Describe the photo $A = c_1B_1 + c_2B_3$ from the checkerboard analogy.

Polynomial Spaces

Let V be the vector space of all cubic or less polynomials $p(x) = c_0 + c_1x + c_2x^2 + c_3x^3$.

37. Find a subspace S of V, $\dim(S) = 2$, which contains the vector $1 + x$.

38. Let S be the subset of V spanned by x, x^2 and x^3. Prove that S is a subspace of V which does not contain the polynomial $1 + x$.

39. Define set S by the conditions $p(0) = 0, p(1) = 0$. Find a basis for S.

40. Define set S by the condition $p(0) = \int_0^1 p(x)dx$. Find a basis for S.

The Space $C(E)$

Define $\vec{f}$ to be the vector package with domain $E = \{x \ : \ -1 \le x \le 1\}$ and equation $y = |x|$. Similarly, $\vec{g}$ is defined by equation $y = x$.

41. Show independence of $\vec{f}, \vec{g}$.

42. Find the dimension of $\mathbf{span}(\vec{f}, \vec{g})$.

43. Let $h(x) = 0$ on $-1 \le x \le 0$, $h(x) = -x$ on $0 \le x \le 1$. Show that $\vec{h}$ is in $C(E)$.

44. Let $h(x) = -1$ on $-1 \le x \le 0$, $h(x) = 1$ on $0 \le x \le 1$. Show that $\vec{h}$ is not in $C(E)$.

45. Let $h(x) = 0$ on $-1 \le x \le 0$, $h(x) = -x$ on $0 \le x \le 1$. Show that $\vec{h}$ is in $\mathbf{span}(\vec{f}, \vec{g})$.

46. Let $h(x) = \tan(\pi x/2)$ on $-1 < x < 1$, $h(1) = h(-1) = 0$. Explain why $\vec{h}$ is not in $C(E)$

The Space $C^1(E)$

Define $\vec{f}$ to be the vector package with domain $E = \{x : -1 \le x \le 1\}$ and equation $y = x|x|$. Similarly, $\vec{g}$ is defined by equation $y = x^2$.

47. Verify that $\vec{f}$ is in $C^1(E)$, but its derivative is not.

48. Show that $\vec{f}, \vec{g}$ are independent in $C^1(E)$.

The Space $C^k(E)$

49. Compute the first three derivatives of $y(x) = e^{-x^2}$ at $x = 0$.

50. Justify that $y(x) = e^{-x^2}$ belongs to $C^k(0,1)$ for all $k \ge 1$.

51. Prove that the span of a finite list of distinct Euler solution atoms (page **??**) is a subspace of $C^k(E)$ for any interval E.

52. Prove that $y(x) = |x|$ is in $C^k(0,1)$ but not in $C^1(-1,1)$.

Solution Space

A differential equations solver finds general solution $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$. Use vector space $V = C^4(E)$ where E is the whole real line.

53. Write the solution set S as the span of four vectors in V.

54. Find a basis for the solution space S of the differential equation. Verify independence using the sampling test or Wronskian test.

55. Find a differential equation $y'' + a_1 y' + a_0 y = 0$ which has solution $y = c_1 + c_2 x$.

56. Find a differential equation $y'''' + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0$ which has solution $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Algebraic Independence Test for Two Vectors

Solve for c_1, c_2 in the independence test for two vectors, showing all details.

57. $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

58. $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Dependence of two vectors

Solve for c_1, c_2 not both zero in the independence test for two vectors, showing all details for dependency of the two vectors.

59. $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

60. $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$

Independence Test for Three Vectors

Solve for the constants c_1, c_2, c_3 in the relation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Report dependent of independent vectors. If dependent, then display a dependency relation.

61. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

62. $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Independence in an Abstract Vector Space

In vector space V, report independence or a dependency relation for the given vectors.

63. Space $V = C(-\infty, \infty)$, $\vec{v}_1 = 1 + x$, $\vec{v}_2 = 2 + x$, $\vec{v}_3 = 3 + x^2$.

64. Space $V = C(-\infty, \infty)$, $\vec{v}_1 = x^{3/5}$, $\vec{v}_2 = x^2$, $\vec{v}_3 = 2x^2 + 3x^{3/5}$

65. Space V is all 3×3 matrices. Let
$\vec{v}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 2 & 5 \\ 0 & 3 & 5 \end{pmatrix}.$

66. Space V is all 2×2 matrices. Let
$\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$
$\vec{v}_3 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}.$

Rank Test

Compute the rank of the augmented matrix to determine independence or dependence of the given vectors.

67. $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

68. $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Determinant Test

Evaluate the determinant of the augmented matrix to determine independence or dependence of the given vectors.

69. $\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$

70. $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Sampling Test for Functions

Invent samples to verify independence.

71. $\cosh(x), \sinh(x)$

72. $x^{7/3}, x\sin(x)$

73. $1, x, \sin(x)$

74. $1, \cos^2(x), \sin(x)$

Sampling Test and Dependence

For three functions f_1, f_2, f_3 to be dependent, constants c_1, c_2, c_3 must be found such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0.$$

The trick is that c_1, c_2, c_3 are not all zero and the relation holds **for all** x. The sampling test method can discover the constants, but it is **unable to prove dependence**!

75. Functions $1, x, 1+x$ are dependent. Insert $x = 0, 1, 2$ and solve for c_1, c_2, c_3, to discover a dependency relation.

76. Functions $1, \cos^2(x), \sin^2(x)$ are dependent. Cleverly choose 3 values of x, insert them, then solve for c_1, c_2, c_3, to discover a dependency relation.

Vandermonde Determinant

77. Let $V = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$. Verify by direct computation the formula

$$|V| = x_2 - x_1.$$

78. Let $V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$. Verify by direct computation the formula

$$|V| = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

Wronskian Test for Functions

Apply the Wronskian Test to verify independence.

79. $\cos(x), \sin(x)$.

80. $\cos(x), \sin(x), \sin(2x)$.

81. $x, x^{5/3}$.

82. $\cosh(x), \sinh(x)$.

Wronskian Test: Theory

83. The functions x^2 and $x|x|$ are continuously differentiable and have zero Wronskian. Verify that they **fail to be dependent** on $-1 < x < 1$.

84. The Wronskian Test can verify the independence of the powers $1, x, \ldots, x^k$. Show the determinant details.

Extracting a Basis

Given a list of vectors in space $V = \mathcal{R}^4$, extract a largest independent subset.

85. $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

86. $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Extracting a Basis

Given a list of vectors in space $V = C(-\infty, \infty)$, extract a largest independent subset.

87. $x, x\cos^2(x), x\sin^2(x), e^x, x + e^x$

88. $1, 2 + x, \frac{x}{1+x^2}, \frac{x^2}{1+x^2}$

Euler Solution Atom

Identify the Euler solution atoms in the given list. Strictly apply the definition: e^x is an atom but $2e^x$ is not.

89. $1, 2 + x, e^{2.15x}, e^{x^2}, \frac{x}{1+x^2}$

90. $2, x^3, e^{x/\pi}, e^{2x+1}, \ln|1 + x|$

Euler Solution Atom Test

Establish independence of set S_1.
Suggestion: First establish an identity $\mathbf{span}(S_1) = \mathbf{span}(S_2)$, where S_2 is an invented list of distinct atoms. The Test implies S_2 is independent. Extract a largest independent subset of S_1, using independence of S_2.

91. Set S_1 is the list $2, 1 + x^2, 4 + 5e^x, \pi e^{2x+\pi}, 10x\cos(x)$.

92. Set S_1 is the list $1 + x^2, 1 - x^2, 2\cos(3x), \cos(3x) + \sin(3x)$.

5.6 Exercises

Basis and Dimension

Compute a basis and the report the dimension of the subspace S.

1. In $\mathcal{R}^3$, S is the solution space of

$$\left|\begin{array}{ccccccc} x_1 & & & + & x_3 & = & 0, \\ & & x_2 & + & x_3 & = & 0. \end{array}\right|$$

2. In $\mathcal{R}^4$, S is the solution space of

$$\left|\begin{array}{cccc} x_1 + 2x_2 + x_3 & & = & 0, \\ & x_4 & = & 0. \end{array}\right|$$

3. In $\mathcal{R}^2$, $S = \textbf{span}(\vec{v}_1, \vec{v}_2)$. Vectors $\vec{v}_1, \vec{v}_2$ are columns of an invertible matrix.

4. Set $S = \textbf{span}(\vec{v}_1, \vec{v}_2)$, in $\mathcal{R}^4$. The vectors are columns in a 4×4 invertible matrix.

5. Set $S = \textbf{span}(\sin^2 x, \cos^2 x, 1)$, in the vector space V of continuous functions.

6. Set $S = \textbf{span}(x, x-1, x+2)$, in the vector space V of all polynomials.

7. Set $S = \textbf{span}(\sin x, \cos x)$, the solution space of $y'' + y = 0$.

8. Set $S = \textbf{span}\left(e^{2x}, e^{3x}\right)$, the solution space of $y'' - 5y' + 6y = 0$.

Euclidean Spaces

9. Let A be 3×2. Why is it impossible for the columns of A to be a basis for $\mathcal{R}^3$?

10. Let A be $m \times n$. What condition on indices m, n implies it is impossible for the columns of A to be a basis for $\mathcal{R}^m$?

11. Find a pairwise orthogonal basis for $\mathcal{R}^3$ which contains $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

12. Display a basis for $\mathcal{R}^4$ which contains the independent columns of $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

13. Let S be a subspace of $\mathcal{R}^{10}$ of dimension 5. Insert a basis for S into an $m \times n$ augmented matrix A. What are m and n?

14. Suppose A and B are 3×3 matrices and let $C = AB$. Assume the columns of A are not a basis for $\mathcal{R}^3$. Is there a matrix B so that the columns of C form a basis for $\mathcal{R}^3$?

15. The term **Hyperplane** is used for an equation like $x_4 = 0$, which in $\mathcal{R}^4$ defines a subspace S of dimension 3. Find a basis for S.

16. Find a 3-dimensional subspace S of $\mathcal{R}^4$ which has no basis consisting of columns of the identity matrix.

Polynomial Spaces

Symbol V is the vector space of all polynomials $p(x)$. Given subspace S of V, find a basis for S and $\dim(S)$.

17. The subset S of $\mathbf{span}(1, x, x^2)$ is defined by $\frac{dp}{dx}(1) = 0$.

18. The subset S of $\mathbf{span}(1, x, x^2, x^3)$ is defined by $p(0) = \frac{dp}{dx}(1) = 0$.

19. The subset S of $\mathbf{span}(1, x, x^2)$ is defined by $\int_0^1 p(x)dx = 0$.

20. The subset S of $\mathbf{span}(1, x, x^2, x^3)$ is defined by $\int_0^1 xp(x)dx = 0$.

Differential Equations

Find a basis for solution subspace S. Assume the general solution of the 4th order linear differential equation is

$$y(x) = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}.$$

21. Subspace S_1 is defined by $y(0) = \frac{dy}{dx}(0) = 0$.

22. Subspace S_2 is defined by $y(1) = 0$.

23. Subspace S_3 is defined by $y(0) = \int_0^1 y(x)dx$.

24. Subspace S_4 is defined by $y(1) = 0$, $\int_0^1 y(x)dx = 0$.

Largest Subset of Independent Vectors

Find a largest independent subset of the given vectors.

25. The columns of $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$.

26. The columns of $\begin{pmatrix} 3 & 1 & 2 & 0 & 5 \\ 2 & 1 & 1 & 0 & 4 \\ 3 & 2 & 1 & 0 & 7 \\ 1 & 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 & 7 \end{pmatrix}$.

27. The polynomials $x, 1+x, 1-x, x^2$.

28. The continuous functions x, e^x, $x + e^x$, e^{2x}.

Pivot Theorem Method

Extract a largest independent set from the columns of the given matrix A. The answer is a list of independent columns of A, called the pivot columns of A.

29. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$

30. $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

31. $\begin{pmatrix} 0 & 2 & 1 & 0 & 1 \\ 1 & 5 & 2 & 0 & 3 \\ 1 & 3 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}$

32. $\begin{pmatrix} 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 5 & 2 & 0 & 3 \\ 0 & 1 & 3 & 1 & 0 & 2 \\ 0 & 2 & 4 & 1 & 0 & 3 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 2 & 4 & 1 & 0 & 3 \end{pmatrix}$

Row and Column Rank

Justify by direct computation that $\mathbf{rank}(A) = \mathbf{rank}\left(A^T\right)$, which means that the row rank equals the column rank.

33. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

34. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Nullspace or Kernel

Find a basis for the nullspace of A, which is also called the kernel of A.

35. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

36. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Row Space

Find a basis for the row space of A. There are two possible answers: (1) The nonzero rows of $\mathbf{rref}(A)$, (2) The pivot columns of A^T. Answers (1) and (2) can differ wildly.

37. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

38. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Column Space

Find a basis for the column space of A, in terms of the columns of A. Normally, we report the pivot columns of A.

39. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

40. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$

Dimension Identities

Let A be an $m \times n$ matrix of rank r. Prove the following dimension identities in Theorem 5.46.

41. $\dim(\mathbf{nullspace}(A)) = n - r$

42. $\dim(\mathbf{colspace}(A)) = r$

43. $\dim(\mathbf{rowspace}(A)) = r$

44. The dimensions of $\mathbf{nullspace}(A)$ and $\mathbf{colspace}(A)$ add to n.

Orthogonal Complement $S^\perp$

Let S be a subspace of vector space V. Define the **Orthogonal complement** by

(1) $$S^\perp = \{\vec{x} \ : \ \vec{x}^T\vec{y} = 0, \ \vec{y} \text{ in } S\}.$$

45. Let $V = \mathcal{R}^3$ and let S be the xy-plane. Compute $S^\perp$. Answer: The z-axis.

46. Prove that $S^\perp$ is a subspace, using the **Subspace Criterion**.

47. Prove that the orthogonal complement of $S^\perp$ is S. In symbols, $\left(S^\perp\right)^\perp = S$.

48. Prove that

$$V = \{\vec{x} + \vec{y} \ : \ \vec{x} \in S, \vec{y} \in S^\perp\}.$$

This relation is called the **Direct Sum** of S and $S^\perp$.

Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix.

49. Write a short proof:
Lemma. Any solution of $A\vec{x} = \vec{0}$ is orthogonal to every row of A.

50. Find the dimension of the kernel and image for both A and A^T. The four answers use symbols $m, n, \textbf{rank}(A)$. The main tool is the rank-nullity theorem.

51. Prove
$\textbf{kernel}(A) = \textbf{Image}\left(A^T\right)^\perp$. Use Exercise 49.

52. Prove
$\textbf{kernel}\left(A^T\right) = \textbf{Image}\,(A)^\perp$.

Fundamental Subspaces

The kernel and image of both A and A^T are called *The Four Fundamental Subspaces* by Gilbert Strang. Let A denote an $n \times m$ matrix.

53. Prove using Exercise 49:
$\textbf{kernel}(A) = \textbf{rowspace}(A)^\perp$

54. Establish these four identities.
$\textbf{kernel}(A) = \textbf{Image}\left(A^T\right)^\perp$
$\textbf{kernel}\left(A^T\right) = \textbf{Image}\,(A)^\perp$
$\textbf{Image}\,(A) = \textbf{kernel}(A^T)^\perp$
$\textbf{Image}\left(A^T\right) = \textbf{kernel}(A)^\perp$

Notation. *kernel* is null space, *image* is column space, symbol $\perp$ is orthogonal complement: see equation (1).

Equivalent Bases

Test the given subspaces for equality.

55. $S_1 = \mathbf{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$,

$S_2 = \mathbf{span}\left(\begin{pmatrix}3\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$

56. $S_3 = \mathbf{span}\left(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right)$,

$S_4 = \mathbf{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)$

57. $S_5 = \mathbf{span}\left(\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}\right)$,

$S_6 = \mathbf{span}\left(\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right)$

58. $S_7 = \mathbf{span}\left(\begin{pmatrix}2\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}\right)$,

$S_8 = \mathbf{span}\left(\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\3\\2\\2\end{pmatrix}\right)$

Chapter 6

Scalar Linear Differential Equations

Contents

6.2 Exercises

General Solution 2nd Order

Solve the constant equation using Theorem 6.1, page 528. Report the general solution using symbols c_1, c_2. Model the solution after Examples **??**–6.3, page **??**.

1. $y'' = 0$
Ans: $y = c_1 + c_2 x$

2. $3y'' = 0$

3. $y'' + y' = 0$

4. $3y'' + y' = 0$

5. $y'' + 3y' + 2y = 0$

6. $y'' - 3y' + 2y = 0$

7. $y'' - y' - 2y = 0$

8. $y'' - 2y' - 3y = 0$

9. $y'' + y = 0$

10. $y'' + 4y = 0$

11. $y'' + 16y = 0$

12. $y'' + 8y = 0$

13. $y'' + y' + y = 0$

14. $y'' + y' + 2y = 0$

15. $y'' + 2y' + y = 0$

16. $y'' + 4y' + 4y = 0$

17. $3y'' + y' + y = 0$

18. $9y'' + y' + y = 0$

19. $5y'' + 25y' = 0$

20. $25y'' + y' = 0$

21. $2y'' + y' - y = 0$

22. $2y'' - 3y' - 2y = 0$

23. $2y'' + 7y' + 3y = 0$

24. $4y'' + 8y' + 3y = 0$

25. $6y'' + 7y' + 2y = 0$

26. $6y'' + y' - 2y = 0$

27. $y'' + 4y' + 8y = 0$

28. $y'' - 2y' + 4y = 0$

29. $y'' + 2y' + 4y = 0$

30. $y'' + 4y' + 5y = 0$

31. $4y'' - 4y' + y = 0$

32. $4y'' + 4y' + y = 0$

33. $9y'' - 6y' + y = 0$

34. $9y'' + 6y' + y = 0$

35. $4y'' + 12y' + 9y = 0$

36. $4y'' - 12y' + 9y = 0$

Initial Value Problem 2nd Order

Solve the given problem, modeling the solution after Example 6.4.

37. $6y'' + 7y' + 2y = 0$, $y(0) = 0$, $y'(0) = -1$

38. $2y'' + 7y' + 3y = 0$, $y(0) = 5$, $y'(0) = -5$

39. $y'' - 2y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

40. $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 1$

41. $9y'' - 6y' + y = 0$, $y(0) = 3$, $y'(0) = 1$

42. $4y'' + 12y' + 9y = 0$, $y(0) = 2$, $y'(0) = 1$

Detecting Euler Solution Atoms

A **Euler solution atom** is defined in Definition 6.1 page 530. Box each list entry that is precisely an atom. Double-box non-atom list entries that are a sum of constants times atoms. Follow Example 6.5 page 535.

43. 1, $e^{x/5}$, -1, $e^{1.1x}$, $2e^x$

44. $-x\cos\pi x$, $x^2\sin 2x$, x^3, $2x^3$

45. e^{2x}, $e^{-x^2/2}$, $\cos^2 2x$, $\sin 1.57x$

46. $x^7e^x\cos 3x$, $x^{10}e^x\sin 4x$

47. $x^7e^x\cosh 3x$, $x^{10}e^{-x}\sinh 5x$

48. $\cosh^2 x$, $x(1+x)$, $x^{1.5}$, $\sqrt{x}e^{-x}$

49. $x^{1/2}e^{x/2}$, $\dfrac{1}{x}e^x$, $e^x(1+x^2)$

50. $\dfrac{x}{1+x}$, $\dfrac{1}{x}(1+x^2)$, $\ln|x|$

Euler Base Atom

An **Euler base atom** is defined in Definition 6.1 page 530. Find the base atom for each Euler solution atom in the given list.

51. $x\cos\pi x$, x^3, $x^{10}e^{-x}\sin 5x$

52. x^6, x^4e^{2x}, $x^2e^{-x/\pi}$, $x^7e^x\cos 1.1x$

Inverse Problems

Find the homogeneous 2nd order differential equation, given the supplied information. Follow Example 6.6.

53. $e^{-x/5}$ and 1 are solutions.
Ans: $5y'' + y' = 0$.

54. e^{-x} and 1 are solutions.

55. $e^x + e^{-x}$ and $e^x - e^{-x}$ are solutions.

56. $e^{2x} + xe^{2x}$ and xe^{2x} are solutions.

57. x and $2+x$ are solutions.

58. $4e^x$ and $3e^{2x}$ are solutions.

59. The characteristic equation is $r^2 + 2r + 1 = 0$.

60. The characteristic equation is $4r^2 + 4r + 1 = 0$.

61. The characteristic equation has roots $r = -2, 3$.

62. The characteristic equation has roots $r = 2/3, 3/5$.

63. The characteristic equation has roots $r = 0, 0$.

64. The characteristic equation has roots $r = -4, -4$.

65. The characteristic equation has complex roots $r = 1 \pm 2i$.

66. The characteristic equation has complex roots $r = -2 \pm 3i$.

Details of proofs

67. **(Theorem 6.1, Background)** Expand the relation $Ar^2 + Br + C = A(r - r_1)(r - r_2)$ and compare coefficients to obtain the sum and product of roots relations

$$\frac{B}{A} = -(r_1 + r_2), \quad \frac{C}{A} = r_1 r_2.$$

68. **(Theorem 6.1, Background)**
Let r_1, r_2 be the two roots of $Ar^2 + Br + C = 0$. The discriminant is $\mathcal{D} = B^2 - 4AC$. Use the quadratic formula to derive these relations for $\mathcal{D} > 0$, $\mathcal{D} = 0$, $\mathcal{D} < 0$, respectively:

$$\begin{aligned} r_1 &= \tfrac{-B+\sqrt{\mathcal{D}}}{2A}, r_2 = \tfrac{-B-\sqrt{\mathcal{D}}}{2A}, \\ r_1 &= r_2 = \tfrac{\sqrt{\mathcal{D}}}{2A}. \\ r_1 &= \tfrac{-B+i\sqrt{-\mathcal{D}}}{2A}, r_2 = \tfrac{-B-i\sqrt{-\mathcal{D}}}{2A}. \end{aligned}$$

69. **(Theorem 6.1, Case 1)**
Let $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$. Assume
$Ar^2 + Br + C = A(r - r_1)(r - r_2)$. Show that y_1, y_2 are solutions of $Ay'' + By' + Cy = 0$.

70. **(Theorem 6.1, Case 2)**
Let $y_1 = e^{r_1 x}$, $y_2 = x\,e^{r_1 x}$. Assume
$Ar^2 + Br + C{=}A(r - r_1)(r - r_1)$.
Show that y_1, y_2 are solutions of $Ay'' + By' + Cy = 0$.

71. **(Theorem 6.1, Case 3)**
Let a, b be real, $b > 0$. Let $y_1 = e^{ax}\cos bx$, $y_2 = e^{ax}\sin bx$. Assume factorization
$Ar^2{+}Br{+}C{=}A(r{-}a{-}ib)(r{-}a{+}ib)$
then show that y_1, y_2 are solutions of $Ay'' + By' + Cy = 0$.

6.3 Exercises

Continuous Coefficients

Determine all intervals J of existence of $y(x)$, according to Picard's theorem.

1. $y'' + y = \ln|x|$

2. $y'' = \ln|x-1|$

3. $y'' + (1/x)y = 0$

4. $y'' + \frac{1}{1+x}y' + \frac{1}{x}y = 0$

5. $x^2y'' + y = \sin x$

6. $x^2y'' + xy' = 0$

Superposition

Verify that $y = c_1y_1 + c_2y_2$ is a solution.

7. $y'' = 0$, $y_1(x) = 1$, $y_2(x) = x$

8. $y'' = 0$, $y_1(x) = 1 + x$, $y_2(x) = 1 - x$

9. $y''' = 0$, $y_1(x) = x$, $y_2(x) = x^2$

10. $y''' = 0$, $y_1(x) = 1 + x$, $y_2(x) = x + x^2$

Structure

Verify that $y = y_h + y_p$ is a solution.

11. $y'' + y = 2$, $y_h(x) = c_1 \cos x + c_2 \sin x$, $y_p(x) = 2$

12. $y'' + 4y = 4$, $y_h(x) = c_1 \cos 2x + c_2 \sin 2x$, $y_p(x) = 1$

13. $y'' + y' = 5$, $y_h(x) = c_1 + c_2e^{-x}$, $y_p(x) = 5x$

14. $y'' + 3y' = 5$, $y_h(x) = c_1 + c_2e^{-3x}$, $y_p(x) = 5x/3$

15. $y'' + y' = 2x$, $y_h(x) = c_1 + c_2e^{-x}$, $y_p(x) = x^2 - 2x$

16. $y'' + 2y' = 4x$, $y_h(x) = c_1 + c_2e^{-2x}$, $y_p(x) = x^2 - x$

Initial Value Problems

Solve for constants c_1, c_2 in the general solution $y_h = c_1y_1 + c_2y_2$.

17. $y'' = 0$, $y_1 = 1$, $y_2 = x$, $y(0) = 1$, $y'(0) = 2$

18. $y'' = 0,\ y_1 = 1 + x,\ y_2 = 1 - x,\ y(0) = 1,\ y'(0) = 2$

19. $y'' + y = 0,\ y_1 = \cos x,\ y_2 = \sin x,\ y(0) = 1,\ y'(0) = -1$

20. $y'' + y = 0,\ y_1 = \sin x,\ y_2 = \cos x,\ y(0) = 1,\ y'(0) = -1$

21. $y'' + 4y = 0,\ y_1 = \cos 2x,\ y_2 = \sin 2x,\ y(0) = 1,\ y'(0) = -1$

22. $y'' + 4y = 0,\ y_1 = \sin 2x,\ y_2 = \cos 2x,\ y(0) = 1,\ y'(0) = -1$

23. $y'' + y' = 0,\ y_1 = 1,\ y_2 = e^{-x},\ y(0) = 1,\ y'(0) = -1$

24. $y'' + y' = 0,\ y_1 = 1,\ y_2 = e^{-x},\ y(0) = 2,\ y'(0) = -3$

25. $y'' + 3y' = 0,\ y_1 = 1,\ y_2 = e^{-3x},\ y(0) = 1,\ y'(0) = -1$

26. $y'' + 5y' = 0,\ y_1 = 1,\ y_2 = e^{-5x},\ y(0) = 1,\ y'(0) = -1$

Recognizing y_h

Extract from the given solution y a particular solution y_p with fewest terms.

27. $y'' + y = x,$
$y = c_1 \cos x + c_2 \sin x + x$

28. $y'' + y = x,$
$y = \cos x + x$

29. $y'' + y' = x,$
$y = c_1 + c_2 e^{-x} + x^2/2 - x$

30. $y'' + y' = x,$
$y = e^{-x} - x + 1 + x^2/2$

31. $y'' + 2y' + y = 1 + x,$
$y = (c_1 + c_2 x)e^{-x} + x - 1$

32. $y'' + 2y' + y = 1 + x,$
$y = e^{-x} + x + xe^{-x} - 1$

Reduction of Order

Given solution y_1, find an independent solution y_2 by reduction of order.

33. $y'' + 2y' = 0,\ y_1(x) = 1$

34. $y'' + 2y' = 0,\ y_1(x) = e^{-2x}$

35. $2y'' + 3y' + y = 0,\ y_1(x) = e^{-x}$

36. $2y'' - y' - y = 0$, $y_1(x) = e^x$

Equilibrium Method

Apply the equilibrium method to find y_p, then find the general solution $y = y_h + y_p$.

37. $2y'' = 3$

38. $y'' + 4y' = 5$

39. $y'' + 3y' + 2y = 3$

40. $y'' - y' - 2y = 2$

41. $y'' + y = 1$

42. $3y'' + y' + y = 7$

43. $6y'' + 7y' + 2y = 5$

44. $y'' - 2y' + 4y = 8$

45. $4y'' - 4y' + y = 8$

46. $4y'' - 12y' + 9y = 18$

6.4 Exercises

Constant Coefficients

Solve for $y(x)$. Proceed as in Examples **??**–6.20.

1. $3y' - 2y = 0$

2. $2y' + 7y = 0$

3. $y'' - y' = 0$

4. $y'' + 2y' = 0$

5. $y'' - y = 0$

6. $y'' - 4y = 0$

7. $y'' + 2y' + y = 0$

8. $y'' + 4y' + 4y = 0$

9. $y'' + 3y' + 2y = 0$

10. $y'' - 3y' + 2y = 0$

11. $y'' + y = 0$

12. $y'' + 4y = 0$

13. $y'' + y' + y = 0$

14. $y'' + 2y' + 2y = 0$

15. $y'' = 0$

16. $y''' = 0$

17. $\frac{d^4y}{dx^4} = 0$

18. $\frac{d^5y}{dx^5} = 0$

19. $y''' + 2y'' = 0$

20. $y''' + 4y' = 0$

21. $\frac{d^4y}{dx^4} + y'' = 0$

22. $\frac{d^5y}{dx^5} + y''' = 0$

6.4 Exercises

Detecting Atoms

Decompose each atom into a base atom times a power of x. If the expression fails to be an atom, then explain the failure.

23. $-x$

24. x

25. $x^2 \cos \pi x$

26. $x^{3/2} \cos x$

27. $x^{1000} e^{-2x}$

28. $x + x^2$

29. $\dfrac{x}{1+x^2}$

30. $\ln |xe^{2x}|$

31. $\sin x$

32. $\sin x - \cos x$

Higher Order

A homogeneous linear constant-coefficient differential equation can be defined by (1) coefficients, (2) the characteristic equation, (3) roots of the characteristic equation. In each case, solve the differential equation.

33. $y''' + 2y'' + y' = 0$

34. $y''' - 3y'' + 2y' = 0$

35. $y^{(4)} + 4y'' = 0$

36. $y^{(4)} + 4y''' + 4y'' = 0$

37. Order 5, $r^2(r-1)^3 = 0$

38. Order 5, $(r^3 - r^2)(r^2 + 1) = 0$.

39. Order 6, $r^2(r^2 + 2r + 2)^2 = 0$.

40. Order 6, $(r^2 - r)(r^2 + 4r + 5)^2 = 0$.

41. Order 10, $(r^4 + r^3)(r^2 - 1)^2(r^2 + 1) = 0$.

42. Order 10, $(r^3 + r^2)(r - 1)^3(r^2 + 1)^2 = 0$.

43. Order 5, roots $r = 0, 0, 1, 1, 1$.

44. Order 5, roots $r = 0, 0, 1, i, -i$.

45. Order 6, roots $r = 0, 0, i, -i, i, -i$.

46. Order 6, roots $r = 0, -1, 1 + i, 1 - i, 2i, -2i$.

47. Order 10, roots $r = 0, 0, 0, 1, 1, -1, -1, -1, i, -i$.

48. Order 10, roots $r = 0, 0, 1, 1, 1, -1, i, -i, i, -i$.

Initial Value Problems

Given in each case is a set of independent solutions of the differential equation. Solve for the coefficients $c_1, c_2, \ldots$ in the general solution, using the given initial conditions.

49. e^x, e^{-x}, $y(0) = 0$, $y'(0) = 1$

50. xe^x, e^x, $y(0) = 1$, $y'(0) = -1$

51. $\cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$

52. $\cos 2x, \sin 2x$, $y(0) = 1$, $y'(0) = 0$

53. $e^x, \cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$

54. $1, \cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$

55. $e^x, xe^x, \cos x, \sin x$, $y(0) = -1$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$

56. $1, x, \cos x, \sin x$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$, $y'''(0) = 0$

57. $1, x, x^2, x^3, x^4$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 1$, $y'''(0) = 3$, $y^{(4)}(0) = 0$

58. $e^x, xe^x, x^2e^x, 1, x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, $y^{(4)}(0) = 0$

Inverse Problem

Find a linear constant-coefficient homogeneous differential equation from the given information. Follow Example 6.21.

59. The characteristic equation is $(r + 1)^3(r^2 + 4) = 0$.

60. The general solution is a linear combination of the Euler solution atoms $e^x, e^{2x}, e^{3x}, \cos x, \sin x$.

61. The roots of the characteristic polynomial are $0, 0, 2 + 3i, 2 - 3i$.

62. The equation has order 4. Known solutions are $e^x + 4\sin 2x$, xe^x.

63. The equation has order 10. Known solutions are $\sin 2x$, $x^7 e^x$.

64. The equation is $my'' + cy' + ky = 0$ with $m = 1$ and c, k positive. A solution is $y(x) = e^{-x/5}\cos(2x - \theta)$ for some angle θ.

Independence of Euler Atoms

65. Apply the independence test page **??** to atoms 1 and x: form equation $0 = c_1 + c_2 x$, then set $x = 0$ to solve for $c_1 = 0$, $c_2 = 0$. This proves Euler atoms $1, x$ are independent.

66. Show that Euler atoms $1, x, x^2$ are independent using the independence test page **??**,

67. A Taylor series is zero if and only if its coefficients are zero. Use this result to give a complete proof that the list $1, \ldots, x^k$ is independent. Hint: a polynomial is a Taylor series.

68. Show that Euler atoms $e^x, xe^x, x^2 e^x$ are independent using the independence test page **??**.

Wronskian Test

Establish independence of the given lists of functions by using the Wronskian test page **??**:

Functions $f_1, f_2, \ldots, f_n$ are independent if $W(x_0) \neq 0$ for some x_0, where $W(x)$ is the $n \times n$ determinant

$$\begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ & \vdots & \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

69. $1, x, e^x$

70. $1, x, x^2, e^x$

71. $\cos x, \sin x, e^x$

72. $\cos x, \sin x, \sin 2x$

Kümmer's Lemma

73. Compute the characteristic polynomials $p(r)$ and $q(r)$ for

$$\begin{aligned} &y'' + 3y' + 2y = 0 \text{ and} \\ &z'' + z' = 0. \end{aligned}$$

Verify the equations are related by $y = e^{-x}z$ and $p(r-1) = q(r)$.

74. Compute the characteristic polynomials $p(r)$ and $q(r)$ for

$$\begin{aligned} &ay'' + by' + cy = 0 \text{ and} \\ &az'' + (2ar_0 + b)z' + \\ &\quad (ar_0^2 + br_0 + c)z = 0. \end{aligned}$$

Verify the equations are related by $y = e^{r_0 x} z$ and $p(r + r_0) = q(r)$.

6.5 Exercises

Independence: Constant Equation

Find solutions y_1, y_2 of the given homogeneous differential equation using Theorem 6.1 page 528. Then apply the Wronskian test page **??** to prove independence, following Example 6.22.

1. $y'' - y = 0$

2. $y'' - 4y = 0$

3. $y'' + y = 0$

4. $y'' + 4y = 0$

5. $4y'' = 0$

6. $y'' = 0$

7. $4y'' + y' = 0$

8. $y'' + y' = 0$

9. $y'' + y' + y = 0$

10. $y'' - y' + y = 0$

11. $y'' + 8y' + 2y = 0$

12. $y'' + 16y' + 4y = 0$

Independence for Euler's Equation

Change variables, $x = e^t$, $u(t) = y(x)$ in $Ax^2y''(x) + Bxy'(x) + Cy(x) = 0$ to obtain a constant-coefficient equation $A\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right) + B\frac{du}{dt} + Au = 0$. Solve for $u(t)$ and then substitute $t = \ln|x|$ to obtain $y(x)$. Find two solutions y_1, y_2 which are independent by the Wronskian test page **??**.

13. $x^2y'' + y = 0$

14. $x^2y'' + 4y = 0$

15. $x^2y'' + 2xy' + y = 0$

16. $x^2y'' + 8xy' + 4y = 0$

Wronskian

Compute the Wronskian, up a constant multiple, without solving the differential equation: Example 6.23 page 572.

17. $y'' + y' - xy = 0$

18. $y'' - y' + xy = 0$

19. $2y'' + y' + \sin(x)y = 0$

20. $4y'' - y' + \cos(x)y = 0$

21. $x^2y'' + xy' - y = 0$

22. $x^2y'' - 2xy' + y = 0$

Variation of Parameters

Find the general solution $y_h + y_p$ by applying a variation of parameters formula: Example 6.24 page 573.

23. $y'' = x^2$

24. $y'' = x^3$

25. $y'' + y = \sin x$

26. $y'' + y = \cos x$

27. $y'' + y' = \ln|x|$

28. $y'' + y' = -\ln|x|$

29. $y'' + 2y' + y = e^{-x}$

30. $y'' - 2y' + y = e^{x}$

6.6 Exercises

Polynomial Solutions

Determine a polynomial solution y_p for the given differential equation.

1. $y'' = x$

2. $y'' = x - 1$

3. $y'' = x^2 - x$

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions

Determine a solution y_p for the given differential equation.

15. $y'' + y = e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

18. $y'' = e^{-2x}$

19. $y'' - y = (x+1)e^{2x}$

20. $y'' - y = (x-1)e^{-2x}$

21. $y'' - y' = (x+3)e^{2x}$

22. $y'' - y' = (x-2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2+3)e^{3x}$

24. $y'' - 3y' + 2y = (x^2-2)e^{-3x}$

Sine and Cosine Solutions

Determine a solution y_p for the given differential equation.

25. $y'' = \sin(x)$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

28. $y'' + y = \cos(x)$

29. $y'' = (x+1)\sin(x)$

30. $y'' = (x+1)\cos(x)$

31. $y'' - y = (x+1)e^x \sin(2x)$

32. $y'' - y = (x+1)e^x \cos(2x)$

33. $y'' - y' - y = (x^2+x)e^x \sin(2x)$

34. $y'' - y' - y = (x^2+x)e^x \cos(2x)$

Undetermined Coefficients Algorithm

Determine a solution y_p for the given differential equation.

35. $y'' = x + \sin(x)$

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2 e^x + xe^x \cos(2x)$

46. $y'' + y' - y = x^2 e^{-x} + xe^x \sin(2x)$

Roots and Related Atoms

Euler atoms A and B are said to be **related** if and only if the derivative lists A, A', ... and B, B', ... share a common Euler atom.

47. Find the roots, listed according to multiplicity, for the atoms 1, x, x^2, e^{-x}, $\cos 2x$, $\sin 3x$, $x\cos \pi x$, $e^{-x}\sin 3x$.

48. Find the roots, listed according to multiplicity, for the atoms 1, x^3, e^{2x}, $\cos x/2$, $\sin 4x$, $x^2 \cos x$, $e^{3x}\sin 2x$.

49. Let $A = xe^{-2x}$ and $B = x^2e^{-2x}$. Verify that A and B are related.

50. Let $A = xe^{-2x}$ and $B = x^2e^{2x}$. Verify that A and B are not related.

51. Prove that atoms A and B are related if and only if their base atoms have the same roots.

52. Prove that atoms A and B are related if and only if they are in the same **group**. See page **??** for the definition of a group of atoms.

Modify a Trial Solution

Apply Rule II to modify the given Rule I trial solution into the shortest trial solution.

53. The characteristic equation has factors r^3, $(r^3 + 2r^2 + 2)$, $(r - 1)^2$, $(r + 1)$, $(r^2 + 4)^3$ and the Rule I trial solution is constructed from atoms 1, x, e^x, xe^x, e^{-x}, $\cos 2x$, $\sin 2x$, $\cos x$, $\sin x$.

54. The characteristic equation has factors r^2, $(r^3 + 3r^2 + 2)$, $(r + 1)$, $(r^2 + 4)^3$ and the Rule I trial solution is constructed from atoms 1, x, e^x, xe^x, e^{-x}, $\cos 2x$, $\sin 2x$.

Annihilators and Laplace Theory

Laplace theory can construct the annihilator of $f(t)$. The example $y'' + 4y = t + 2t^3$ is used to discuss the techniques. Formulas to be justified: $p(s) = \mathcal{L}(f)/\mathcal{L}(y)$ and $q(s) = \mathbf{denom}(\mathcal{L}(f(t)))$.

55. **(Transfer Function)** Find the characteristic polynomial $p(r)$ for the homogeneous equation $y'' + 4y = 0$. The transfer function for $y'' + 4y = f(t)$ is $\mathcal{L}(y)/\mathcal{L}(f)$, which equals $1/p(s)$.

56. **(Laplace of $y_p(t)$)**
The Laplace of $y(t)$ for problem
$y'' + 4y = f(t)$, $y(0) = y'(0) = 0$ must equal the Laplace of $f(t)$ times the transfer function. Justify and explain what it has to do with finding y_p.

57. **(Annihilator of $f(t)$)**
Let $f(t) = t + 2t^3$. Verify that $\mathcal{L}(f(t)) = \dfrac{s^2+12}{s^4}$, which is a proper fraction $v(s)/q(s)$. Then explain why the annihilator of $f(t)$ has characteristic polynomial $q(r)$.

58. **(Laplace Theory finds y_p)**
Show that the problem $y'' + 4y = t + 2t^3$, $y(0) = y'(0) = 0$ has Laplace transform

$$\mathcal{L}(y) = \frac{s^2+12}{(s^2+4)s^4} = \frac{v(s)}{p(s)q(s)}.$$

Explain why $y(t)$ must be a solution of the constant-coefficient homogeneous differential equation having characteristic polynomial $p(r)q(r)$.

Annihilator Method Justified

The method of annihilators can be justified by successive differentiation of a non-homogeneous differential equation. It is carried out here, for exposition efficiency, for the non-homogeneous equation $y'' + 4y = x + 2x^3$. Then its right side is $f(x) = x + 2x^3$ and the homogeneous equation is $y'' + 4y = 0$.

59. **(Homogeneous equation)**
Verify that $y'' + 4y = 0$ has characteristic polynomial $p(r) = r^2 + 4$.

60. **(Annihilator)**
Verify that $y^{(4)} = 0$ is the annihilator for $f(x) = x + 2x^3$, with characteristic polynomial $q(r) = r^4$.

61. **(Composite Equation)**
Differentiate four times across the equation $y'' + 4y = f(x)$ to obtain $y^{(6)} + 4y^{(4)} = f^{(4)}(x)$. Argue that $f^{(4)}(x) = 0$ because $y^{(4)} = 0$ is an annihilator of $f(x)$. Then argue that $w(r) = p(r)q(r)$ is the characteristic polynomial of the equation $y^{(6)} + 4y^{(4)} = 0$. This proves that y_p is a solution of $y^{(6)} + 4y^{(4)} = 0$.

62. **(General Solution)**
Solve the homogeneous composite equation $y^{(6)} + 4y^{(4)} = 0$ using its characteristic polynomial $w(r)$.

63. **(Extraneous Atoms)**
Argue that the general solution from the previous exercise contains two terms constructed from atoms derived from roots of the polynomial $p(r) = 0$. These terms can be removed to obtain the shortest expression for y_p.

64. (Particular Solution)
Report the form of the shortest particular solution of $y'' + 4y = f(x)$, according to the previous exercise.

6.7 Exercises

Simple Harmonic Motion

Determine the model equation $mx''(t) + kx(t) = 0$, the natural frequency $\omega = \sqrt{k/m}$, the period $2\pi/\omega$ and the solution $x(t)$ for the following spring–mass systems.

1. A mass of 4 Kg attached to a spring of Hooke's constant 20 Newtons per meter starts from equilibrium plus 0.05 meters with velocity 0.

2. A mass of 2 Kg attached to a spring of Hooke's constant 20 Newtons per meter starts from equilibrium plus 0.07 meters with velocity 0.

3. A mass of 2 Kg is attached to a spring that elongates 20 centimeters due to a force of 10 Newtons. The motion starts at equilibrium with velocity -5 meters per second.

4. A mass of 4 Kg is attached to a spring that elongates 20 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity -8 meters per second.

5. A mass of 3 Kg is attached to a coil spring that compresses 2 centimeters when 1 Kg rests on the top coil. The motion starts at equilibrium plus 3 centimeters with velocity 0.

6. A mass of 4 Kg is attached to a coil spring that compresses 2 centimeters when 2 Kg rests on the top coil. The motion starts at equilibrium plus 4 centimeters with velocity 0.

7. A mass of 5 Kg is attached to a coil spring that compresses 1.5 centimeters when 1 Kg rests on the top coil. The motion starts at equilibrium plus 3 centimeters with velocity -5 meters per second.

8. A mass of 4 Kg is attached to a coil spring that compresses 2.2 centimeters when 2 Kg rests on the top coil. The motion starts at equilibrium plus 4 centimeters with velocity -8 meters per second.

9. A mass of 5 Kg is attached to a spring that elongates 25 centimeters due to a force of 10 Newtons. The motion starts at equilibrium with velocity 6 meters per second.

10. A mass of 5 Kg is attached to a spring that elongates 30 centimeters due to a force of 15 Newtons. The motion starts at equilibrium with velocity 4 meters per second.

Phase–amplitude Form

Solve the given differential equation and report the general solution. Solve for the constants c_1, c_2. Report the solution in phase–amplitude form

$$x(t) = A\cos(\omega t - \alpha)$$

with $A > 0$ and $0 \le \alpha < 2\pi$.

11. $x'' + 4x = 0$,
$x(0) = 1$, $x'(0) = -1$

12. $x'' + 4x = 0$,
$x(0) = 1$, $x'(0) = 1$

13. $x'' + 16x = 0$,
$x(0) = 2$, $x'(0) = -1$

14. $x'' + 16x = 0$,
$x(0) = -2$, $x'(0) = -1$

15. $5x'' + 11x = 0$,
$x(0) = -4$, $x'(0) = 1$

16. $5x'' + 11x = 0$,
$x(0) = -4$, $x'(0) = -1$

17. $x'' + x = 0$,
$x(0) = 1$, $x'(0) = -2$

18. $x'' + x = 0$,
$x(0) = -1$, $x'(0) = 2$

19. $x'' + 36x = 0$,
$x(0) = 1$, $x'(0) = -4$

20. $x'' + 64x = 0$,
$x(0) = -1$, $x'(0) = 4$

Pendulum

The formula

$$\frac{P_1}{P_2} = \frac{R_1}{R_2}\sqrt{\frac{L_1}{L_2}}$$

is valid for the periods P_1, P_2 of two pendulums of lengths L_1, L_2 located at distances R_1, R_2 from the center of the earth. The formula implies that a pendulum can be used to find the radius of the earth at a location. It is also useful for designing a pendulum clock adjustment screw.

21. Derive the formula, using $\omega = \sqrt{g/L}$, period $P = 2\pi/\omega$ and the gravitational relation $g = GM/R^2$.

22. A pendulum clock taken on a voyage loses 2 minutes a day compared to its exact timing at home. Determine the altitude change at the destination.

23. A pendulum clock with adjustable length L loses 3 minutes per day when $L = 30$ inches. What length L adjusts the clock to perfect time?

24. A pendulum clock with adjustable length L loses 4 minutes per day when $L = 30$ inches. What fineness length F is required for a 1/4–turn of the adjustment screw, in order to have 1/4–turns of the screw set the clock to perfect time plus or minus one second per day?

Torsional Pendulum

Solve for $\theta_0(t)$.

25. $\theta_0''(t) + \theta_0(t) = 0$

26. $\theta_0''(t) + 4\theta_0(t) = 0$

27. $\theta_0''(t) + 16\theta_0(t) = 0$

28. $\theta_0''(t) + 36\theta_0(t) = 0$

Shockless Auto

Find the period and frequency of oscillation of the car on four springs. Use the model $mx''(t) + kx(t) = 0$.

29. Assume the car plus occupants has mass 1650 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter.

30. Assume the car plus occupants has mass 1850 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter.

31. Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 18000$ Newtons per meter.

32. Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 16000$ Newtons per meter.

Rolling Wheel on a Spring

Solve the rolling wheel model $mx''(t) + \frac{2}{3}\,kx(t) = 0$ and also the frictionless model $mx''(t) + kx(t) = 0$, each with the given initial conditions. Graph the two solutions on one set of axes.

33. $m = 1$, $k = 4$,
$x(0) = 1$, $x'(0) = 0$

34. $m = 5$, $k = 18$,
$x(0) = 1$, $x'(0) = 0$

35. $m = 11$, $k = 18$,
$x(0) = 0$, $x'(0) = 1$

36. $m = 7$, $k = 18$,
$x(0) = 0$, $x'(0) = 1$

6.8 Exercises

Forced Undamped Vibration

Solve the given equation.

1. $x'' + 100x = 20\cos(5t)$

2. $x'' + 16x = 100\cos(10t)$

3. $x'' + \omega_0^2 x = 100\cos(\omega t)$, when the internal frequency ω_0 is twice the external frequency ω.

4. $x'' + \omega_0^2 x = 5\cos(\omega t)$, when the internal frequency ω_0 is half the external frequency ω.

Black Box in the Trunk

5. Construct an example $x'' + \omega_0^2 x = F_1\cos(\omega t)$ with a solution $x(t)$ having beats every two seconds.

6. A solution $x(t)$ of $x'' + 25x = 100\cos(\omega t)$ has beats every two seconds. Find ω.

Rotating Drum

Solve the given equation.

7. $x'' + 100x = 500\omega^2\cos(\omega t)$, $\omega \neq 10$.

8. $x'' + \omega_0^2 x = 5\omega^2\cos(\omega t)$, $\omega \neq \omega_0$.

Harmonic Oscillations

Express the general solution as a sum of two harmonic oscillations of different frequencies, each oscillation written in phase-amplitude form.

9. $x'' + 9x = \sin 4t$

10. $x'' + 100x = \sin 5t$

11. $x'' + 4x = \cos 4t$

12. $x'' + 4x = \sin t$

Beats: Convert and Graph

Write each linear combination as $x(t) = C\sin at \sin bt$. Then graph the slowly-varying envelope curves and the curve $x(t)$.

13. $x(t) = \cos 4t - \cos t$

14. $x(t) = \cos 10t - \cos t$

15. $x(t) = \cos 16t - \cos 12t$

16. $x(t) = \cos 25t - \cos 23t$

Beats: Solve, find Envelopes

Solve each differential equation with $x(0) = x'(0) = 0$ and determine the slowly-varying envelope curves.

17. $x'' + x = 99 \cos 10t$.

18. $x'' + 4x = 252 \cos 10t$.

19. $x'' + x = 143 \cos 10t$.

20. $x'' + 256x = 252 \cos 2t$.

Waves and Superposition

Graph the individual waves x_1, x_2 and then the superposition $x = x_1 + x_2$. Report the apparent period of the superimposed waves.

21. $x_1(t) = \sin 22t$, $x_2(t) = 2 \sin 20t$

22. $x_1(t) = \cos 16t$, $x_2(t) = 4 \cos 20t$

23. $x_1(t) = \cos 16t$, $x_2(t) = 4 \sin 16t$

24. $x_1(t) = \cos 25t$, $x_2(t) = 4 \cos 27t$

Periodicity

25. Let $x_1(t) = \cos 25t$, $x_2(t) = 4 \cos 27t$. Their sum has period $T = m\frac{2\pi}{25} = n\frac{2\pi}{27}$ for some integers m, n. Find all m, n and the least period T.

26. Let $x_1(t) = \cos \omega_1 t$, $x_2(t) = \cos \omega_2 t$. Find a condition on ω_1, ω_2 which implies that the sum $x_1 + x_2$ is periodic.

27. Let $x(t) = \cos(t) - \cos(\sqrt{2}t)$. Explain without proof, from a graphic, why $x(t)$ is not periodic.

28. Let $x(t) = \cos(5t) + \cos(5\sqrt{2}t)$. Is $x(t)$ is periodic? Explain without proof.

Rotating Drum

Let $x(t)$ and $x_p(t)$ be defined as in Example 6.46, page 637.

29. Re-compute the amplitude of solution $x_p(t)$ when Hooke's constant $k = 1$ (instead of 10).

30. Plot $x(t)$ when $x(0) = x'(0) = 0$. Verify that the amplitude of vibration is about 0.13.

Musical Instruments

Melodious tones are superpositions of harmonics $\sin(n\omega t)$, with $n =$ an integer, $\omega =$ fundamental frequency. Tones $\sin(\omega t)$, $\sin(2\omega t)$ are an **Octave** apart. **Equal temperament** divides an octave into 12 equal intervals.

31. **(Equal Temperament)** Equation $x(t) = \sin 220\pi t$ could represent a tuning fork of frequency 110 Hz. Find the 12 frequencies of equal temperament for the octave $110 - 220$ Hz.

32. **(Flute or Noise)** Equation $x(t) = \sin 220\pi t + 2 \sin 330\pi t$ could represent a tone from a flute or just a dissonant, unpleasing sound. Which is it?

33. **(Guitar)** Air inside a guitar vibrates a little like air in a bottle when you blow across the top. Consider a flask of volume $V = 1$ liter, neck length $L = 5$ cm and neck cross-section $S = 3$ cm^2. The vibration has model $x'' + f^2 x = 0$ with $f = c\sqrt{\frac{S}{VL}}$, where $c = 343$ m/s is the speed of sound in air. Compute $\frac{f}{2\pi}$ and $\lambda = \frac{2\pi c}{f}$, the frequency and wavelength. The answers are about 130 Hz and 2.6 meters, a low sound.

34. **(Helmholtz Resonance)** Repeat the previous exercise calculations, using a flask with neck diameter 2.0 cm and neck length 3 cm. The tone should be lower and the wavelength longer.

Seismoscope

35. Verify that (**??**) and (**??**), page **??**, have the same initial conditions when $u(0) = u'(0) = 0$ (the ground does not move at $t = 0$). Conclude that answers x_p, x_p^* in this situation are identical.

36. A **release test** begins by starting a vibration with $u = 0$. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. Explain how to find β, Ω_0 in the equation $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$, using Exercises 69 and 70, *infra*.

Free Damped Motion

Classify the homogeneous equation $mx'' + cx' + kx = 0$ as **over-damped**, **critically damped** or **under-damped**. Then solve the equation for the general solution $x(t)$.

37. $m = 1, c = 2, k = 1$

38. $m = 1, c = 4, k = 4$

39. $m = 1, c = 2, k = 3$

40. $m = 1, c = 5, k = 6$

41. $m = 1, c = 2, k = 5$

42. $m = 1, c = 12, k = 37$

43. $m = 6, c = 17, k = 7$

44. $m = 10, c = 31, k = 15$

45. $m = 25, c = 30, k = 9$

46. $m = 9, c = 30, k = 25$

47. $m = 9, c = 24, k = 41$

48. $m = 4, c = 12, k = 34$

Cafe and Pet Door

Classify as a cafe door model and/or a pet door model. Solve the equation for the general solution and identify as oscillatory or non-oscillatory.

49. $x'' + x' = 0$

50. $x'' + 2x' + x = 0$

51. $x'' + 2x' + 5x = 0$

52. $x'' + x' + 3x = 0$

53. $9x'' + 24x' + 41x = 0$

54. $6x'' + 17x' = 0$

55. $9x'' + 24x' = 0$

56. $6x'' + 17x' + 7x = 0$

Classification

Classify $mx'' + cx' + kx = 0$ as **over-damped**, **critically damped** or **under-damped** without solving the differential equation.

57. $m = 5, c = 12, k = 34$

58. $m = 7, c = 12, k = 19$

59. $m = 5, c = 10, k = 3$

60. $m = 7, c = 12, k = 3$

61. $m = 9, c = 30, k = 25$

62. $m = 25, c = 80, k = 64$

Critically Damped

The equation $mx'' + cx' + kx = 0$ is critically damped when $c^2 - 4mk = 0$. Establish the following results for $c > 0$.

63. The mass undergoes no oscillations, because

$$x(t) = (c_1 + c_2 t)e^{-\frac{ct}{2m}}.$$

64. The mass passes through $x = 0$ at most once.

Over-Damped

Equation $mx'' + cx' + kx = 0$ is defined to be over-damped when $c^2 - 4mk > 0$. Establish the following results for $c > 0$.

65. The mass undergoes no oscillations, because if r_1, r_2 are the roots of $mr^2 + cr + c = 0$, then

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

66. The mass passes through equilibrium position $x = 0$ at most once.

Under-Damped

Equation $mx'' + cx' + kx = 0$ is defined to be under-damped when $c^2 - 4mk < 0$. Establish the following results.

67. The mass undergoes infinitely many oscillations. If $c = 0$, then the oscillations are harmonic.

68. The solution $x(t)$ can be factored as an exponential function $e^{-\frac{ct}{2m}}$ times a harmonic oscillation. In symbols:

$$x(t) = e^{-\frac{ct}{2m}} \left(A\cos(\omega t - \alpha)\right).$$

Experimental Methods

Assume model $mx'' + cx' + kx = 0$ is oscillatory. The results apply to find nonnegative constants m, c, k from one experimentally known solution $x(t)$.

69. Let $x(t)$ have consecutive maxima at $t = t_1$ and $t = t_2$. Then $t_2 - t_1 = T = \frac{2\pi}{\omega}$, the pseudo-period of $x(t)$.

70. Let (t_1, x_1) and (t_2, x_2) be two consecutive maximum points of the graph of $x(t)$. Let T be the pseudo-period of $x(t)$. Then

$$\ln \frac{x_1}{x_2} = \frac{c}{2mT}.$$

71. **(Bike Trailer)** Assume *fps* units. A trailer equipped with one spring and one shock has mass $m = 100$ in the model $mx'' + cx' + kx = 0$. Find c and k from this experimental data: two consecutive maxima of $x(t)$ are $(0.35, 10/12)$ and $(1.15, 8/12)$.
Hint: Use exercises 69 and 70.

72. **(Auto)** Assume *fps* units. An auto weighing 2.4 tons is equipped with four identical springs and shocks. Each spring-shock module has damped oscillations satisfying $mx'' + cx' + kx = 0$. Find m. Then find c and k from this experimental data: two consecutive maxima of $x(t)$ are $(0.3, 3/12)$ and $(0.7, 2/12)$.
Hint: Use exercises 69 and 70.

Structure of Solutions

Establish these results for the damped spring-mass system $mx'' + cx' + kx = 0$.

73. **(Monotonic Factor)** Let the equation be critically damped or over-damped. Prove that

$$x(t) = e^{-pt} f(t)$$

where $p \geq 0$ and $f(t)$ is monotonic (f' one-signed).

74. **(Harmonic Factor)** Let the equation be under-damped. Prove that

$$x(t) = e^{-pt} f(t)$$

where $p > 0$ and $f(t) = A\cos(\omega t - \alpha)$ is a harmonic oscillation.

75. **(Limit Zero and Transients)** A term appearing in a solution is called **transient** if it has limit zero at $t = \infty$. Prove that positive damping $c > 0$ implies that the homogeneous solution satisfies $\lim_{t\to\infty} x(t) = 0$.

76. **(Steady-State)** An **observable** or **steady-state** is the collection of terms appearing in a solution which do not vanish at $t = \infty$. Assume $mx'' + cx' + kx = 25\cos 2t$ has a solution

$$x(t) = 2te^{-t} - \cos 2t + \sin 2t.$$

Find the transient and steady-state terms.

Damping Effects

Construct a figure on $0 \le t \le 2$ with two curves, to illustrate the effect of removing the dashpot. Curve 1 is the solution of $mx'' + cx' + kx = 0$, $x(0) = x_0$, $x'(0) = v_0$. Curve 2 is the solution of $my'' + ky = 0$, $y(0) = x_0$, $y'(0) = v_0$.

77. $m = 2, c = 12, k = 50,$
$x_0 = 0, v_0 = -20$

78. $m = 1, c = 6, k = 25,$
$x_0 = 0, v_0 = 20$

79. $m = 1, c = 8, k = 25,$
$x_0 = 0, v_0 = 60$

80. $m = 1, c = 4, k = 20,$
$x_0 = 0, v_0 = 4$

Envelope and Pseudo-period

Plot on one graphic the envelope curves and the solution $x(t)$, over two pseudo-periods. Use initial conditions $x(0) = 0$, $x'(0) = 4$.

81. $x'' + 2x' + 5x = 0$

82. $x'' + 2x' + 26x = 0$

83. $2x'' + 12x' + 50x = 0$

84. $4x'' + 8x' + 20x = 0$

6.9 Exercises

Beats

Each equation satisfies the beats relation $\omega \neq \omega_0$. Find the general solution. See Example 6.53, page 662.

1. $x'' + 100x = 10 \sin 9t$

2. $x'' + 100x = 5 \sin 9t$

3. $x'' + 25x = 5 \sin 4t$

4. $x'' + 25x = 5 \cos 4t$

Pure Resonance

Each equation satisfies the pure resonance relation $\omega = \omega_0$. Find the general solution. See Example 6.53, page 662.

5. $x'' + 4x = 10 \sin 2t$

6. $x'' + 4x = 5 \sin 2t$

7. $x'' + 16x = 5 \sin 4t$

8. $x'' + 16x = 10 \sin 4t$

Practical Resonance

For each model, find the **tuned practical resonance** frequency Ω and the **resonant amplitude** C:

$$\begin{aligned}&\Omega = \sqrt{k/m - c^2/(2m^2)},\\ &C = F_0/\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}\end{aligned}$$

9. $x'' + 2x' + 17x = 100 \cos(4t)$

10. $x'' + 2x' + 10x = 100 \cos(4t)$

11. $x'' + 4x' + 5x = 10 \cos(2t)$

12. $x'' + 2x' + 6x = 10 \cos(2t)$

Transient Solution

Identify from superposition $x = x_h + x_p$ a shortest particular solution, given one particular solution.

13. $x'' - 2x' + 10x = 18e^t \cos(3t)$,
$x = 100e^t \cos(3t) + 3te^t \sin(3t)$

14. $x'' - 4x' + 13x = 54e^{2t}\cos(3t)$,
$x = -5e^{2t}\sin(3t) + 4e^{2t}\cos(3t) + 9te^{2t}\sin(3t)$

15. $x'' - 2x' + 2x = 2e^{t}\cos(t)$,
$x = -5e^{t}\sin(t) + te^{t}\sin(t) + 4e^{t}\cos(t)$

16. $x'' - 2x' + 17x = 16e^{t}\cos(4t)$,
$x = -3e^{t}\cos(4t) + 2te^{t}\sin(4t) + 10e^{t}\sin(4t)$

Steady-State Periodic Solution

Consider the model $mx'' + cx' + kx = F_0\cos(\omega t)$ of external frequency ω. Compute the unique steady-state solution $A\cos(\omega t) + B\sin(\omega t)$ and its amplitude $C(\omega) = \sqrt{A^2 + B^2}$. Graph the ratio $100C(\omega)/C(\Omega)$ on $0 < \omega < \infty$, where Ω is the tuned practical resonance frequency.

17. $x'' + 2x' + 17x = 100\cos(4t)$

18. $x'' + 2x' + 10x = 100\cos(4t)$

19. $x'' + 4x' + 5x = 10\cos(2t)$

20. $x'' + 2x' + 6x = 10\cos(2t)$

21. $x'' + 4x' + 5x = 5\cos(2t)$

22. $x'' + 2x' + 5x = 5\cos(1.5t)$

Phase-Amplitude

Solve for a particular solution in the form $x(t) = C\cos(\omega t - \alpha)$.

23. $x'' + 6x' + 13x = 174\sin(5t)$

24. $x'' + 8x' + 25x = 100\cos(t) + 260\sin(t)$

Chapter 7

Topics in Linear Differential Equations

Contents

7.2 Exercises

Higher Order Factored

Solve the higher order equation with the given characteristic equation. Display a table of distinct roots, multiplicities and solution atoms. Verify the general solution y with a computer algebra system, if possible.

1. $(r-1)(r+2)(r-3)^2 = 0$

2. $(r-1)^2(r+2)(r+3) = 0$

3. $(r-1)^3(r+2)^2r^4 = 0$

4. $(r-1)^2(r+2)^3r^5 = 0$

5. $r^2(r-1)^2(r^2+4r+6)=0$

6. $r^3(r-1)(r^2+4r+6)^2=0$

7. $(r-1)(r+2)(r^2+1)^2=0$

8. $(r-1)^2(r+2)(r^2+1)=0$

9. $(r-1)^3(r+2)^2(r^2+4)=0$

10. $(r-1)^4(r+2)(r^2+4)^2=0$

Higher Order Unfactored

Completely factor the given characteristic equation, then report the general solution. Display a table of distinct roots, multiplicities and solution atoms. Check the answer in a computer algebra system, if possible.

11. $(r-1)(r^2-1)^2(r^2+1)^3=0$

12. $(r+1)^2(r^2-1)^2(r^2+1)^2=0$

13. $(r+2)^2(r^2-4)^2(r^2+16)^2=0$

14. $(r+2)^3(r^2-4)^4(r^2+5)^2=0$

15. $(r^3-1)^2(r-1)^2(r^2-1)=0$

16. $(r^3-8)^2(r-2)^2(r^2-4)=0$

17. $(r^2-4)^3(r^4-16)^2=0$

18. $(r^2+8)(r^4-64)^2=0$

19. $(r^2-r+1)(r^3+1)^2=0$

20. $(r^2+r+1)^2(r^3-1)=0$

Higher Order Equations

The exercises study properties of Euler atoms and nth order linear differential equations.

21. **(Euler's Theorem)**
Explain why the derivatives of atom x^3e^x satisfy a higher order equation with characteristic equation $(r-1)^4=0$.

22. **(Euler's Theorem)**
Explain why the derivatives of atom $x^3\sin x$ satisfy a higher order equation with characteristic equation $(r^2+1)^4=0$.

23. (Kümmer's Change of Variable)
Consider a fourth order equation with characteristic equation $(r-a)^4 = 0$ and general solution y. Define $y = ue^{ax}$. Find the differential equation for u.

24. (Kümmer's Change of Variable)
A polynomial $u = c_0 + c_1 x + c_2 x^2$ satisfies $u''' = 0$. Define $y = ue^{ax}$. Prove that y satisfies a third order equation and determine its characteristic equation.

25. (Ziebur's Derivative Lemma)
Let y be a solution of a higher order constant-coefficient linear equation. Prove that the derivatives of y satisfy the same differential equation.

26. (Ziebur's Lemma: atoms)
Let $y = x^3 e^x$ be a solution of a higher order constant-coefficient linear equation. Prove that Euler atoms e^x, xe^x, $x^2 e^x$ are solutions of the same differential equation.

27. (Ziebur's Atom Lemma)
Let y be an Euler atom solution of a higher order constant-coefficient linear equation. Prove that the Euler atoms extracted from the expressions $y, y', y'', \ldots$ are solutions of the same differential equation.

28. (Differential Operators)
Let y be a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5 = 0$. Explain why y''' is a solution of a differential equation with characteristic equation $(r-1)^3(r+2)^6(r^2+4)^5 r^3 = 0$.

29. (Higher Order Algorithm)
Let atom $x^2 \cos x$ appear in the general solution of a linear higher order equation. Find the pair of complex conjugate roots that constructed this atom, and the multiplicity k. Report the $2k$ atoms which must also appear in the general solution.

30. (Higher Order Algorithm)
Let Euler atom $xe^x \cos 2x$ appear in the general solution of a linear higher order equation. Find the pair of complex conjugate roots that constructed this atom and estimate the multiplicity k. Report the $2k$ atoms which are expected to appear in the general solution.

31. (Higher Order Algorithm)
Let a higher order equation have characteristic equation $(r-9)^3(r-5)^2(r^2+4)^5 = 0$. Explain why the general solution is a sum of constants times atoms.

32. (Higher Order Algorithm)
Explain why a higher order equation has general solution a sum of constants times atoms.

7.3 Exercises

Operator Arithmetic

Compute the operator and solve the corresponding differential equation.

1. $D(D+1)+D$

2. $D(D+1)+D(D+2)$

3. $D(D+1)^2$

4. $D(D^2+1)^2$

5. $D^2(D^2+4)^2$

6. $(D-1)((D-1)^2+1)^2$

Operator Properties.

7. **(Operator Composition)** Multiply $P = D^2 + D$ and $Q = 2D + 3$ to get $R = 2D^3 + 5D^2 + 3D$. Then compute $P(Qy)$ and $Q(Py)$ for $y(x)$ 3-times differentiable, and show both equal Ry.

8. **(Kernels)**
The operators $(D-1)^2(D+2)$ and $(D-1)(D+2)^2$ share common factors. Find the solution atoms shared by the corresponding differential equations.

9. **(Operator Multiply)**
The differential equation $(D^2+2D+1)y = 0$ is formally differentiated four times. Find its operator and solve the equation. What does this have to do with operator multiply?

10. **(Non-homogeneous Equation)** The differential equation $(D^5+4D^3)y = 0$ can be viewed as $(D^2+4)u = 0$ and $u = D^3y$. On the other hand, y is a linear combination of the atoms generated from the characteristic equation $r^3(r^2+4) = 0$. Use these facts to find a particular solution of the non-homogeneous equation $y''' = 3\cos 2x$.

Kümmer's Change of Variable

Kümmer's change of variable $y = ue^{ax}$ changes a y-differential equation into a u-differential equation. It can be used as a basis for solving homogeneous nth order linear constant coefficient differential equations.

11. Supply details: $y = ue^{ax}$ changes $y'' = 0$ into $u'' + 2au' + a^2u = 0$.

12. Supply details: $y = ue^{ax}$ changes $(D^2+4D)y = 0$ into $((D+a)^2 + 4(D+a))u = 0$.

13. Supply details: $y = ue^{ax}$ changes the differential equation $D^n y = 0$ into $(D + a)^n u = 0$.

14. Kümmer's substitution $y = ue^{ax}$ changes the differential equation $(D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0$ into $(F^n + a_{n-1}F^{n-1} + \cdots + a_0)u = 0$, where $F = D + a$. Write the proof.

7.4 Exercises

Variation of Parameters

Solve the higher order equation given by its characteristic equation and right side $f(x)$. Display the Cauchy kernel $\mathcal{K}(x)$ and a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to evaluate integrals, if possible.

1. $(r-1)(r+2)(r-3)^2 = 0$,
$f(x) = e^x$

2. $(r-1)^2(r+2)(r+3) = 0$,
$f(x) = e^x$

3. $(r-1)^3(r+2)^2r^4 = 0$,
$f(x) = x + e^{-2x}$

4. $(r-1)^2(r+2)^3r^5 = 0$,
$f(x) = x + e^{-2x}$

5. $r^2(r-1)^2(r^2+4r+6) = 0$,
$f(x) = x + e^x$

6. $r^3(r-1)(r^2+4r+6)^2 = 0$,
$f(x) = x^2 + e^x$

7. $(r-1)(r+2)(r^2+1)^2 = 0$,
$f(x) = \cos x + e^{-2x}$

8. $(r-1)^2(r+2)(r^2+1) = 0$,
$f(x) = \sin x + e^{-2x}$

9. $(r-1)^3(r+2)^2(r^2+4) = 0$,
$f(x) = \cos 2x + e^x$

10. $(r-1)^4(r+2)(r^2+4)^2 = 0$,
$f(x) = \sin 2x + e^x$

Undetermined Coefficient Method

A higher order equation is given by its characteristic equation and right side $f(x)$. Display (a) a trial solution, (b) a system of equations for the undetermined coefficients, and (c) a particular solution $y_p(x)$ with fewest terms. Use a computer algebra system to solve for undetermined coefficients, if possible.

11. $(r-1)(r+2)(r-3)^2 = 0$,
$f(x) = e^x$

12. $(r-1)^2(r+2)(r+3)=0$,
$f(x)=e^x$

13. $(r-1)^3(r+2)^2r^4=0$,
$f(x)=x+e^{-2x}$

14. $(r-1)^2(r+2)^3r^5=0$,
$f(x)=x+e^{-2x}$

15. $r^2(r-1)^2(r^2+4r+6)=0$,
$f(x)=x+e^x$

16. $r^3(r-1)(r^2+4r+6)^2=0$,
$f(x)=x^2+e^x$

17. $(r-1)(r+2)(r^2+1)^2=0$,
$f(x)=\cos x+e^{-2x}$

18. $(r-1)^2(r+2)(r^2+1)=0$,
$f(x)=\sin x+e^{-2x}$

19. $(r-1)^3(r+2)^2(r^2+4)=0$,
$f(x)=\cos 2x+e^x$

20. $(r-1)^4(r+2)(r^2+4)^2=0$,
$f(x)=\sin 2x+e^x$

7.5 Exercises

Cauchy-Euler Equation

Find solutions y_1, y_2 of the given homogeneous differential equation which are independent by the Wronskian test, page **??**.

1. $x^2y'' + y = 0$

2. $x^2y'' + 4y = 0$

3. $x^2y'' + 2xy' + y = 0$

4. $x^2y'' + 8xy' + 4y = 0$

Variation of Parameters

Find a solution y_p using a variation of parameters formula.

5. $x^2y'' = e^x$

6. $x^3y'' = e^x$

7. $y'' + 9y = \sec 3x$

8. $y'' + 9y = \csc 3x$

7.6 Exercises

Cauchy Kernel

Find the Cauchy kernel $\mathcal{K}(x, t)$ for the given homogeneous differential equation.

1. $y'' - y = 0$

2. $y'' - 4y = 0$

3. $y'' + y = 0$

4. $y'' + 4y = 0$

5. $4y'' + y' = 0$

6. $y'' + y' = 0$

7. $y'' + y' + y = 0$

8. $y'' - y' + y = 0$

Variation of Parameters

Find the general solution $y_h + y_p$ by applying a variation of parameters formula.

9. $y'' = x^2$

10. $y'' = x^3$

11. $y'' + y = \sin x$

12. $y'' + y = \cos x$

13. $y'' + y' = \ln |x|$

14. $y'' + y' = -\ln |x|$

15. $y'' + 2y' + y = e^{-x}$

16. $y'' - 2y' + y = e^{x}$

7.7 Exercises

Polynomial Solutions

Determine a polynomial solution y_p for the given differential equation. Apply Theorem 7.8, page 714, and model the solution after Examples 7.5, 7.6, 7.7 and 7.8.

1. $y'' = x$

2. $y'' = x - 1$

3. $y'' = x^2 - x$

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions

Determine a solution y_p for the given differential equation. Apply Theorem 7.9, page 714, and model the solution after Example 7.9.

15. $y'' + y = e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

18. $y'' = e^{-2x}$

19. $y'' - y = (x + 1)e^{2x}$

20. $y'' - y = (x-1)e^{-2x}$

21. $y'' - y' = (x+3)e^{2x}$

22. $y'' - y' = (x-2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2+3)e^{3x}$

24. $y'' - 3y' + 2y = (x^2-2)e^{-3x}$

Sine and Cosine Solutions

Determine a solution y_p for the given differential equation. Apply Theorem 7.10, page 715, and model the solution after Examples 7.10 and 7.11.

25. $y'' = \sin(x)$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

28. $y'' + y = \cos(x)$

29. $y'' = (x+1)\sin(x)$

30. $y'' = (x+1)\cos(x)$

31. $y'' - y = (x+1)e^x\sin(2x)$

32. $y'' - y = (x+1)e^x\cos(2x)$

33. $y'' - y' - y = (x^2+x)e^x\sin(2x)$

34. $y'' - y' - y = (x^2+x)e^x\cos(2x)$

Undetermined Coefficients Algorithm

Determine a solution y_p for the given differential equation. Apply the polynomial algorithm, page **??**, and model the solution after Example 7.12.

35. $y'' = x + \sin(x)$

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2e^x + xe^x\cos(2x)$

46. $y'' + y' - y = x^2e^{-x} + xe^x\sin(2x)$

Additional Proofs

The exercises below fill in details in the text.

47. (Theorem 7.8)
Supply the details in the proof of Theorem 7.8 for case 1. In particular, give the details for back-substitution.

48. (Theorem 7.8)
Supply the details in the proof of Theorem 7.8 for case 2. In particular, give the details for back-substitution and explain fully why it is possible to select $y_0 = 0$.

49. (Theorem 7.8)
Supply the details in the proof of Theorem 7.8 for case 3. In particular, explain why back-substitution leaves y_0 and y_1 undetermined, and why it is possible to select $y_0 = y_1 = 0$.

50. (Superposition)
Let Ly denote $ay'' + by' + cy$. Show that solutions of $Lu = f(x)$ and $Lv = g(x)$ add to give $y = u + v$ as a solution of $Ly = f(x) + g(x)$.

51. (Easily Solved Equations)
Let Ly denote $ay'' + by' + cy$. Let $Ly_k = f_k(x)$ for $k = 1, \ldots, n$ and define $y = y_1 + \cdots + y_n$, $f = f_1 + \cdots + f_n$. Show that $Ly = f(x)$.

Chapter 8

Laplace Transform

Contents

8.2 Exercises

Laplace method

Solve the given initial value problem using Laplace's method.

1. $y' = -2$, $y(0) = 0$.

2. $y' = 1$, $y(0) = 0$.

3. $y' = -t$, $y(0) = 0$.

4. $y' = t$, $y(0) = 0$.

5. $y' = 1 - t$, $y(0) = 0$.

6. $y' = 1 + t$, $y(0) = 0$.

7. $y' = 3 - 2t$, $y(0) = 0$.

8. $y' = 3 + 2t$, $y(0) = 0$.

9. $y'' = -2$, $y(0) = y'(0) = 0$.

10. $y'' = 1$, $y(0) = y'(0) = 0$.

11. $y'' = 1 - t$, $y(0) = y'(0) = 0$.

12. $y'' = 1 + t$, $y(0) = y'(0) = 0$.

13. $y'' = 3 - 2t$, $y(0) = y'(0) = 0$.

14. $y'' = 3 + 2t$, $y(0) = y'(0) = 0$.

Exponential order

Show that $f(t)$ is of exponential order, by finding a constant $\alpha \geq 0$ in each case such that $\lim_{t\to\infty} \dfrac{f(t)}{e^{\alpha t}} = 0$.

15. $f(t) = 1 + t$

16. $f(t) = e^t \sin(t)$

17. $f(t) = \sum_{n=0}^{N} c_n t^n$, for any choice of the constants $c_0, \ldots, c_N$.

18. $f(t) = \sum_{n=1}^{N} c_n \sin(nt)$, for any choice of the constants $c_1, \ldots, c_N$.

Existence of transforms

Let $f(t) = te^{t^2} \sin(e^{t^2})$. Establish these results.

19. The function $f(t)$ is not of exponential order.

20. The Laplace integral of $f(t)$, $\int_0^\infty f(t)e^{-st}dt$, converges for all $s > 0$.

Jump Magnitude

For f piecewise continuous, define the **jump** at t by

$$J(t) = \lim_{h\to 0+} f(t+h) - \lim_{h\to 0+} f(t-h).$$

Compute $J(t)$ for the following f.

21. $f(t) = 1$ for $t \geq 0$, else $f(t) = 0$

22. $f(t) = 1$ for $t \geq 1/2$, else $f(t) = 0$

23. $f(t) = t/|t|$ for $t \neq 0$, $f(0) = 0$

24. $f(t) = \sin t/|\sin t|$ for $t \neq n\pi$, $f(n\pi) = (-1)^n$

Taylor series

The series relation $\mathcal{L}(\sum_{n=0}^{\infty} c_n t^n) = \sum_{n=0}^{\infty} c_n \mathcal{L}(t^n)$ often holds, in which case the result $\mathcal{L}(t^n) = n!s^{-1-n}$ can be employed to find a series representation of the Laplace transform. Use this idea on the following to find a series formula for $\mathcal{L}(f(t))$.

25. $f(t) = e^{2t} = \sum_{n=0}^{\infty}(2t)^n/n!$

26. $f(t) = e^{-t} = \sum_{n=0}^{\infty}(-t)^n/n!$

Transfer of Radiance

The differential equation $\frac{d}{dr}N + \alpha N = N^*$ models laser beam radiance (absorption and scattering out of the beam) in a medium like water, where r is the distance from the source.

27. Solve $\frac{d}{dr}N + 2N = 1, N(0) = 20$ by Laplace's method.
Ans: $N(r) = \frac{1}{2} + \frac{39}{2}\,\mathrm{e}^{-2\,r}$.
Hint: Obtain $\mathcal{L}(N(t)) = \frac{1+20\,s}{s(s+2)} = \frac{1}{2s} + \frac{39}{2(s+2)}$ using $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ from the Forward Table page **??**.

28. Solve $\frac{d}{dr}N + 2N = 1 - e^{-r}, N(0) = 25$ by any method.
Ans: $N(r) = \frac{1}{2} - \mathrm{e}^{-r} + \frac{51}{2}\,\mathrm{e}^{-2\,r}$.
Hint: A particular solution is $N_p = \frac{1}{2} - e^{-r}$. Superposition applies. See also Example 8.11, Section 8.3.

Piecewise-Defined Functions

29. Define a piecewise continuous function $f(t)$ on $[-1, 1]$ that agrees with $\frac{\sin(t)}{|t|}$ except at $t = 0$. Suggestion: use Taylor expansion $\sin(t) = t - t^3/6 + \cdots$ to define continuous functions f_1, f_2 on $-\infty < t < \infty$.

30. Explain in detail why $1/t$ is not piecewise continuous on $[-1, 1]$.

31. Find $\mathcal{L}(f(t))$, given
$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ 0 & \text{otherwise.} \end{cases}$

32. Find $\mathcal{L}(\mathbf{pulse}(t, a, b))$, given
$\mathbf{pulse}(t, a, b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise.} \end{cases}$

33. Define

$$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ 2 & 3 \le t < 4, \\ 0 & \text{otherwise.} \end{cases}$$

Find the weights c_1, c_2 such that
$f(t) = c_1\,\mathbf{pulse}(t, 1, 2) +$
$c_2\,\mathbf{pulse}(t, 3, 4)$.

34. Let
$$f(t) = \cos(t)\,\mathbf{pulse}(t, 0, \pi)+ \\ (\sin(t) - 1)\,\mathbf{pulse}(t, \pi, 2\pi)$$
Write f as a piecewise-defined function and graph it.

Piecewise Continuous Definition

Let $g(t)$ be zero for $t < 0$ and have on $t \geq 0$ at most finitely many points of discontinuity, at which finite right and left hand limits exist.

This definition is an alternative way to define *piecewise continuous*, crafted for Laplace theory.

35. Let t_1, t_2 be consecutive points of discontinuity of g. Define a function $g_1(t)$ continuous on $-\infty < t < \infty$ such that $g(t) = g_1(t)$ on $t_1 \leq t \leq t_2$.

The whole real line is the required domain of g_1, which must be defined using g itself and right and left hand limit values of g.

36. Let t_1, t_2, t_3 be consecutive points of discontinuity of g. Invent functions $g_1(t)$, $g_2(t)$ continuous on $-\infty < t < \infty$ such that $g(t) = g_1(t)$ on $t_1 \leq t \leq t_2$ and $g(t) = g_2(t)$ on $t_2 \leq t \leq t_3$.

37. Define g_1, g_2 as above. Compute the **jump** at $t = t_2$, $J(t_2) = g(t_2 + 0) - g(t_2 - 0)$ in terms of g_1, g_2.

38. Using the preceding steps, prove that g is piecewise continuous according to the definition given in the text.

8.3 Exercises

Laplace Transform Forward Table

Using the basic Laplace table and linearity properties of the transform, compute $\mathcal{L}(f(t))$. Do not use the direct Laplace transform!

1. $\mathcal{L}(2t)$

2. $\mathcal{L}(4t)$

3. $\mathcal{L}(1 + 2t + t^2)$

4. $\mathcal{L}(t^2 - 3t + 10)$

5. $\mathcal{L}(\sin 2t)$

6. $\mathcal{L}(\cos 2t)$

7. $\mathcal{L}(e^{2t})$

8. $\mathcal{L}(e^{-2t})$

9. $\mathcal{L}(t + \sin 2t)$

10. $\mathcal{L}(t - \cos 2t)$

11. $\mathcal{L}(t + e^{2t})$

12. $\mathcal{L}(t - 3e^{-2t})$

13. $\mathcal{L}((t + 1)^2)$

14. $\mathcal{L}((t + 2)^2)$

15. $\mathcal{L}(t(t + 1))$

16. $\mathcal{L}((t + 1)(t + 2))$

17. $\mathcal{L}(\sum_{n=0}^{10} t^n/n!)$

18. $\mathcal{L}(\sum_{n=0}^{10} t^{n+1}/n!)$

19. $\mathcal{L}(\sum_{n=1}^{10} \sin nt)$

20. $\mathcal{L}(\sum_{n=0}^{10} \cos nt)$

Laplace Backward Table

Solve the given equation for the function $f(t)$. Use the basic table and linearity properties of the Laplace transform.

21. $\mathcal{L}(f(t)) = s^{-2}$

22. $\mathcal{L}(f(t)) = 4s^{-2}$

23. $\mathcal{L}(f(t)) = 1/s + 2/s^2 + 3/s^3$

24. $\mathcal{L}(f(t)) = 1/s^3 + 1/s$

25. $\mathcal{L}(f(t)) = 2/(s^2 + 4)$

26. $\mathcal{L}(f(t)) = s/(s^2 + 4)$

27. $\mathcal{L}(f(t)) = 1/(s - 3)$

28. $\mathcal{L}(f(t)) = 1/(s + 3)$

29. $\mathcal{L}(f(t)) = 1/s + s/(s^2 + 4)$

30. $\mathcal{L}(f(t)) = 2/s - 2/(s^2 + 4)$

31. $\mathcal{L}(f(t)) = 1/s + 1/(s - 3)$

32. $\mathcal{L}(f(t)) = 1/s - 3/(s - 2)$

33. $\mathcal{L}(f(t)) = (2 + s)^2/s^3$

34. $\mathcal{L}(f(t)) = (s + 1)/s^2$

35. $\mathcal{L}(f(t)) = s(1/s^2 + 2/s^3)$

36. $\mathcal{L}(f(t)) = (s + 1)(s - 1)/s^3$

37. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{1+n}$

38. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{2+n}$

39. $\mathcal{L}(f(t)) = \sum_{n=1}^{10} \dfrac{n}{s^2 + n^2}$

40. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} \dfrac{s}{s^2 + n^2}$

Laplace Table Extension

Compute the indicated Laplace integral using the extended Laplace table, page **??**.

41. $\mathcal{L}(u(t - 2) + 2u(t))$

42. $\mathcal{L}(u(t - 3) + 4u(t))$

43. $\mathcal{L}(u(t-\pi)(u(t)+u(t-1)))$

44. $\mathcal{L}(u(t-2\pi)+3u(t-1)u(t-2))$

45. $\mathcal{L}(\delta(t-2))$

46. $\mathcal{L}(5\delta(t-\pi))$

47. $\mathcal{L}(\delta(t-1)+2\delta(t-2))$

48. $\mathcal{L}(\delta(t-2)(5+u(t-1)))$

49. $\mathcal{L}(\textbf{floor}(3t))$

50. $\mathcal{L}(\textbf{floor}(2t))$

51. $\mathcal{L}(5\,\textbf{sqw}(3t))$

52. $\mathcal{L}(3\,\textbf{sqw}(t/4))$

53. $\mathcal{L}(4\,\textbf{trw}(2t))$

54. $\mathcal{L}(5\,\textbf{trw}(t/2))$

55. $\mathcal{L}(t+t^{-3/2}+t^{-1/2})$

56. $\mathcal{L}(t^3+t^{-3/2}+2t^{-1/2})$

Inverse Laplace, Extended Table

Solve the given equation for $f(t)$, using the extended Laplace integral table.

57. $\mathcal{L}(f(t))=e^{-s}/s$

58. $\mathcal{L}(f(t))=5e^{-2s}/s$

59. $\mathcal{L}(f(t))=e^{-2s}$

60. $\mathcal{L}(f(t))=5e^{-3s}$

61. $\mathcal{L}(f(t))=\dfrac{e^{-s/3}}{s(1-e^{-s/3})}$

62. $\mathcal{L}(f(t))=\dfrac{e{-}2s}{s(1-e^{-2s})}$

63. $\mathcal{L}(f(t))=\dfrac{4\tanh(s)}{s}$

64. $\mathcal{L}(f(t))=\dfrac{5\tanh(3s)}{2s}$

65. $\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{3s^2}$

66. $\mathcal{L}(f(t)) = \dfrac{5\tanh(2s)}{11s^2}$

67. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s}}$

68. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s^3}}$

8.4 Exercises

First Order Linear DE

Display the Laplace method details which verify the supplied answer.

The first two exercises use forward and backward Laplace tables plus the first shifting theorems. The others require a calculus background in partial fractions.

1. $x' + x = e^{-t}$, $x(0) = 1$;
$x(t) = (1 + t)e^{-t}$.

2. $x' + 2x = -e^{-2t}$, $x(0) = 1$;
$x(t) = (1 - t)e^{-2t}$.

3. $x' + x = 1$, $x(0) = 1$; $x(t) = 1$.

4. $x' + 4x = 4$, $x(0) = 1$; $x(t) = 1$.

5. $x' + x = t$, $x(0) = -1$; $x(t) = t - 1$.

6. $x' + x = t$, $x(0) = 1$;
$x(t) = t - 1 + 2e^{-t}$.

Second Order Linear DE

Display the Laplace method details which verify the supplied answer.

The first 4 exercises require only forward and backward Laplace tables and the first shifting theorems. The others require methods in partial fractions beyond a calculus background.

7. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 1$; $x(t) = \cos t + \sin t$.

8. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 2$; $x(t) = \cos t + 2\sin t$.

9. $x'' + 2x' + x = 0$, $x(0) = 0$, $x'(0) = 1$; $x(t) = te^{-t}$.

10. $x'' + 2x' + x = 0$, $x(0) = 1$, $x'(0) = -1$; $x(t) = e^{-t}$.

11. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$; $x(t) = e^{-t}$.

12. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -2$; $x(t) = e^{-2t}$.

13. $x'' + 3x' = 0$, $x(0) = 5$, $x'(0) = 0$; $x(t) = 5$.

14. $x'' + 3x' = 0$, $x(0) = 1$, $x'(0) = -3$; $x(t) = e^{-3t}$.

15. $x'' + x = 1$, $x(0) = 1$, $x'(0) = 0$; $x(t) = 1$.

16. $x'' = 2$, $x(0) = 0$, $x'(0) = 0$; $x(t) = t^2$.

Forward Integral Rule

The rule is $\mathcal{L}\left(\int_0^t g(r)dr\right) = \frac{1}{s}\mathcal{L}(g(t))$

17. Relate this rule to the convolution rule with $f(t) = 1$.

18. Compute $\mathcal{L}\left(\int_0^t \sin(r)dr\right)$.

19. Compute $\mathcal{L}\left(\int_0^t (r+1)^3\,dr\right)$.

20. Compute $\mathcal{L}\left(\int_0^t \textsf{\textbf{sqw}}(r)dr\right)$, where **sqw** is the square wave of period 2. Use the Extended Laplace Table.

Backward Integral Rule

Apply rule $\frac{1}{s}\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t g(r)dr\right)$
and Lerch's theorem to solve for $f(t)$.

21. $\mathcal{L}(f(t)) = \frac{1}{s(s^2+1)}$

22. $\mathcal{L}(f(t)) = \frac{1}{s}\frac{s+1}{s^2+1}$

23. $\mathcal{L}(f(t)) = \frac{1}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$

24. $\mathcal{L}(f(t)) = \frac{1}{s}\frac{e^{-s}}{s}$
Hint: $\mathcal{L}(u(t-a)) = \frac{1}{s}e^{-as}$.

The s–Integral Rule

Identity $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \mathcal{L}(f(t))\,ds$
requires piecewise continuous $f(t)$ of exponential order with $\lim_{t\to 0+}\frac{f(t)}{t} = L$.

25. Prove the identity.

26. Compute $\mathcal{L}\left(\frac{\sin(t)}{t}\right)$.

Forward First Shifting Rule

Apply $\mathcal{L}(f(t)e^{at}) = \mathcal{L}(f(t))|_{s\to s-a}$ to find the Laplace transform.

27. $\mathcal{L}(te^t)$

28. $\mathcal{L}(te^t + e^{2t})$

29. $\mathcal{L}(\sin(t)e^t)$

30. $\mathcal{L}(\sin(2t)e^{2t} + \cos(t)e^{t})$

31. $\mathcal{L}(t\cosh(2t))$ using identity
$\cosh(w) = \frac{1}{2}e^{w} + \frac{1}{2}e^{-w}$.

32. $\mathcal{L}((t+1)^3\, e^t)$

Backward First Shifting Rule

Apply $\mathcal{L}(f(t))|_{s\to s-a} = \mathcal{L}(f(t)e^{at})$ and Lerch's theorem to solve for $f(t)$.

33. Explain for $\mathcal{L}(t^2)\big|_{s\to s-4}$ the rule
Erase a shift $|_{s\to s-a}$ *by inserting* e^{at} *inside the scope of* $\mathcal{L}$.

34. $\mathcal{L}(f(t)) = \left.\frac{s}{s^2+1}\right|_{s\to s-1}$

35. $\mathcal{L}(f(t)) = \frac{s-1}{(s-1)^2+4}$

36. $\mathcal{L}(f(t)) = \frac{8}{(s+1)^2+4}$

37. $\mathcal{L}(f(t)) = \frac{s+1}{s^2+2s+5}$

38. $\mathcal{L}(f(t)) = \frac{4}{s^2+8s+17}$

39. $\mathcal{L}(f(t)) = \frac{2}{(s+1)^2}$

40. $\mathcal{L}(f(t)) = \frac{1}{(s+2)^{101}}$

Forward s-Differentiation

Apply $\mathcal{L}((-t)f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$ to find the Laplace transform.

41. Explain for $\mathcal{L}((-t)\cos(t))$ the rule
Multiplying by $(-t)$ *differentiates the Laplace transform..*

42. $\mathcal{L}((-t)\sin(2t))$

43. $\mathcal{L}((-t)\sinh(2t))$, using identity
$\sinh(w) = \frac{1}{2}e^{w} - \frac{1}{2}e^{-w}$.

44. $\mathcal{L}(te^t\sin(2t) + te^{2t}\cos(t))$

Backward s-Differentiation

Apply $\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}((-t)f(t))$ and Lerch's theorem to solve for $f(t)$.

45. Explain for $\frac{d}{ds}\mathcal{L}(\cos(t))$ the rule
Erase $\frac{d}{ds}$ *by inserting factor* $(-t)$ *inside the scope of* $\mathcal{L}$.

46. $\mathcal{L}(f(t)) = \frac{d}{ds}\frac{s}{s^2+4}$

47. $\mathcal{L}(f(t)) = \frac{d^2}{ds^2}\frac{1}{(s+1)^5}$

48. $\mathcal{L}(f(t)) = \frac{d^3}{ds^3}\frac{s+1}{s^2+2s+5}$

Unit Step and Pulse

Define

$$\textbf{pulse}(t,a,b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{else,} \end{cases}$$

which is a tool for encoding and decoding piecewise-defined functions.

49. Prove the identity
$\textbf{pulse}(t,a,b){=}u(t-a) - u(t-b)$,
where u is the **unit step**.

50. Prove the Laplace formula
$\mathcal{L}(\textbf{pulse}(t,a,b)){=}\frac{e^{-at}-e^{-bt}}{s}$

51. Verify that $f(t)$ defined by
$\begin{cases} 2 & 1 \le t < 2, \\ 0 & \text{else} \end{cases} + \begin{cases} 3 & 3 \le t < 4, \\ 0 & \text{else} \end{cases}$
encodes to representation
$2\,\textbf{pulse}(t,1,2){+}3\,\textbf{pulse}(t,3,4)$.

52. Decode $f(t)$ into a piecewise–defined function and graph it by hand, no computer, given $f(t)$ is
$e^t\,\textbf{pulse}(t,1,3){+}e^{-t}\,\textbf{pulse}(t,4,6)$

53. Decode $f(t)$ into a piecewise–defined function and graph it, no computer, given $f(t)$ is the sum
$\sum_{n=1}^{3} |\sin(n\pi t)|\,\textbf{pulse}(t,2n,2n{+}1)$

54. Encode as a combination of pulses
$$f(t){=}\begin{cases} 1 & 1 \le t < 2, \\ -2 & 3 \le t < 4, \\ 1 & 5 \le t < 6, \\ 0 & \text{else,} \end{cases}$$
showing all encoding details. Ans: $f(t){=}\,\textbf{pulse}(t,1,2){-}2\,\textbf{pulse}(t,3,4)$
$+\,\textbf{pulse}(t,5,6)$.

Alternate Second Shifting Rule

$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(w)|_{w=t+a}\right)$. No Laplace here. The focus is on function notation and finding $g(t+a) = g(w)|_{w=t+a}$, which means *substitute* $w = t + a$ *into the* $g(w)$*–formula.*

55. Let $g(t) = te^{-t}$. Verify identity $g(w)|_{w=t+2} = e^{-2}(te^{-t} + 2e^{-t})$.

56. Let $g(t) = t^3$. Verify identity $g(w)|_{w=t+2} = 8 + 12t + 6t^2 + t^3$.

57. Typical polynomial $g(w) = 1 + 2w^2 + 3w^4$ upon substitution $w = t + a$ requires expansions for $(t+a)^2$ and $(t+a)^4$. Pascal's Triangle can be useful. Find the answer for $g(t + a) = g(w)|_{w=t+a}$.

58. Polynomial $1+2w^2+3w^4$ upon substitution $w = t-b$ is a Taylor polynomial expansion
$f(t) = \sum_{n=0}^{4} \frac{f^{(n)}(b)}{n!} (t - b)^n$.
Find the MaClaurin expansion
$f(t) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} t^n$.

Forward Second Shifting Rule

$\mathcal{L}(g(t)u(t - a)) = e^{-as}\mathcal{L}(g(t + a))$
Find $\mathcal{L}(f(t))$, where u is the unit step.

59. $f(t) = u(t - \pi)$

60. $f(t) = e^t\, u(t - 1)$

61. $f(t) = t^3u(t - \pi)$

62. $f(t) = e^t\,\mathbf{pulse}(t, 1, 2)$, where
$\mathbf{pulse}(t, a, b){=}u(t - a){-}u(t - b)$.

63. $f(t) = te^tu(t - 2)$

64. $f(t) = t\sin(t)u(t - \pi)$

Backward Second Shifting Rule

$e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t - a)u(t - a))$
Find $f(t)$ using the rule and Lerch's theorem, giving both a piecewise–defined display and a unit step or pulse formula.

65. $\mathcal{L}(f(t)) = \frac{1}{s}e^{-3s}$
Ans: $f(t){=}u(t - 3){=}\begin{cases} 1 & t \ge 3, \\ 0 & \text{else,} \end{cases}$

66. $\mathcal{L}(f(t)) = \dfrac{1}{s^2}e^{3-3s}$

67. $\mathcal{L}(f(t)) = \dfrac{4}{s^2 + 8s + 17}e^{-2s}$

68. $\mathcal{L}(f(t)) = \dfrac{4 + s}{s^2 + 8s + 17}e^{-3s}$

69. $\mathcal{L}(f(t)) = \left(\dfrac{1}{s^2}+\dfrac{2}{s^3}\right) e^{-2s}$

70. $\mathcal{L}(f(t)) = \dfrac{1}{(s-4)^2}\, e^{-2s}$

Trigonometric Formulas

Supply the details in Example 8.21.

71. $\mathcal{L}(t\sin at) = \dfrac{2sa}{(s^2+a^2)^2}$

72. $\mathcal{L}(t^2\sin at) = \dfrac{6s^2a-a^3}{(s^2+a^2)^3}$

Exponential Formulas

Supply the details in Example 8.22.

73. $\mathcal{L}(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$

74. $\mathcal{L}(te^{at}\sin bt) = \dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

Hyperbolic Functions

Supply the details in Example 8.23.

75. $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

76. $\mathcal{L}(t\cosh at) = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

Waves

Use Laplace ideas from Examples **??** and 8.25. Each $f(t)$ can be expressed as a **pulse train**, which is an expression $\sum_{n=1}^{\infty} f_n(t)\,\mathbf{pulse}(t,a_i,b_i)$ to which the second shifting theorem applies.

77. Find $\mathcal{L}(f(t))$ for the square wave
$f(t)=\sum_{n=0}^{\infty}(-1)^n\,\mathbf{pulse}(t,n,n+1)$

78. Define pulse train
$f(t)=\sum_{n=0}^{\infty} f_n(t)\,\mathbf{pulse}(t,n,n+1)$,
$f_{2n}(t)=t-2n$, $f_{2n+1}(t)=2-t+2n$. Show that $f(t+2)=f(t)$ and
$f(t)=\begin{cases} t & 0\le t<1, \\ 2-t & 1\le t\le 2. \end{cases}$

79. Find $\mathcal{L}(f(t))$ for
$f(t) = \begin{cases} |\sin(2t)| & 0 \le t \le \pi, \\ 0 & \pi \le t \le 2\pi, \end{cases}$
and $f(t + 2\pi) = f(t)$.

80. Find $\mathcal{L}(f(t))$ for
$f(t) = \begin{cases} 1 & 0 \le t \le \pi, \\ |\sin(t)| & \pi \le t \le 2\pi, \end{cases}$
and $f(t + 2\pi) = f(t)$.

81. Given $f(t) = \frac{1}{2}(|\sin t| + \sin t)$, called the **Half–wave rectification** of the sine wave, derive $\mathcal{L}(f(t)) = \frac{1}{s^2+1} \coth\left(\frac{\pi s}{2}\right)$

82. Solve for 2–periodic function $f(t)$:
$\mathcal{L}(f(t)) = \dfrac{1}{s} \tanh\left(\dfrac{s}{2}\right)$.
Ans: $\mathbf{pulse}(t,0,1) - \mathbf{pulse}(t,1,2),\ 0 \le t \le 2$. Use the extended Laplace table.

8.5 Exercises

Partial Fraction Mistakes

1. How many real constants appear in the partial fraction expansion of the fraction $\dfrac{s+1}{s^2(s+2)(s+3)^2}$?

2. How many real constants appear in the partial fraction expansion of $\dfrac{s+1}{s^2(s^2+4)(s^2+2s+5)^2}$?

3. Guido expanded $\dfrac{s+1}{s(s+2)(s+3)^2}$
to get $\dfrac{a}{s}+\dfrac{b}{s+2}+\dfrac{c}{(s+3)^2}$.
What is the mistake?

4. Helena made this expansion:
$\dfrac{s+1}{s(s+2)}=\dfrac{a}{s}+\dfrac{b}{s+2}+\dfrac{c}{s+3}$
The expansion is correct! Explain how you know that $c = 0$ without computing anything.

This example explains why fractions on the right have denominators dividing the denominator on the left.

5. Marco made an expansion:
$\dfrac{s+1}{s(s^2+4)}=\dfrac{a}{s}+\dfrac{b}{s+2}+\dfrac{c}{s-2}$
Explain why it is a mistake.

This example explains why sanity checks have more than one item to check.

6. Violeta made an expansion
$\dfrac{s+2}{s(s-2)(s+2)}=\dfrac{a}{s}+\dfrac{b}{s-2}+\dfrac{c}{s+2}$
Explain why $c = 0$ without computing anything.

This example explains why common factors of numerator and denominator should be removed.

7. Find the mistake in expansion
$\dfrac{(s+2)^2}{s(s-2)}=\dfrac{a}{s}+\dfrac{b}{s-2}$
This example explains why the degree of the numerator and denominator are checkpoints.

8. Is there a mistake here?
$$\frac{(s+2)^2}{s^2(s-2)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s-2}$$

Sampling Method

Apply the sampling method (a *failsafe method*) to verify the given equation.

9. $\dfrac{s}{s^2-1} = \dfrac{1/2}{s-1} + \dfrac{1/2}{s+1}$

10. $\dfrac{s}{s^4-1} = \dfrac{1/4}{s-1} + \dfrac{1/4}{s+1} + \dfrac{-s/2}{s^2+1}$

Method of Atoms

Apply the method of atoms to verify the given equation.

11. $\dfrac{2s}{s^2-1} = \dfrac{1}{s-1} + \dfrac{1}{s+1}$

12. $\dfrac{4s}{s^4-1} = \dfrac{1}{s-1} + \dfrac{1}{s+1} + \dfrac{-2s}{s^2+1}$

Heaviside's 1890 Shortcut

Apply Heaviside's shortcut to verify the given equation.

13. $\dfrac{2s}{s^2-4} = \dfrac{1}{s-2} + \dfrac{1}{s+2}$

14. $\dfrac{s+4}{s^3+4s} = \dfrac{1}{s} + \dfrac{-s+1}{s^2+4}$

Residues and Poles

Compute the residue for the given pole.

15. Residue at $s = 2$ for $\dfrac{2s}{s^2-4}$.

16. Residue at $s = 0$ for $\dfrac{s+4}{s^3+16s}$.

Scalar Differential Equations

The **transfer function** of $x'' + x = f(t)$ is $H(s) = \frac{1}{s^2+1}$. A common definition is $H(s) = \mathcal{L}(f(t))$ divided by $\mathcal{L}(x(t))$, assuming $x(0) = x'(0) = 0$.

17. Verify for $x'' + x = e^{-t}$ with $x(0) = 0$, $x'(0) = 0$ that $\mathcal{L}(x) = \frac{1}{s+1}\,\frac{1}{s^2+1}$. Then compute $H(s)$.

18. Explain the transfer function equation
$H(s) = \frac{1}{\text{characteristic equation}}$.

19. Solve $\mathcal{L}(x(t)){=}\frac{1}{s+1}\,\frac{1}{s^2+1}$ by Heaviside cover–up for output $x(t) = \frac{1}{2}(e^{-t} - \cos t + \sin t)$.

20. Given $x'' + x = te^{-t}$, $x(0) = x'(0) = 0$, show all steps to find
$\mathcal{L}(x(t)) = \frac{1}{(s+1)^2}\,\frac{1}{s^2+1}$.

First Order System

Using Example 8.29 as a guide, solve the system for $x_1(t)$ by Laplace's method.

21. $\begin{cases} x_1'{=}x_2, \\ x_2'{=}4x_1 + 12e^{-t}, \\ x_1(0){=}x_2(0){=}0. \end{cases}$
Ans: $x_1(t){=}e^{2t} + 3e^{-2t} - 4e^{-t}$.

22. $\begin{cases} x_1'{=}x_2, \\ x_2'{=}x_3, \\ x_3'{=}4x_1 - 4x_2 + x_3 + 10e^{-t}, \\ x_1(0){=}x_2(0){=}x_3(0){=}0. \end{cases}$
Ans: $x_1(t){=}e^{t} - e^{-t} - \sin(2t)$.

Second Order System

Using Example 8.29 as a guide, compute $x(t), y(t)$.

23. $\mathcal{L}(x(t)){=}\frac{3s^2+2}{(s-1)(s^2+4)}$,
$\mathcal{L}(y(t)){=}\frac{10}{(s-1)(s^2+4)}$.
Ans: $x{=}2\cos(2t){+}\sin(2t){+}e^t$,
$y{=}-2\cos(2t){-}\sin(2t){+}2e^t$

24. $\mathcal{L}(x(t)){=}\frac{2s^2+4}{(s+1)(s^2+1)}$,
$\mathcal{L}(y(t)){=}\frac{2}{(s+1)(s^2+1)}$.
Ans: $x{=}-\cos(t){+}\sin(t){+}3e^{-t}$,
$y{=}-\cos(t){+}\sin(t){+}e^{-t}$.

8.6 Exercises

Unit Step and Heaviside

1. The unit step $\mathbf{u}(t)$ is defined on the whole real line. Is it piecewise continuous on the whole line?

2. Is there a continuous function on the real line that agrees with the Heaviside function except at $t = 0$?

3. The piecewise continuous function $\mathbf{pulse}(t, a, b)$ is defined everywhere. Redefine $\mathbf{pulse}(t, a, b)$ using $H(t)$ instead of $\mathbf{u}(t)$.

4. Write $f(t) = \mathsf{floor}(t)\,\mathbf{u}(t)$ as a sum of terms, each of which has the form $g(t)\,\mathbf{pulse}(t, a, b)$.

Dirac Impulse

5. Verify $\int_{-\infty}^{\infty} \frac{\mathbf{pulse}(t,a,b)}{b-a}\,dt = 1$.

6. Verify by direct integration that $f(t) = 10\,\mathbf{pulse}(t, -0.001, 0.001)$ represents a simple impulse of 10 at $t = 0$ of duration 0.002. Graph it without using technology.

7. Find $\mathcal{L}(\delta(t-1) + \delta(t-2))$.

8. Find $\mathcal{L}(10\,\delta(t-1) - 5\,\delta(t-2))$.

9. Solve for $f(t)$ in terms of δ:
$\mathcal{L}(f(t)) = 10e^{-s}$

10. Solve for $f(t)$ in terms of δ:
$\mathcal{L}(f(t)) = 10e^{-s} + \frac{s}{s^2+1}\,e^{-2s}$

11. Find $\mathcal{L}\left(\sum_{i=1}^{10}(1+i)\delta(t-i)\right)$.

12. A sequence of camshaft impulses happening periodically in a finite time interval have transform $\mathcal{L}(f(t)) = \sum_{i=1}^{N} e^{-c_i\,s}$. Find the idealized impulse train f.

Riemann–Stieltjes Integral

Evaluate the integrals either directly from the definition or else by using Theorem 8.15.

13. $\int_0^2 d\,\mathbf{u}(t-1)$

14. $\int_0^\infty d\,\mathbf{u}(t-2)$

15. $\int_0^2 \tanh(t^2+1)\,d\,\mathbf{u}(t-1)$

16. $\int_0^\infty \frac{t}{1+t^2}\,d\,\mathbf{u}(t-2)$

8.7 Exercises

Oscillatory and Non–oscillatory

Assume $x'' + px' + qx = 0$ with p, q nonnegative.

Parameter p is imagined as a set screw adjustment on a screen door dashpot, larger p meaning more damping effect.

Parameter q is the Hooke's constant for the spring restoring force.

1. Let $q = 100$, $p = 99$. Verify that the equation is over–damped in two ways:
(1) Graph $x(t)$;
(2) Justify that $r^2 + pr + q = 0$ has real negative roots.

2. Let $q = 100$. The case which is called *critically–damped* happens at exactly one value $p = p^*$ between 0 and 99. Compute p^* numerically. Graph $x(t)$ using $q = 100$, $p = p^*$, $x(0) = 0$, $x'(0) = 1$.

3. Let $q = 100$. Verify that $p = 0$ produces the **harmonic oscillator** $x'' + \omega^2 x = 0$, $\omega = 10$.

Small set screw changes from $p = 0$ to $p > 0$ are still oscillatory. Under–damped means weak dashpot reaction.

4. Let $q = 100$, $p = 2$. Justify oscillatory under–damped from the graph of $x(t)$ and also by solving $r^2 + pr + q = 0$.

Simplistic Dirac Impulse

Define $g(t) = 7\,e^{-153800\,t}\,\mathbf{u}(t)$ and
$f(t, a) = \frac{1}{a}\,(u(t) - u(t - a))$, $a > 0$.
The impulse of force h is $\int_{-\infty}^{\infty} h(t)\,dt$.

5. Compute the impulse for $f(t, a)$.
Ans: 1.

6. Plot $f(t, a)$ for $a = 0.1$, 0.001, 0.0001.

7. Calculate the impulse for $g(t)$.
Ans: About 45.4 times 10^{-6}.

8. Try to find an **RC** discharge circuit with 10 volt *emf* and output $g(t)$.

Circuit response $g(t)$ simulates Dirac impulsive force $\frac{45.5}{1000000}\delta(t)$.

Parameters: Over–Damped

Find $a, b, \omega = \sqrt{ab}, \zeta = \frac{a+b}{2\omega}$ given the plot and two dots on the graph.

9. Step input Figure 9, dots
$(1, 0.1998)$, $(4, 0.4819)$.
Ans: $a = 1.0000$, $b = 1.9997$, $\omega = 1.4141$, $\zeta = 1.0607$.

10. Impulse input Figure 10, dots
$(0.5, 0.1193)$, $(2, 0.0585)$.
Ans: $a = 0.9991$, $b = 2.0021$, $\omega = 1.4143$, $\zeta = 1.0610$.

Parameters: Under–Damped

Find $a, b, \omega = \sqrt{a^2 + b^2}, \zeta = \frac{a}{\omega}$ given the plot and two dots on the graph.

11. Zero input like Figure 11, but consecutive maxima at $(2.5107, 0.0257)$, $(4.6051, 0.0032)$.
Ans: Approximately $a = 1$, $b = 3$.

12. Step input like Figure 13, but steady–state $y_0 = 1/26$ and consecutive maxima at $(0.6283, 0.0205)$, $(1.8850, 0.0058)$.
Ans: Approximately $a = 1$, $b = 5$.

Chapter 9

Eigenanalysis

Contents

9.2 Exercises

Eigenanalysis

Classify as true or false. If false, then explain.

1. The purpose of eigenanalysis is to discover a new coordinate system.

2. Eigenanalysis can discover an opportunistic change of coordinates.

3. A matrix can have eigenvalue 0.

4. Eigenvalues are scale factors, imagined to be measurement units.

5. Eigenvectors are directions.

6. For each eigenvalue of a matrix A, there always exists at least one eigenpair.

7. If A^{-1} has eigenvalue λ, then A has eigenvalue $1/\lambda$.

8. Eigenvectors cannot be $\vec{\mathbf{0}}$.

9. The transpose of A has the same eigenvalues as A.

10. Eigenpairs $(\lambda, \vec{\mathbf{v}})$ of A satisfy the equation $(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Eigenpairs of a Diagonal Matrix

Find eigenpairs of A without computation. Use Theorem 9.7.

11. $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

13. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

14. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

15. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

16. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Fourier Replacement

Let symbols c_1, c_2 represent arbitrary constants. Let 2×2 matrix A have Fourier replacement equation

$$A\left(c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = 2c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 5c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

17. Display the eigenpairs of A.

18. Display the replacement equation if the eigenvalues $2, -5$ are replaced by $1, 0$.

19. Display the eigenpair packages P, D such that $AP = PD$.

20. Find A.

Eigenanalysis Facts

Mark as true or false, then explain your answer.

21. If matrix A has all eigenvalues zero, then A is the zero matrix.

22. If 2×2 matrix A has all eigenvalues zero, then Fourier's replacement equation is

$$A\,(c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2) = \vec{\mathbf{0}}.$$

23. There are infinitely many 2×2 matrices A with complex eigenvalues $1 + i, 1 - i$.

24. A real 2×2 matrix A with eigenvalues $2 + 3i, 2 - 3i$ cannot have a real eigenvector.

25. A real 2×2 matrix A with real eigenvalues has only real eigenvectors.

Eigenpair Packages and equation $AP = PD$

26. Suppose A has eigenpair packages. Explain why there are so many different answers for P, D.

27. Suppose $AP = PD$ and $AQ = QD$ hold (same diagonal matrix D). Does $P = Q$?

28. Find one choice of P and D for $A = 2 \times 2$ diagonal matrix.

29. Given $A = 3 \times 3$ zero matrix, find one choice of P and D with column one of P equal to $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Matrix Eigenanalysis Method

30. The eigenvalues of $\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ satisfy a quadratic equation. Find the equation and solve for the eigenvalues.

31. Find the eigenvalues of $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

32. Find all eigenpairs of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

33. A triangular $n \times n$ matrix with distinct diagonal entries has n eigenpairs. Provide a detailed proof for the case $n = 3$.

34. Find all eigenpairs of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

35. A triangular $n \times n$ matrix may not have n eigenpairs. Provide a series of examples for dimensions $n = 2, 3, 4, 5$.

36. Prove that equations $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ and $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have exactly the same solutions $\vec{\mathbf{x}}$.

37. Cite basic linear algebra theorems to prove that $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a nonzero solution $\vec{\mathbf{x}}$ if and only if λ is a root of the characteristic equation $|A - \lambda I| = 0$.

Basis of Eigenvectors

The problem $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ has a standard general solution $\vec{x}$ with invented symbols $t_1, t_2, t_3, \ldots$. **Strang's special solutions** are defined to be the vector partial derivatives of $\vec{x}$ with respect to the invented symbols.

38. Why are Strang's special solutions independent?

39. Prove that linear combinations of Strang's special solutions provide all possible solutions of $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

Independence of Eigenvectors

Eigenvectors of matrix A for eigenvalue λ are the nonzero solutions of $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

40. Invent a 2×2 example A with eigenpairs $\big(2, (1, 1)\big)$, $\big(2, (5, 5)\big)$. Then explain why eigenvectors for eigenvalue λ are not unique.

41. Explain: *For a given eigenvalue λ, there are infinitely many eigenvectors.*

42. Explain: *Each solution $\vec{\mathbf{x}}$ of $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ is a linear combination of Strang's special solutions for $B = A - \lambda I$.*

Eigenspaces

Let $\mathcal{B}(\lambda)$ denote some basis of eigenvectors for the eigenpair equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$. The **eigenspace** for λ is the subspace $\mathbf{span}(\mathcal{B}(\lambda))$.

43. Explain: The eigenspace of λ does not depend on the choice of basis.

44. Every nonzero vector in eigenspace $\mathbf{span}(\mathcal{B}(\lambda))$ is an eigenvector of A for eigenvalue λ. Provide details of proof.

45. Justify that $\mathbf{span}(\mathcal{B}(\lambda))$ is a vector subspace of $\mathcal{R}^n$ with one possible basis being Strang's special solutions for $B = A - \lambda I$.

46. Find a 4×4 matrix A with only one eigenvalue $\lambda = 1$ such that eigenspace $\mathcal{B}(\lambda)$ (defined above) has dimension two.

47. Find a 10×10 matrix A with only one eigenvalue $\lambda = 1$ such that eigenspace $\mathcal{B}(\lambda)$ (defined above) has dimension two.

Independence of Unions of Eigenvectors

Denote by $\mathcal{B}(\lambda)$ some basis for the eigenpair equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$.

48. Define U_1 to be the union of lists $\mathcal{B}(\lambda_1)$, $\mathcal{B}(\lambda_2)$ and define U_2 to be the union of lists $\mathcal{B}(\lambda_3)$, $\mathcal{B}(\lambda_4)$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is a list of distinct eigenvalues of A. Prove that subspaces $V_1 = \mathbf{span}(U_1)$ and $V_2 = \mathbf{span}(U_2)$ intersect in only the zero vector.

49. Complete the details of the induction proof of Theorem 9.5, using the textbook details for $k = 3$.

50. Let U^* be a subset of the list U of independent vectors in Theorem 9.5. Explain why U^* is an independent set.

51. Let B_i be a subset of the list of independent vectors in $\mathcal{B}(\lambda_i)$, $i = 1, \ldots, p$. Explain why the union U^* of $B_1, \ldots, B_p$ is an independent set.

Diagonalization Theory

52. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

(a) Find Strang's special solutions for each eigenvalue.
(b) Compare to Theorem 9.7.

53. Let $\vec{\mathbf{v}}_!, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ be independent vectors in $\mathcal{R}^3$. Explain why $(0, \vec{\mathbf{v}}_1)$, $(0, \vec{\mathbf{v}}_2)$, $(0, \vec{\mathbf{v}}_3)$ is a complete set of eigenpairs for the 3×3 zero matrix. Does this contradict Theorem 9.7?

54. Write a proof of Theorem 9.7.

55. State Theorem 9.7 for $n \times n$ diagonal matrices and outline a proof.

Non-diagonalizable Matrices

Verify that the matrix is not diagonalizable by using the equation $AP = PD$.

56. $A = \begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix}$

57. $A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$

Distinct Eigenvalues

Find the eigenvalues.

58. $A = \begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix}$ Ans: $8, -3$

59. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ Ans: $0, 5$

60. $A = \begin{pmatrix} 2 & 6 & 2 \\ 9 & 3 & 9 \\ 1 & 3 & 1 \end{pmatrix}$ Ans: $0, 12, -6$

61. $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$ Ans: $0, 1, 3$

62. $A = \begin{pmatrix} 7 & 12 & 6 \\ 2 & 2 & 2 \\ -7 & -12 & -6 \end{pmatrix}$ Ans: $0, 1, 2$

63. $A = \begin{pmatrix} 2 & 2 & -6 \\ -3 & -4 & 3 \\ -3 & -4 & -1 \end{pmatrix}$ Ans: $0, 1, 4$

Computing 2×2 Eigenpairs

64. Verify eigenpairs: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$,
$\left(-1, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$, $\left(5, \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}\right)$

65. Verify eigenpairs: $\begin{pmatrix} 1 & 6 \\ 2 & -3 \end{pmatrix}$,
$\left(-5, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$

66. Verify eigenpairs: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$,
$\left(5, \frac{1}{2}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$, $\left(-1, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$

67. Verify eigenpairs: $\begin{pmatrix} 7 & 4 \\ -1 & 3 \end{pmatrix}$,
$\left(5, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right)$, only one eigenpair

Computing 2×2 Complex Eigenpairs

68. Verify eigenpairs: $\begin{pmatrix} -2 & -6 \\ 3 & 4 \end{pmatrix}$,
$\left(1+3i, \begin{pmatrix} -1+i \\ 1 \end{pmatrix}\right)$, $\left(1-3i, \begin{pmatrix} -1-i \\ 1 \end{pmatrix}\right)$

69. Verify eigenpairs: $\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$,
$\left(2+3i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$, $\left(2-3i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

70. Let a, b be real with $b \neq 0$. Assume $n \times n$ real matrix A has eigenpair $(a+ib, \vec{\mathbf{v}})$. Replace i by $-i$ throughout expression $\vec{\mathbf{v}}$ to obtain vector $\vec{\mathbf{w}}$. Prove that $(a-ib, \vec{\mathbf{w}})$ is an eigenpair.

71. Explain: Eigenpairs of a 2×2 real matrix A with complex eigenvalues are computed with just one row-reduction sequence.

Computing 3×3 Eigenpairs

72. Show algorithm steps to compute eigenpairs of $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Answers: $\left(1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

73. Show algorithm steps to compute eigenpairs of $A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 4 & -4 & -1 \end{pmatrix}$.

Answers:
$\left(1, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right)$, $\left(-1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$,
$\left(-1, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

74. Suppose A is row-reduced to a triangular form B. Are the eigenvalues of B also the eigenvalues of A? Give a proof or a counter-example.

75. Suppose $A - \lambda I$ is row-reduced to a triangular form B. Explain: The eigenvalues of A are the roots λ of $|B| = 0$.

Decomposition $A = PDP^{-1}$

Compute the eigenpairs. If diagonalizable, then display D, P and Fourier's replacement equation.

76. $A = \begin{pmatrix} 7 & 4 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
Ans: only 2 eigenpairs

.

77. $A = \begin{pmatrix} 1 & 6 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
Ans: $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \begin{pmatrix} 3 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
Fourier equation: $AP\vec{\mathbf{c}} = PD\vec{\mathbf{c}}$.

Diagonalization

Report **diagonalizable** or not and explain why.

78. $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}$
Ans: diagonalizable

79. $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
Ans: not diagonalizable

Non-diagonalizable Matrices

80. Verify $A = \begin{pmatrix} 1 & 2 \\ -8 & 9 \end{pmatrix}$ is not diagonalizable.

81. Verify $A = \begin{pmatrix} 1 & 2 & 0 \\ -8 & 9 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ is not diagonalizable.

82. Invent a 3×3 matrix which has exactly one eigenpair.

83. Invent a 4×4 matrix which has exactly two eigenpairs.

Fourier's Heat Model

Define
$\vec{\mathbf{v}}_1 = \sin \pi x, \vec{\mathbf{v}}_2 = \sin 2\pi x, \vec{\mathbf{v}}_3 = \sin 3\pi x$
considered as vectors in the vector space V of twice continuously differentiable functions on $0 \le x \le 1$.

84. Verify that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent vectors in V.

85. Verify that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ vanish at $x = 0$ and $x = 1$.

86. Define $u(x) = \sin \pi x$ (from $\vec{\mathbf{v}}_1$). Explain: Function u satisfies differential equation $\dfrac{d^2u}{dx^2} + \pi^2 u = 0$.

87. Write vector expression
$$c_1 e^{-\pi^2 t}\vec{\mathbf{v}}_1 + c_2 e^{-4\pi^2 t}\vec{\mathbf{v}}_2 + c_3 e^{-9\pi^2 t}\vec{\mathbf{v}}_3$$
as a scalar function $u(t, x)$. Find initial heat distribution $u(0, x)$. Explain how Fourier replacement (re-scaling) constructs future state $u(t, x)$ from initial state $u(0, x)$.

9.3 Exercises

Discrete Dynamical Systems

Define matrix A via equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 1 & 0 \\ 3 & 4 & 3 \\ 2 & 5 & 7 \end{pmatrix} \vec{\mathbf{x}} \tag{1}$$

1. Find eigenpair packages of A.
Answers:
$D=\begin{pmatrix} .5 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P=\begin{pmatrix} -1 & 1 & 1 \\ 0 & -4 & 5 \\ 1 & 3 & 9 \end{pmatrix}$

2. Explain: A is a **transition matrix**.[1]

3. Assume $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ has period one year. Find the system state after two years.

4. Explain: $A^n\vec{\mathbf{x}}$ is the system state after n periods.

Market Shares

Define matrix A via equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix} \vec{\mathbf{x}} \tag{2}$$

5. Verify the eigenpairs of A using software.

6. Compute A^2, A^3, A^4 using software. Predict the limit of A^n as n approaches infinity.

7. Compute with software (rounded)

$$A^{10}=\begin{pmatrix} .30 & .30 & .30 \\ .37 & .38 & .37 \\ .32 & .32 & .33 \end{pmatrix} \tag{3}$$

8. Let $\vec{\mathbf{x}}=\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Compute
$A^{10}\vec{\mathbf{x}} = \begin{pmatrix} 0.30 \\ 0.37 \\ 0.33 \end{pmatrix}$ (rounded)
in two ways by calculator:
(1) Fourier replacement (**??**).
(2) Matrix multiply using (3).

[1] Perron-Frobenius theory extensions in the literature apply to transition matrices. See the Weierstrass Proof exercises.

Stochastic Matrices

Reference: Perron-Frobenius proof on page **??**.

9. Establish the identity $|A - \lambda I| = |A^T - \lambda I|$.

10. Explain why A and A^T have the same eigenvalues but not necessarily the same eigenvectors.

11. Verify $\mathbf{max_r}(A) = \langle \vec{\mathbf{w}}|\vec{\mathbf{w}}|\cdots|\vec{\mathbf{w}}\rangle$, where $\vec{\mathbf{w}}$ has components $w_i = \max\{a_{ij}, 1 \le j \le n\}$.

12. Verify $\mathbf{max_r}(A) = D\mathcal{O}$, where D is the diagonal matrix of row maxima and $\mathcal{O}$ is the matrix of all ones.

Perron-Frobenius Theorem

Let $A > 0$ be $n \times n$ stochastic with unique eigenpair $(1, \vec{\mathbf{w}})$, all $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. Assume $\vec{\mathbf{v}} \ge \vec{\mathbf{0}}$, $\sum_{i=1}^n v_i = 1$ and $\delta = \min_{i,j} a_{ij}$.

13. Apply inequality $\mathbf{min_r}(A^n)\vec{\mathbf{v}} \le A^n\vec{\mathbf{v}} \le \mathbf{max_r}(A^n)\vec{\mathbf{v}}$ to prove $\lim_{n\to\infty} A^n\vec{\mathbf{v}} = \left(\sum_{i=1}^n v_i\right)\vec{\mathbf{w}}$.

14. Verify Euclidean norm inequality
$\|A^{k+1}\vec{\mathbf{v}} - \vec{\mathbf{w}}\| \le \sqrt{n}\,(1-\delta)^k$

Weierstrass Proof

These exercises establish existence of an eigenpair $(1, \vec{\mathbf{v}})$ for stochastic A having only nonnegative entries.

Weierstrass Compactness Theorem

A sequence of vectors $\{\vec{\mathbf{v}}_i\}_{i=1}^\infty$ contained in a closed, bounded set K in $\mathcal{R}^n$ has a subsequence converging in the vector norm of $\mathcal{R}^n$ to some vector $\vec{\mathbf{v}}$ in K.

Define set K to be all vectors $\vec{\mathbf{v}}$ with nonnegative components adding to 1. Let $\vec{\mathbf{v}}_0$ be any element of K. Assume stochastic A with $a_{ij} \ge 0$ and define $\vec{\mathbf{v}}_N = \frac{1}{N}\sum_{j=0}^{N-1} A^j\vec{\mathbf{v}}_0$.

15. Verify K is closed and bounded in $\mathcal{R}^n$. Then prove $\lambda\vec{\mathbf{x}} + (1-\lambda)\vec{\mathbf{y}}$ is in K for $0 \le \lambda \le 1$ and $\vec{\mathbf{x}}, \vec{\mathbf{y}}$ in K.

16. Prove identity
$\vec{\mathbf{v}}_{N+1} = \lambda\vec{\mathbf{v}}_N + (1-\lambda)A^N\vec{\mathbf{v}}_0$
where $\lambda = \frac{N}{N+1}$ and then prove by induction that $\vec{\mathbf{v}}_N$ is in K.

17. Verify all hypotheses in the Weierstrass theorem applied to $\{\vec{\mathbf{v}}_N\}_{N=0}^\infty$. Applying the theorem produces a subsequence $\{\vec{\mathbf{v}}_{N_p}\}_{p=1}^\infty$ limiting to some $\vec{\mathbf{v}}$ in K.

18. Verify identity
$\vec{\mathbf{v}}_N - A\vec{\mathbf{v}}_N = \frac{1}{N}(\vec{\mathbf{v}}_0 - A^N\vec{\mathbf{v}}_0)$.

19. Explain why $A\vec{\mathbf{v}} = \lim_{p\to\infty} A\vec{\mathbf{v}}_{N_p}$. Then prove $\vec{\mathbf{v}} = A\vec{\mathbf{v}}$.

20. The claimed eigenpair $(1, \vec{\mathbf{v}})$ has been found, provided $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$. Explain why $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$.

Coupled Systems

Find the coefficient matrix A. Identify as coupled or uncoupled and explain why.

21. $x' = 2x + 3y$, $y' = x + y$

22. $x' = 3y$, $y' = x$

23. $x' = 3x$, $y' = 2y$

24. $x' = 3x$, $y' = 2y$, $z' = z$

Solving Uncoupled Systems

Solve for the general solution.

25. $x' = 3x$, $y' = 2y$

26. $x' = 3x$, $y' = 2y$, $z' = z$

Change of Coordinates

Given the change of coordinates $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$, find the matrix B for the inverse change $\vec{\mathbf{x}} = B\vec{\mathbf{y}}$.

27. $\vec{\mathbf{y}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{\mathbf{x}}$

28. $\vec{\mathbf{y}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\mathbf{x}}$

Constructing Coupled Systems

Given the uncoupled system and change of coordinates $\vec{\mathbf{y}} = P\vec{\mathbf{x}}$, find the coupled system.

29. $x_1' = 2x_1$, $x_2' = 3x_2$, $P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

30. $x_1' = x_1$, $x_2' = -x_2$, $P = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

Uncoupling a System

Change the given coupled system into an uncoupled system using the eigenanalysis change of variables $\vec{\mathbf{y}} = P\vec{\mathbf{x}}$.

31. $x_1' = 2x_1,\ x_2' = x_1 + x_2,\ x_3' = x_3$

Ans: $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ y_1' = 2y_1,\ y_2' = y_2,\ y_3' = y_3$

32. $x_1' = x_1 + x_2,\ x_2' = x_1 + x_2,\ x_3' = x_3$

Ans: $P = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ y_1' = 0,\ y_2' = 2y_2,\ y_3' = y_3$

Solving Coupled Systems

Report the answers for $x(t), y(t)$.

33. $x' = -x - 2y,\ y' = -4x + y$

34. $x' = 8x - y,\ y' = -2x + 7y$

Eigenanalysis and Footballs

The exercises study the ellipsoid
$17x^2 + 8y^2 - 12xy + 80z^2 = 80$.

35. Let $A = \begin{pmatrix} 17 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 80 \end{pmatrix}$. Expand equation $\vec{\mathbf{W}}^T A \vec{\mathbf{W}} = 80$, where $\vec{\mathbf{W}}$ has components x, y, z.

36. Find the eigenpairs of
$A = \begin{pmatrix} 17 & -6 & 0 \\ -6 & 8 & 0 \\ 0 & 0 & 80 \end{pmatrix}$.

37. Verify the semi-axis lengths $1, 4, 2$.

38. Verify that the ellipsoid has semi-axis unit directions
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\ \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

The Ellipse and Eigenanalysis

The exercises study the ellipse
$2x^2 + 4xy + 5y^2 = 24$.

39. Let $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$. Expand equation $\vec{\mathbf{W}}^T A \vec{\mathbf{W}} = 24$, where $\vec{\mathbf{W}} = \begin{pmatrix} x \\ y \end{pmatrix}$.

40. Find the eigenpairs of $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$.

41. Verify the semi-axis lengths $2, 2\sqrt{6}$.

42. Verify that the ellipse has semi-axis unit directions $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Orthogonal Triad Computation

The exercises fill in details from page **??**.
The ellipsoid equation: $x^2{+}4y^2{+}xy{+}4z^2{=}16$ or $\vec{\mathbf{x}}^T A\vec{\mathbf{x}}{=}16$,

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

43. Find the characteristic equation of A. Then verify the roots are 4, $5/2 + \sqrt{10}/2$, $5/2 - \sqrt{10}/2$.

44. Show the steps from **rref** to second eigenvector $\vec{\mathbf{x}}_2$:

$$\mathbf{rref} = \begin{pmatrix} 1 & 3{-}\sqrt{10} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\vec{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{10}{-}3 \\ 1 \\ 0 \end{pmatrix}$$

9.4 Exercises

Diagonalization

Find the eigenpair packages P and D in the relation $AP = PD$.

1. $A = \begin{pmatrix} -4 & 2 \\ 0 & -1 \end{pmatrix}$

2. $A = \begin{pmatrix} 7 & 5 \\ 10 & -7 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

5. $A = \begin{pmatrix} -1 & 0 & 3 \\ 3 & 4 & -9 \\ -1 & 0 & 3 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

7. $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

8. $A = \begin{pmatrix} 4 & 0 & 0 & 1 \\ 12 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 21 & -6 & 1 & 0 \end{pmatrix}$

Jordan's Theorem

Given matrices P and T, verify Jordan's relation $AP = PT$.

9. $A = \begin{pmatrix} -4 & 2 \\ 0 & -1 \end{pmatrix}$, $P = I$, $T = A$.

10. $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Cayley-Hamilton Theorem

11. Verify that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies

$$A^2 = -(a+d)A - (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

12. Verify $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{20} = \begin{pmatrix} 1 & 0 \\ 40 & 1 \end{pmatrix}$ by induction using Cayley-Hamilton.

Gram-Schmidt Process

Find the Gram–Schmidt orthonormal basis from the given independent set.

13. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

Ans: Columns of I.

14. $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}.$

15. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$

16. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$

Ans: Columns of I.

Gram-Schmidt on Polynomials

Define $V = \mathbf{span}(1, x, x^2)$ with inner product $\int_0^1 f(x)g(x)dx$. Find the Gram–Schmidt orthonormal basis.

17. 1, $1 + x$, x^2

18. $1 - x$, $1 + x$, $1 + x^2$

Gram-Schmidt: Coordinate Map

Define $V = \mathbf{span}(1, x, x^2)$ with inner product $\int_0^1 f(x)g(x)dx$. The coordinate map is

$$T: \; c_1 + c_2 x + c_3 x^2 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

19. Find the images of $1 - x$, $1 + x$, $1 + x^2$ under T.

20. Assume column vectors $\vec{\mathbf{x}}_1$, $\vec{\mathbf{x}}_2$, $\vec{\mathbf{x}}_3$ in $\mathcal{R}^3$ orthonormalize under Gram-Schmidt to $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$. Are the pre-images $T^{-1}(\vec{\mathbf{u}}_1)$, $T^{-1}(\vec{\mathbf{u}}_2)$, $T^{-1}(\vec{\mathbf{u}}_3)$ orthonormal in V?

Shadow Projection

Compute shadow vector $(\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})\vec{\mathbf{u}}$ for direction $\vec{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$. Illustrate with a hand–drawn figure.

21. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Ans: $-\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

22. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

23. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Ans: $\sqrt{5}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

24. $\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

Orthogonal Projection

Find an orthonormal basis $\{\vec{\mathbf{u}}_k\}_{k=1}^n$ for $V = \mathbf{span}(1+x, x, x+x^2)$, inner product $\int_0^1 f(x)g(x)dx$. Then compute the orthogonal projection $\vec{\mathbf{p}} = \sum_{k=1}^n (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}}_k)\vec{\mathbf{u}}_k$.

25. $\vec{\mathbf{x}} = 1 + x + x^2$

26. $\vec{\mathbf{x}} = 1 + 2x + x^2$

Orthogonal Projection: Theory

27. **P**rove that the orthogonal projection on $V = \{\vec{\mathbf{Y}}\}$ is the vector shadow projection:
$\mathbf{proj}_{\vec{Y}}(\vec{\mathbf{x}}) = d\vec{\mathbf{u}} = \mathbf{Proj}_V(\vec{\mathbf{x}})$.

28. **Gram-Schmidt Construction**
Define $\vec{\mathbf{x}}_j^{\perp} = \vec{\mathbf{x}}_j - \mathbf{Proj}_{W_{j-1}}(\vec{\mathbf{x}}_j)$,
and $W_{j-1} = \mathbf{span}(\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_{j-1})$.
Prove these properties.

(a) Subspace W_{j-1} is equal to the Gram-Schmidt $V_{j-1} = \mathbf{span}(\vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_j)$.

(b) Vector $\vec{\mathbf{x}}_j^\perp$ is orthogonal to all vectors in W_{j-1}.

(c) The vector $\vec{\mathbf{x}}_j^\perp$ is not zero.

(d) The Gram-Schmidt vector is

$$\vec{\mathbf{u}}_j = \frac{\vec{\mathbf{x}}_j^\perp}{\|\vec{\mathbf{x}}_j^\perp\|}.$$

Near Point Theorem

Find the near point to the subspace V.

29. $\vec{\mathbf{x}} = \begin{pmatrix}1\\1\end{pmatrix}$, $V = \mathbf{span}\left(\begin{pmatrix}1\\2\end{pmatrix}\right)$

30. $\vec{\mathbf{x}} = \begin{pmatrix}1\\1\end{pmatrix}$, $V = \mathbf{span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$

31. $\vec{\mathbf{x}} = \begin{pmatrix}1\\1\\0\end{pmatrix}, V = \mathbf{span}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$

32. $\vec{\mathbf{x}} = \begin{pmatrix}1\\0\\1\end{pmatrix}, V = \mathbf{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$

QR-Decomposition

Give A, find an orthonormal matrix Q and an upper triangular matrix R such that $A = QR$.

33. $A = \begin{pmatrix}5 & 9\\1 & 7\\1 & 5\\3 & 5\end{pmatrix}$, Ans: $R = \begin{pmatrix}6 & 12\\0 & 6\end{pmatrix}$

34. $A = \begin{pmatrix}2 & 1\\2 & 0\\2 & 0\\2 & 1\end{pmatrix}$, Ans: $R = \begin{pmatrix}4 & 1\\0 & 1\end{pmatrix}$

35. $A = \begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 0\\1 & 0 & 0\end{pmatrix}$, Ans: $R = \begin{pmatrix}2 & 1 & 0\\0 & 1 & 0\end{pmatrix}$

36. $A=\begin{pmatrix}1&0&0\\1&1&1\\1&1&1\\1&0&0\end{pmatrix}$, Ans: $R=\begin{pmatrix}2&1&1\\0&1&1\end{pmatrix}$

Linear Least Squares: 3×2

Let $A=\begin{pmatrix}2&0\\0&2\\1&1\end{pmatrix}$, $\vec{\mathbf{b}}=\begin{pmatrix}1\\0\\5\end{pmatrix}$.

37. Find the normal equations for $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

38. Solve $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ by least squares.

Linear Least Squares: 4×3

Let $A=\begin{pmatrix}4&0&1\\1&0&1\\0&1&0\\1&1&1\end{pmatrix}$, $\vec{\mathbf{b}}=\begin{pmatrix}3\\0\\0\\0\end{pmatrix}$.

39. Find the normal equations for $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

40. Solve $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ by least squares.

Orthonormal Diagonal Form

Let $A = A^T$. The **spectral theorem** implies $AQ = QD$ where D is diagonal and Q has orthonormal columns. Find Q and D.

41. $A=\begin{pmatrix}7&2\\2&4\end{pmatrix}$

42. $A=\begin{pmatrix}1&5\\5&1\end{pmatrix}$

43. $A=\begin{pmatrix}1&5&0\\5&1&0\\0&0&2\end{pmatrix}$

Ans: Eigenvalues $2, -4, 6$, orthonormal eigenvectors

$\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$

44. $A=\begin{pmatrix}1&5&0\\5&1&1\\0&1&1\end{pmatrix}$

Eigenpairs of Symmetric Matrices

45. Let $A=\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$. Show that $2, 2, 5$ are eigenvalues and find three eigenpairs.

46. Let $A=\begin{pmatrix} 5 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{pmatrix}$. Then $|A-\lambda I|=(4-\lambda)^2(7-\lambda)$. Find three eigenpairs.

47. Let $A=\begin{pmatrix} 6 & -1 & 1 \\ -1 & 6 & -1 \\ 1 & -1 & 6 \end{pmatrix}$. Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ correspond to $\lambda = 5, 5, 8$. Find a diagonal matrix D and an orthogonal matrix Q with $AQ = QD$.

48. Matrix A for $\lambda = 1, 1, 4$ has orthogonal eigenvectors
$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
Find A and verify $A = A^T$.

Singular Value Decomposition

Find the SVD $A = U\Sigma V^T$.

49. $A=\begin{pmatrix} -1 & 1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$.

Ans: U=3×3, V=2×2. Matrix
$\Sigma=\begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$=$3 \times 2$, the size of A.

50. $A=\begin{pmatrix} -1 & 1 \\ -2 & 2 \\ 1 & 1 \end{pmatrix}$.

Ans: $\sigma_1 = \sqrt{10}, \sigma_2 = \sqrt{2}$.

51. $A=\begin{pmatrix} -3 & 3 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$.

52. $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$.

Ellipse and the SVD

Repeat Example 9.17, page 883 for the given ellipse equation.

53. $85x^2 30xy + 10y^2 = 2500$

54. $340x^2 60xy + 10y^2 = 2500$

Mapping and the SVD

Reference: Example 9.18, page 885.

Let $\vec{\mathbf{w}} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2$,

$U = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$, $V = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$,

$A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$. Then $A = U\Sigma V^T$.

55. Verify $\|\vec{\mathbf{w}}\|^2 = \vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = c_1^2 + c_2^2$.

56. Verify $V^T \vec{\mathbf{w}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ from the general identity $V^T V = I$. Then show that $\Sigma V^T \vec{\mathbf{w}} = \begin{pmatrix} 10c_1 \\ 5c_2 \end{pmatrix}$.

Therefore, coordinate map $\vec{\mathbf{w}} \to \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ undergoes re-scaling by 10 in direction $\vec{\mathbf{v}}_1$ and 5 in direction $\vec{\mathbf{v}}_2$.

57. Find the angle θ of rotation for U and the angle ϕ of rotation for V^T.

58. Assume $|\vec{\mathbf{w}}| = 1$, a point on the unit circle. Is $A\vec{\mathbf{w}}$ on an ellipse with semi-axes 10 and 5? Justify your answer geometrically, no proof expected. Check your answer with a computer plot.

Four Fundamental Subspaces

Compute matrices S_1, S_2 such that the column spaces of S_1, S_2 are the nullspaces of A and A^T. Verify the orthogonality relations of the four subspaces from the matrix identities $AS_1 = 0$, $A^T S_2 = 0$.

59. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$. Answer:

$S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $S_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

60. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}$. Answer:

$$S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ S_2 = \begin{pmatrix} -1 & -1 \\ -2 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

61. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \end{pmatrix}$ Answer:

$$S_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},\ S_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

62. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ Answer:

$$S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},\ S_2 = \begin{pmatrix} 2 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Fundamental Theorem of Linear Algebra

Strang's Theorem says that the four subspaces built from $n \times m$ matrix A and $m \times n$ matrix A^T satisfy

$$\begin{aligned} &\mathbf{colspace}(A^T) \perp \mathbf{nullspace}(A), \\ &\mathbf{colspace}(A) \perp \mathbf{nullspace}(A^T). \end{aligned}$$

Let $r = \mathbf{rank}(A) = \mathbf{rank}(A^T)$. The four subspace dimensions are:

$$\begin{aligned} &\dim(\mathbf{colspace}(A)) = r, \\ &\dim(\mathbf{nullspace}(A)) = n - r, \\ &\dim(\mathbf{colspace}(A^T)) = r, \\ &\dim(\mathbf{nullspace}(A^T)) = m - r. \end{aligned}$$

63. Explain why $\dim(\mathbf{colspace}(A)) = \dim(\mathbf{colspace}(A^T)) = r$ from the Pivot Theorem.

64. Suppose A is 10×4. What are the dimensions of the four subspaces?

65. Invent a 4×4 matrix A where one of the four subspaces is the zero vector alone.

66. Prove that the only vector in common with $\mathbf{rowspace}(A)$ and $\mathbf{nullspace}(A)$ is the zero vector.

67. Prove that each vector $\vec{\mathbf{x}}$ in $\mathcal{R}^n$ can be uniquely written as $\vec{\mathbf{x}} = \vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ where $\vec{\mathbf{x}}_1$ is in $\mathbf{colspace}(A^T)$ and $\vec{\mathbf{x}}_2$ is in $\mathbf{nullspace}(A)$.

68. Prove that each vector $\vec{\mathbf{y}}$ in $\mathcal{R}^m$ can be uniquely written as $\vec{\mathbf{y}} = \vec{\mathbf{y}}_1 + \vec{\mathbf{y}}_2$ where $\vec{\mathbf{y}}_1$ is in **colspace**(A) and $\vec{\mathbf{y}}_2$ is in **nullspace**(A^T).

Chapter 10

Phase Plane Methods

Contents

10.2 Exercises

Autonomous Planar Systems.

Consider

(1)
$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

1. (Vector-Matrix Form) System (1) can be written in vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

2. (Picard's Theorem) Picard's vector existence-uniqueness theorem applies to system (1) with initial data $x(0) = x_0$, $y(0) = y_0$. Show the details.

Trajectories Don't Cross.

3. **(Theorem 10.1 Details)** Compute $\frac{dy}{dt} = g(x_1(t+c), y_1(t+c))$, then show that $y'(t) = g(x(t), y(t))$ in the proof of Theorem 10.1.

4. **(Orbits Can Cross)** The example

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3y^{2/3}$$

has infinitely many orbits crossing at $x = y = 0$. Exhibit two distinct orbits which cross at $x = y = 0$. Does this example contradict Theorem 10.1?

Equilibria. A point (x_0, y_0) is called an **Equilibrium** provided $x(t) = x_0$, $y(t) = y_0$ is a solution of the dynamical system.

5. Justify that $(1,-1), (-1,1)$ are the only equilibria for the system $x' = x+y$, $y' = 1 - x^2$.

6. Display the details which justify that $(0,0), (90,0), (0,60), (80,20)$ are all equilibria for the system $x'(t) = x(-2x-y+180)$, $y'(t) = y(-x-2y+120)$.

Practical Methods for Computing Equilibria.

7. **(Murray System)** The biological system

$$x' = x(6-2x-y), y' = y(4-x-y)$$

has equilibria $(0,0)$, $(3,0)$, $(0,4)$, $(2,2)$. Justify the four answers.

8. **(Nullclines)** Curves along which either $x' = 0$ or $y' = 0$ are called **nullclines**. The biological system

$$x' = x(6-2x-y), y' = y(4-x-y)$$

has nullclines $x = 0$, $y = 0$, $6-2x-y = 0$, $4-x-y = 0$. Justify the four answers.

9. **(Nullclines by Computer)** Produce a graphical display of the nullclines of the Murray System above. Maple code to produce a nullcline plot is as follows

```
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=0..130,y=0..80;
plots[contourplot](eqns,wind,
  contours=[0]);
```

10. **(Isoclines by Computer)** Level curves $f(x,y) = c$ are called **Isoclines**. `Maple` will plot level curves $f(x,y) = -2$, $f(x,y) = 0$, $f(x,y) = 2$ using the nullcline code above, with replacement `contours=[-2,0,2]`. Produce an isocline plot for the Murray System above with these same contours.

11. (Implicit Plot) Equilibria can be found graphically by an implicit plot. `Maple` code:

```
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=0..130,y=0..80;
plots[implicitplot](eqns,wind);
```

Produce the implicit plot. Is it the same as the nullcline plot?

Rabbit-Fox System.

12. (Predator-Prey) Consider a rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

Argue why extinction of the rabbits ($x = 0$) implies extinction of the foxes ($y = 0$).

13. (Predator-Prey) The rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(40-y), \\ y' &= \frac{1}{100}y(x-40), \end{aligned}$$

has extinction of the foxes ($y = 0$) implying Malthusian population explosion of the rabbits ($\lim_{t=\infty} x(t) = \infty$). Explain.

Trout System.

Consider

$$\begin{aligned} x'(t) &= x(-2x-y+180), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

14. (Carrying Capacity) Show details for calculation of the carrying capacities $x = 80$, $y = 20$.

15. (Stability) Equilibrium point $x = 80$, $y = 20$ is stable. Explain this statement using geometry from Figure 10 and the definition of stability.

Phase Portraits.

Consider

$$\begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

16. (Equilibria) Solve for x, y in the system

$$\begin{aligned} 0 &= x + y, \\ 0 &= 1 - x^2, \end{aligned}$$

for equilibria $(1, -1)$, $(-1, 1)$.

17. (Graph Window) Explain why $-2 \le x \le 2$, $-2 \le y \le 2$ is a suitable window.

18. (Grid Points) Draw a 5×5 grid on the graph window $|x| \le 2, |y| \le 2$. Label the equilibria.

19. (Direction Field) Draw direction field arrows on the 5×5 grid of the previous exercise. They coincide with the tangent direction $\vec{v} = x'\vec{\imath} + y'\vec{\jmath} = (x + y)\vec{\imath} + (1 - x^2)\vec{\jmath}$, where (x, y) is the grid point. The arrows may not touch.

20. (Threaded Orbits) On the direction field of the previous exercise, draw orbits (*threaded solution curves*), using the rules:

1. Orbits don't cross.
2. Orbits pass direction field arrows with nearly matching tangent.

Phase Plot by Computer. Use a computer algebra system or a numerical workbench to produce phase portraits for the given dynamical system. A graph window should contain all equilibria.

21. (Rabbit-Fox System I)

$$\begin{aligned} x' &= \frac{1}{200} x(30 - y), \\ y' &= \frac{1}{100} y(x - 40). \end{aligned}$$

22. (Rabbit-Fox System II)

$$\begin{aligned} x' &= \frac{1}{100} x(50 - y), \\ y' &= \frac{1}{200} y(x - 40). \end{aligned}$$

23. (Trout System I)

$$\begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

24. **(Trout System II)**

$$\begin{aligned} x'(t) &= x(-2x - y + 200), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

Stability Inequalities.
The signs of $x'(t)$ and/or $y'(t)$ can predict stability or instability. Consider an equilibrium point (x_0, y_0) and all solutions $x(t), y(t)$ satisfying for H small the inequalities

$$|x(0) - x_0| \le H, \quad |y(0) - y_0| \le H.$$

25. **(Instability: Repeller)** Prove that $x'(t) > 0$ and $y'(t) > 0$ for all small $H > 0$ implies instability at x_0, y_0.

26. **(Stability: Attractor)** Prove that $x'(t) < 0$ and $y'(t) < 0$ for all small $H > 0$ implies stability at x_0, y_0.

27. **(Instability in x)** Prove that $x'(t) > 0$ for all small $H > 0$ implies instability at x_0, y_0.

28. **(Instability in y)** Prove that $y'(t) > 0$ for all small $H > 0$ implies instability at x_0, y_0.

Geometric Stability.

29. **(Attractor)** Imagine a dust particle in a fluid draining down a funnel, whose trace is a space curve. Project the space curve onto the plane orthogonal to the centerline of the funnel. Is this planar orbit stable at centerline position in the sense of the definition?

30. **(Repeller)** Imagine a paint droplet from a paint spray can, which traces a space curve. Project the space curve onto the plane orthogonal to the spray orifice direction. Is this planar orbit stable at centerline position in the sense of the definition?

Algebraic Stability.

31. **(Rabbit–Fox Stability)** Provide algebraic details for stability of equilibrium $x = 40$, $y = 30$ for the system

$$\begin{aligned} x' &= \frac{1}{200} x(30 - y), \\ y' &= \frac{1}{100} y(x - 40). \end{aligned}$$

32. **(Rabbit–Fox Instability)** Provide algebraic details for instability of equilibrium $x = 0$, $y = 0$ for the system

$$\begin{aligned} x' &= \frac{1}{100}x(50 - y), \\ y' &= \frac{1}{200}y(x - 40). \end{aligned}$$

10.3 Exercises

Planar Constant Linear Systems

1. (Picard's Theorem) Explain why planar solutions don't cross, by appeal to Picard's existence-uniqueness theorem for $\frac{d}{dt}\vec{u} = A\vec{u}$.

2. (Equilibria) System $\frac{d\vec{u}}{dt} = A\vec{u}$ always has solution $\vec{u}(t) = \vec{0}$, so there is always one equilibrium point. Give an example of a matrix A for which there are infinitely many equilibria.

Putzer's Formula

3. (Cayley-Hamilton) Define matrices $\vec{\mathbf{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\vec{\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Given matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, expand left and right sides to verify the **Cayley-Hamilton identity**
$A^2 - (c + d)A + (ad - bc)\vec{\mathbf{I}} = \vec{\mathbf{0}}$.

4. (Complex Roots) Verify the Putzer solution $\vec{u} = \Phi(t)\vec{u}(0)$ of $\vec{u}' = A\vec{u}$ for complex roots $\lambda_1 = \overline{\lambda_2} = a + bi$, $b > 0$, where $\Phi(t)$ is

$$e^{at}\left(\cos(bt)\, I + (A - aI)\frac{\sin(bt)}{b}\right).$$

5. (Distinct Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\vec{u}.$$

6. (Real Equal Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix}\vec{u}.$$

7. (Complex Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\vec{u}.$$

Continuity and Redundancy

8. (Real Equal Eigenvalues) Show that limiting $\lambda_2 \to \lambda_1$ in the Putzer formula for distinct eigenvalues gives Putzer's formula for real equal eigenvalues.

9. (Complex Eigenvalues) Assume $\lambda_1 = \overline{\lambda_2} = a+ib$ with $b > 0$. Then Putzer's first formula holds. Show the third formula details for $\Phi(t)$:

$$e^{at}\left(\cos(bt)\,I + (A - aI)\frac{\sin(bt)}{b}\right).$$

Illustrations

10. (Distinct Eigenvalues) Show the details for the solution of

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}\vec{u}.$$

11. (Complex Eigenvalues) Show the details for the solution of

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix}\vec{u}.$$

Isolated Equilibria

12. (Determinant Expansion) Verify that $|A - \lambda I|$ equals

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

13. (Infinitely Many Equilibria) Explain why $A\vec{u} = \vec{0}$ has infinitely many solutions when $\det(A) = 0$.

Classification of Equilibria

14. (Rotating Figures) When sines and cosines appear in the Euler atoms, the phase portrait at $(0,0)$ rotates around the origin. Explain precisely why this is true.

15. (Non-Rotating Figures) When sines and cosines do not appear in the Euler atoms, the phase portrait at $(0,0)$ has no rotation. Give a precise explanation.

Attractor and Repeller

16. (Classification) Which of spiral, center, saddle, node can be an attractor or a repeller?

17. (Attractor) Prove that $(0,0)$ is an attractor if and only if the Euler atoms have limit zero at $t = \infty$.

18. (Repeller) Prove that $(0,0)$ is a repeller if and only if the Euler atoms have limit zero at $t = -\infty$.

19. (Center) A center is neither an attractor nor a repeller. Explain, using Euler atoms.

Phase Portrait Linear

Show the classification details for spiral, center, saddle, proper node, improper node. Include a drawing which identifies eigenvector directions, where such information applies. In all exercises, $' = \frac{d}{dt}$.

20. (Spiral)

$$\begin{aligned} x' &= 2x + 3y, \\ y' &= -3x + 2y. \end{aligned}$$

21. (Center)

$$\begin{aligned} x' &= 3y, \\ y' &= -3x. \end{aligned}$$

22. (Saddle)

$$\begin{aligned} x' &= 3x, \\ y' &= -5y. \end{aligned}$$

23. (Proper Node)

$$\begin{aligned} x' &= 2x, \\ y' &= 2y. \end{aligned}$$

24. (Improper Node: Degenerate)

$$\begin{aligned} x' &= 2x + y, \\ y' &= 2y. \end{aligned}$$

25. (Improper Node: $\lambda_1 \neq \lambda_2$)

$$\begin{aligned} x' &= 2x + y, \\ y' &= 3y. \end{aligned}$$

10.4 Exercises

Almost Linear Systems. Find all equilibria (x_0, y_0) of the given nonlinear system. Then compute the Jacobian matrix $A = J(x_0, y_0)$ for each equilibria.

1. (Spiral and Saddle)

$$\begin{aligned} \frac{d}{dt}x &= x + 2y, \\ \frac{d}{dt}y &= 1 - x^2. \end{aligned}$$

2. (Saddle and Two Spirals)

$$\begin{aligned} \frac{d}{dt}x &= x - 3y + 2xy, \\ \frac{d}{dt}y &= 4x - 6y - xy - x^2. \end{aligned}$$

3. (Spiral, Saddle)

$$\begin{aligned} \frac{d}{dt}x &= 3x - 2y - x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y. \end{aligned}$$

4. (Center and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y - xy. \end{aligned}$$

5. (Proper Node and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= y - xy. \end{aligned}$$

6. (Improper Degenerate Node, Spiral and Two Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3 + y^3, \\ \frac{d}{dt}y &= y + 3xy. \end{aligned}$$

7. (Improper Node and a Saddle)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

8. (Proper Node and a Saddle)

$$\begin{aligned} \frac{d}{dt}x &= 2x + y^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Phase Portrait Almost Linear. Linear library phase portraits can be locally pasted atop the equilibria of an almost linear system, with limitations. Apply the theory for the following examples. Complete the phase diagram by computer, thereby resolving the possible mutation of a center or node into a spiral. Label eigenvector directions, where it makes sense.

9. (Center and Three Saddles)

$$\begin{aligned} \tfrac{d}{dt}x &= x - y + x^2 - y^2, \\ \tfrac{d}{dt}y &= 2x - y - xy. \end{aligned}$$

10. (Proper Node and 3 Saddles)

$$\begin{aligned} \tfrac{d}{dt}x &= x - y + x^2 - y^2, \\ \tfrac{d}{dt}y &= y - xy. \end{aligned}$$

11. (Improper Degenerate Node, Spiral and Two Saddles)

$$\begin{aligned} \tfrac{d}{dt}x &= x - y + x^3 + y^3, \\ \tfrac{d}{dt}y &= y + 3xy. \end{aligned}$$

12. (Improper Node and a Saddle)

$$\begin{aligned} \tfrac{d}{dt}x &= x - y + x^3, \\ \tfrac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

13. (Proper Node and a Saddle)

$$\begin{aligned} \tfrac{d}{dt}x &= 2x + y^3, \\ \tfrac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Classification of Almost Linear Equilibria. With computer assist, find and classify the nonlinear equilibria.

14. (Co-existing Species)

$$\begin{aligned} x'(t) &= x(t)(24 - 2x(t) - y(t)), \\ y'(t) &= y(t)(30 - 2y(t) - x(t)). \end{aligned}$$

15. (Doomsday-Extinction)

$$\begin{aligned} x'(t) &= x(t)(x(t) - y(t) - 4), \\ y'(t) &= y(t)(x(t) + y(t) - 8). \end{aligned}$$

Almost Linear Geometry. A separatrix is a union of curves and equilibria with orbits limiting to it. With computer assist, make a plot of threaded curves which identify one or more separatrices near the equilibrium.

16. (Saddle $(-1, 1)$)

$$\begin{aligned} \tfrac{d}{dt}x &= x + y, \\ \tfrac{d}{dt}y &= 1 - x^2. \end{aligned}$$

17. (Saddle $(-1/5, -2/5)$**)**

$$\begin{aligned}\frac{d}{dt}x &= 3x - 2y - x^2 - y^2,\\ \frac{d}{dt}y &= 2x - y.\end{aligned}$$

18. (Saddle $(-2/3, \sqrt[3]{4/3})$**)**

$$\begin{aligned}\frac{d}{dt}x &= 2x + y^3,\\ \frac{d}{dt}y &= 2y + 3xy.\end{aligned}$$

19. (Degenerate Improper Node)

$$\begin{aligned}\frac{d}{dt}x &= x - y + x^3 + y^3,\\ \frac{d}{dt}y &= y + 3xy, \quad \text{at } (0,0).\end{aligned}$$

Rayleigh and van der Pol. Each example below has a unique periodic orbit surrounding an equilibrium point that is the limit at $t = \infty$ of any other orbit. Verify the spiral repeller at $(0,0)$ in the attached figure, from the linearized problem at $(0,0)$ and **Paste Theorem** 10.4. Create phase portraits with computer assist for the linear and nonlinear problems.

20. (Lord Rayleigh 1877, Clarinet Reed Model)

$$\begin{aligned}\frac{d}{dt}x &= y,\\ \frac{d}{dt}y &= -x + y - y^3.\end{aligned}$$

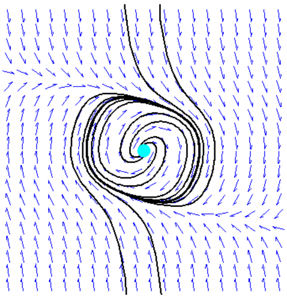

Figure 1. Clarinet Reed.

21. (van der Pol 1924, Radio Oscillator Circuit Model)

$$\begin{aligned}\frac{d}{dt}x &= y,\\ \frac{d}{dt}y &= -x + (1 - x^2)y.\end{aligned}$$

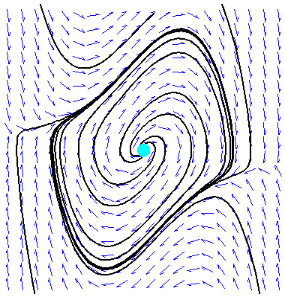

Figure 2. Oscillator Circuit.

10.5 Exercises

Predator-Prey Models.

Consider the system

$$\begin{aligned} x'(t) &= \frac{1}{250}(1-2y(t))x(t), \\ y'(t) &= \frac{3}{500}(2x(t)-1)y(t). \end{aligned}$$

1. (System Variables) The system has vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

2. (System Parameters) Identify the values of $a, b, c, d, p, q,$ as used in the textbook's predator-prey system.

3. (Identify Predator and Prey) Which of $x(t), y(t)$ is the predator?

4. (Switching Predator and Prey) Give an example of a predator-prey system in which $x(t)$ is the predator and $y(t)$ is the prey.

Implicit Solution Predator-Prey.

These exercises prove the equation

$$a\,\ln|y| + b\,\ln|x| - q\,x - p\,y = C.$$

5. (First Order Equation) Verify from the chain rule of calculus the first order equation

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{y}{x}\frac{qx-b}{a-py}.$$

6. (Separated Variables) Verify

$$\left(\frac{a}{y} - p\right) dy = \left(q - \frac{b}{x}\right) dx.$$

7. (Quadrature) Integrate the equation of the previous exercise to obtain

$$a\,\ln|y| - p\,y = q\,x - b\,\ln|x| = C.$$

Then re-arrange to obtain the reported implicit solution.

8. (Energy Function) Define $E(t) = a\ln|u| - pu$. Show that $dE/du = (a - pu)/u$. Then show that $dE/du < 0$ for $a > 0$, $p > 0$ and $a/p < u < \infty$.

Linearized Predator-Prey System. Consider

$$\begin{aligned} x'(t) &= (100 - 2y(t))x(t), \\ y'(t) &= (2x(t) - 160)y(t). \end{aligned}$$

9. (Find Equilibria) Verify equilibria $(0,0), (80,50)$.

10. (Jacobian Matrix) Compute $J(x,y)$ for each x,y. Then find $J(0,0)$ and $J(80,50)$.

11. (Transit Time) Find the trip time of an orbit about $(0,0)$ for system $\frac{d}{dt}\vec{v} = \begin{pmatrix} 0 & -160 \\ 100 & 0 \end{pmatrix} \vec{v}$, the linearization about $(80,50)$.

12. (Paste Theorem) Describe the local figures expected near equilibria in the nonlinear phase portrait.

Rabbits and Foxes. Consider

$$\begin{aligned} x'(t) &= \frac{1}{200} x(t)(50 - y(t)), \\ y'(t) &= \frac{1}{100} y(t)(x(t) - 40). \end{aligned}$$

13. (Equilibria) Verify equilibria $(0,0)$, $(40,50)$, showing all details.

14. (Jacobian) Compute Jacobian $J(x,y)$, then $J(0,0)$ and $J(40,50)$.

15. (Rabbit Oscillation) The linear and nonlinear scenes for $x(t)$ must approximate each other. Find estimates for the max and min of rabbits and their period of oscillation, for the nonlinear system.

Pesticides. Consider the system

$$\begin{aligned} x'(t) &= (10 - y(t))x(t) - s_1 x(t), \\ y'(t) &= (x(t) - 20)y(t) - s_2 y(t). \end{aligned}$$

16. (Equilibria) Show details for computing the pesticide system equilibria $(0,0)$, $(20 + s_2, 10 - s_1)$, where s_1, s_2 are the pesticide death rates.

17. (Average Populations) Explain: A field biologist should count, on the average, populations of about $20 + s_2$ prey and $10 - s_1$ predators.

Survival of One Species. Consider

$$\begin{aligned} x'(t) &= x(t)(24 - x(t) - 2y(t)), \\ y'(t) &= y(t)(30 - y(t) - 2x(t)). \end{aligned}$$

18. (Interactions) Show that doubling either x or y causes the interaction term $2xy$ to double.

19. (Equilibria) Find all equilibria.

20. (Linearization) Find the linearized systems $\frac{d}{dt}\vec{v} = J(x_0, y_0)\vec{v}$ for each equilibrium point (x_0, y_0).

21. (Nonlinear Classification) Classify each equilibrium point (x_0, y_0) as center, spiral, node, saddle, using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

Co-existence

Find the equilibria, then classify them as node, saddle, spiral, center using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

22. (Node, Saddle, Saddle, Node)

$$\begin{aligned} x' &= x(144 - 2x - 3y), \\ y' &= y(90 - 6y - x). \end{aligned}$$

23. (Node, Saddle, Saddle, Node)

$$\begin{aligned} x' &= 2x(144 - 4x - 3y), \\ y' &= y(90 - 6y - 2x). \end{aligned}$$

Explosion and Extinction

Find the equilibria, then classify them as node, saddle, spiral, center using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

24. (Node, Saddle, Saddle, Spiral)

$$\begin{aligned} x' &= x(x - 2y - 4), \\ y' &= y(x + 2y - 8). \end{aligned}$$

25. (Node, Saddle, Saddle, Spiral)

$$\begin{aligned} x' &= x(x - y - 4), \\ y' &= y(x + y - 6). \end{aligned}$$

10.6 Exercises

Linear Mechanical Models

Consider the unforced linear model $mx'' + cx + kx = 0$, where m, c, k are mass, dashpot constant, Hooke's constant. Assume below that m, c, k are positive.

1. **(Dynamical System Form)** Write the scalar problem as $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$. Explicit definitions of $\vec{\mathbf{u}}(t)$ and A are expected.

2. **(Attractor to $\vec{\mathbf{u}} = \vec{\mathbf{0}}$)** Explain why $\lim_{t\to\infty} \vec{\mathbf{u}}(t) = \vec{\mathbf{0}}$, giving citations to theorems in this book.

3. **(Isolated Equilibrium)** Prove that $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$ has a unique equilibrium at $\vec{\mathbf{u}} = \vec{\mathbf{0}}$. Then explain why the equilibrium is isolated.

4. **(Phase Plots)** Classify the cases of **over-damped** and **under-damped** as a stable node or a stable spiral for $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$ at equilibrium $\vec{\mathbf{u}} = \vec{\mathbf{0}}$. Why are classifications *center* and *saddle* impossible?

Nonlinear Spring-Mass System

Consider the general model $x'' + F(x(t)) = 0$ with the assumptions on page **??**.

5. **(Harmonic Oscillator)** Let $F(x) = \omega^2 x$ with $\omega > 0$. Show F is odd and $F(0) = 0$. Then find the general solution $x(t)$ for $x'' + F(x) = 0$.

6. **(Taylor Series)** Show that an odd function $F(x)$ with Maclaurin series $\sum_{n=0}^{\infty} a_n x^n$ has all even order terms zero, that is, $a_n = 0$ for n even.

Soft and Hard Springs

Classify as a hard or soft spring. Then write the conservation law for the equation.

7. $x'' + x + x^3 = 0$

8. $x'' + x - x^3 = 0$

Hard spring

9. Prove that a hard spring has exactly one equilibrium $x = y = 0$.

10. Substitute $x = x(t), y = x'(t)$ into $z = y^2 + x^2 + x^4$ to obtain $z(t)$. Function $z(t)$ has a minimum when $\frac{dz}{dt} = 0$. Reduce this equation to $x'' + x + 2x^3 = 0$.

Soft Spring

Consider the soft spring $x'' + kx - \beta x^3 = 0$.

11. **(Equilibria)** Verify the three equilibria $(0,0), (0, \sqrt{k}\beta), (0, -\sqrt{k}\beta)$.

12. **(Saddles)** Verify by linearization and the **Paste Theorem** that equilibria $(0, \sqrt{k}\beta), (0, -\sqrt{k}\beta)$ are nonlinear saddles.

13. **(Center or Spiral)** The **Paste Theorem** says that $(0,0)$ is a nonlinear center or spiral. Make a computer phase portrait to verify for $k = 1$ and $\beta = 2$ Figure 37, page 968.

14. **(Mass at Rest)** Verify that the only solutions with the mass at rest are the equilibria.

15. **Phase Portrait** Solve for the equilibria of $x'' + 4x - x^3 = 0$. Draw a phase portrait similar to Figure 37, page 968.

16. **Separatrix** The energy equation for $x''+4x-x^3 = 0$ is $\frac{1}{2}y^2+2x^2-\frac{1}{4}x^4 = E$. Substitute the saddle equilibria to find $E = 4$. Plot implicitly the energy equation curve. The separatrix is the union of the two saddle equilibria and this implicit curve.

Damped Nonlinear Pendulum

Consider $\frac{d^2\theta(t)}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{L}\sin(\theta(t)) = 0$, which has vector-matrix form $\vec{\mathbf{u}}' = \vec{\mathbf{G}}(\vec{\mathbf{u}}(t))$.

17. Display both $\vec{\mathbf{u}}$ and $\vec{\mathbf{G}}$.

18. Find the Jacobian matrix of $\vec{\mathbf{G}}$ with respect to $\vec{\mathbf{u}}$.

Undamped Nonlinear Pendulum

Consider $\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) = 0$, having vector-matrix form $\vec{\mathbf{u}}' = \vec{\mathbf{F}}(\vec{\mathbf{u}}(t))$.

19. Find the Jacobian matrix of $\vec{\mathbf{F}}$ with respect to $\vec{\mathbf{u}}$.

20. Solve $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \vec{\mathbf{0}}$ for $\vec{\mathbf{u}}$, showing all details.

21. Evaluate the Jacobian matrix at the roots of $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \vec{\mathbf{0}}$.

22. Plot $y^2 + \frac{4g}{L}\sin^2(x/2) = 4\frac{g}{L}$
implicitly for $\frac{g}{L} = 10$. The separatrix is this curve plus equilibria.

Chapter 11

Systems of Differential Equations

Contents

11.2 Exercises

Solving 2×2 Systems

1. Solve $x_1' = 2x_1 + x_2$, $x_2' = x_2$. Ans: $x_1 = -c_2\,\mathrm{e}^t + c_1\,\mathrm{e}^{2\,t}, x_2 = c_2\,\mathrm{e}^t$

2. Discuss how to solve $\vec{\mathbf{x}}' = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \vec{\mathbf{x}}$.

Triangular 2×2 Matrix A

3. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \vec{\mathbf{x}}$.

4. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} \vec{\mathbf{x}}$.

Non-Triangular 2×2 Matrix A

5. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \vec{\mathbf{x}}$.

6. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Diagonal A

7. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

8. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Lower Triangular

9. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

10. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Upper Triangular

11. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

12. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \vec{\mathbf{x}}$.

Jordan's $n \times n$ Variable Change

Let $A = PTP^{-1}$ with T upper triangular and P invertible. Define change of variable $\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t)$. Then:

13. If $\vec{\mathbf{x}}(t)$ solves $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$, then $\vec{\mathbf{y}}(t) = P^{-1}\vec{\mathbf{x}}(t)$ solves $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$.

14. If $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$, then $\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t)$ solves $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$.

Convert Scalar Linear 2nd Order to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

15. $x'' + 2x' + x = \sin t$

16. $2x'' + 3x' + 8x = 4\cos t$

Convert Second Order Scalar System to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

17. $x'' = x + y,\ y'' = x - y$

18. $x'' = x + y + \sin t,\ y'' + y = x + \cos t$

Convert Coupled Spring-Mass System to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

19. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$

20. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \vec{\mathbf{x}}$

Convert Higher Order Linear Equations to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

21. $x''' = x$

22. $\dfrac{d^4y}{dx^4} + 16y = 0$

Convert Scalar Continuous-Coefficient Equation to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

23. $x^2y'' + 3xy' + 2y = 0$

24. $y''' + xy'' + x^2y + y = 0$

Convert Forced Higher Order Equation to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

25. $\dfrac{d^4y}{dx^4} = y''' + y + \sin t$

26. $\dfrac{d^6y}{dx^6} = \dfrac{d^4y}{dx^4} + y + \cos t$

Convert 2nd Order System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{G}}(t)$

27. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

28. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \vec{\mathbf{x}} + e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Convert Damped 2nd Order System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{G}}(t)$

29. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{\mathbf{x}}' + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

30. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \vec{\mathbf{x}} + \vec{\mathbf{x}}' + e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

11.3 Exercises

Linear Systems

Convert to matrix notation.

1. $x_1' = 2x_1 + x_2 + e^t$,
$x_2' + x_1 - 2x_2 = \sinh(t)$

2. $x_1' = x_1 + x_2 + x_3$,
$x_2' + x_1 - 2x_2 + x_3 = \ln|1 + t^2|$,
$x_3' = x_2 + x_3 + \cosh(t)$

Existence-Uniqueness

3. Apply Gronwall's inequality:
$|y(t)| \le 4 + \int_0^t (1 + r^2)|y(r|\,dr,\ t \ge 0$.

4. Solve with $x_1(0) = x_2(0) = 0$:
$x_1' = e^t x + e^{-t} x_2$,
$x_2' = \ln|1 + \sinh^2(t)|\, x_1 + x_2$

5. Find the interval on which the solution is defined:
$x_1' = tx_1 + x_2,\ x_2' = x_1 + \tan(t)\, x_2$

6. Let matrix A be 2×2 constant. Find A, given $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has general solution $x_1 = c_1 e^t + c_2 e^{2t},\ x_2 = 5c_1 2e^t + 4c_2 e^{2t}$.

7. Let $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ have two solutions : $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} e^t \\ e^t \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

8. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

9. Let constant matrix A be 10×10. Two solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ have equal value at $t = 100$. Are they the same solution?

10. Solutions y_1, y_2 of $y' + p(x)y = q(x)$ are zero at $x = -2$. What assumptions on p, q imply $y_1 \equiv y_2$?

Superposition

11. Explain: e^t is a solution of $y'' - y = 0$ because $\cosh(t)$, $\sinh(t)$ are a solution basis.

12. Explain: $e^t + 10$ is a solution of $y'' - y = -10$, therefore 10 is a particular solution.

13. The shortest solution of $y' + y = 100$ is $y = 100$. Explain why.

14. Let $x_1' = 2x_1$, $x_2' = -x_2$. Report the matrix form $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ and the vector general solution.

15. Let 2-dimensional $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have general solution $x_1 = c_1e^t + c_2e^{3t}$, $x_2 = (c_1 + c_2)e^t + 2c_2e^{3t} + \cos(t)$. Find formulas for vectors $\vec{\mathbf{x}}_h$ and $\vec{\mathbf{x}}_p$.

16. Let $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have two solutions $x_1 = e^t + e^{3t}$, $x_2 = 2e^t + \sin(t)$ and $x_1 = e^{3t}$, $x_2 = e^{3t} + \sin(t)$. Find a solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

Superposition $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

17. Let $\vec{\mathbf{u}}_1(t), \ldots, u_k(t)$ be solutions of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$. Let $c_1, \ldots, c_k$ be constants. Prove: $\vec{\mathbf{u}}(t) = \sum_{i=1}^{k} c_i\vec{\mathbf{u}}_i(t)$ is a solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$.

18. Find the **standard basis** $\vec{\mathbf{w}}_1(t), \vec{\mathbf{w}}_2(t)$:
$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}}$

19. Let matrix A be 2×2. For $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$, find $\vec{\mathbf{x}}_h(t)$, $\vec{\mathbf{x}}_p(t)$:
$x_1 = c_1 + c_2t + e^t$, $x_e = (c_1 = c_2)t + e^{2t}$

20. Let matrix $A(t)$ be 2×2. Let $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have two solutions $\begin{pmatrix} 1 + e^t \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 + e^{-t} \\ -1 \end{pmatrix}$. Find a solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$.

General Solution

21. Assume A is 2×2 and $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has solutions $e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Find the general solution and explain.

22. Assume $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Prove that zero is not a solution.

23. Assume $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_0 = \text{constant}$. Find an equation for $\vec{\mathbf{x}}_0$.

24. Find the vector general solution:
$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

25. Given 3 $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ with scalar general solution $x_1 = c_1 + c_2t + c_3t^2$, $x_2 = c_2 + c_3t$, $x_3 = c_3$, find the vector general solution.

26. Given 3 $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ with scalar general solution $x_1 = c_1 + c_2 t + c_3 t^2$, $x_2 = c_2 + c_3 t$, $x_3 = c_3$, find $A(t)$.

27. Find the vector general solution:
$$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

28. Find the vector general solution:
$$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Independence

29. Assume A is 2×2 and $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has solutions $e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Prove they are independent directly from the definition.

30. Compute the Wronskian:
$$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Abel-Liouville Formula

31. Apply Abel's Independence Test:
$$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

32. Let $\Phi(t)$ an invertible matrix satisfying $\Phi'(t) = A\Phi(t)$. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

33. Let $\Phi(t)$ an invertible matrix satisfying $\Phi'(t) = A\Phi(t)$. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

34. Let $\Phi(t)$ any matrix satisfying $\Phi'(t) = A\Phi(t)$. Assume the determinant of $\Phi(t_0)$ is nonzero. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

35. Let $\Phi(t)$ any matrix satisfying $\Phi'(t) = A\Phi(t)$. Let C be a constant matrix. Prove that the columns of $\Phi(t)C$ are solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

36. Assume continuous coefficients:
$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y = 0$
Prove from the Abel-Liouville formula for the companion system
that the Wronskian $W(t)$ of
solutions $y_1, \ldots, y_n$ satisfies
$W' + p_{n-1}(t) W = 0$.

Initial Value Problem

37. Let matrix A be 3×3. Assume $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ has scalar general solution $x_1 = c_1 e^t + c_2 e^{-t} + t$, $x_2 = (c_1 + c_2)e^t + c_3 e^{2t}$, $x_3 = (c_1 + c_2)e^t - 2c_2 e^{-t} + c_3 e^{2t} + t$. Given initial conditions $x_1(0) = x_2(0) = 0$, $x_3(0) = 1$, solve for c_1, c_2, c_3.

38. Let matrix A be 3×3. Assume $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ has scalar general solution $x_1 = c_1 + c_2 t + c_3 t^2 + e^t$, $x_2 = c_2 + c_3 t + e^{2t}$, $x_3 = c_3$. Find the **vector** particular solution $\vec{\mathbf{x}}$ for initial conditions $x_1(0) = x_2(0) = 0$, $x_3(0) = 1$.

Equilibria

39. Find all equilibria:
$$\vec{\mathbf{x}}' = \begin{pmatrix} \cos(t) & \cos(t) \\ 2 & 2 \end{pmatrix} \vec{\mathbf{x}}$$

40. Find all equilibria:
$$\vec{\mathbf{x}}' = \begin{pmatrix} \sin(t) & \sin^2(t) \\ 2 & 2 \end{pmatrix} \vec{\mathbf{x}}$$

11.4 Exercises

Matrix Exponential.

1. **(Picard)** Let A be real 2×2. Write out the two initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$ of e^{At}.

2. **(Picard)** Let A be real 3×3. Write out the three initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$, $\vec{\mathbf{w}}_3(t)$ of e^{At}.

3. Let A be real 2×2. Show that $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{u}}_0$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{u}}_0$.

4. Let A be real $n \times n$. Show that $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$.

Matrix Exponential 2×2.

Find e^{At} from representation $e^{At} = \left\langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \right\rangle$. Use first-order scalar methods.

5. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

6. $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

7. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

8. $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.

Matrix Exponential Identities.

Verify from exponential identities.

9. $e^{A}\, e^{-A} = I$

10. $e^{-A} = \left(e^{A}\right)^{-1}$

11. $A = \dfrac{d}{dt}\, e^{At}$ at $t = 0$

12. If $A^3 = \mathbf{0}$, then $e^{A} = I + A + \frac{1}{2}A^2$.

13. Let A be 2×2 diagonal and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Prove $N^2 = \mathbf{0}$ and $e^{At+Nt} = e^{At}(I + Nt)$.

14. Let A be 3×3 diagonal and $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Prove $N^3 = \mathbf{0}$ and $e^{At+Nt} = e^{At}(I + Nt + N^2 \dfrac{t^2}{2})$.

15. $e^{\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix} t} = \begin{pmatrix} e^t & e^{2t}-e^t \\ 0 & e^{2t}\end{pmatrix}$

16. $e^{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} t} = \begin{pmatrix} e^t & te^t \\ 0 & e^t\end{pmatrix}$

Putzer's Spectral Formula.

17. Apply Picard-Lindelöf theory to conclude that $r_1, \ldots, r_n$ are everywhere defined,

18. Prove that $P_1, \ldots, P_k$ commute.

Putzer's Formula 2×2 .

19. Find a formula for $\dfrac{d}{dt} e^{At}$ for a 2×2 matrix A with eigenvalues $1, 2$.

20. Let 2×2 matrix A have duplicate eigenvalues $0, 0$. Compute r_1, r_2 and then report e^{At}.

Putzer: Real Distinct.
Find the matrix exponential.

21. $A = \begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}$

22. $A = \begin{pmatrix}1 & 0\\ 2 & 3\end{pmatrix}$

Putzer: Real Equal.
Find the matrix exponential.

23. $A = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

24. $A = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

Putzer: Complex Eigenvalues.
Find the matrix exponential.

25. $A = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

26. $A = \begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}$

How to Remember Putzer's 2×2 Formula.

27. Find $\lim_{\lambda \to \lambda_1} \dfrac{e^{\lambda t} - e^{\lambda_1 t}}{\lambda - \lambda_1}$.

28. Let matrix A be 2×2 real. Take the real part: $e^{At} = I + \dfrac{e^{it} - e^{-it}}{2i} A$.

Classical $n \times n$ Spectral Formula. Find e^{At}.

29. $A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

30. $A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Proofs of Matrix Exponential Properties.

31. Let $A\vec{\mathbf{u}} = B\vec{\mathbf{u}}$ for all vectors $\vec{\mathbf{u}}$. Prove $A = B$.

32. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Compute the first four Picard iterates for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$.

Special Cases e^{At}.

33. Show the details to solve
$x_1' = 2x_1 + x_3,$
$x_2' = 3x_2 + x_3,$
$x_3' = 4x_3,$
$x_1(0) = 1, x_2(0) = x_3(0) = 0.$

34. Let $A = \mathbf{diag}(1, 2, 3, 4)$. Find e^{At}.

35. Let $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A = \mathbf{diag}(B, B)$. Find e^{At}.

36. Let $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $A = \mathbf{diag}(B, B)$. Find e^{At}.

11.5 Exercises

Determinant $|A - rI|$

Justify these statements.

1. Subtract r from the diagonal of A to form $|A - rI|$.

2. If A is 2×2, then $|A - rI|$ is a quadratic.

3. If A is 3×3, then $|A - rI|$ is a cubic.

4. Expansion of $|A - rI|$ by the cofactor rule often preserves factorizations.

5. If A is triangular, then $|A - rI|$ is the product of diagonal entries.

6. The *combo, mult* and *swap* rules for determinants are generally counter-productive for expansion of $|A - rI|$.

Characteristic Polynomial

Show expansion details for $|A - rI|$.

7. $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$.
Ans: $(2 - r)(4 - r)$

8. $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$.
Ans: $(2 - r)(5 - r)(7 - r)$

Eigenanalysis Method: 2×2

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

9. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

10. $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Eigenanalysis Method: 3×3

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

11. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

12. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Eigenanalysis Method: $n \times n$

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

13. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

14. $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

e^{At} for Simple Eigenvalues

Find a^{At} using classical spectral theory. Check by computer.

15. $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

16. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

e^{At} for Multiple Eigenvalues

Find a^{At} using classical spectral theory. Check by computer.

17. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

18. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Cayley-Hamilton Theorem

Prove the identity by applying the Cayley-Hamilton Theorem.

19. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a_0 = |A| = ad - bc$, $a_1 = \mathbf{trace}(A) = a + d$. Then

$$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

20. Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$. Then:

$$(2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

CHZ Theorem: Scalar Form

21. Write Theorem 11.27 proof details for $n = 3$.

22. Write Theorem 11.27 proof details for any n.

CHZ Theorem: Vector Form

23. Write Theorem 11.28 proof details for $n = 2$.

24. Write Theorem 11.28 proof details for $n = 3$.

CHZ Identity: Vandermonde

Find matrix $D = \left\langle \vec{\mathbf{d}}_1 | \cdots | \vec{\mathbf{d}}_n \right\rangle$ using Theorems 11.29, 11.31, given $\vec{\mathbf{x}}(0) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

25. $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$. Ans: $W(0)^T, D =$ $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & c_1 \\ 2c_1 + c_2 & -2c_1 \end{pmatrix}$

26. $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Ans: $W(0)^T, D =$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}, \begin{pmatrix} c_1 & 0 & 0 \\ -2c_1 & 2c_1 + c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$

CHZ and Eigenvectors

Supply details for the following.

27. Find a scalar 3rd order linear differential equation that has e^t, e^{2it}, e^{-2it} as solutions. Apply theorems to conclude that the Wronskian of the exponentials is invertible for every t.

28. Assume $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ are independent exponentials . Apply theorems to conclude that the Wronskian of the exponentials is invertible for every t.

29. If $\vec{\mathbf{d}}_1 e^t + \vec{\mathbf{d}}_2 e^{-t} + \vec{\mathbf{d}}_3 e^{2t} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} e^{3t}$, then $\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_2 = \vec{\mathbf{d}}_3 = \vec{\mathbf{0}}$.

30. Independence of atoms applied to the n-vector equation $\vec{\mathbf{d}}_1 e^t + \vec{\mathbf{d}}_2 e^{-t} = c_1 \vec{\mathbf{v}}_1 e^t + c_2 \vec{\mathbf{v}}_2 e^{-t}$ implies $\vec{\mathbf{d}}_1 = c_1 \vec{\mathbf{v}}_1$ and $\vec{\mathbf{d}}_2 = c_2 \vec{\mathbf{v}}_2$.

31. There is a 2×2 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ for which CHZ vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ are not eigenvectors of A.

32. Let A be the 3×3 identity matrix. For $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, two of the CHZ vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3$ are zero.

Eigenvectors by Matrix Multiply Find the eigenvectors of A by Theorem 11.33. Report the choice of $\vec{\mathbf{U}}$.

33. $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$. Ans: $\vec{\mathbf{U}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

34. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Ans: $\vec{\mathbf{U}} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

CHZ 2×2 Matrix Shortcut Find the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ using Theorem 11.36.

35. $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $r = -2, 4$

36. $A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$, $r = 1 \pm 3i$

CHZ Scalar 2×2 Shortcut Find the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ using Theorem 11.35.

37. $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, $r = -3, 5$

38. $A = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$, $r = 1 \pm 4i$

Putzer's 2×2 Spectral Formula Verify the identity.

39. $A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}$

$$e^{At} = e^{5t} I + \frac{e^{5t} - e^{2t}}{3} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

40. $A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$

$$e^{At} = e^{-2t} I + \frac{e^{3t} - e^{-2t}}{5} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$$

41. $A = \begin{pmatrix} 0 & 1 \\ -16 & 8 \end{pmatrix}$

$e^{At} = e^{4t}I + te^{4t}\begin{pmatrix} -4 & 1 \\ -16 & 4 \end{pmatrix}$

42. $A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$, $e^{At} =$

$e^{3t}\cos(2t)I + e^{3t}\sin(2t)\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

11.6 Exercises

Jordan block definition. Write out the Jordan form matrix explicitly.

1. $\mathbf{diag}(B(7,2), B(5,3))$

Answer: $\begin{pmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$

2. $\mathbf{diag}(B(0,2), B(4,3))$

3. $\mathbf{diag}(B(-1,1), B(-1,2), B(5,3))$

4. $\mathbf{diag}(B(1,1), B(5,2), B(5,3))$

Jordan form definition. Which are Jordan forms and which are not? Explain.

5. $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$

6. $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{pmatrix}$

8. $\begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$

Decoding $A = PJP^{-1}$. Decode $A = PJP^{-1}$ in each case, displaying explicitly the Jordan chain relations.

9. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$,

$J = \mathbf{diag}(-4, B(4,2), 2, 2)$

10. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$,
$J = \textbf{diag}(-4, 4, 4, 0, 0)$

Geometric and algebraic multiplicity.

Determine $\textbf{GeoMult}(\lambda)$ and $\textbf{AlgMult}(\lambda)$.

11. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$, $\lambda = 4$

12. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$, $\lambda = 4$

Generalized eigenvectors.

Find all generalized eigenvectors and represent $A = PJP^{-1}$. Check the answer in a computer algebra system.

13. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$,
Answer: $J = \textbf{diag}(-4, 2, 2, 4, 4)$,
$P = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

14. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$,
Answer: $J = \textbf{diag}(-4, 4, 4, 0, 0)$,
$P = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 1 & -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \end{pmatrix}$

15. $A = \begin{pmatrix} 0 & 2 & -2 & -2 \\ 2 & 0 & -2 & -4 \\ 2 & 2 & -4 & -2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$,

Ans: $J = \textbf{diag}(0, -4, -2, -2)$,

$P = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & -4 & 0 \\ 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

16. $A = \begin{pmatrix} -2 & 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(2, 2, B(2,3))$,

$P = \begin{pmatrix} 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

17. $A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(B(2,3), B(2,2))$,

$P = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

18. $A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(B(2,4), 2)$,

$P = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Number of Jordan Blocks. Outlined here is the derivation of

$$s(j) = 2k(j-1) - k(j-2) - k(j).$$

Definitions:

- $s(j)$= number of blocks $B(\lambda, j)$
- $N = A - \lambda I$
- $k(j) = \dim(\mathbf{kernel}(N^j))$
- $L_j = \mathbf{kernel}(N^{j-1})^\perp$ relative to $\mathbf{kernel}(N^j)$
- $\ell(j) = \dim(L_j)$
- p minimizes
  $\mathbf{kernel}(N^p) = \mathbf{kernel}(N^{p+1})$

19. Verify $k(j) \le k(j+1)$ from

$$\mathbf{kernel}(N^j) \subset \mathbf{kernel}(N^{j+1}).$$

20. Verify the direct sum formula

$$\mathbf{kernel}(N^j) = \mathbf{kernel}(N^{j-1}) \oplus L_j.$$

Then $k(j) = k(j-1) + \ell(j)$.

21. Given $N^j\vec{\mathbf{v}} = \vec{\mathbf{0}}$, $N^{j-1}\vec{\mathbf{v}} \ne \vec{\mathbf{0}}$, define $\vec{\mathbf{v}}_i = N^{j-i}\vec{\mathbf{v}}$, $i = 1, \ldots, j$. Show that these are independent vectors satisfying Jordan chain relations $N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$, $N\vec{\mathbf{v}}_{i+i} = \vec{\mathbf{v}}_i$.

22. A block $B(\lambda, p)$ corresponds to a Jordan chain $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p$ constructed from the Jordan decomposition. Use $N^{j-1}\vec{\mathbf{v}}_j = \vec{\mathbf{v}}_1$ and $\mathbf{kernel}(N^p) = \mathbf{kernel}(N^{p+1})$ to show that the number of such blocks $B(\lambda, p)$ is $\ell(p)$. Then for $p > 1$, $s(p) = k(p) - k(p-1)$.

23. Show that $\ell(j-1) - \ell(j)$ is the number of blocks $B(\lambda, j)$ for $2 < j < p$. Then

$$s(j) = 2k(j-1) - k(j) - k(j-2).$$

24. Test the formulas above on the special matrices

$$A = \mathsf{\mathbf{diag}}(B(\lambda,1), B(\lambda,1), B(\lambda,1)),$$

$$A = \mathsf{\mathbf{diag}}(B(\lambda,1), B(\lambda,2), B(\lambda,3)),$$

$$A = \mathsf{\mathbf{diag}}(B(\lambda,1), B(\lambda,3), B(\lambda,3)),$$

$$A = \mathsf{\mathbf{diag}}(B(\lambda,1), B(\lambda,1), B(\lambda,3)),$$

Computing Jordan m-chains. Find the Jordan m-chain formulas for the given eigenvalue. Then solve them to find the generalized eigenvectors.

25. $A = \begin{pmatrix} -2 & 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$, $\lambda = 2$

26. $A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$, $\lambda = 2$

Generalized Eigenspace Basis.

Let A be $n \times n$ with distinct eigenvalues λ_i, $n_i = \textsf{\textbf{AlgMult}}(\lambda_i)$ and $E_i = \textbf{kernel}((A - \lambda_i I)^{n_i})$, $i = 1, \ldots, k$. Assume a Jordan decomposition $A = PJP^{-1}$.

27. Let Jordan block $B(\lambda, j)$ appear in J. Prove that a Jordan chain corresponding to this block is a set of j independent columns of P.

28. Let $\mathcal{B}_\lambda$ be the union of all columns of P originating from Jordan chains associated with Jordan blocks $B(\lambda, j)$. Prove that $\mathcal{B}_\lambda$ is an independent set.

29. Verify that $\mathcal{B}_\lambda$ has $\textsf{\textbf{AlgMult}}(\lambda)$ basis elements.

30. Prove that $E_i = \textbf{span}\,(\mathcal{B}_{\lambda_i})$ and $\dim(E_i) = n_i$, $i = 1, \ldots, k$.

Direct Sum Decomposition.

31. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Let $\lambda = 2$. Compute $k = \textsf{\textbf{AlgMult}}(\lambda)$ and a basis of generalized eigenvectors for the subspace $\textbf{kernel}((A - \lambda I)^k)$.

32. Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$, $\vec{\mathbf{y}} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 9 \end{pmatrix}$.
Find $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2$ in decomposition $\vec{\mathbf{y}} = \vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ in Theorem 11.42.

Exponential Matrices. Compute the exponential matrix e^{At} on paper. Check the answer using `maple`.

33. $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

34. $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Nilpotent matrices. Find the nilpotency of N.

35. $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

36. $N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Real Jordan Form

Find Jordan decomposition $A = PJP^{-1}$ where J and P are real matrices.

37. $A = \begin{pmatrix} -2 & 6 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$. Answer:
$\lambda = -2, 4 \pm i$,
$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 4 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

38. $A = \begin{pmatrix} -31 & -10 & 18 \\ -15 & -5 & 10 \\ -54 & -20 & 32 \end{pmatrix}$. Answer:
$\lambda = -4, \pm 5i$
$J = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -5 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 0 & 3 \\ 3 & -6 & 6 \end{pmatrix}$

Solving $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Solve for $\vec{\mathbf{x}}$ in the differential equation.

39. $\vec{\mathbf{x}}' = \begin{pmatrix} -2 & 6 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \vec{\mathbf{x}}$.

40. $\vec{\mathbf{x}}' = \begin{pmatrix} -31 & -10 & 18 \\ -15 & -5 & 10 \\ -54 & -20 & 32 \end{pmatrix} \vec{\mathbf{x}}$.

Numerical Instability

Show directly that Jordan form J of A satisfies $\lim_{\epsilon \to 0+} J(\epsilon) \neq J(0)$.

41. $A = \begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix}$

42. $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & \epsilon & 1 \\ 0 & 0 & 0 \end{pmatrix}$

11.7 Exercises

Variation of Parameters

Let $A(t) = \frac{1}{a(t)}\begin{pmatrix} 0 & 1 \\ -c(t) & -b(t)\end{pmatrix}$,
$\vec{\mathbf{F}}(t) = \frac{1}{a(t)}\begin{pmatrix} 0 \\ f(t)\end{pmatrix}$, $\vec{\mathbf{x}} = \begin{pmatrix} u(t) \\ u'(t)\end{pmatrix}$.

1. Verify equivalence of $a(t)u'' + b(t)u' + c(t)u = f(t)$ and $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

2. For $u'' + 100u = \sin(t)$, find $A(t)$ and $\vec{\mathbf{F}}(t)$.

3. For $u'' = f(t)$, find $A(t)$ and $\vec{\mathbf{F}}(t)$.

4. For $u'' = f(t)$, let $u_1 = 1$, $u_2 = t$, $\Phi(t) = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2'\end{pmatrix}$. Verify $|\Phi(t)| = 1$, then find $A(t) = \Phi'(t)\Phi^{-1}(t)$.

5. State Theorem 11.46 for $n = 2$, then explain how it applies to this special case.

6. Prove Theorem 11.47 using the previous exercise.

Variation of Parameters: Scalar 2nd Order

Let $a(t)u'' + b(t)u' + c(t)u = 0$ have two independent solutions u_1, u_2.
Define $\Psi(t) = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2'\end{pmatrix}$. Then:

7. Matrix $\Psi(t)$ has an inverse.

8. Matrix $\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$ is invertible and $\Phi(t_0) = I$.

9. Let $\Psi(t) = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$. Define
$\begin{pmatrix} u \\ v\end{pmatrix} = \Psi(t)\int_0^t \Psi^{-1}(s)f(s)ds$.
Then u is a particular solution of $u'' = f(t)$.

10. Let $\Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t}\end{pmatrix}$. Define
$\begin{pmatrix} u \\ v\end{pmatrix} = \Psi(t)\int_0^t \Psi^{-1}(s)f(s)ds$.
Then u is a particular solution of $u'' - u = f(t)$.

Variation of Parameters

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3\end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ using $\vec{\mathbf{x}}_p = \int_0^t e^{A(t-s)}\vec{\mathbf{F}}(s)ds$ and computer assist.

11. $\vec{\mathbf{F}}(t) = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{\mathbf{x}}_p = e^t \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

12. $\vec{\mathbf{F}}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$,

$\vec{\mathbf{x}}_p = e^{-t} \begin{pmatrix} -1 \\ -\frac{1}{4} \end{pmatrix}$

Undetermined Coefficients

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ by undetermined coefficients. Assume

$\vec{\mathbf{x}}_h(t) = (c_1 + c_2)e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

13. $\vec{\mathbf{F}}(t) = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

$\vec{\mathbf{x}}_p = \begin{pmatrix} e^{-t} + 3te^t - e^t \\ e^t - e^{-t} \end{pmatrix}$

14. $\vec{\mathbf{F}}(t) = 2 \begin{pmatrix} \cos t \\ e^t \end{pmatrix}$,

$\vec{\mathbf{x}}_p = \begin{pmatrix} 2te^t + \sin(t) - \cos(t) + e^{-t} \\ e^t - e^{-t} \end{pmatrix}$

Undetermined Coefficients

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ by undetermined coefficients. Assume $\vec{\mathbf{x}}_h(t) = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{pmatrix}$.

15. $\vec{\mathbf{F}}(t) = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{\mathbf{x}}_p = e^t \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

16. $\vec{\mathbf{F}}(t) = 4 \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$, $\vec{\mathbf{x}}_p = e^{-t} \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

17. $\vec{\mathbf{F}}(t) = 10 \begin{pmatrix} \cos t \\ e^t \end{pmatrix}$,

$\vec{\mathbf{x}}_p = \begin{pmatrix} -4\cos(t) + 2\sin(t) \\ -5e^t \end{pmatrix}$

18. $\vec{\mathbf{F}}(t) = 2e^t \begin{pmatrix} \cos t \\ 1 \end{pmatrix}$,

$\vec{\mathbf{x}}_p = e^t \begin{pmatrix} -\cos(t) + \sin(t) \\ -1 \end{pmatrix}$

11.8 Exercises

Euler's Substitution: $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$

1. Change variables: $\vec{\mathbf{u}} = e^{rt}\vec{\mathbf{w}}$. Answer: $\vec{\mathbf{w}}' = (C - rI)\vec{\mathbf{w}}$

2. Prove: $(\lambda, \vec{\mathbf{v}})$ is an eigenpair of C if and only if $(0, \vec{\mathbf{v}})$ is an eigenpair of $C - \lambda I$.

3. Let $|C - \lambda I|$ have factor λ^2. Let $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ have solution $\vec{\mathbf{u}} = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2$. Prove: $C\vec{\mathbf{d}}_2 = \vec{\mathbf{0}}$, $C\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_2$. Are $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ eigenvectors of C? Discuss.

4. Let $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{\mathbf{u}} = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2$. Let $\vec{\mathbf{u}}$ solve $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$. Find $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ in terms of arbitrary constants c_1, c_2.

Euler's Substitution: $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$

5. Change variables: $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{y}}$. Answer: $\vec{\mathbf{y}}'' = (A - r^2 I)\vec{\mathbf{y}}$

6. Prove: $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{v}}$ is a nonzero solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ if and only if $(r^2, \vec{\mathbf{v}})$ is an eigenpair of A.

Repeated Root: $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, eigenvalues $0, 0$.

7. Verify: Matrix A is a Jordan block with generalized eigenvectors the columns of I.

8. Prove: $x_1 = c_1 + c_2 t + c_3 \frac{t^2}{2} + c_4 \frac{t^3}{6}$, $x_2 = c_3 + c_4 t$ for arbitrary constants c_1 to c_4.

9. Prove: The solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ is a vector linear combination of atoms $1, t, t^2, t^3$.

10. Let $\vec{\mathbf{x}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 t + \vec{\mathbf{d}}_3 \frac{t^2}{2} + \vec{\mathbf{d}}_4 \frac{t^3}{6}$. Assume $\vec{\mathbf{x}}$ solves $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$. Prove: $A\vec{\mathbf{d}}_3 = A\vec{\mathbf{d}}_4 = \vec{\mathbf{0}}$, $A\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_3$, $A\vec{\mathbf{d}}_2 = \vec{\mathbf{d}}_4$. These are generalized eigenvector chains for eigenvalue zero.

CHZ Method

11. Given a 3×3 matrix A, supply proof details for the Cayley-Hamilton-Ziebur structure theorem.

12. Solve $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ by CHZ for any 3×3 diagonal matrix.

13. Solve $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ by CHZ for any $n \times n$ diagonal matrix.

14. Invent a non-diagonal 3×3 example $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ and solve it by CHZ.

Conversion

Given $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, let $\vec{\mathbf{u}} = \begin{pmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{pmatrix}$. Display system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$.

15. $A = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$

16. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

Eigenanalysis $\lambda \le 0$

Display the general solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

17. $A = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$

18. $A = \begin{pmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \\ 5 & 0 & -1 \end{pmatrix}$

Earthquakes

Apply formulas page **??** to find homogenous solution $\vec{\mathbf{x}}_h$, the amplitude of $\vec{\mathbf{x}}_p(t)$ and the natural frequencies ω_j. Assume $F(t) = F_0 \cos(\omega t)$.

19. Three-floor problem, $k/m = 10$.

20. Four-floor problem, $k/m = 10$.

Two Masses

Assume MKS units. Let $m_1 = 2$, $m_2 = 0.5$, $k_1 = 75$, $k_2 = 25$ in system:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2[x_2 - x_1] \\ m_2 x_2'' &= -k_2[x_2 - x_1] \end{aligned}$$

21. Convert the system to the form $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

22. Show details for finding the vector solution $\vec{\mathbf{x}}(t)$.

Three Rail Cars: $k{=}2m$

Assume MKS units. Consider

$$\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -2 \end{pmatrix} \vec{\mathbf{x}}$$

23. Show eigenpair details for the 3×3 matrix.

24. Find the vector solution $\vec{\mathbf{x}}(t)$.

Three Rail Cars: Disengagement

For $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, assume FPS units and

$$A = \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}$$

Suppose the springs disengage upon full expansion. Let the cars engage at $t = 0$ with $x_1 = x_2 = x_3 = 0$.

25. Verify A has eigenvalues $\lambda = -16, 0, -4$ and corresponding eigenvectors
$\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

26. For $x_1{=}x_2{=}x_3{=}0$ at $t{=}0$, verify:
$x_1(t) = c_2 t + c_1 \sin(2t) - c_3 \sin(4t)$,
$x_2(t) = c_2 t + 3c_3 \sin(4t)$,
$x_3(t) = c_2 t - c_1 \sin(2t) - c_3 \sin(4t)$

27. Let $x_1' = 0$, $x_2' = 0$, $x_3' = 48$ at $t = 0$. Verify disengagement time $t_1 = \pi/2$ and determine the car velocities thereafter.

28. Let $x_1' = 144$, $x_2' = 48$, $x_3' = 48$ at $t = 0$. Verify disengagement time $t_1 = \pi/2$ and determine the car velocities thereafter.

Dynamic Dashpot

Assume conventions for Figure 26 and system

$$\begin{aligned} m_s X'' &= -k_1 X - d_1 X' - k_2(Y - X) \\ &\quad - d_2(Y' - X') + F(t), \\ m_b Y'' &= k_2(Y - X) + d_2(Y' - X') \end{aligned}$$

29. Assume $Y = 0$, ideal suspension. Derive:

$$\begin{aligned} &m_s X'' = -k_1 X - d_1 X' + F(t), \\ &d_2 X' + k_2 X = 0 \end{aligned}$$

30. Assume $Y = 0$, ideal suspension and $X(0) = 0.015$ meters. Find $X(t)$ and $F(t)$.

11.9 Exercises

Planar Methods

Apply the Euler, Heun and RK4 methods. Compare with the exact solution in a table.

1. $x' = x$, $y' = -y$, $x(0) = 2$, $y(0) = 2$. $h = 0.1$, 10 steps

2. $x' = -3x + y$, $y' = x - 3y$, $x(0) = 2$, $y(0) = 0$, $h = 0.1$, 10 steps

3. $x' = -x + y$, $y' = -x - y$, $x(0) = 0$, $y(0) = 3$, $h = 0.2$, 5 steps

4. $x' = 2x - 4y$, $y' = x - 3y$, $x(0) = 4$, $y(0) = 0$, $h = 0.1$, 10 steps

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, 2×2

Apply vector Euler, Heun and RK4 methods for 10 steps with $h = 0.1$.

5. $\vec{\mathbf{u}}' = \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

6. $\vec{\mathbf{u}}' = \begin{pmatrix} -3u_1 + u_2 \\ u_1 - 3u_2 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$

Apply vector Euler, Heun and RK4 methods for 10 steps with $t_0 = 0$, $h = 0.1$. Compare results for the last step.

7. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $\vec{\mathbf{F}} = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$,

$\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Ans Euler: $3.99, -4.12$

8. $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{F}} = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}$,

$\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans RK4: $3.03, -4.46, 0.0$

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, 3×3

Apply vector Euler, Heun and RK4 methods for 10 steps with $h = 0.1$.

9. $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans Heun: $1.86, -3.00, 0.00$

10. $A = \begin{pmatrix} 1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans RK4: $-1.17, -3.27, 0.00$

Chapter 12

Series Methods and Approximations

Contents

12.2 Exercises

Series Convergence

Find R, the radius of convergence.

1. $\sum_{k=2}^{\infty} \frac{x^k}{k\ln(k)}$

2. $\sum_{k=1}^{\infty} a_k x^k$, $a_{2n} = 2$, $a_{2n+1} = 4$.

Series Properties

Compute the series given by the indicated operation(s).

3. $\frac{d}{dx} \sum_{k=2}^{\infty} \frac{x^k}{k \ln(k)}$

4. $4 \sum_{k=1}^{\infty} \frac{1}{1+k} x^k + \sum_{k=2}^{\infty} \frac{1}{1+k^2} x^k$

MaClaurin Series

Find the MaClaurin series expansion.

5. $f(x) = \frac{1}{1+x^3}$ for $|x| < 1$.

6. $f(x) = \arctan(x)$, using $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$.

7. $f(x) = \left(\frac{3}{2}\right)^x$ for all x.

8. $f(x) = \int_0^x \frac{\sin t}{t}\, dt$, called the **Sine Integral**.

9. $f(x)$ is the solution of $f' = 1 + xf$, $f(0) = 0$.

10. The first 4 terms, $f(x) = \tan x$.

Taylor Series

Find the series expansion about the given point.

11. $f(x) = \ln |1 - x|$, at $x = 0$.

12. $f(x) = \frac{1}{x^2}$, at $x = 1$.

12.3 Exercises

Differentiation

Verify using term–by–term differentiation. Document all series and calculus steps.

1. $\frac{d}{dx}\sum_{n=1}^{\infty}\frac{1}{n}\,x^n = \sum_{n=1}^{\infty} x^n$.
Is this valid for $x = -1$?

2. $\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\,x^{2n+1} = \sum_{n=0}^{\infty}(-1)^n\,x^{2n}$.

Subscripts

Perform a change of variables to verify the identity.

3. $\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{k=2}^{\infty} c_{k-2}\,x^k$

4. $\sum_{n=2}^{\infty} n(n-1)c_n(x-x_0)^{n-2} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}\,(x-x_0)^k$

5. $-1+x+\sum_{n=2}^{\infty}(-1)^{n+1}\,x^n = \sum_{k=0}^{\infty}(-1)^{k+1}\,x^k$

6. $\sum_{n=0}^{\infty}\frac{1}{n+1}\,x^n + \sum_{m=1}^{\infty}\frac{1}{m+2}\,x^m = 1+\sum_{k=1}^{\infty}\frac{2k+1}{(k+1)(k+2)}\,x^k$

Linearity

Find the power series of the given function.

7. $\cos(x) + 2\sin(x)$

8. $e^x + \sin(x)$

Cauchy Product

Find the power series.

9. $(1+x)\sin(x)$

10. $\frac{(}{\sin}(x)e^x$

Recursion Relations

Solve the given recursion.

11. $x_{k+1} = 2x_k$

12. $x_{k+1} = 2x_k + 1$

13. $x_{k+2} = 2x_k + 1$

14. $x_{k+3} = 2x_k + 1$

12.4 Exercises

First Order Series Method

Solve by power series.

1. $y' - 4y = 0$

2. $y' - xy = 0$

Second Order Series Method

Solve by power series using the Airy equation example.

3. $y'' = 4y$

4. $y'' + y = 0$

Taylor Series Method

Solve by Taylor series, finding the first 8 terms.

5. $y' = 16y$

6. $y'' = y$

7. $y' = (1 + x)y$

8. $y'' = (2 + x)y$

12.5 Exercises

Standard Form

Convert to form $y'' + p(x)y' + q(x)y = 0$. Find the singular points and ordinary points.

1. $(x+1)y'' + xy' + y = 0$

2. $x^2y'' + 3xy' + 4y = 0$

3. $x(1+x)y'' + xy' + (1+x)y = 0$

4. $xy'' = (1+x)y' + e^x y$

Ordinary Point Method

Find a power series solution, following the method in the text for $y'' - 2xy' + y = 0$. Use a CAS or mathematical workbench to check the answer.

5. $y'' + xy' = 0$

6. $y'' + x^2y' + y = 0$

12.6 Exercises

Regular Singular Point

Test the equation for regular singular points.

1. $x^2y'' + xy' + y = 0$

2. $x^2(x-1)y'' + \sin(x)y' + y = 0$

3. $x^3(x^2-1)y'' - x(x+1)y' + (1-x)y = 0$

4. $x^3(x-1)y'' + (x-1)y' + 2xy = 0$

Indicial Equation

Each equation is an Euler differential equation $ax^2y'' + bxy' + cy = 0$ with a, b, c replaced by power series. Find the Euler differential equation and the indicial equation.

5. $x^2y'' - 2x(x+1)y' + (x-1)y = 0$
Ans: $x^2y'' - 2xy' - y = 0$, $r(r-1) - 2r - 1 = 0$.

6. $x^2y'' - 2xy' + y = 0$
Ans: The same equation, $r(r-1) - 2r + 1 = 0$.

7. $xy'' + (1-x)y' + 2y = 0$

8. $x^2y'' - 2xy' + (2 + \sin x)y = 0$

Frobenius Solutions

Find two linearly independent solutions. Follow Examples 1, 2, 3 for cases (a), (b), (c) in the Frobenius Theorem.

9. $2x^2y'' + xy' - y = 0$

10. $4x^2y'' + (2x-7)y' + 6y = 0$

11. $4x^2(x+1)y'' + x(3x-1)y' + y = 0$

12. $3x^2y'' + xy' - (1+x)y = 0$

13. $x^2y'' + 3xy' + (1+x)y = 0$

14. $xy'' + (1-x)y' + 3y = 0$

15. $x^2y'' + x(x-1)y' + (1-x)y = 0$

16. $xy'' + (2x+3)y' + 4y = 0$

12.7 Exercises

Values of J_0 and J_1

Use series representations and identities to compute the decimal values of the following functions. Use a computer algebra system to check the answers.

1. $J_0(1)$

2. $J_1(1)$

3. $J_0(1/2)$

4. $J_1(1/2)$

Bessel Function Properties

Prove the following relations by expanding LHS and RHS in series.

5. $J_0'(x) = -J_1(x)$

6. $J_1'(x) = J_0(x) - \dfrac{1}{x} J_1(x)$

7. $(x^p J_p(x))' = x^p J_{p-1}(x)$,
$p \geq 1$

8. $\left(x^{-p} J_p(x)\right)' = -x^{-p} J_{p+1}(x)$,
$p \geq 0$

Bessel Function Recursion Proofs

Add and subtract the expanded equations of the previous exercises.

9. $J_{p+1} = \dfrac{2p}{x} J_{p+1}(x) - J_{p-1}(x)$,
$p \geq 1$

10. $J_{p+1}(x) = -2J_p'(x) + J_{p-1}(x)$,
$p \geq 1$

Recurrence Relations

Use results of the previous exercises.

11. Express J_3 and J_4 in terms of J_0 and J_1.

12. $\int_0^\infty J_{n+1}(x)dx = \int_0^\infty J_{n-1}(x)dx$
for integers $n > 0$

13. $\int_0^\infty J_n(x)dx = 1$ for integers $n > 1$

14. $\displaystyle\int_0^\infty \frac{J_n(x)dx}{x} = \frac{1}{n}$
for integers $n > 0$

12.8 Exercises

Equivalent Legendre Equations

Prove the following are equivalent to
$(1-x^2)y''-2xy'+(n+1)(n+2)y=0$

1. $((1-x^2)y')' + n(n+1)y = 0$

2. Let $x = \cos\theta$, $' = \frac{d}{d\theta}$, then
$\sin\theta y'' + \cos\theta y' + n(n+1)\sin\theta y = 0$.

Proof of Bonnet's Recursion

Define

$$y_1 = (n+1)P_{n+1},$$
$$y_2 = (2n+1)xP_n - nP_{n-1}.$$

The next 4 exercises prove $y_1 = y_2$, which establishes Bonnet's recursion

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$$

3. Use identity $P_k(1) = 1$ to prove $y_1(1) = n+1$ and $y_2(1) = n+1$.

4. Explain why y_1 is a polynomial solution of
$(1-x^2)y''-2xy'+(n+1)(n+2)y=0$.

5. Prove that y_2 is also a solution of
$(1-x^2)y''-2xy'+(n+1)(n+2)y=0$. Suggestion: use the differential equations for P_n, P_{n-1}.

6. Any polynomial solution of $(1-x^2)y''-2xy'+(n+1)(n+2)y=0$ is a scalar multiple of P_{n+1} (page **??**). Prove that y_1 and y_2 both equal $(n+1)P_{n+1}$, hence $y_1 = y_2$, proving that $(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$.

Boundary Data at $x = \pm 1$

Use these identities:
(1) $(a+b)^k = \sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r}$
(2) $(uv)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$

7. Let $y = x - 1$ in Rodrigues' formula to prove
$P_n(y+1) = \left(\dfrac{d}{dy}\right)^n \left(y^2 + 2y\right)^n$

8. Verify from identity (1):
$\left(y^2+2y\right)^n = \sum_{r=0}^{n} \binom{n}{r} 2^r y^{2n-r}$

9. Prove $P_n(1) = 1$ from the previous exercises.

10. Assume $P_n(-x) = (-1)^n P_n(x)$ and $P_n'(1) = \dfrac{n(n+1)}{2}$. Prove
$P_n(-1) = (-1)^n$ and
$P_n'(-1) = (-1)^n \dfrac{n(n+1)}{2}$.

12.9 Exercises

Legendre series. Establish the following results.

1. Prove $\int_{-1}^{1} |P_n(x)|^2 dx = \frac{2}{2n+1}$.

2. Let $(f, g) = \int_0^\pi f(x)g(x)\sin(x)dx$. Show that the sequence $\{P_n(\cos x)\}$ is orthogonal on $0 \le x \le \pi$ with respect to inner product (f, g).

3. Let $F(x) = \sin^3(x) - \sin(x)\cos(x)$. Expand F as a Legendre series $F(x) = \sum_{n=0}^{\infty} c_n P_n(\cos x)$.

Chebyshev Polynomials. Define

$$T_n(x) = \cos(n \arccos(x)).$$

These are called **Chebyshev polynomials**. Define
$(f, g) = \int_{-1}^{1} f(x)g(x)(1 - x^2)^{-1/2}dx$.

4. Show that $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$.

5. Show that $T_3(x) = 4x^3 - 3x$.

6. Prove that (f, g) satisfies the abstract properties of an inner product.

7. Show that T_n is a solution of the **Chebyshev equation** $(1 - x^2)y'' - xy' + n^2y = 0$.

8. Prove that $\{T_n\}$ is orthogonal relative to the weighted inner product (f, g).

Hermite Polynomials. Define the Hermite polynomials by $H_0(x) = 1$,
$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$.
Define the inner product
$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2/2}dx$.

9. Verify $H_1(x) = x$, $H_2(x) = x^2 - 1$.

10. Show that $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$.

11. Prove that $H_n'(x) = nH_{n-1}(x)$.

12. Prove that $H_{n+1}(x) = xH_n(x) - H_n'(x)$.

13. Show that H_n is a polynomial solution of **Hermite's equation** $y'' - xy' + ny = 0$.

14. Prove that (f, g) defines an inner product.

15. Show that the sequence $\{H_n(x)\}$ is orthogonal with respect to (f, g).

Laguerre Polynomials. Define the Laguerre polynomials by $L_0(x) = 1$, $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x})$. Define $(f, g) = \int_0^\infty f(x)g(x)e^{-x}dx$.

16. Show that $L_1(x) = 1 - x$, $L_2(x) = 1 - 2x + x^2/2$.

17. Show that $L_3(x) = 1 - 3x + 3x^2/2 - x^3/6$.

18. Prove that (f, g) satisfies the abstract properties for an inner product.

19. Show that L_0, L_1, L_2 are orthogonal with respect to the inner product (f, g), using direct integration methods.

20. Show that $L_n(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!k!}(-x)^n$.

21. Show that $\{L_n\}$ is an orthogonal sequence with respect to (f, g).

22. Find an expression for a polynomial solution to **Laguerre's equation** $xy'' + (1 - x)y' + ny = 0$ using Frobenius theory.

23. Show that **Laguerre's equation** is satisfied for $y = L_n$.

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