



DeepLearning.AI

# Math for Machine Learning

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## Linear algebra - Week 4

Bases

Span

Orthogonal and orthonormal bases

Orthogonal and orthonormal matrices



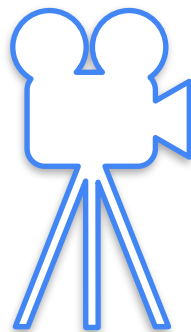
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# Determinants and Eigenvectors

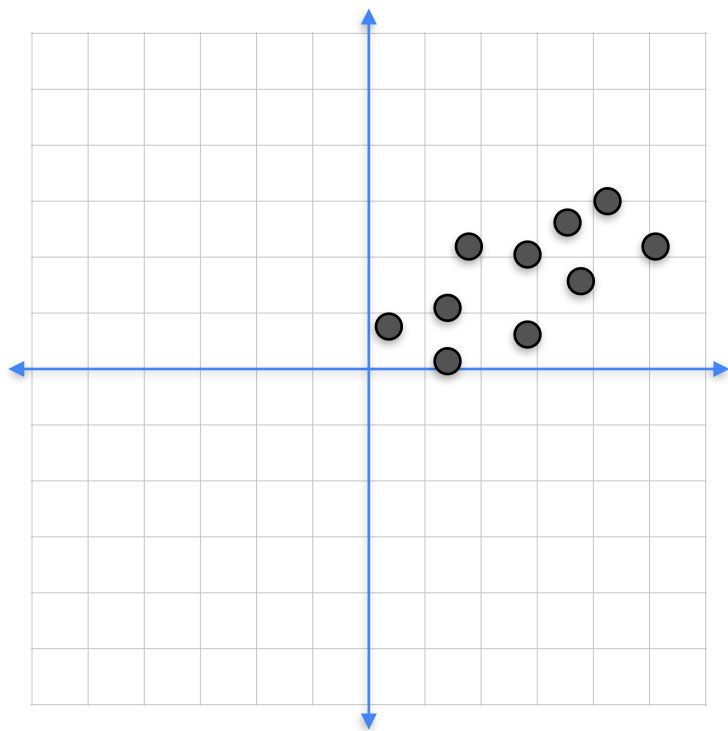
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## **Machine learning motivation**

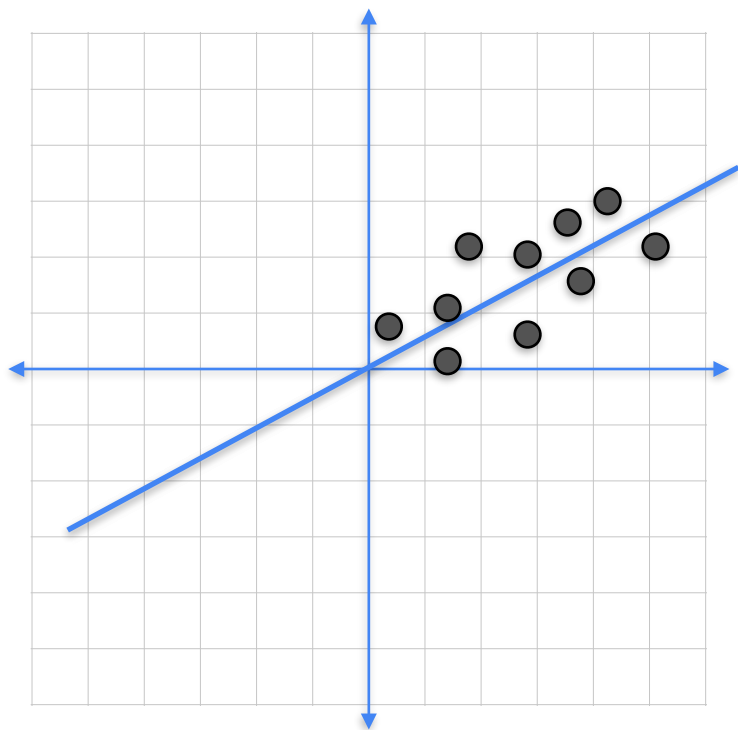
# PCA



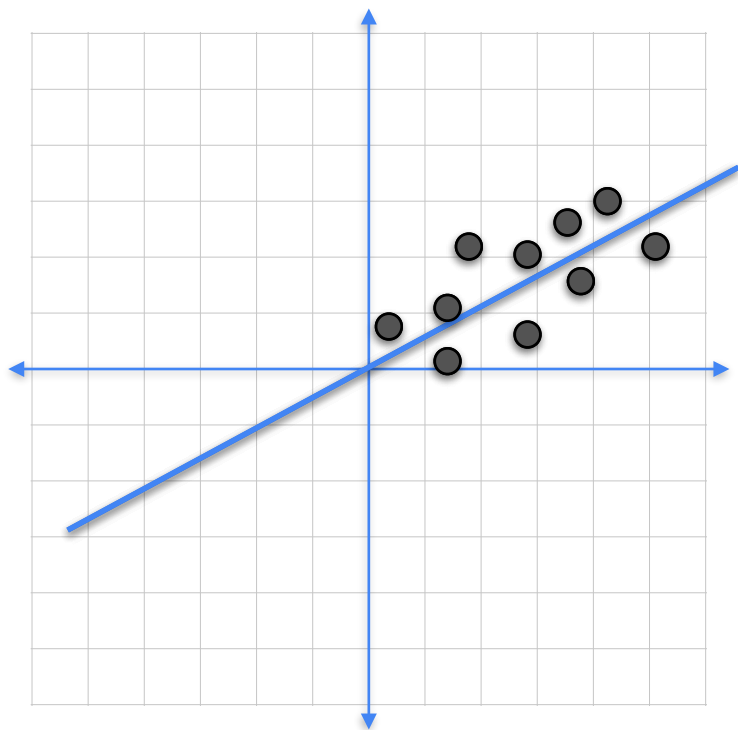
# Principal Component Analysis



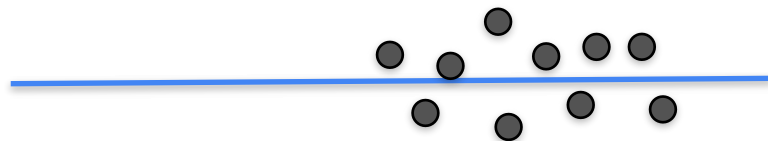
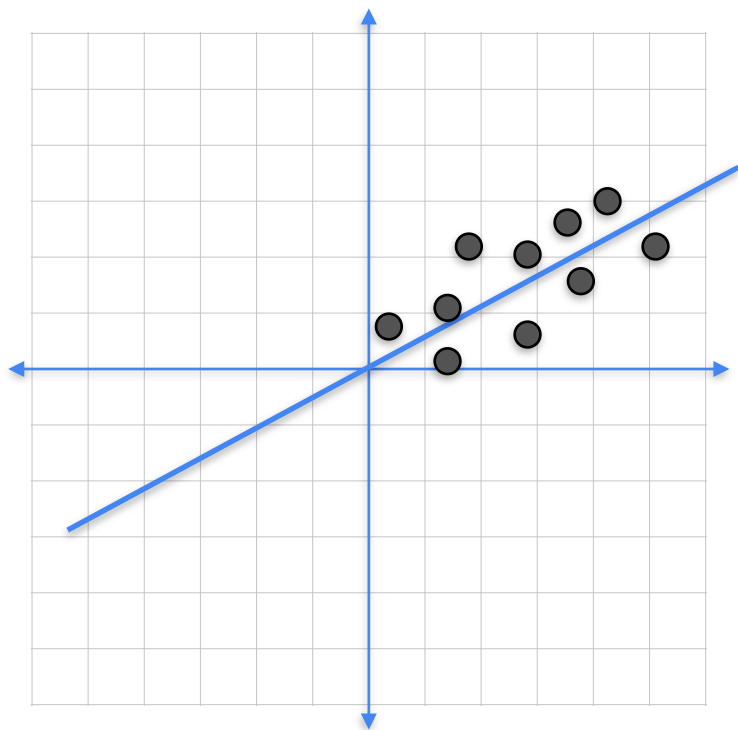
# Principal Component Analysis



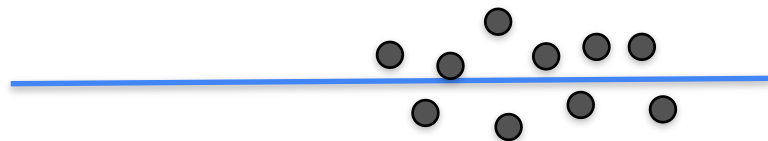
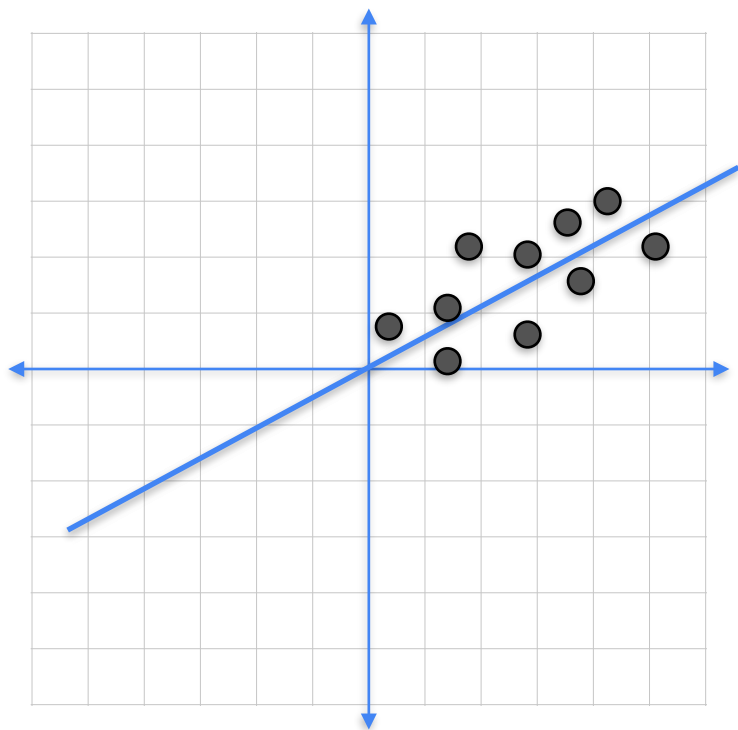
# Principal Component Analysis



# Principal Component Analysis

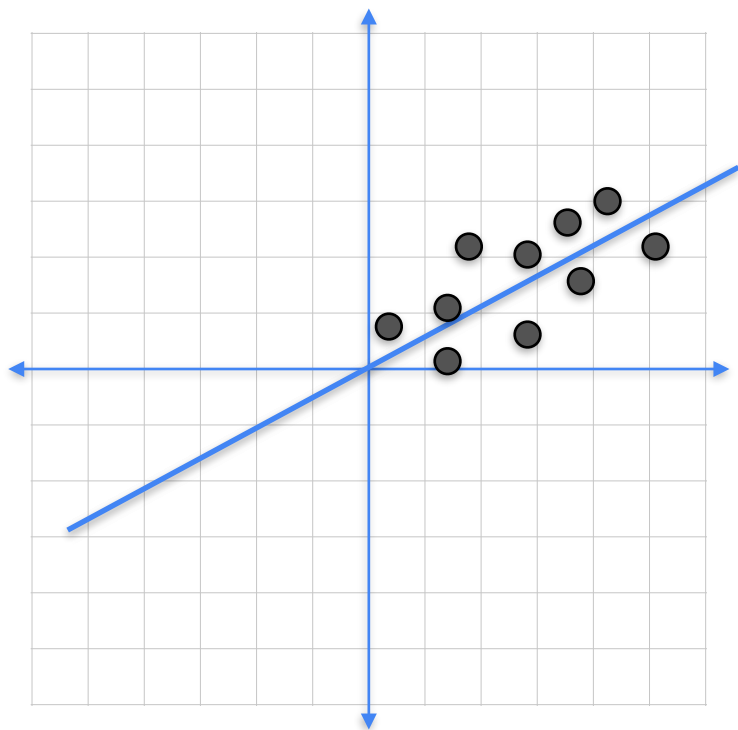


# Principal Component Analysis

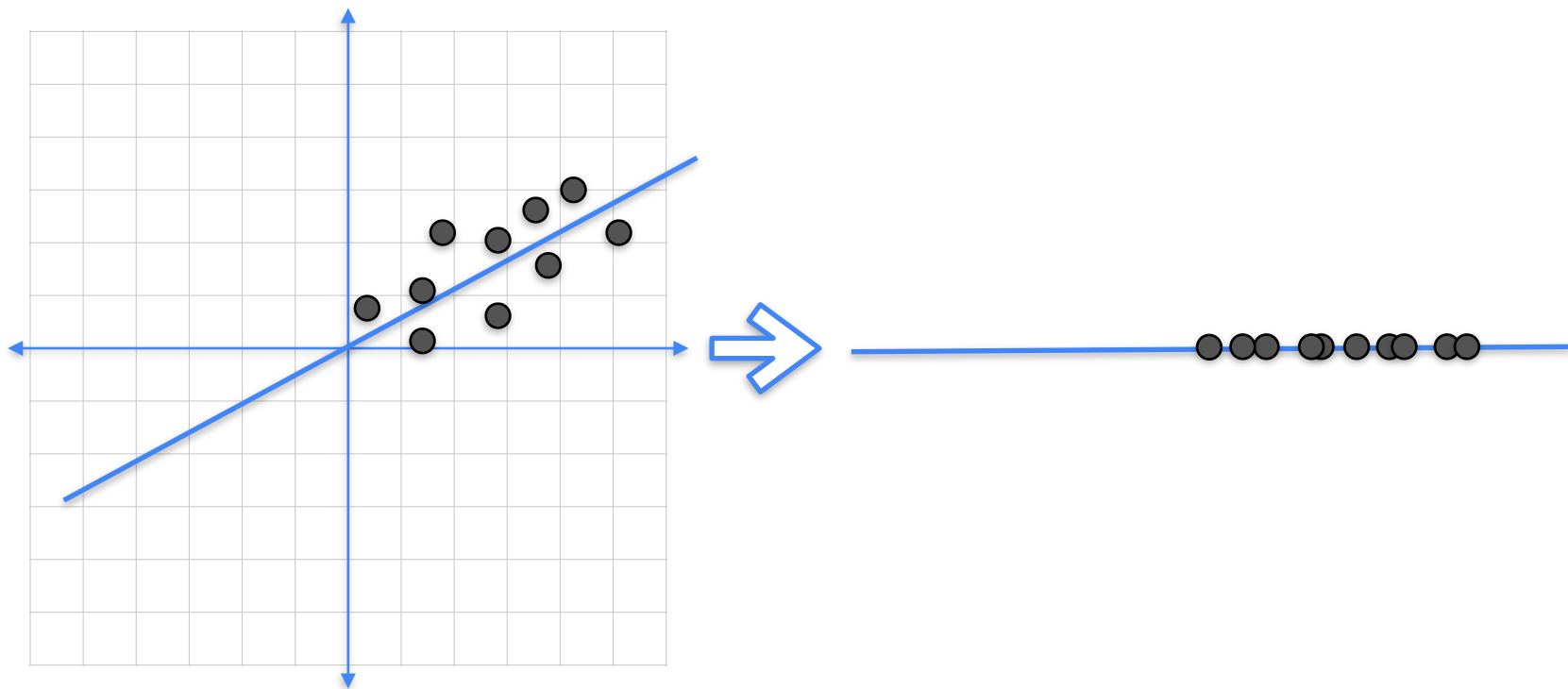




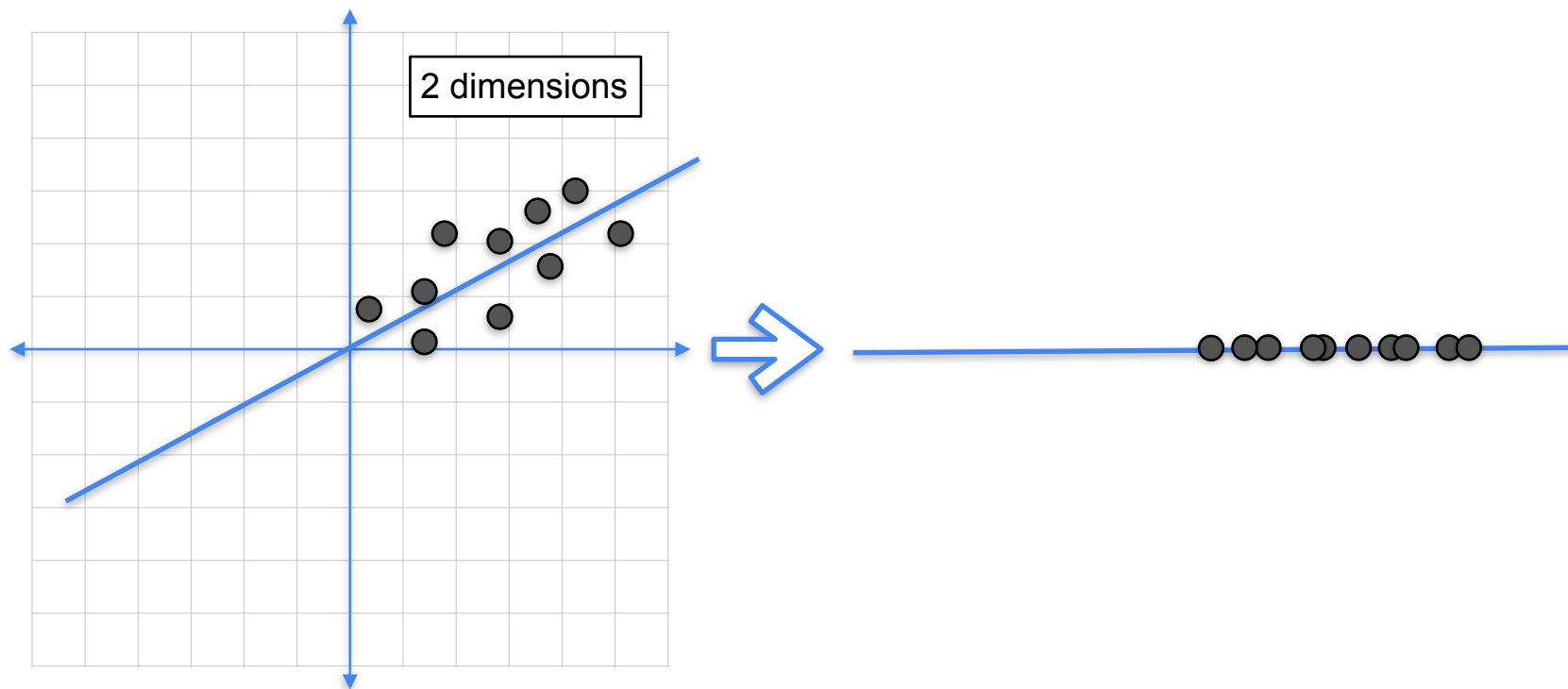
# Principal Component Analysis



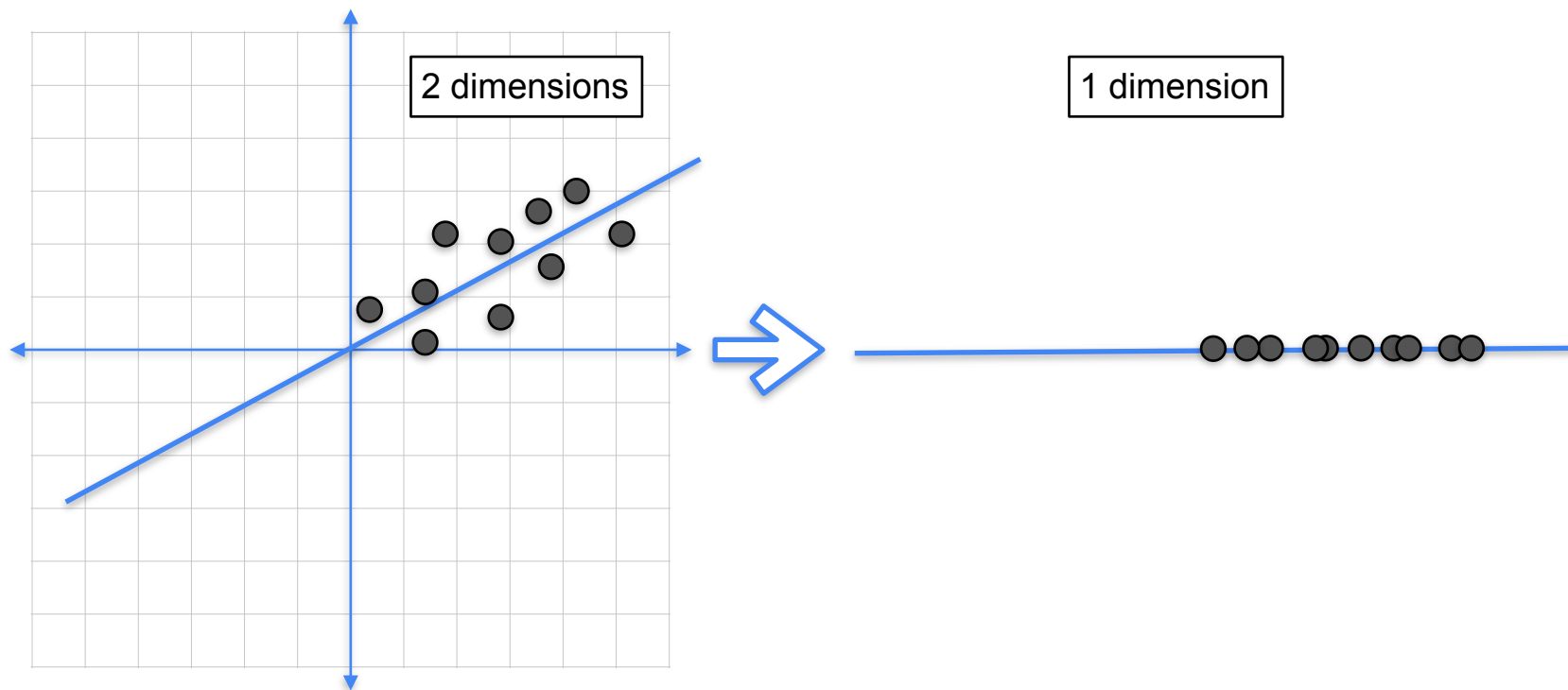
# Principal Component Analysis



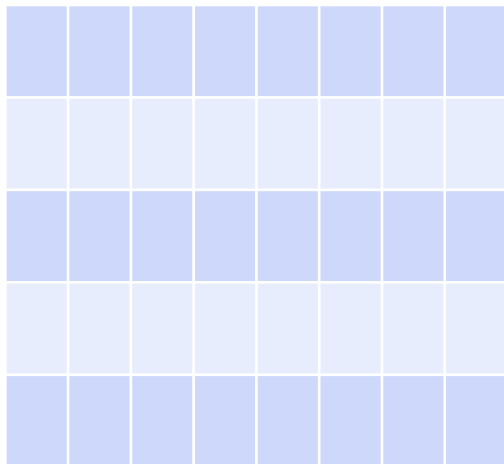
# Principal Component Analysis



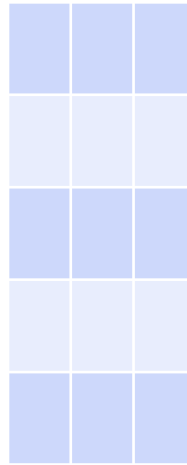
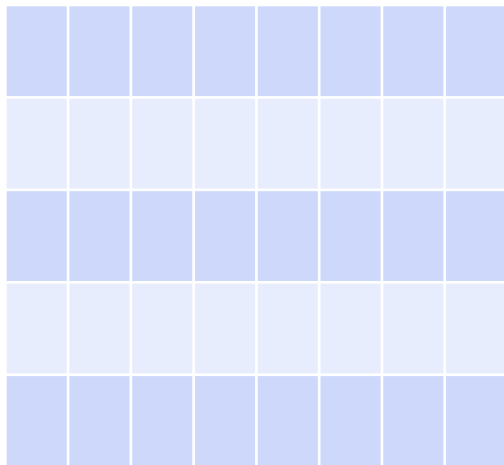
# Principal Component Analysis



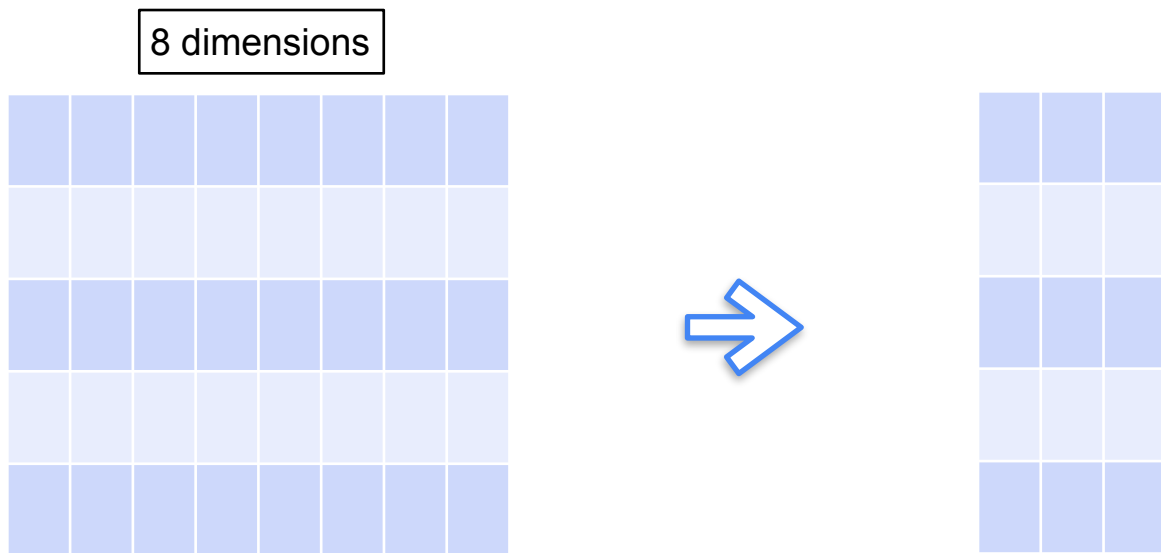
# Principal Component Analysis



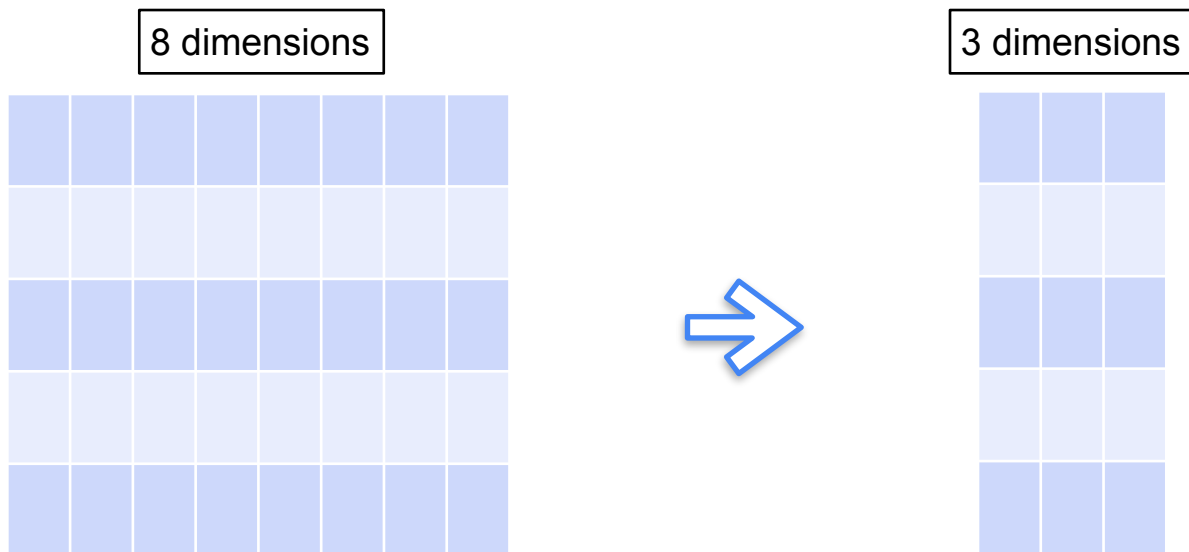
# Principal Component Analysis



# Principal Component Analysis



# Principal Component Analysis







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# Determinants and Eigenvectors

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**Singularity and rank of linear transformations**

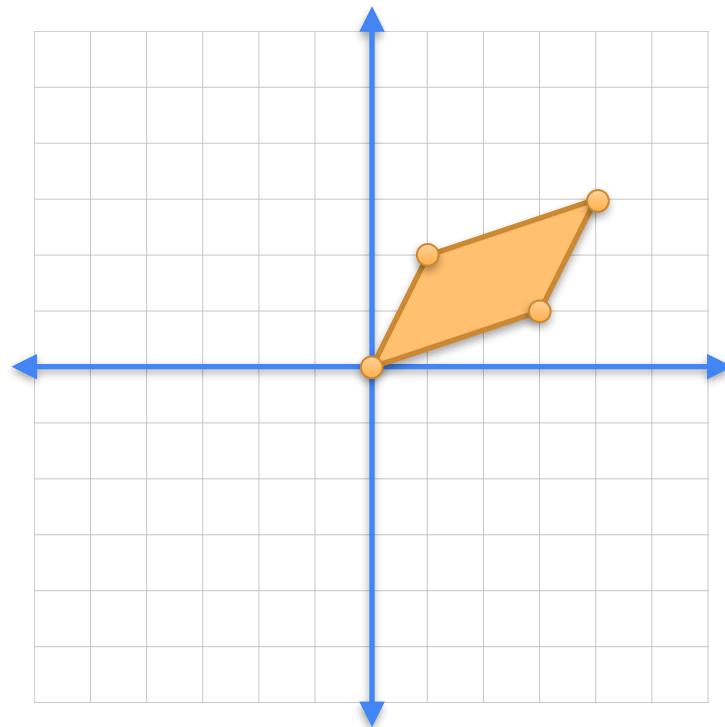
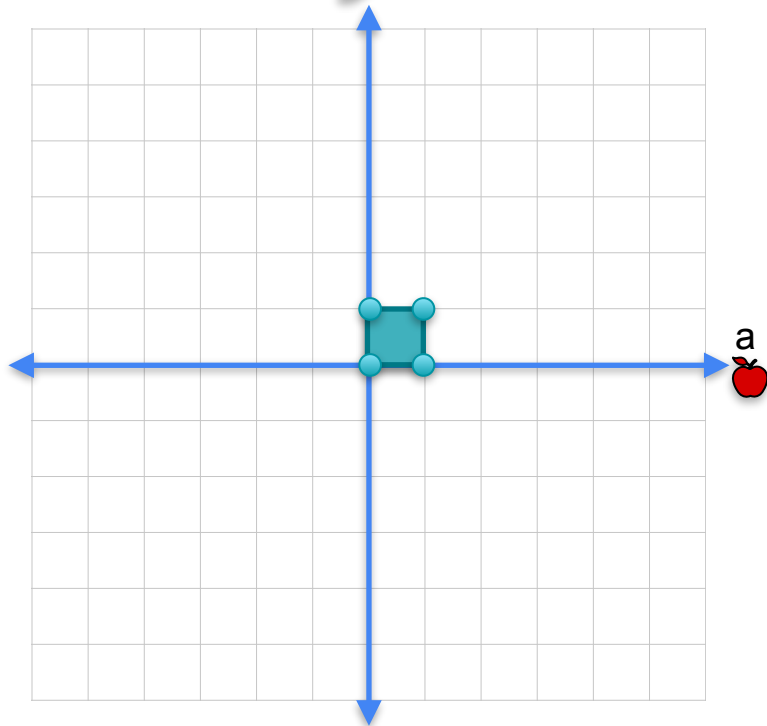
# Non-singular transformation

 b





3	1
1	2

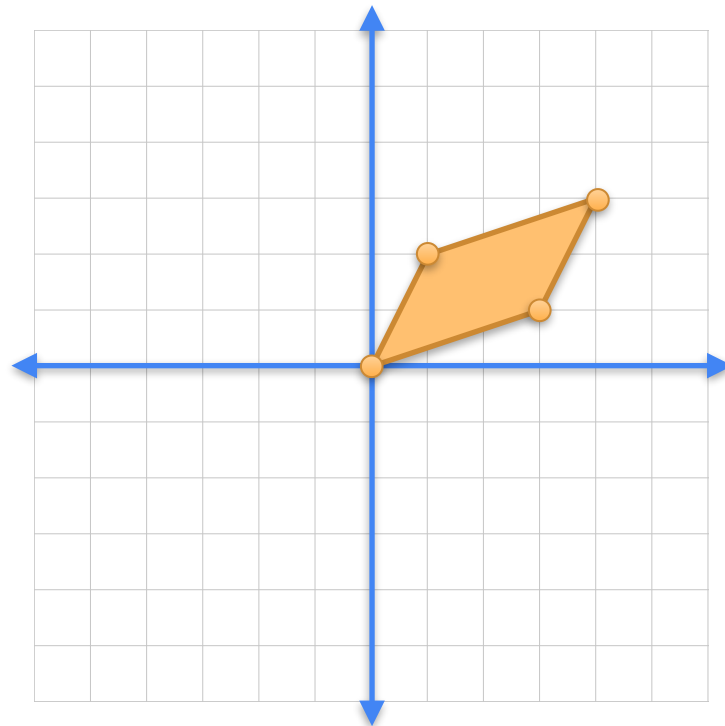
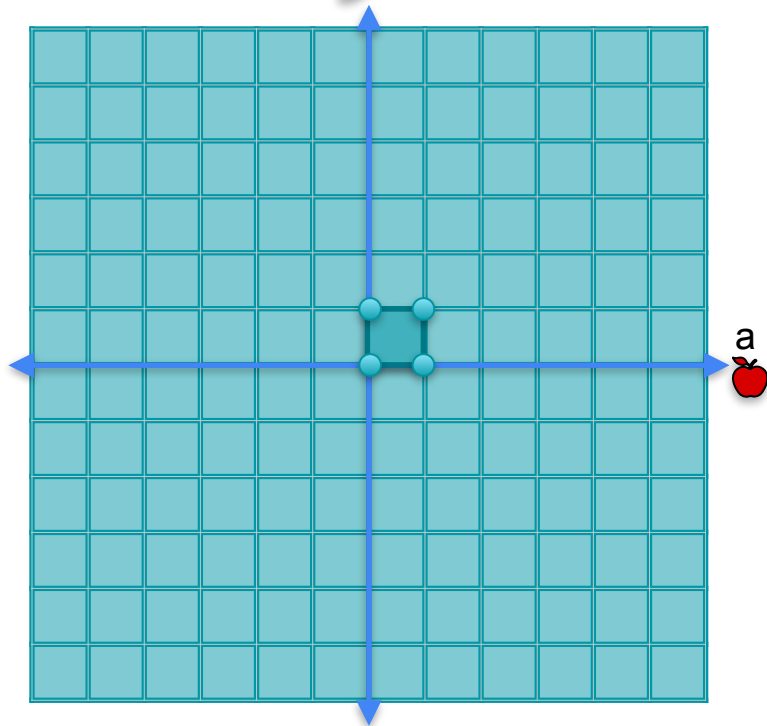


# Non-singular transformation

 b

3	1
1	2



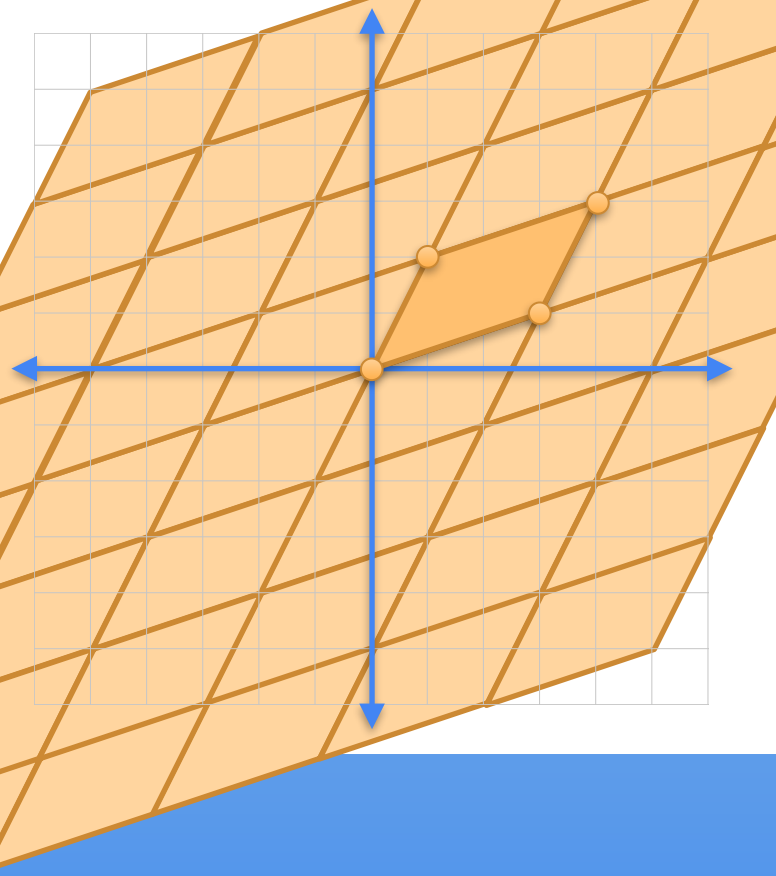
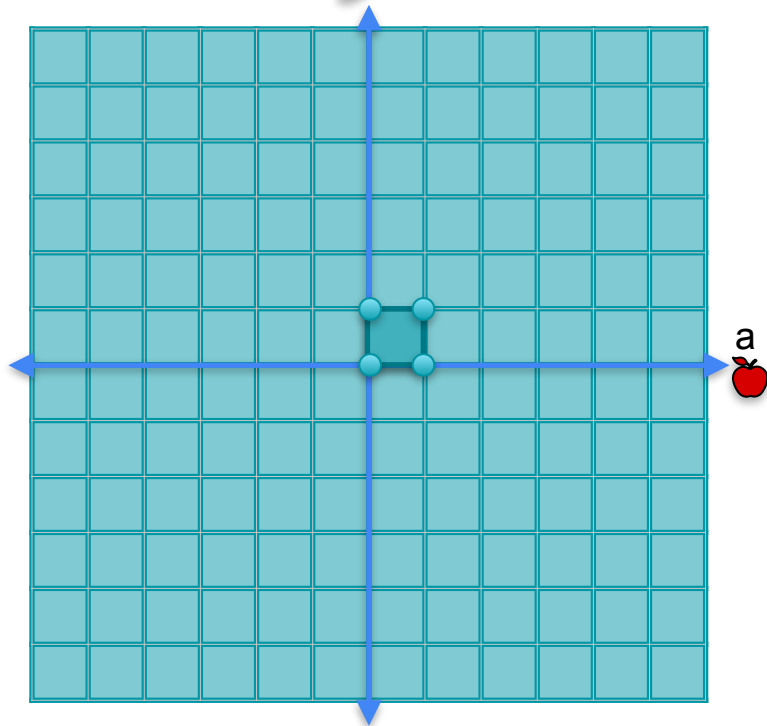
# Non-singular transformation

 b





3	1
1	2



# Singular transformation

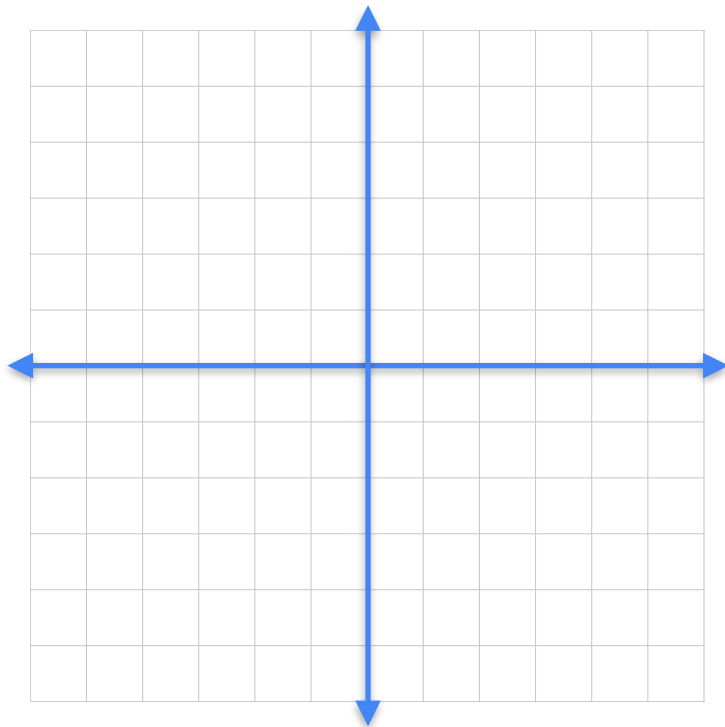
 b





1	1
2	2

a  

# Singular transformation

b

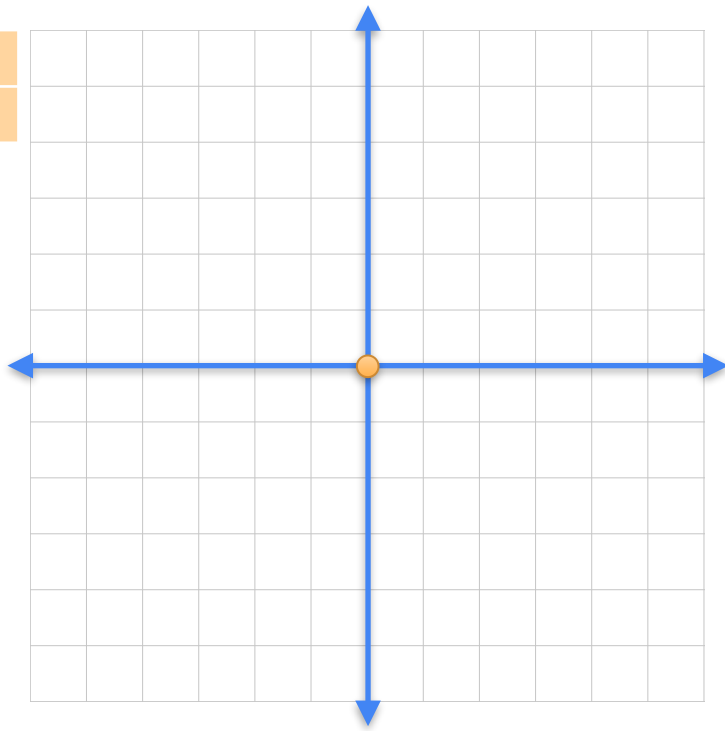
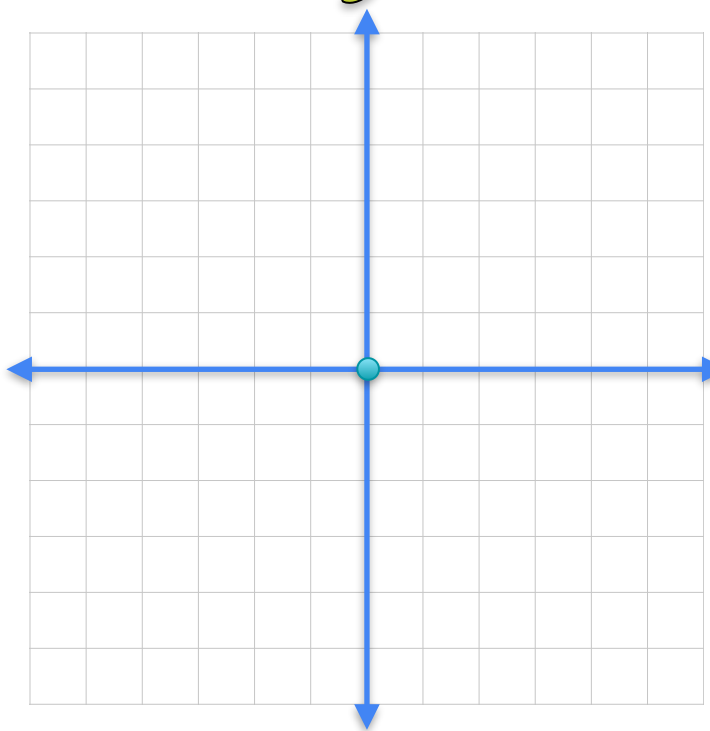
a

1	1	0	0
2	2	0	0

 $=$ 

0	0
0	0

$(0,0) \rightarrow (0,0)$



# Singular transformation

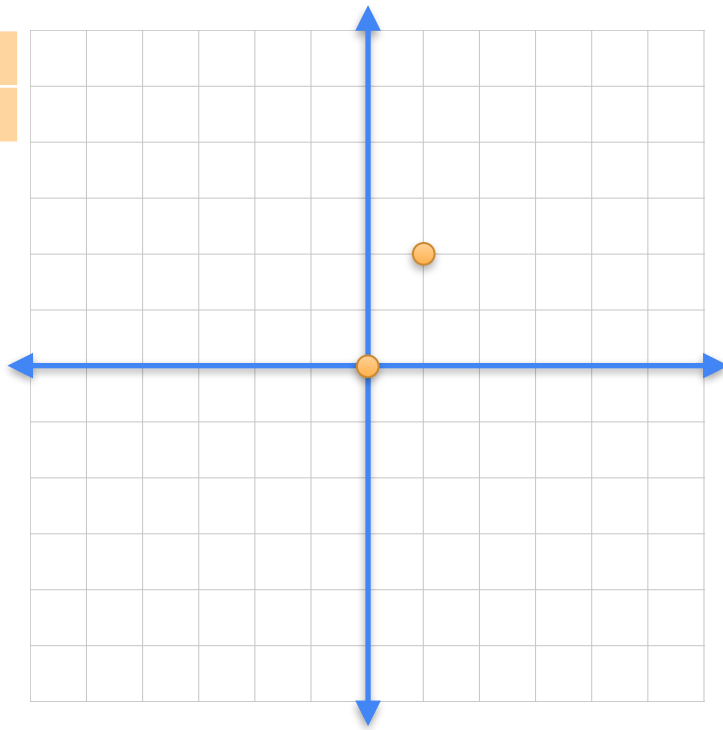
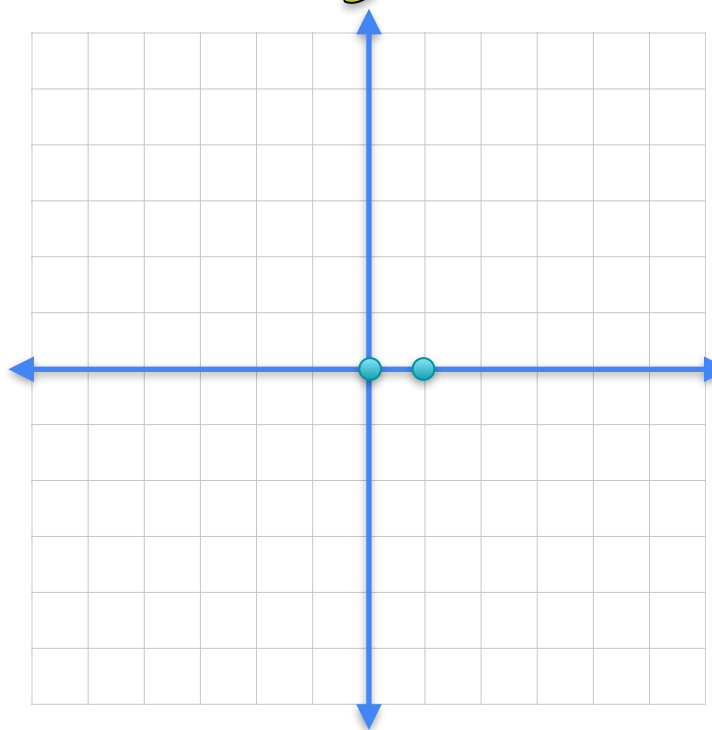
b

a

1	1	1	=	1
2	2	0		2

$(0,0) \rightarrow (0,0)$

$(1,0) \rightarrow (1,2)$



# Singular transformation

b

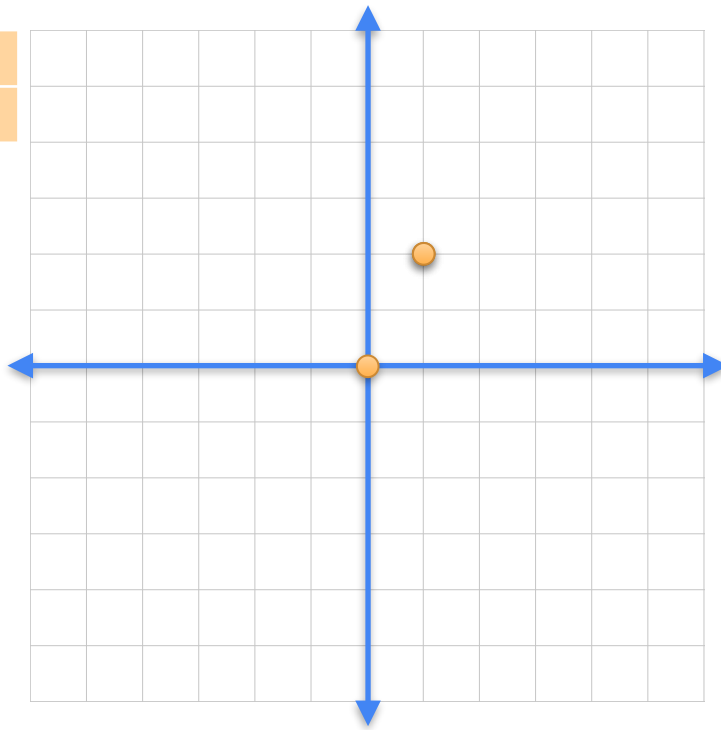
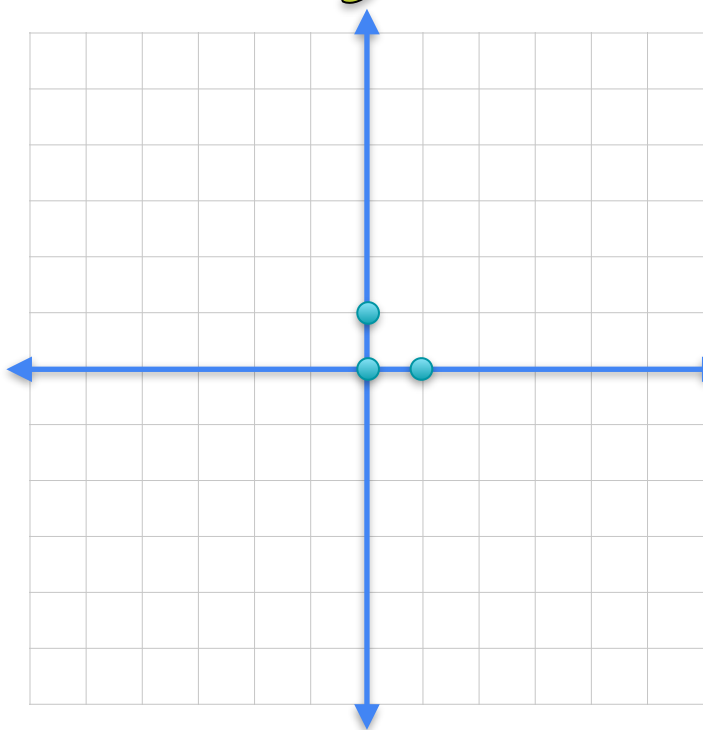
a

1	1	0	1
2	2	1	2

$(0,0) \rightarrow (0,0)$

$(1,0) \rightarrow (1,2)$

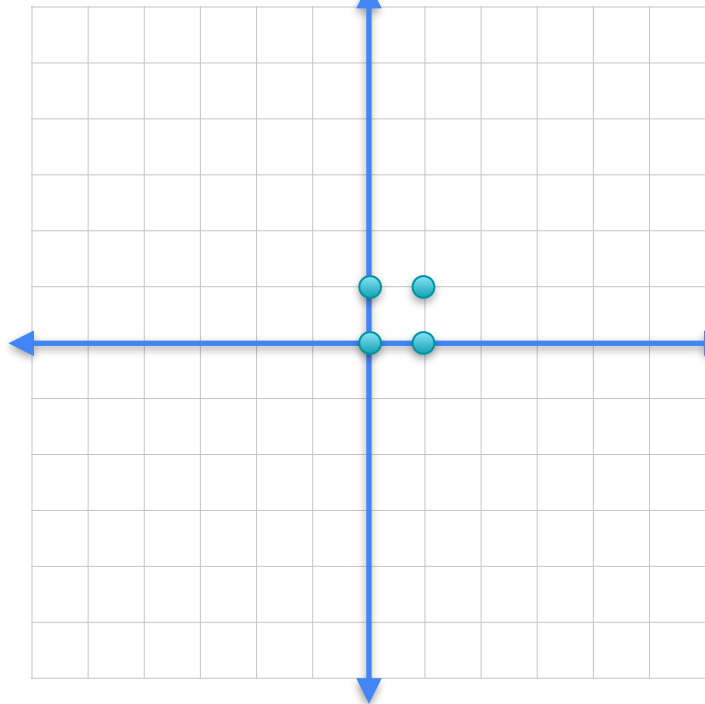
$(0,1) \rightarrow (1,2)$





# Singular transformation

b



🍎 🍌

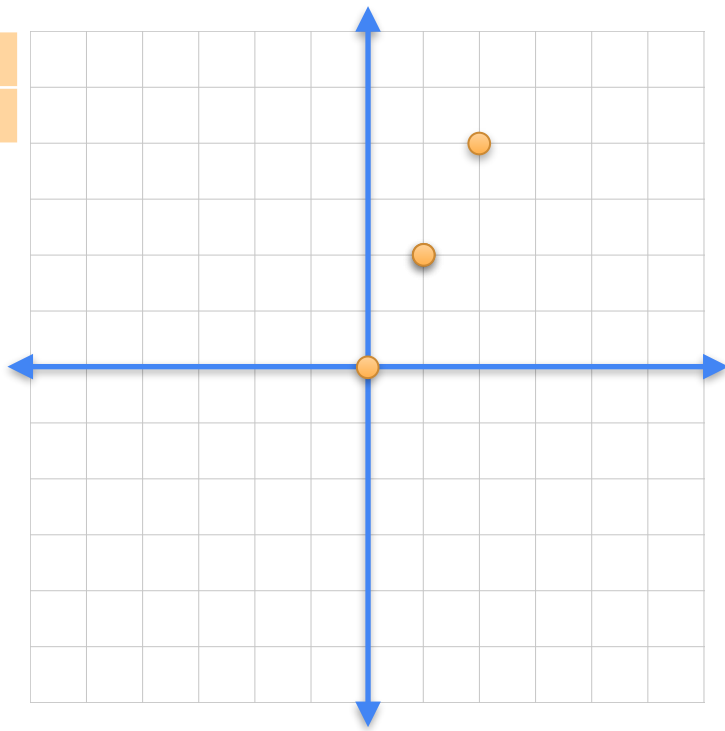
1	1	1	=	2
2	2	1		4

(0,0) → (0,0)

(1,0) → (1,2)

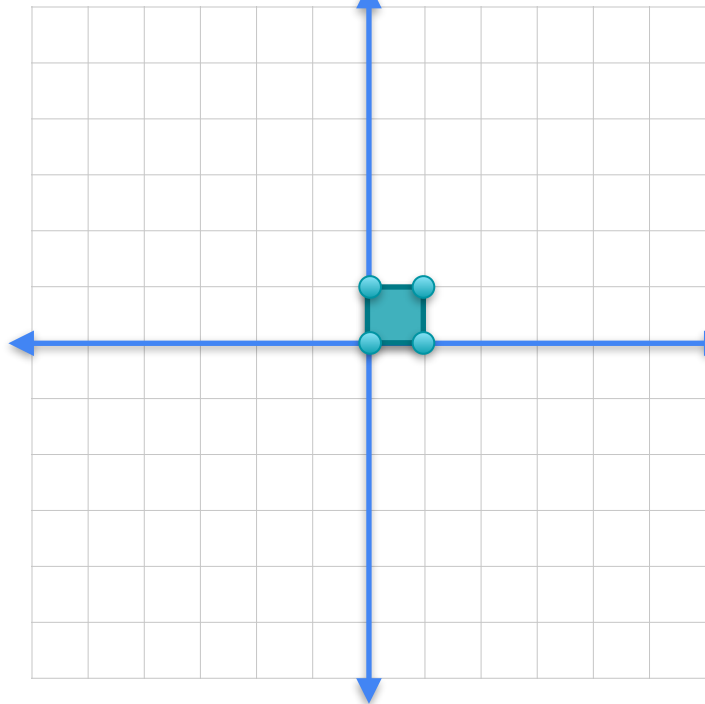
(0,1) → (1,2)

(1,1) → (2,4)



# Singular transformation

b

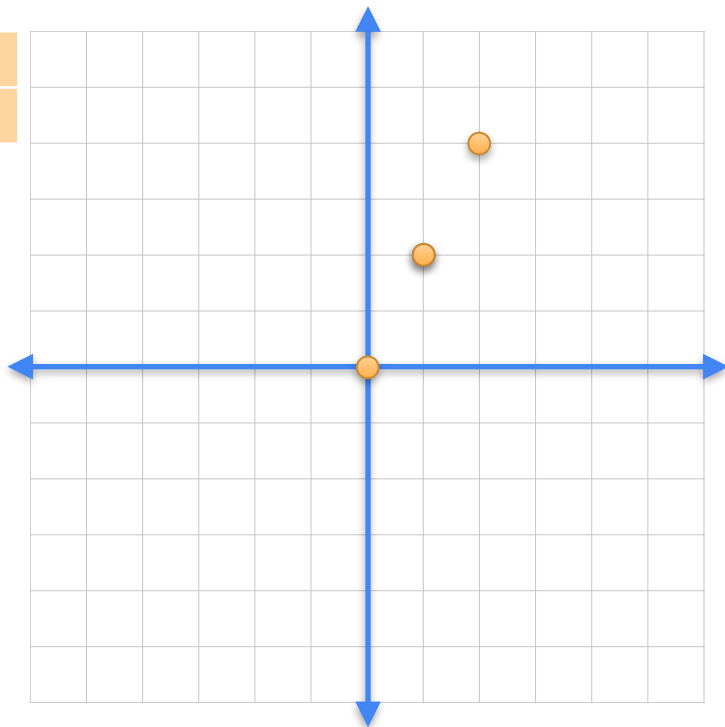


🍎 🍌

1	1	1	=	2
2	2	1		4

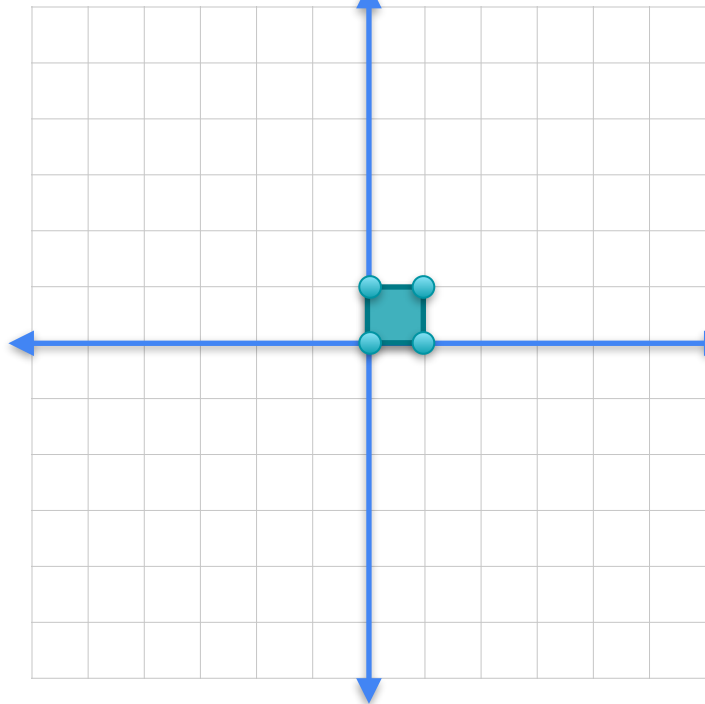
a

(0,0) → (0,0)  
(1,0) → (1,2)  
(0,1) → (1,2)  
(1,1) → (2,4)



# Singular transformation

b

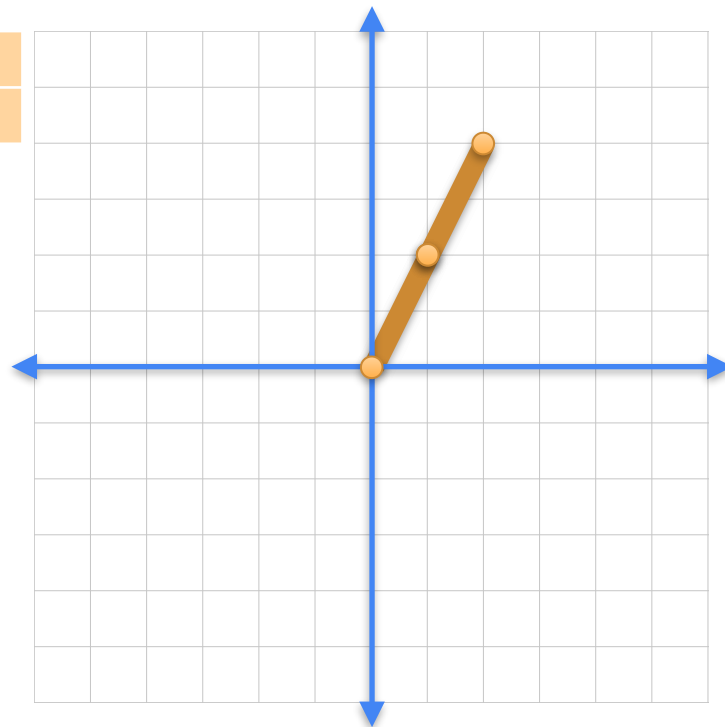


🍎 🍌

1	1	1	=	2
2	2	1		4

a

(0,0) → (0,0)  
(1,0) → (1,2)  
(0,1) → (1,2)  
(1,1) → (2,4)



# Singular transformation

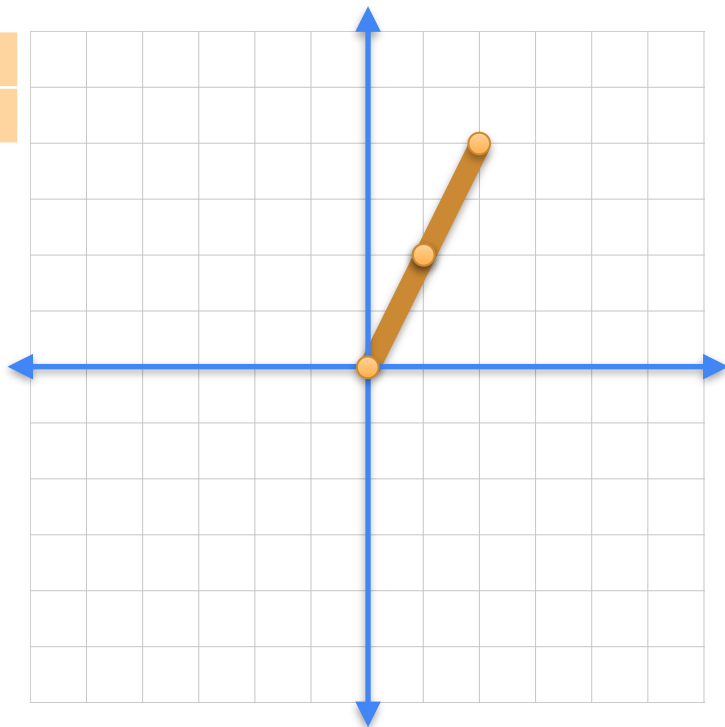
b

🍎 🍌

1	1	1	=	2
2	2	1		4

a

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$



# Singular transformation

 b

1	1	1	=	2
2	2	1		4

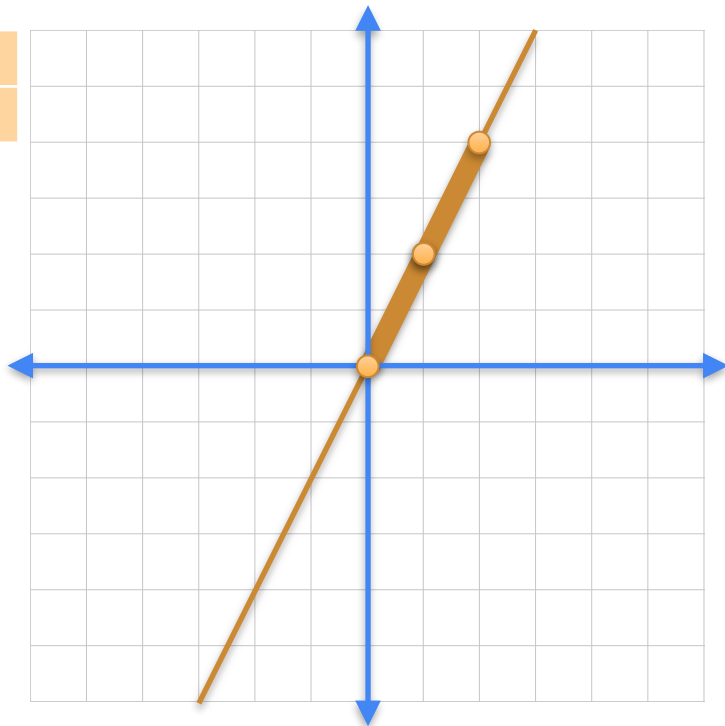
$(0,0) \rightarrow (0,0)$

$(1,0) \rightarrow (1,2)$

$(0,1) \rightarrow (1,2)$

$(1,1) \rightarrow (2,4)$

 a

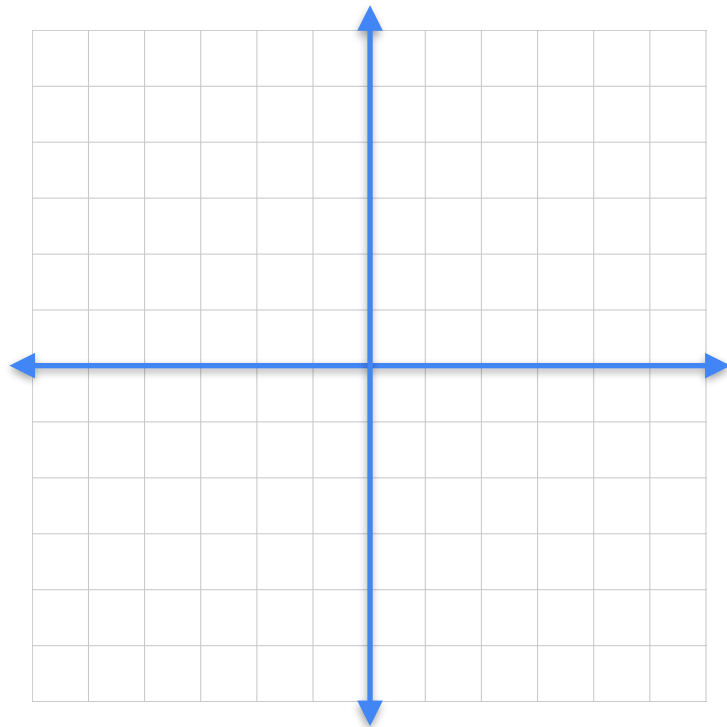
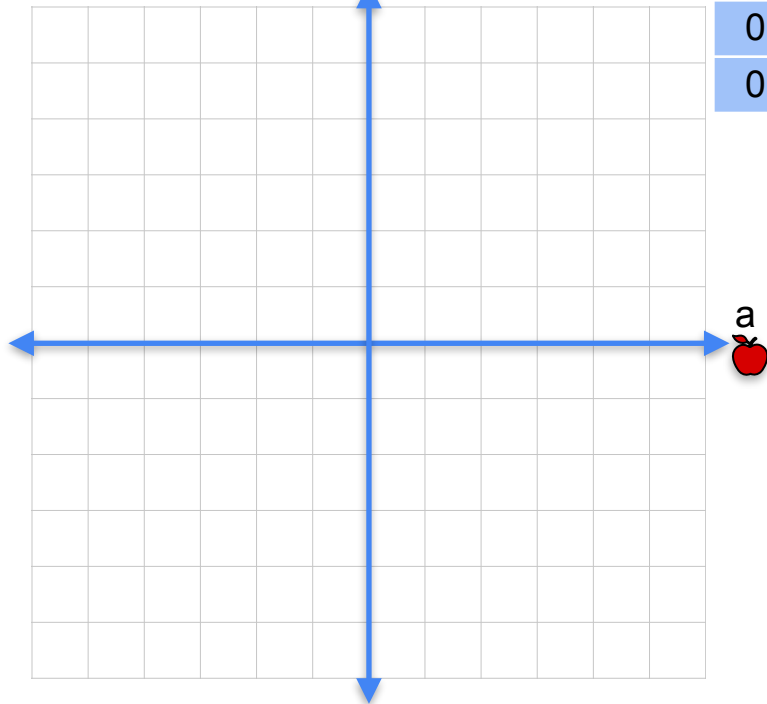


# Singular transformation

 b

0	0
0	0



# Singular transformation

 b





0

0

a

0

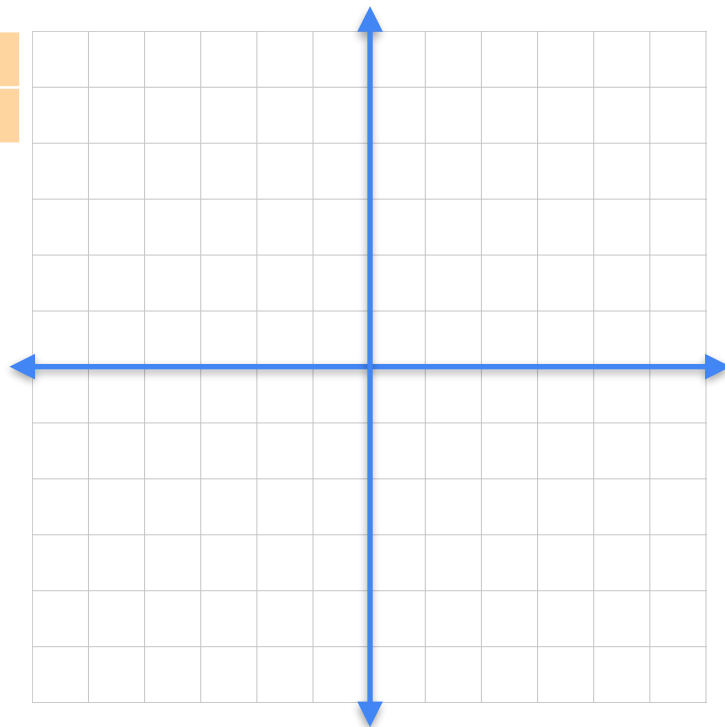
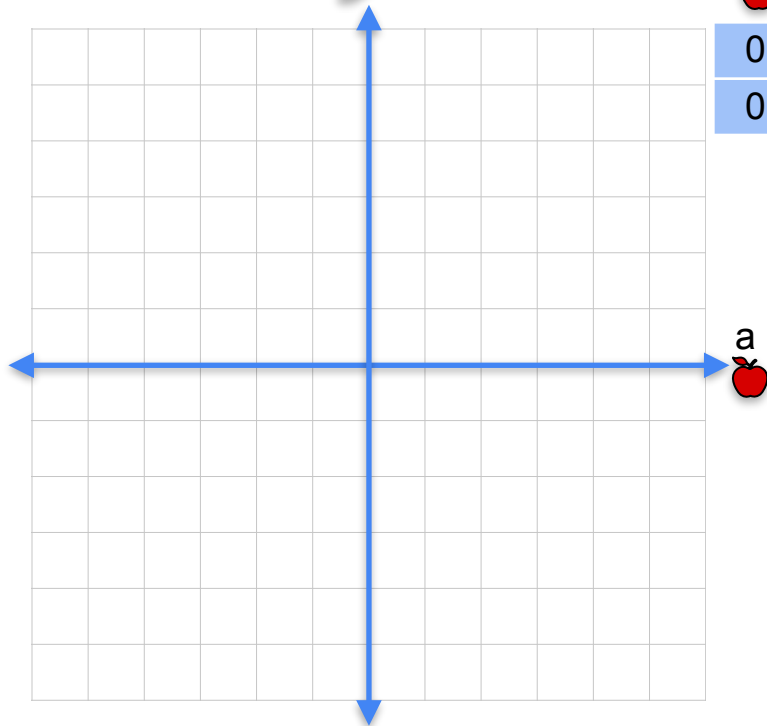
0

0

b

0

=



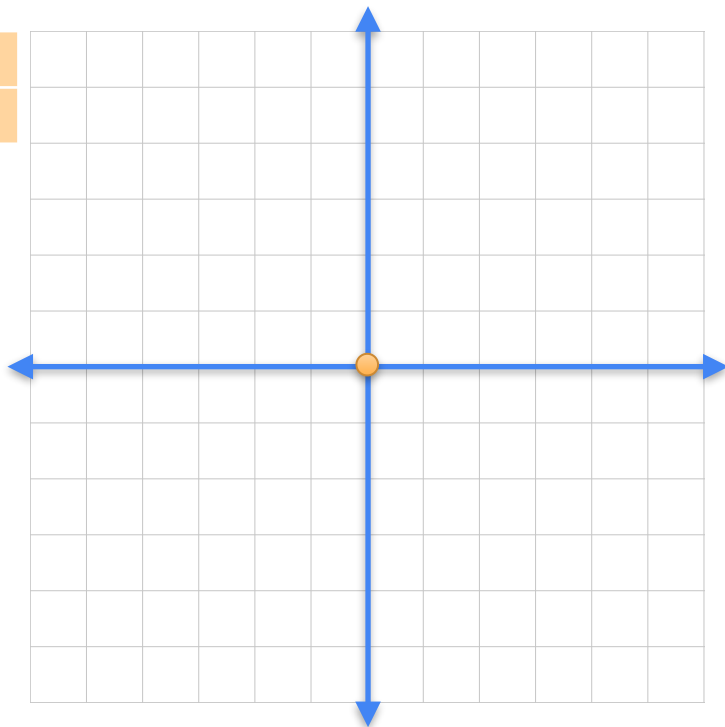
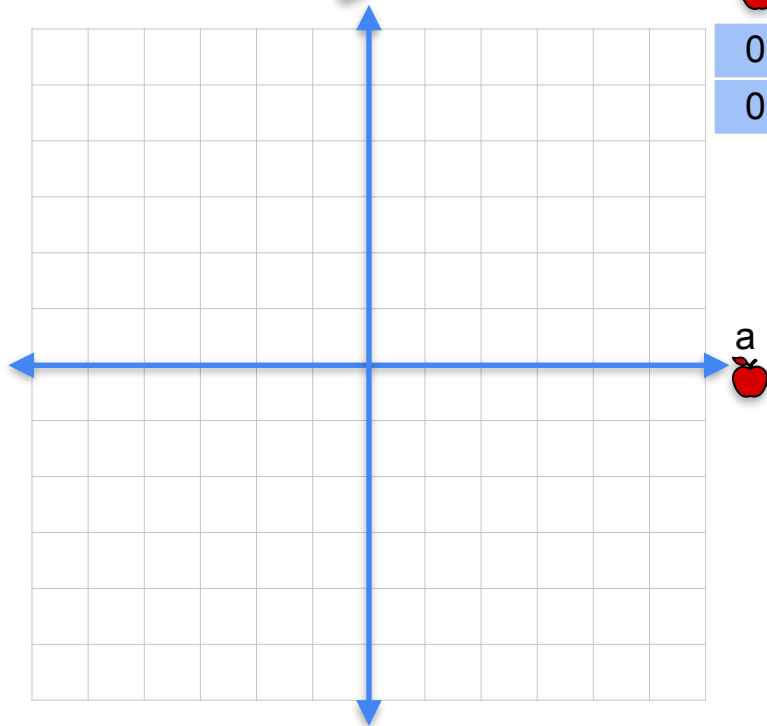
# Singular transformation

 b



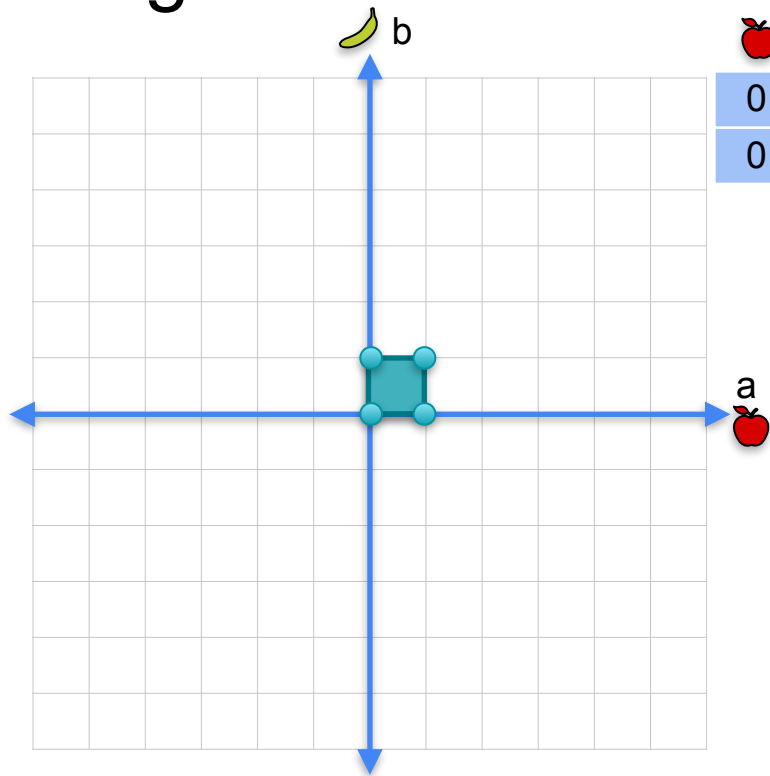




0	0	a	0
0	0	b	0





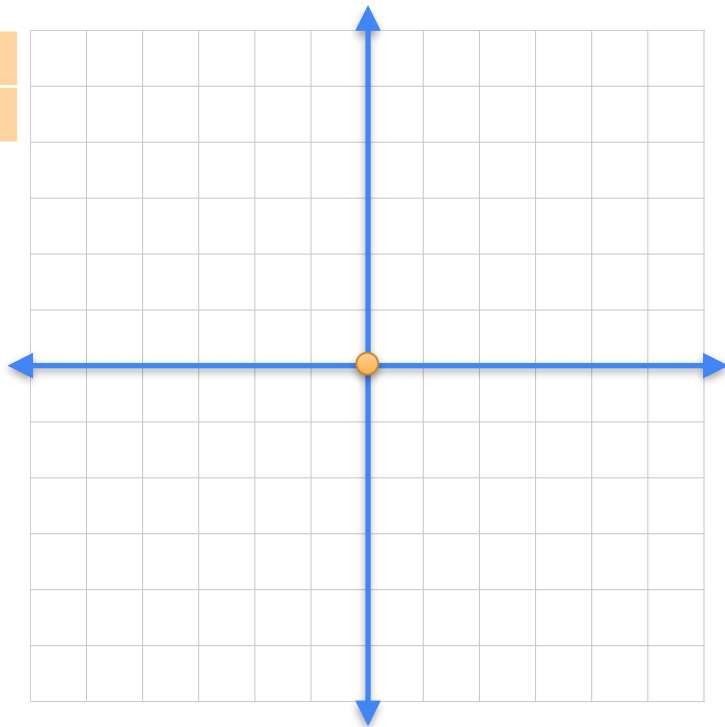
# Singular transformation



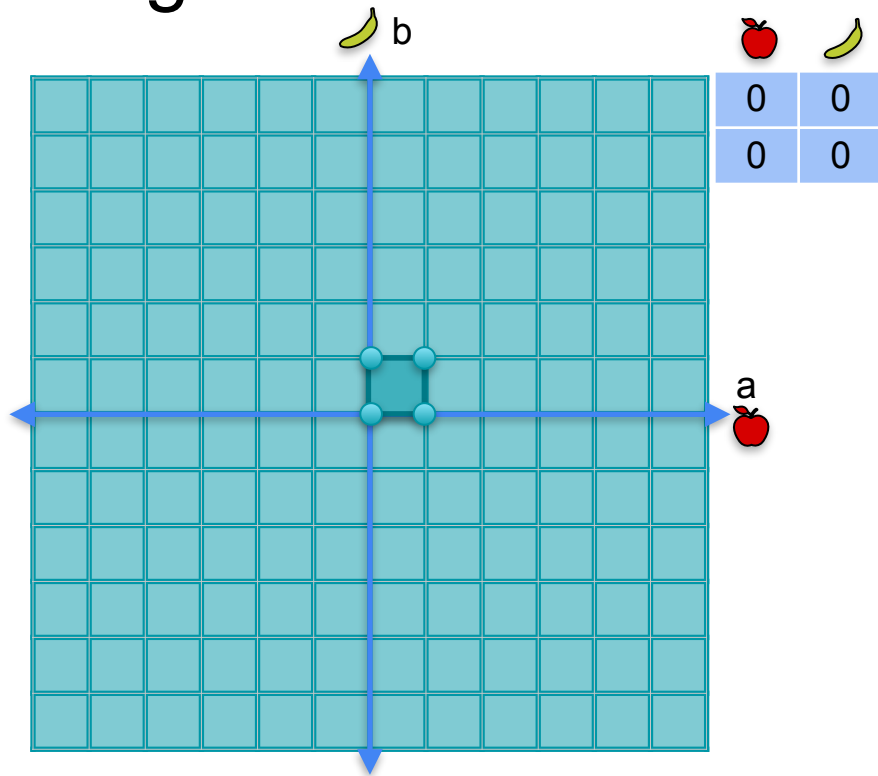
		
0	0	a
0	0	b

 $=$ 

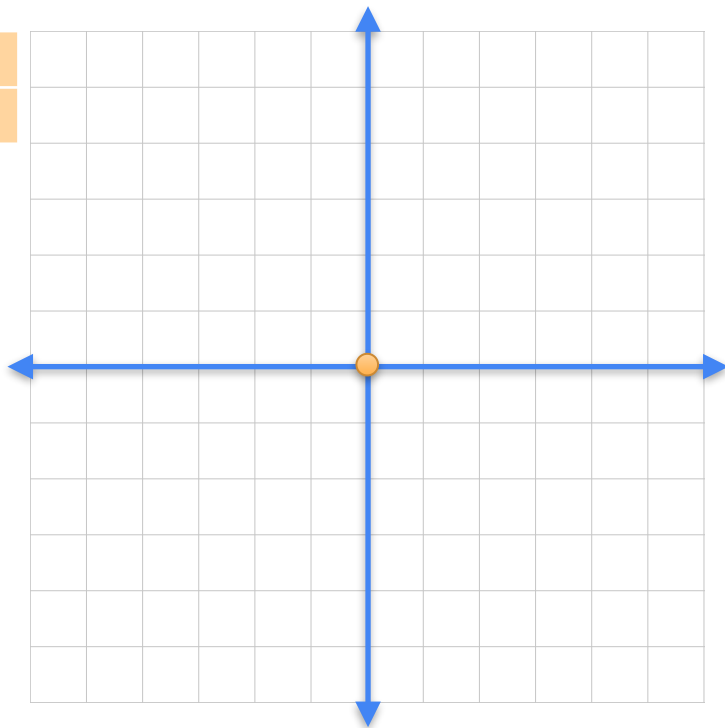
0
0



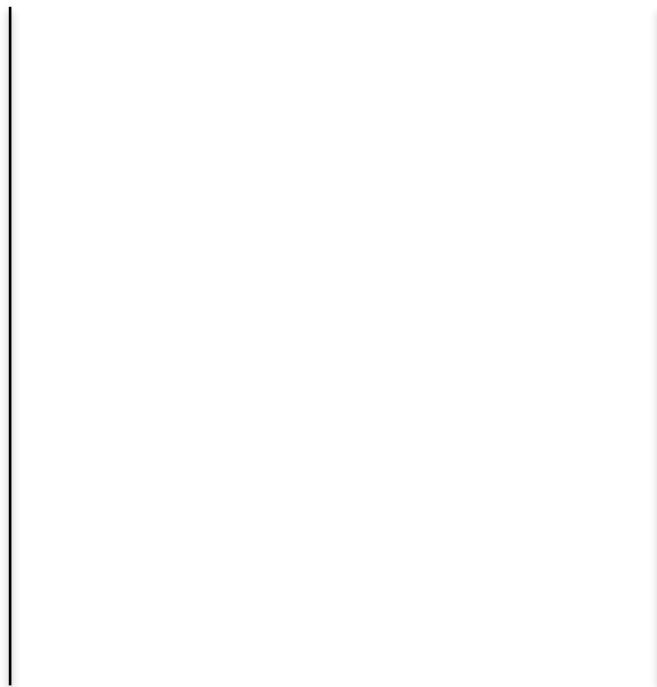
# Singular transformation



0	0	a	=	0
0	0	b		0



# Singular and non-singular transformations

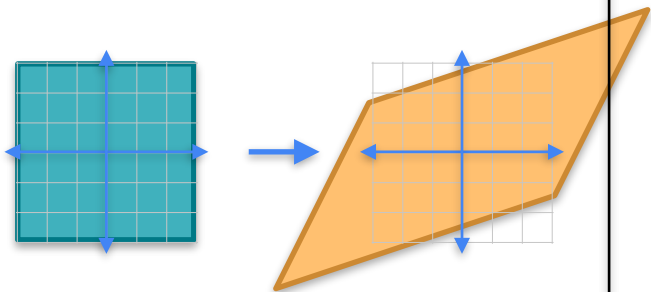


# Singular and non-singular transformations

Non-singular





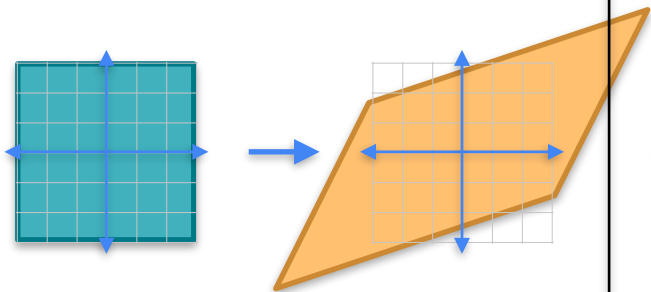
3	1
1	2





# Singular and non-singular transformations

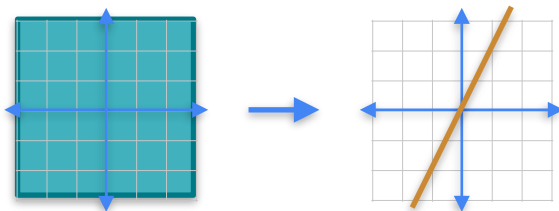
Non-singular

	
3	1
1	2





Singular

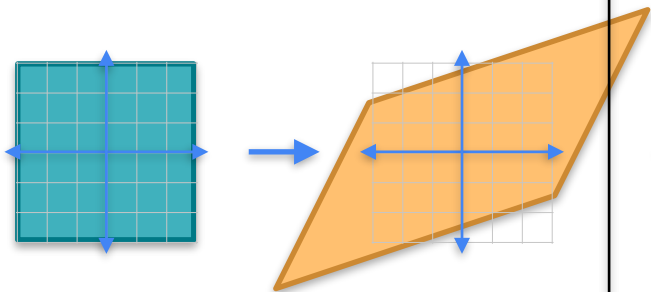
	
1	1
2	2





# Singular and non-singular transformations

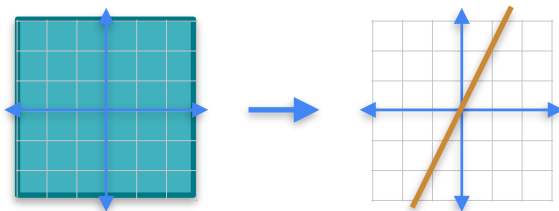
Non-singular

	
3	1
1	2





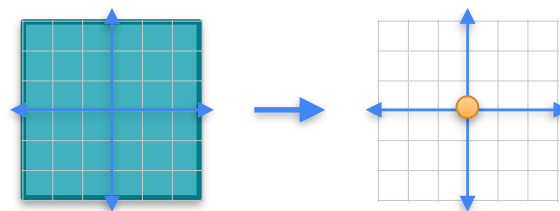
Singular

	
1	1
2	2





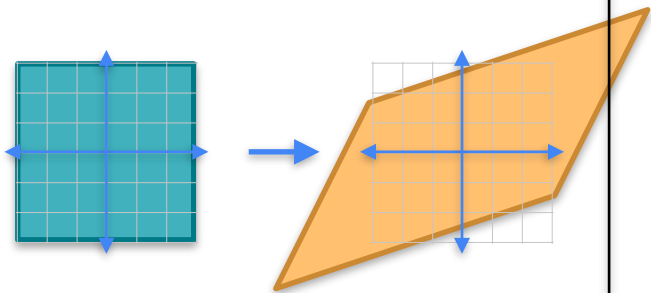
Singular



	
0	0
0	0

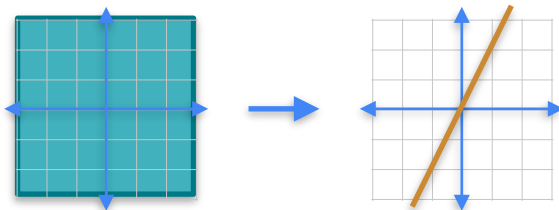




# Rank of linear transformations

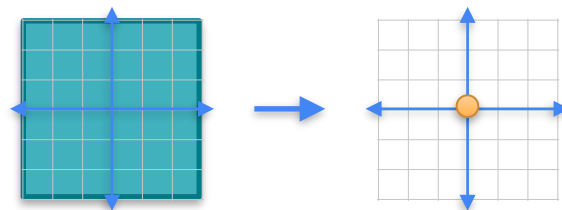
	
3	1
1	2





	
1	1
2	2

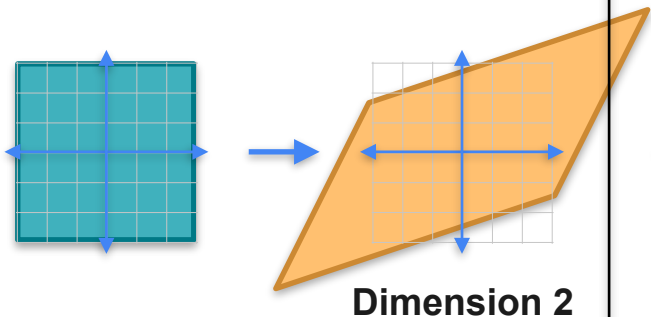




	
0	0
0	0

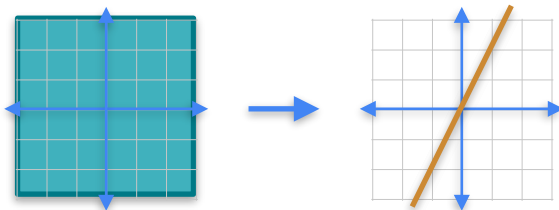




# Rank of linear transformations

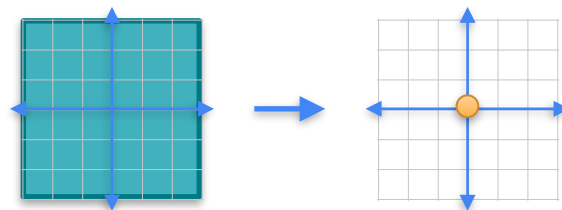
	
3	1
1	2



	
1	1
2	2





	
0	0
0	0

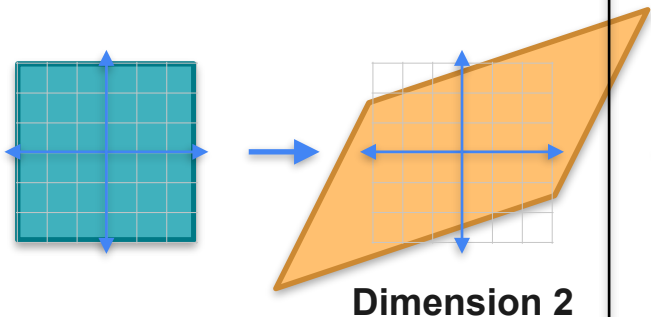






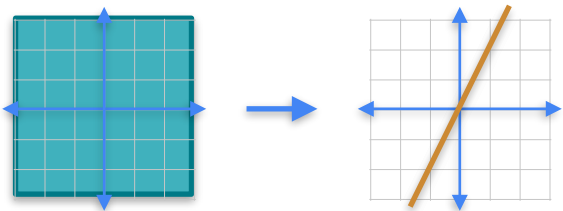
# Rank of linear transformations



Rank 2

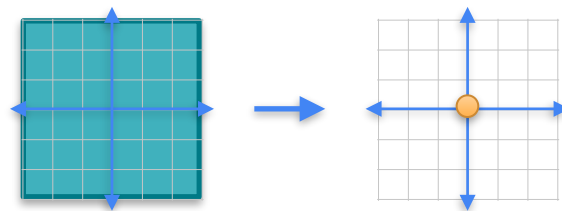
	
3	1
1	2



	
1	1
2	2





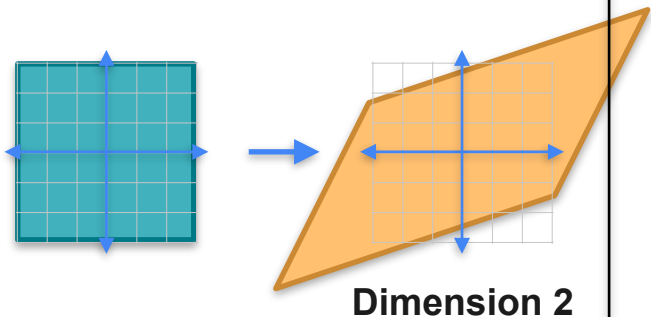
	
0	0
0	0





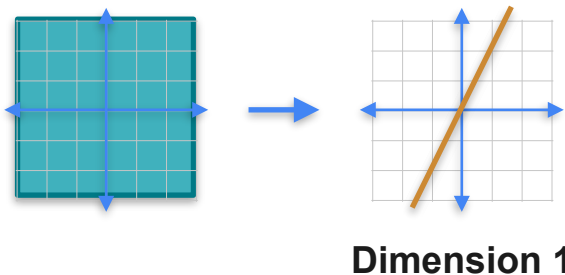
# Rank of linear transformations



Rank 2

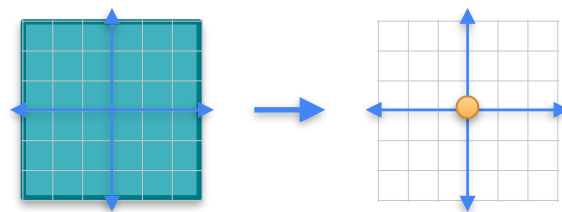
	
3	1
1	2



	
1	1
2	2





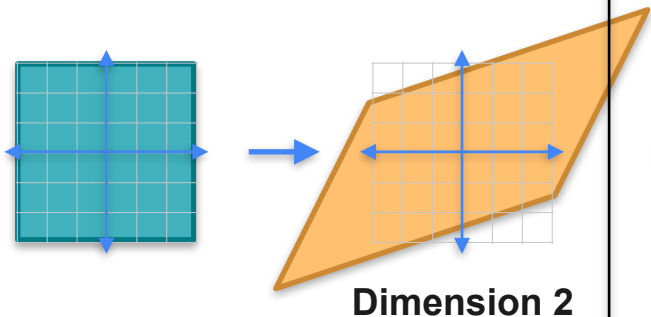
	
0	0
0	0





# Rank of linear transformations

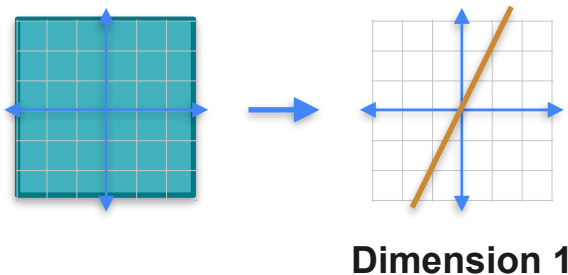
Rank 2



	
3	1
1	2

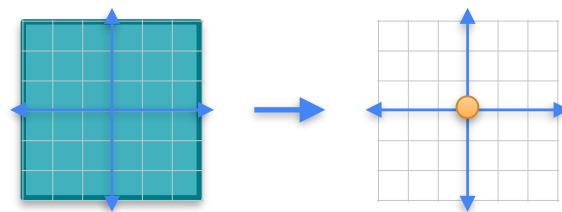


Rank 1

	
1	1
2	2





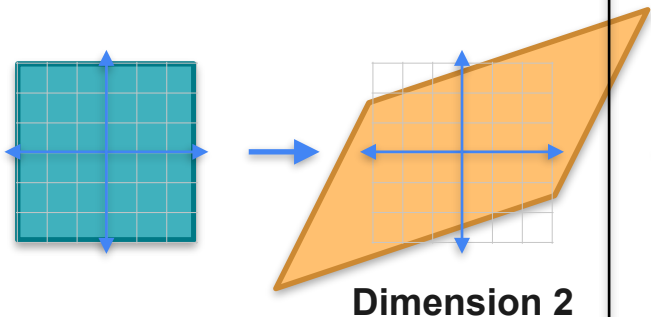
	
0	0
0	0





# Rank of linear transformations

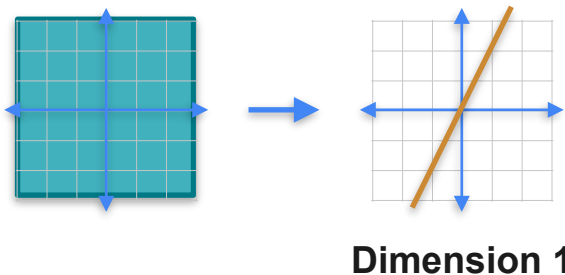
Rank 2



	
3	1
1	2

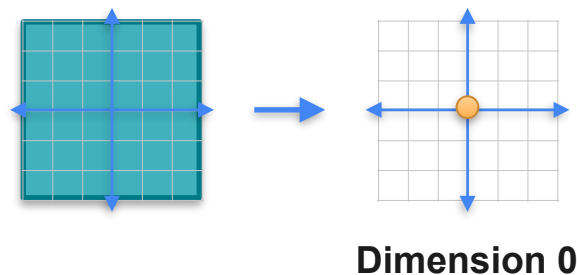


Rank 1

	
1	1
2	2





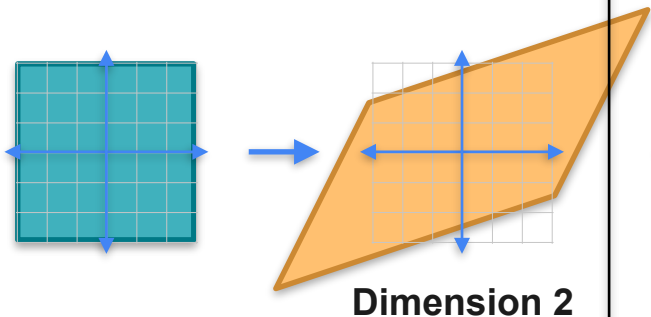
	
0	0
0	0





# Rank of linear transformations

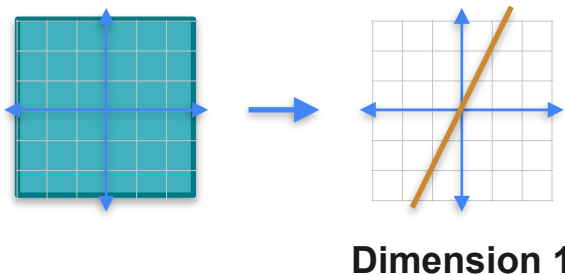
Rank 2

	
3	1
1	2





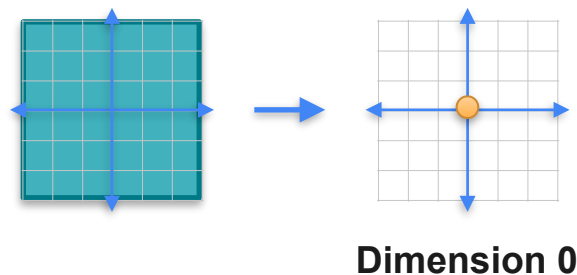
Rank 1

	
1	1
2	2



Rank 0

	
0	0
0	0





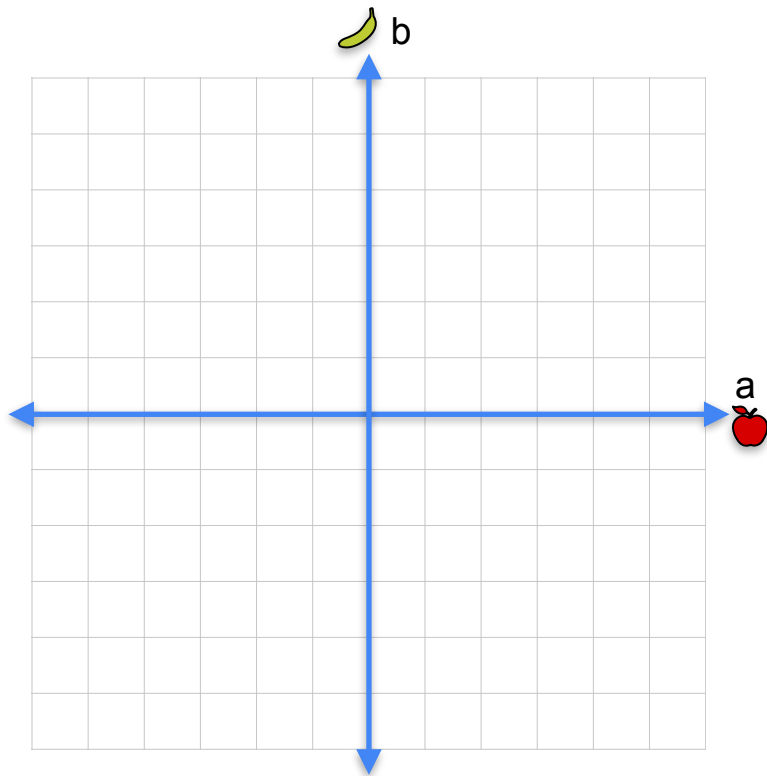
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# Determinants and Eigenvectors

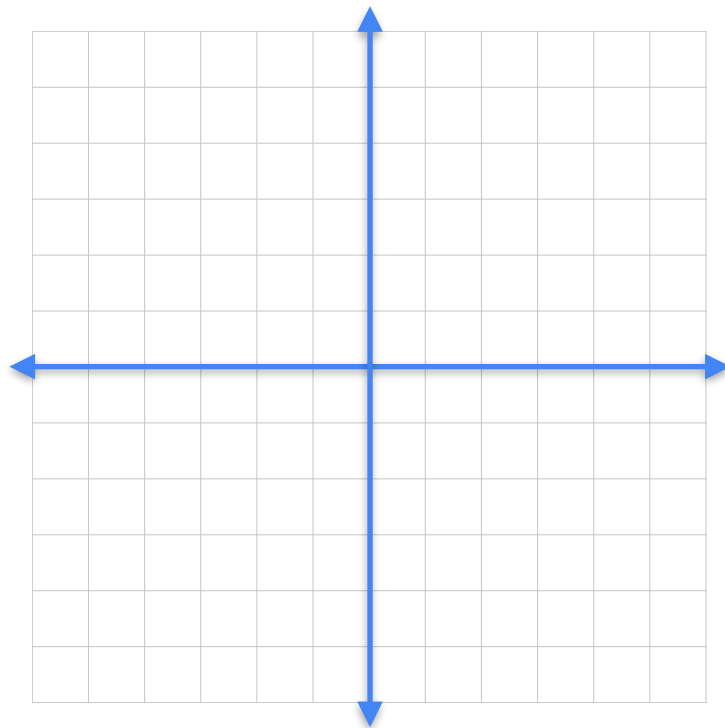
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## **Determinant as an area**

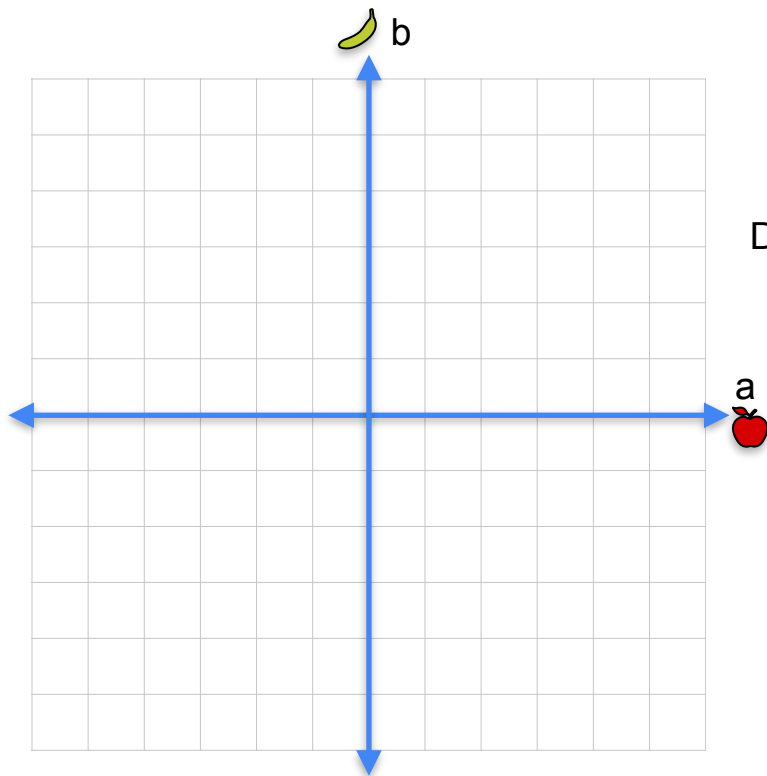
# Determinant as an area





 3	 1
1	2



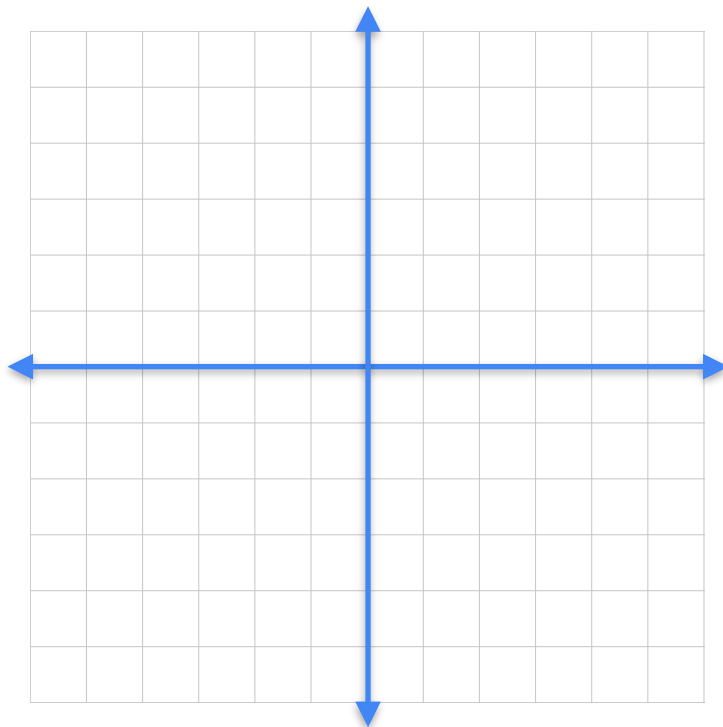
# Determinant as an area



	
3	1
1	2

$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

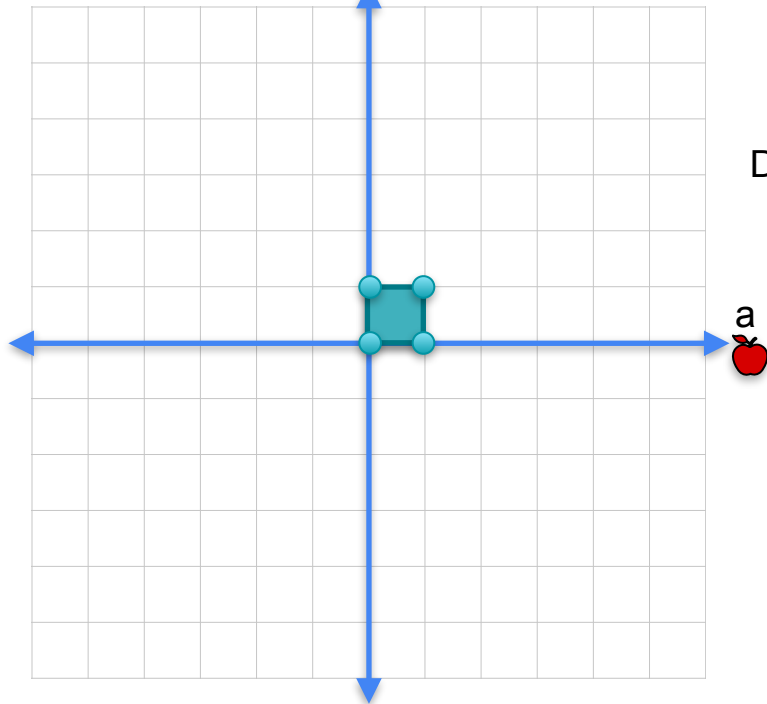
$$\text{Det} = 5$$







# Determinant as an area

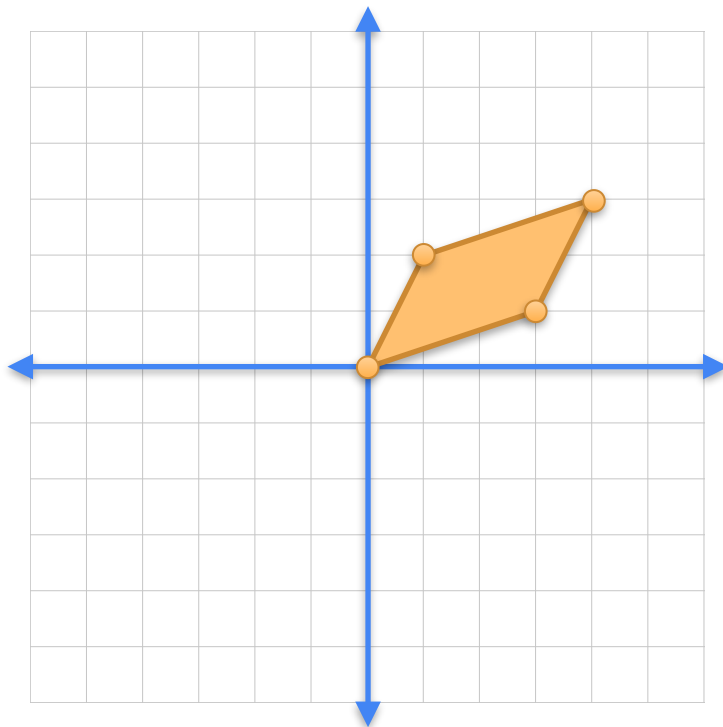
 b



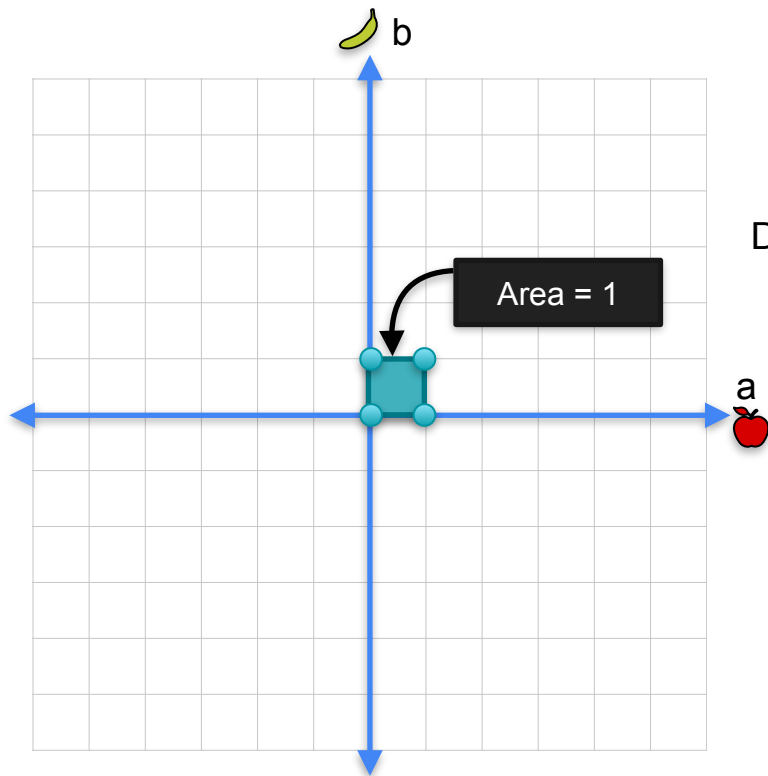
	
3	1
1	2



$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$



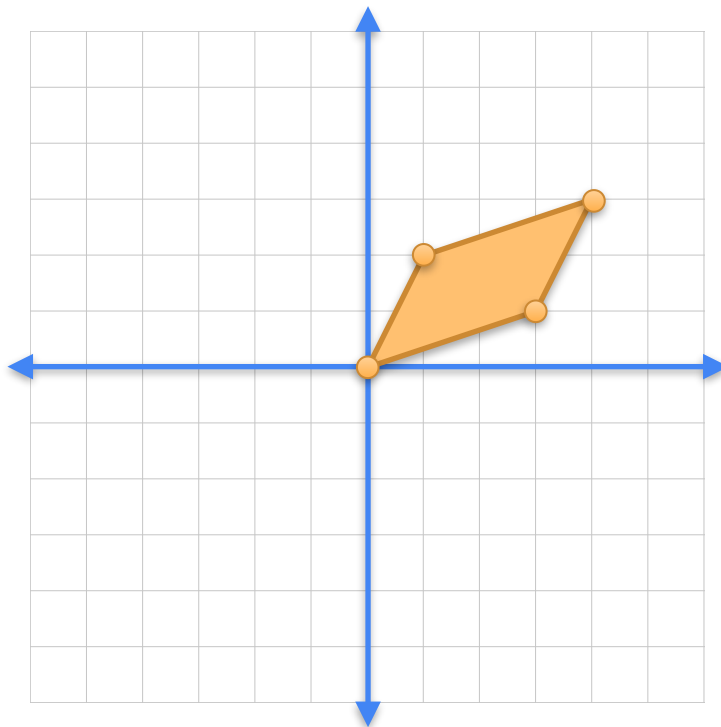
# Determinant as an area



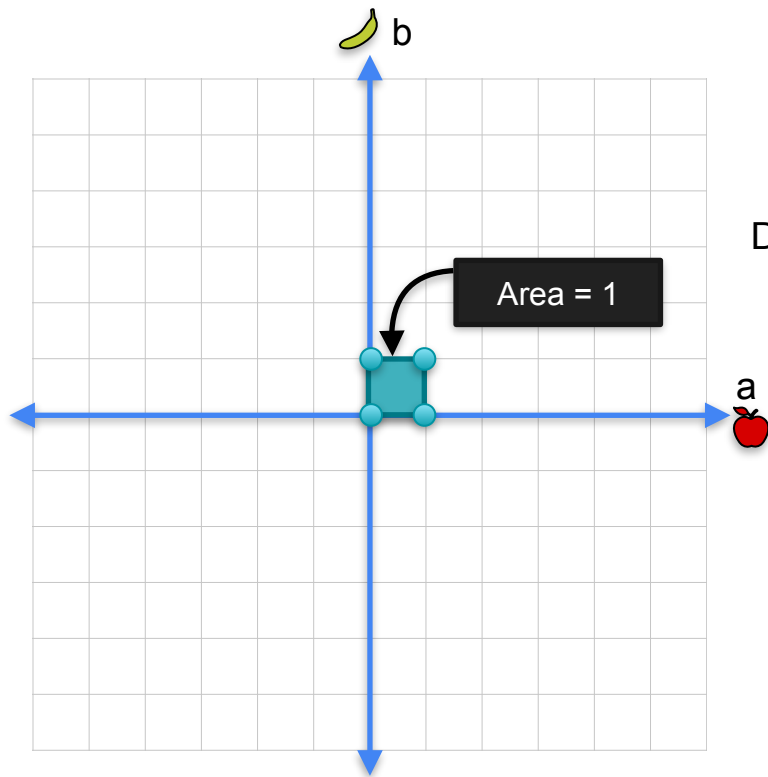
 3	 1
1	2



$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$



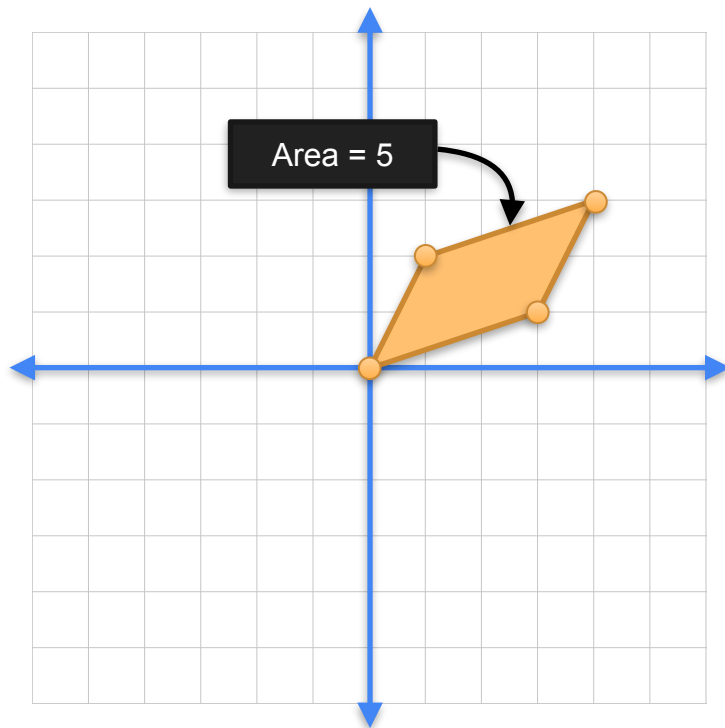
# Determinant as an area



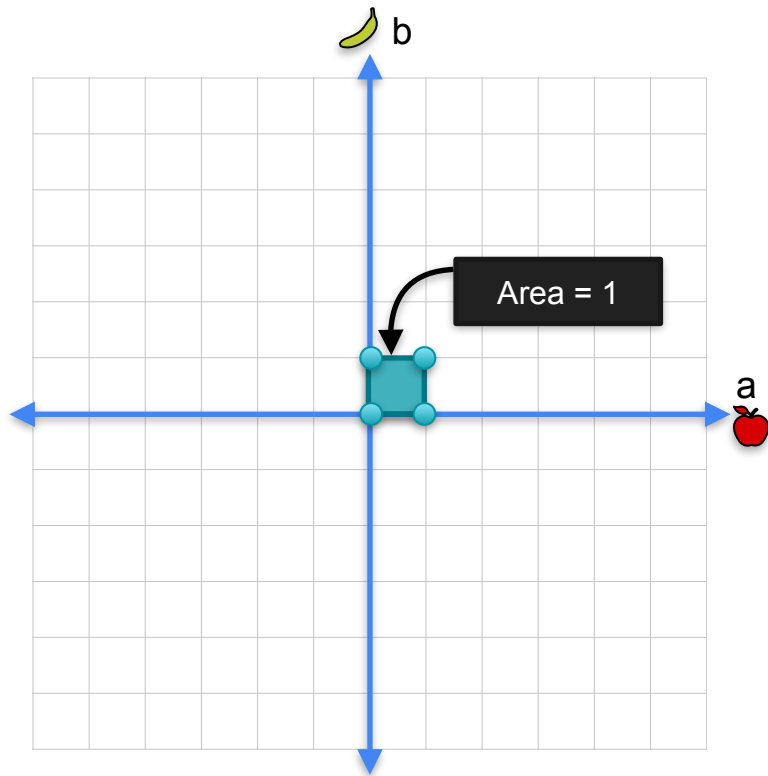
 3	 1
1	2



$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

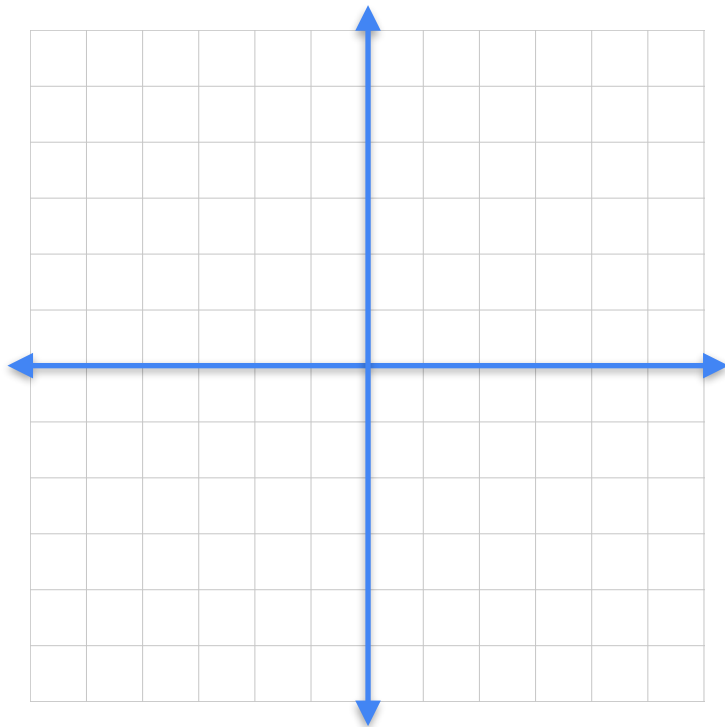
$$\text{Det} = 5$$



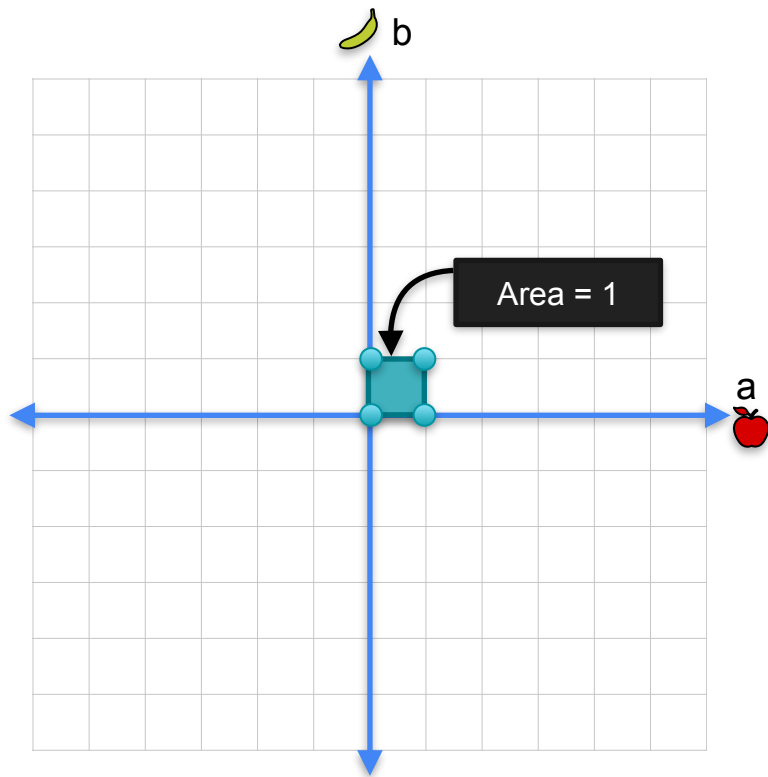
# Determinant as an area





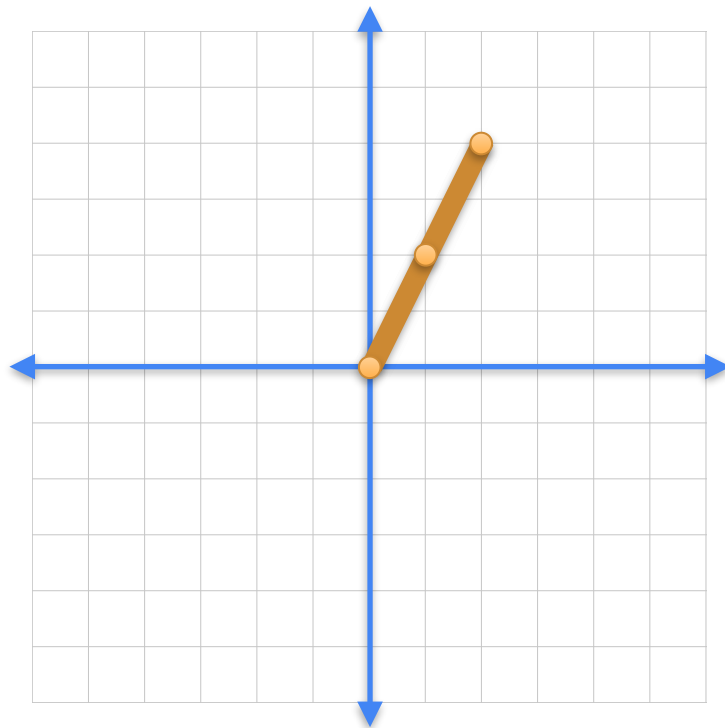
 1	 1
2	2



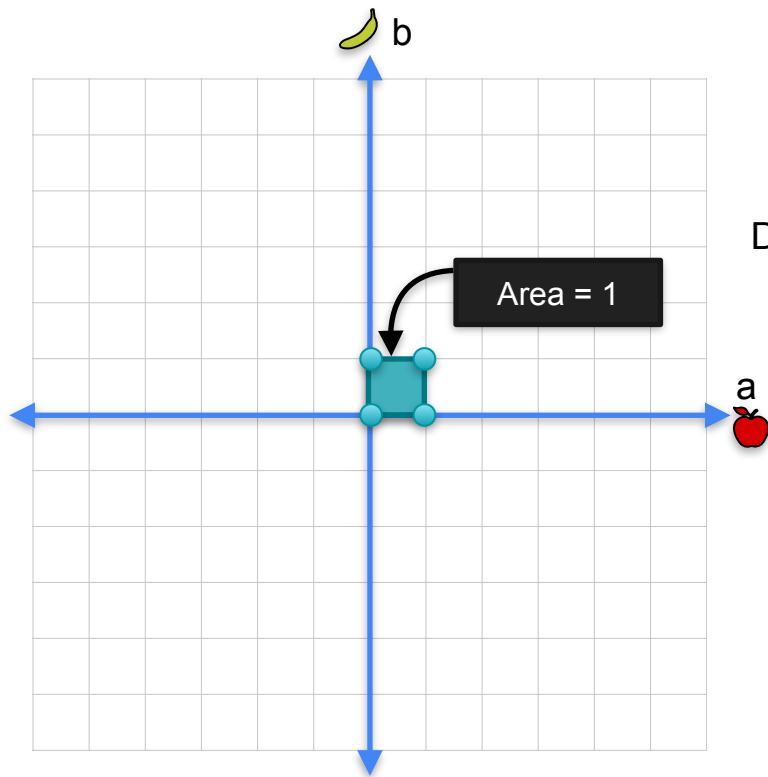
# Determinant as an area





	
1	1
2	2



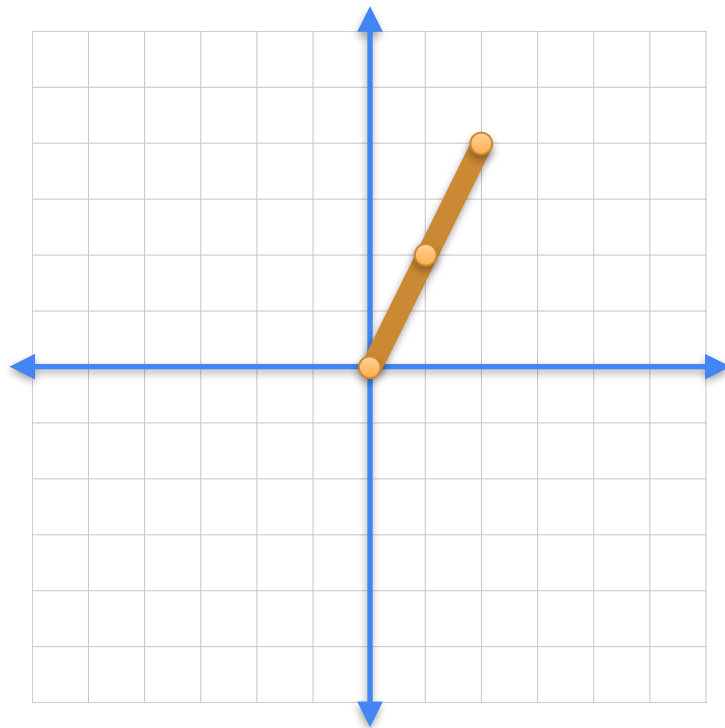
# Determinant as an area



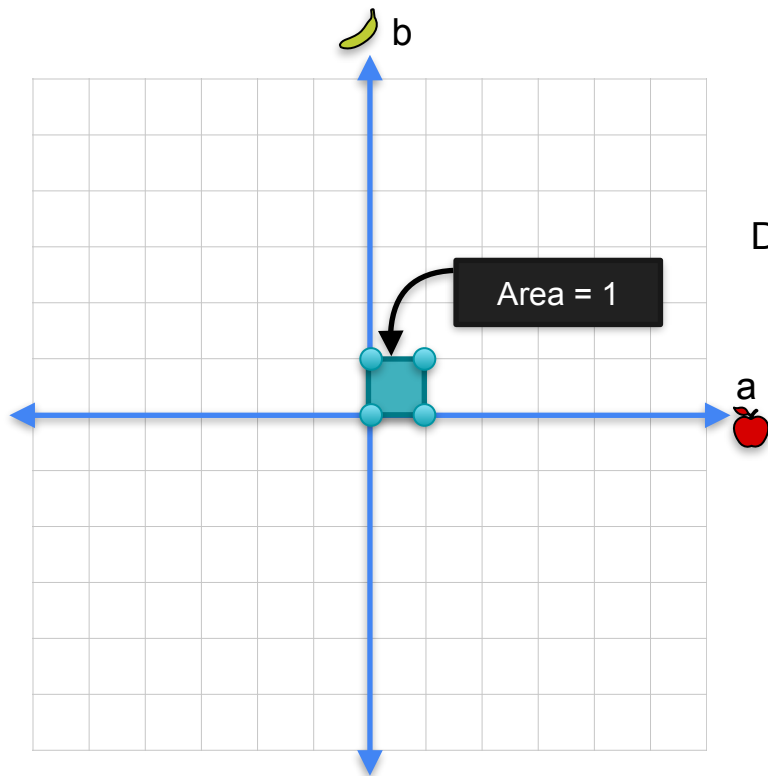
	
1	1
2	2



$$\text{Det} = 1 \cdot 2 - 1 \cdot 2$$

$$\text{Det} = 0$$



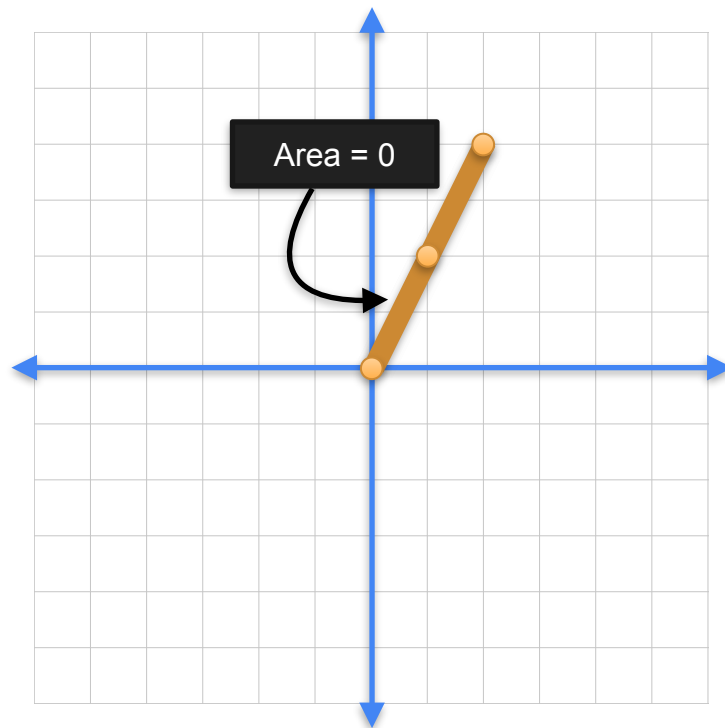
# Determinant as an area



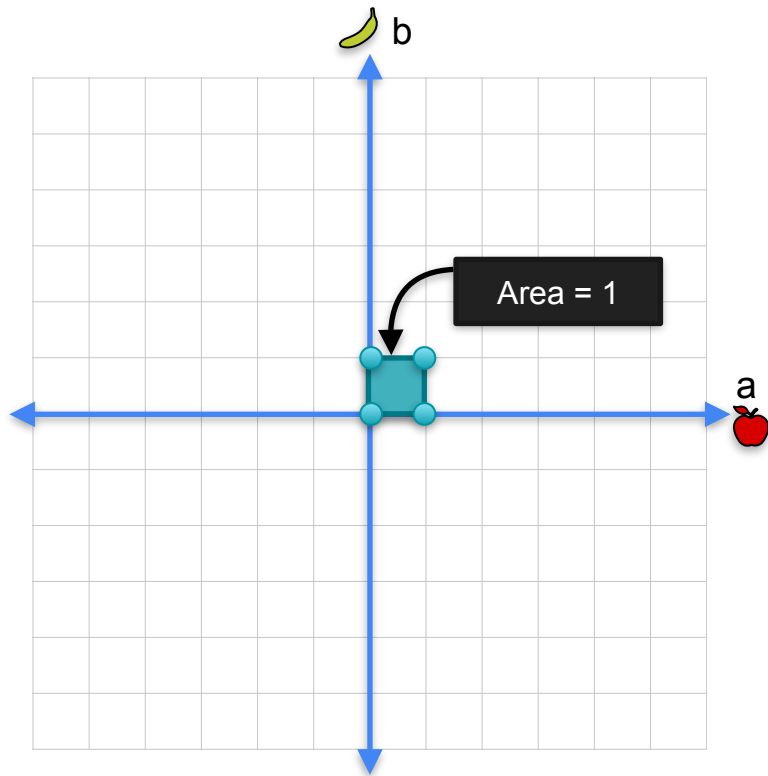
 1	 1
2	2

$$\text{Det} = 1 \cdot 2 - 1 \cdot 2$$

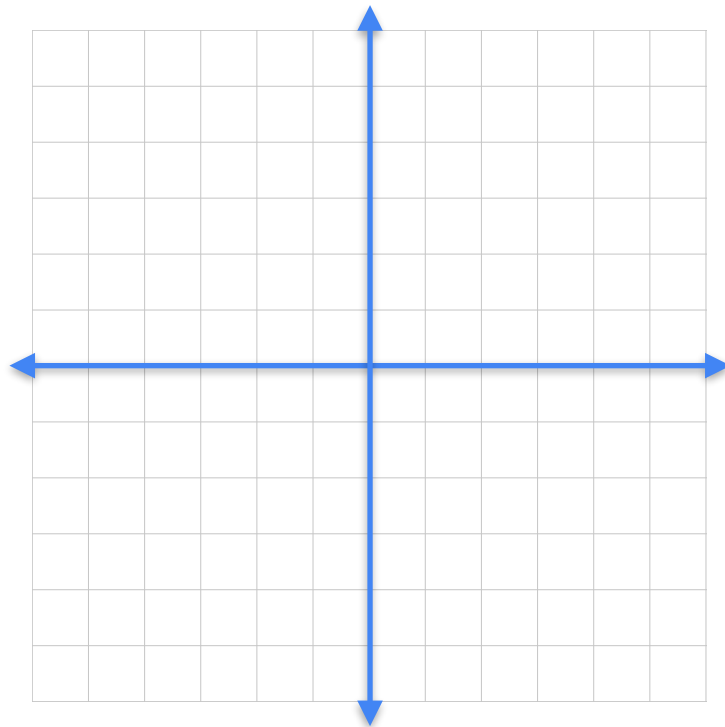
$$\text{Det} = 0$$



# Determinant as an area

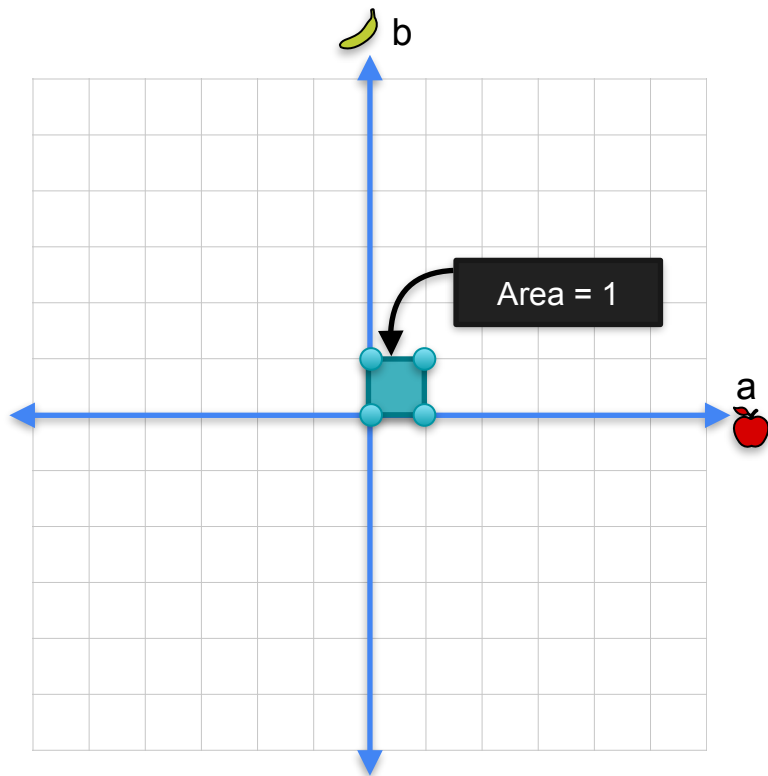


🍏	🍌
0	0
0	0

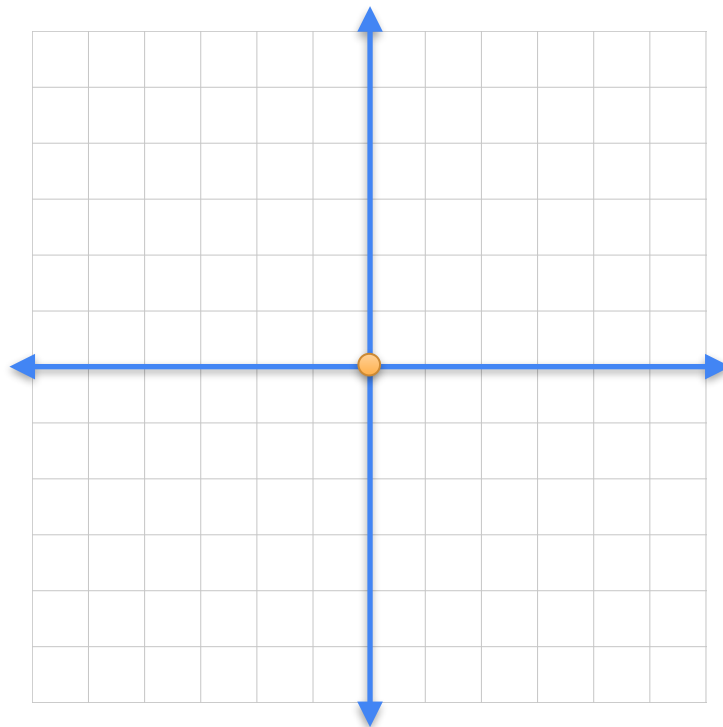




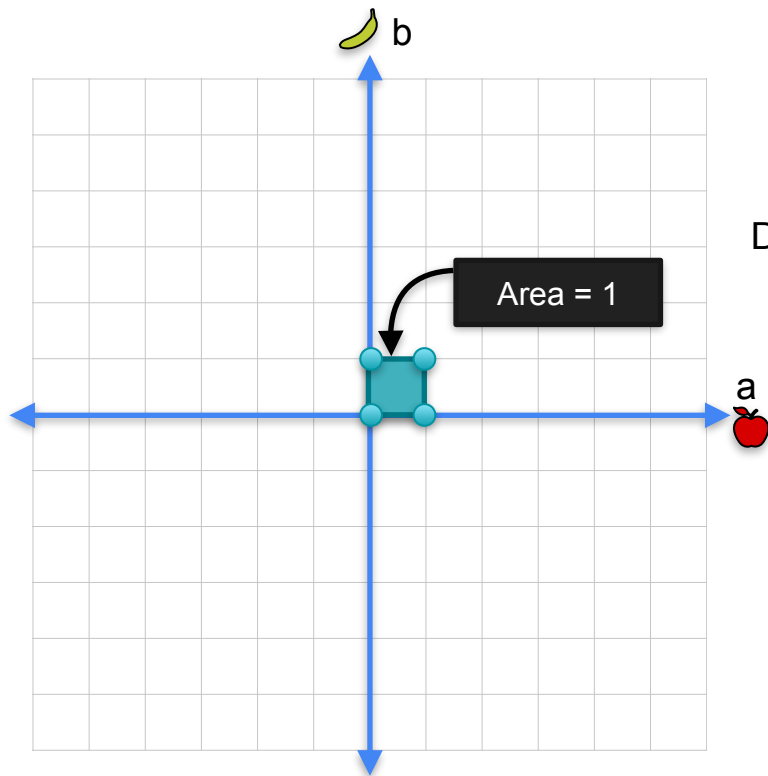
# Determinant as an area





🍏	🍌
0	0
0	0



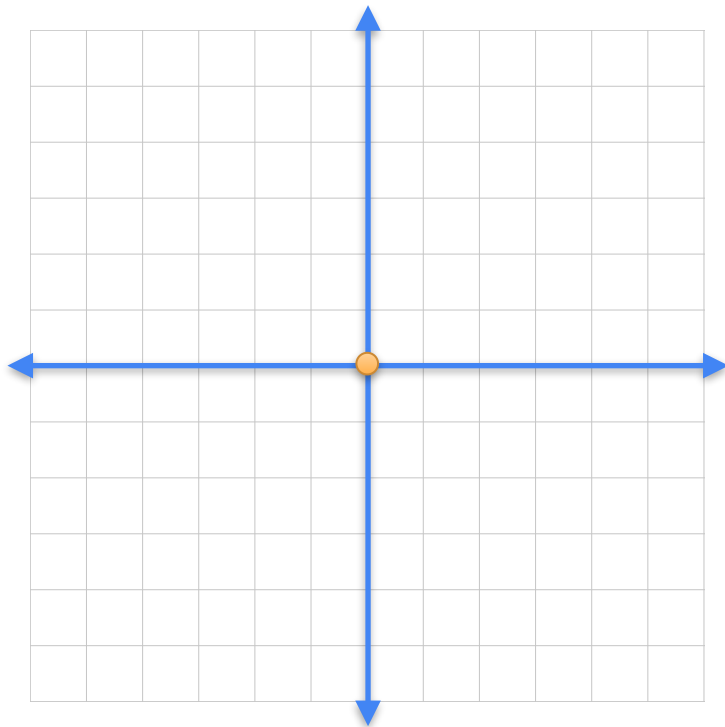
# Determinant as an area



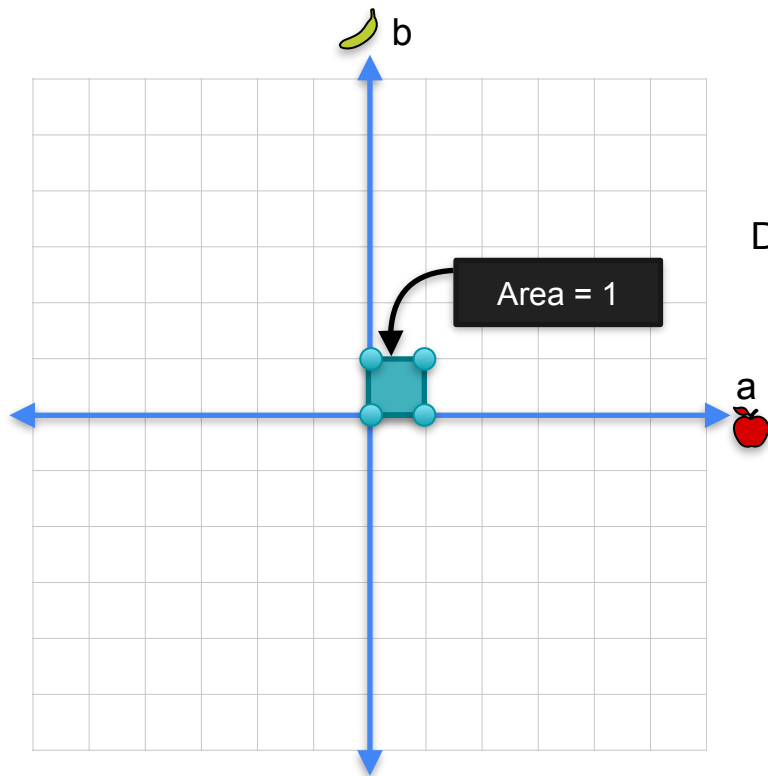
	
0	0
0	0



$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$



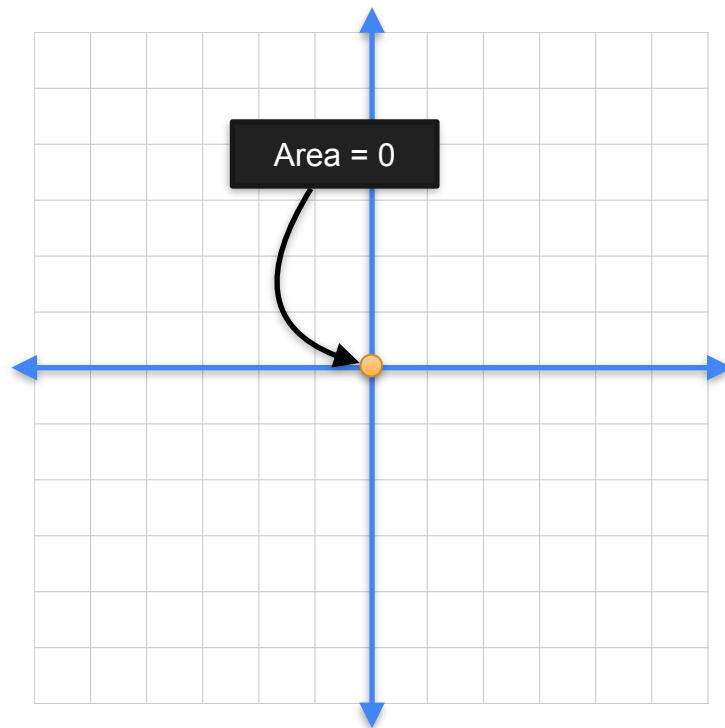
# Determinant as an area



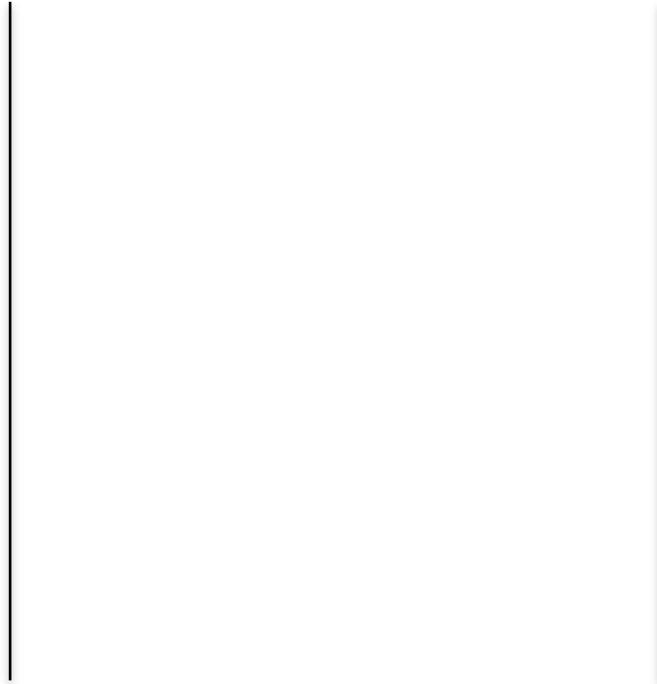
	
0	0
0	0

$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$



# Determinant as an area



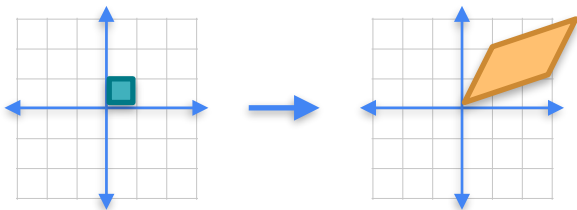
# Determinant as an area

Non-singular



3	1
1	2

Determinant = 5



Area = 5

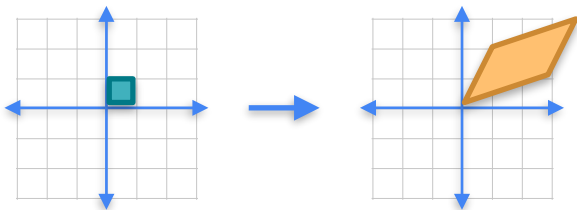
# Determinant as an area

Non-singular



3	1
1	2

Determinant = 5



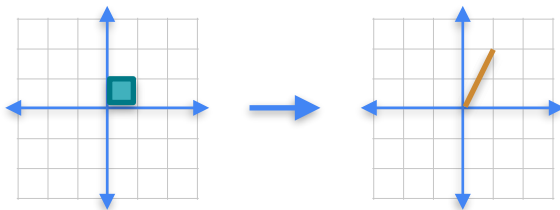
Area = 5

Singular



1	1
2	2



Determinant = 0



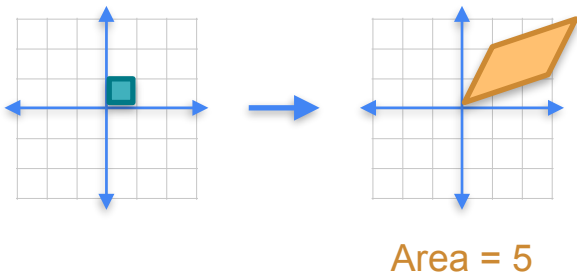
Area = 0

# Determinant as an area



Non-singular

	
3	1
1	2

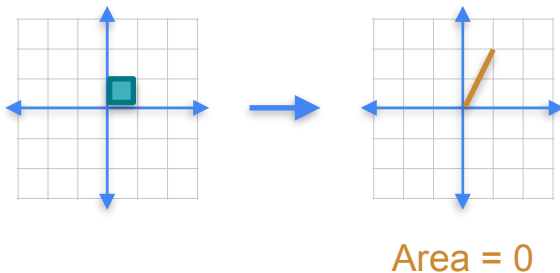
Determinant = 5





Singular

	
1	1
2	2

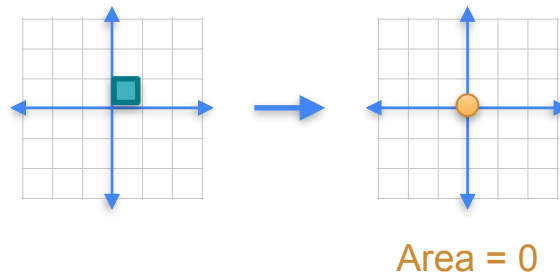
Determinant = 0





Singular

	
0	0
0	0



Determinant = 0



# Negative determinants?





3	1
1	2



1	3
2	1





# Negative determinants?



3	1
1	2



$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$



1	3
2	1



# Negative determinants?



3	1
1	2

$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$

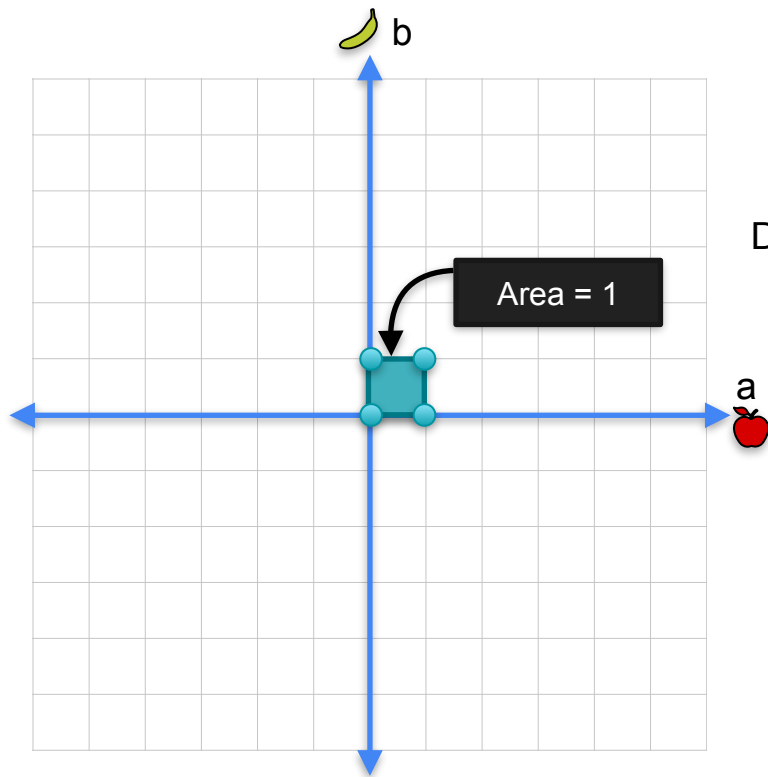




1	3
2	1

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

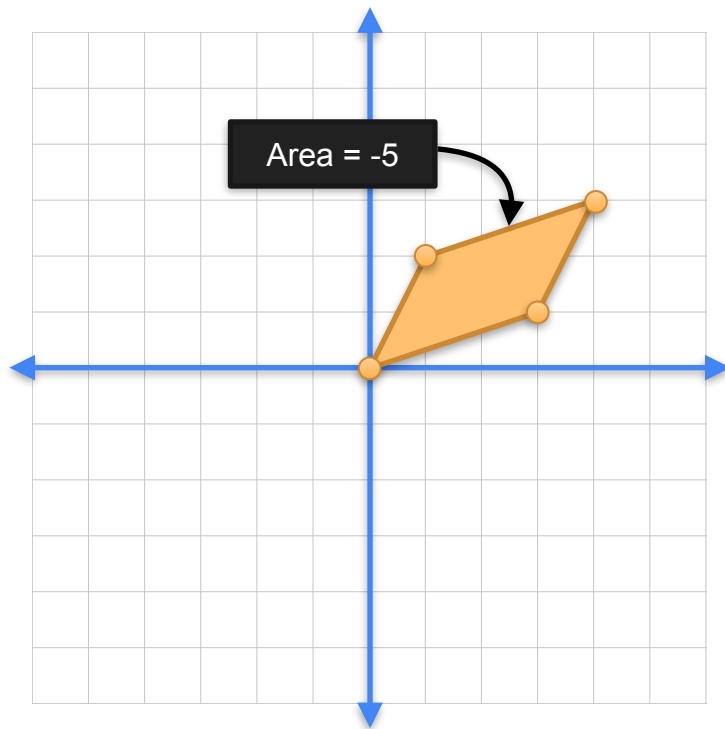
# Determinant as an area



 1	 3
2	1

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



# Determinant as an area

 b

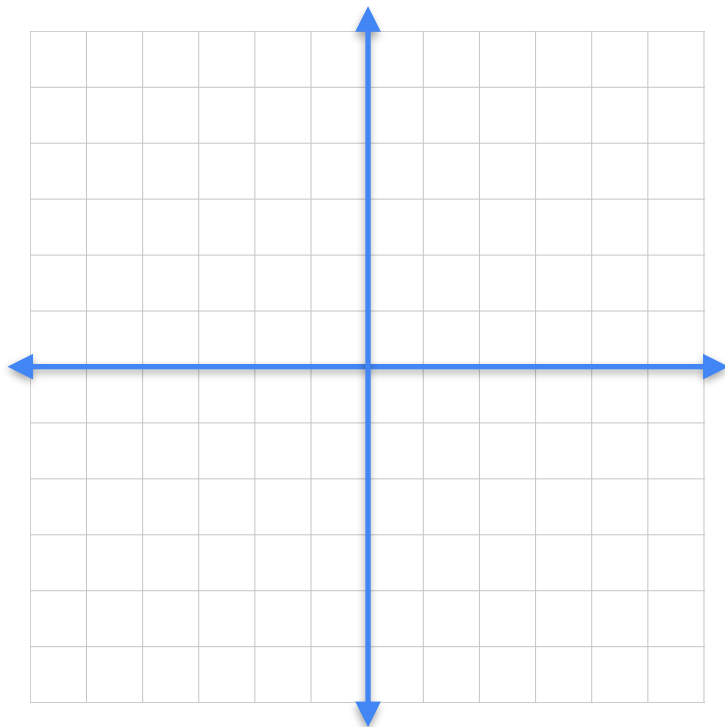
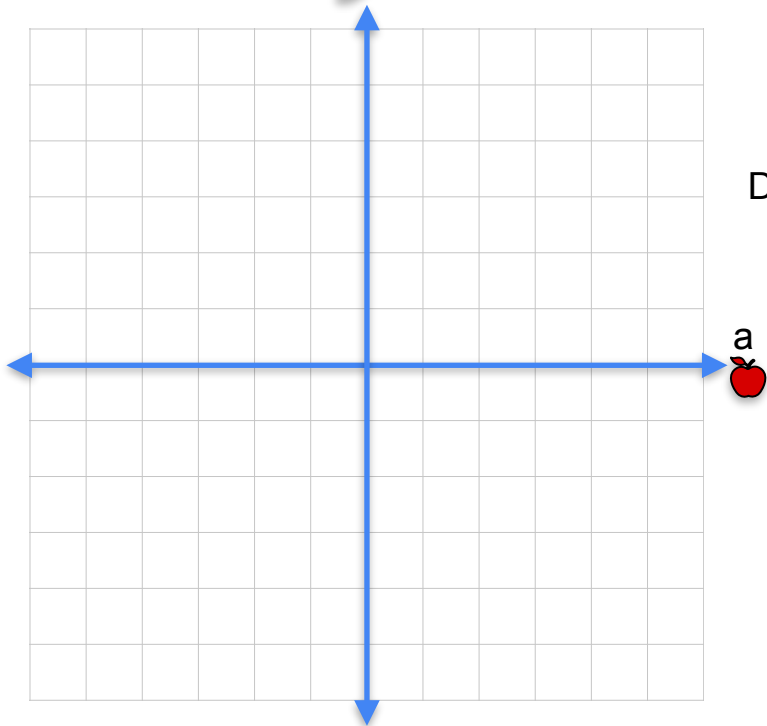
 

1	3
2	1

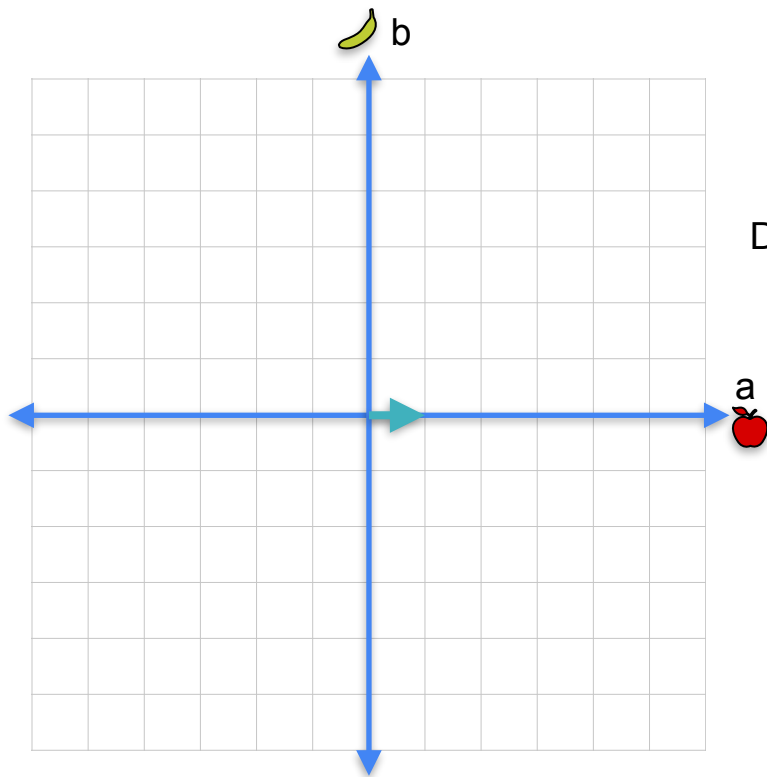
$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$



$$\text{Det} = -5$$

a 



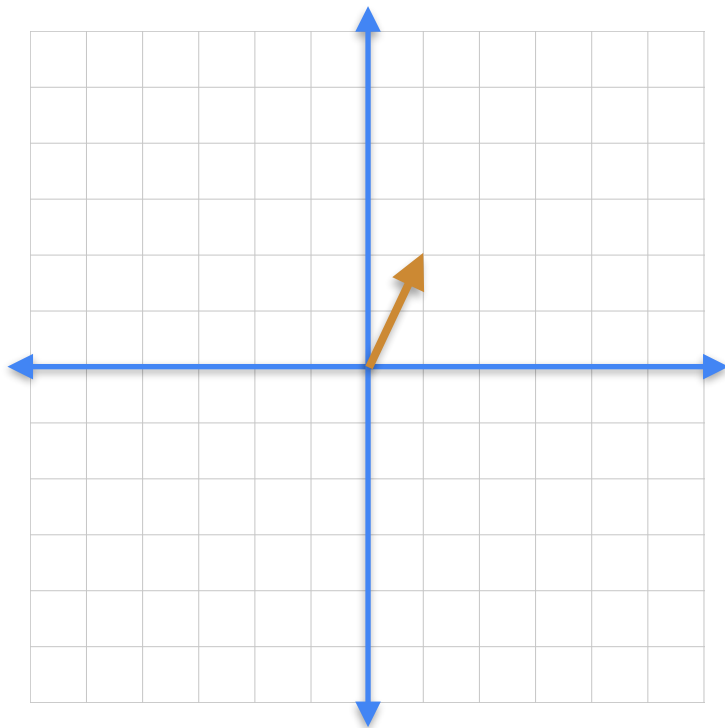
# Determinant as an area



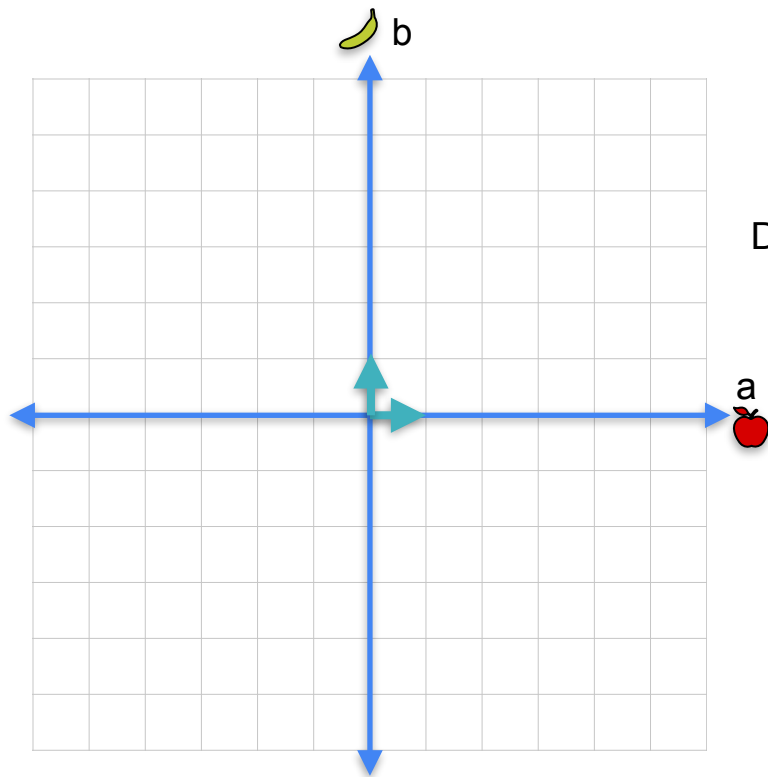
 1	 3
2	1



$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



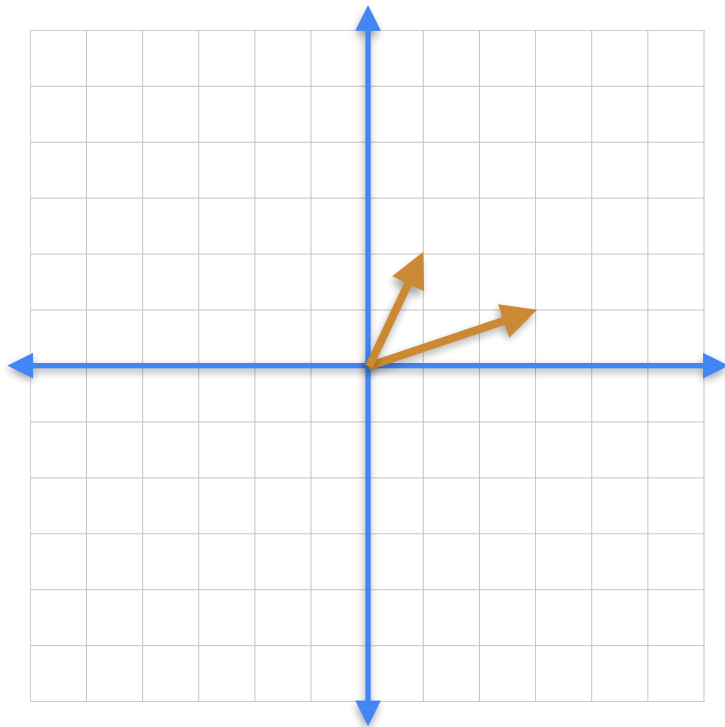
# Determinant as an area



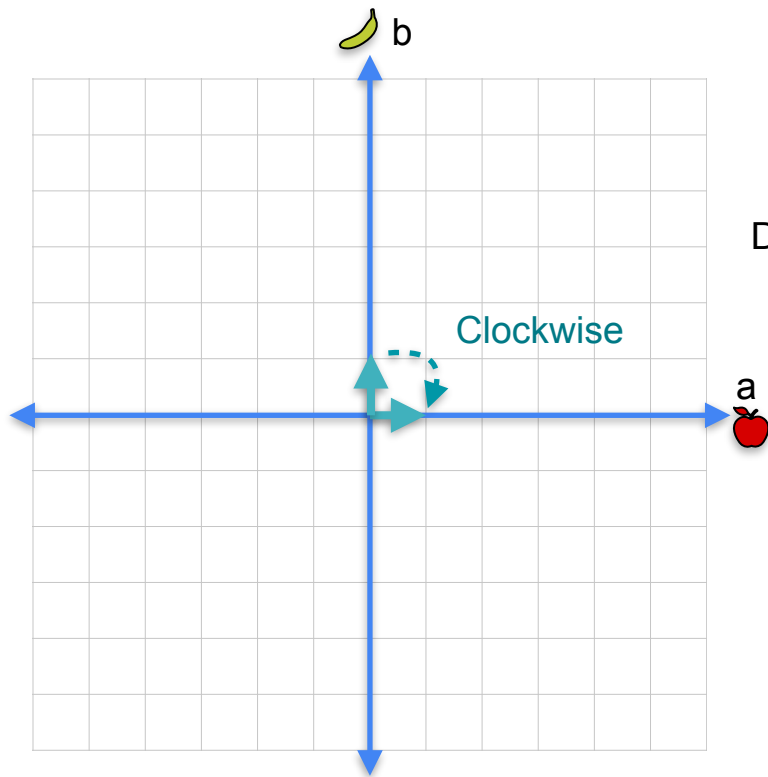
	
1	3
2	1



$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



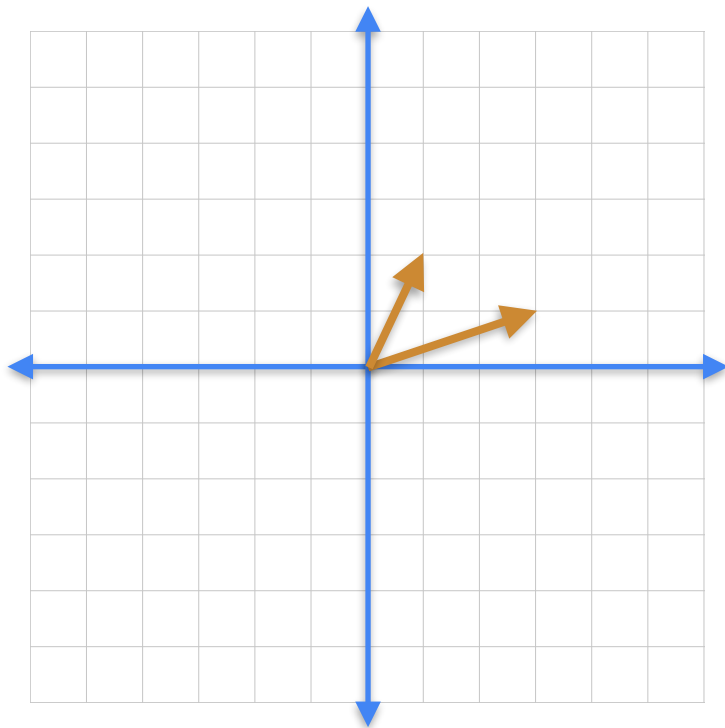
# Determinant as an area



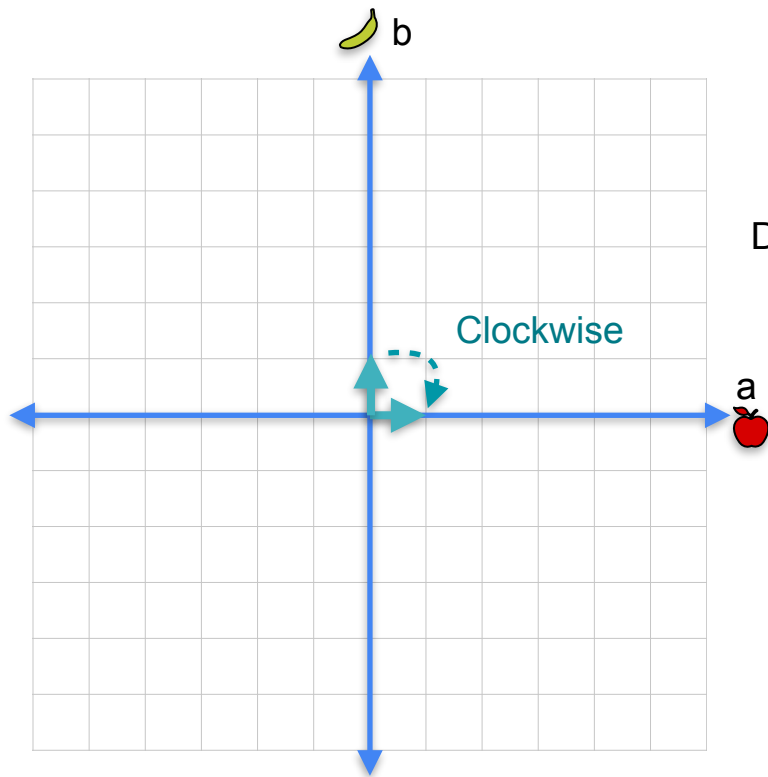
	
1	3
2	1



$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



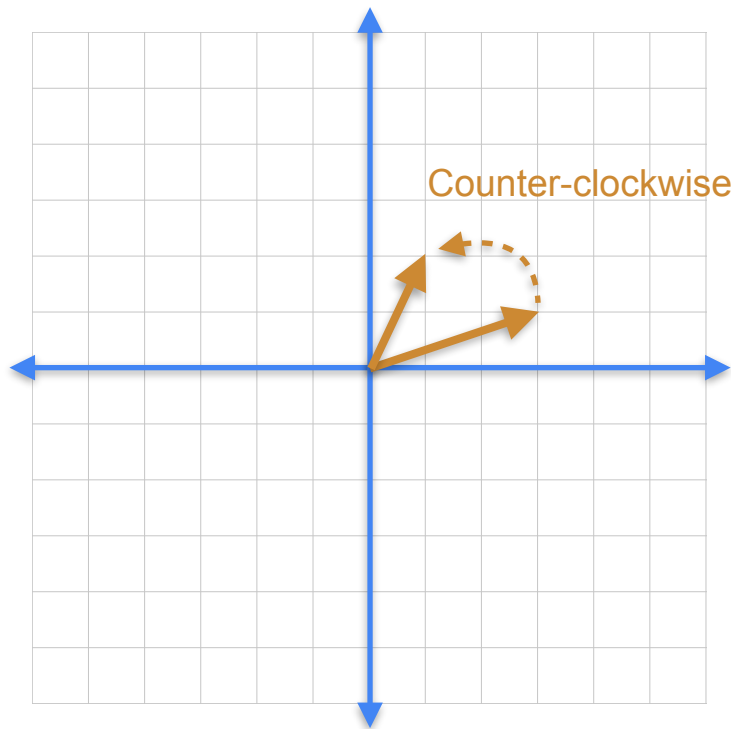
# Determinant as an area



	
1	3
2	1

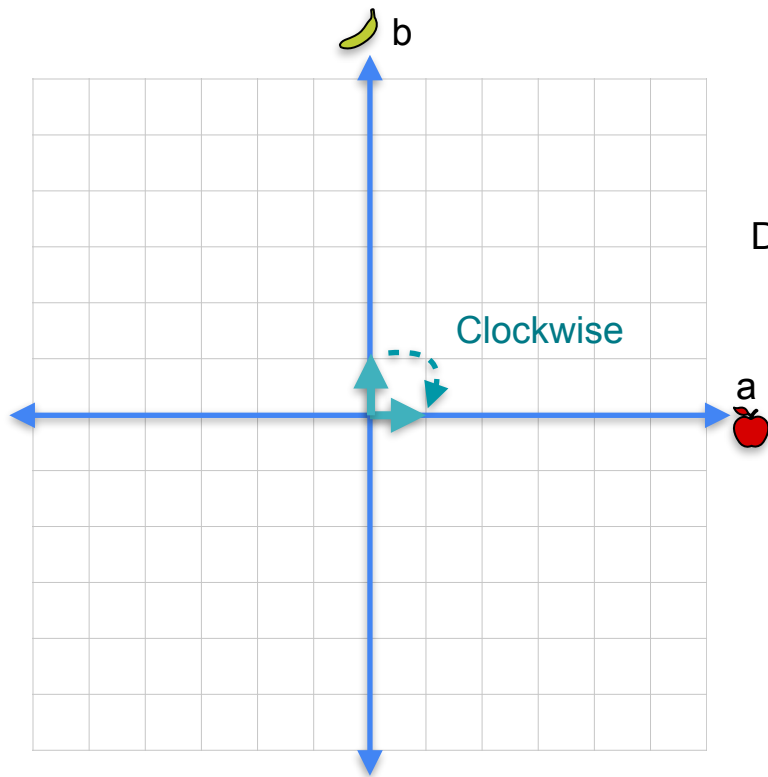
$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$



$$\text{Det} = -5$$





# Determinant as an area

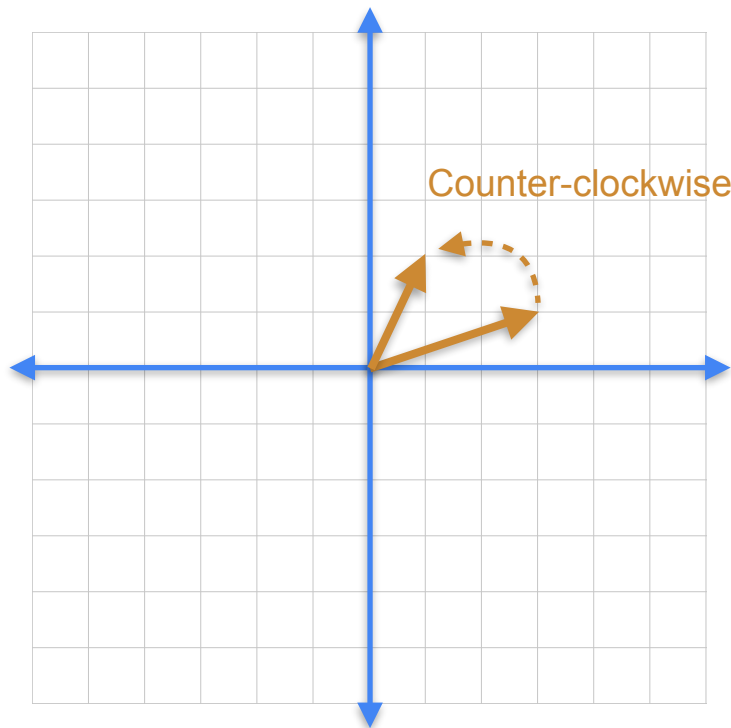


 1	 3
2	1

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

Negative





DeepLearning.AI

# Determinants and Eigenvectors

---

## **Determinant of a product**

# Determinant of a product

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 7 & 6 \end{bmatrix}$$

# Determinant of a product

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 7 & 6 \end{bmatrix}$$

$$\det = 5$$

$$3 \cdot 2 - 1 \cdot 1$$

# Determinant of a product

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 7 & 6 \end{bmatrix}$$

$$\det = 5$$

$$3 \cdot 2 - 1 \cdot 1$$

$$\det = 8$$

$$5 \cdot 2 - 2 \cdot 1$$

# Determinant of a product

3	1	5	2	=	16	8
1	2	1	2		7	6
det = 5		det = 8			det = 40	
$3 \cdot 2 - 1 \cdot 1$		$5 \cdot 2 - 2 \cdot 1$			$16 \cdot 6 - 8 \cdot 7$	

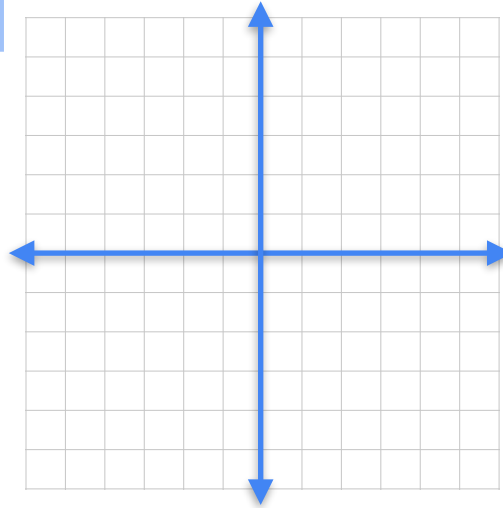
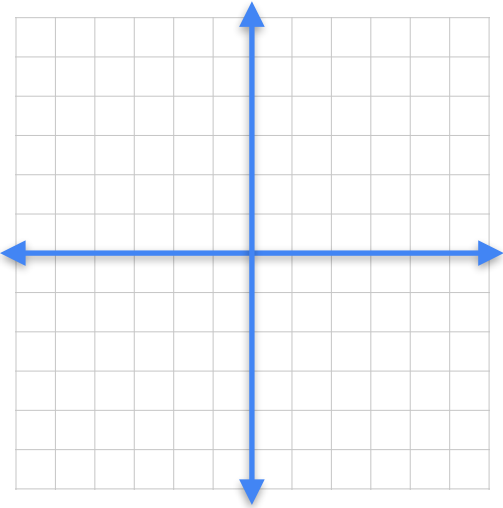
# Determinant of a product

$$\det(AB) = \det(A) \det(B)$$

# Determinant of a product

3	1
1	2

Det = 5

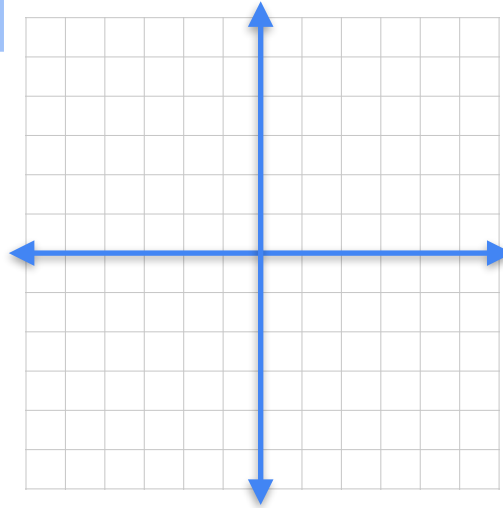
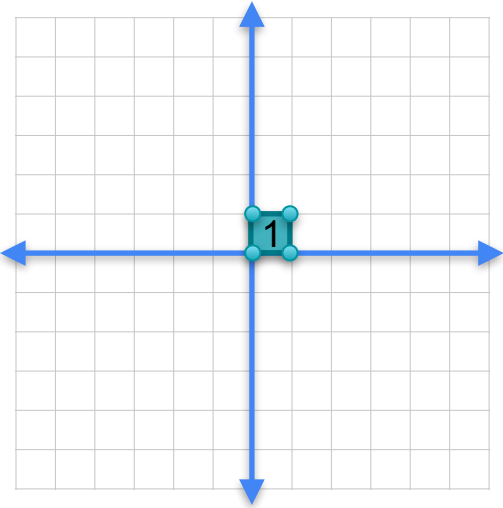




# Determinant of a product

3	1
1	2

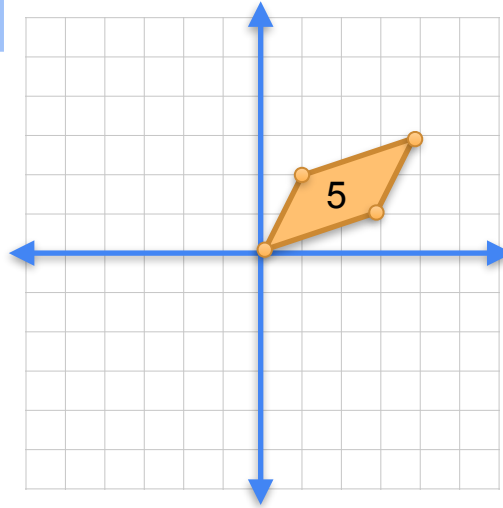
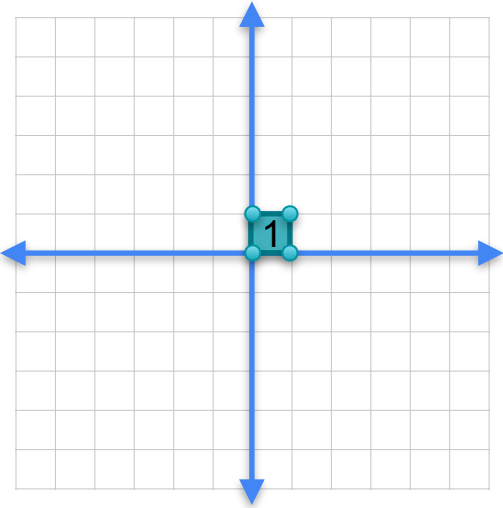
Det = 5



# Determinant of a product

3	1
1	2

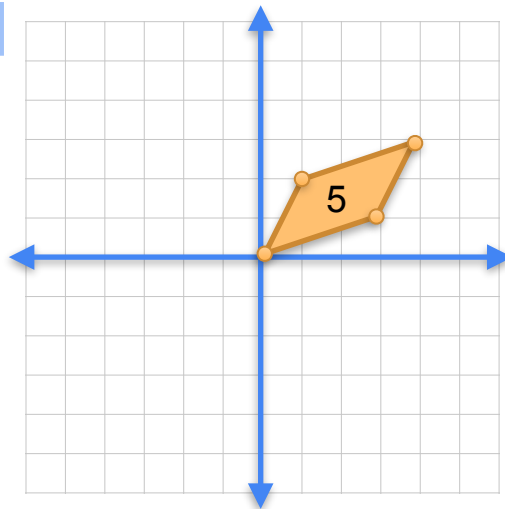
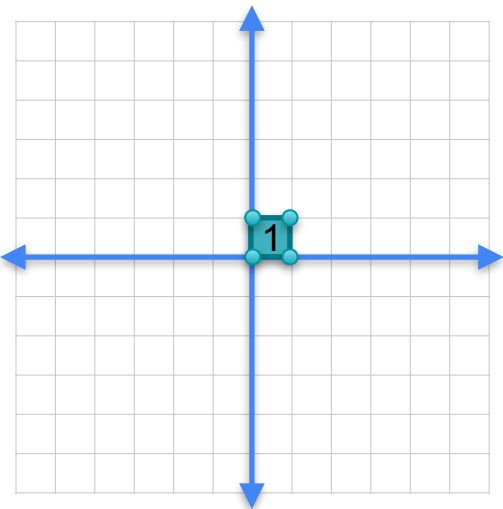
Det = 5



# Determinant of a product

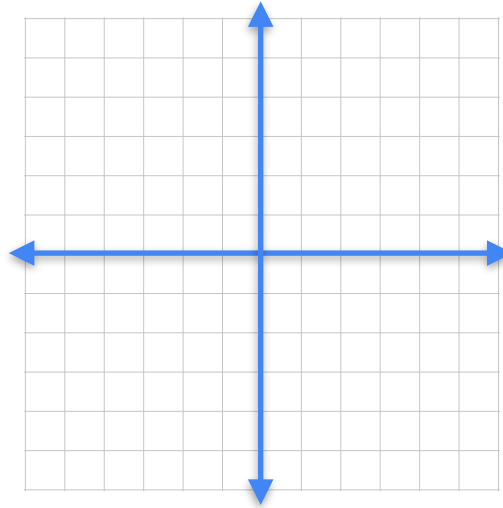
3	1
1	2

Det = 5



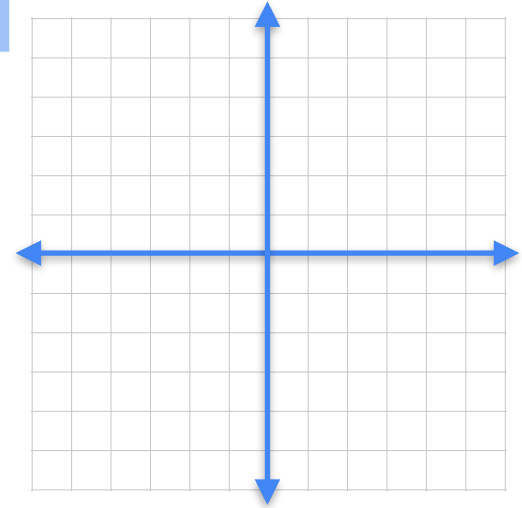
Area blows up by 5

# Determinant of a product

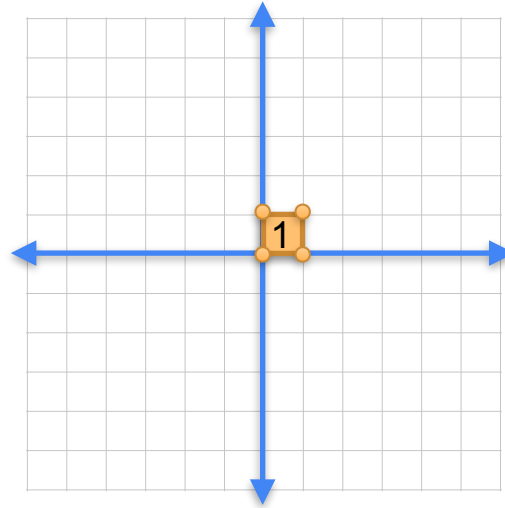


1	1
-2	1

Det = 3

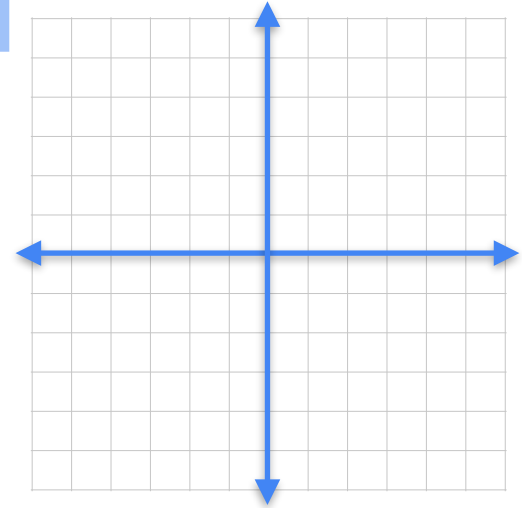


# Determinant of a product

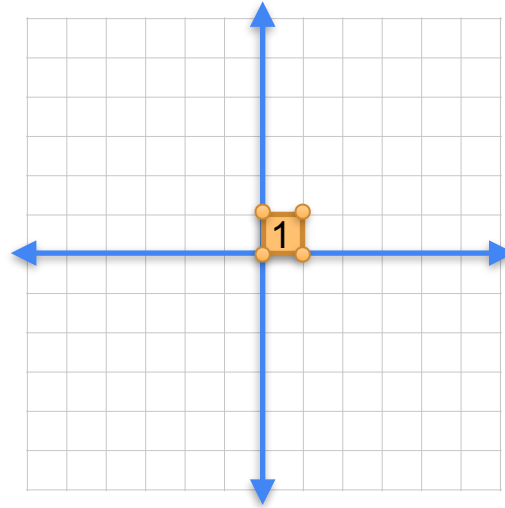


1	1
-2	1

Det = 3

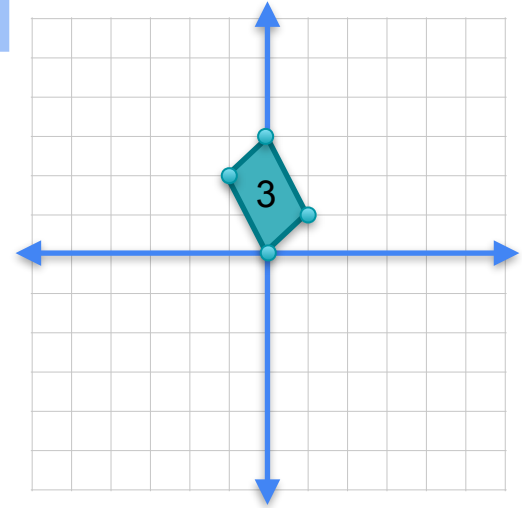


# Determinant of a product

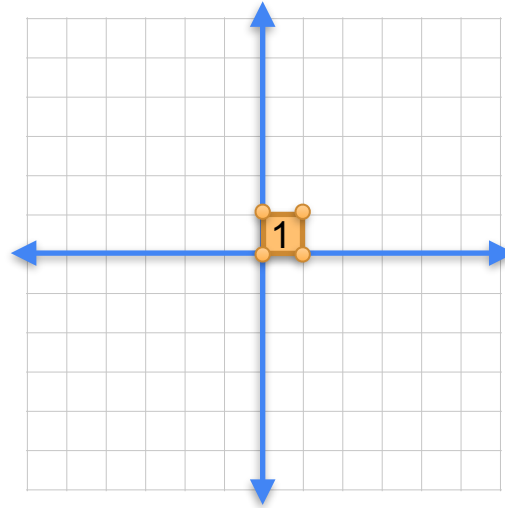


1	1
-2	1

Det = 3

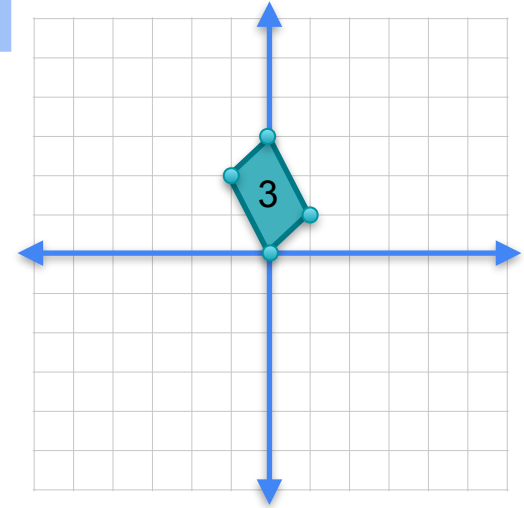


# Determinant of a product



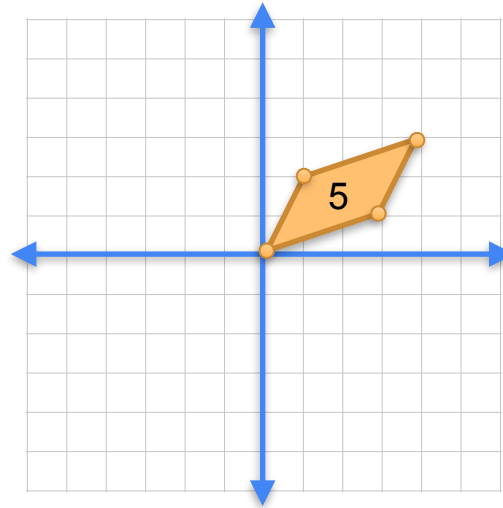
1	1
-2	1

Det = 3



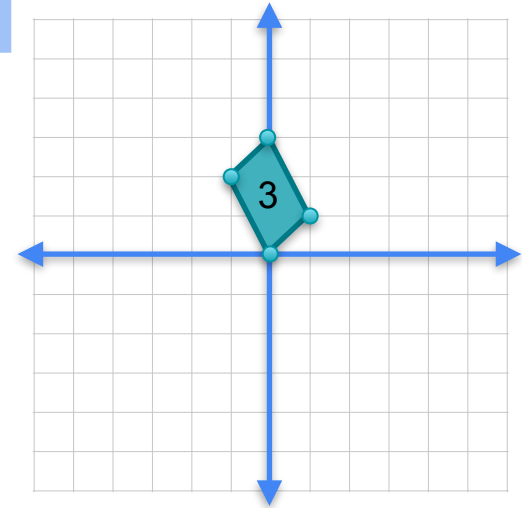
Area blows up by 3

# Determinant of a product



1	1
-2	1

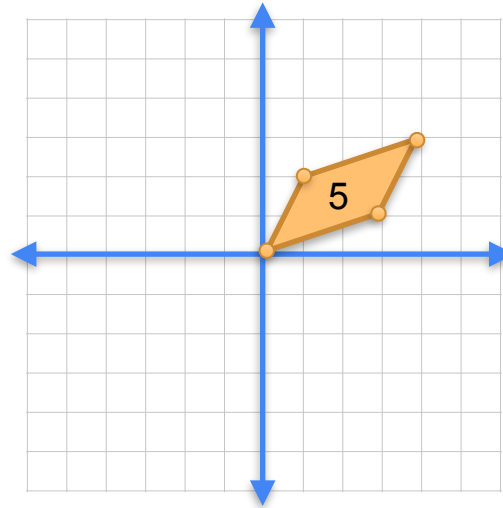
Det = 3



Area blows up by 3

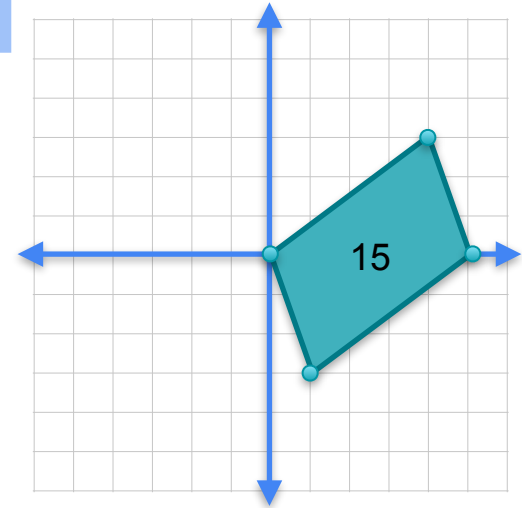


# Determinant of a product



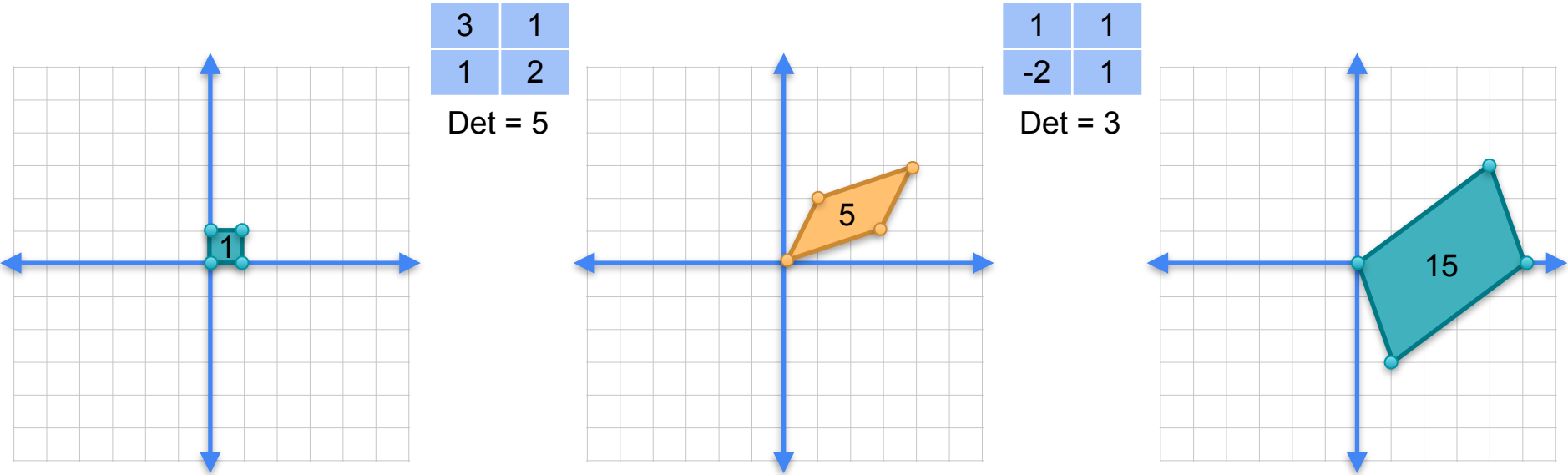
1	1
-2	1

Det = 3

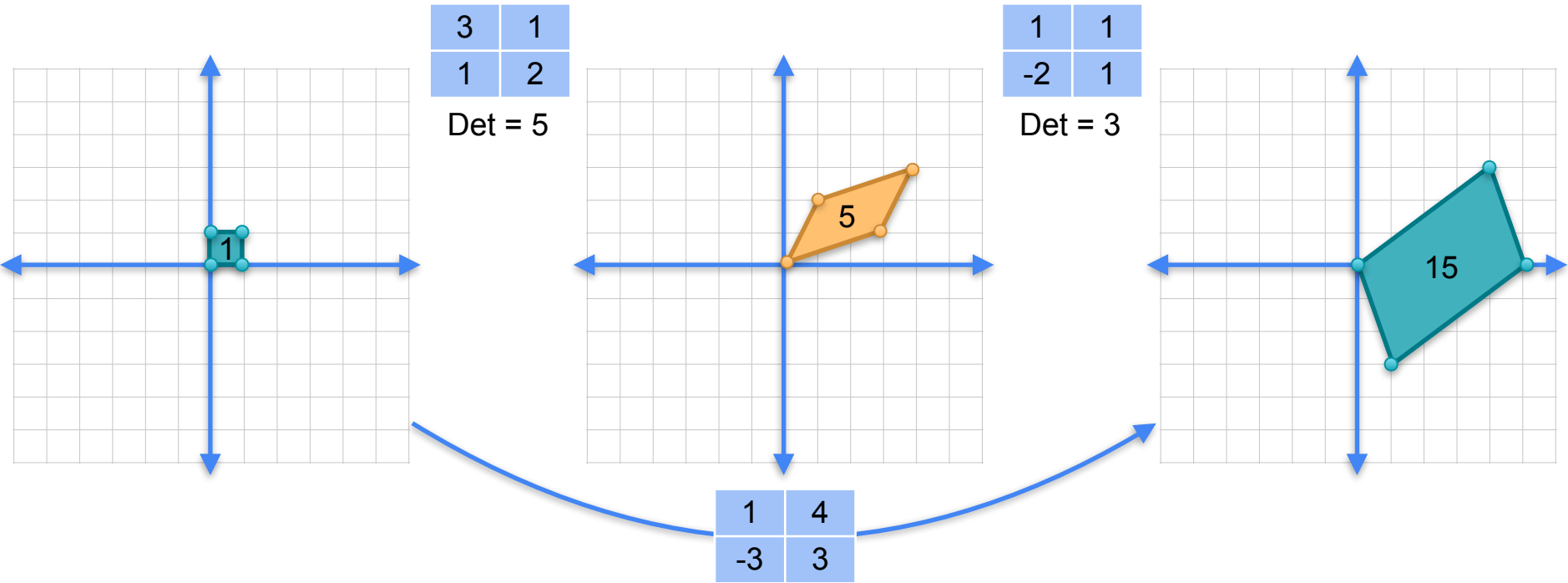


Area blows up by 3

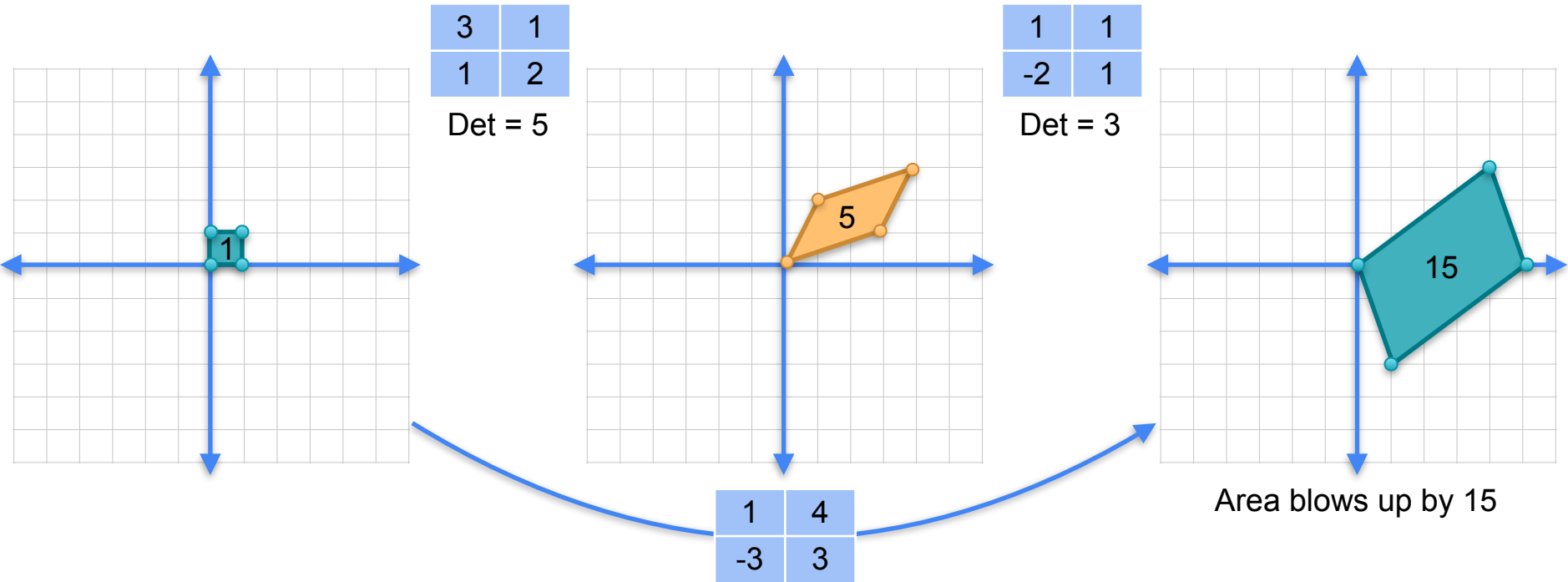
# Determinant of a product



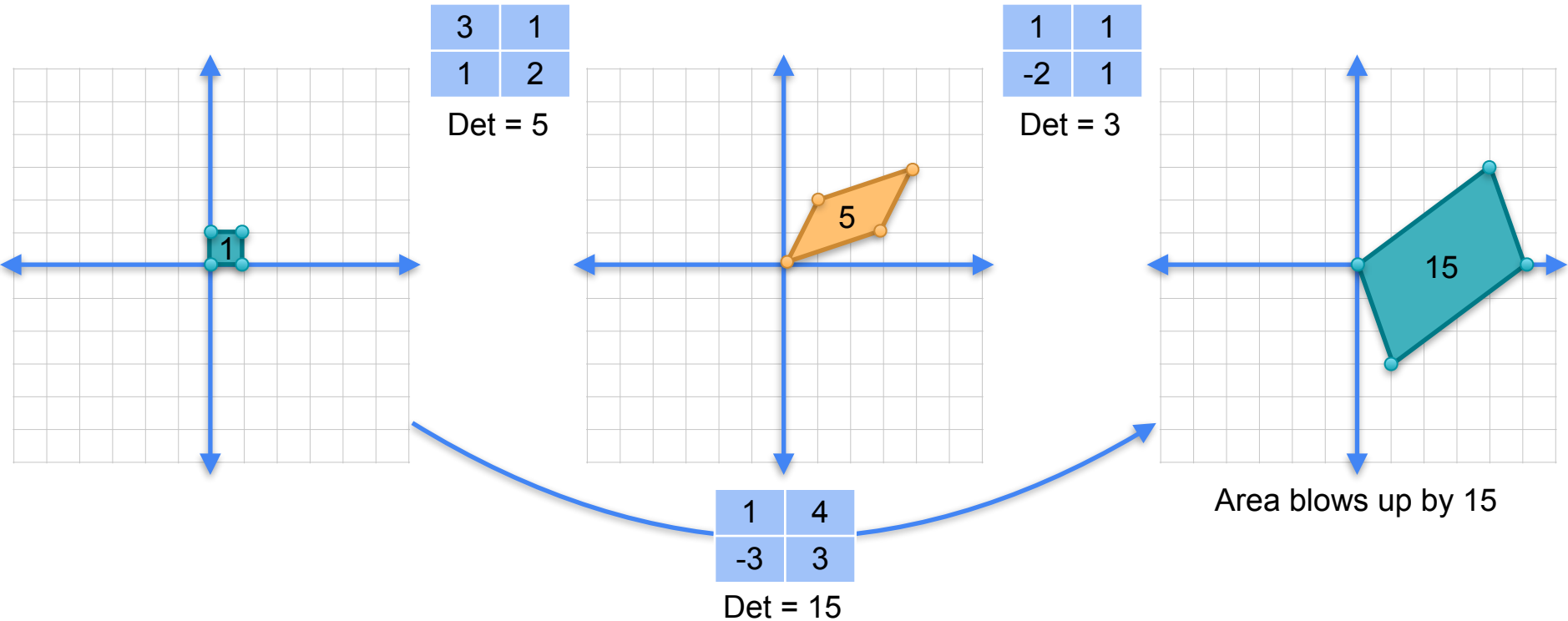
# Determinant of a product



# Determinant of a product



# Determinant of a product



# Quiz

- The product of a singular and a non-singular matrix (in any order) is:
  - Singular
  - Non-singular
  - Could be either one

# Solution

- If  $A$  is non-singular and  $B$  is singular, then  $\det(AB) = \det(A) \times \det(B) = 0$ , since  $\det(B) = 0$ . Therefore  $\det(AB) = 0$ , so  $AB$  is **singular**.

# When one factor is zero



# When one factor is zero

5

# When one factor is zero

$$5 \cdot 0$$

When one factor is zero

$$5 \cdot 0 = 0$$

# When one factor is singular...

Non-singular		Singular		Singular												
<table><tr><td>3</td><td>1</td></tr><tr><td>1</td><td>2</td></tr></table>	3	1	1	2		<table><tr><td>1</td><td>2</td></tr><tr><td>1</td><td>2</td></tr></table>	1	2	1	2	=	<table><tr><td>4</td><td>8</td></tr><tr><td>3</td><td>6</td></tr></table>	4	8	3	6
3	1															
1	2															
1	2															
1	2															
4	8															
3	6															
Det = 5		Det = 0		Det = 0												

# If one factor is singular...

3	1
1	2

Det = 5

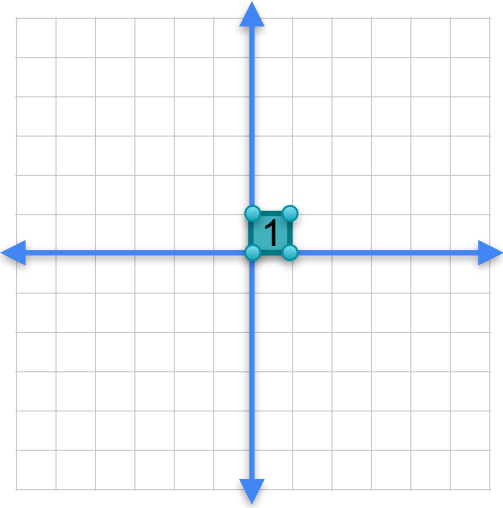
1	2
1	2

Det = 0

4	8
3	6

Det = 0

# If one factor is singular...



3	1
1	2

Det = 5

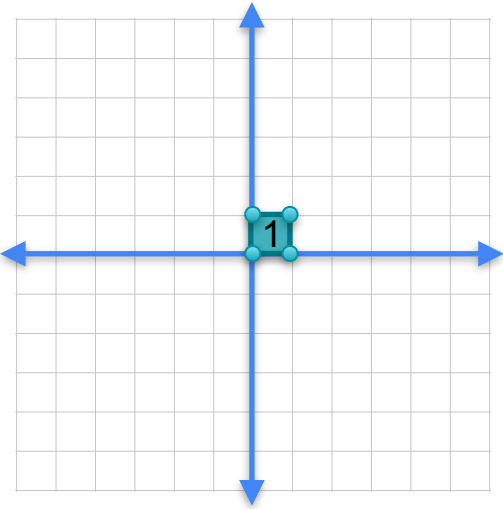
1	2
1	2

Det = 0

4	8
3	6

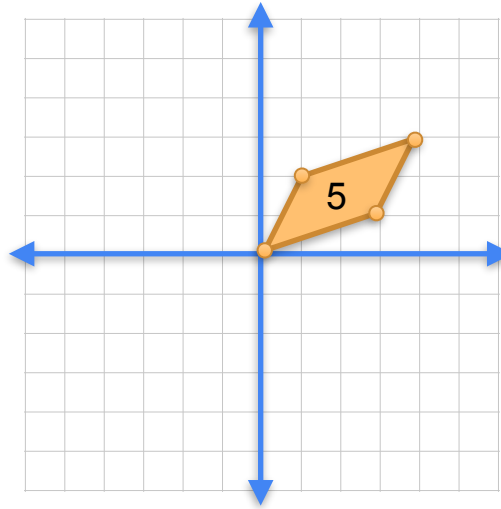
Det = 0

# If one factor is singular...



3	1
1	2

Det = 5



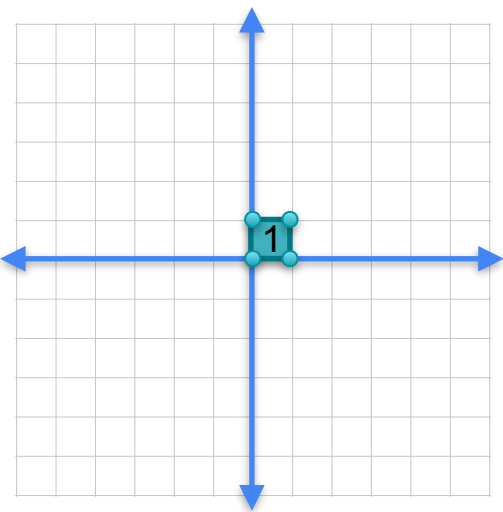
1	2
1	2

Det = 0

4	8
3	6

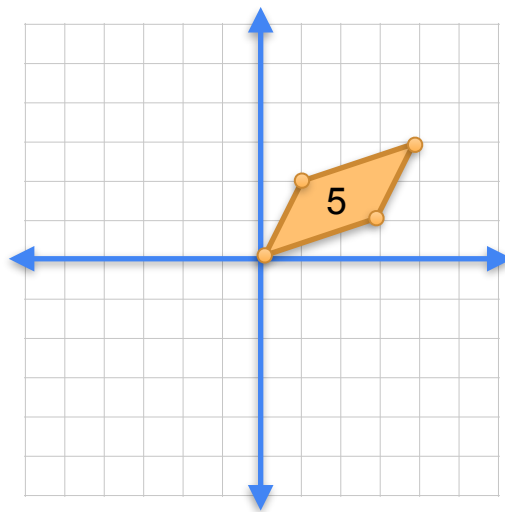
Det = 0

# If one factor is singular...



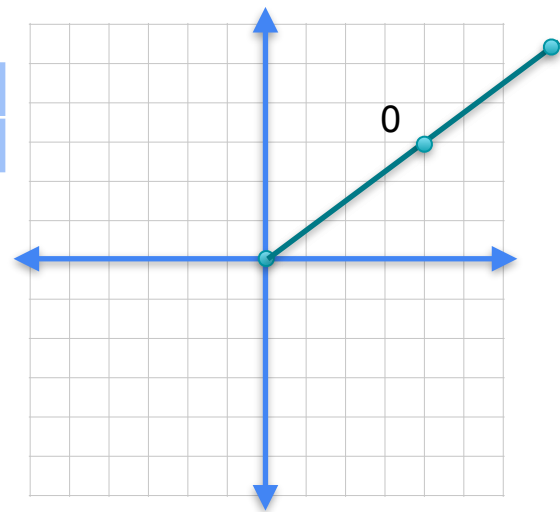
3	1
1	2

Det = 5



1	2
1	2

Det = 0

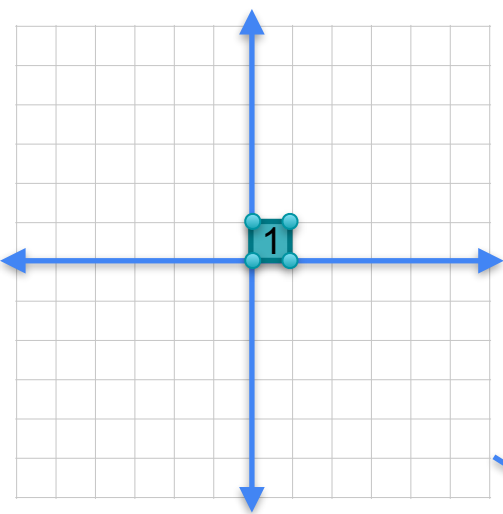


4	8
3	6

Det = 0

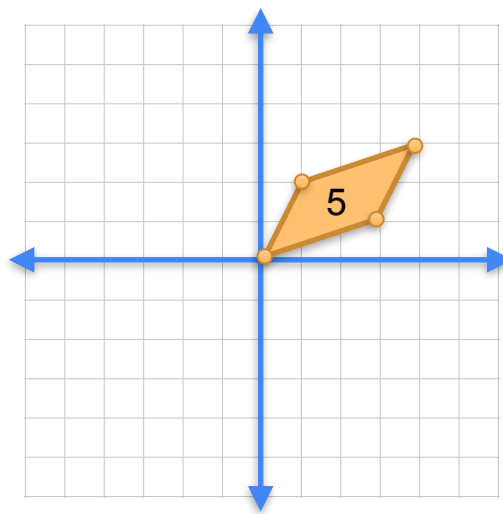


# If one factor is singular...



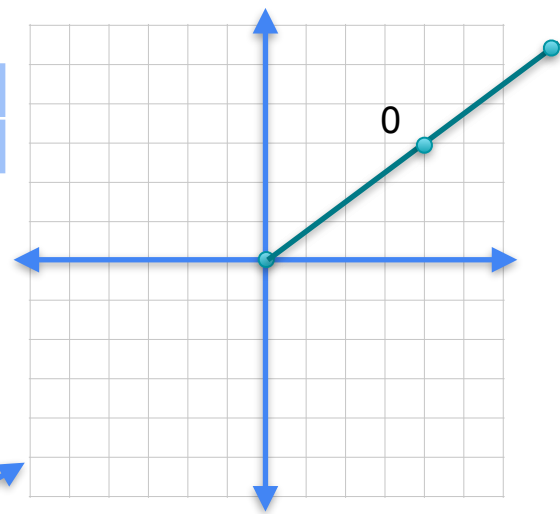
3	1
1	2

Det = 5



1	2
1	2

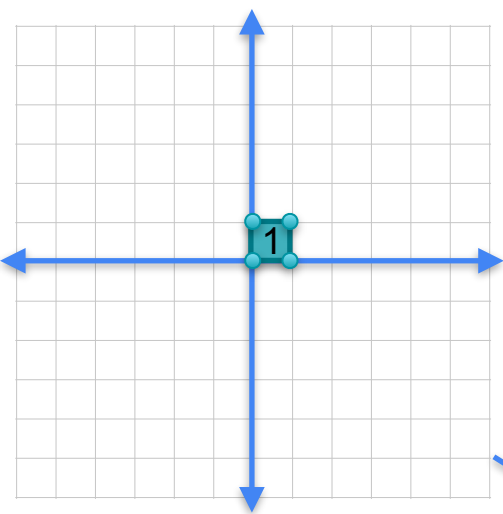
Det = 0



4	8
3	6

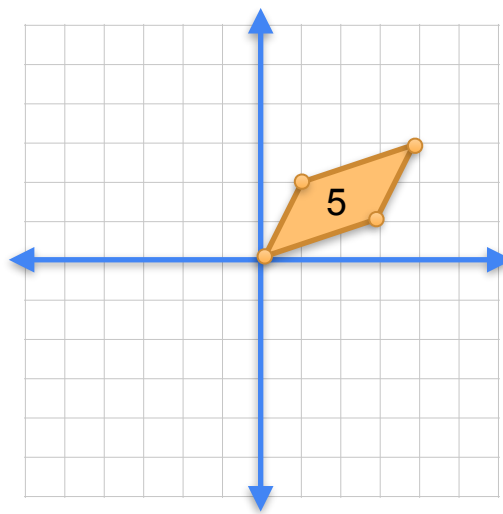
Det = 0

# If one factor is singular...



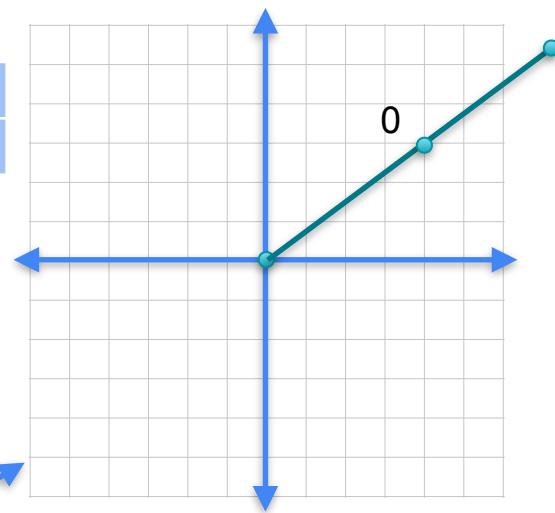
3	1
1	2

Det = 5



1	2
1	2

Det = 0



Area blows up by 0

4	8
3	6

Det = 0



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# Determinants and Eigenvectors

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## **Determinant of inverse**

# Quiz

- Find the determinants of the following matrices

0.4	-0.2
-0.2	0.6

0.25	-0.25
-0.125	0.625

# Solution

$$\text{Det} \begin{array}{|c|c|} \hline 0.4 & -0.2 \\ \hline -0.2 & 0.6 \\ \hline \end{array} = (0.4)(0.6) - (-0.2)(-0.2) = 0.2$$

$$\text{Det} \begin{array}{|c|c|} \hline 0.25 & -0.25 \\ \hline -0.125 & 0.625 \\ \hline \end{array} = (0.25)(0.625) - (-0.125)(-0.25) = 0.125$$

# Determinant of an inverse

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

$$\det = 5$$



# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

$$\det = 5$$

$$\det = 0.2$$

$$5^{-1} = 0.2$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

$$5^{-1} = 0.2$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

det = 8

$$5^{-1} = 0.2$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

det = 8

det = 0.125

$$5^{-1} = 0.2$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

$$\det = 5$$

$$\det = 0.2$$

$$5^{-1} = 0.2$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

$$\det = 8$$

$$\det = 0.125$$

$$8^{-1} = 0.125$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

$$5^{-1} = 0.2$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

det = 8

det = 0.125

$$8^{-1} = 0.125$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

$$5^{-1} = 0.2$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

det = 8

det = 0.125

$$8^{-1} = 0.125$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

det = 0



# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

$$\det = 5$$

$$\det = 0.2$$

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$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

$$\det = 8$$

$$\det = 0.125$$

$$8^{-1} = 0.125$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$\det = 0$$

$$\det = ???$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

$$\det = 5$$

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$$5^{-1} = 0.2$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

$$\det = 8$$

$$\det = 0.125$$

$$8^{-1} = 0.125$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$\det = 0$$

$$\det = ???$$

$$0^{-1} = ???$$

# Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

# Why?

# Why?

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

# Why?

$$\det(AB) = \det(A) \det(B)$$

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Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AA^{-1}) = \det(A) \det(A^{-1})$$


# Why?

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AB) = \det(A) \det(B)$$

$$\det(AA^{-1}) = \det(A) \det(A^{-1})$$


$$\det(I) = \det(A) \det(A^{-1})$$



# Why?

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AB) = \det(A) \det(B)$$

$$\det(AA^{-1}) = \det(A) \det(A^{-1})$$

$$\det(I) = \det(A) \det(A^{-1})$$

$$\frac{1}{\det(A)}$$

# Why?

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AB) = \det(A) \det(B)$$

$$\det(AA^{-1}) = \det(A) \det(A^{-1})$$

$$\begin{array}{ccc} & \uparrow & \\ \det(I) & = & \det(A) \det(A^{-1}) \\ \uparrow & & \uparrow \\ 1 & & \frac{1}{\det(A)} \end{array}$$

# Determinant of the identity matrix

$$\det \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} = 1 \cdot 1 - 0 \cdot 0 = 1$$

# Determinant of the identity matrix

$$\det \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\det(I) = 1$$



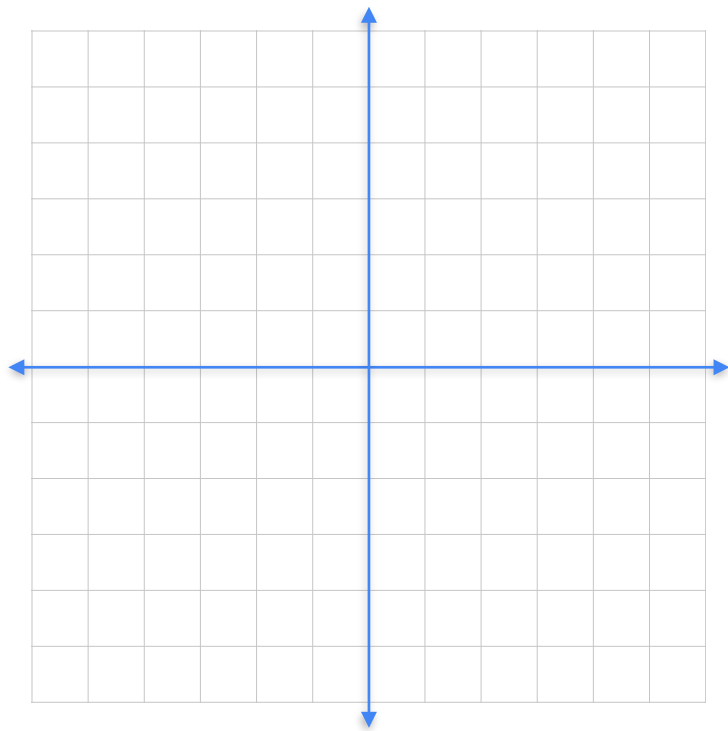
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# Determinants and Eigenvectors

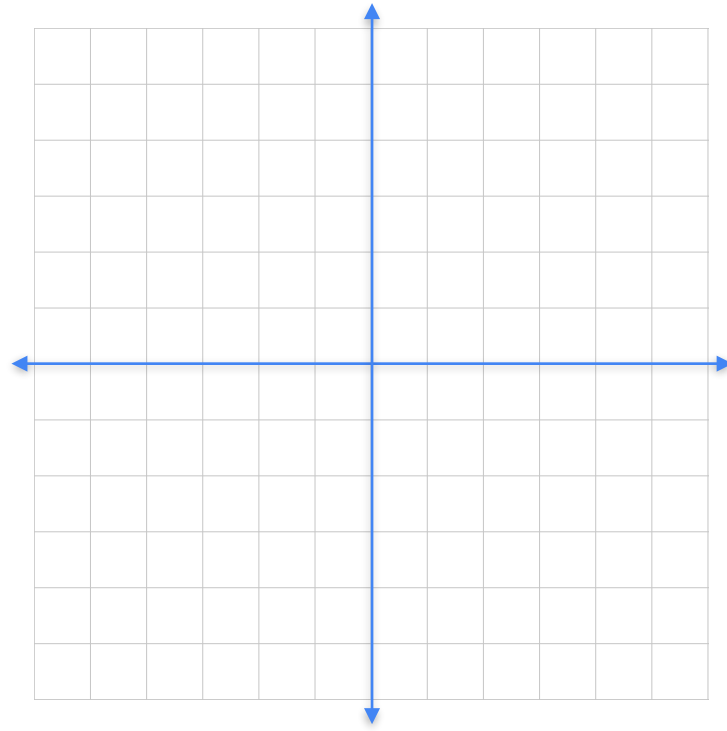
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## **Bases**

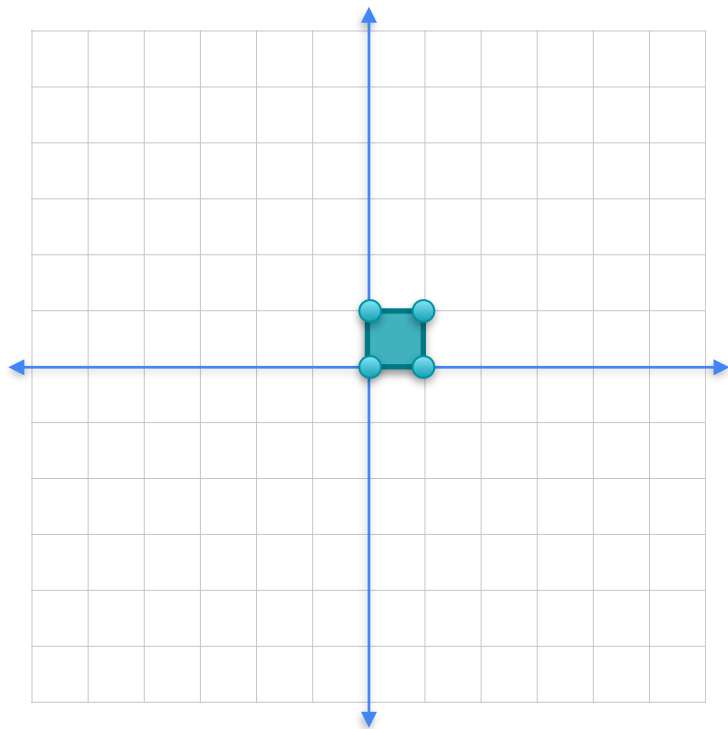
# Bases



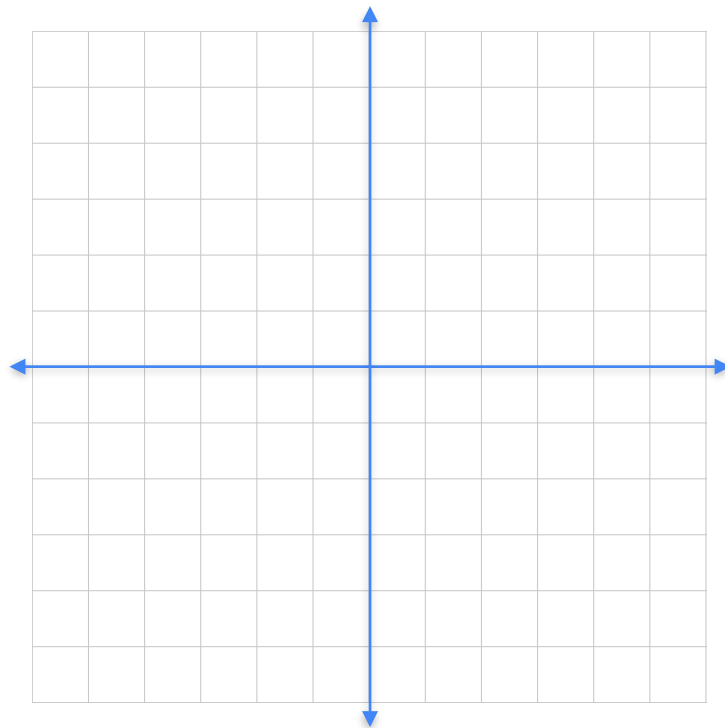
3	1
1	2



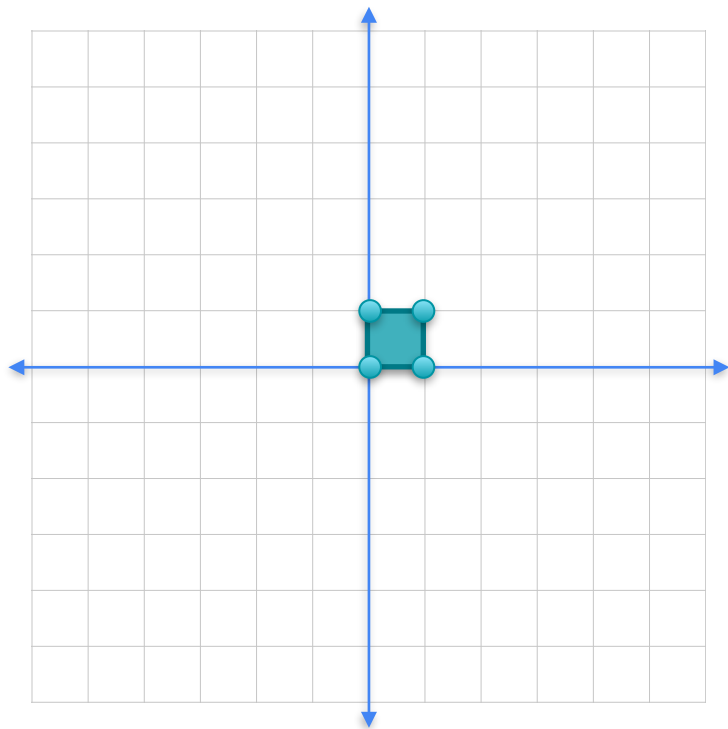
# Bases



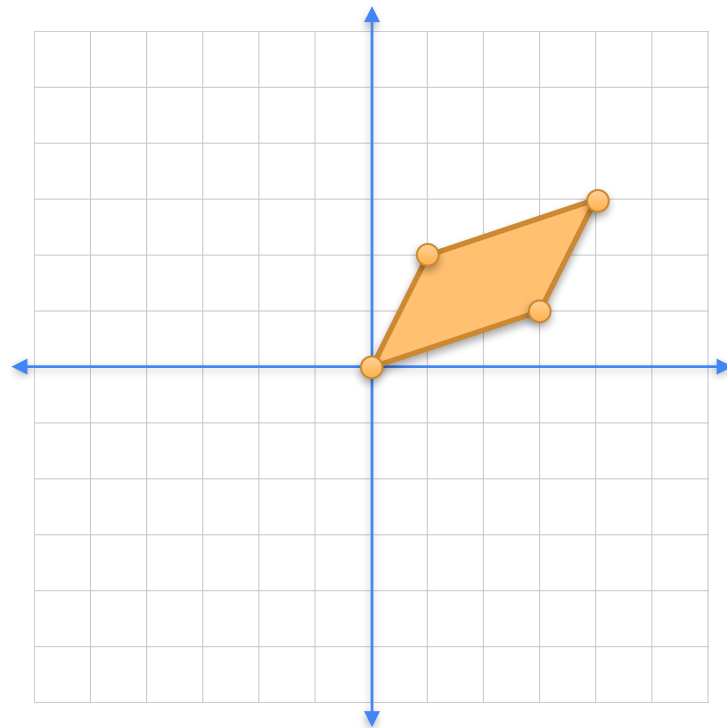
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1	2



# Bases

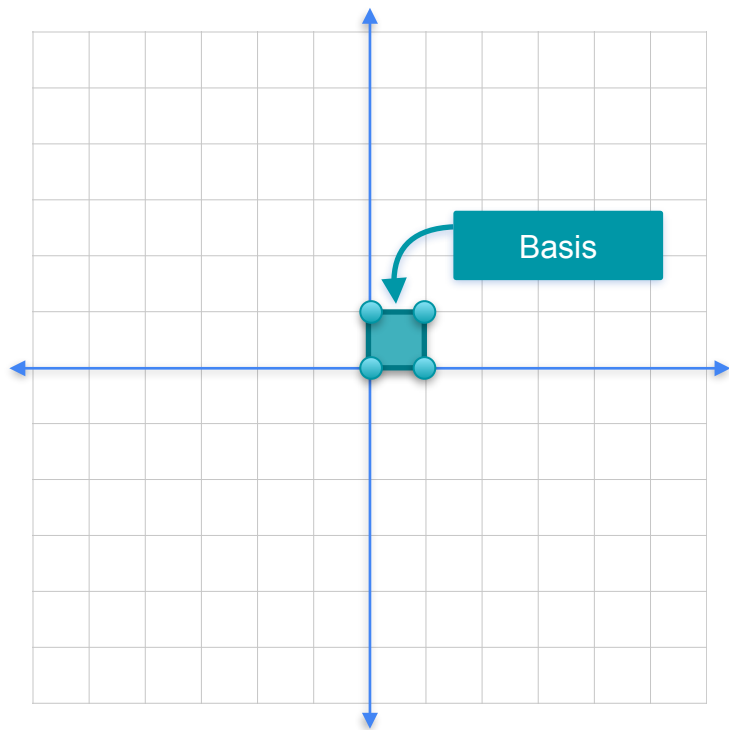


3	1
1	2

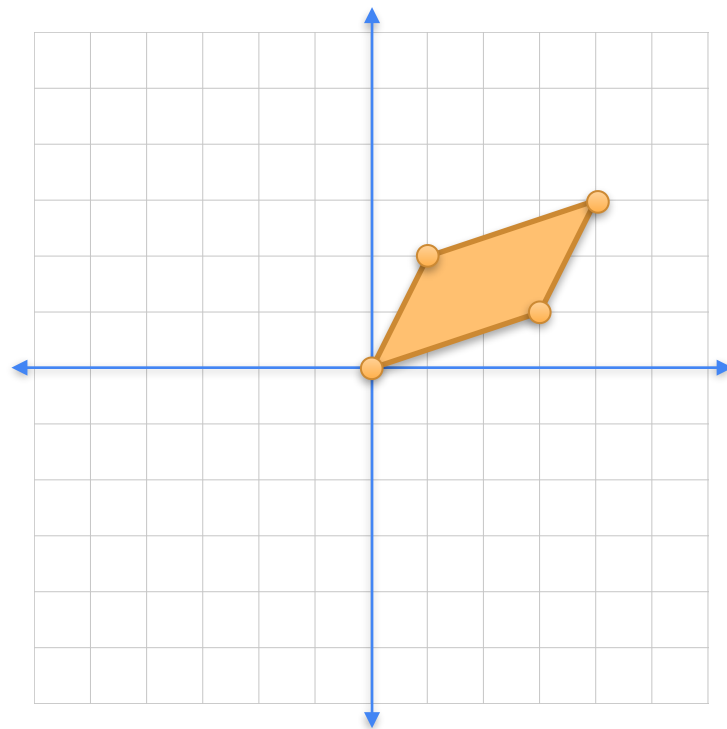




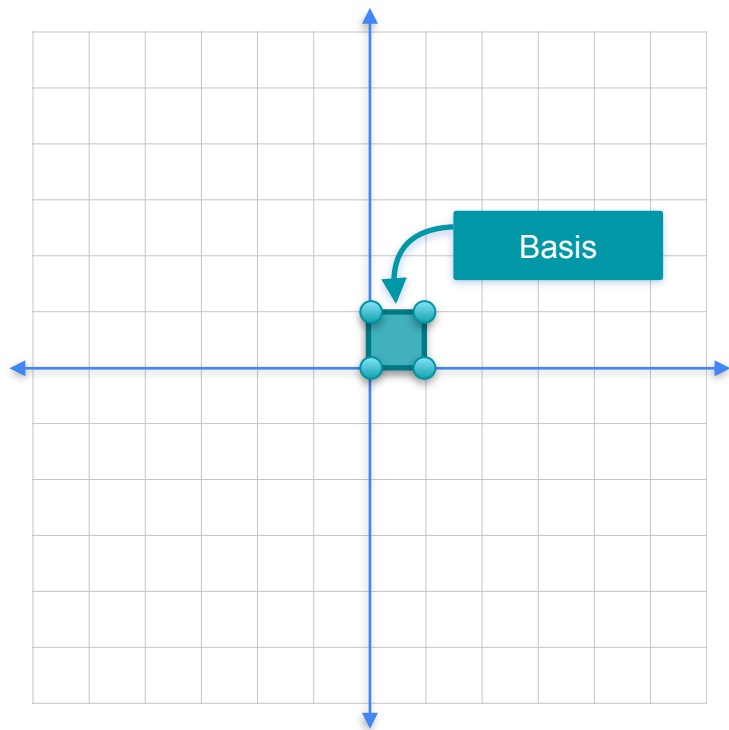
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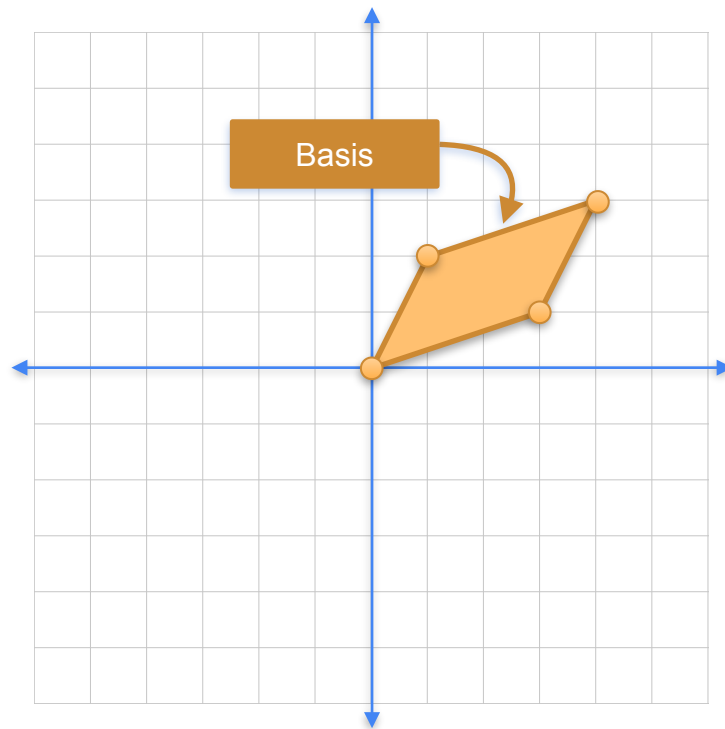
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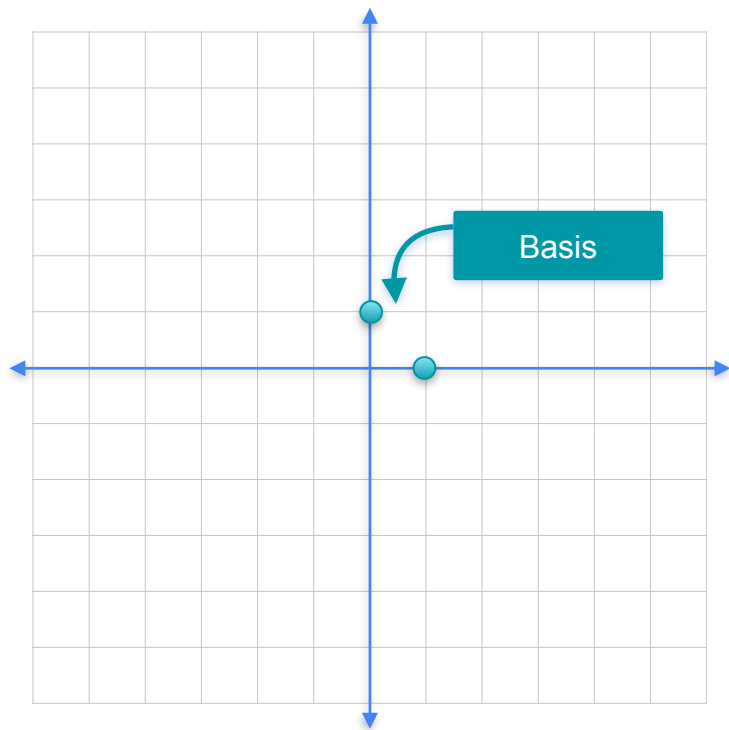
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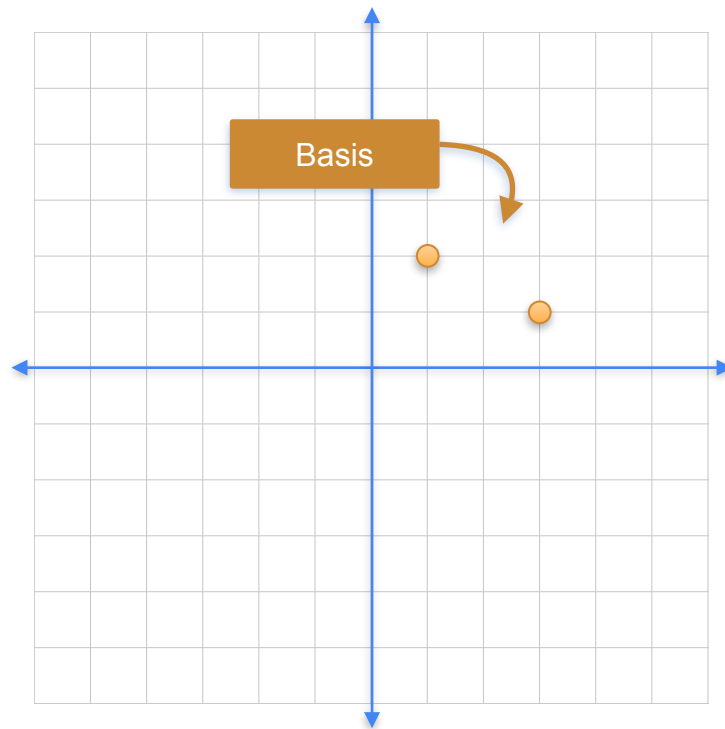
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1	2



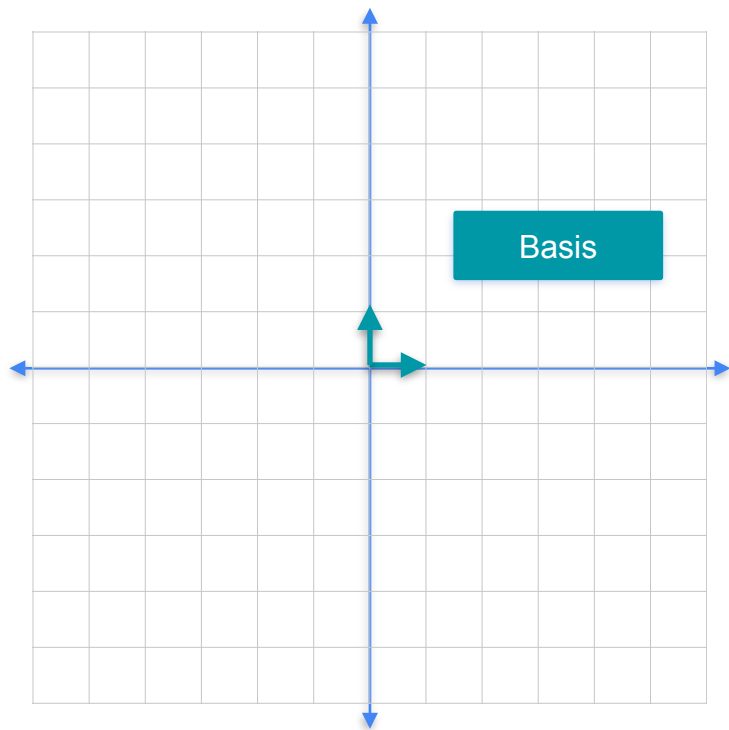
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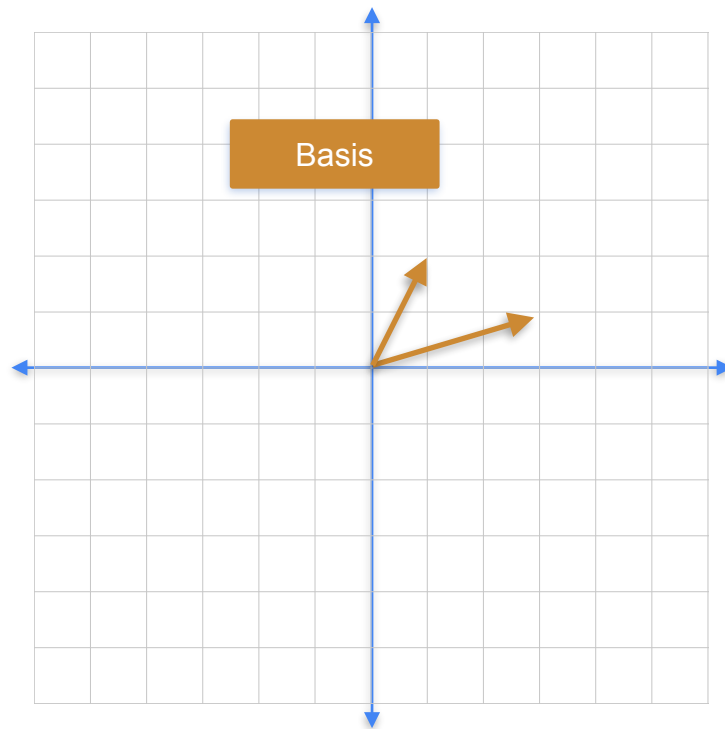
3	1
1	2



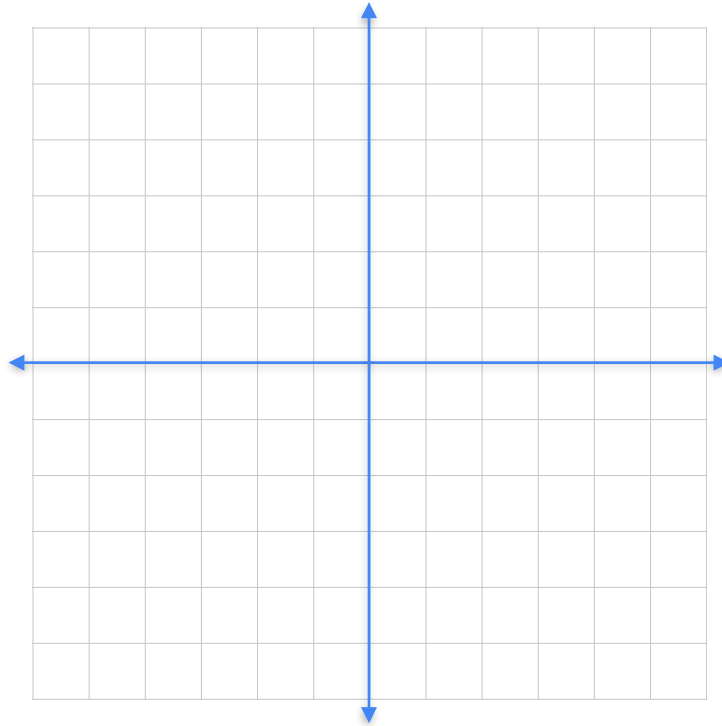
# Bases



3	1
1	2

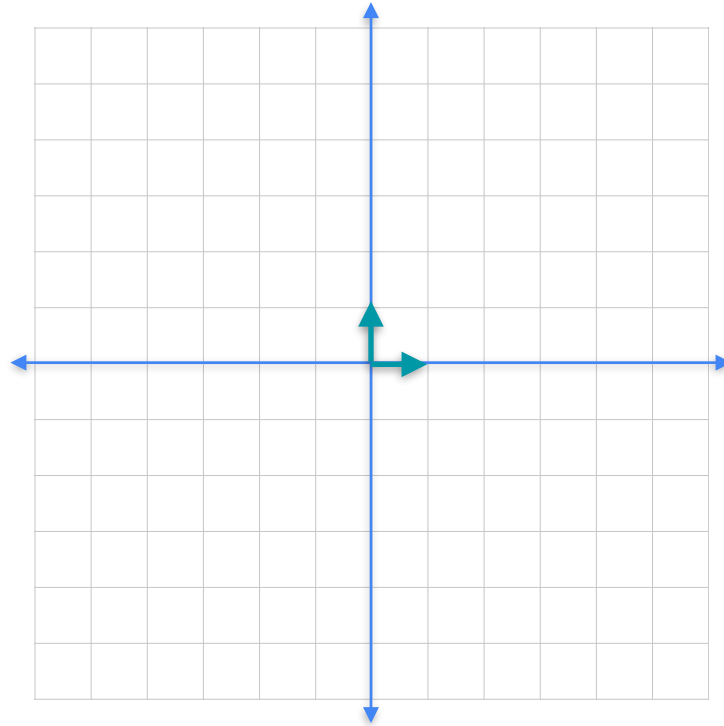


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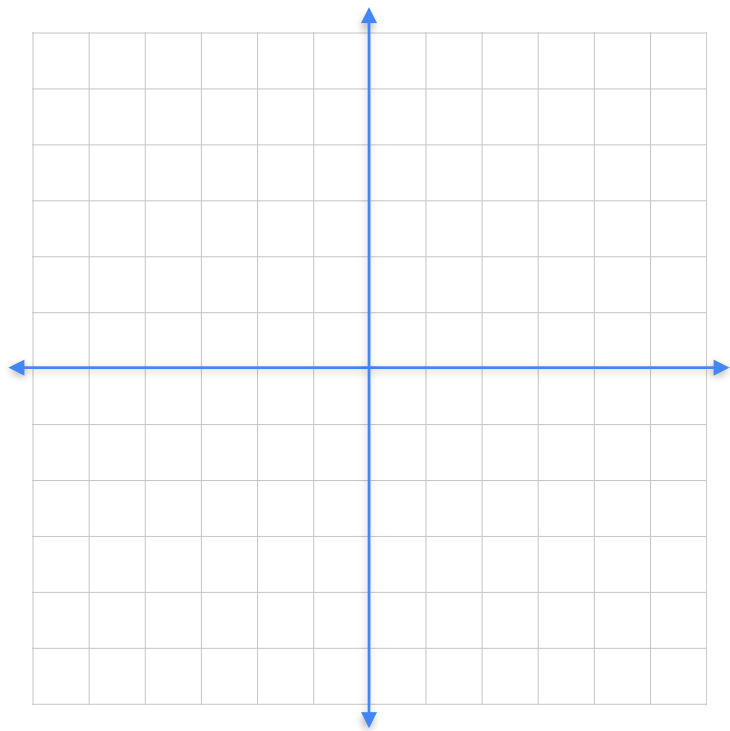
## Bases


# Bases

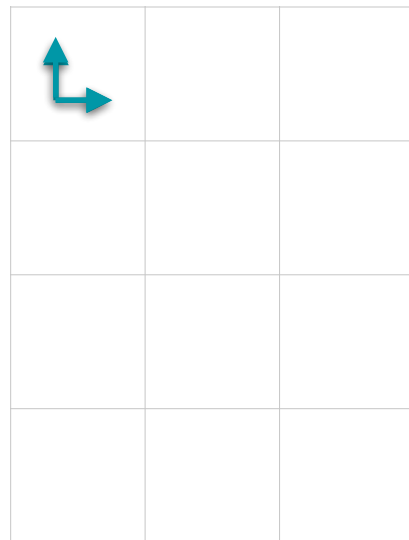


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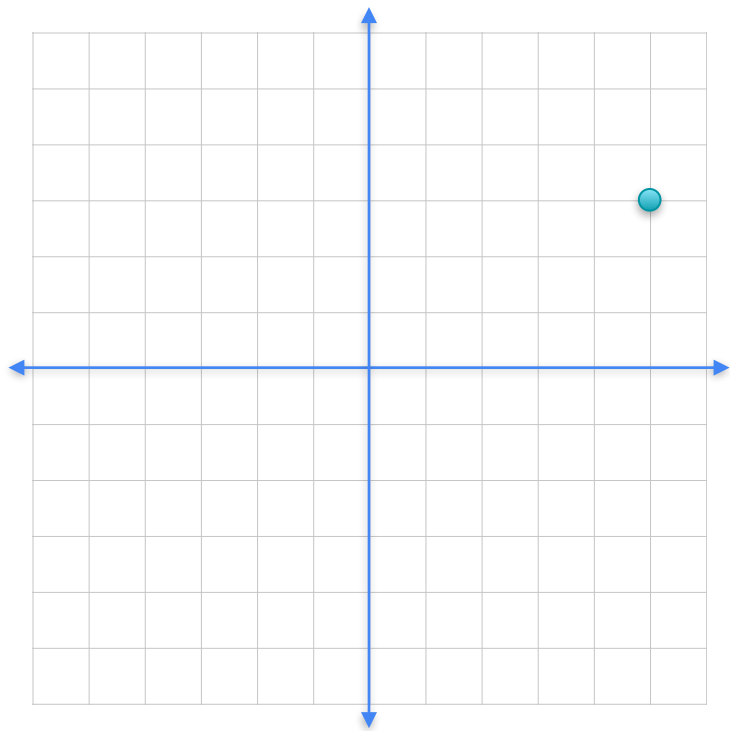

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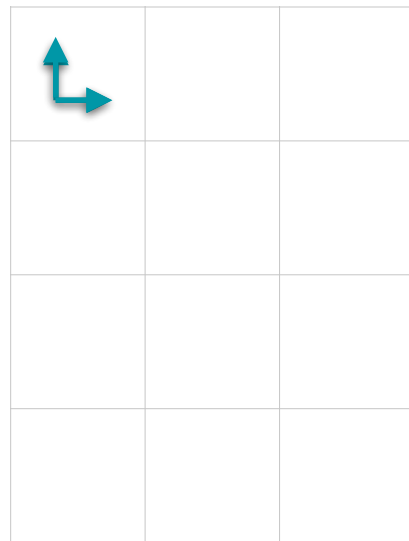
## Bases



# Bases

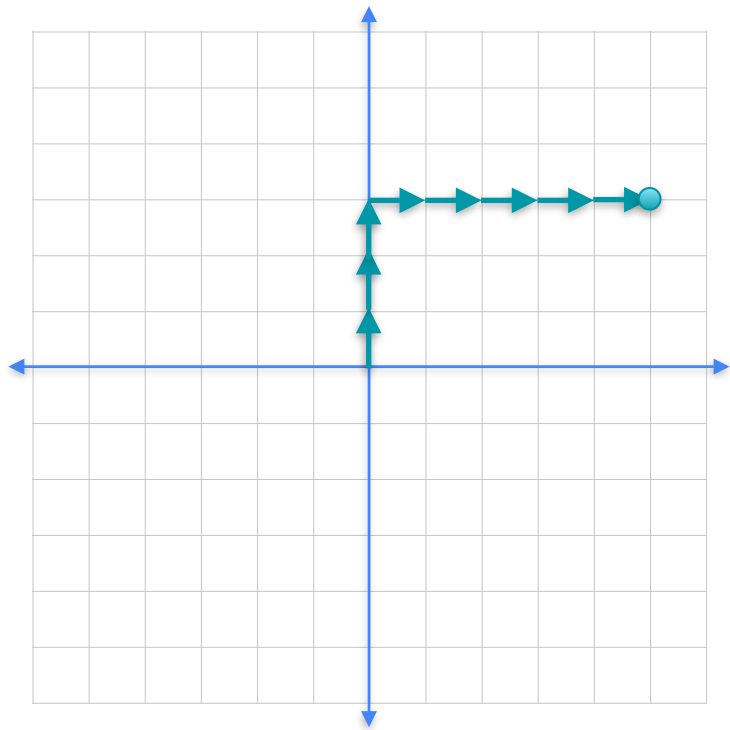


## Bases

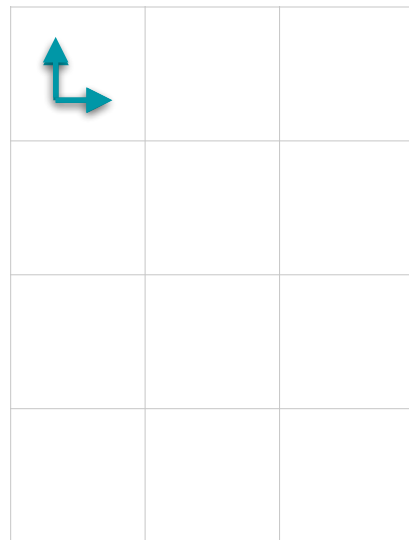




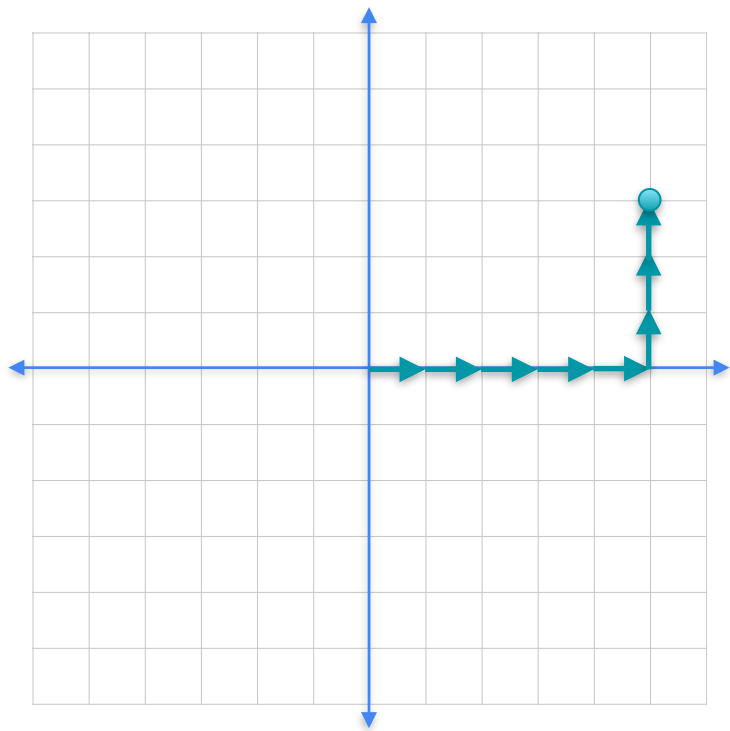
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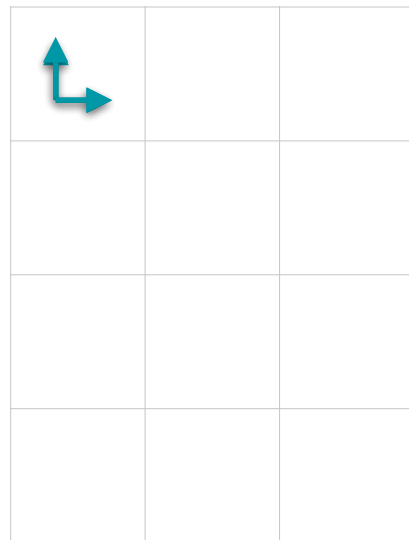
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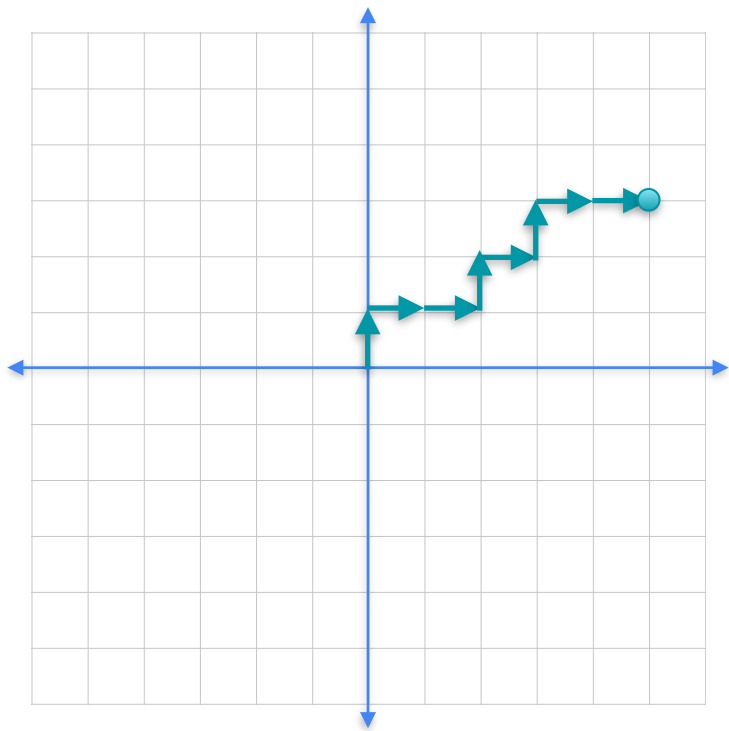
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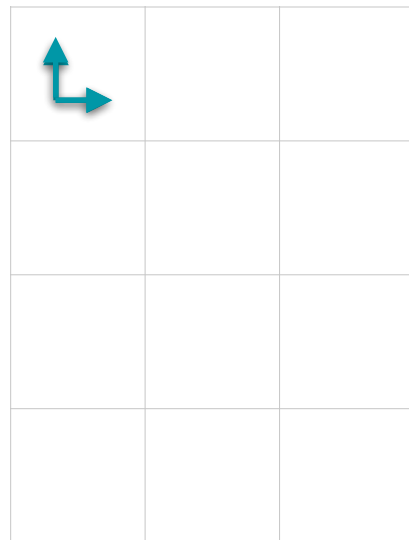
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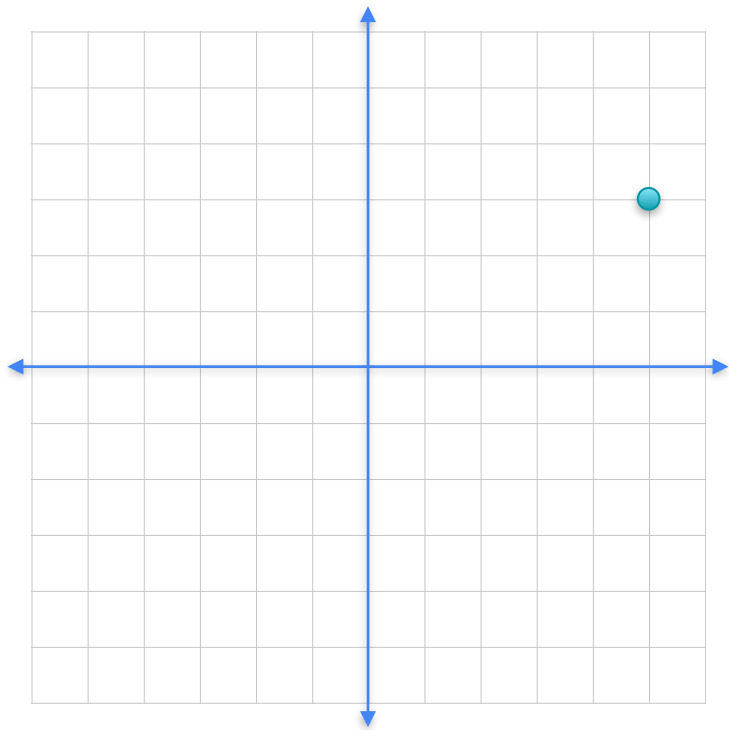
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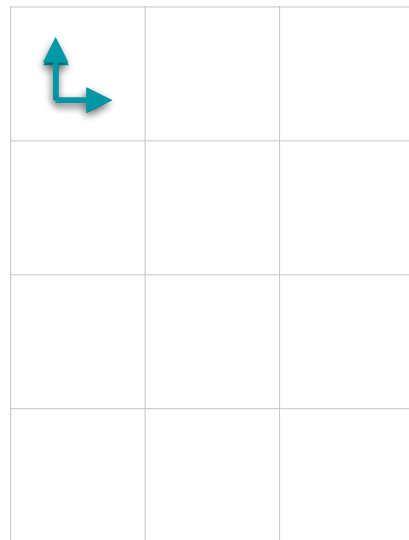
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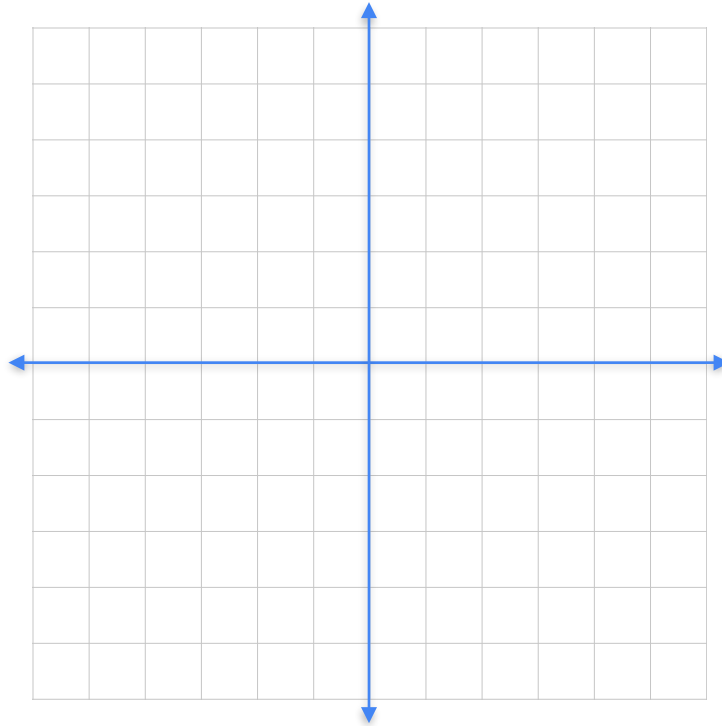
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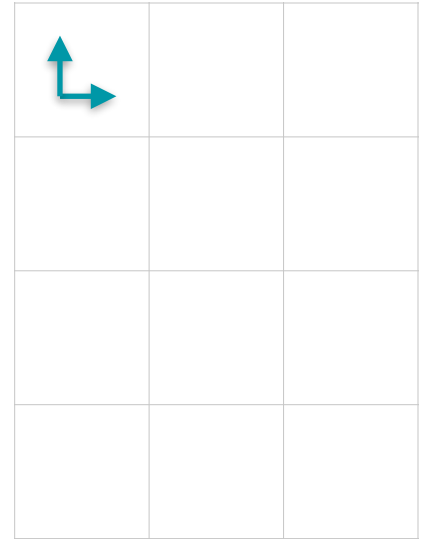
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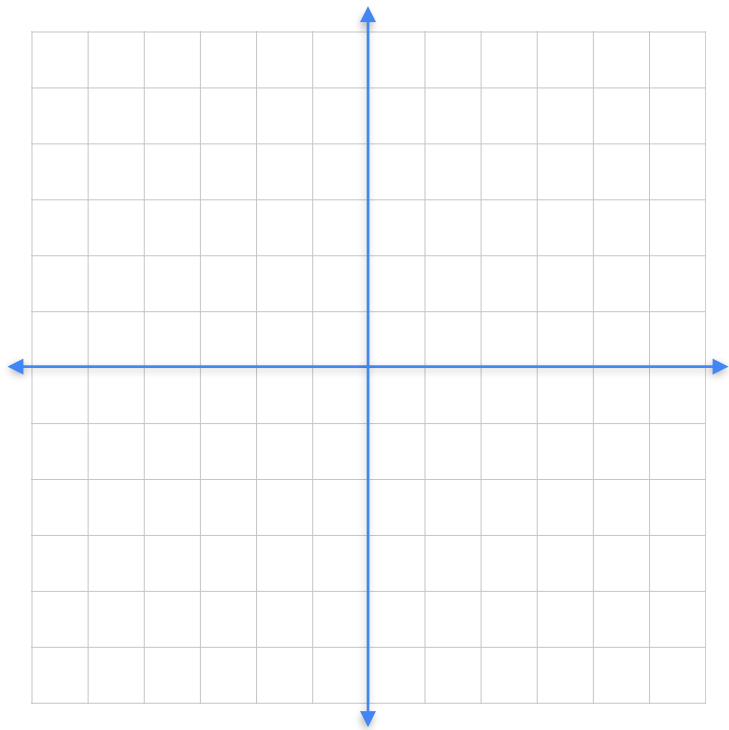
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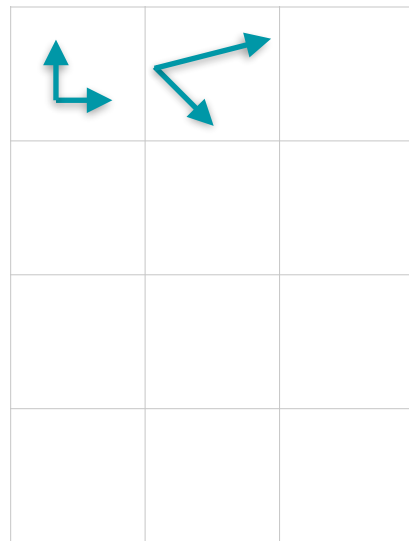
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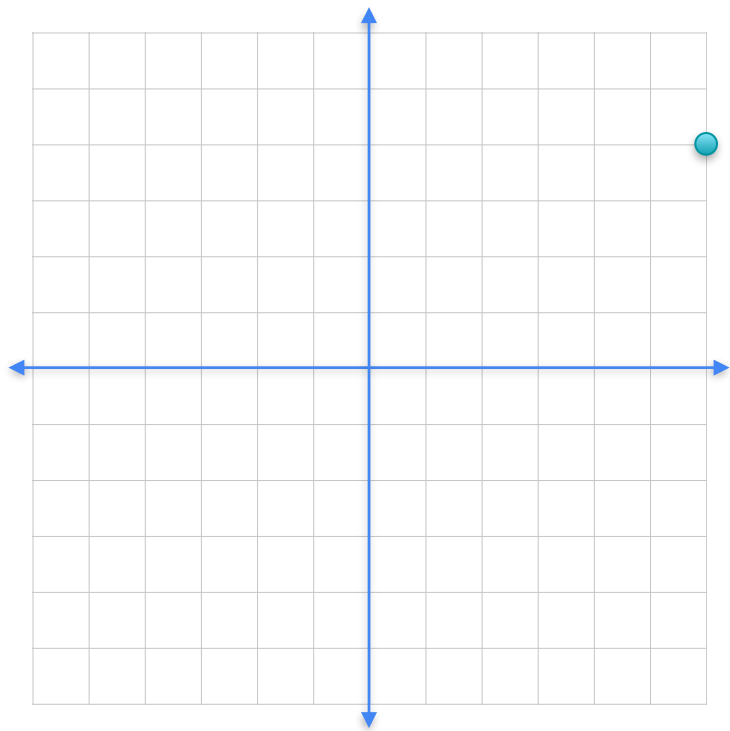
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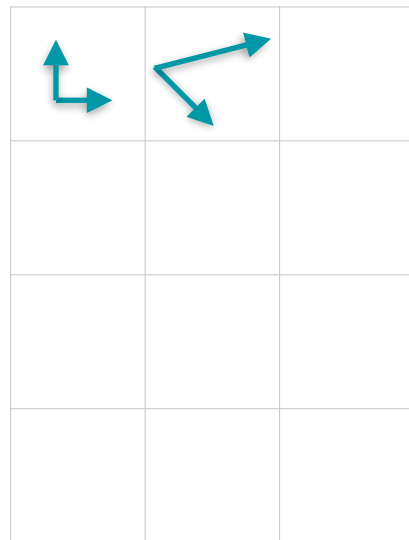
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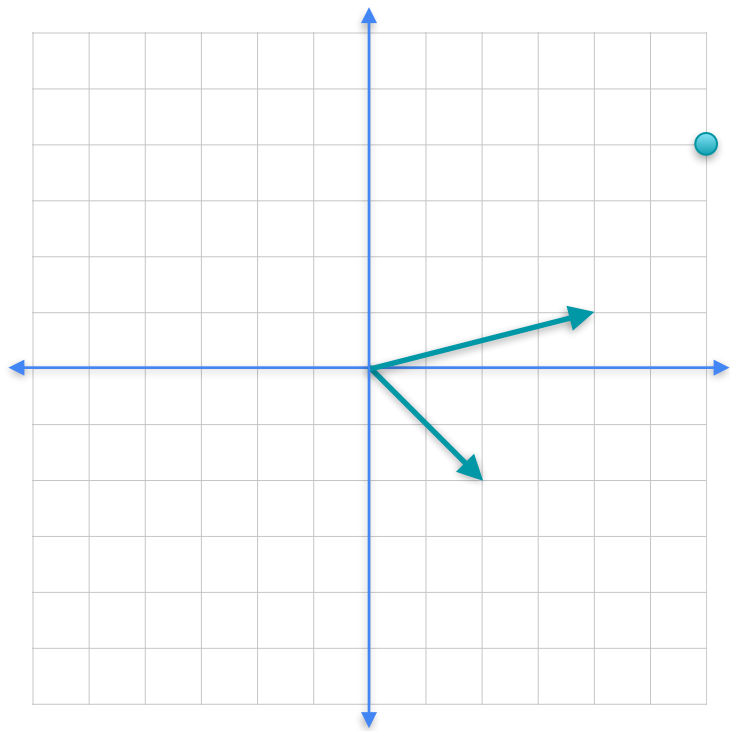
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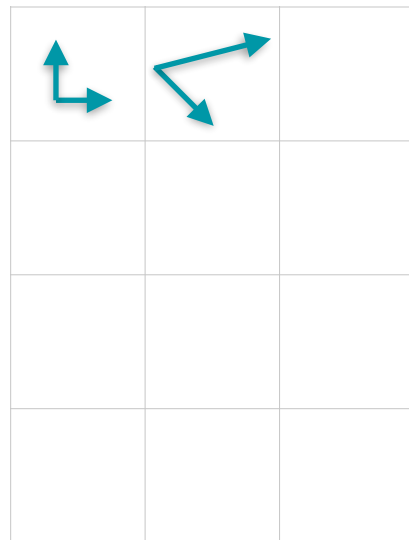
**Bases**



# Bases

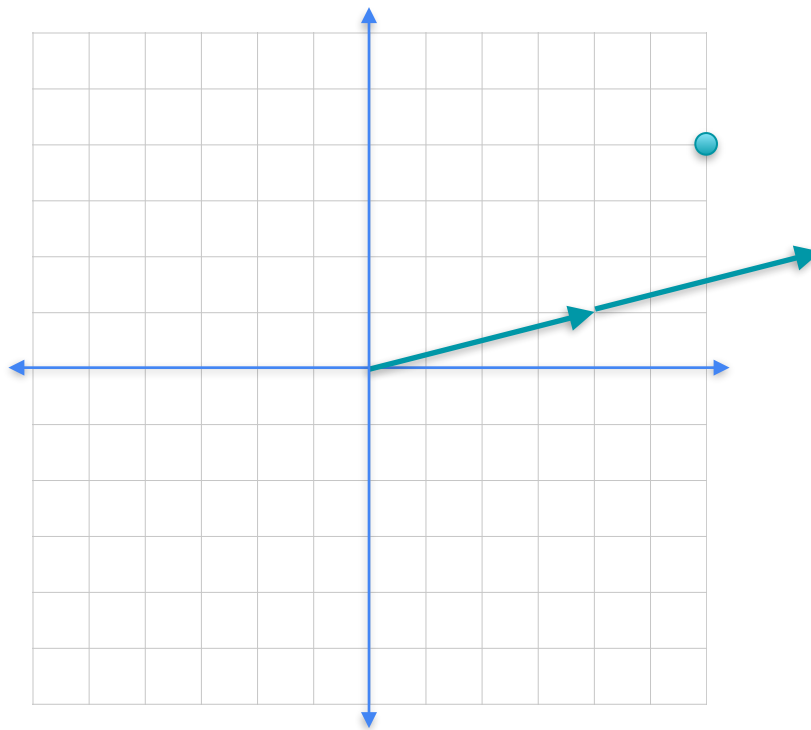


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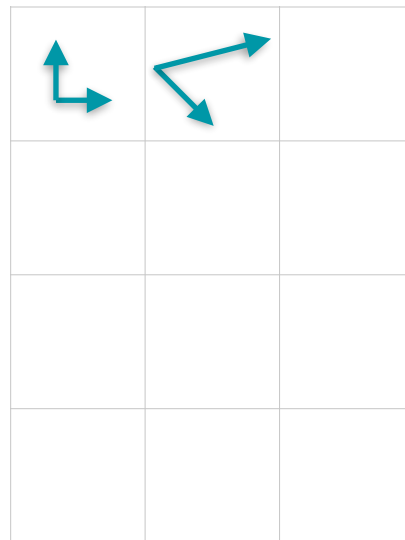




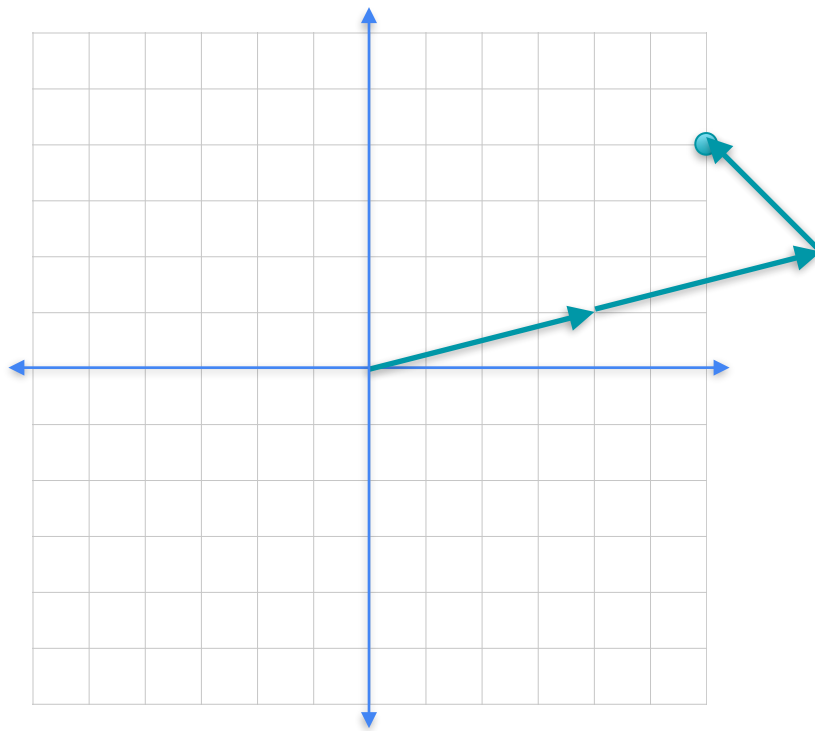
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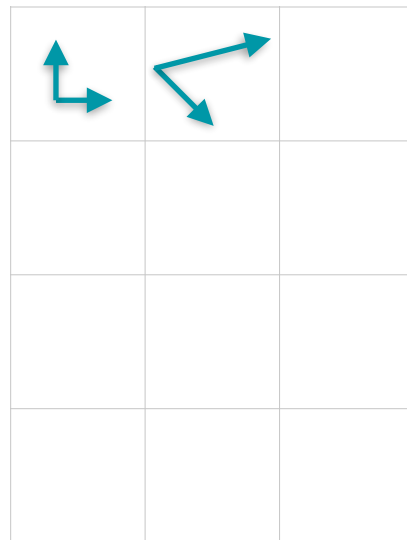
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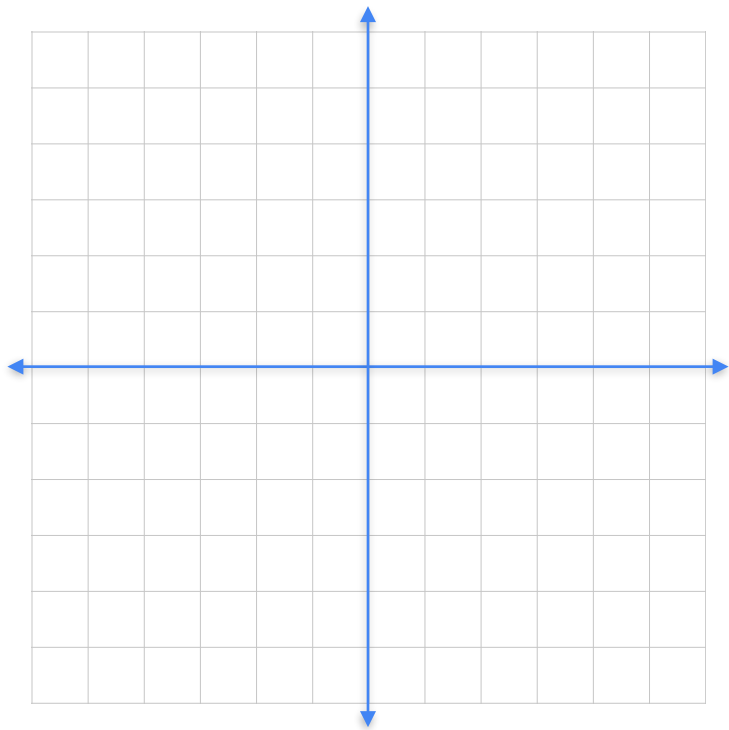
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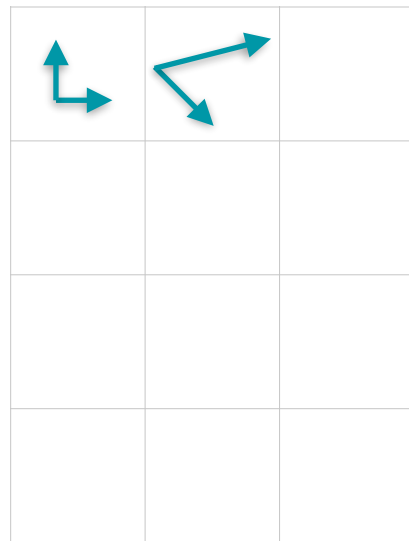
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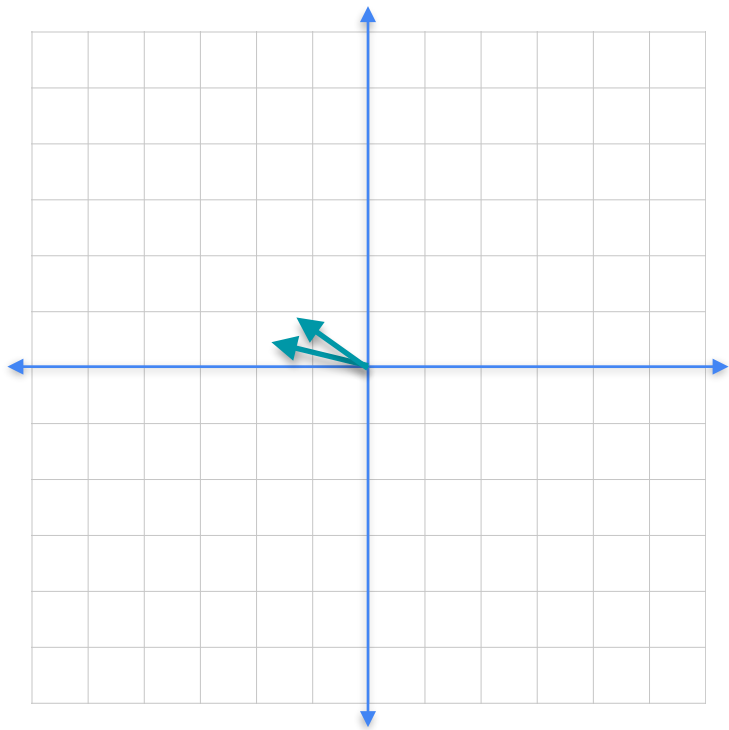
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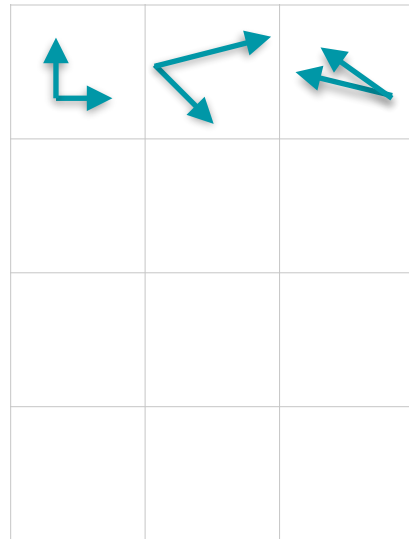
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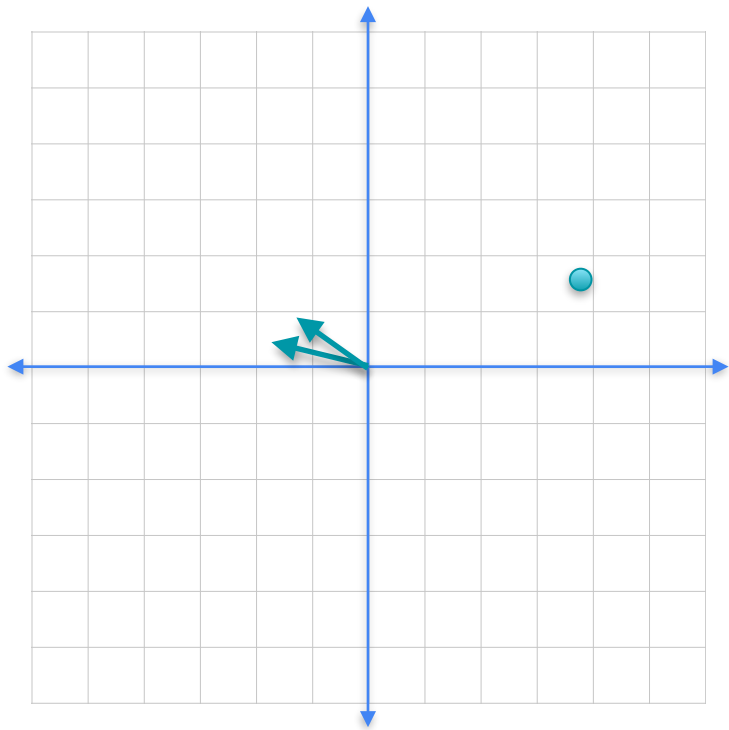
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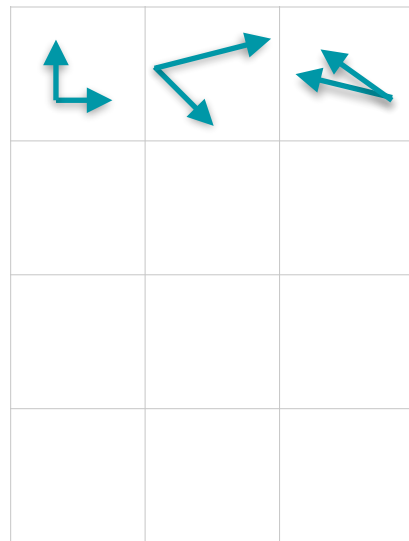
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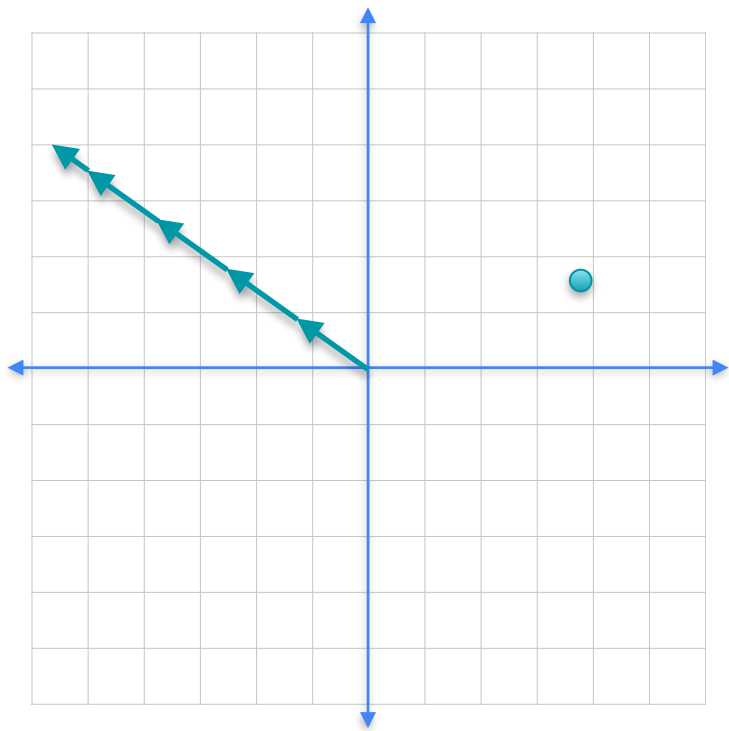
# Bases



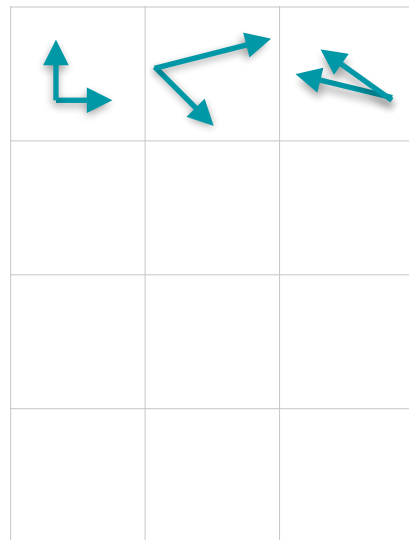
**Bases**



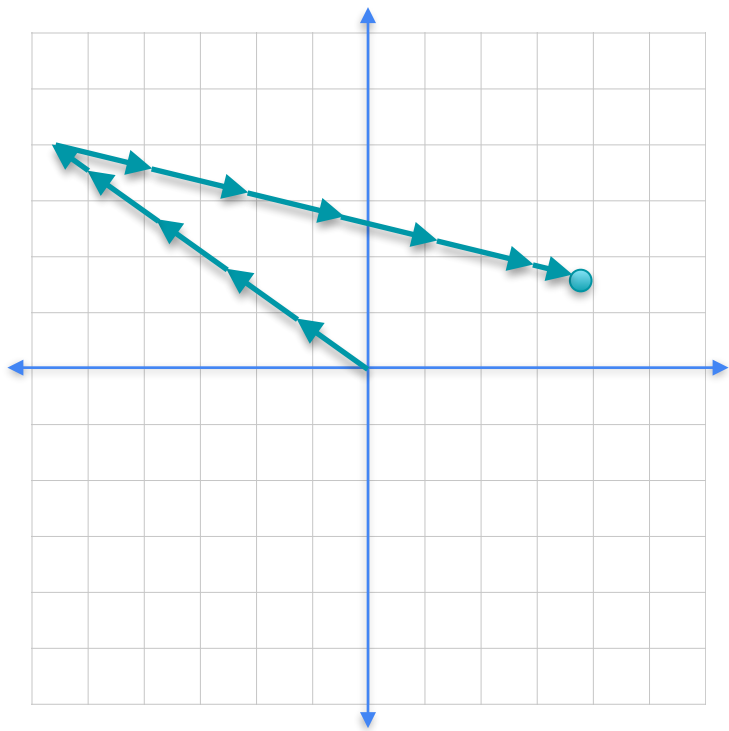
# Bases



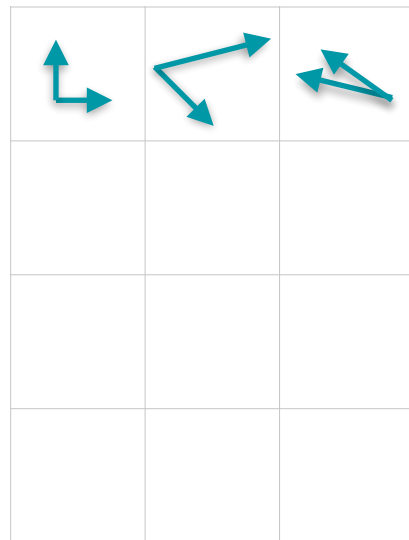
**Bases**



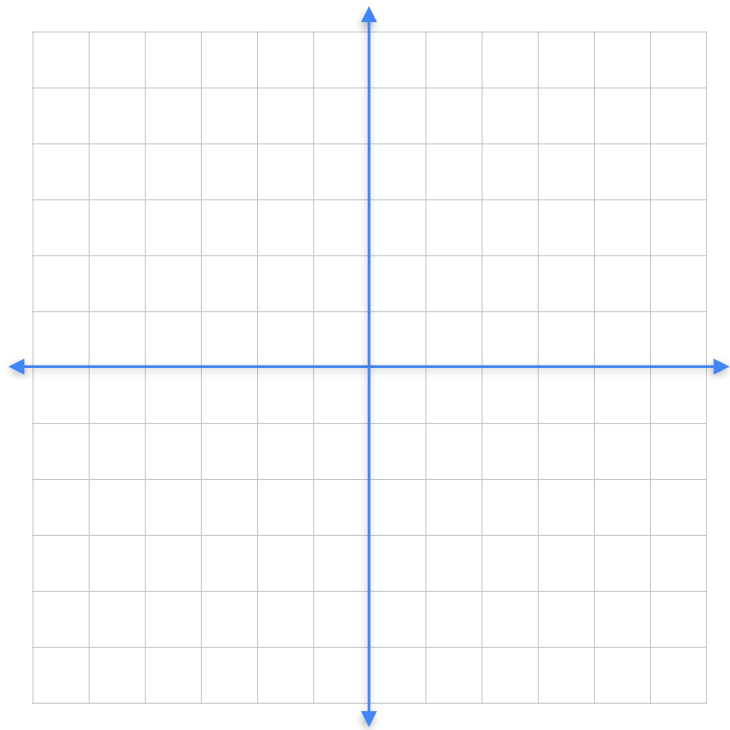
# Bases



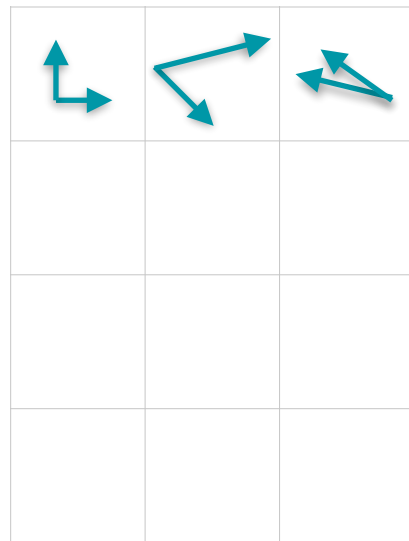
**Bases**



# Bases

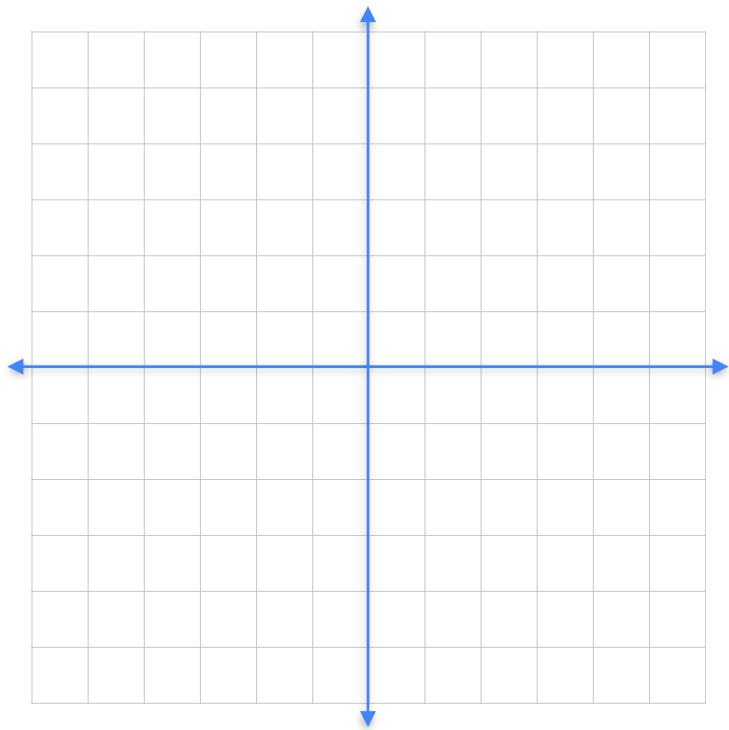


**Bases**

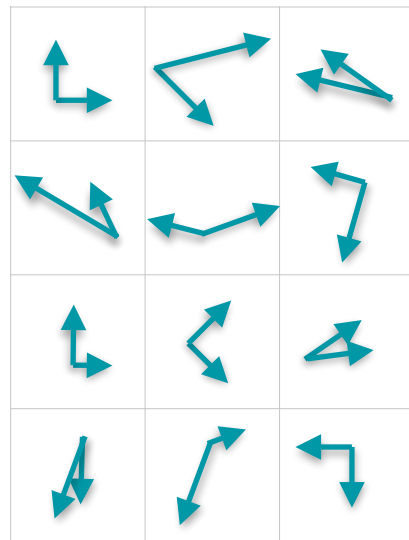




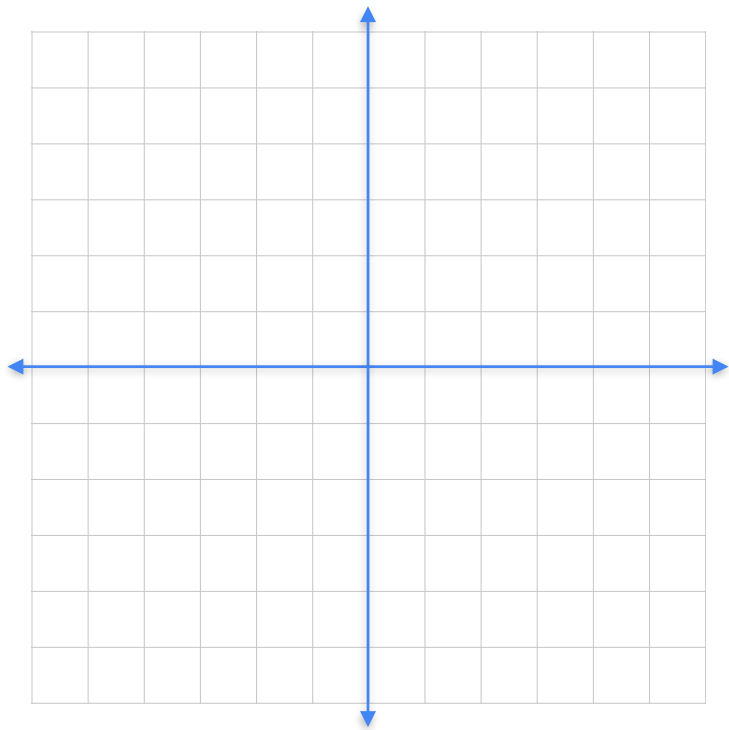
# Bases



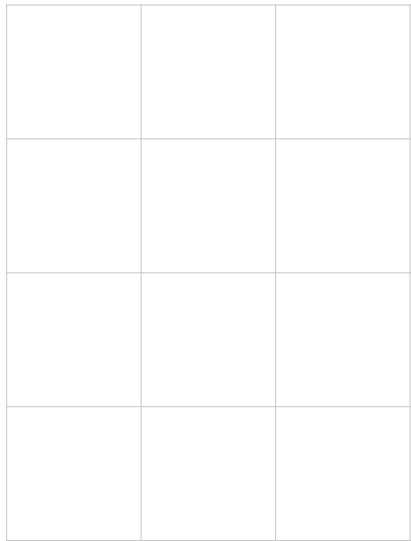
**Bases**



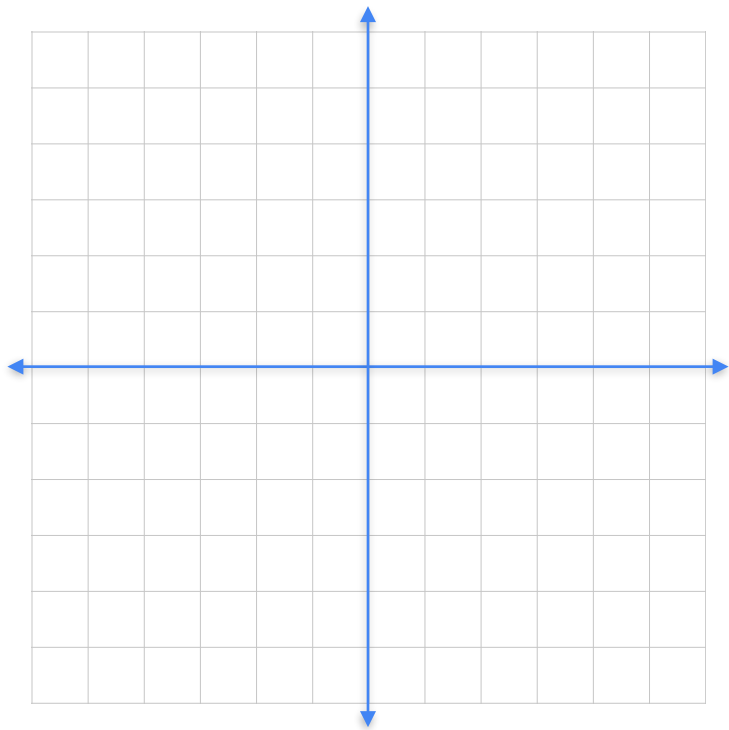
# What is not a basis?



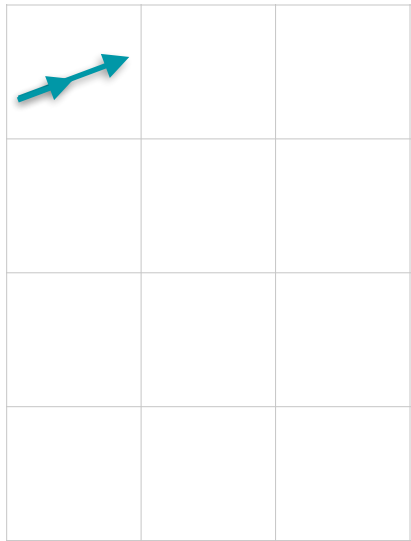
**Not bases**



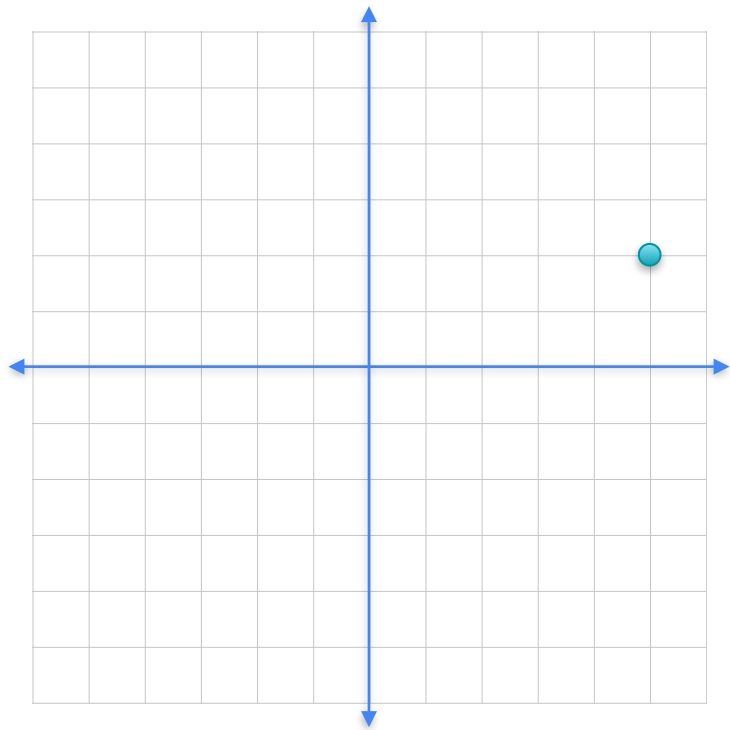
# What is not a basis?



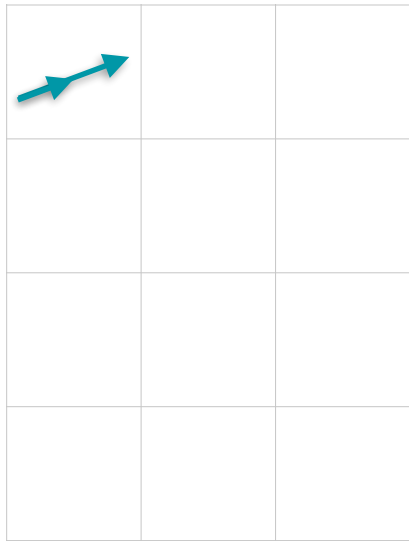
**Not bases**



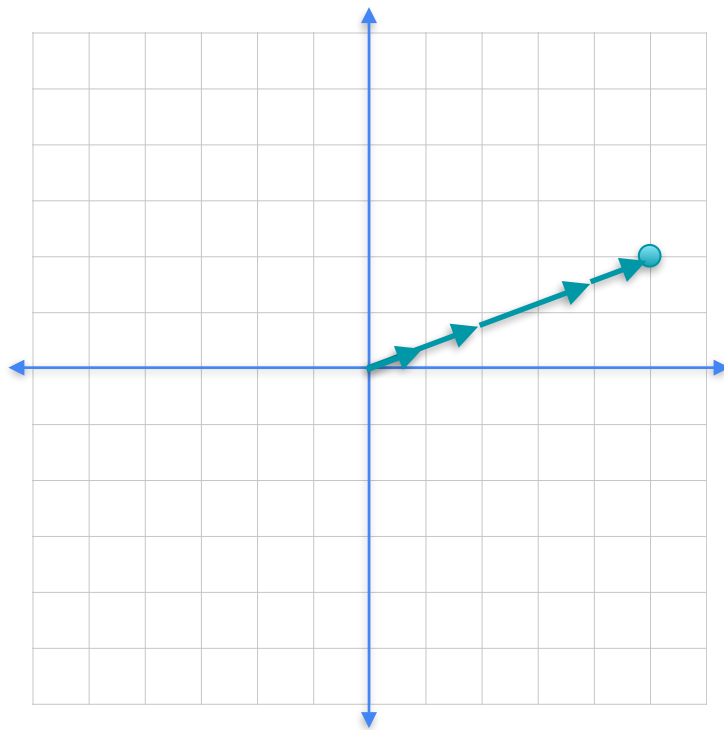
# What is not a basis?



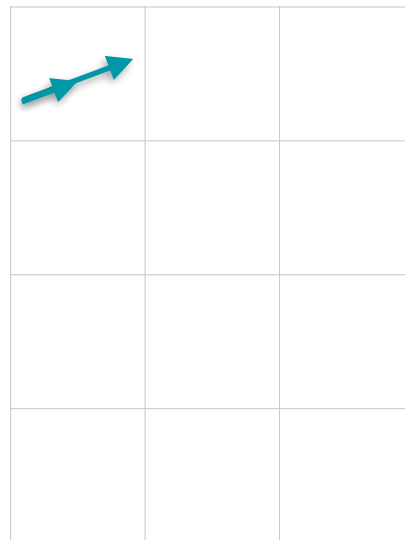
**Not bases**



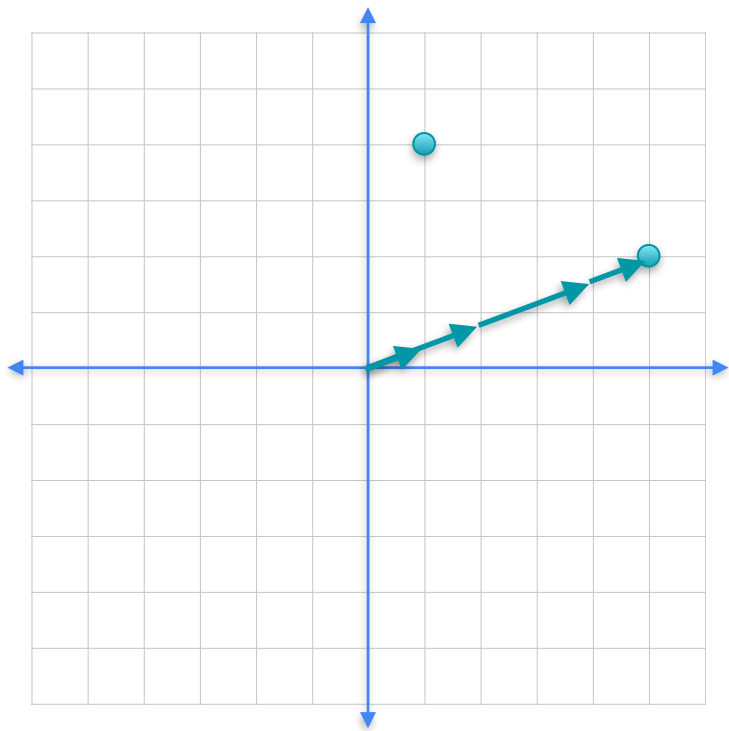
# What is not a basis?



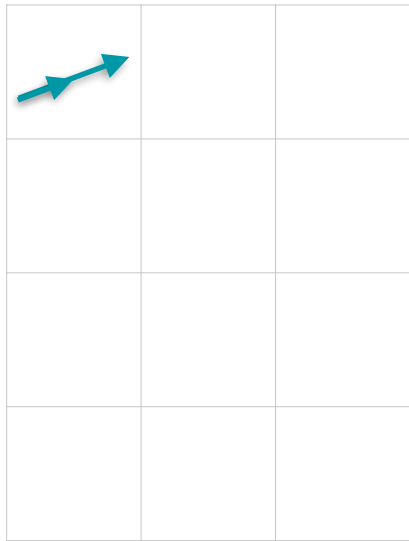
**Not bases**



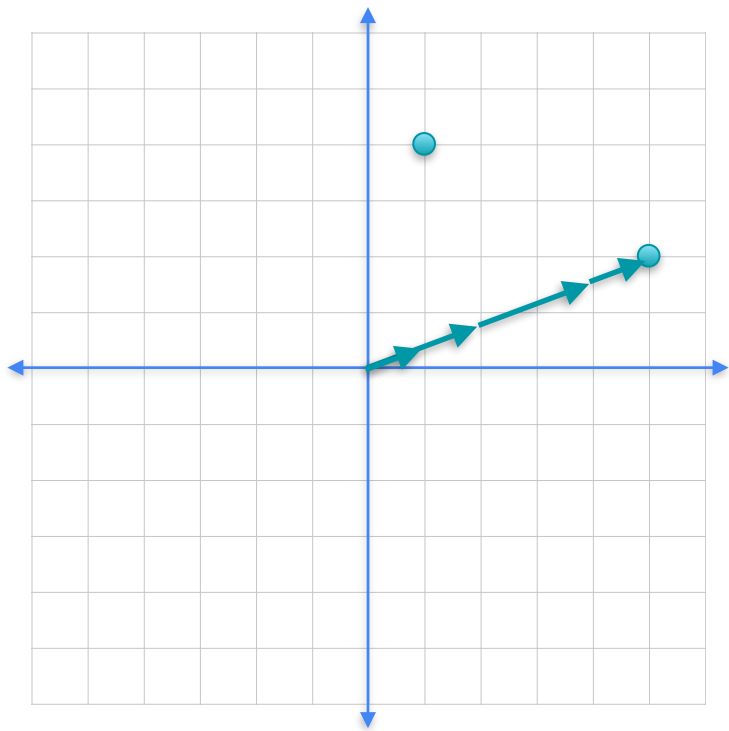
# What is not a basis?



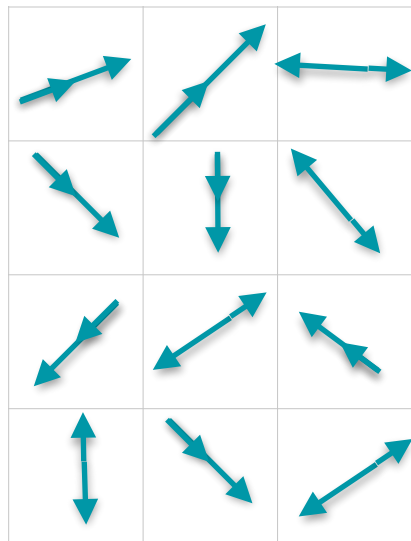
**Not bases**



# What is not a basis?



Not bases





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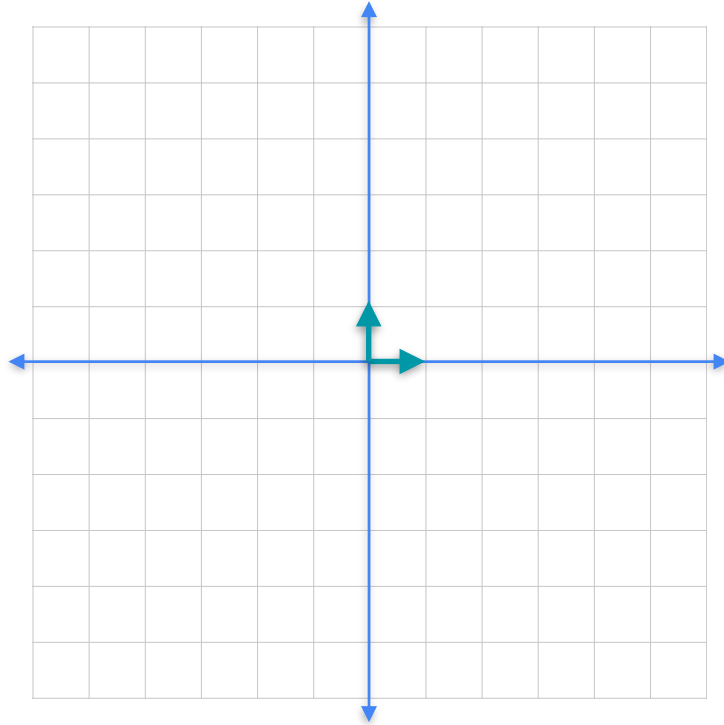
# Determinants and Eigenvectors

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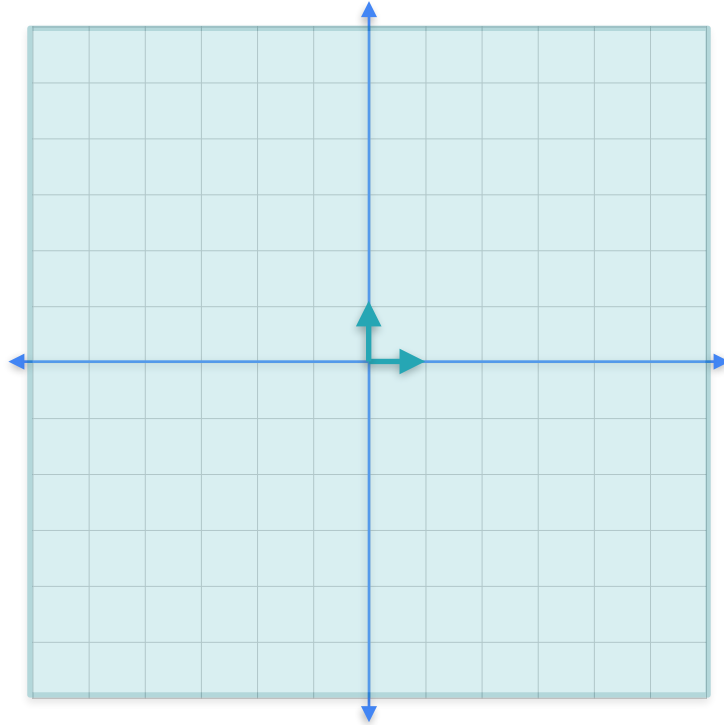
## Span



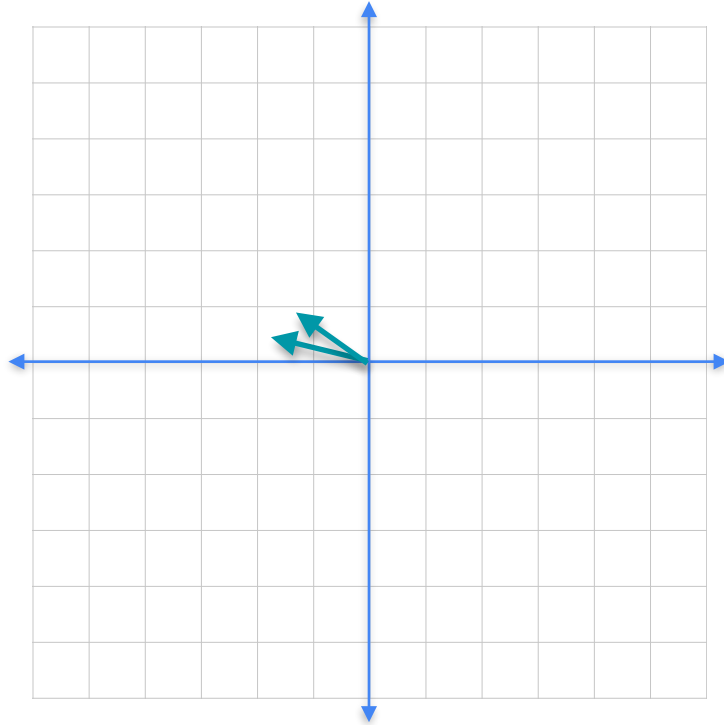
# Span



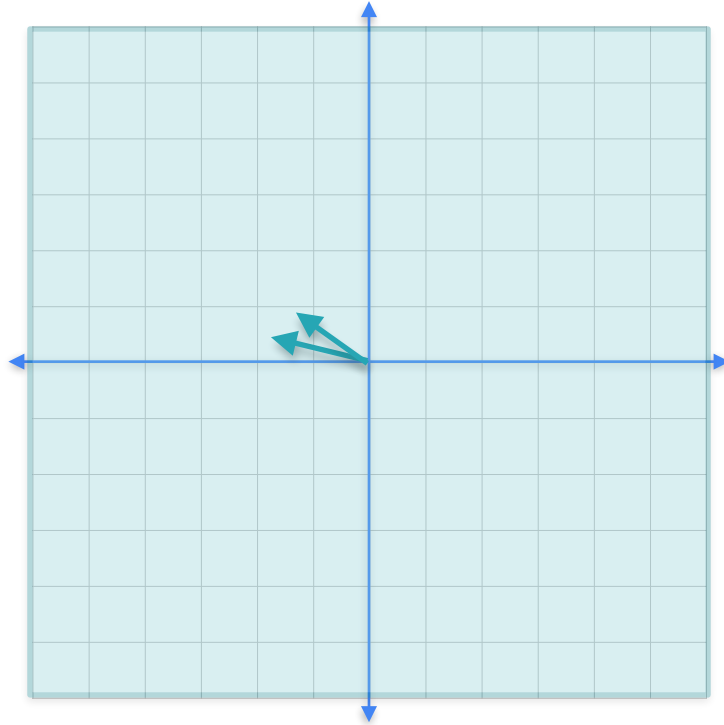
# Span



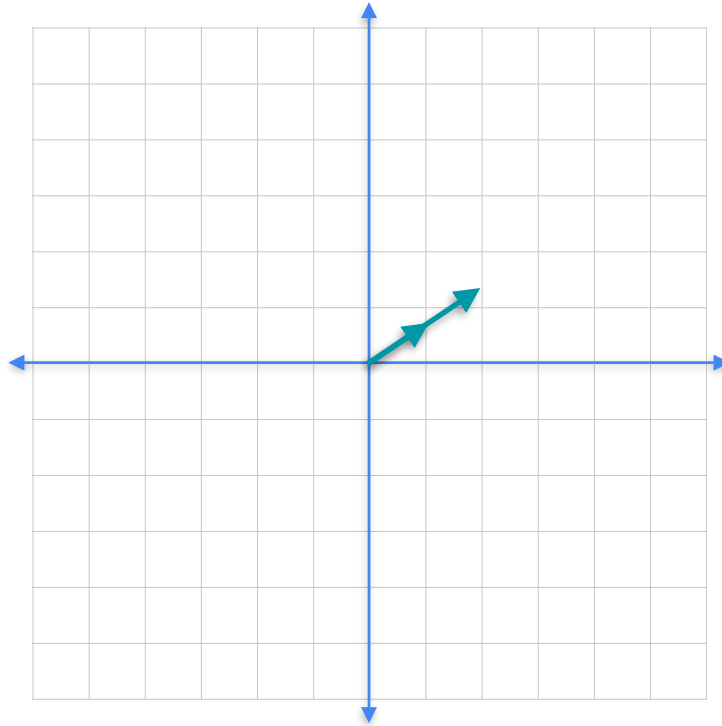
# Span



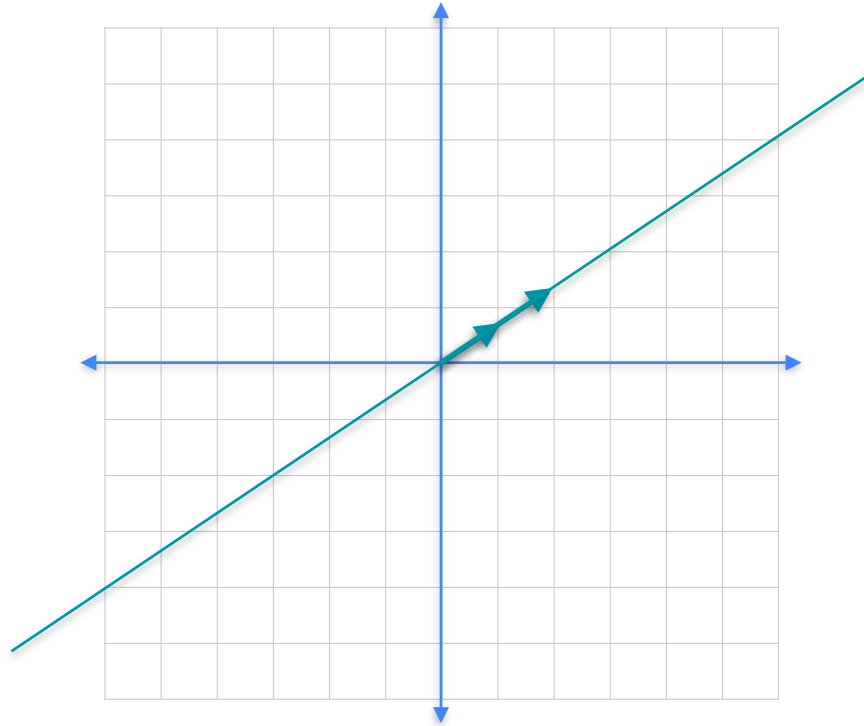
# Span



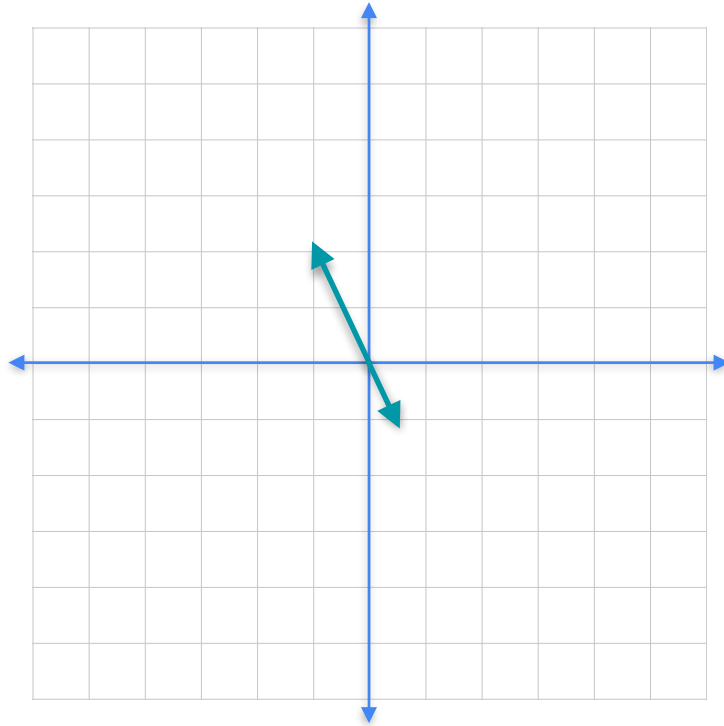
# Span



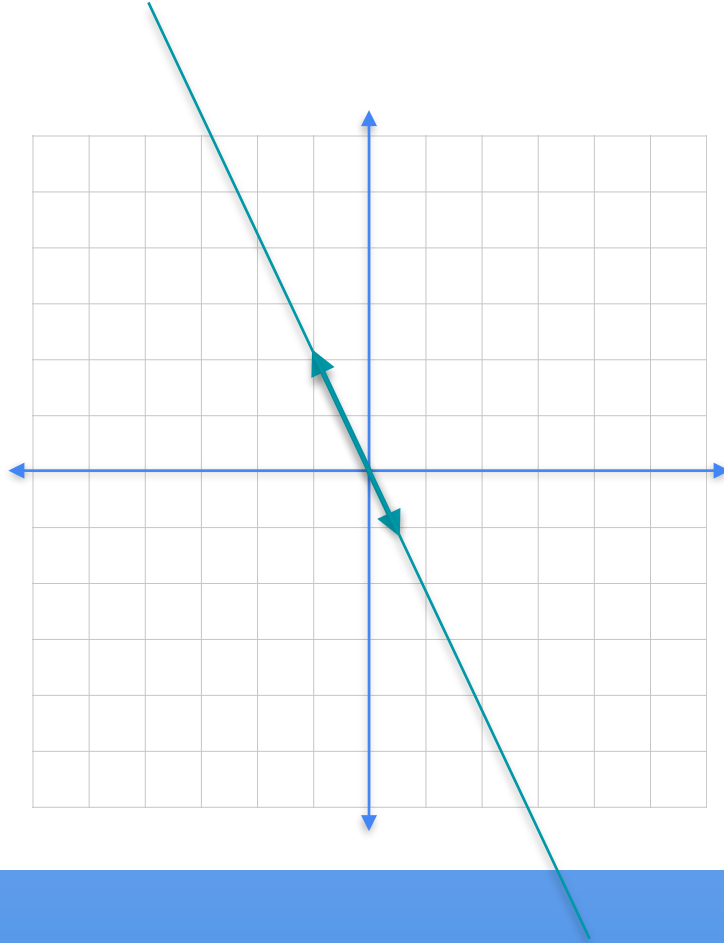
# Span



# Span

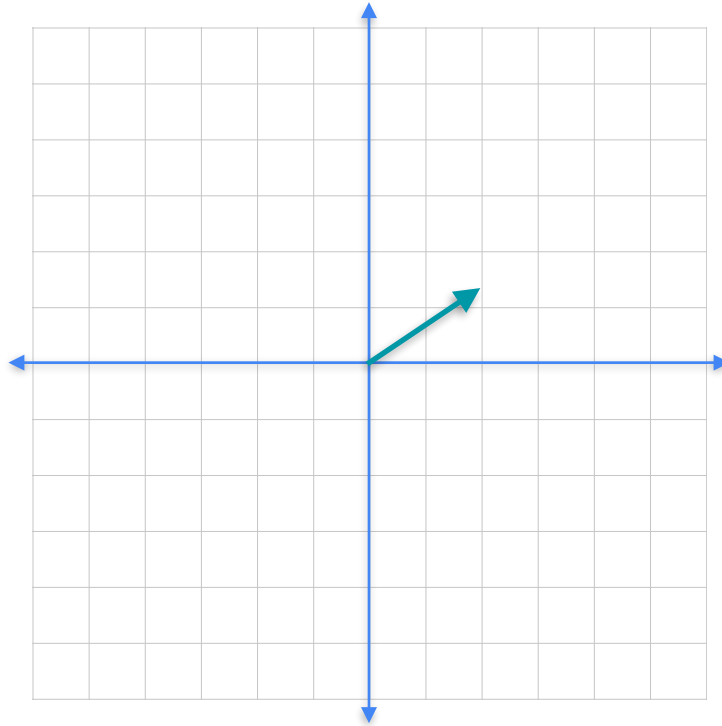


# Span

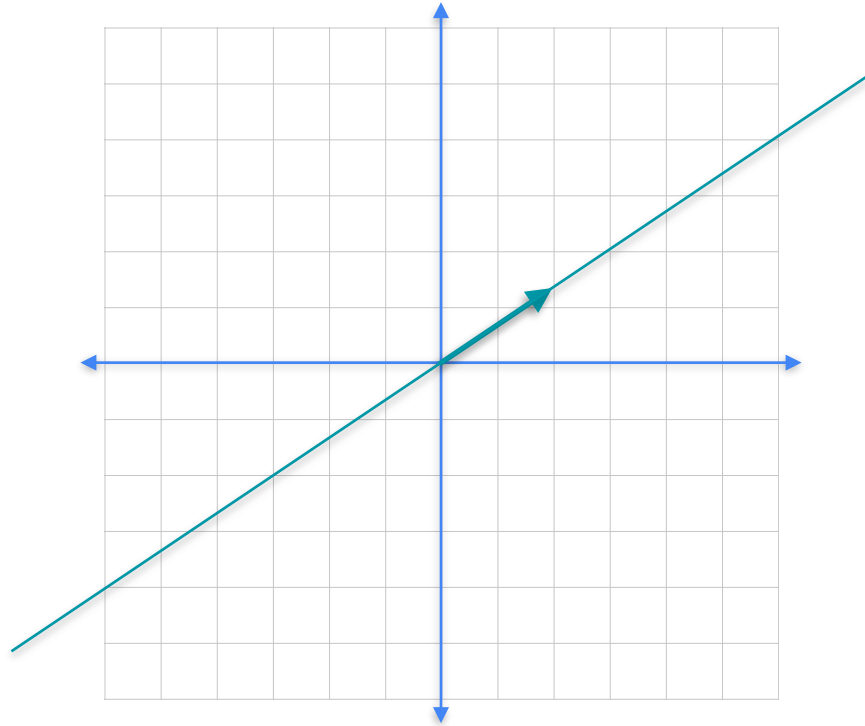




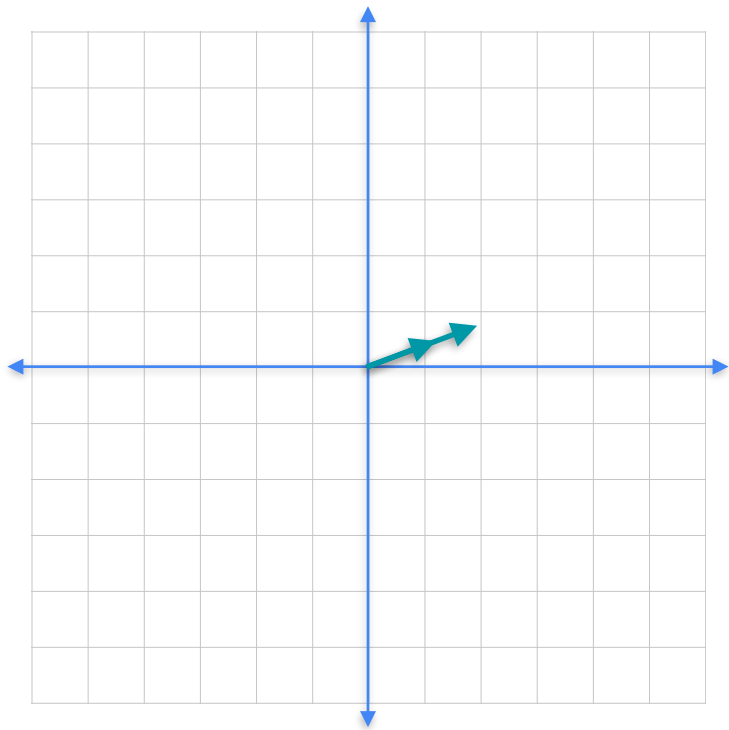
# Span



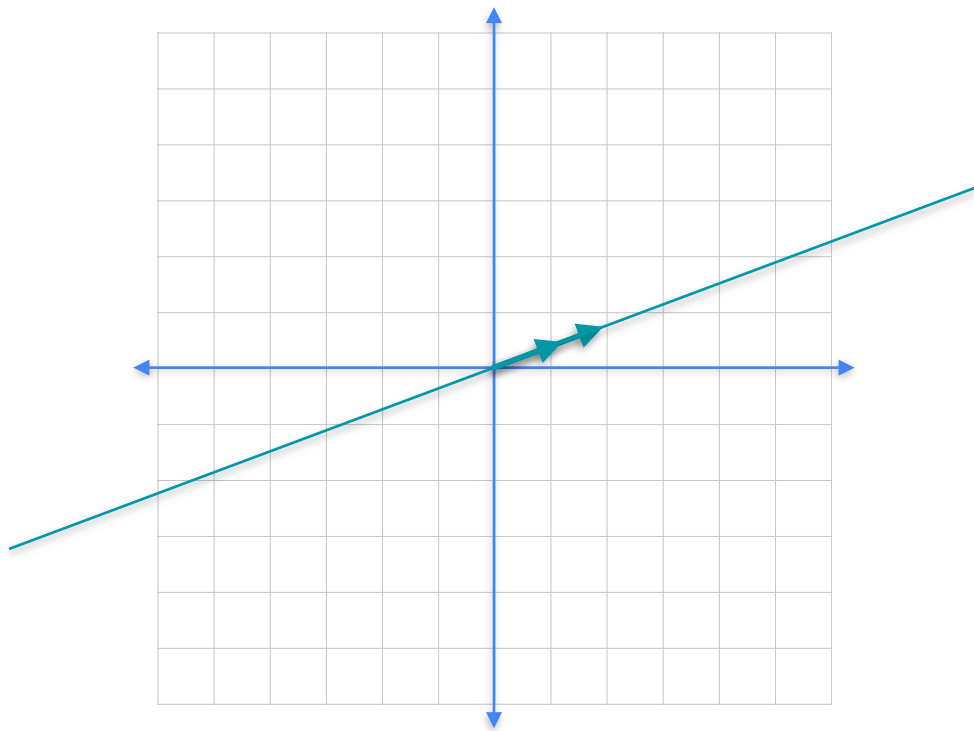
# Span



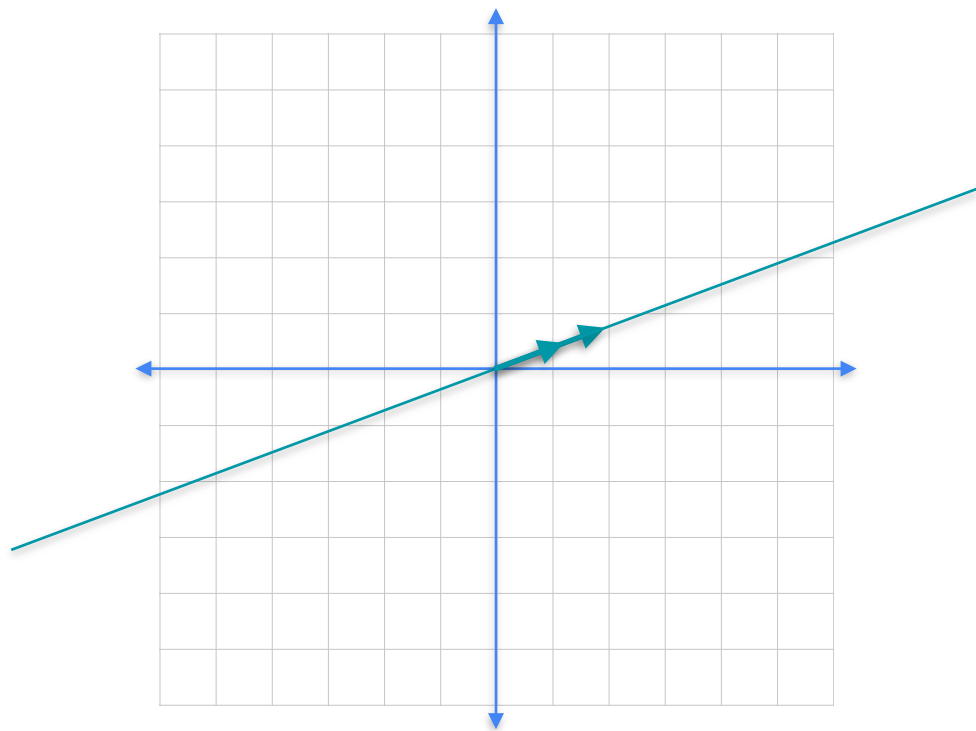
# Is this a basis?



# Is this a basis?



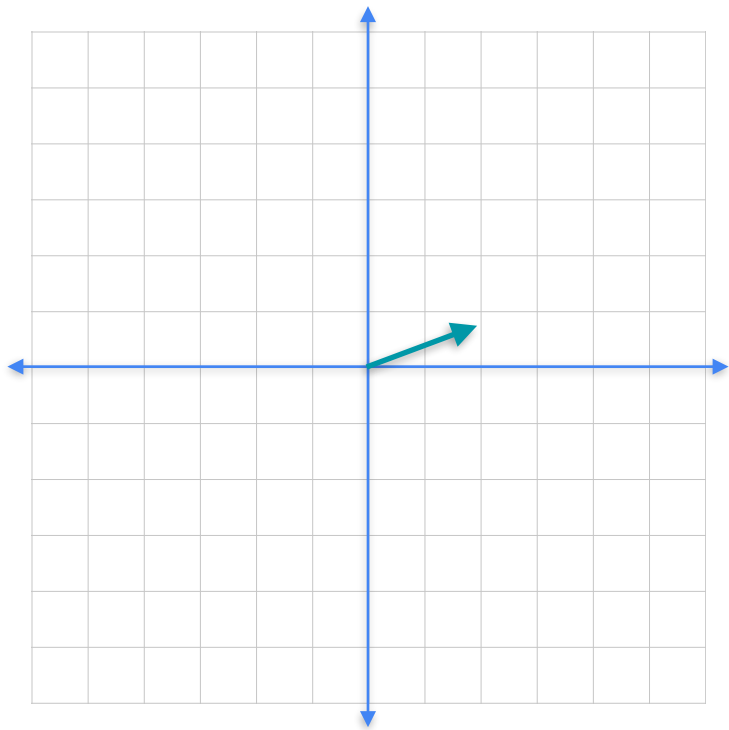
# Is this a basis?



**No**

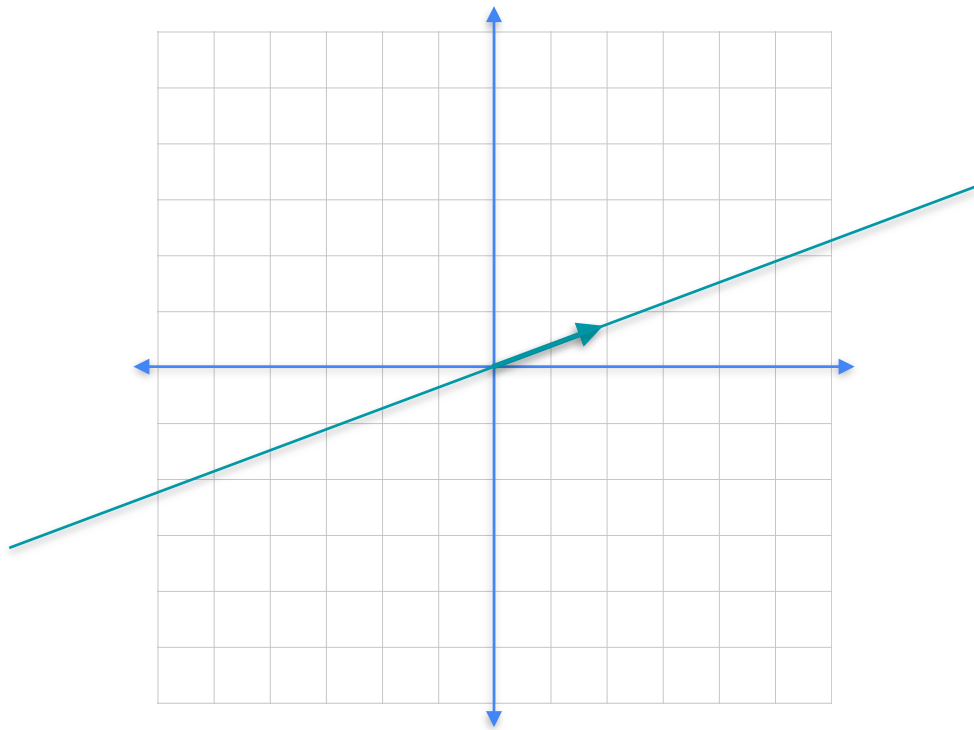
# Is this a basis for something?

**Bases**

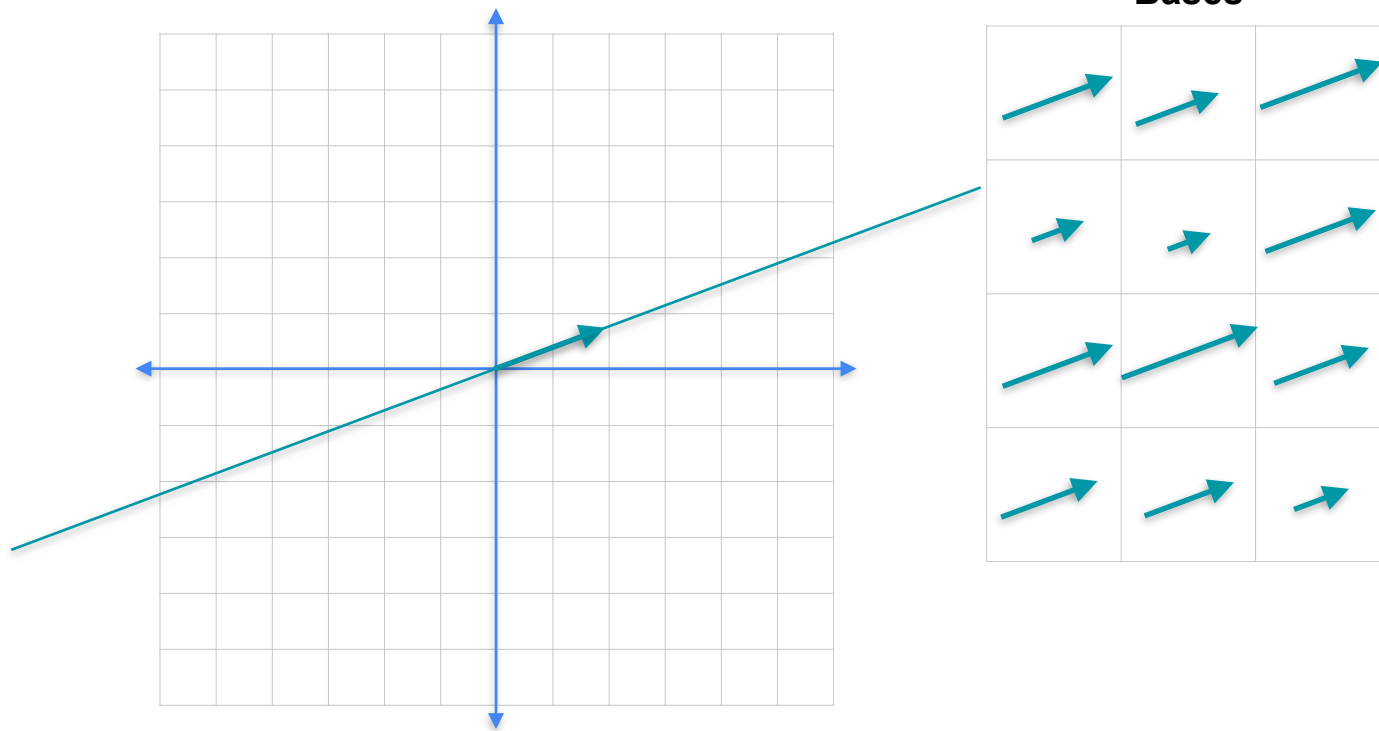


# Is this a basis for something?

**Bases**

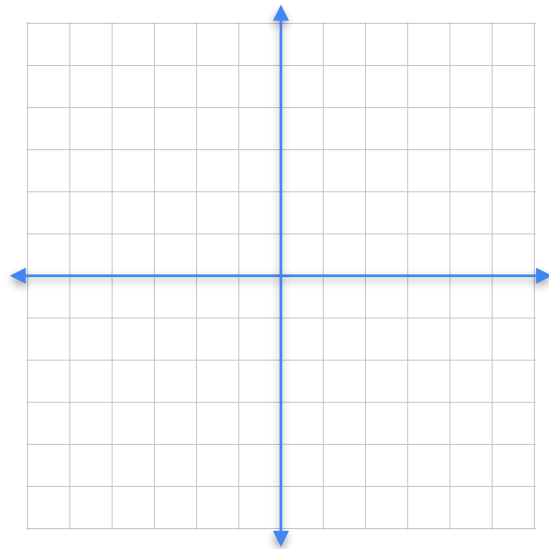
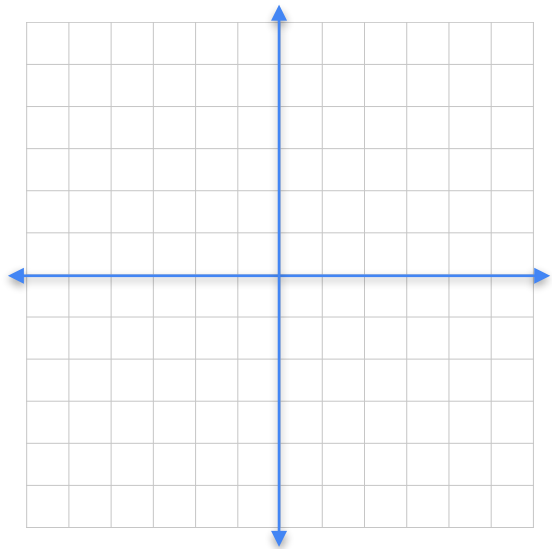


# Is this a basis for something?

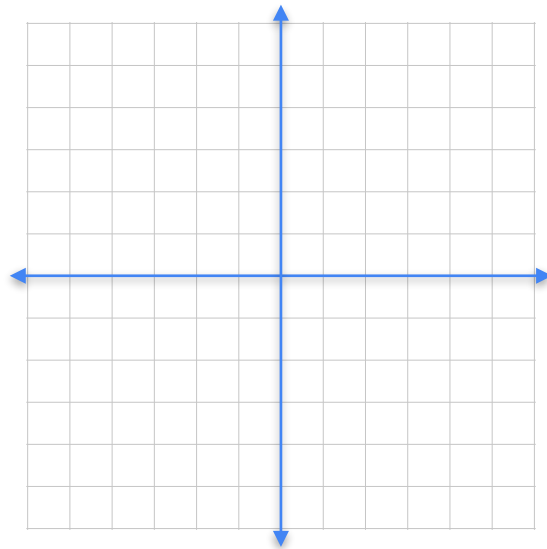
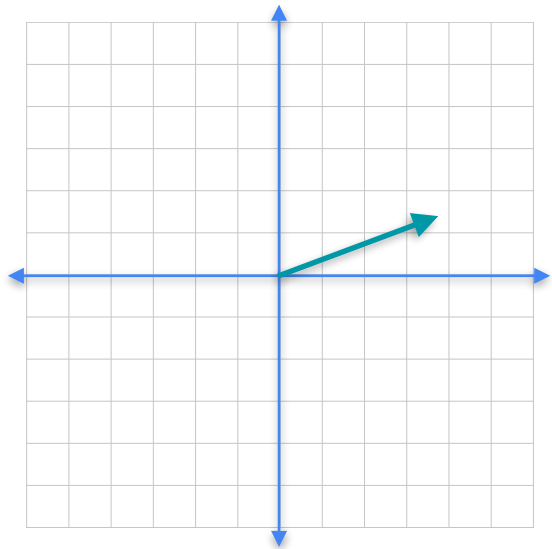




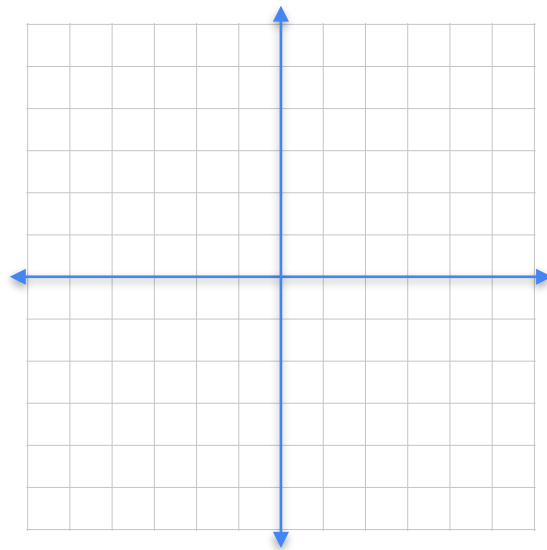
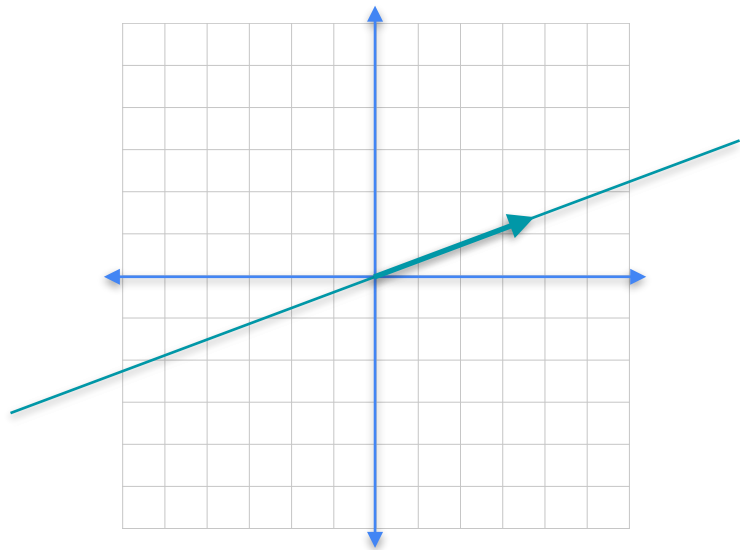
# A basis is a minimal spanning set



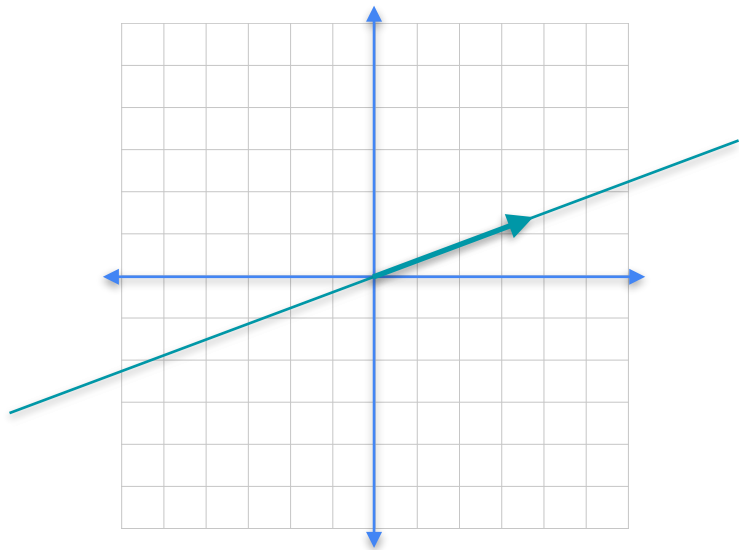
# A basis is a minimal spanning set



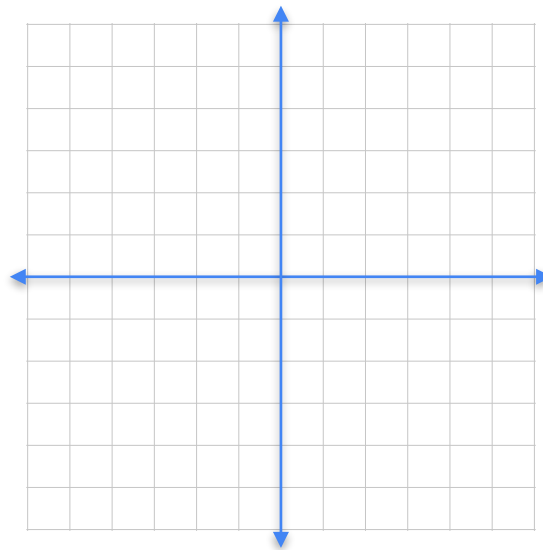
# A basis is a minimal spanning set



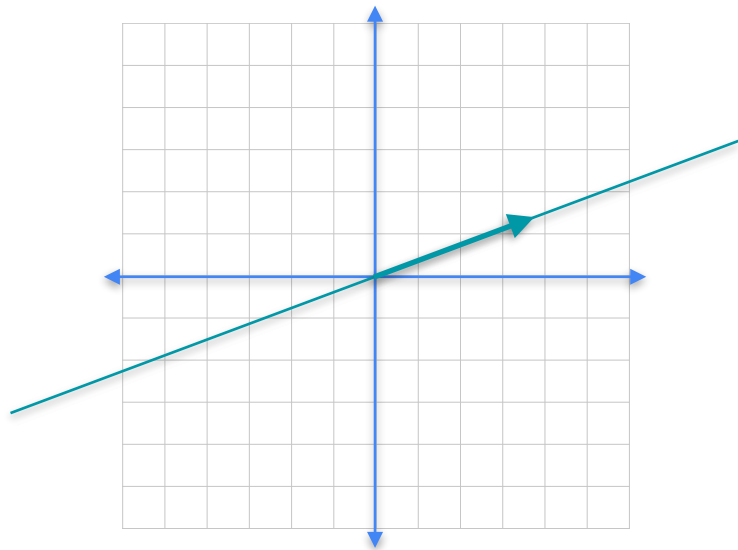
# A basis is a minimal spanning set



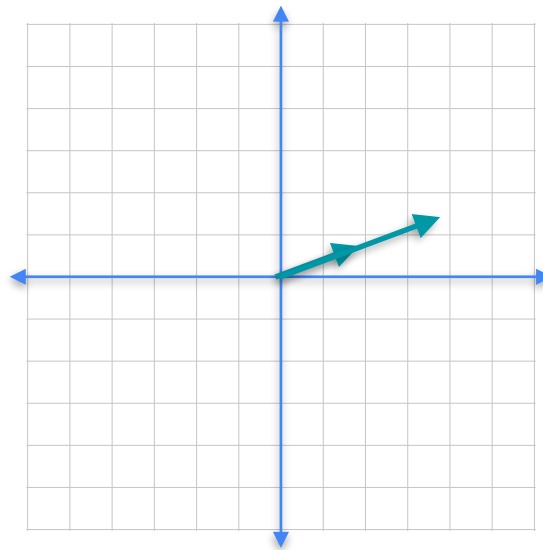
Basis



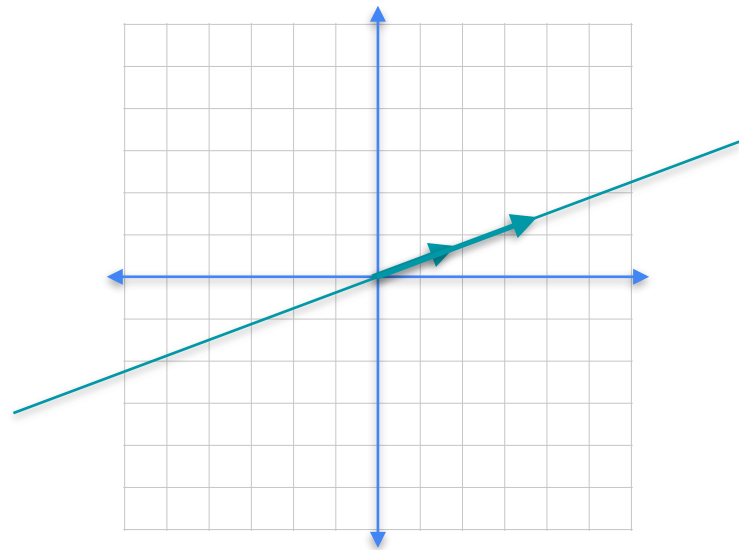
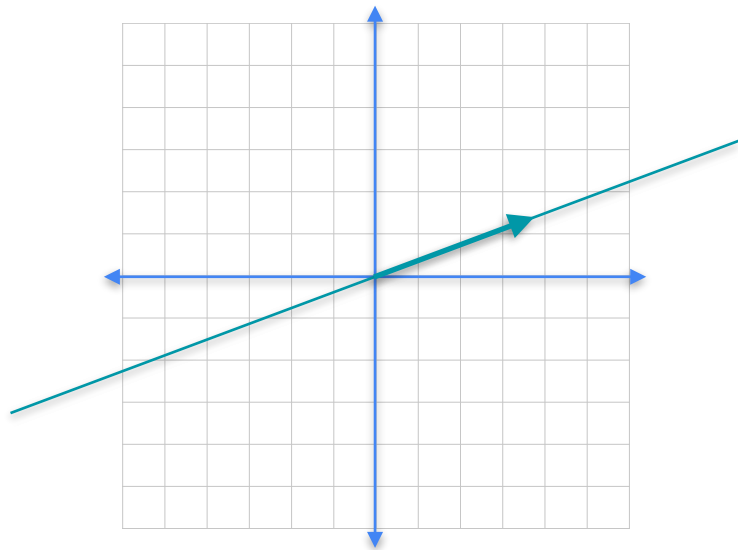
# A basis is a minimal spanning set



Basis

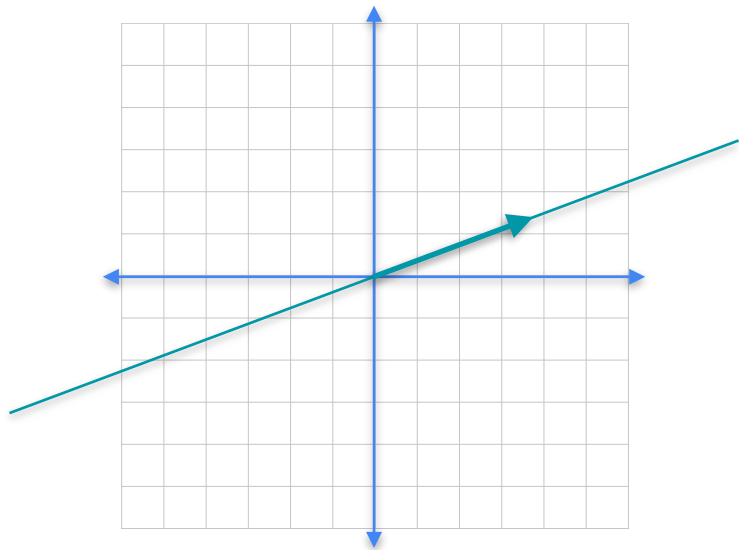


# A basis is a minimal spanning set

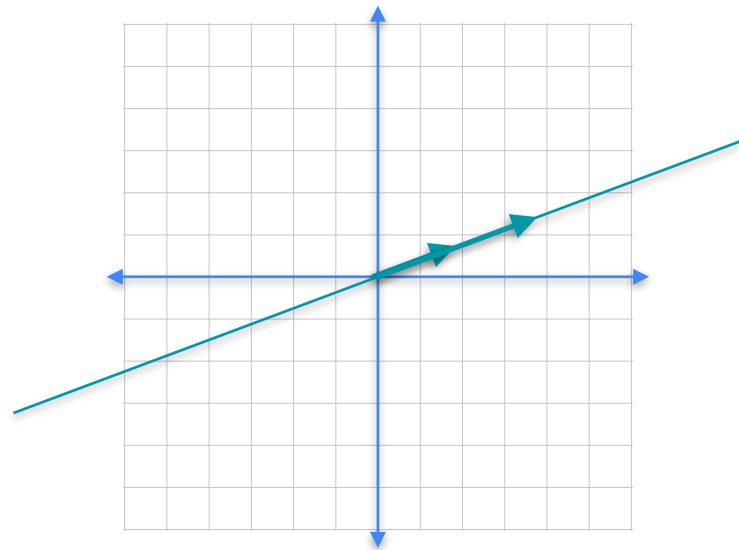


Basis

# A basis is a minimal spanning set

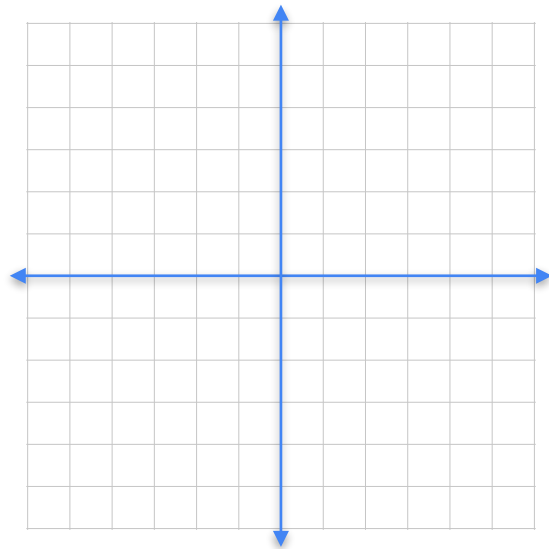
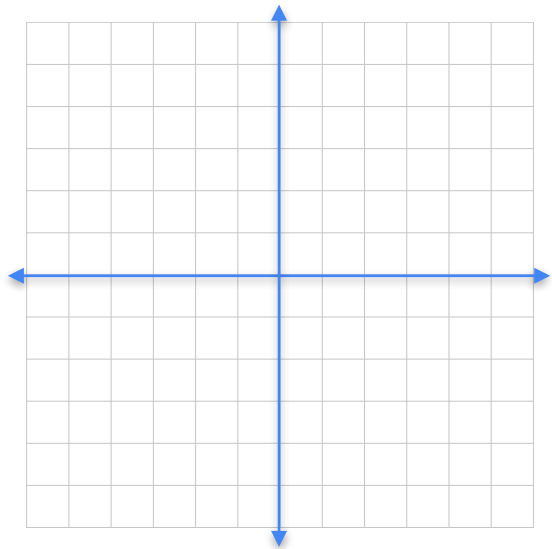


Basis



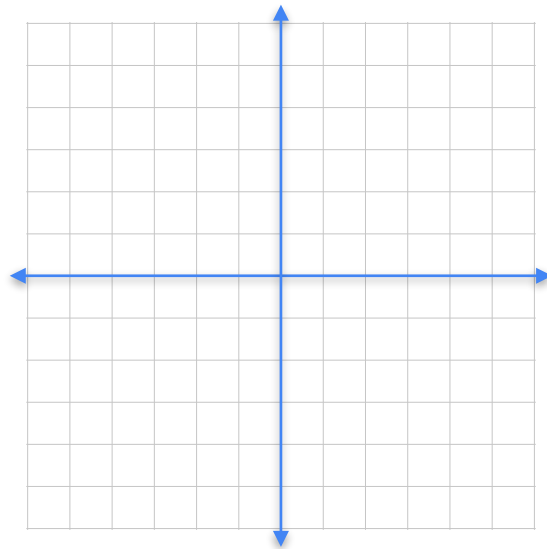
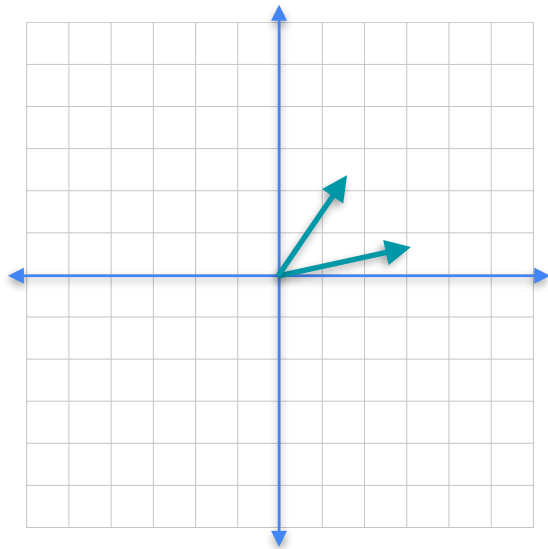
Not a basis

# A basis is a minimal spanning set

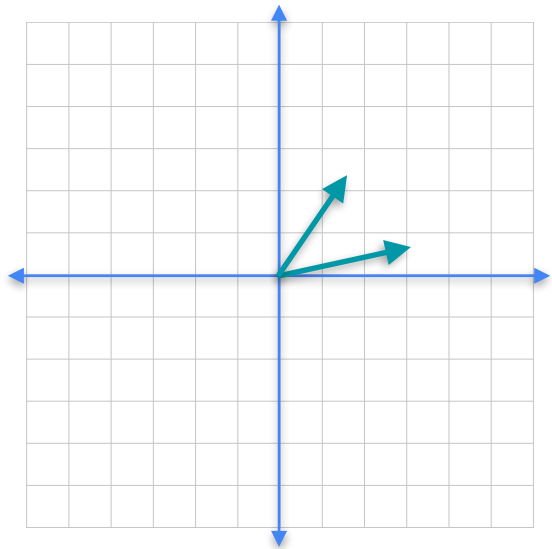




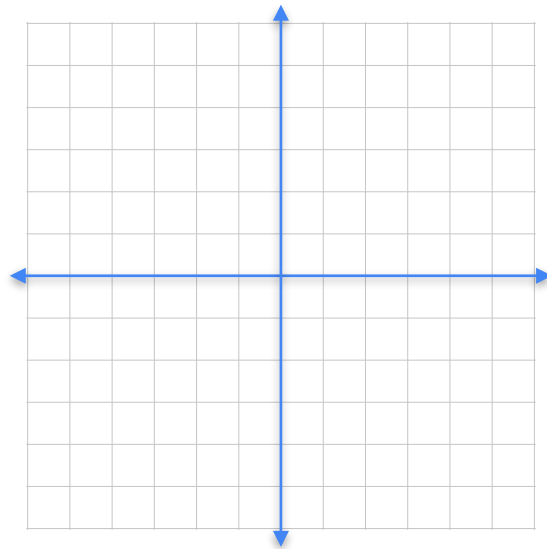
# A basis is a minimal spanning set



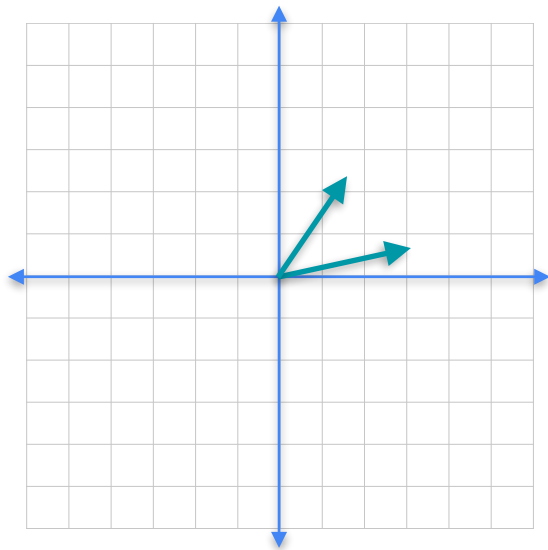
# A basis is a minimal spanning set



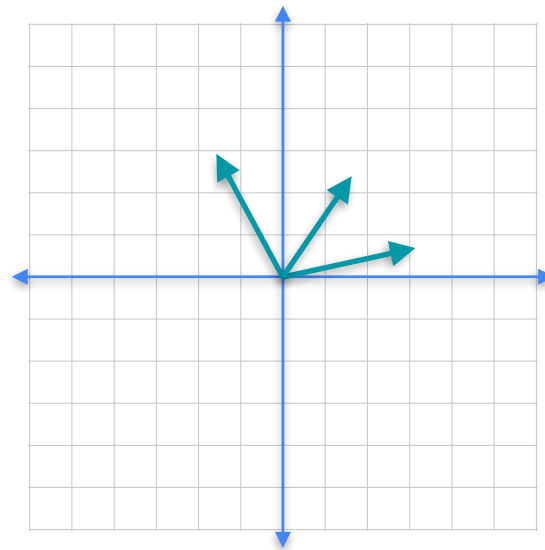
Basis



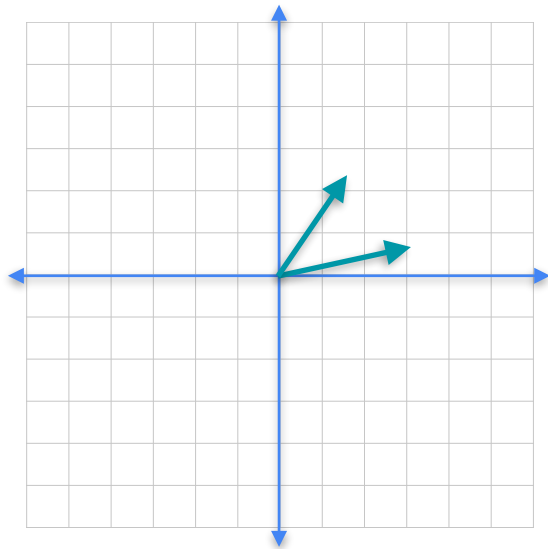
# A basis is a minimal spanning set



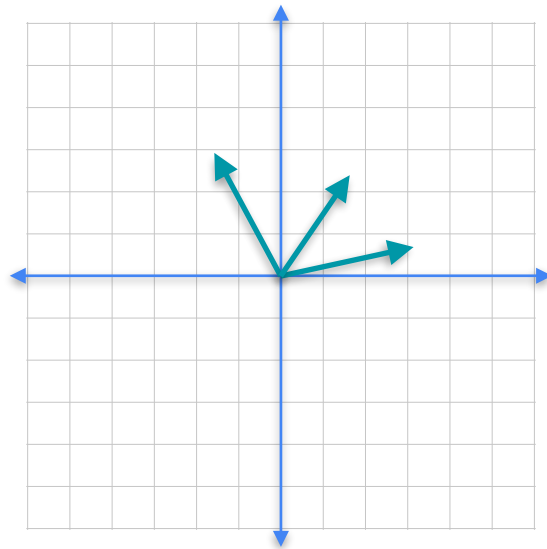
Basis



# A basis is a minimal spanning set

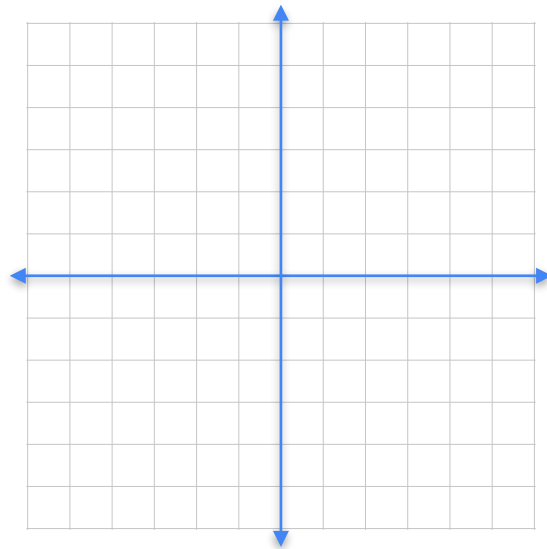
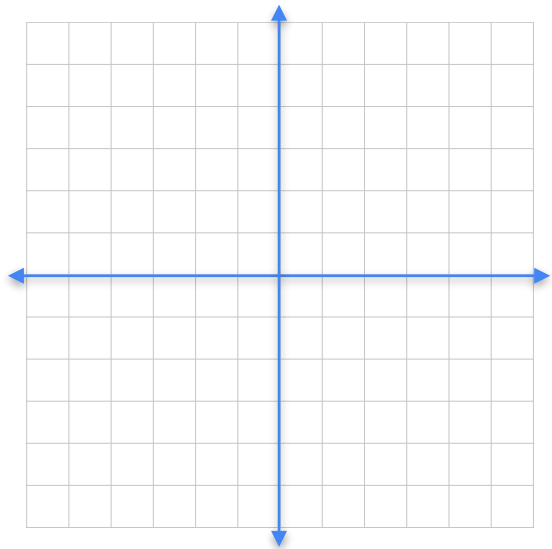


Basis

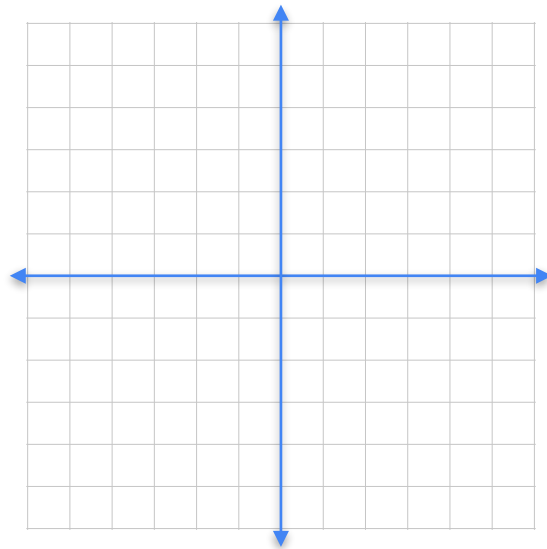
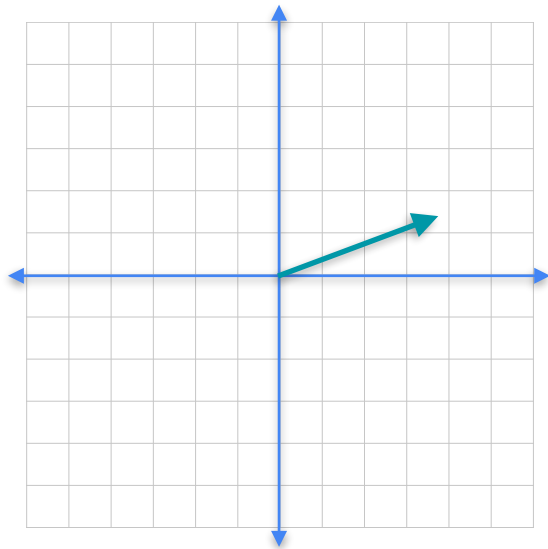


Not a basis

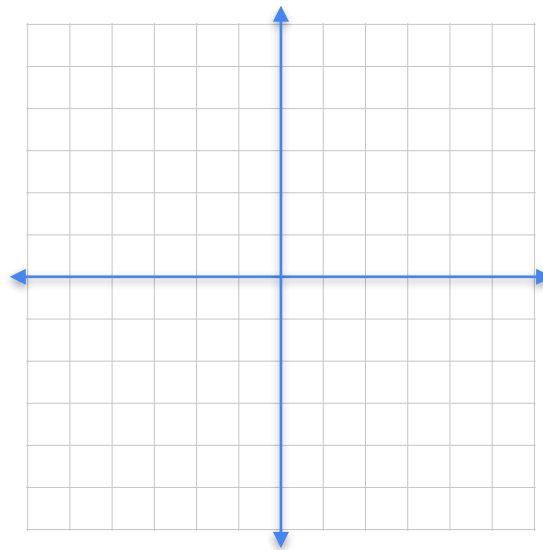
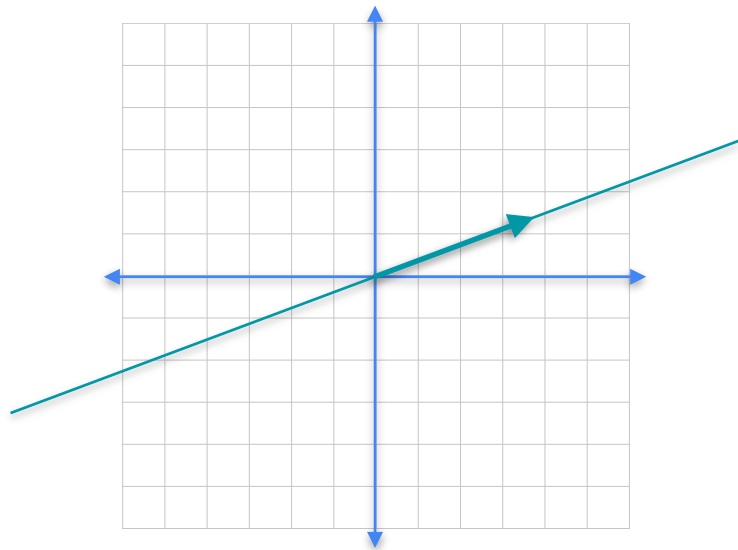
# Number of elements in the basis is the dimension



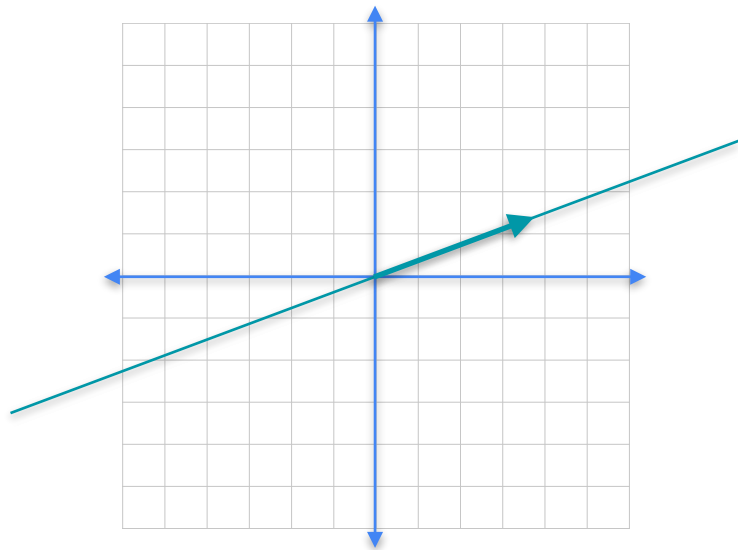
# Number of elements in the basis is the dimension



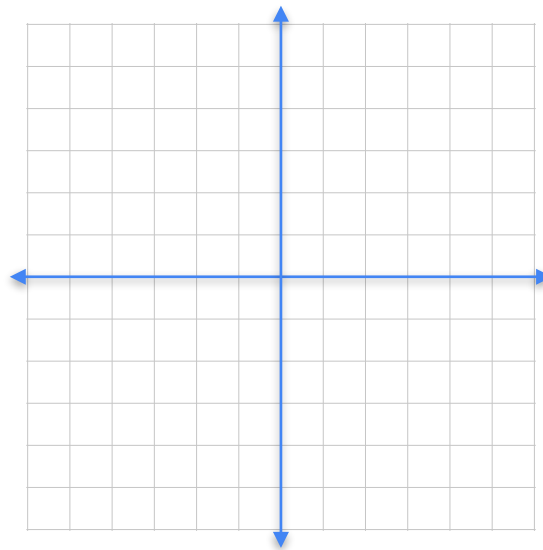
# Number of elements in the basis is the dimension



# Number of elements in the basis is the dimension

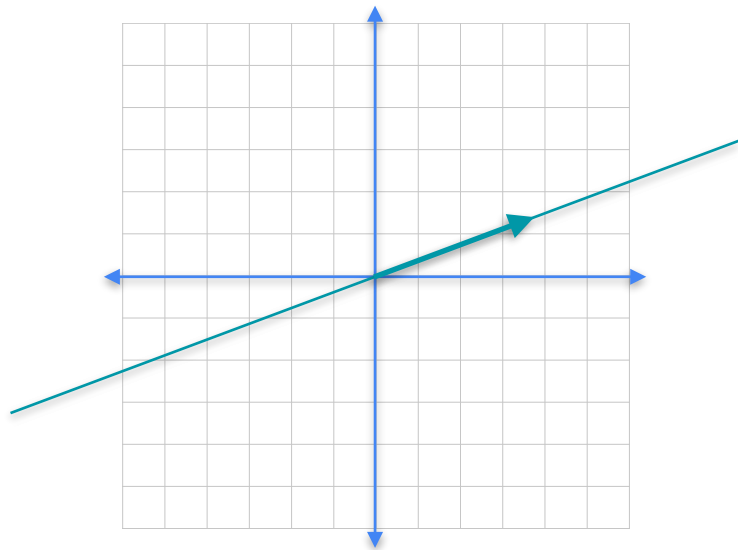


1 element  
Dimension = 1

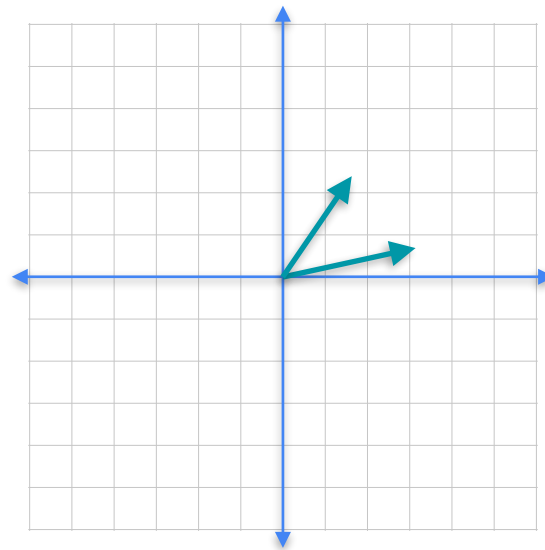




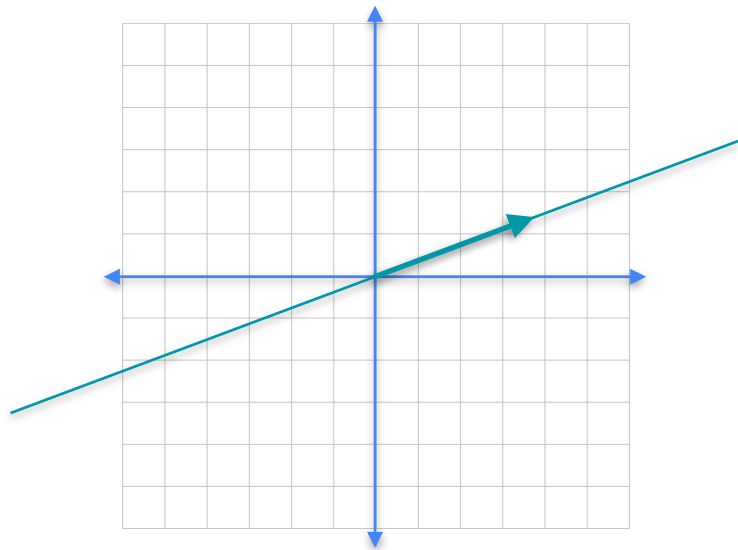
# Number of elements in the basis is the dimension



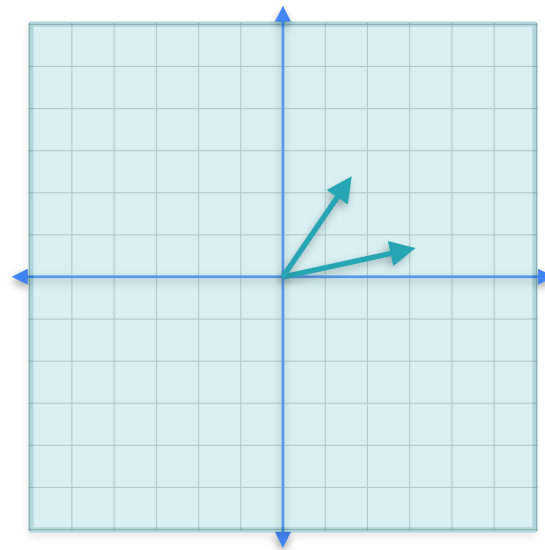
1 element  
Dimension = 1



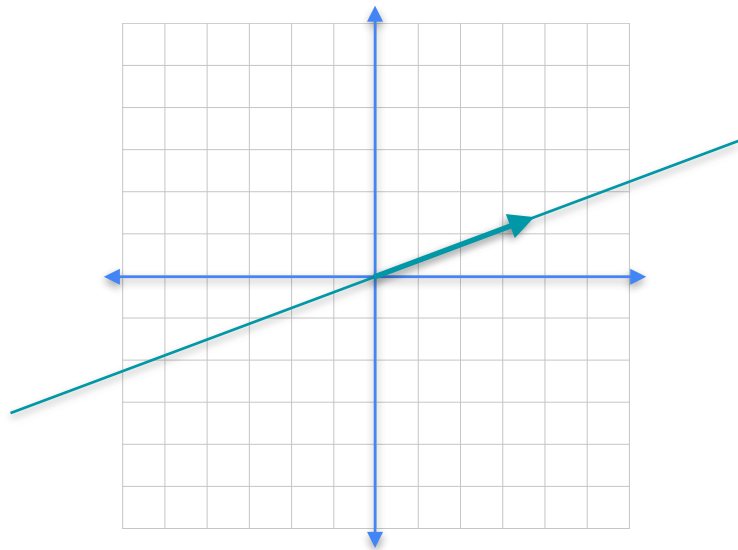
# Number of elements in the basis is the dimension



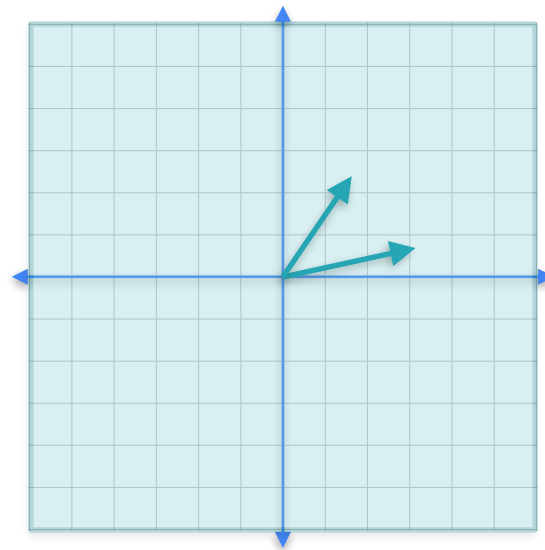
1 element  
Dimension = 1



# Number of elements in the basis is the dimension



1 element  
Dimension = 1



2 elements in the basis  
Dimension = 2



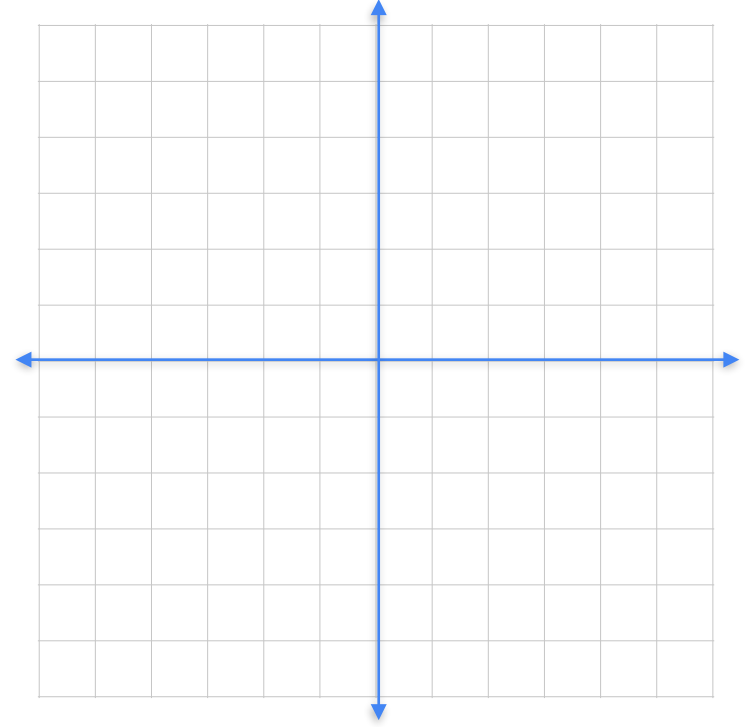
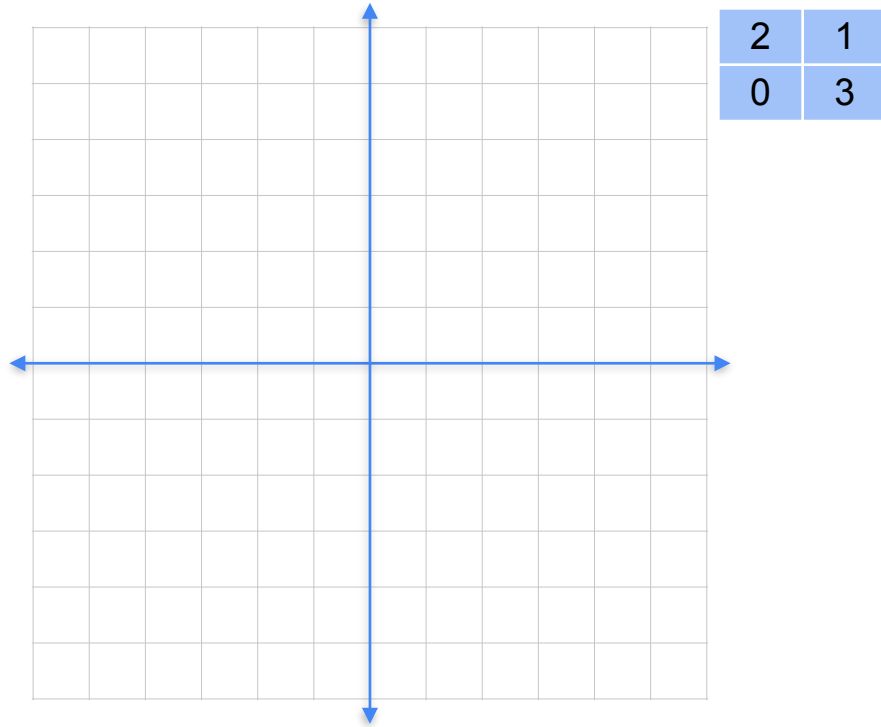
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# Determinants and Eigenvectors

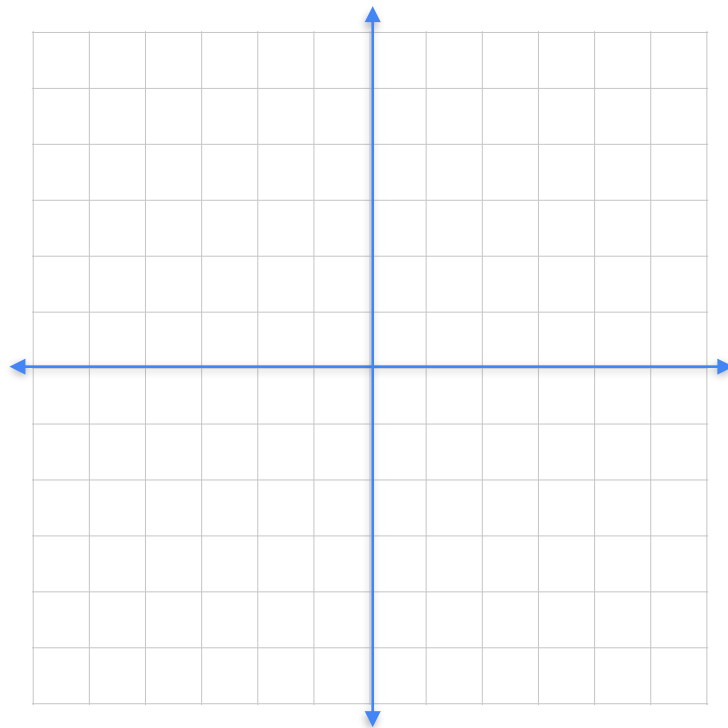
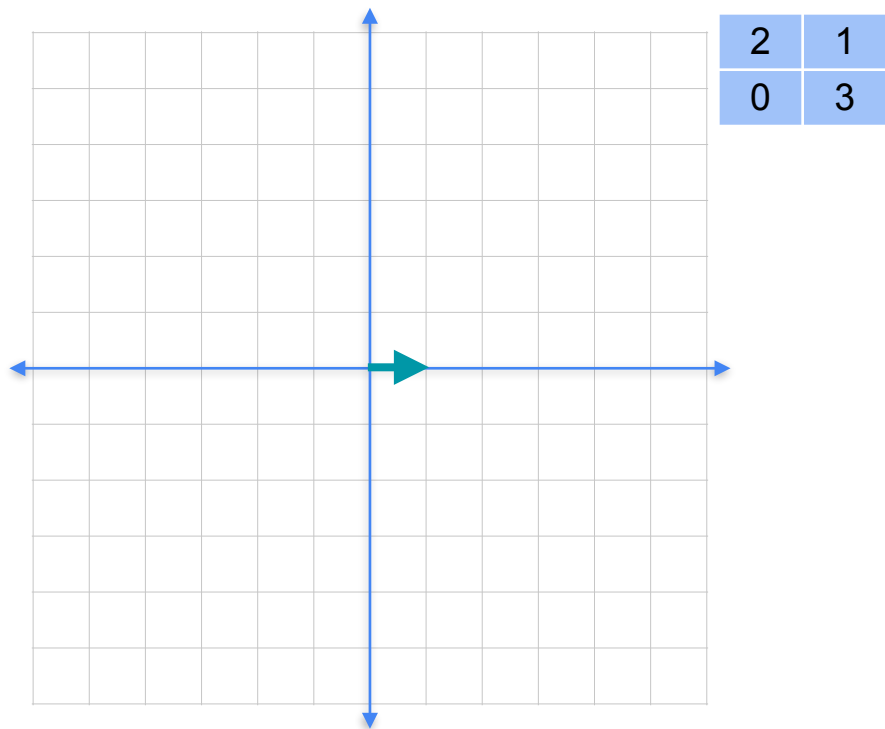
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## **Eigenbasis**

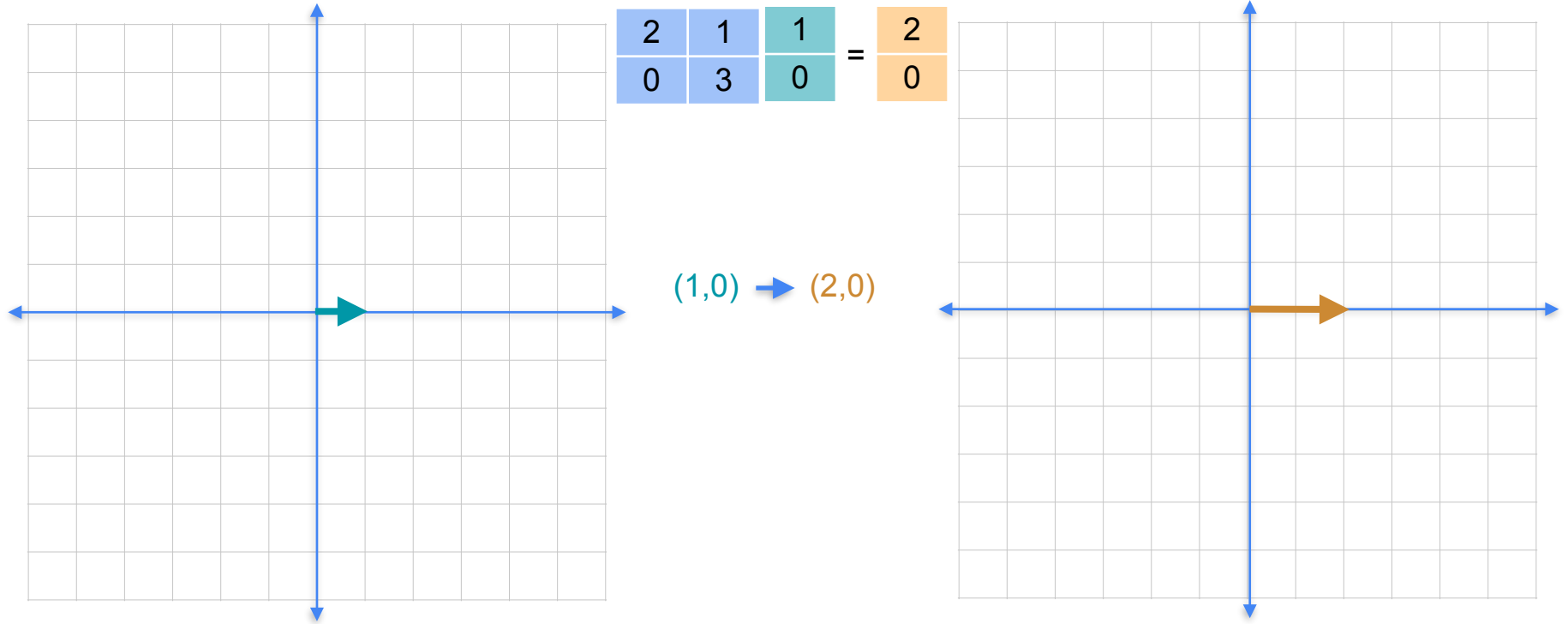
# Basis



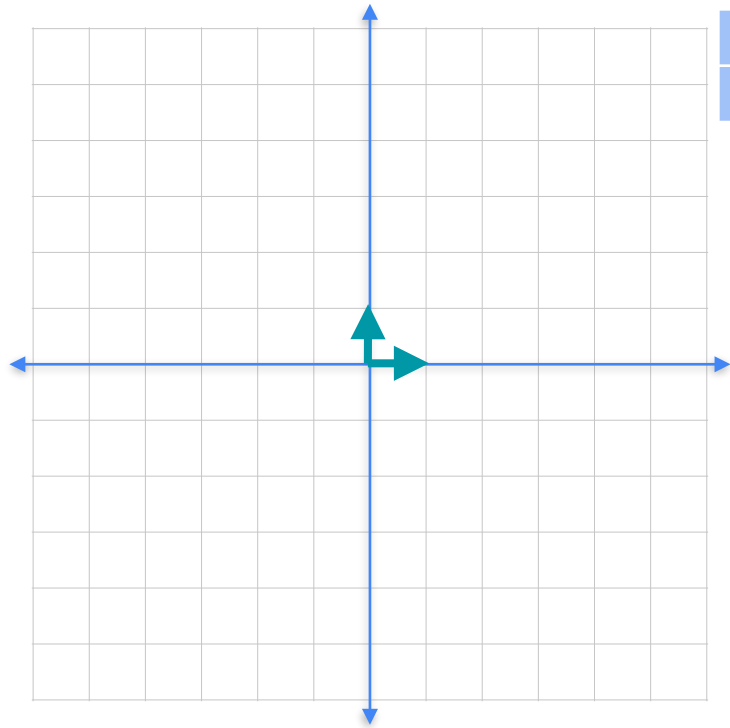
# Basis



# Basis

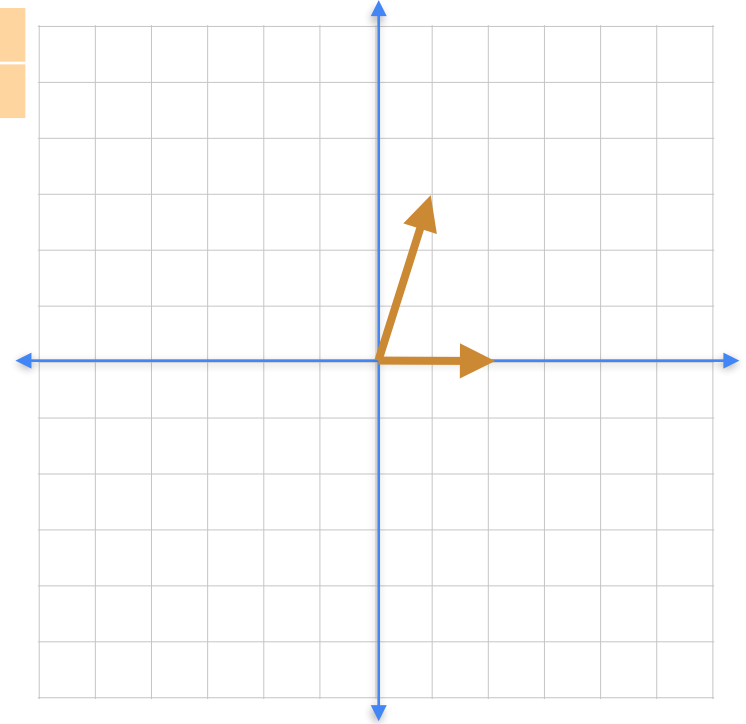


# Basis



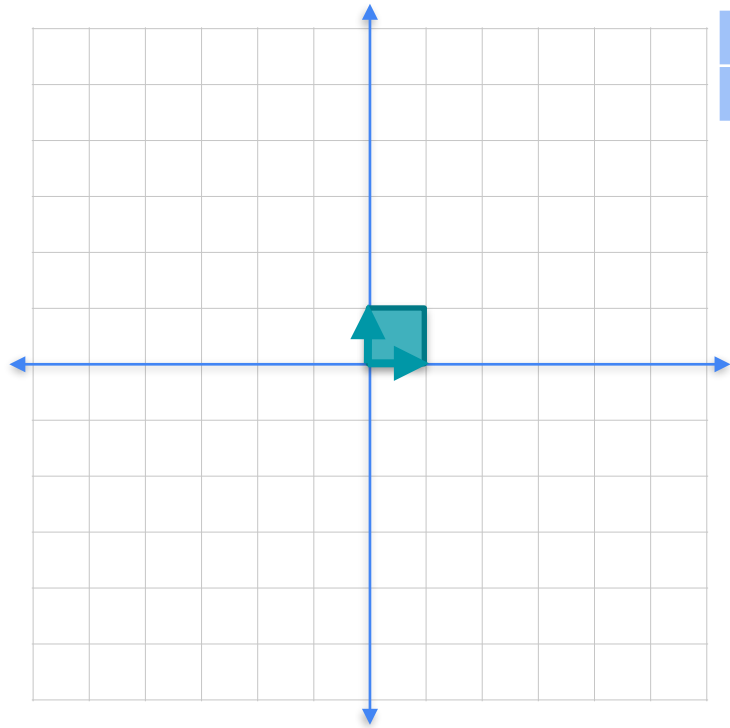
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (0,1) &\rightarrow (1,3) \end{aligned}$$



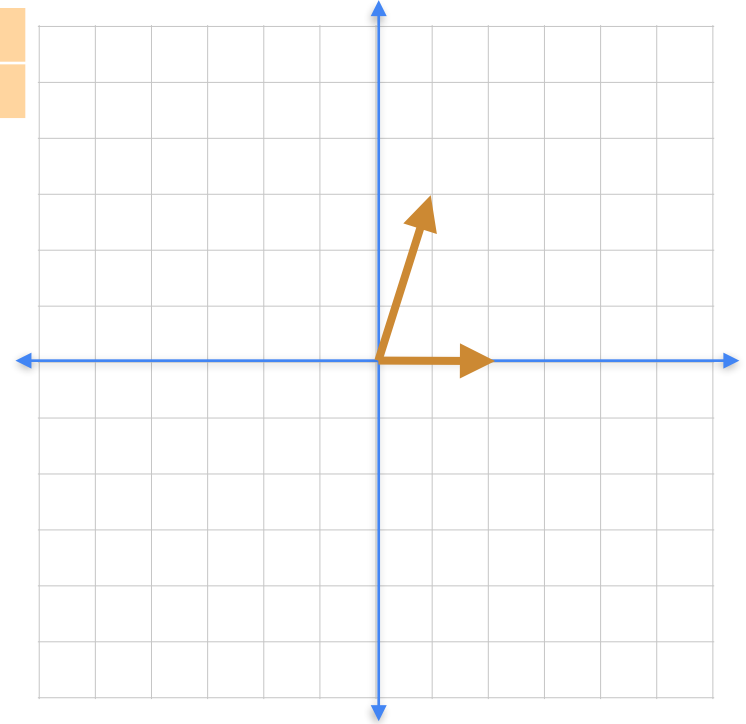


# Basis

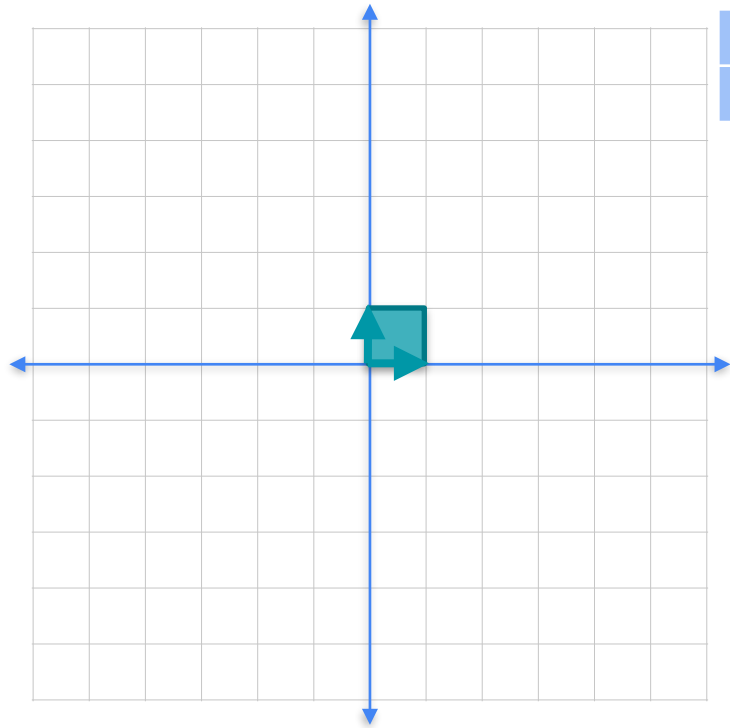


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (0,1) &\rightarrow (1,3) \end{aligned}$$

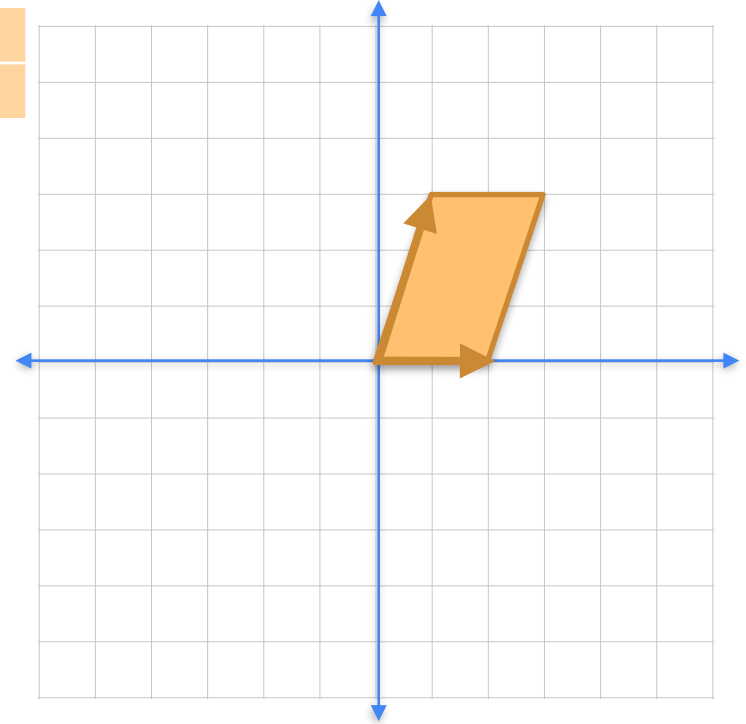


# Basis

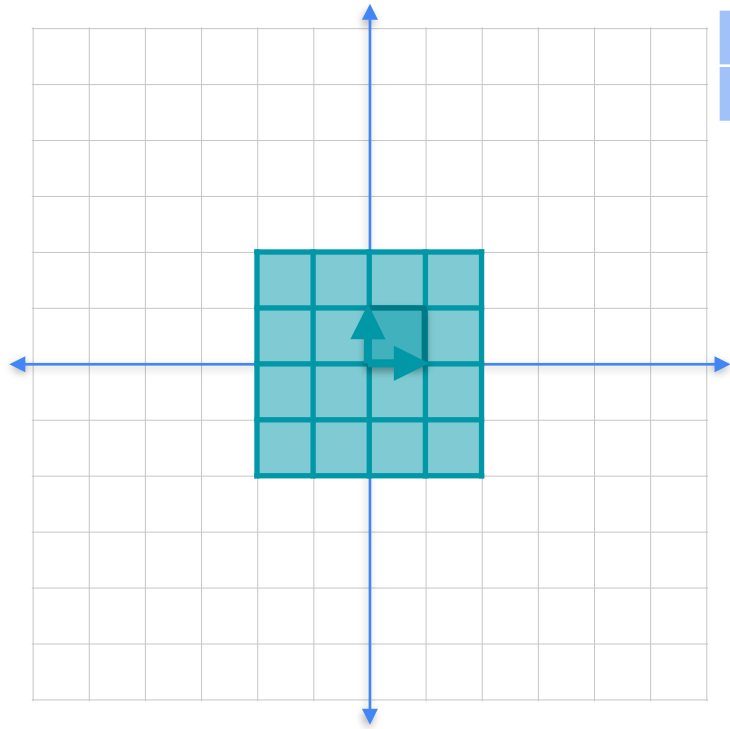


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (0,1) &\rightarrow (1,3) \end{aligned}$$

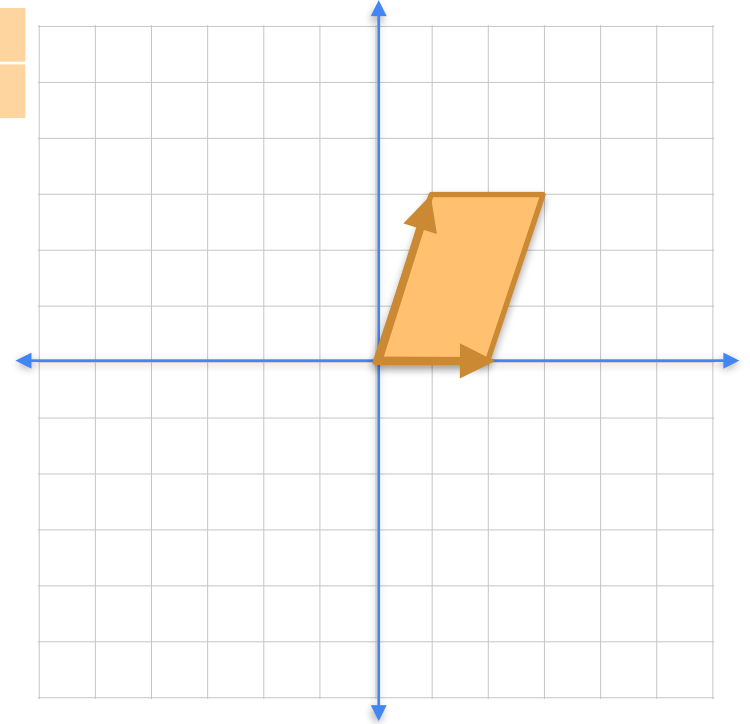


# Basis

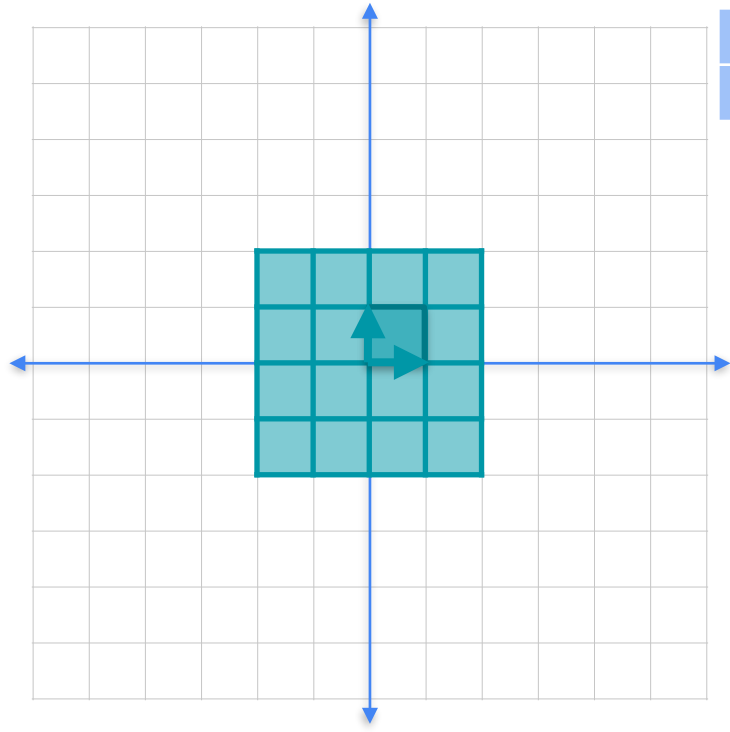


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (0,1) &\rightarrow (1,3) \end{aligned}$$

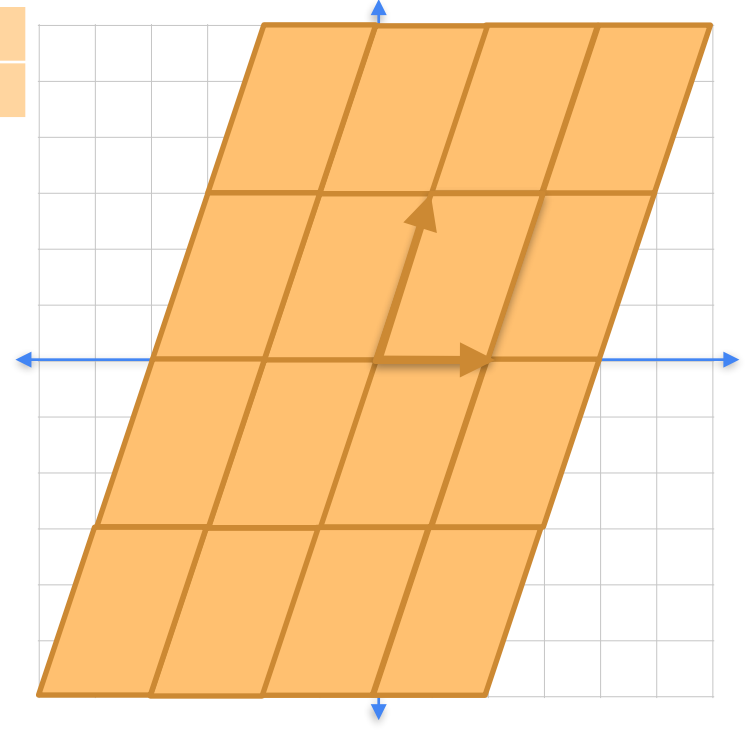


# Basis

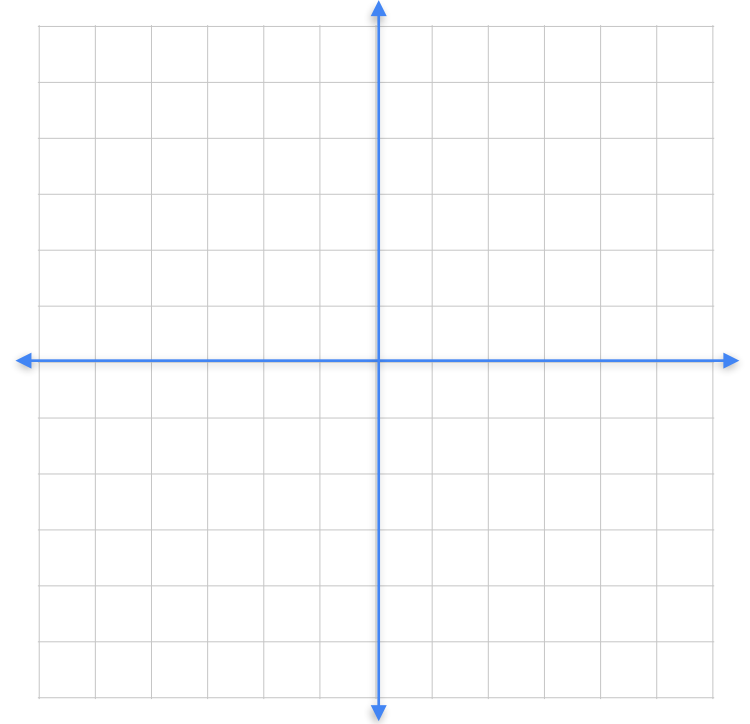
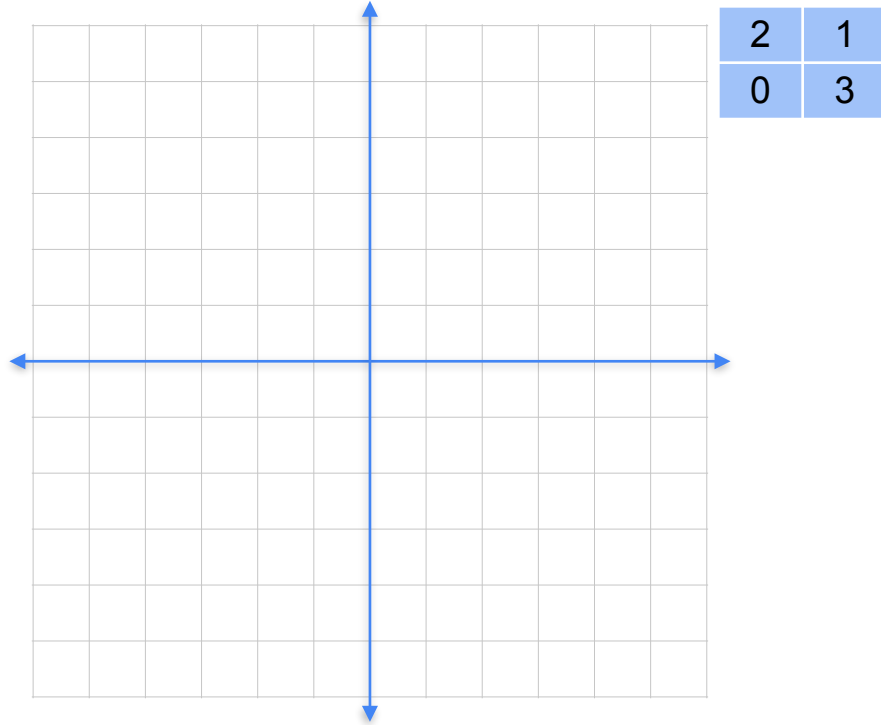


2	1	0	=	1
0	3	1		3

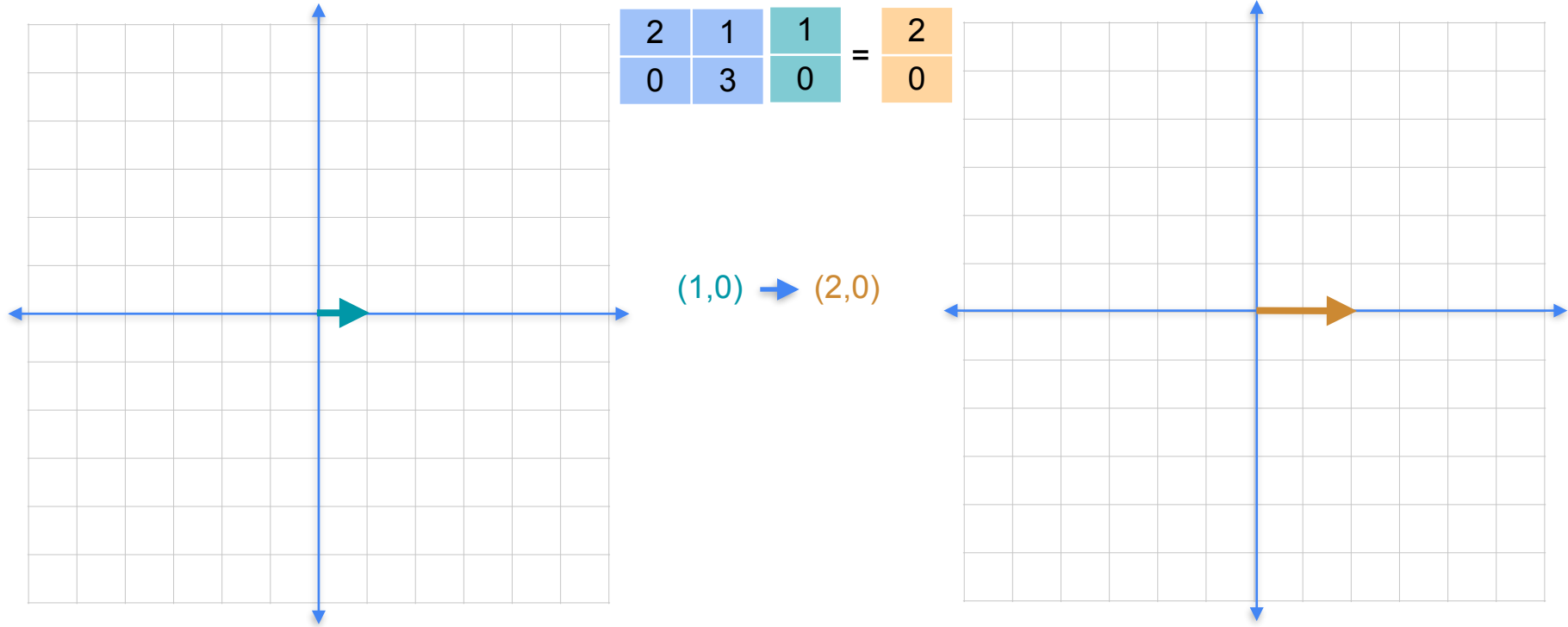
$$\begin{aligned}(1,0) &\rightarrow (2,0) \\ (0,1) &\rightarrow (1,3)\end{aligned}$$



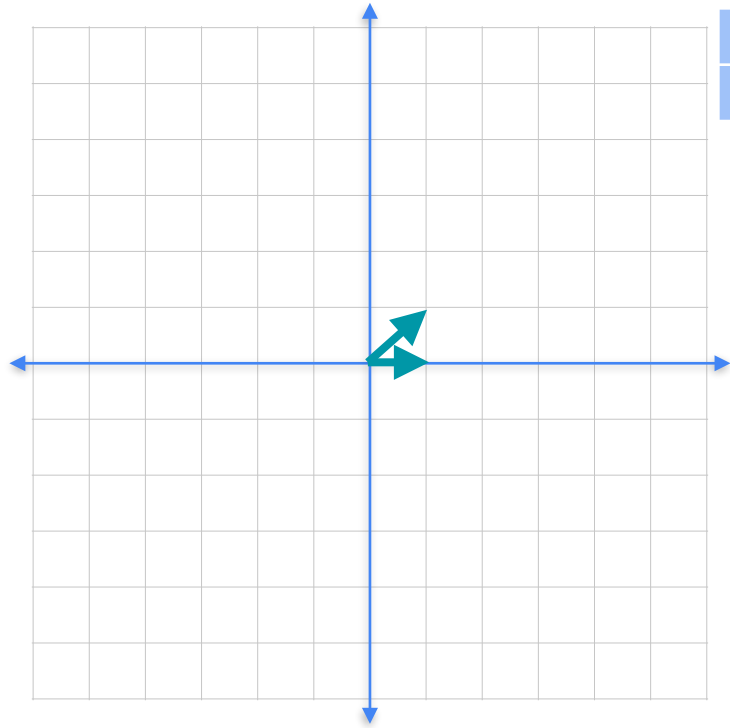
# Eigenbasis



# Eigenbasis

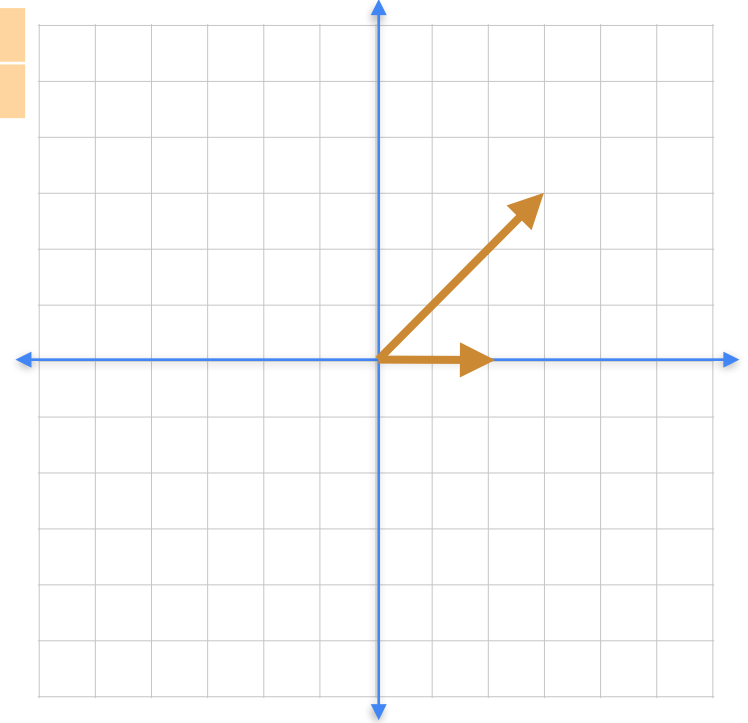


# Eigenbasis

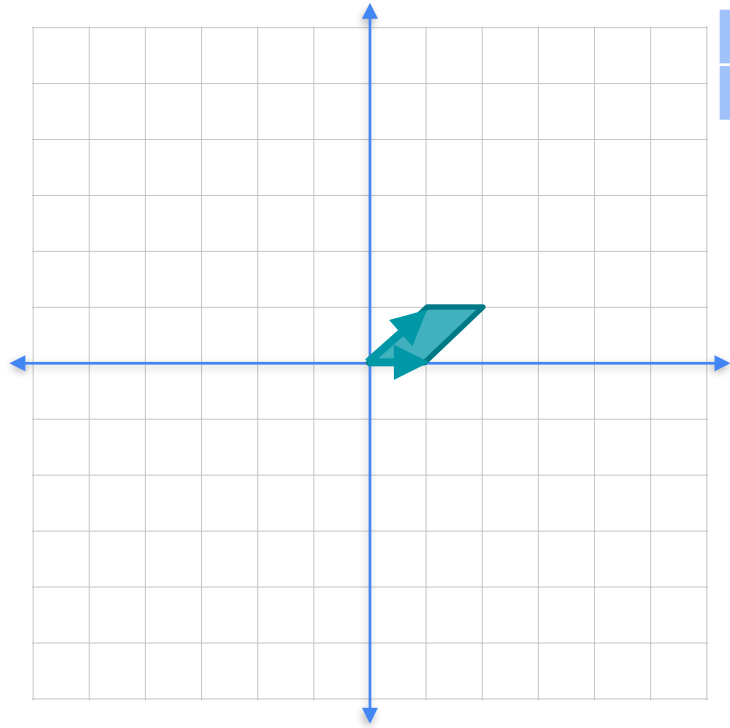


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (1,1) &\rightarrow (3,3) \end{aligned}$$

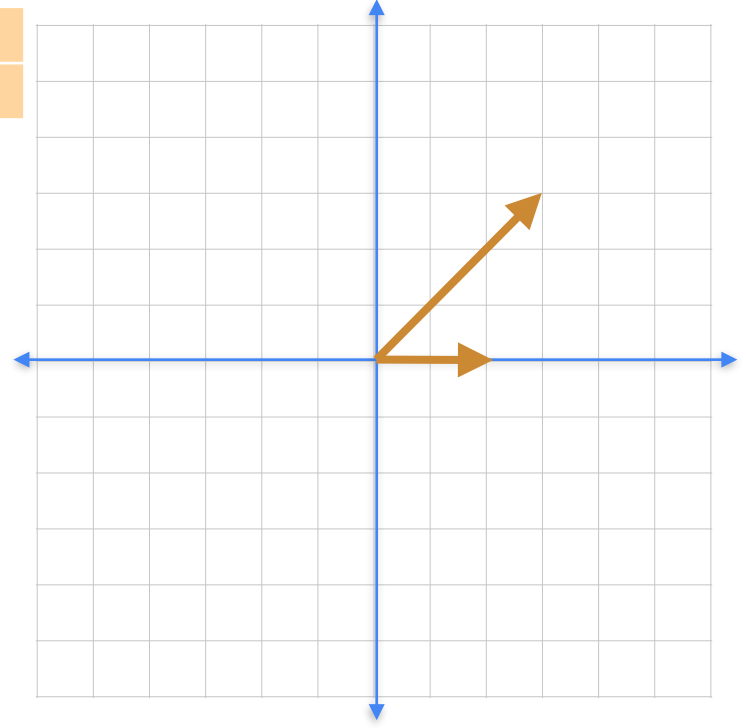


# Eigenbasis



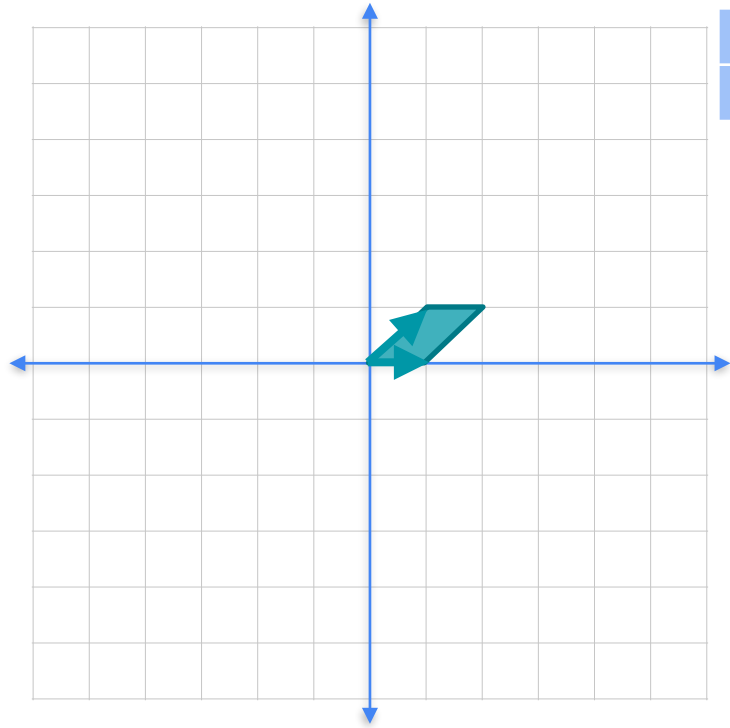
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (1,1) &\rightarrow (3,3) \end{aligned}$$



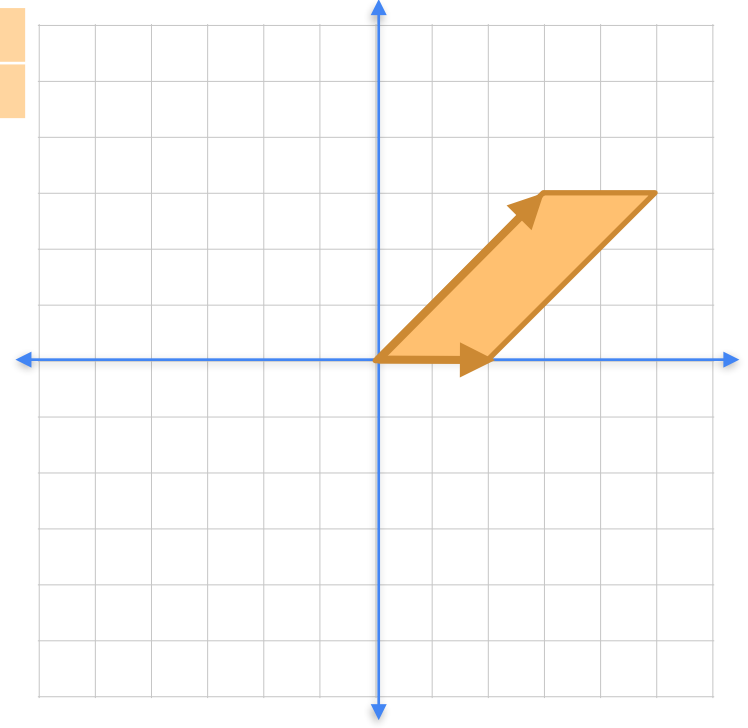


# Eigenbasis

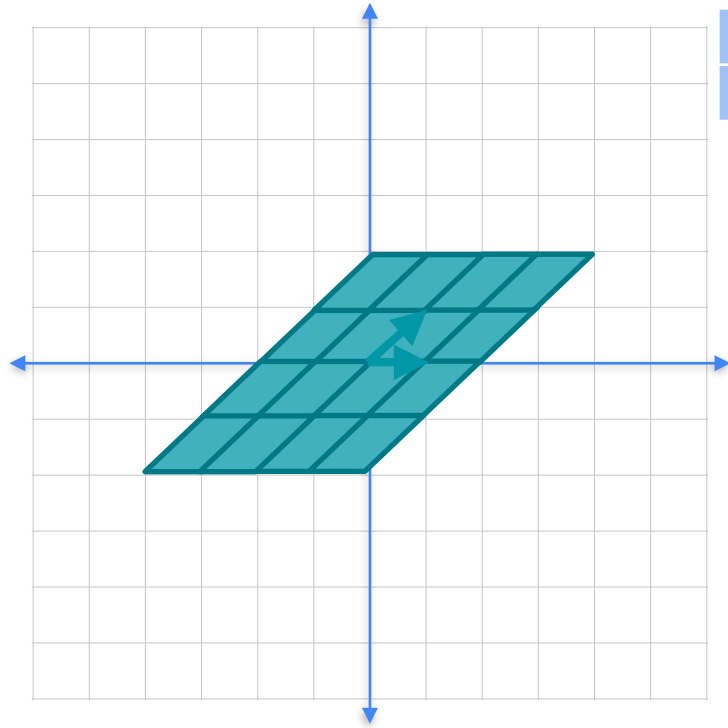


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (1,1) &\rightarrow (3,3) \end{aligned}$$

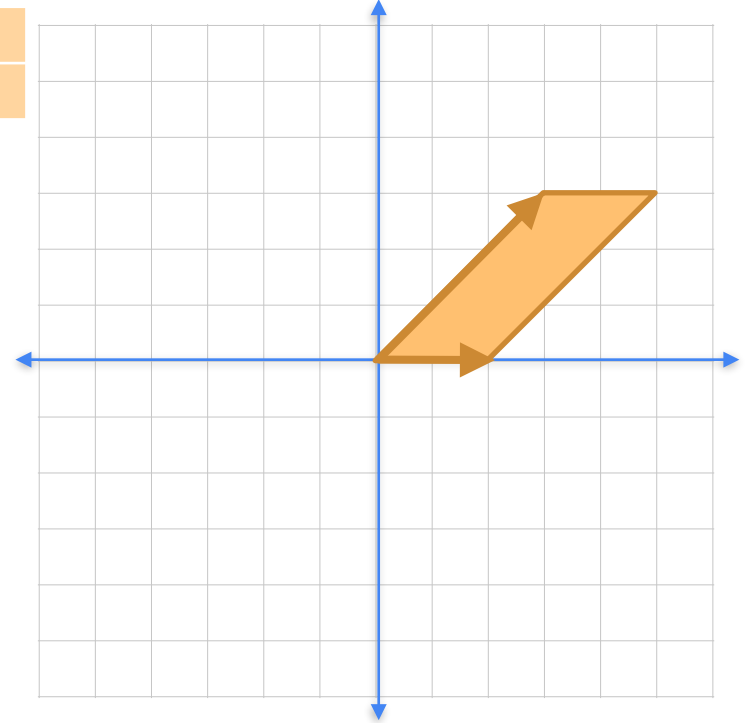


# Eigenbasis

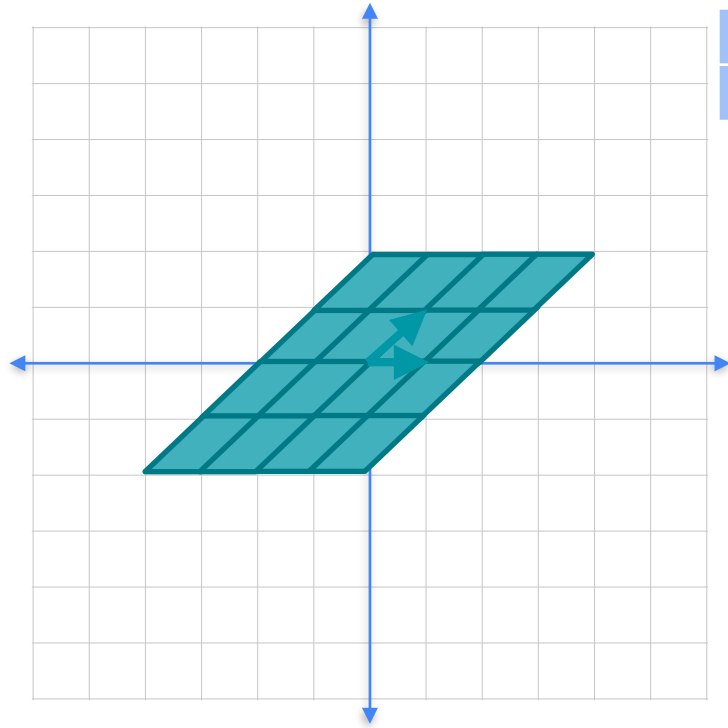


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (1,1) &\rightarrow (3,3) \end{aligned}$$

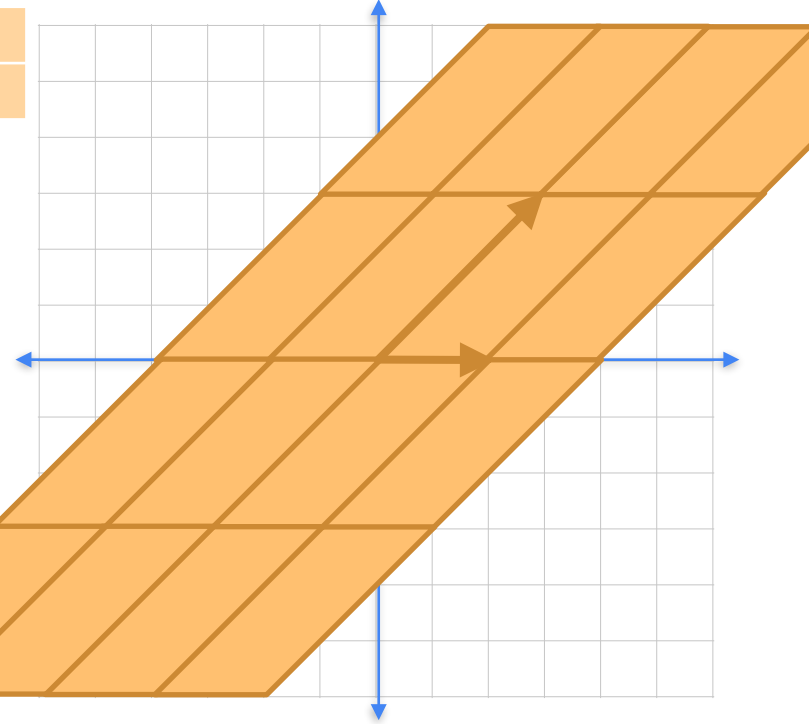


# Eigenbasis

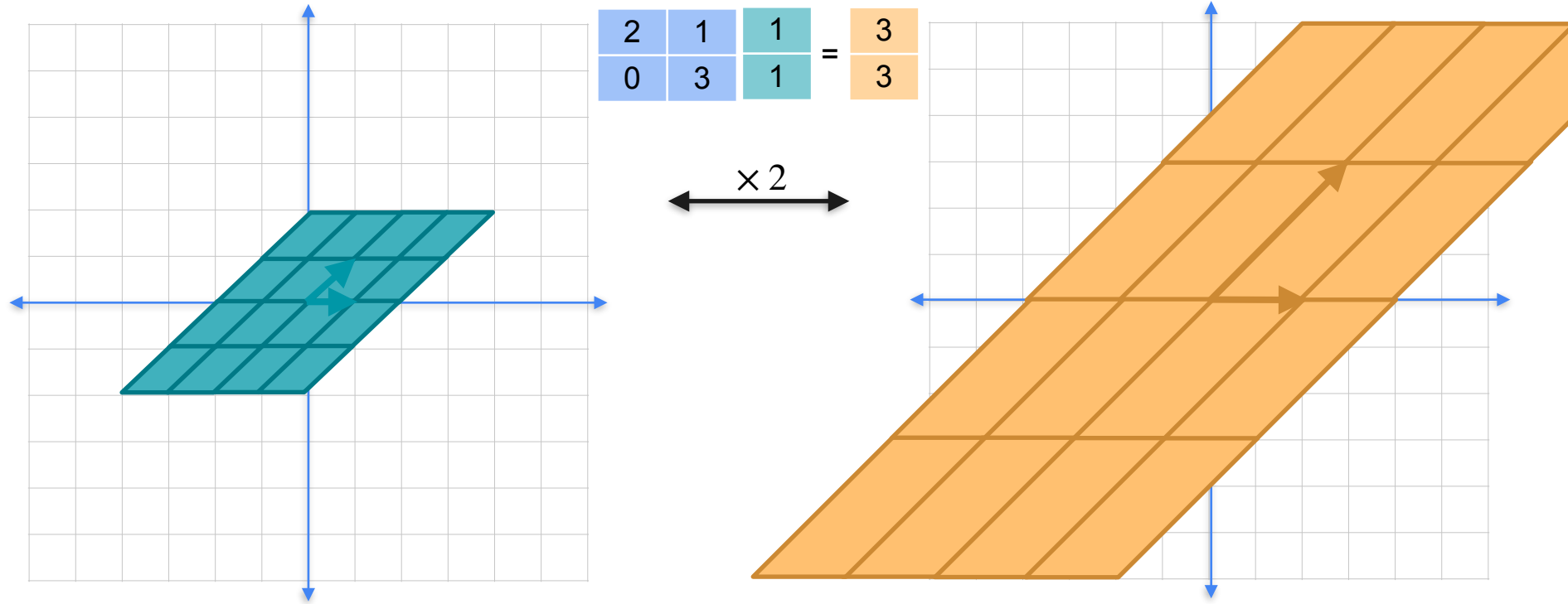


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

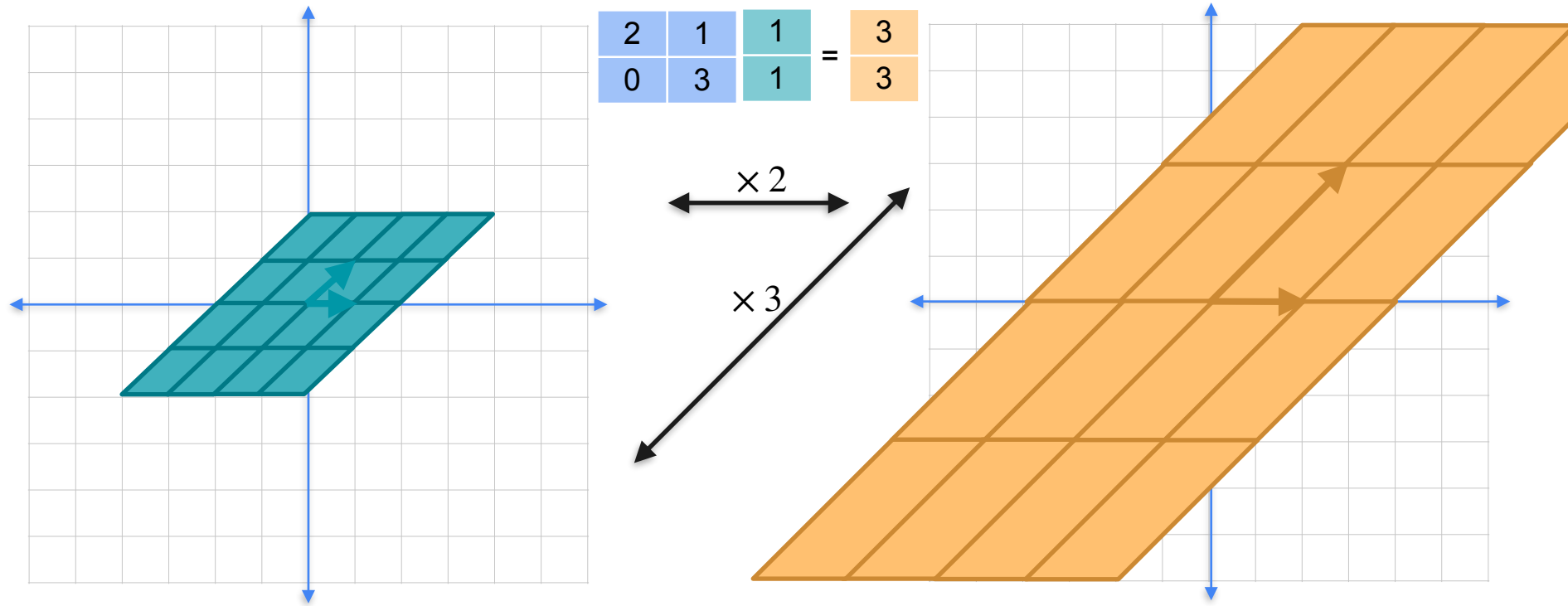
$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (1,1) &\rightarrow (3,3) \end{aligned}$$



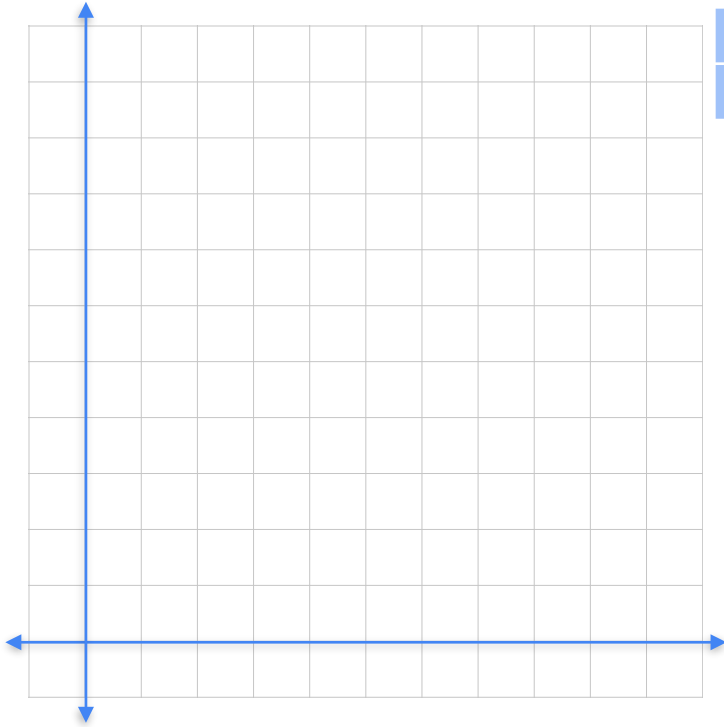
# Eigenbasis



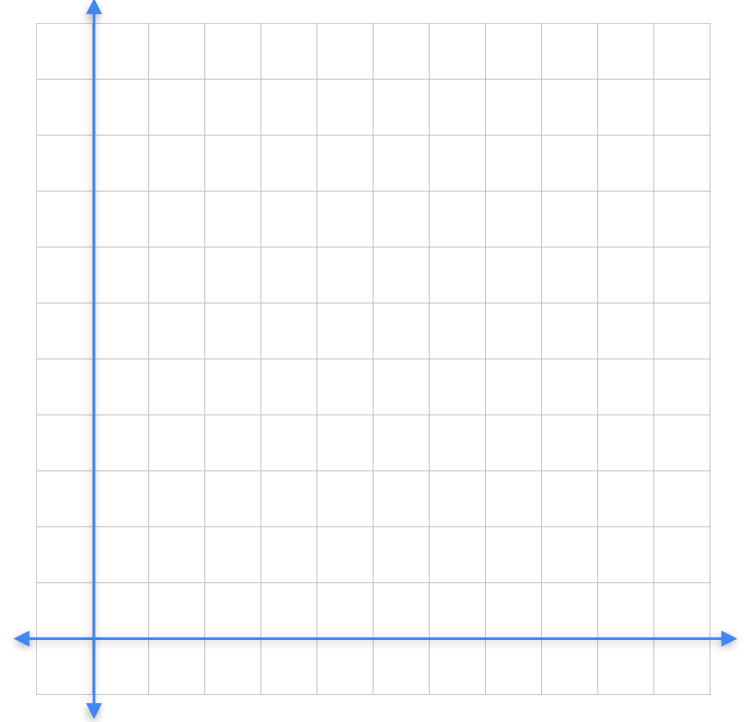
# Eigenbasis



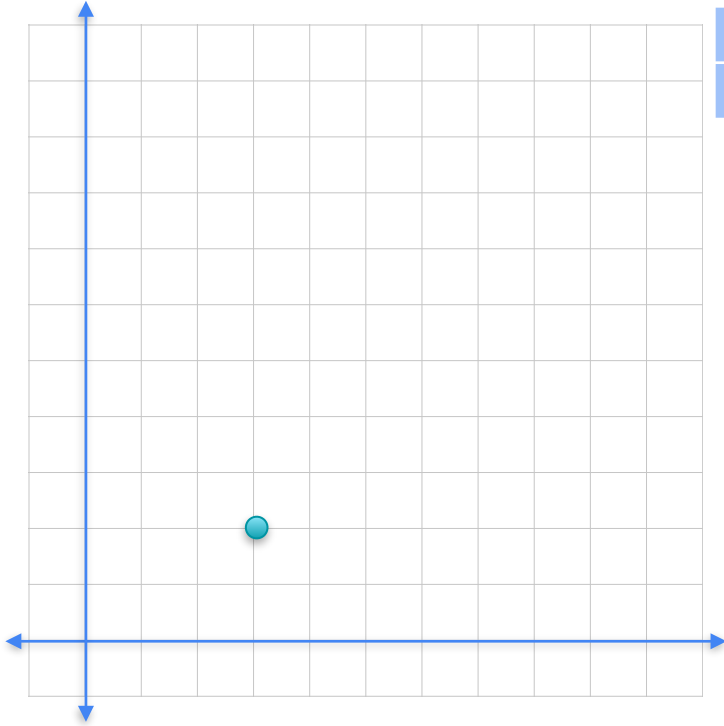
# Eigenbasis



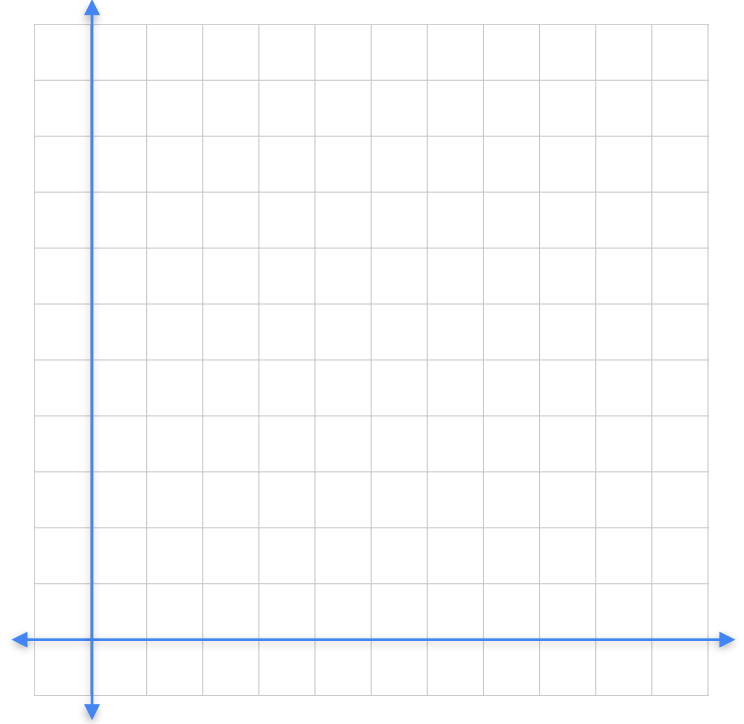
2	1
0	3



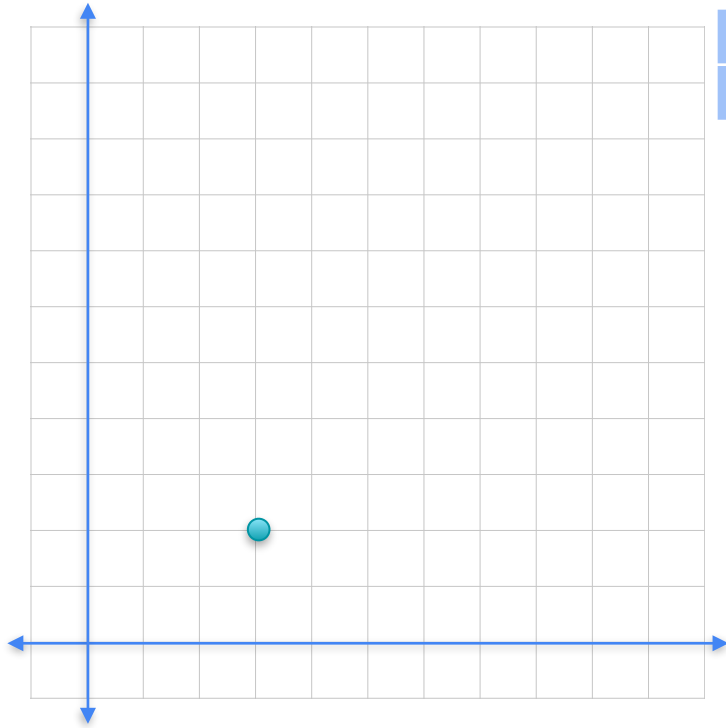
# Eigenbasis



2	1
0	3

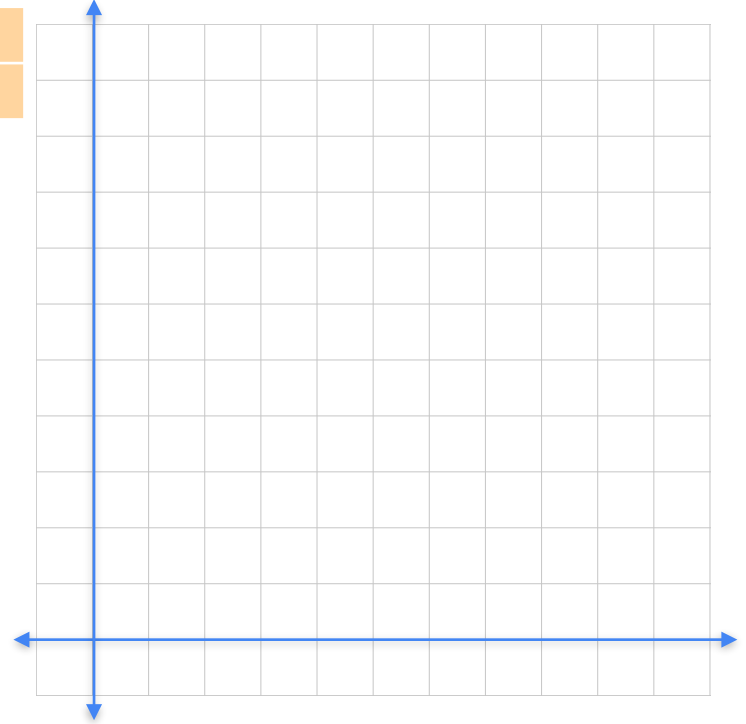


# Eigenbasis



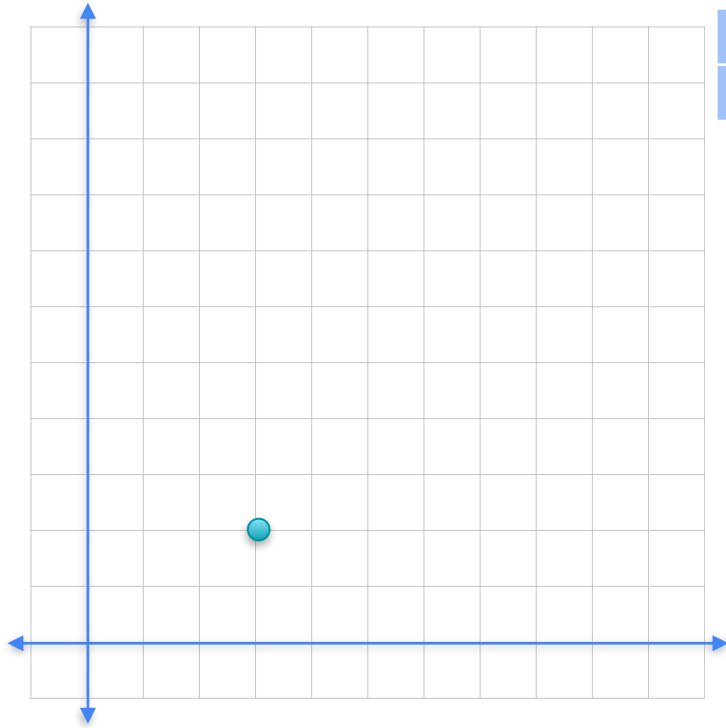
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$(3, 2) \rightarrow (8, 6)$$



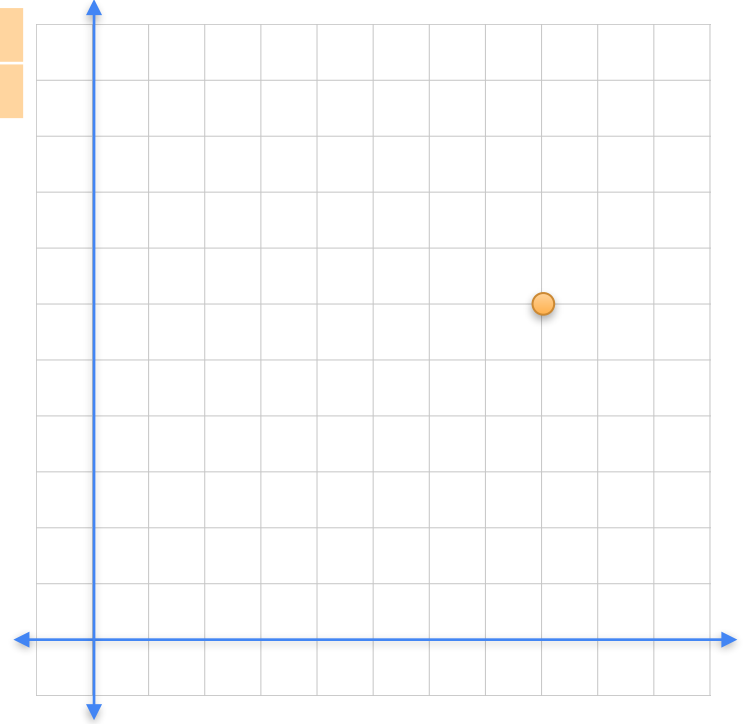


# Eigenbasis

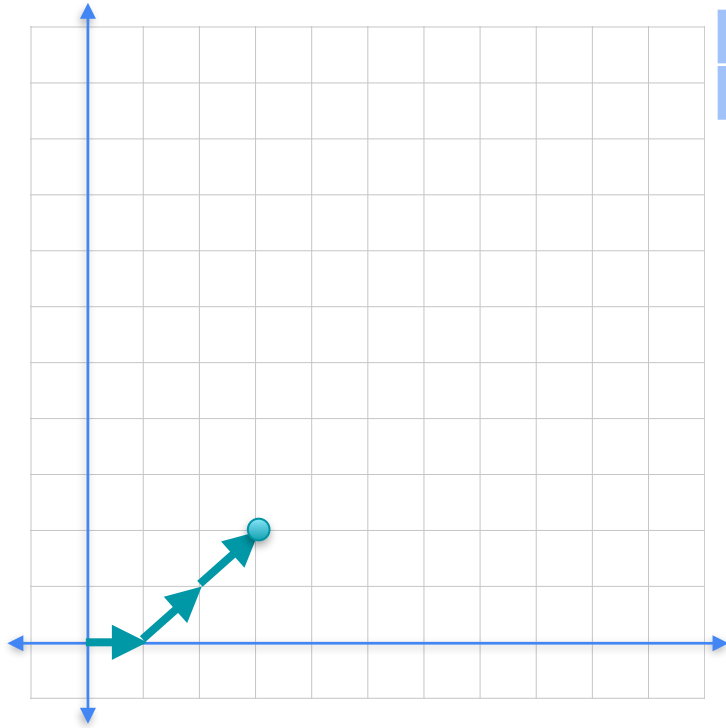


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$(3, 2) \rightarrow (8, 6)$$

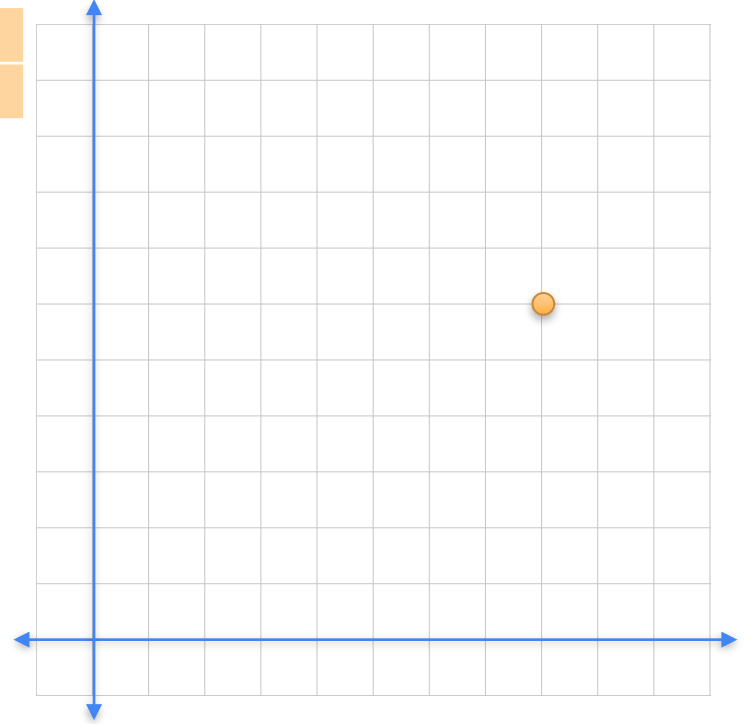


# Eigenbasis

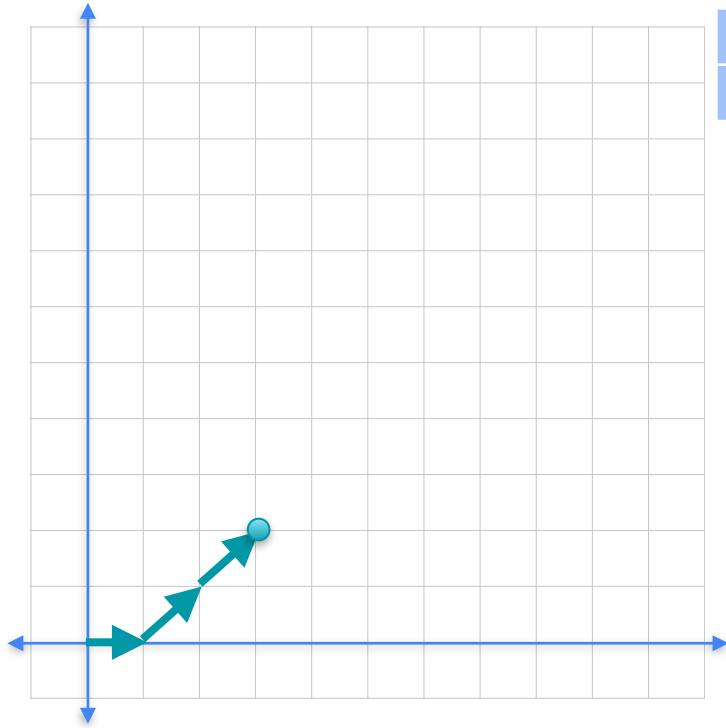


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$(3,2) \rightarrow (8,6)$$

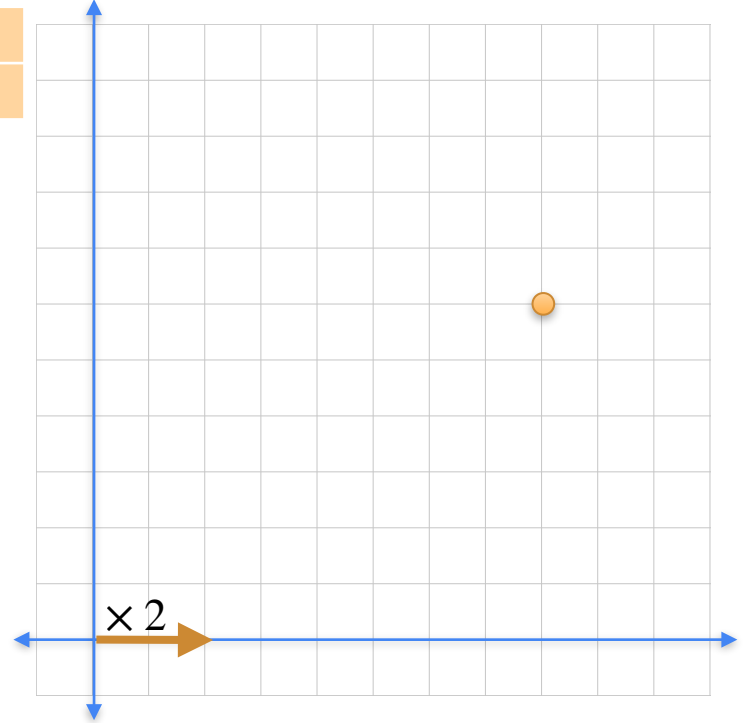


# Eigenbasis

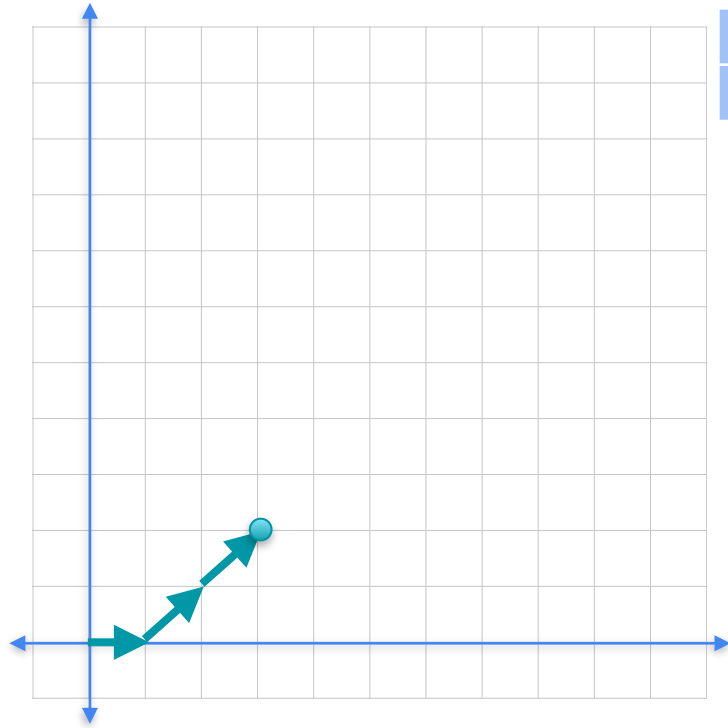


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$(3,2) \rightarrow (8,6)$$

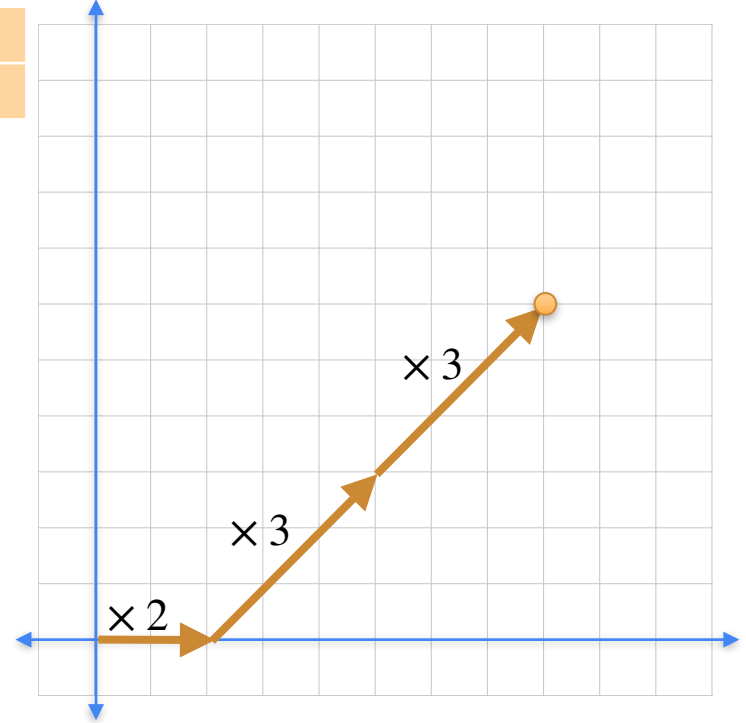


# Eigenbasis



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$(3,2) \rightarrow (8,6)$$





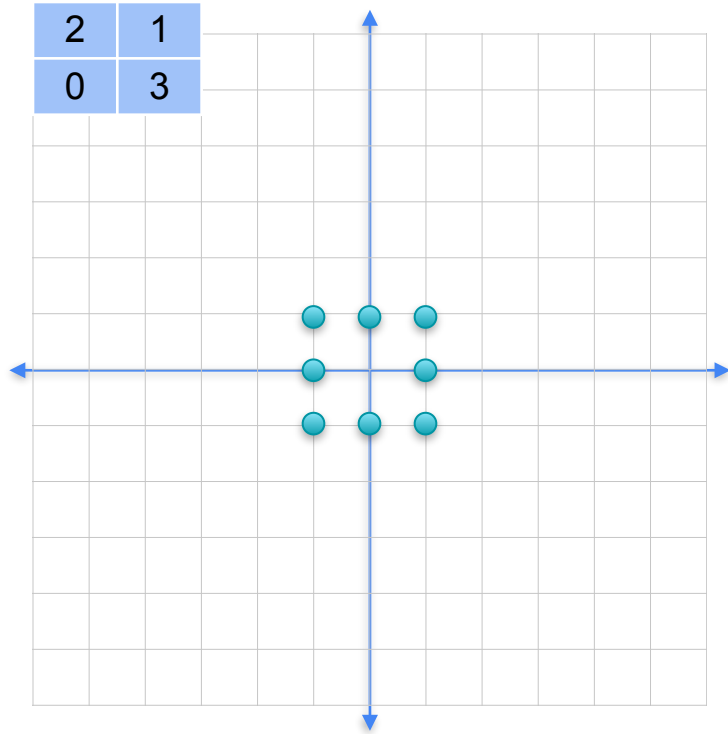
DeepLearning.AI

# Determinants and Eigenvectors

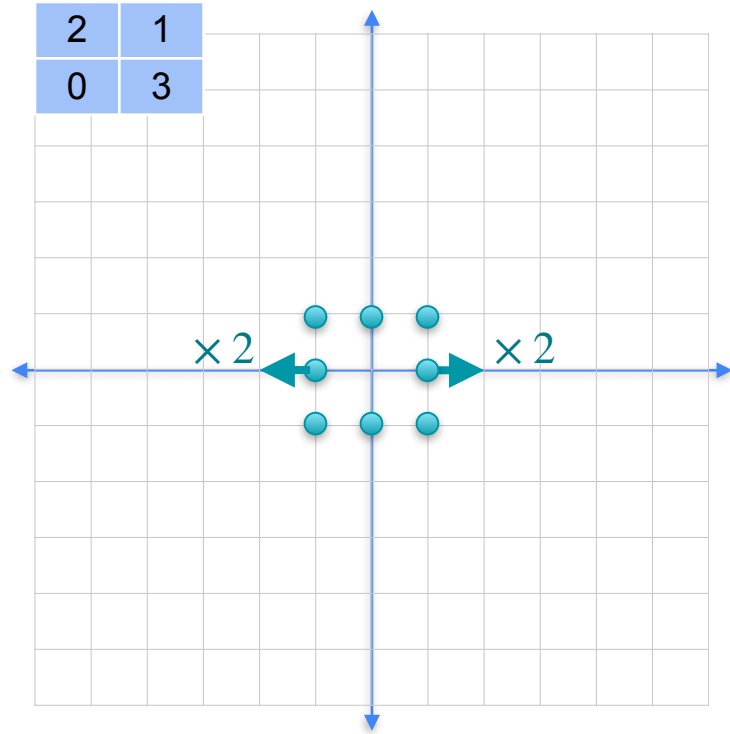
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## **Eigenvalues and eigenvectors**

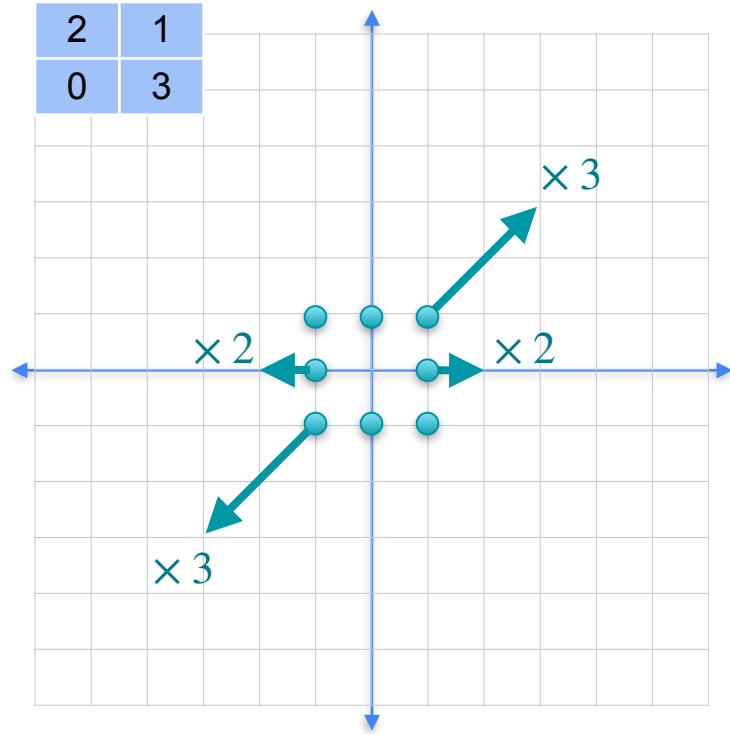
# Finding eigenvalues



# Finding eigenvalues

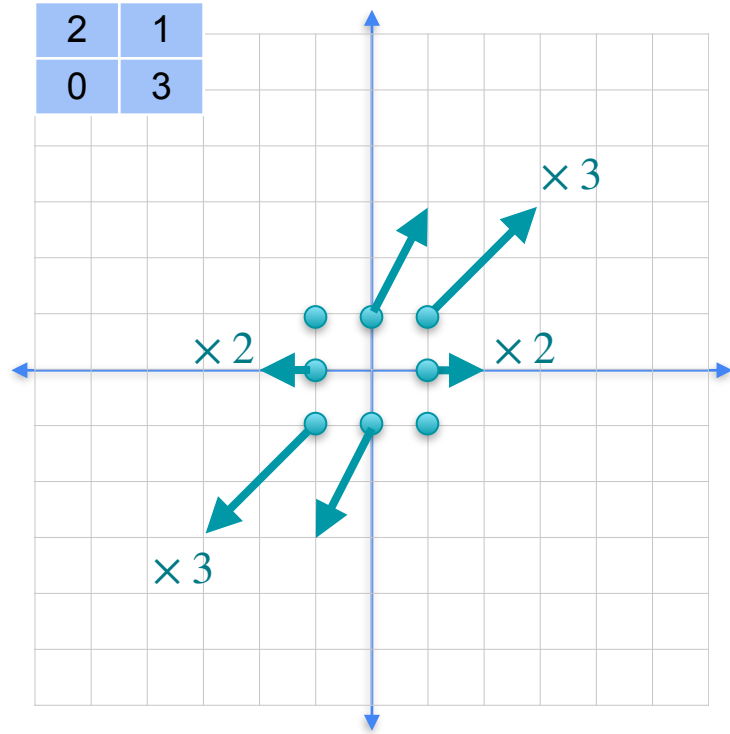


# Finding eigenvalues

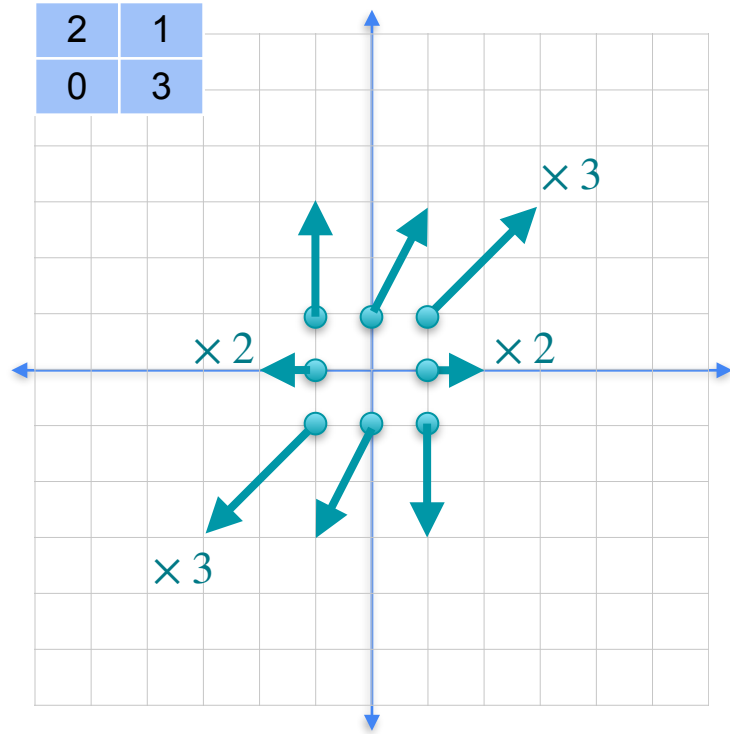




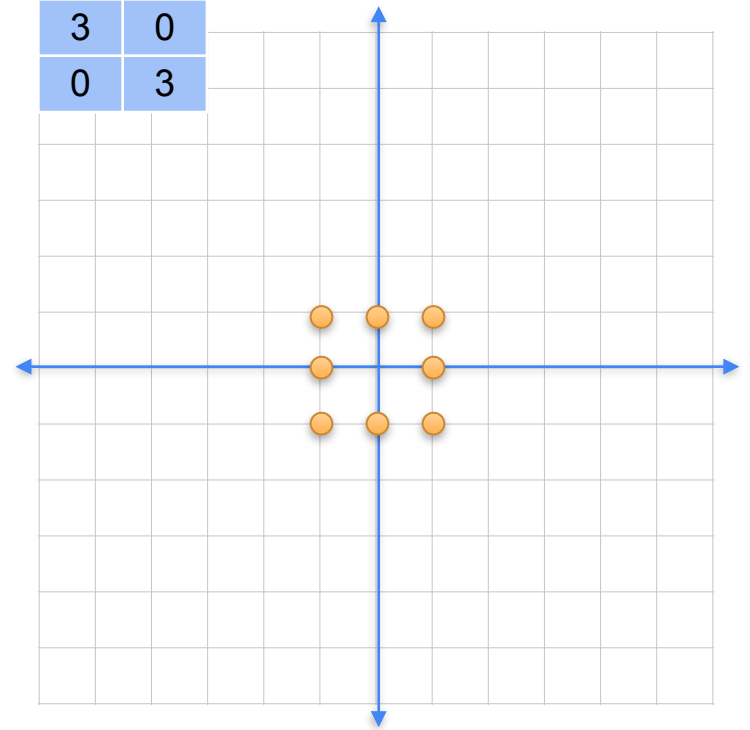
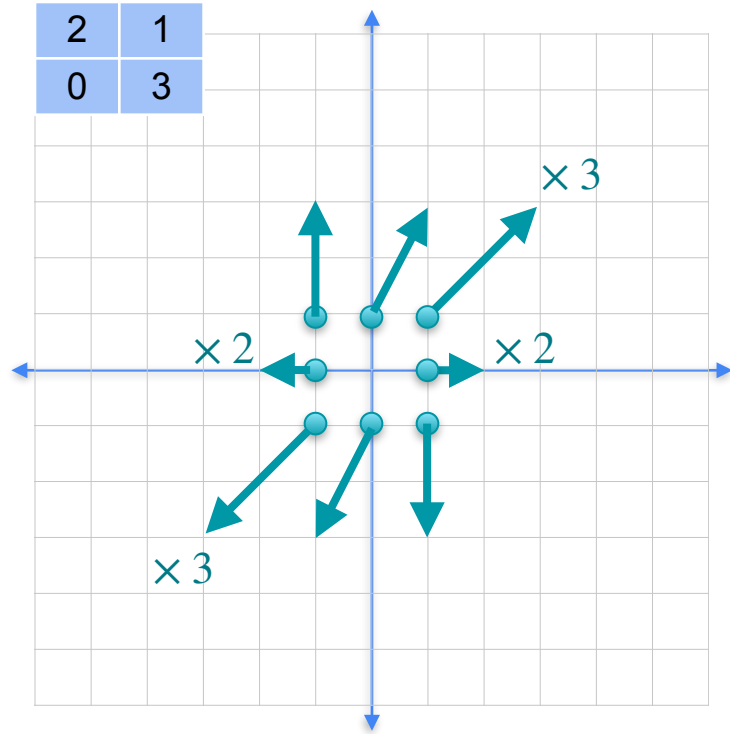
# Finding eigenvalues



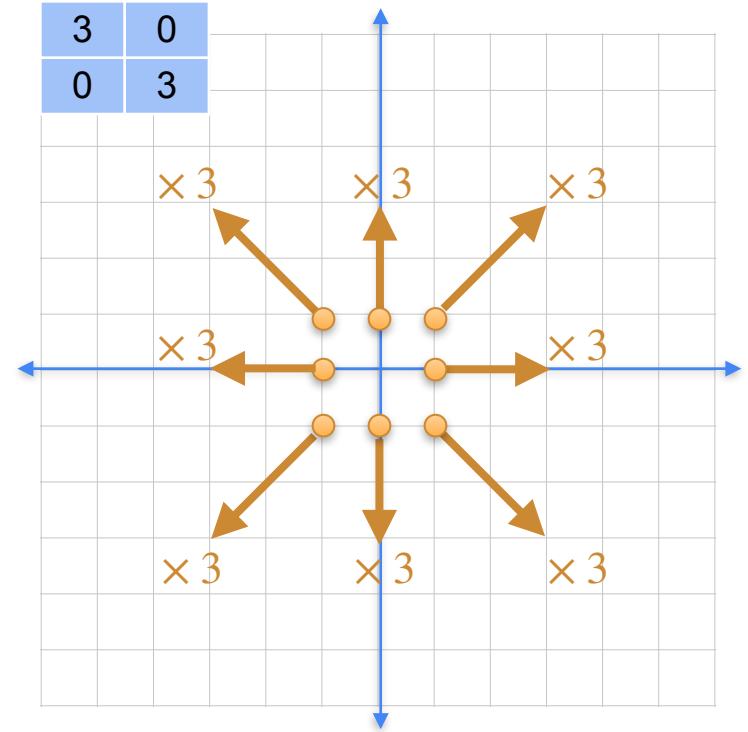
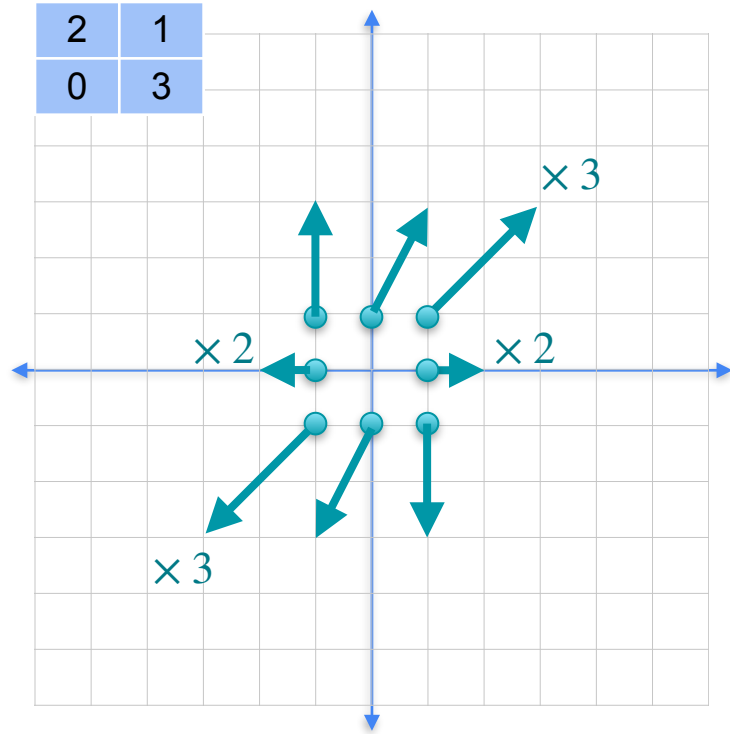
# Finding eigenvalues



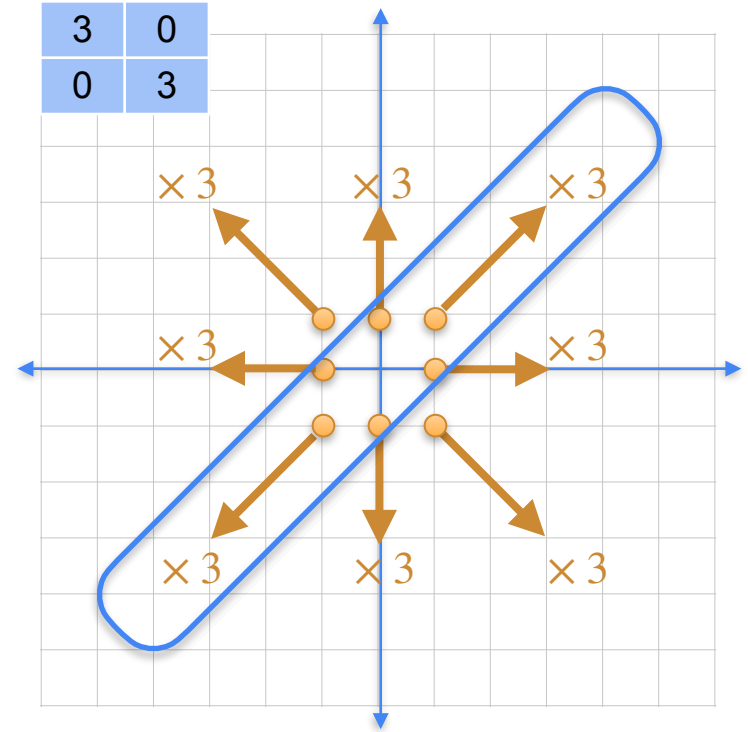
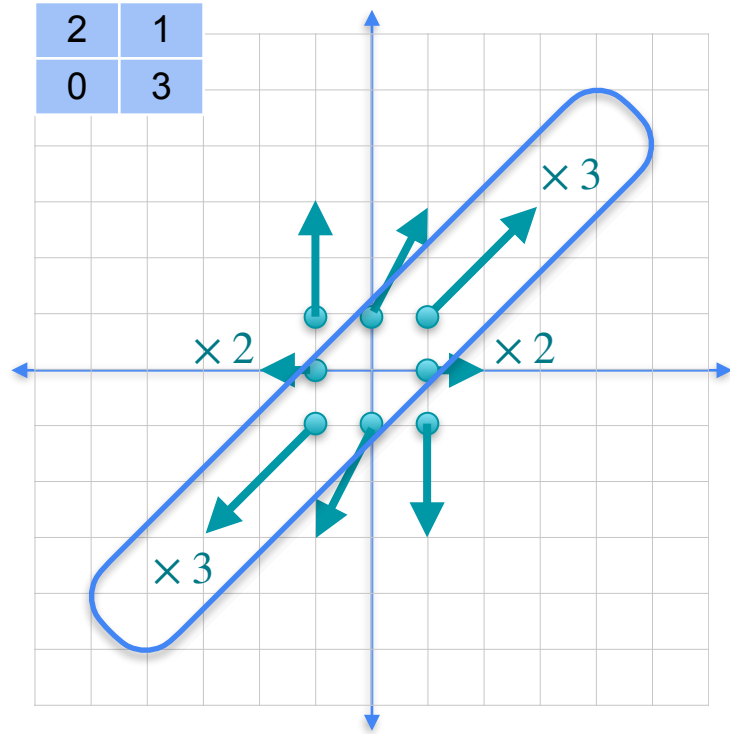
# Finding eigenvalues



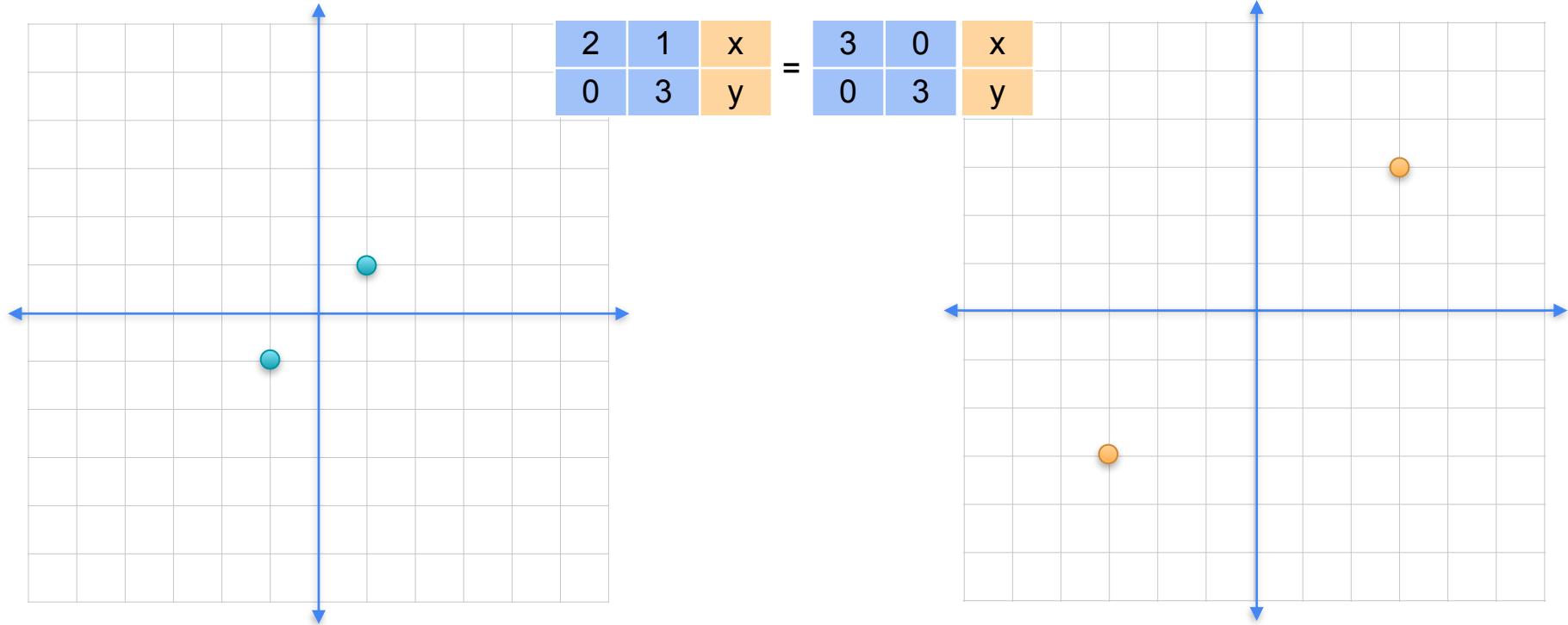
# Finding eigenvalues



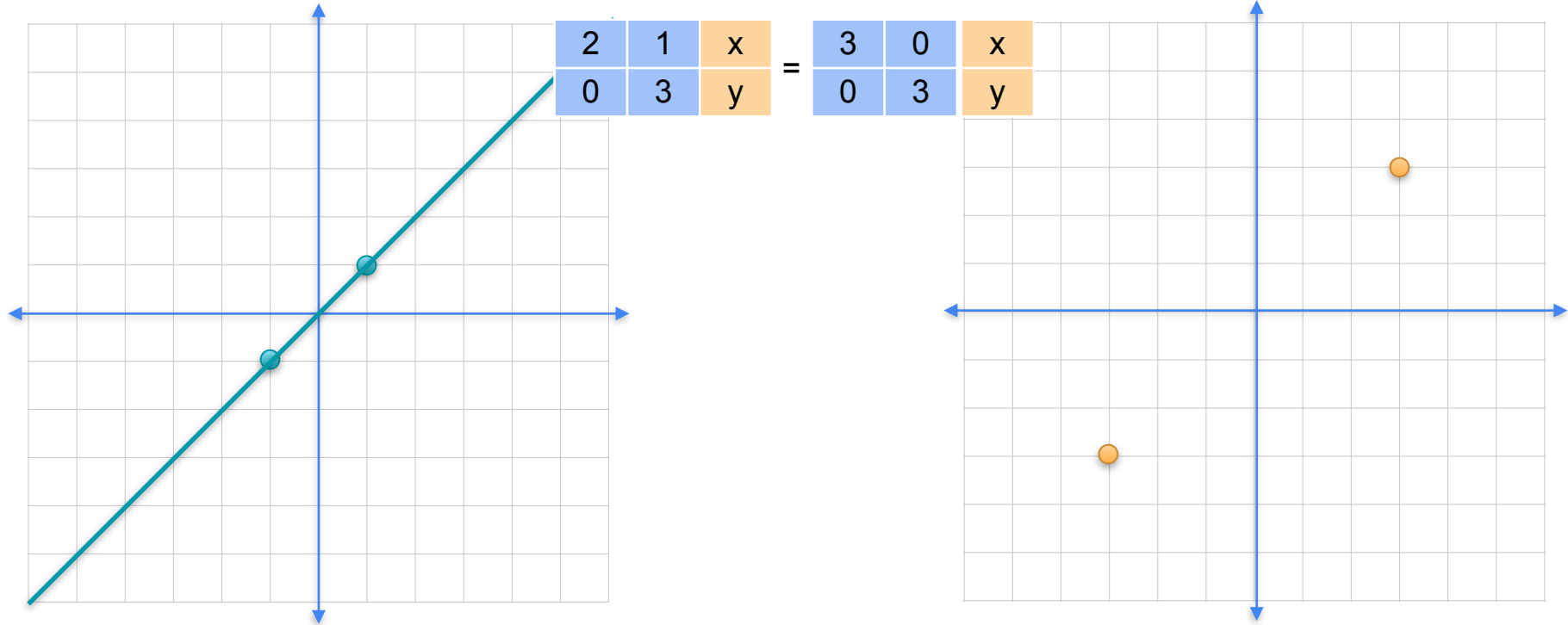
# Finding eigenvalues



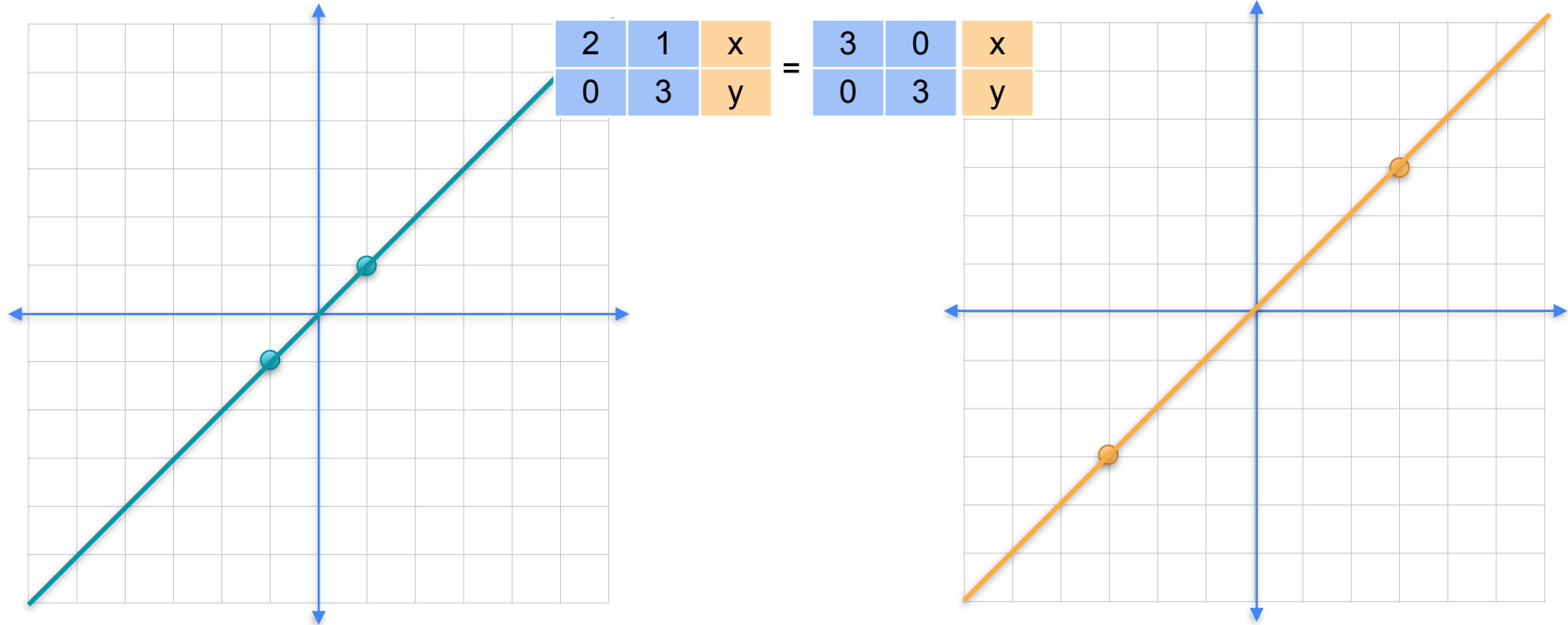
# Finding eigenvalues



# Finding eigenvalues

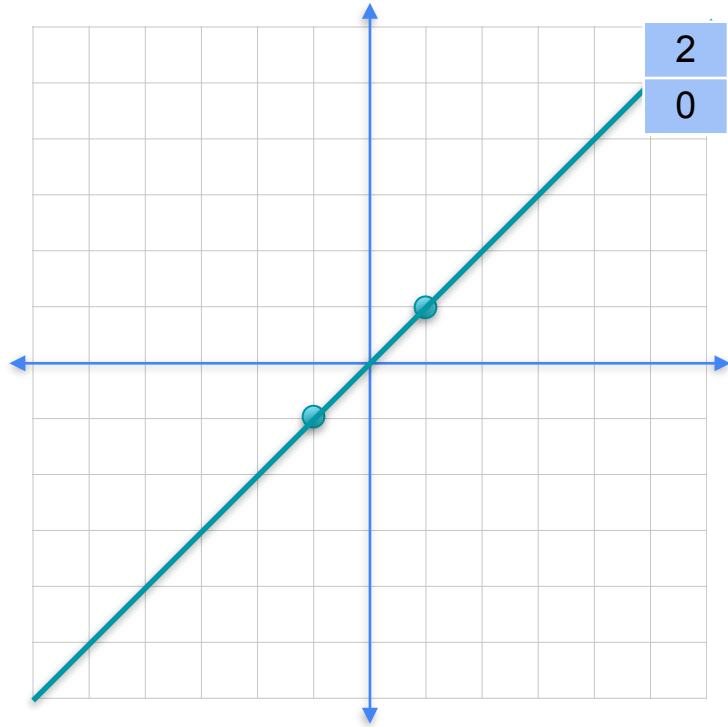


# Finding eigenvalues



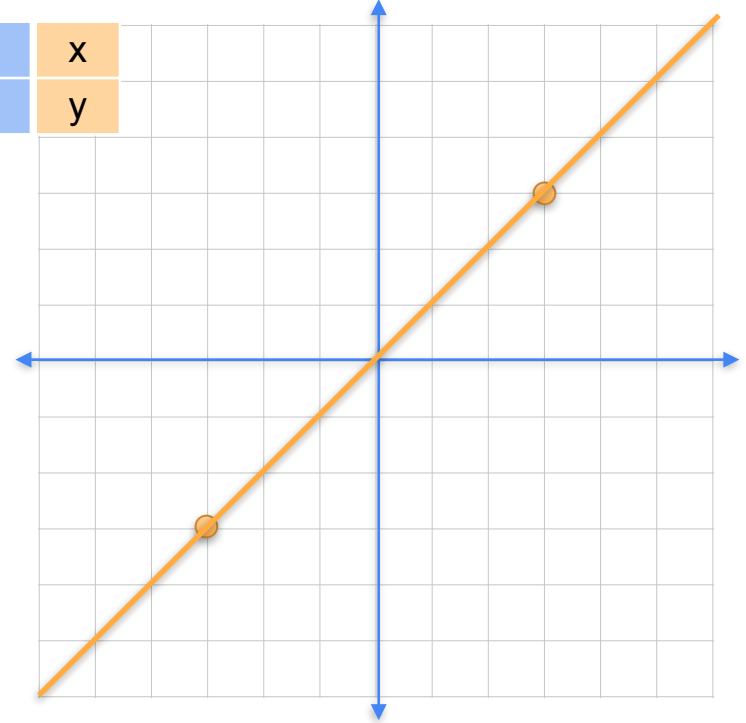


# Finding eigenvalues

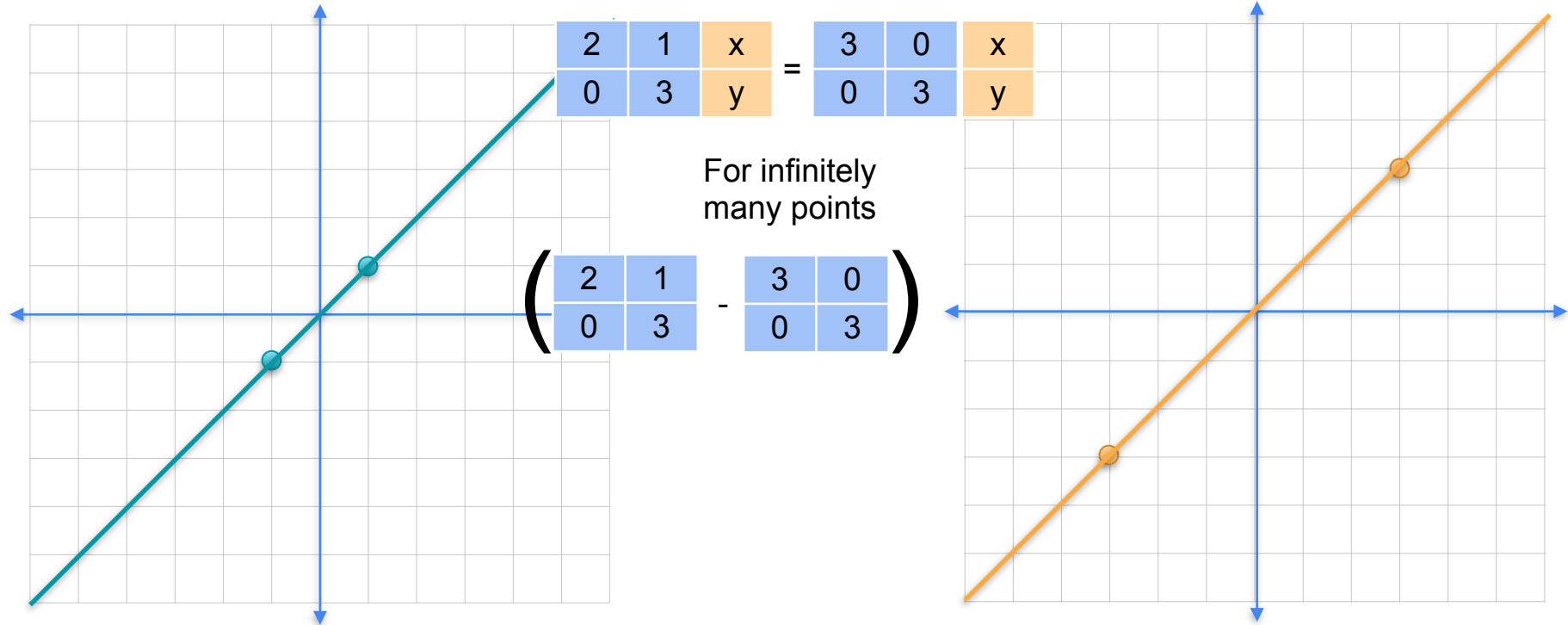


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

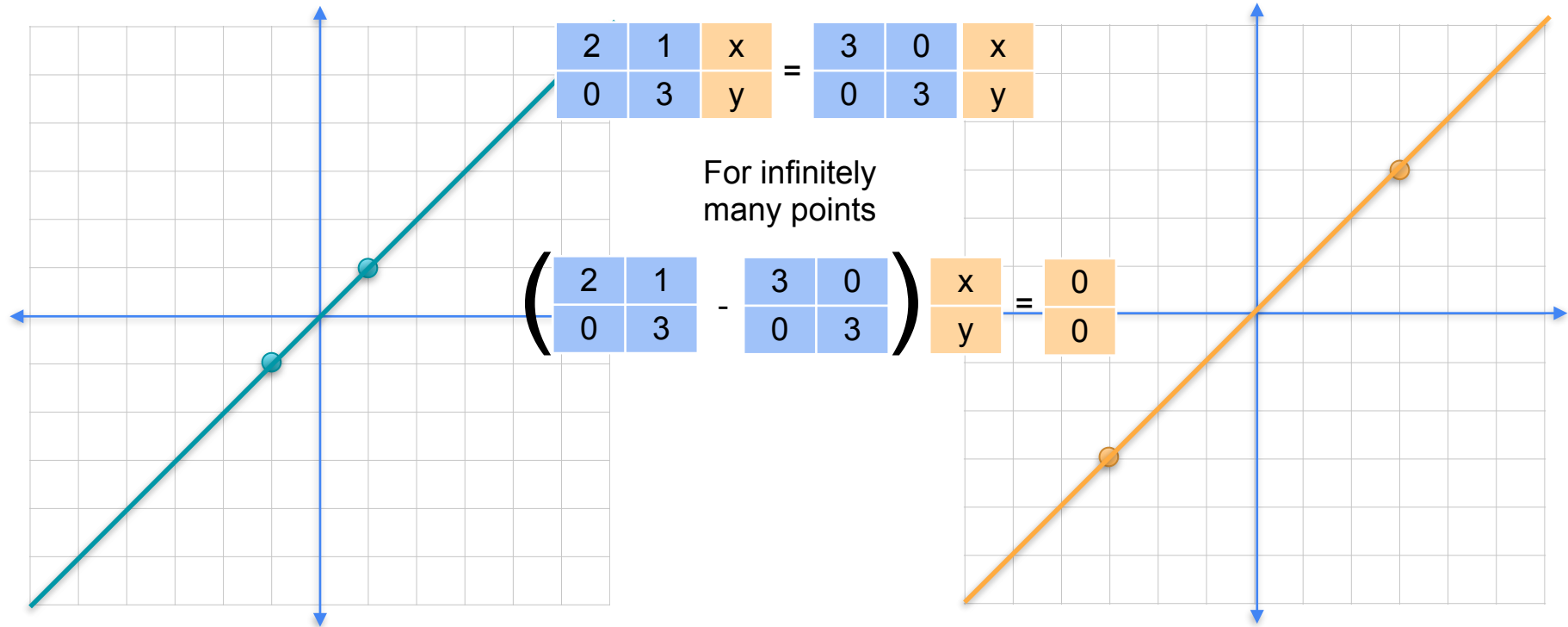
For infinitely many points



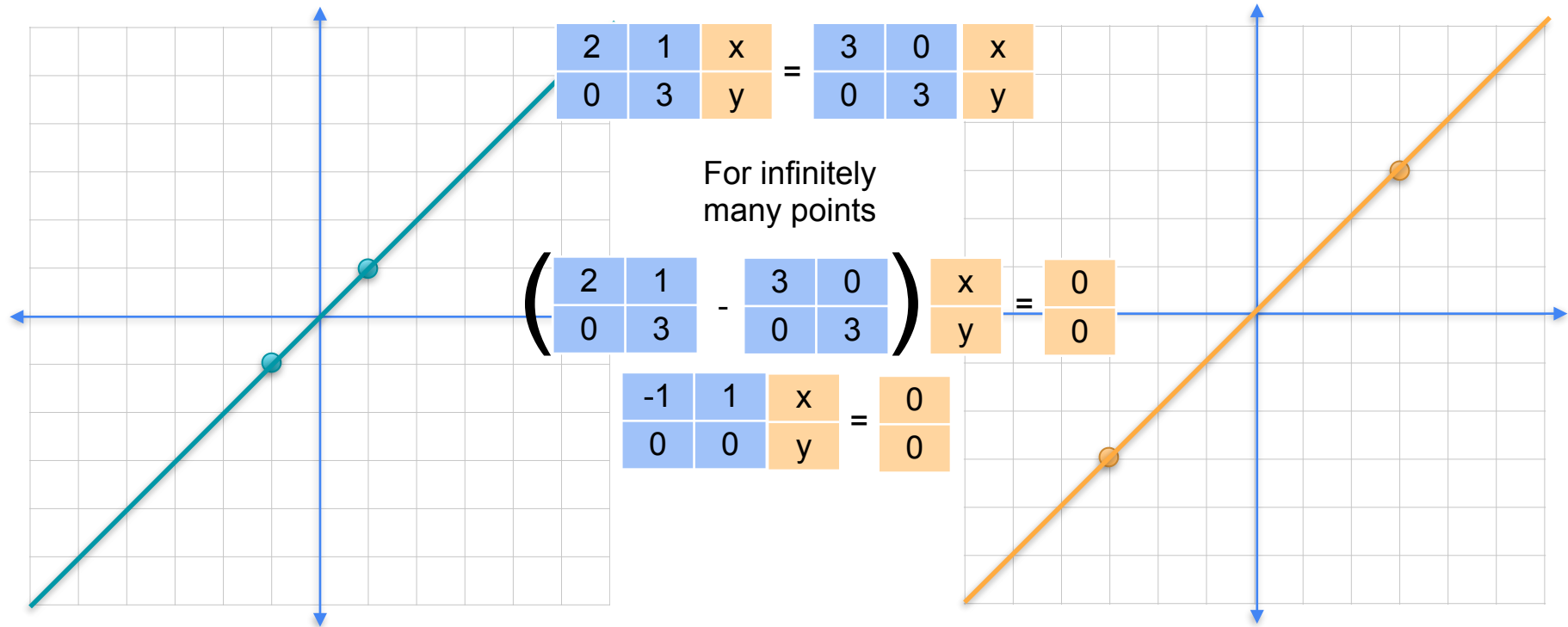
# Finding eigenvalues



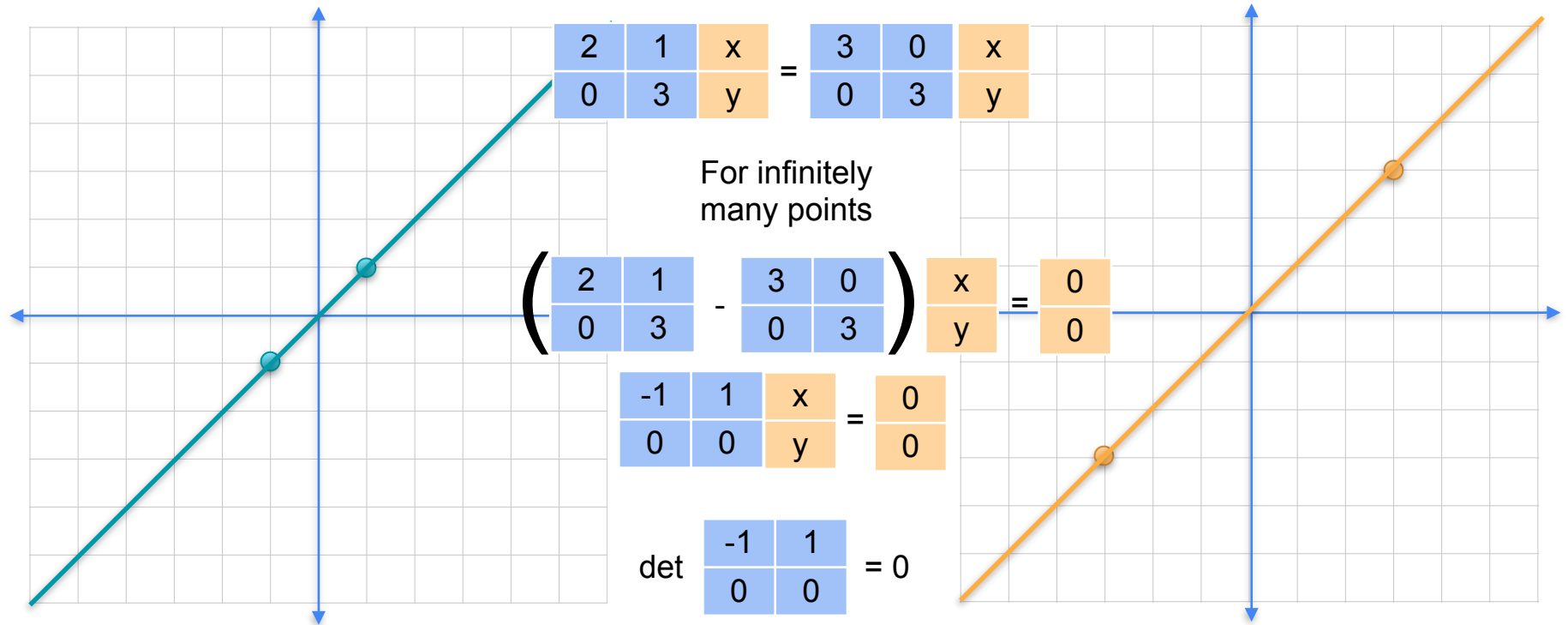
# Finding eigenvalues



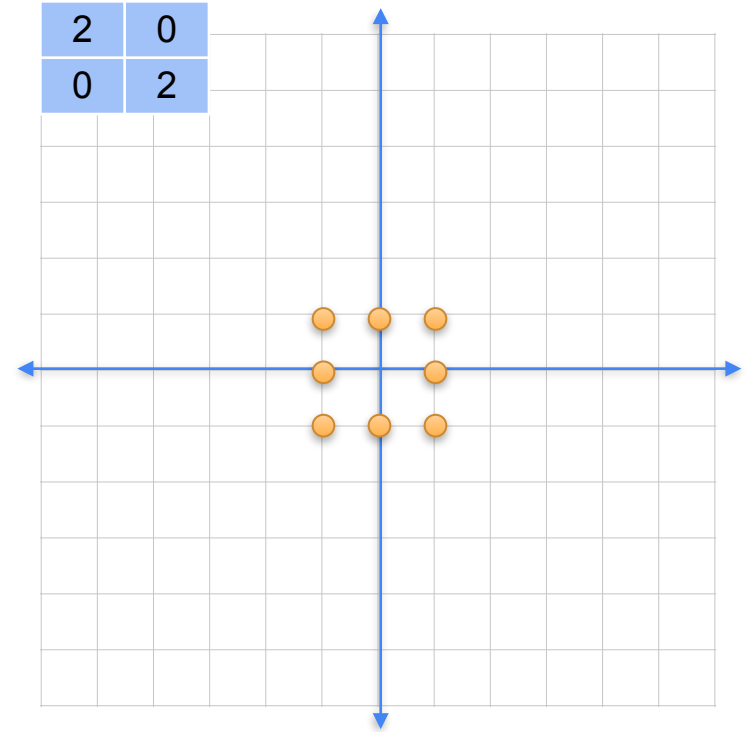
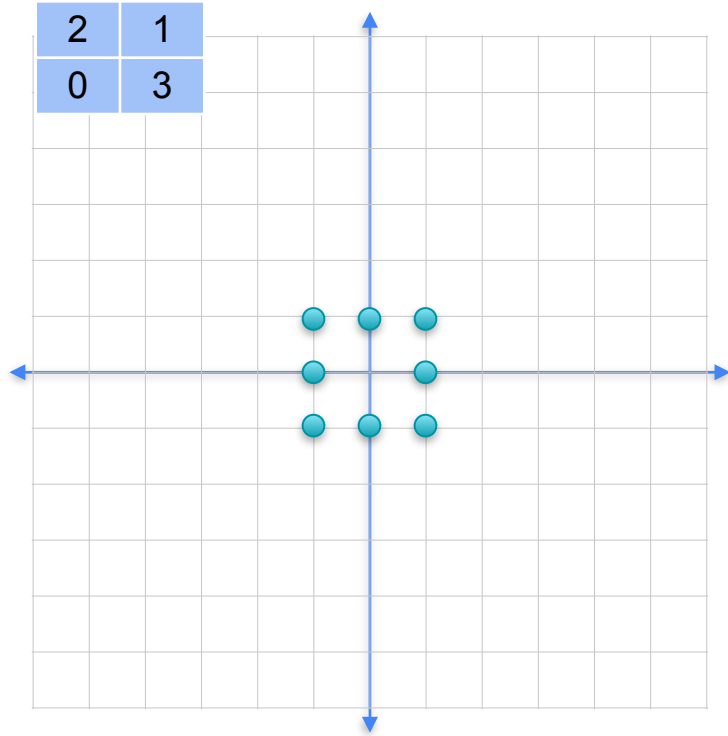
# Finding eigenvalues



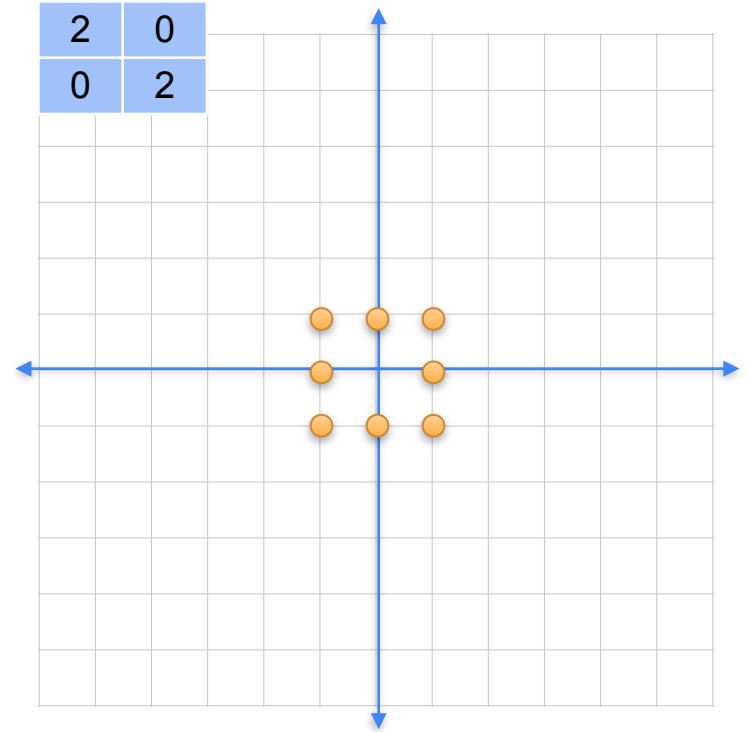
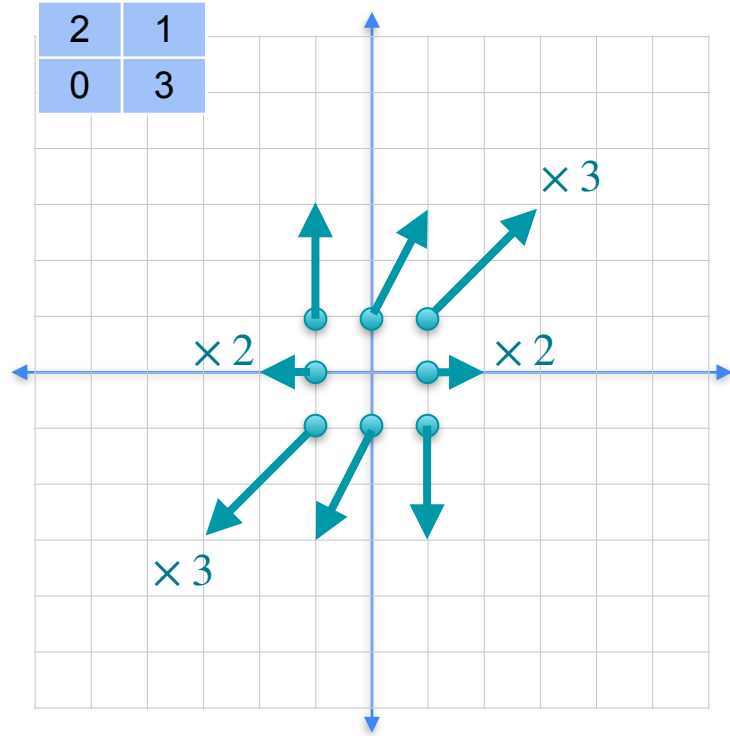
# Finding eigenvalues



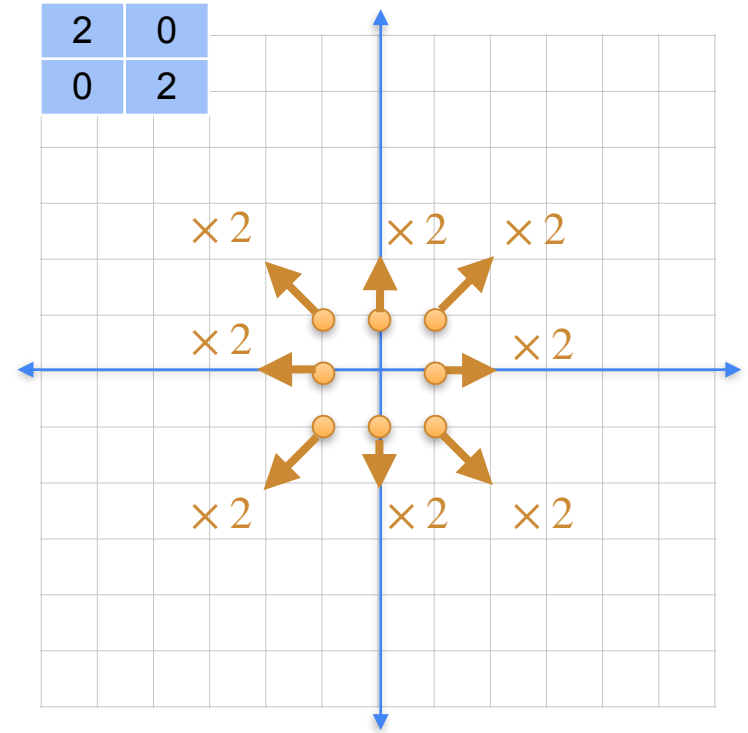
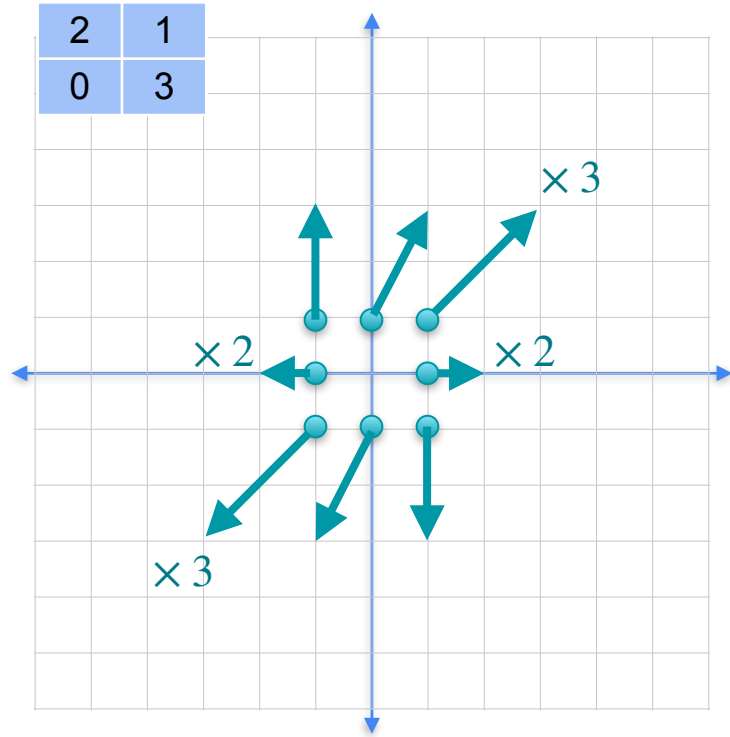
# Finding eigenvalues



# Finding eigenvalues

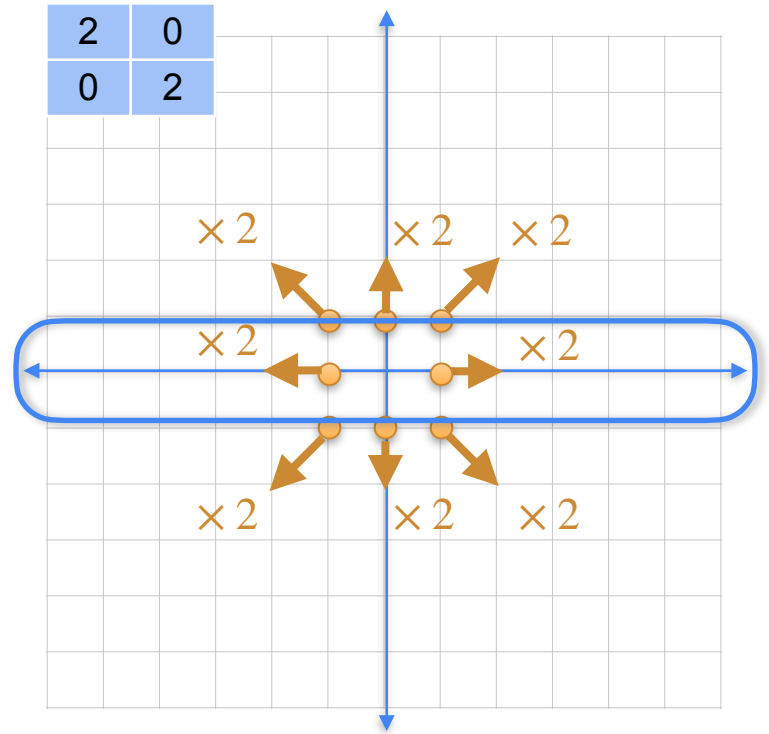
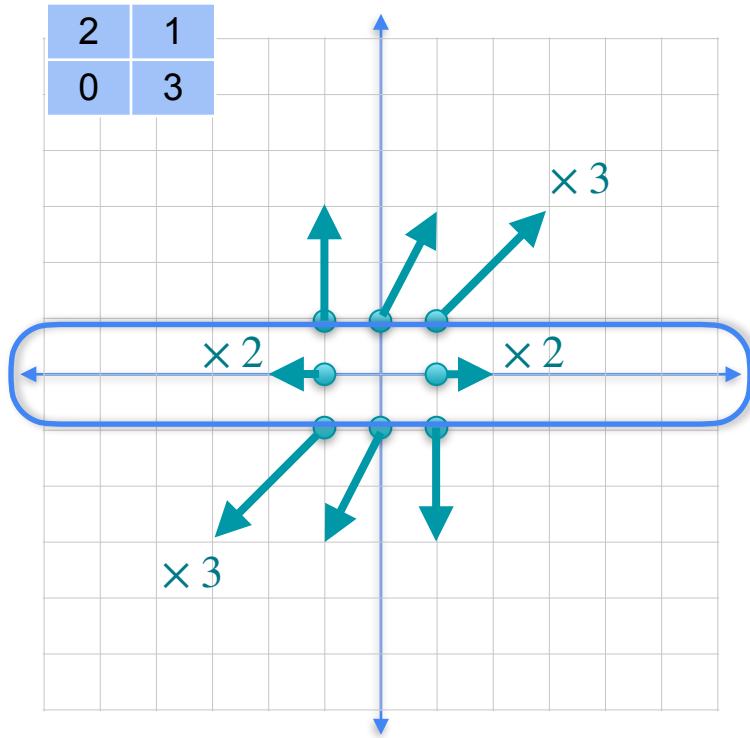


# Finding eigenvalues

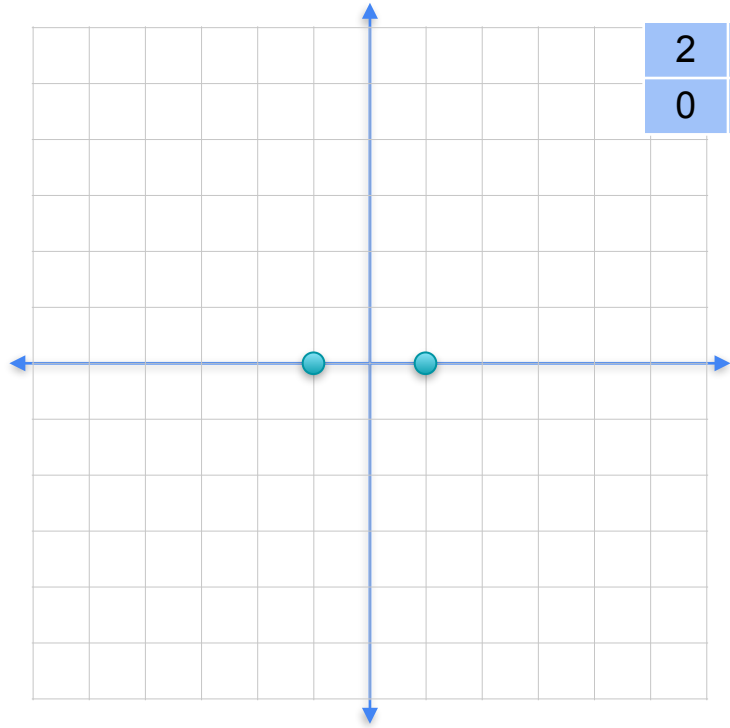




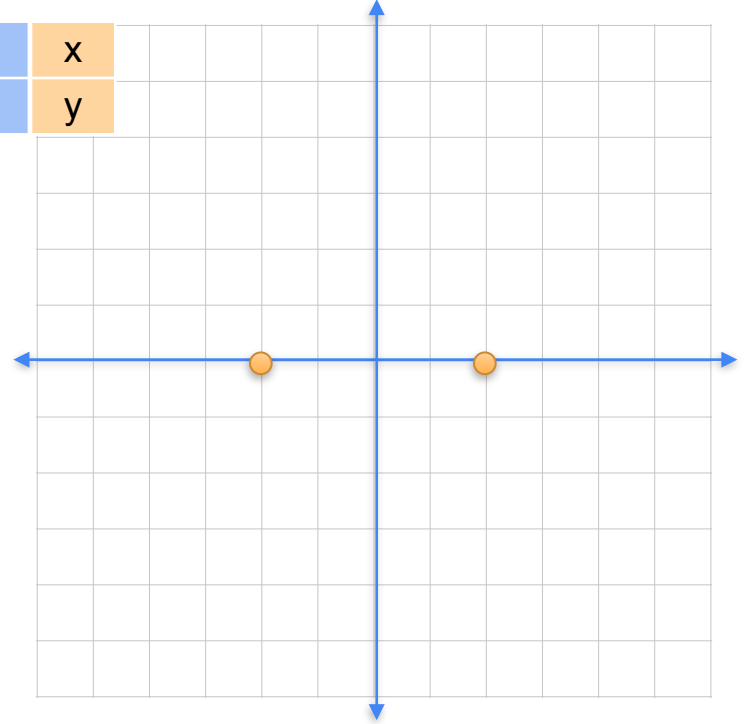
# Finding eigenvalues



# Finding eigenvalues

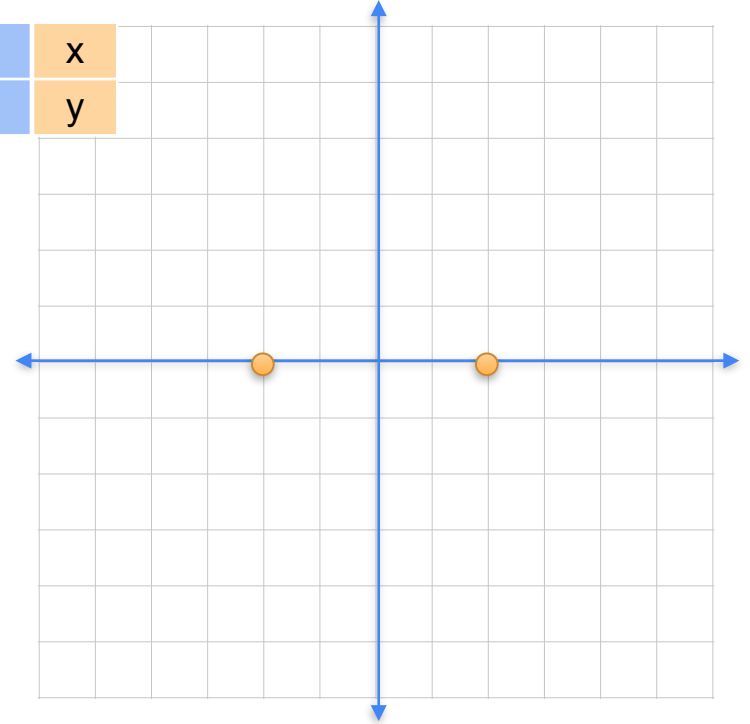
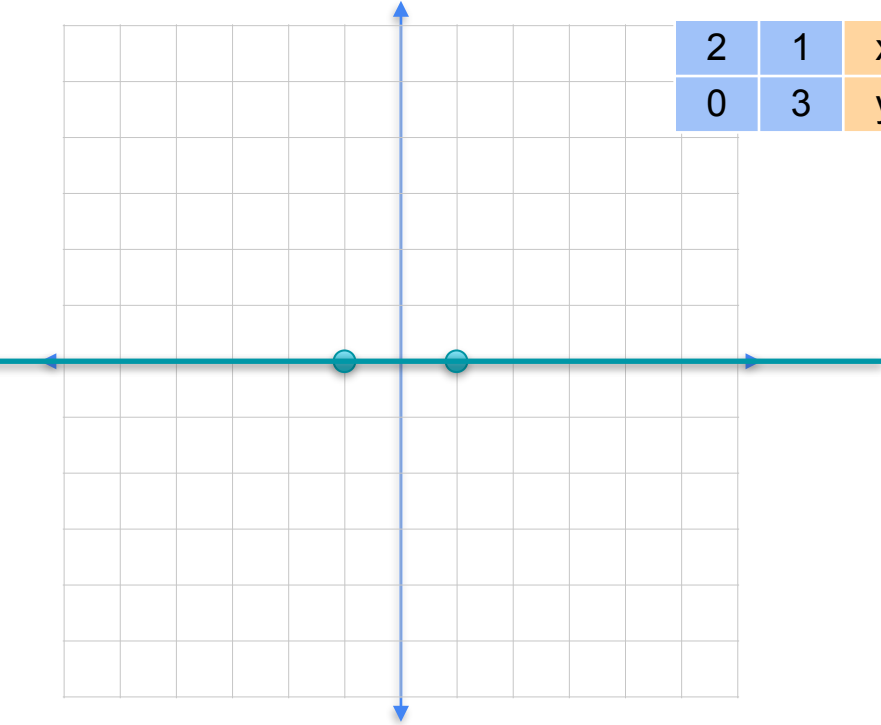


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



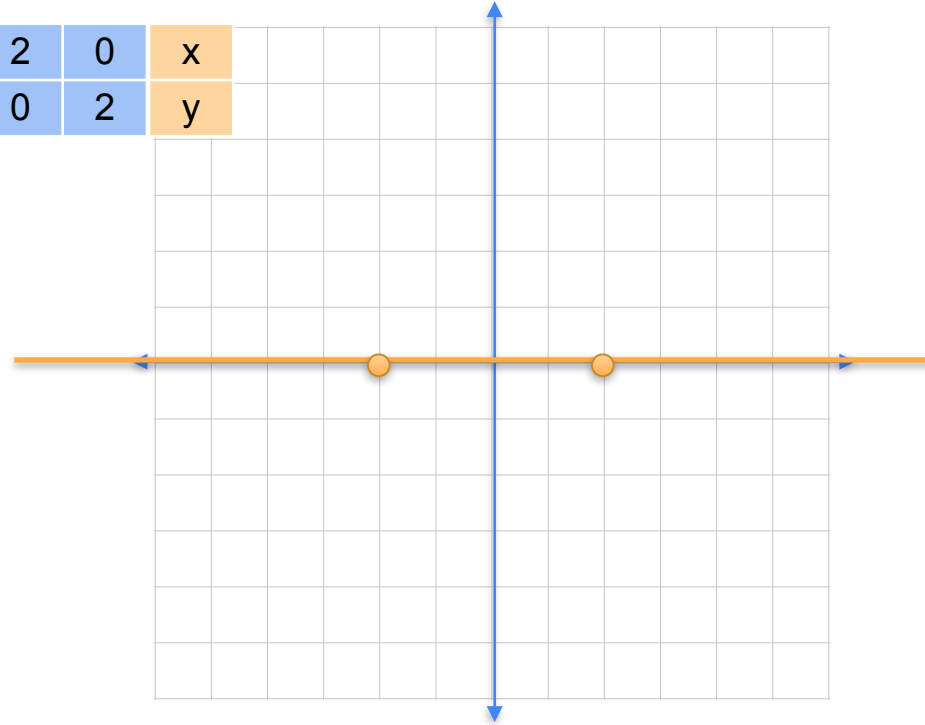
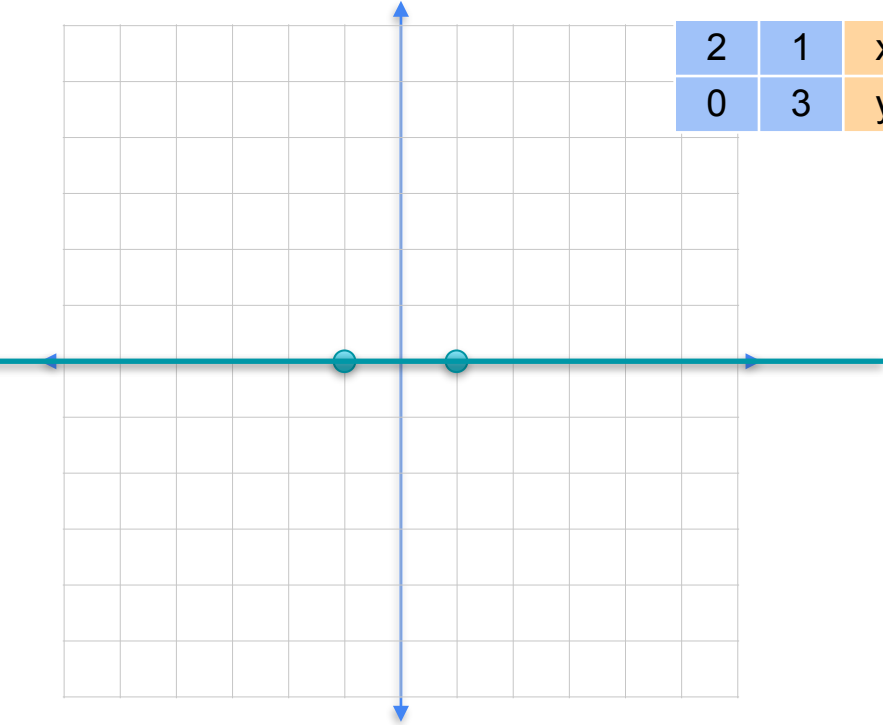
# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



# Finding eigenvalues

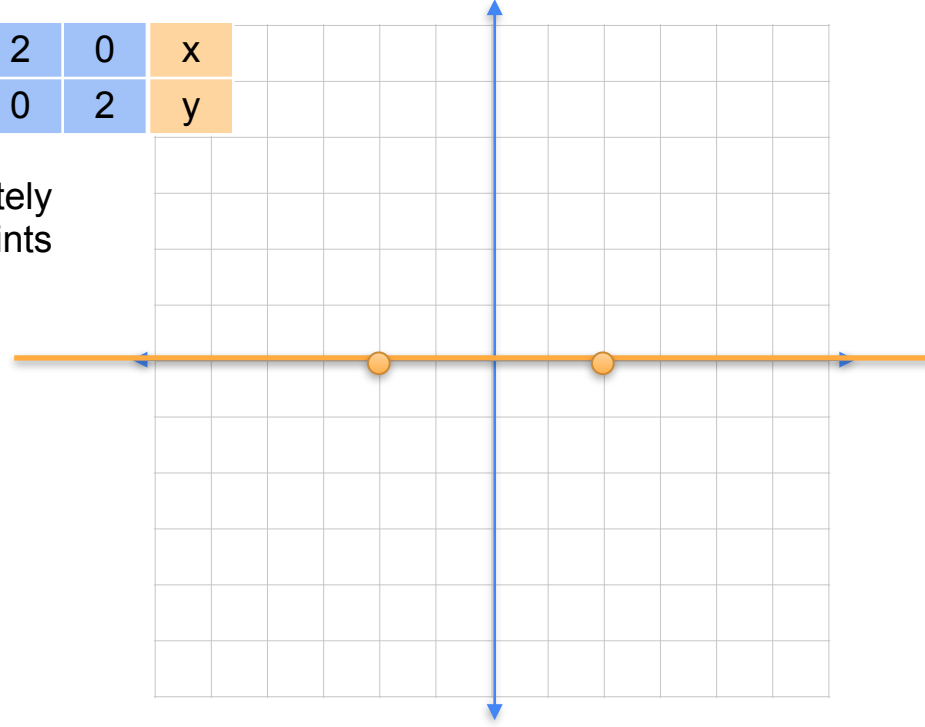
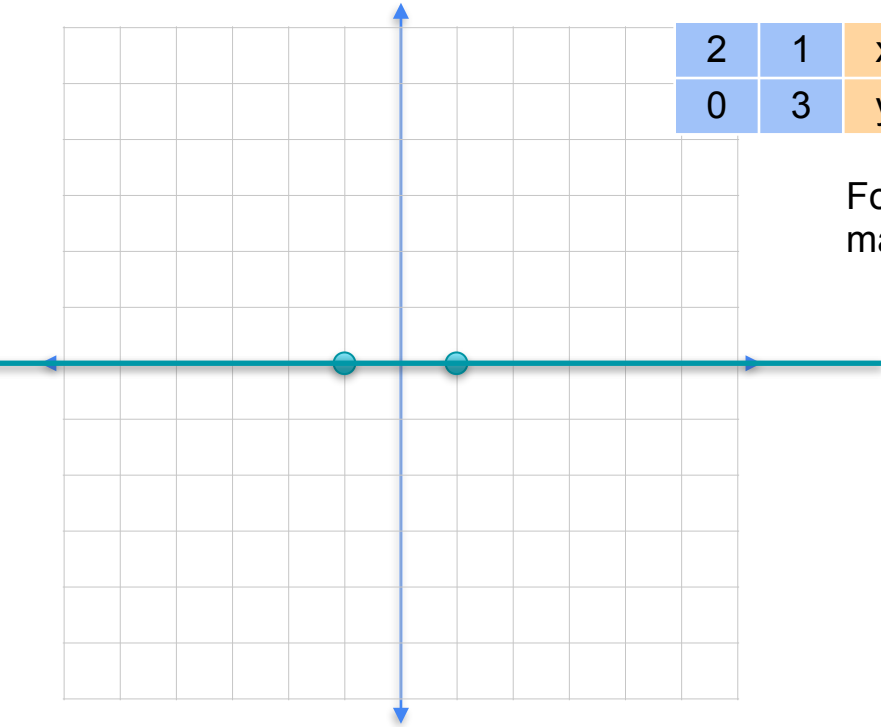
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely  
many points

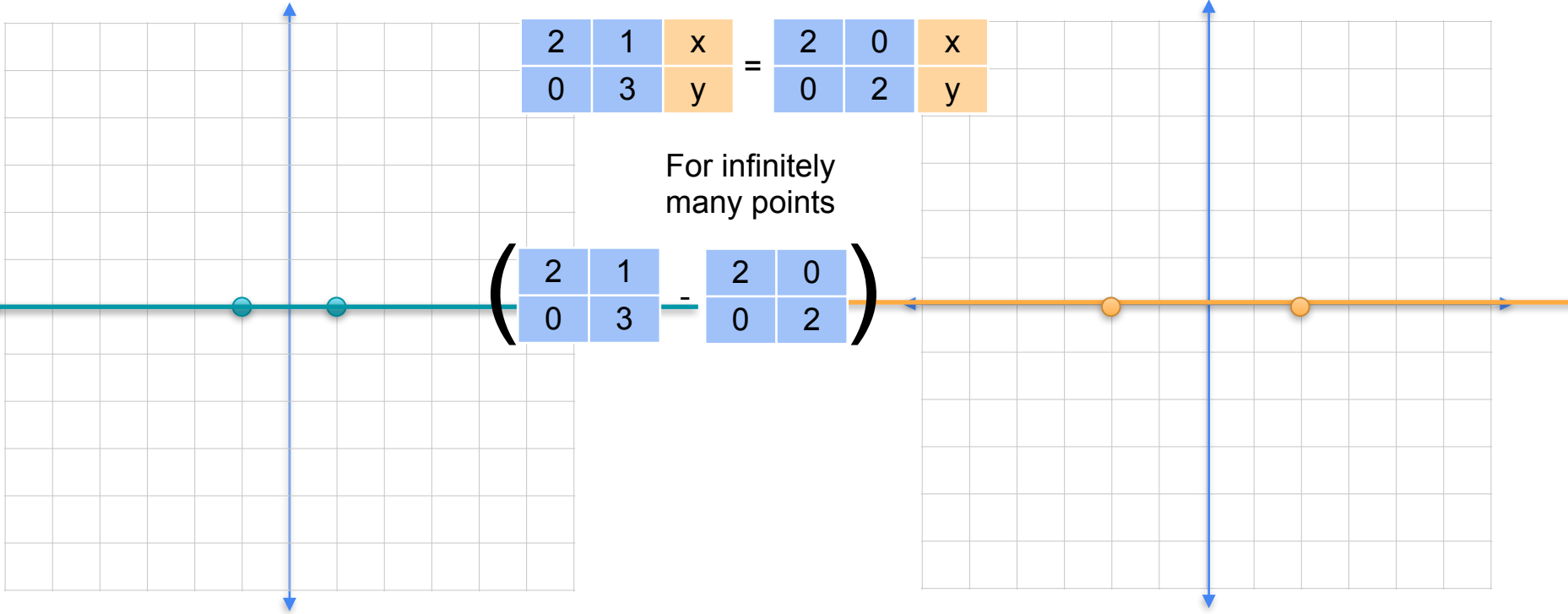


# Finding eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For infinitely  
many points

$$\left( \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right)$$

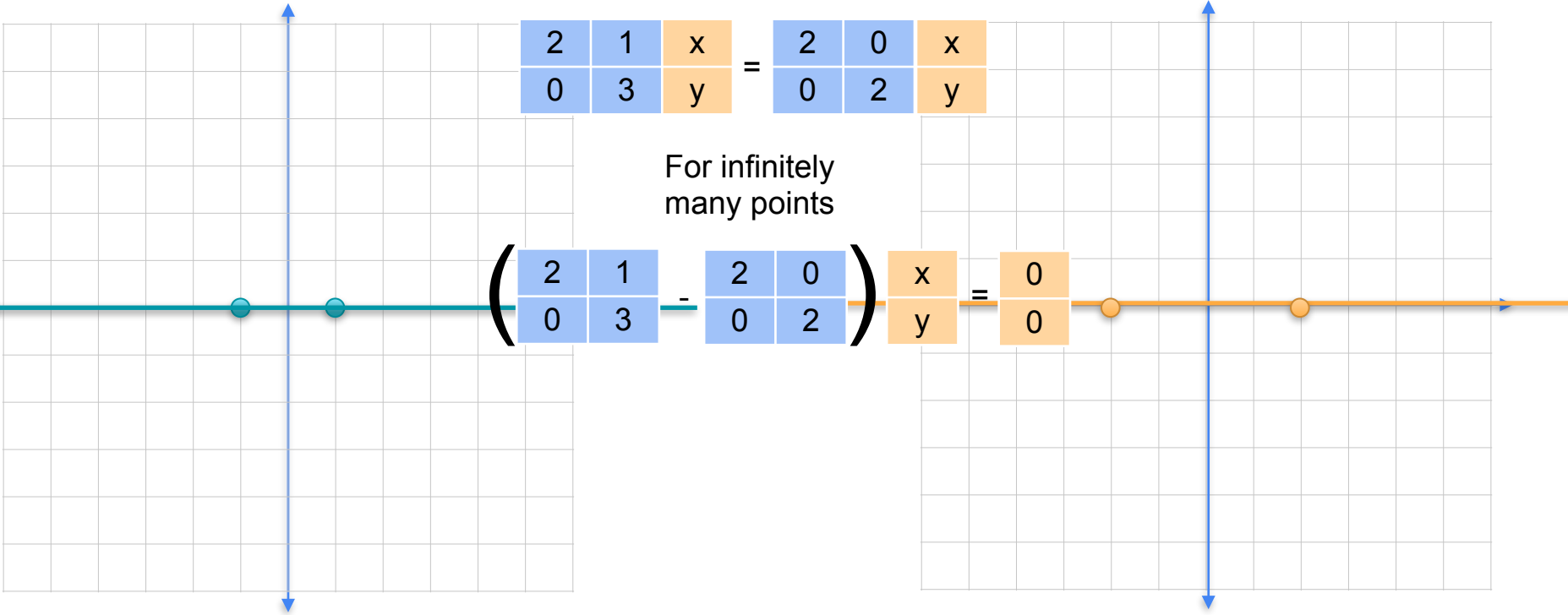


# Finding eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For infinitely  
many points

$$\left( \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



# Finding eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For infinitely many points

$$\left( \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



# Finding eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For infinitely many points

$$\left( \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0$$

# Finding eigenvalues

2	1
0	3

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

2	1
0	3

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

2	1
0	3

$\lambda$	0
0	$\lambda$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \lambda & 0 & x \\ \hline 0 & \lambda & y \\ \hline \end{array}$$

For infinitely many (x,y)

$$\begin{array}{|c|c|c|} \hline 2-\lambda & 1 & x \\ \hline 0 & 3-\lambda & y \\ \hline \end{array} = \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array}$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \lambda & 0 & x \\ \hline 0 & \lambda & y \\ \hline \end{array}$$

For infinitely many (x,y)

$$\begin{array}{|c|c|c|} \hline 2-\lambda & 1 & x \\ \hline 0 & 3-\lambda & y \\ \hline \end{array} = \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline \end{array}$$

Has infinitely many solutions



# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\begin{aligned} \lambda &= 2 \\ \lambda &= 3 \end{aligned}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$



# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$x = 1$$

$$0x + 3y = 2y$$

$$y = 0$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

$$x = 1$$

$$y = 1$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

$$x = 1$$

$$y = 1$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

# Quiz

- Find the eigenvalues and eigenvectors of this matrix:

9	4
4	3

# Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

9	4
4	3

- The characteristic polynomial is

$$\det \begin{array}{|c|c|} \hline 9-\lambda & 4 \\ \hline 4 & 3-\lambda \\ \hline \end{array} = (9-\lambda)(3-\lambda) - 4 \cdot 4 = 0$$

- Which factors as  $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$

- The solutions are  $\lambda = 11$   
 $\lambda = 1$





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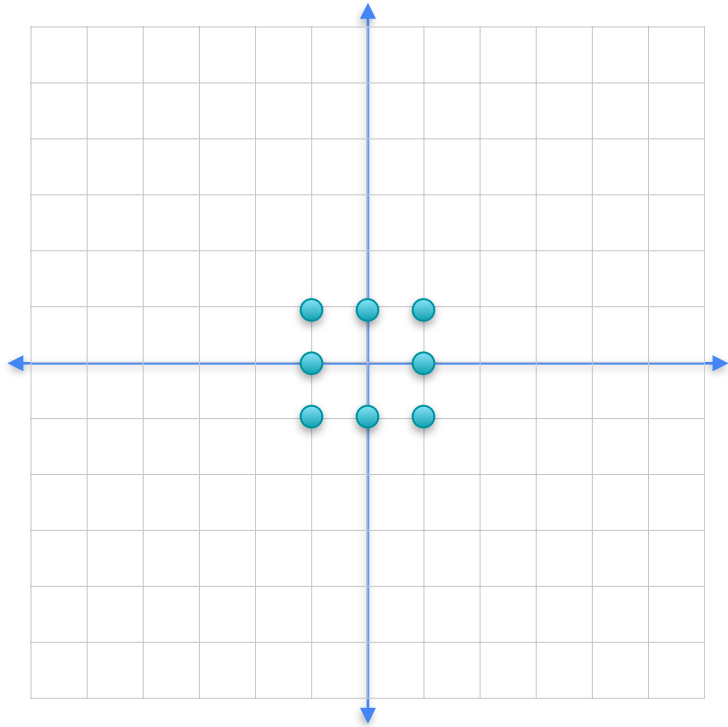
# Determinants and Eigenvectors

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## **Conclusion**

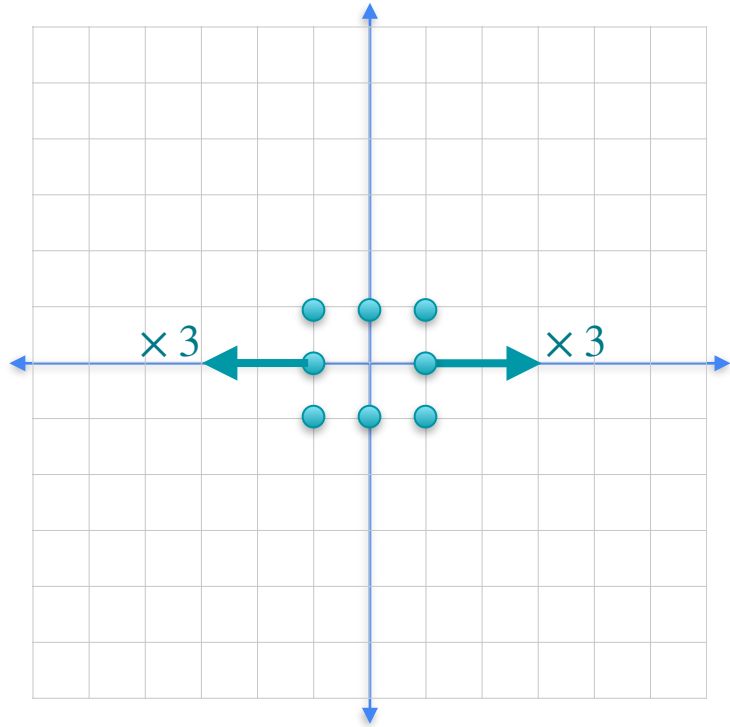
# Find the eigenvalues

2	1
0	3



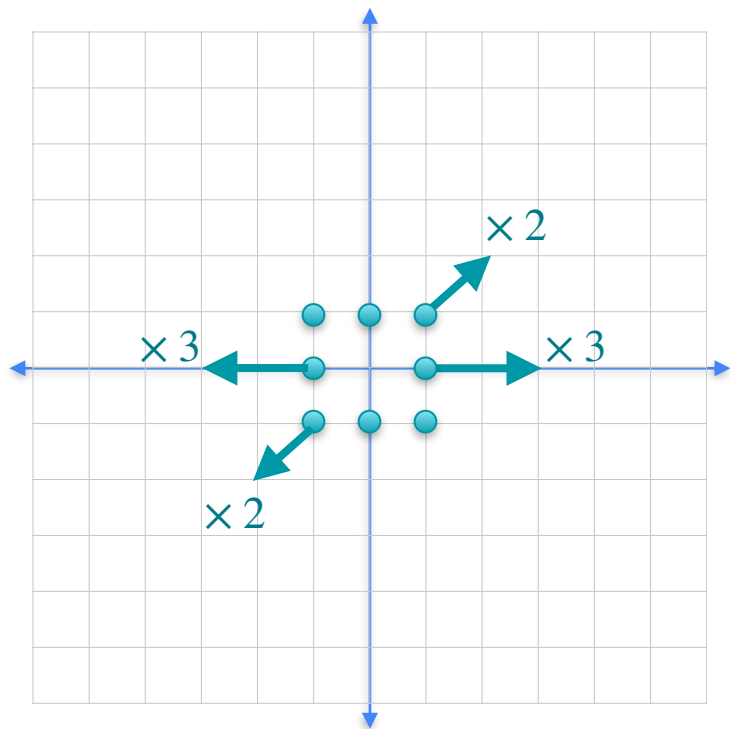
# Find eigenvalues

2	1
0	3



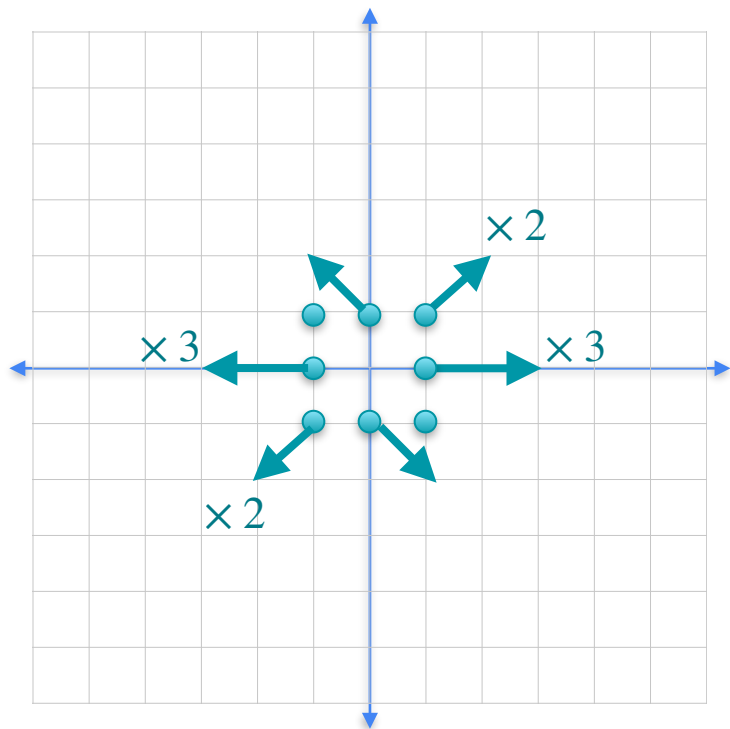
# Find eigenvalues

2	1
0	3



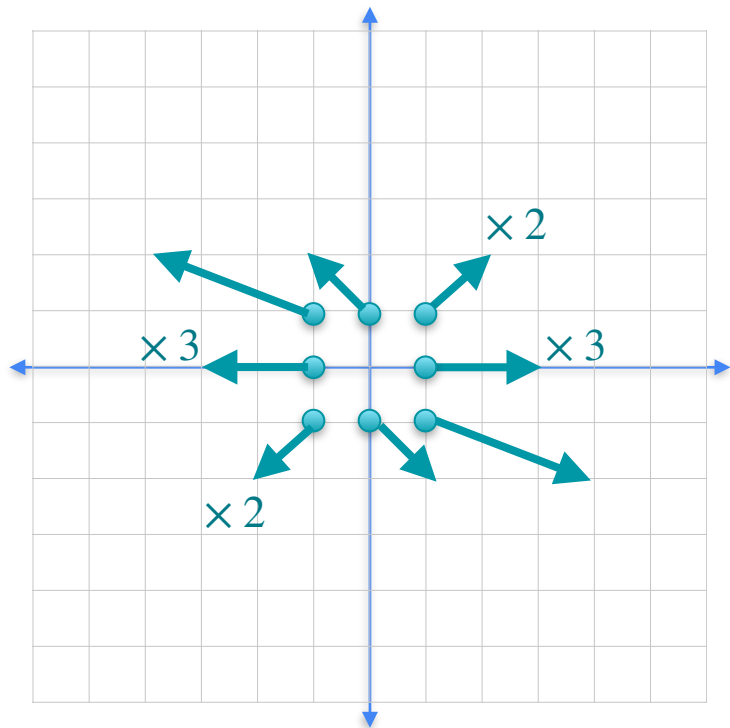
# Find eigenvalues

2	1
0	3



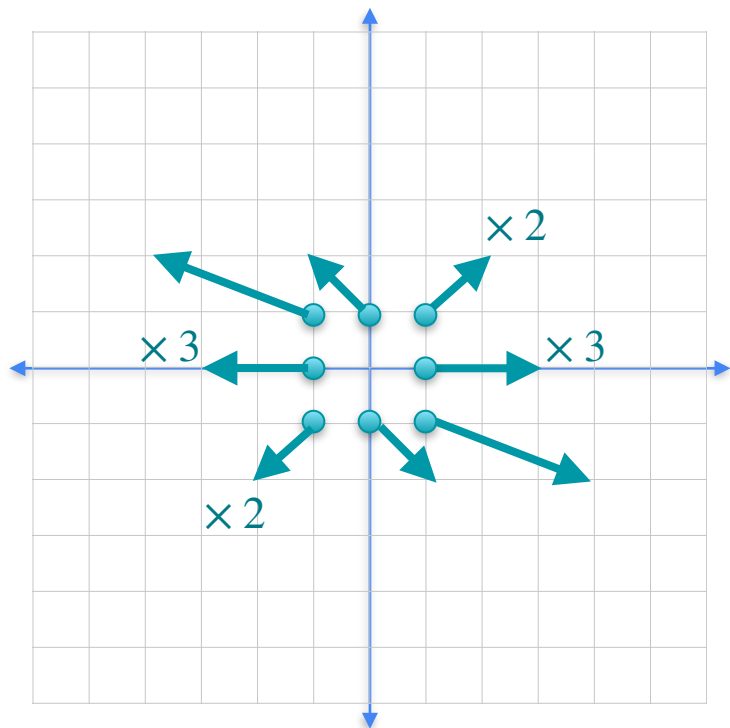
# Find eigenvalues

2	1
0	3

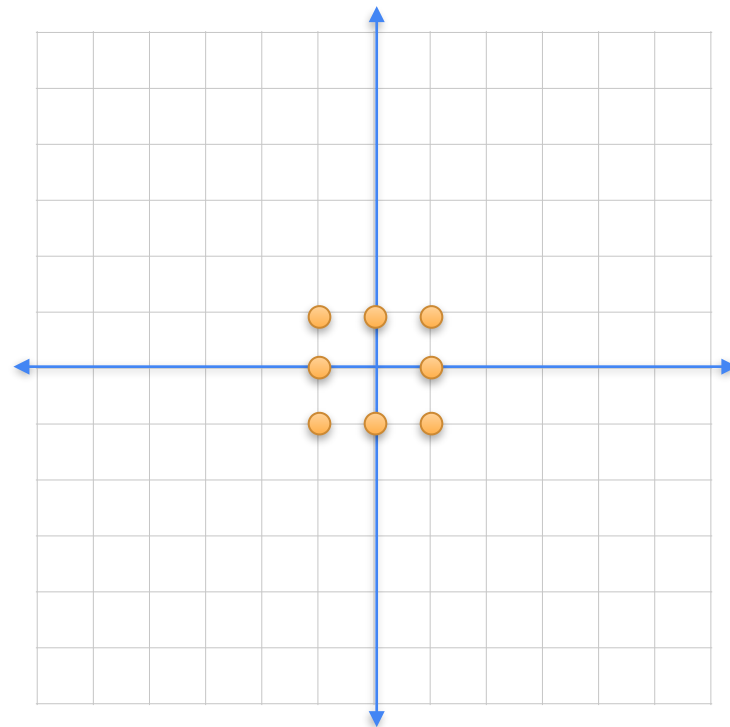


# Find eigenvalues

2	1
0	3

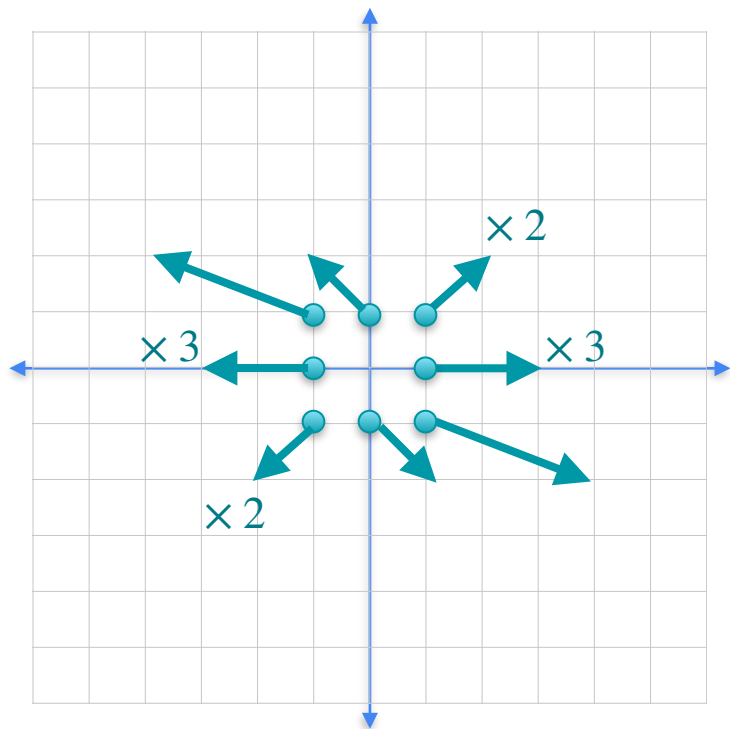


3	0
0	3

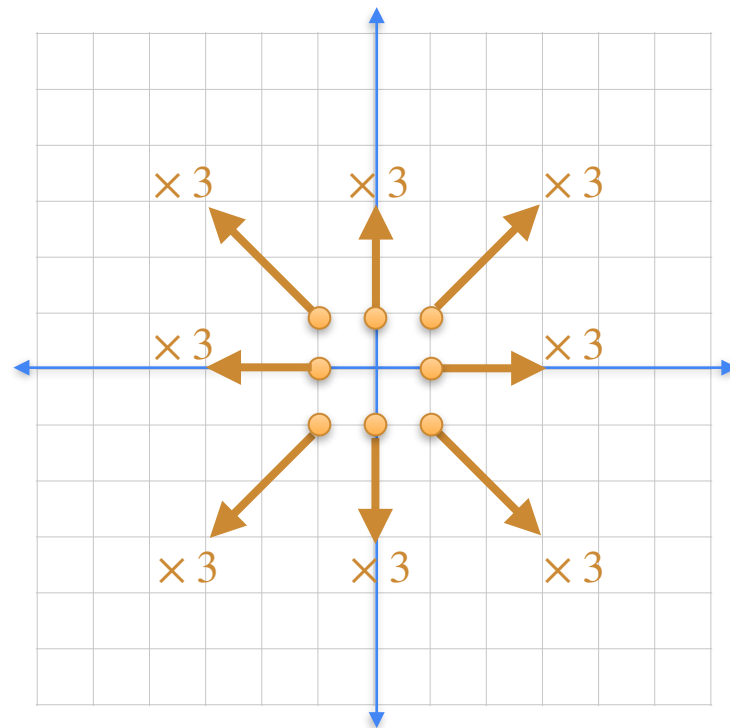


# Find eigenvalues

2	1
0	3



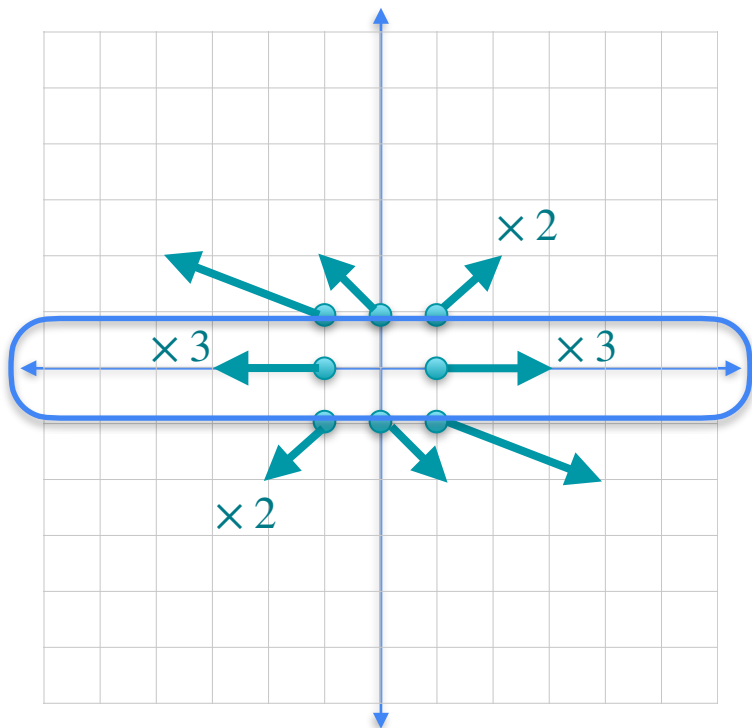
3	0
0	3



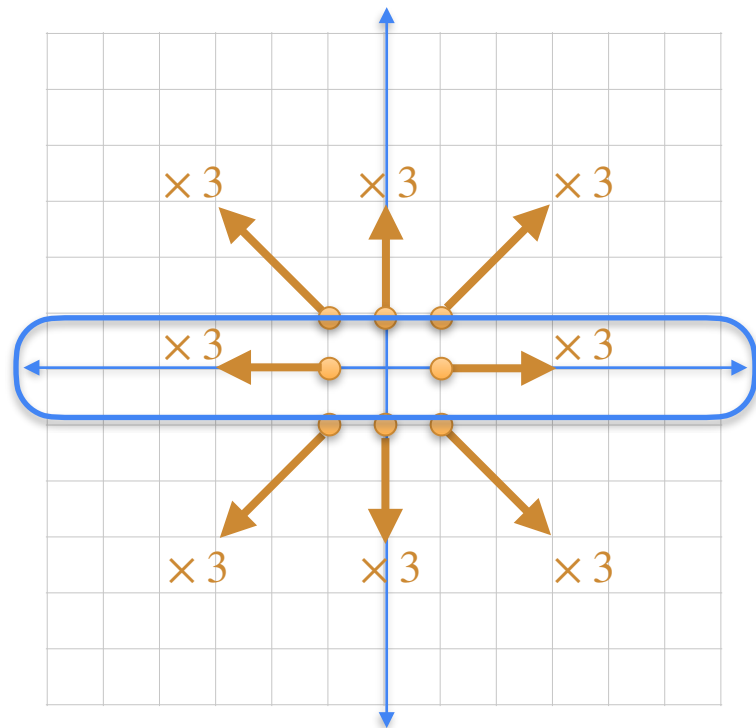


# Find eigenvalues

2	1
0	3

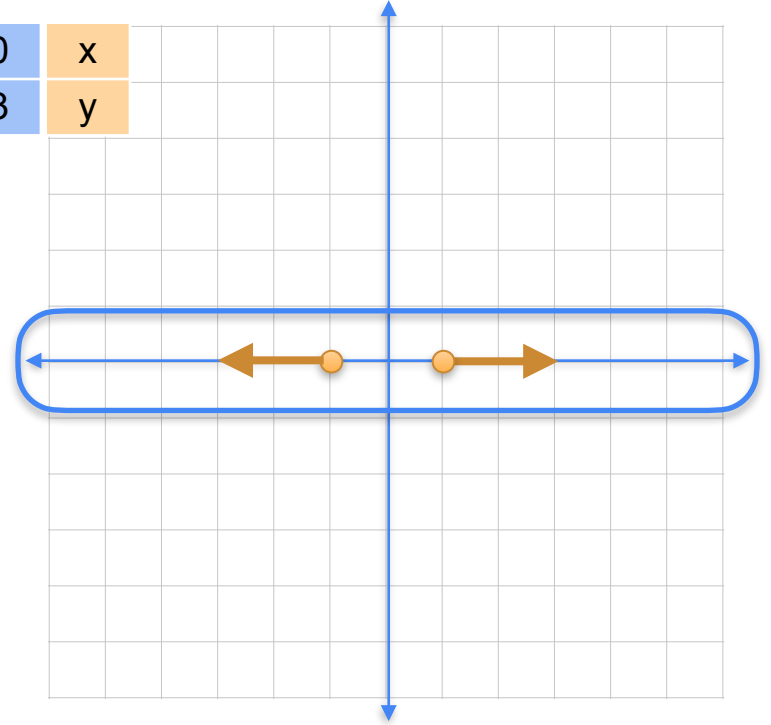
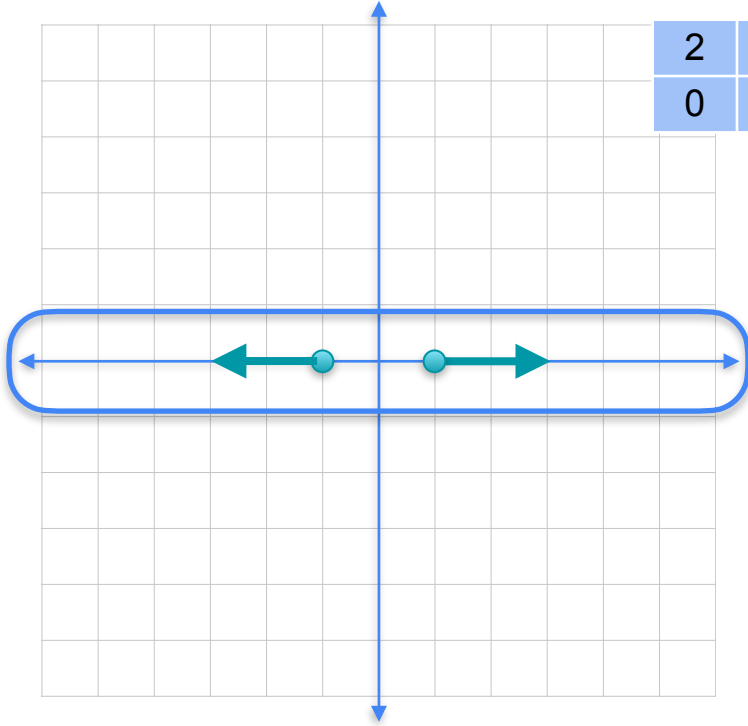


3	0
0	3



# Finding eigenvalues

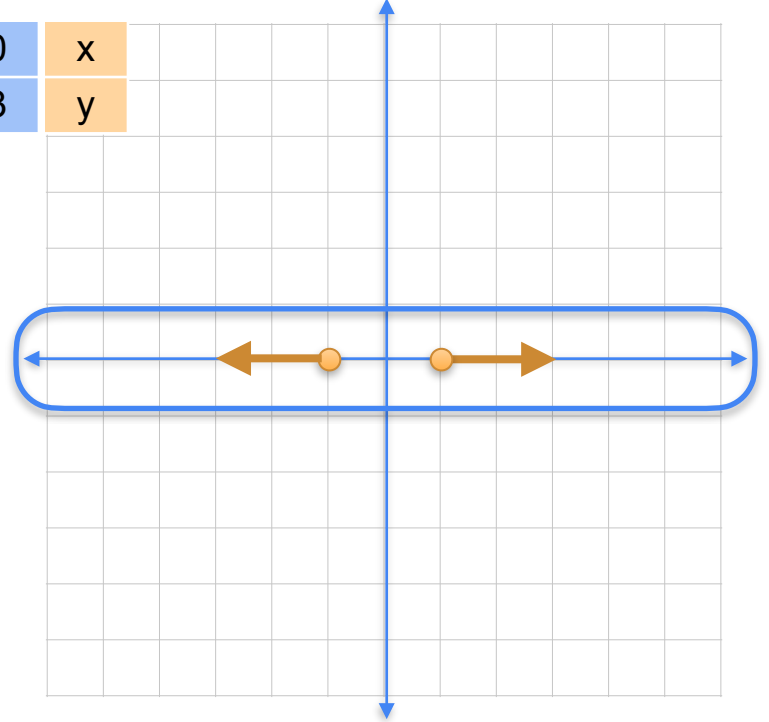
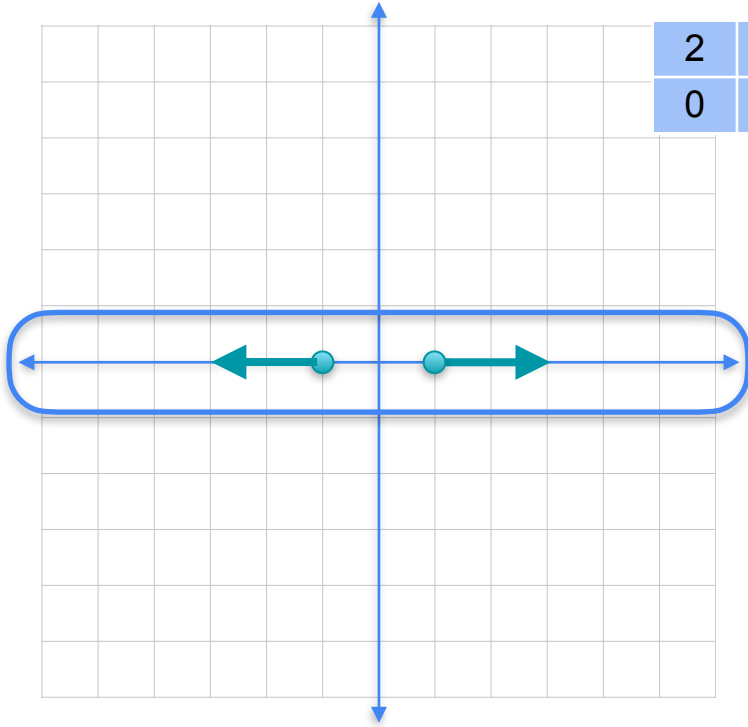
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

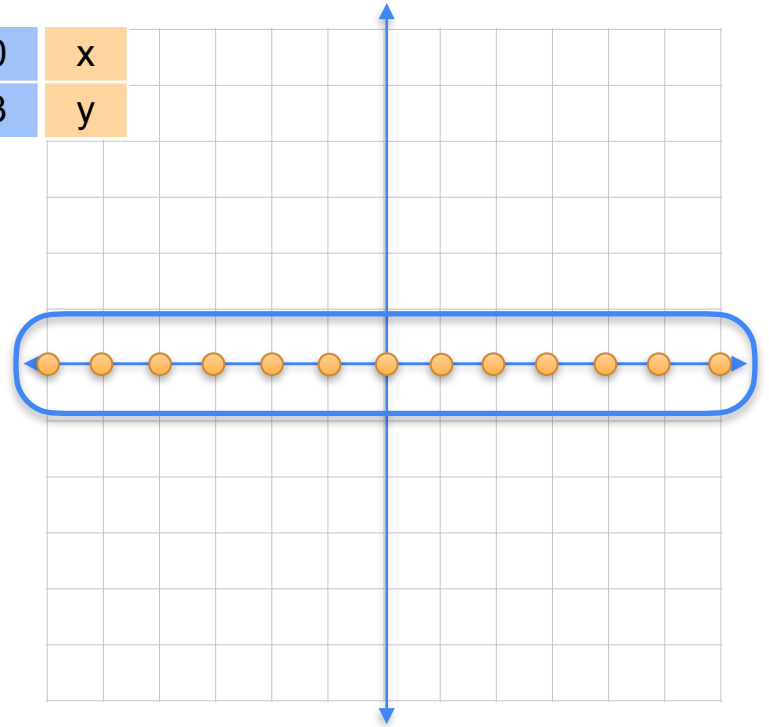
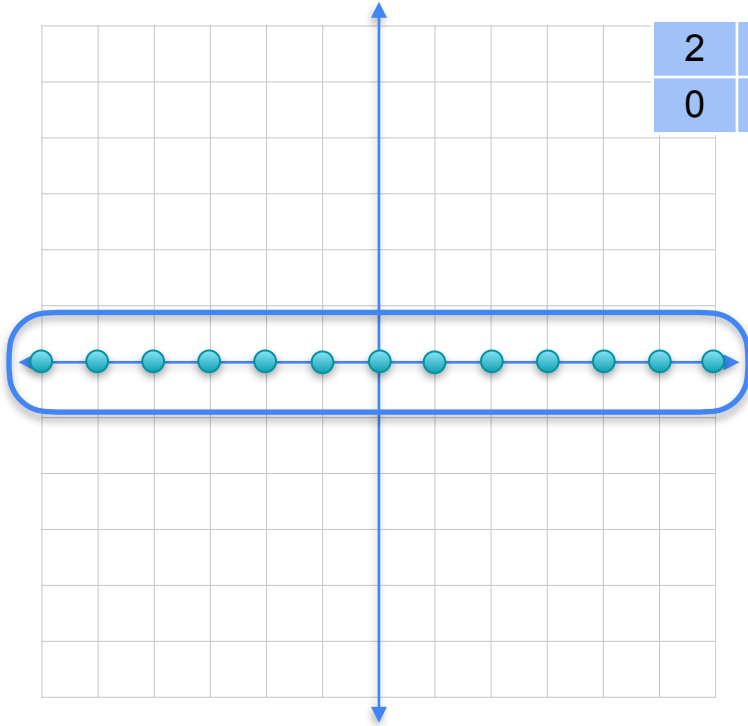
For infinitely many points



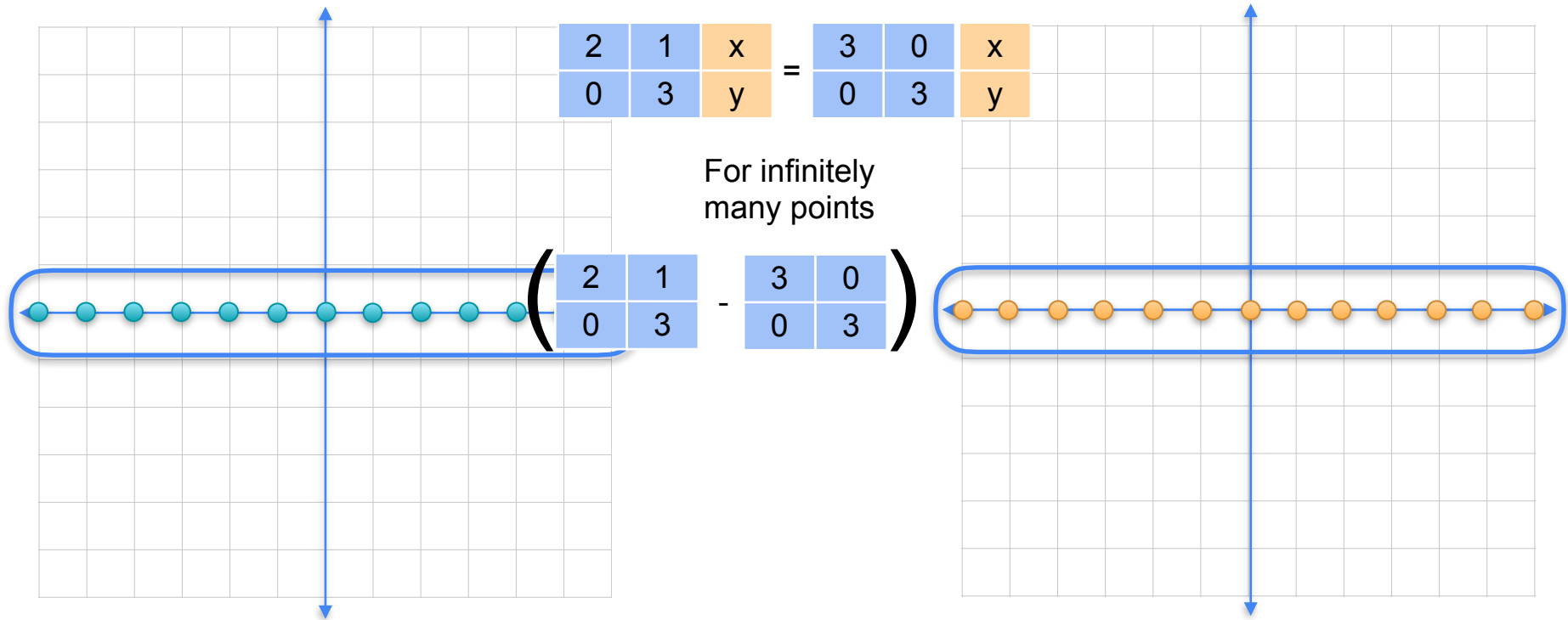
# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

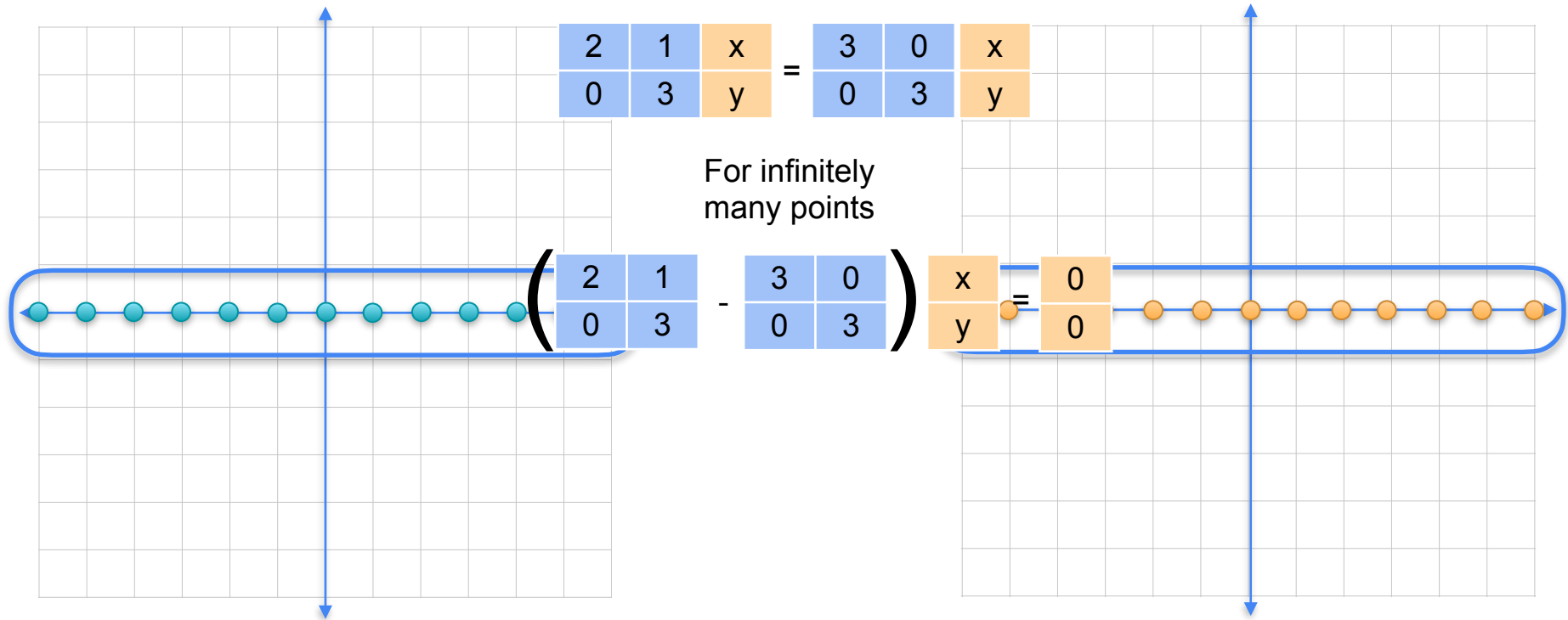
For infinitely  
many points



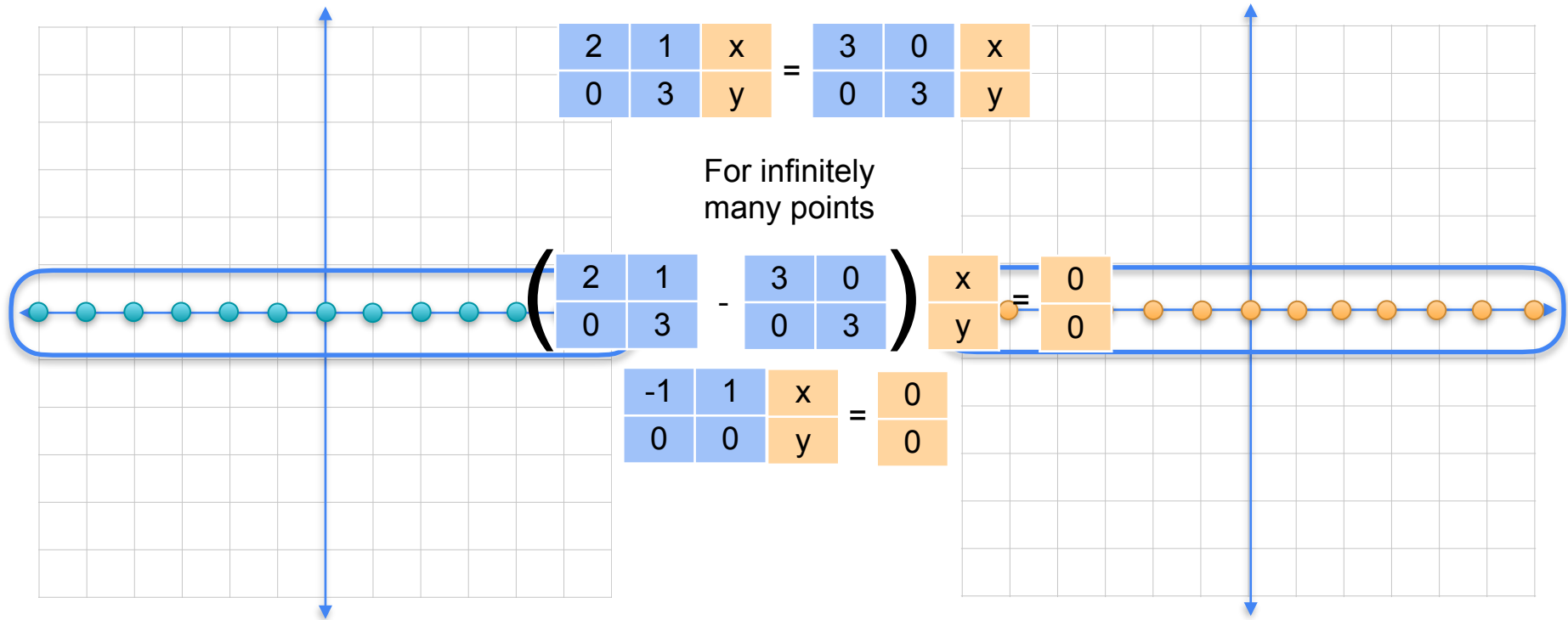
# Finding eigenvalues



# Finding eigenvalues



# Finding eigenvalues



# Finding eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For infinitely many points

$$\left( \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

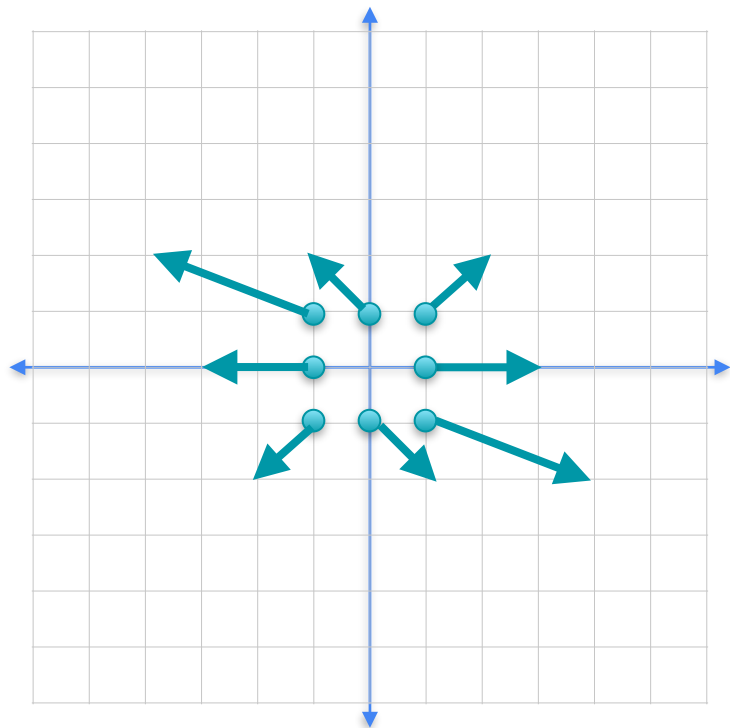
$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

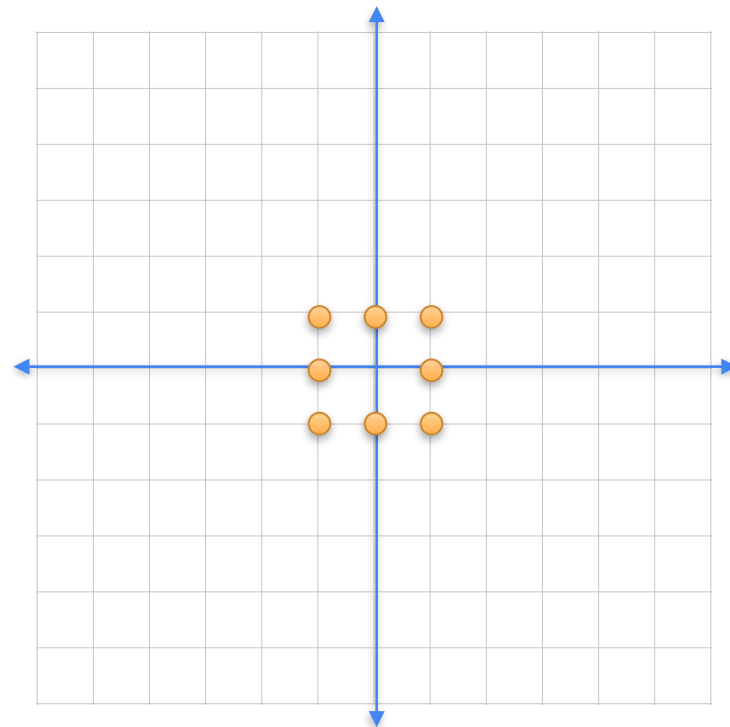


# Find eigenvalues

2	1
0	3

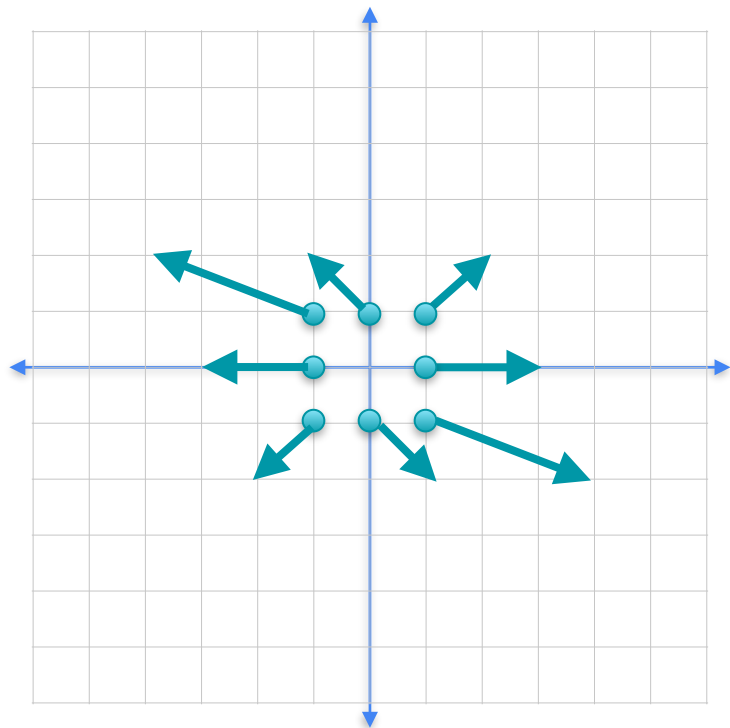


2	0
0	2

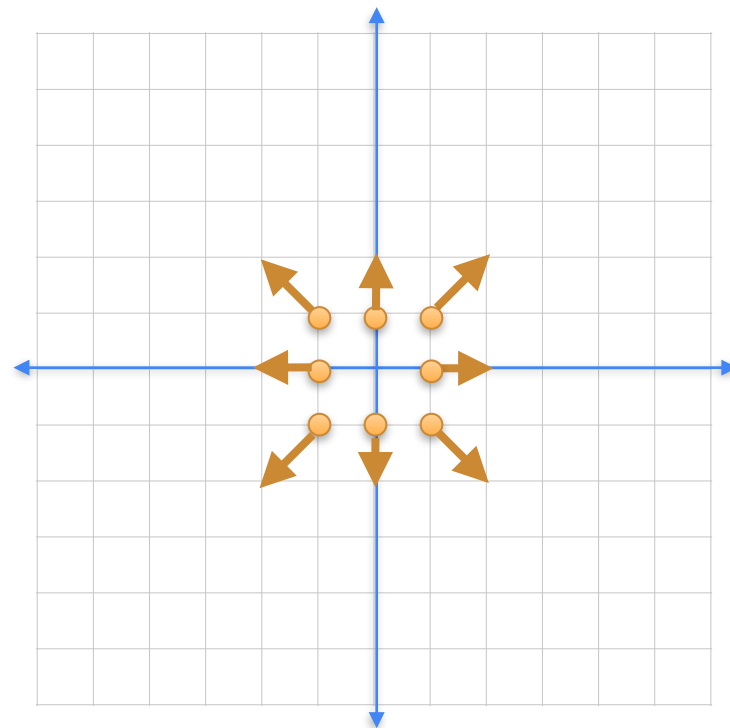


# Find eigenvalues

2	1
0	3

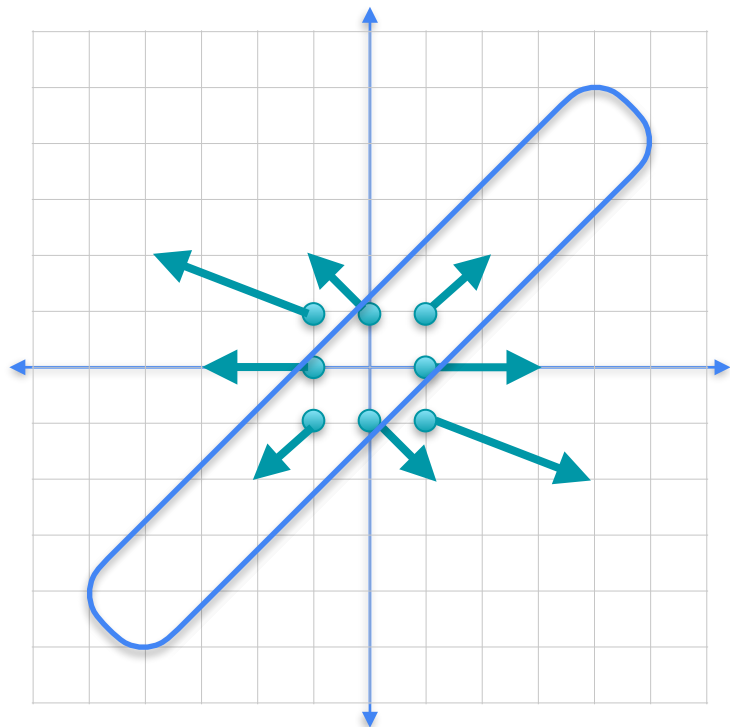


2	0
0	2

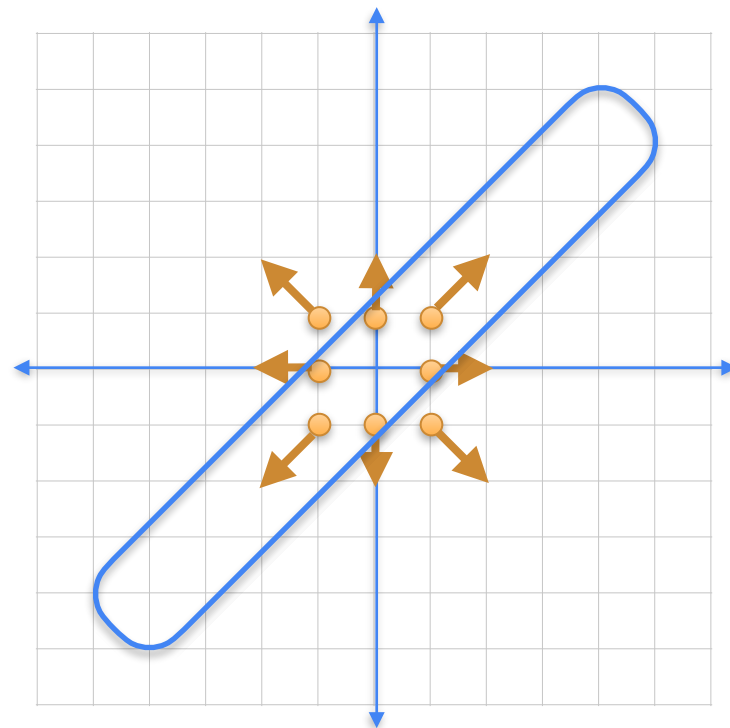


# Find eigenvalues

2	1
0	3

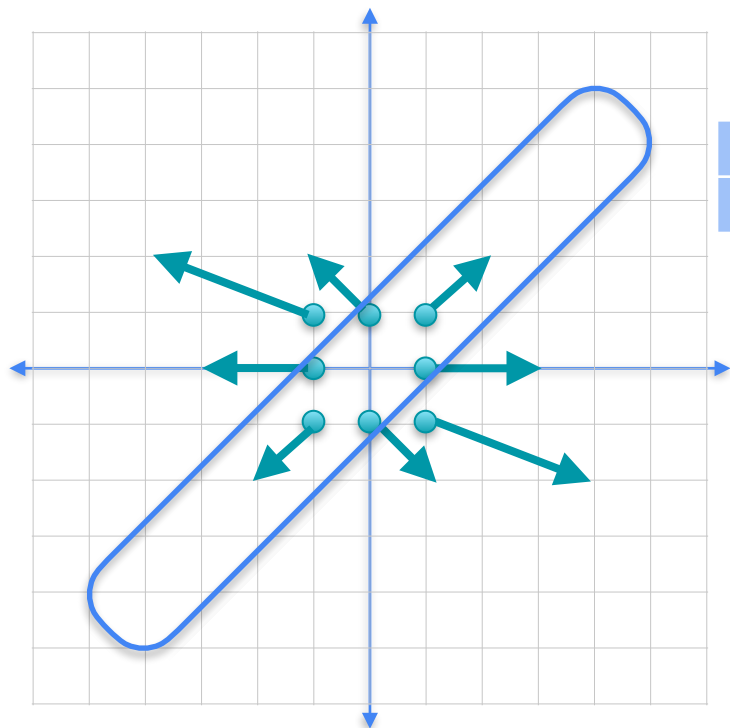


2	0
0	2



# Find eigenvalues

2	1
0	3

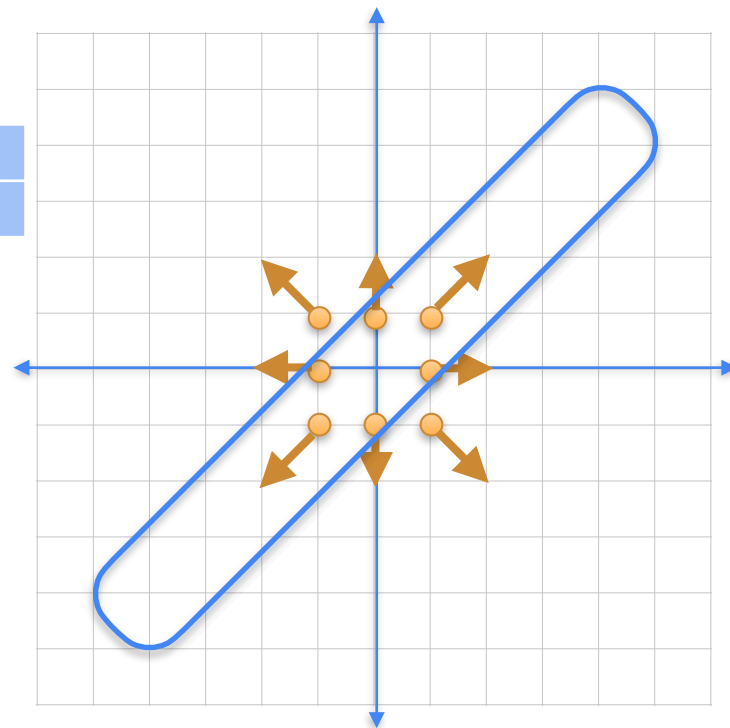


2	1
0	3

-

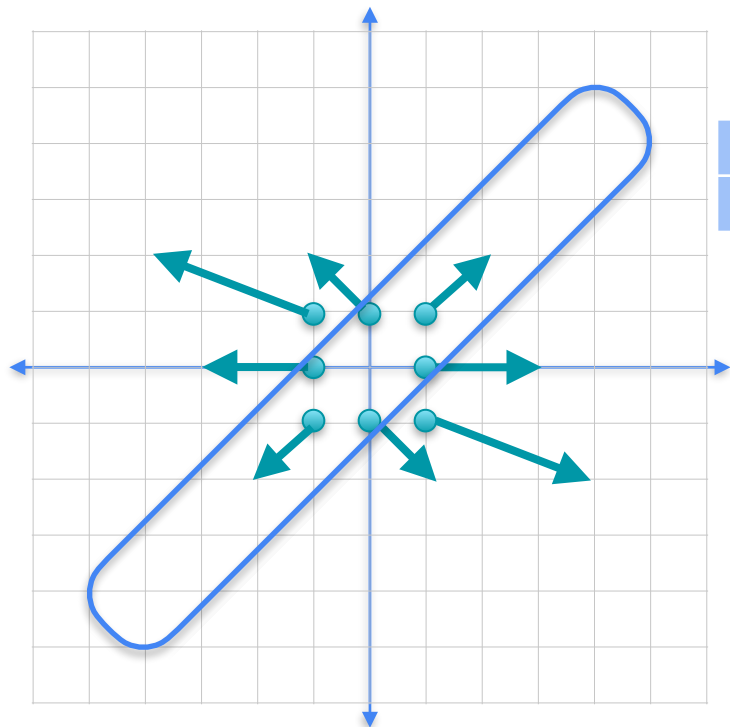
2	0
0	2

2	0
0	2



# Find eigenvalues

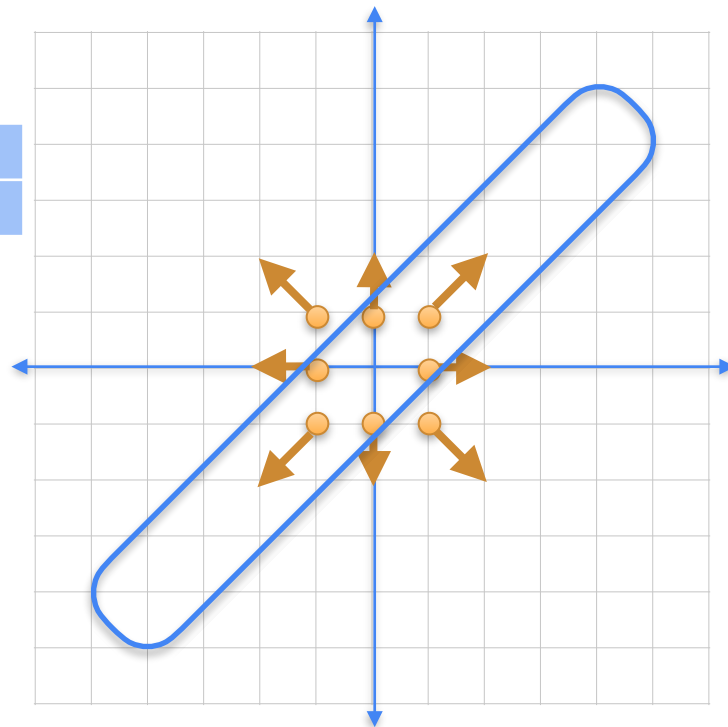
2	1
0	3



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

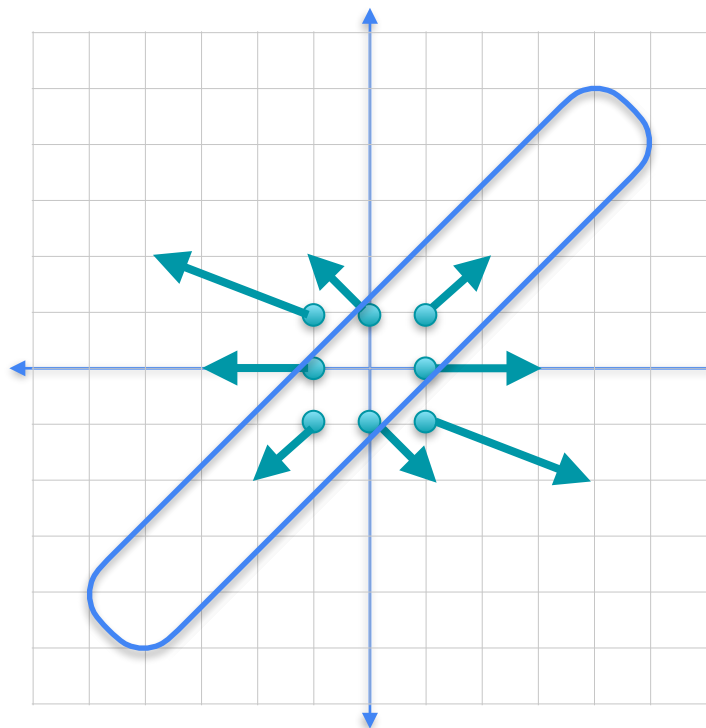
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

2	0
0	2



# Find eigenvalues

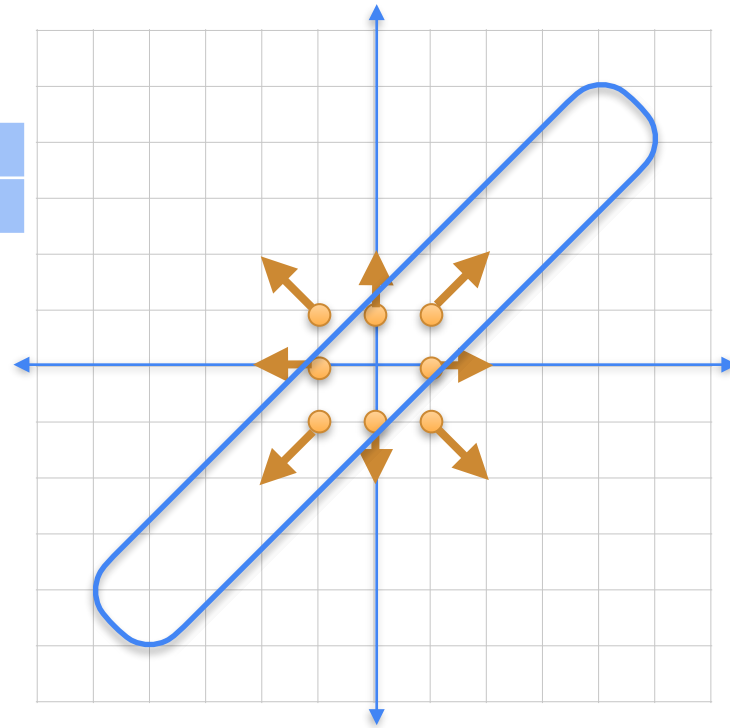
2	1
0	3



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

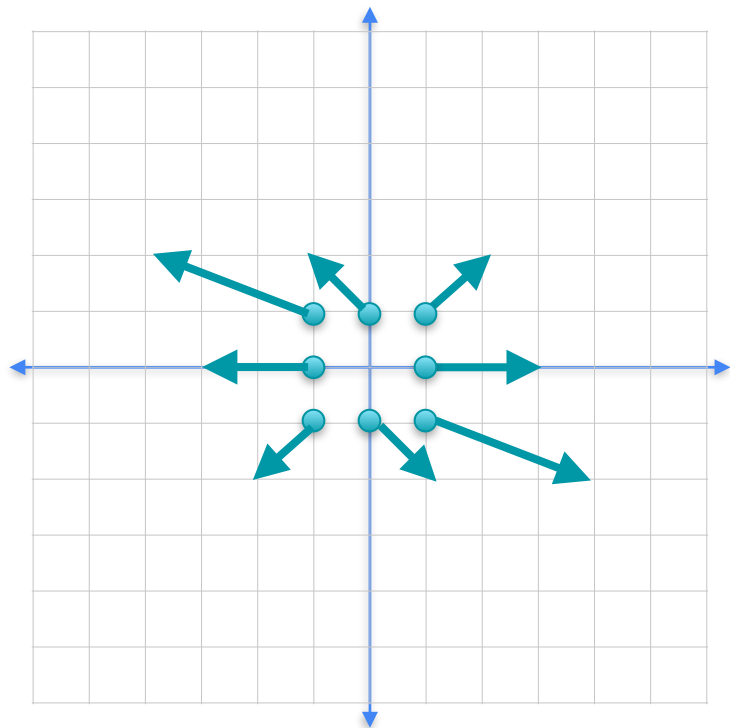
$$\det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$$

2	0
0	2

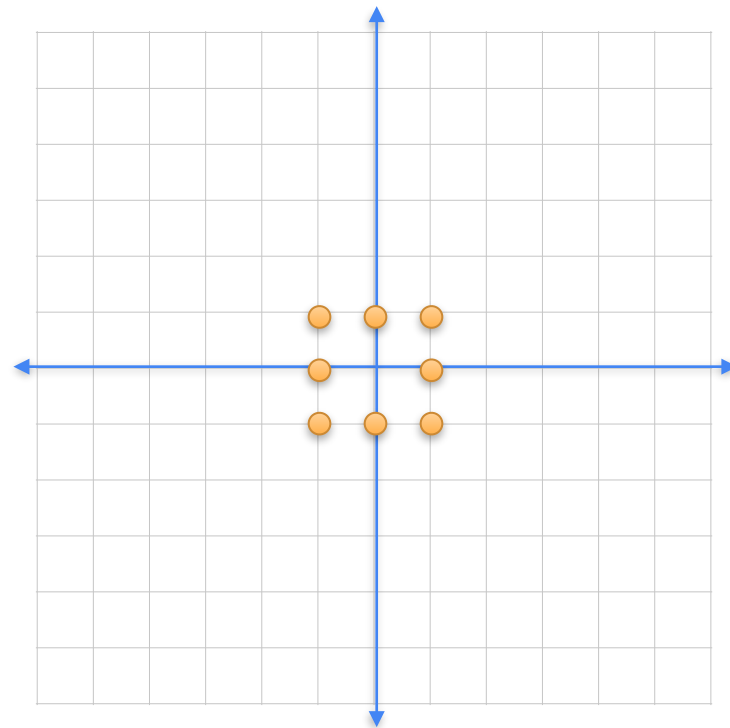


# Find eigenvalues

2	1
0	3

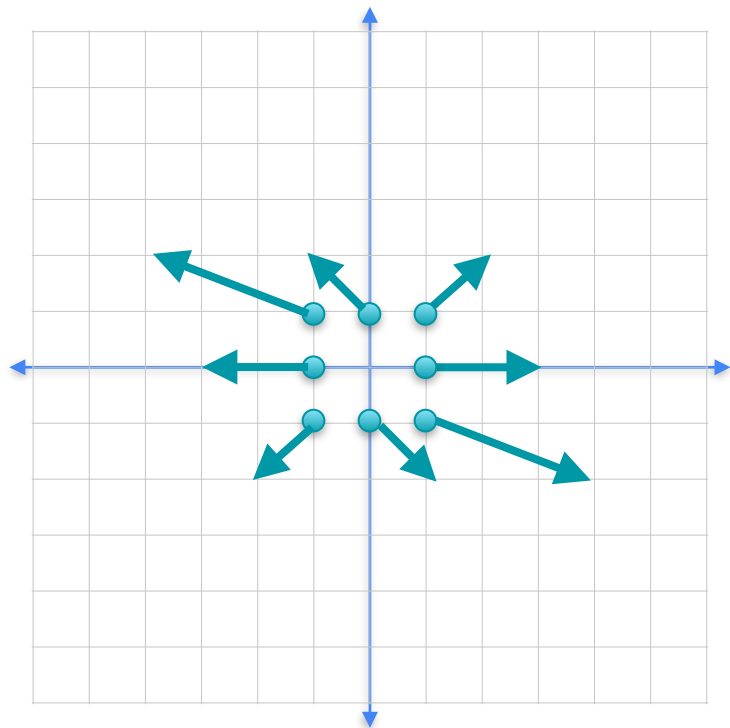


4	0
0	4

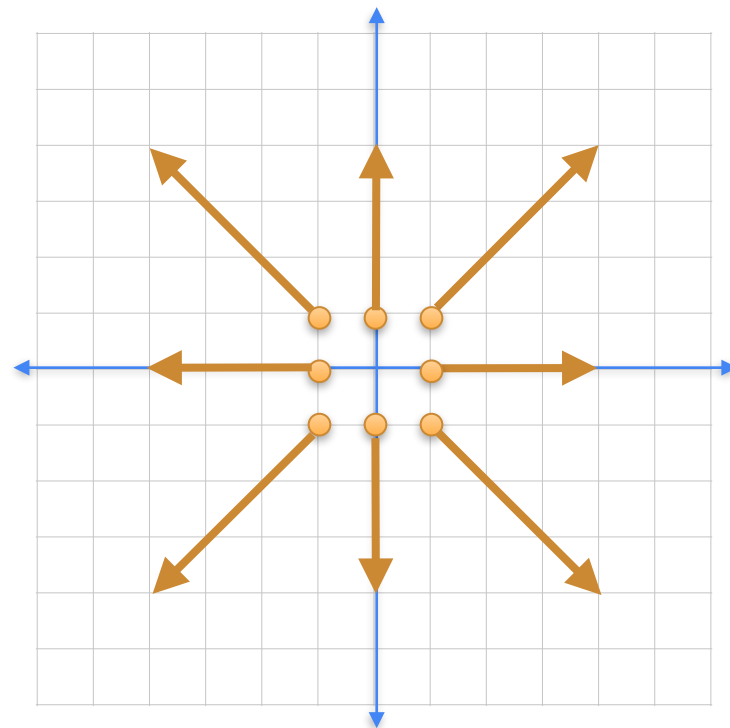


# Find eigenvalues

2	1
0	3



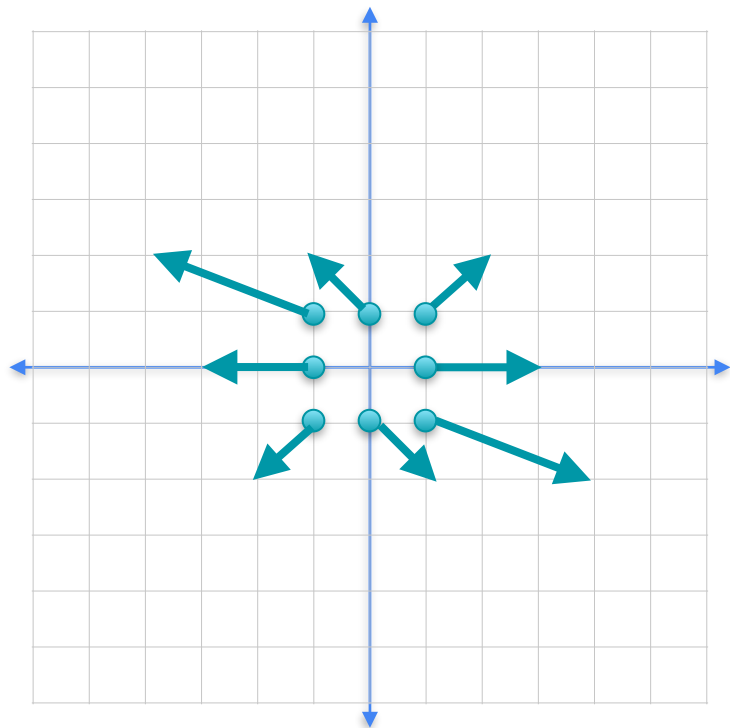
4	0
0	4



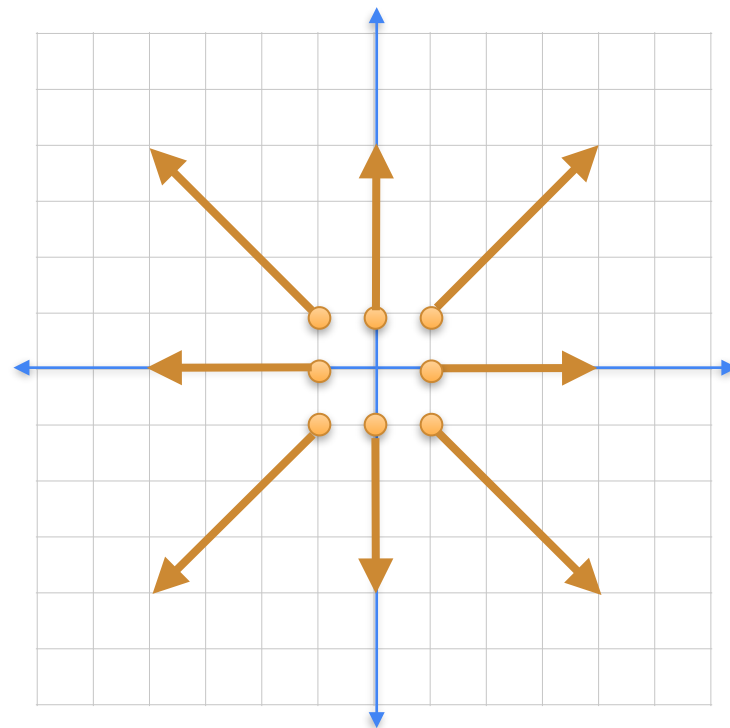


# Find eigenvalues

2	1
0	3

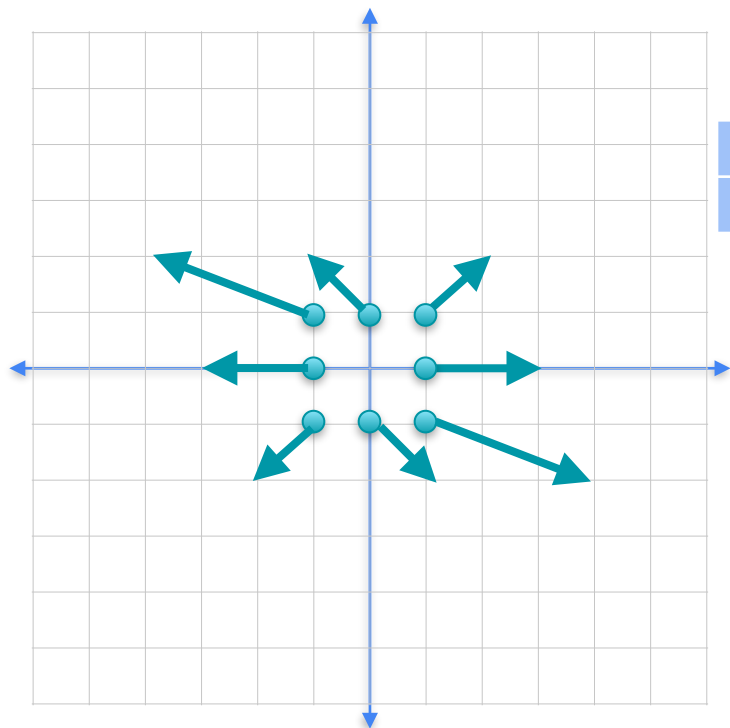


4	0
0	4



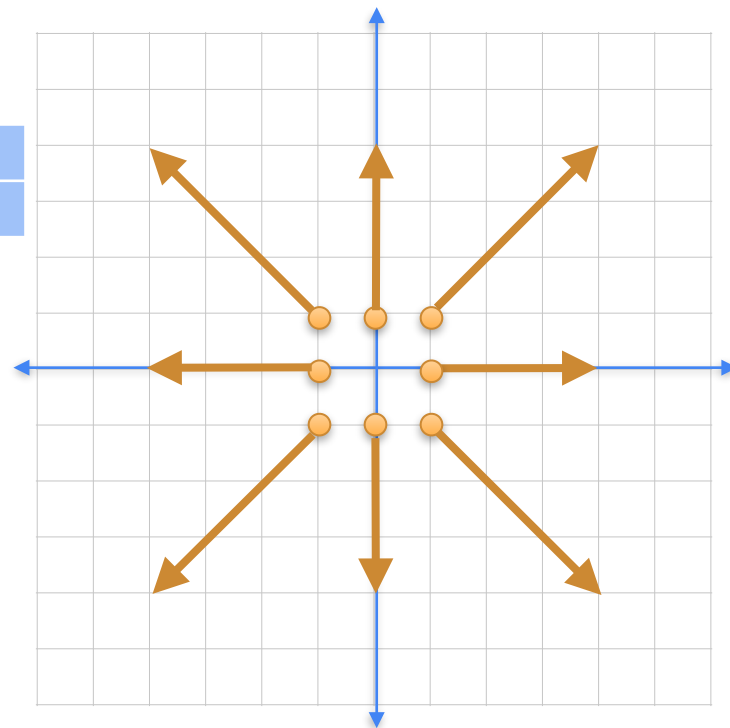
# Find eigenvalues

2	1
0	3



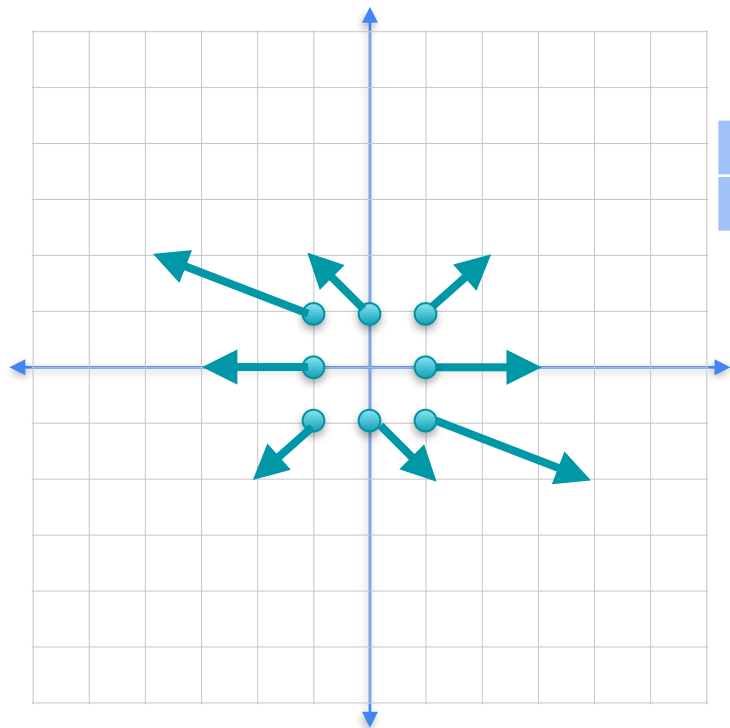
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

4	0
0	4



# Find eigenvalues

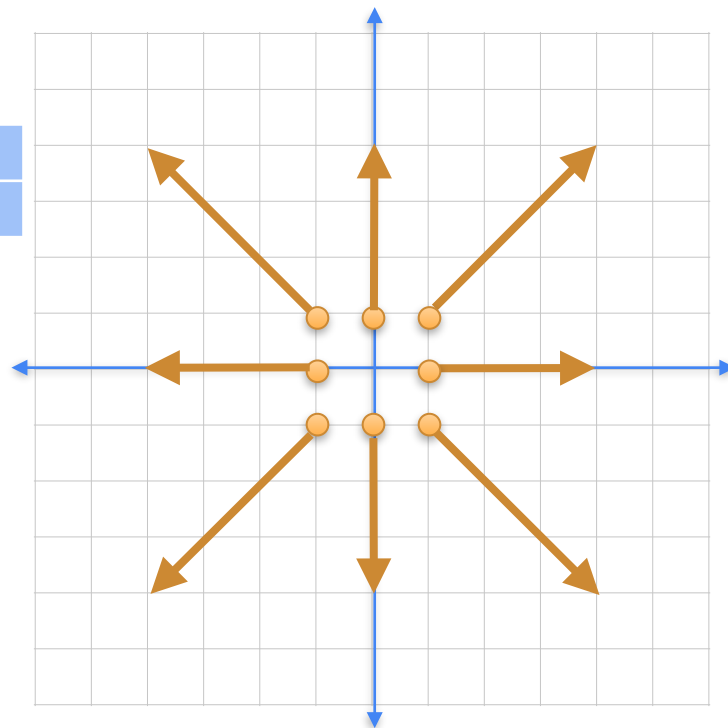
2	1
0	3



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

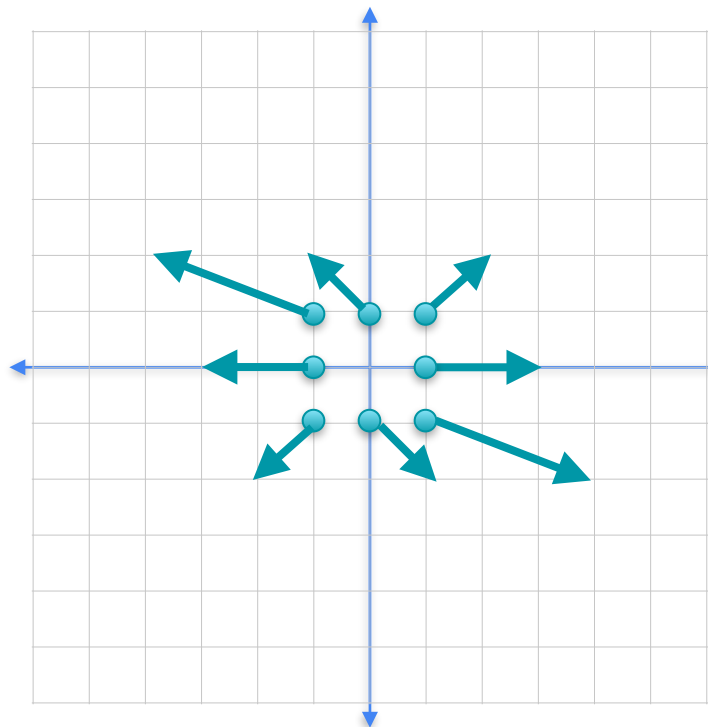
$$\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

4	0
0	4



# Find eigenvalues

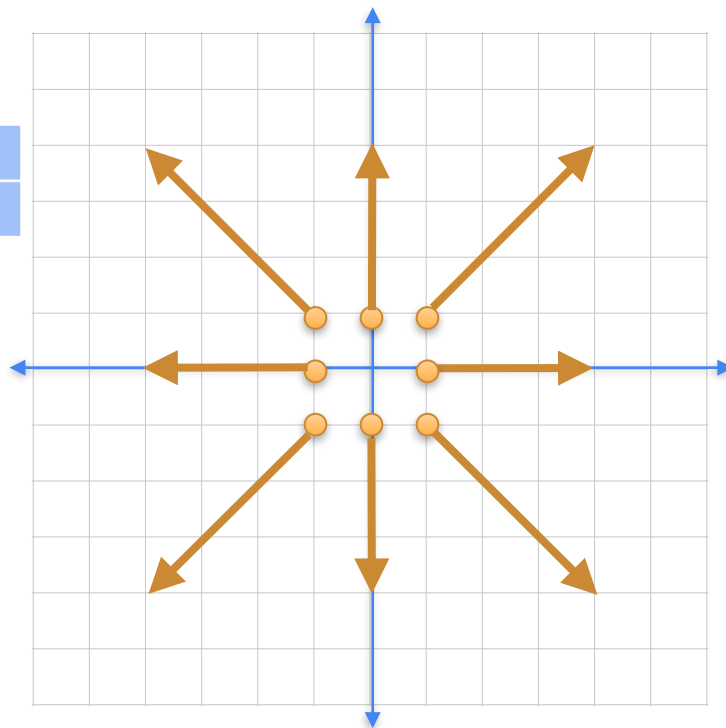
2	1
0	3



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\det \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \neq 0$$

4	0
0	4



# Finding eigenvalues

2	1
0	3

# Finding eigenvalues

2	1
0	3

$\lambda$	0
0	$\lambda$

# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

# Finding eigenvalues

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\lambda = 2$$

$$\lambda = 3$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\lambda = 2$$

$$\lambda = 3$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\begin{aligned} \lambda &= 2 \\ \lambda &= 3 \end{aligned}$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\begin{aligned} \lambda &= 2 \\ \lambda &= 3 \end{aligned}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$



# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$x = 1$$

$$0x + 3y = 2y$$

$$y = 0$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

$$x = 1$$

$$y = 1$$



# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

$$x = 1$$

$$y = 1$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

# Quiz

- Find the eigenvalues and eigenvectors of this matrix:

9	4
4	3

# Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

9	4
4	3



- The characteristic polynomial is

$$\det \begin{array}{|c|c|} \hline 9-\lambda & 4 \\ \hline 4 & 3-\lambda \\ \hline \end{array} = (9-\lambda)(3-\lambda) - 4 \cdot 4 = 0$$

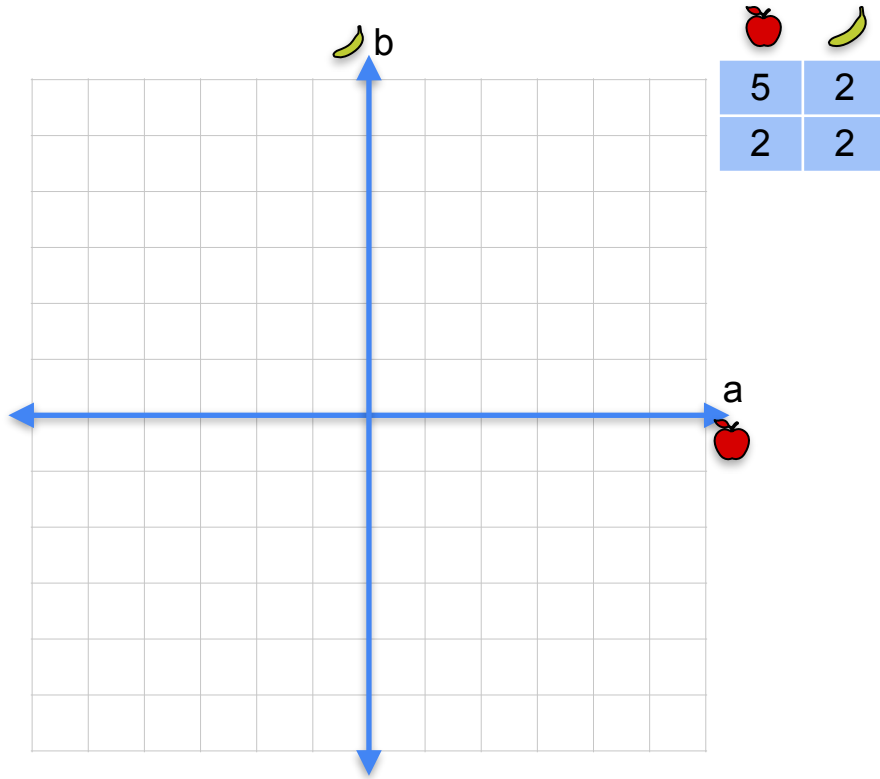
- Which factors as  $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$

- The solutions are  $\lambda = 11$   
 $\lambda = 1$

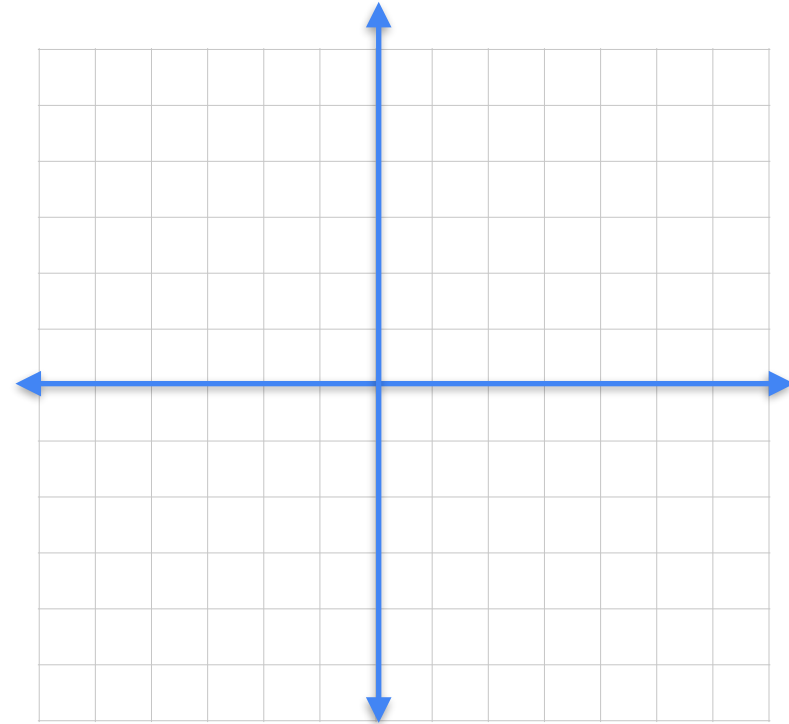
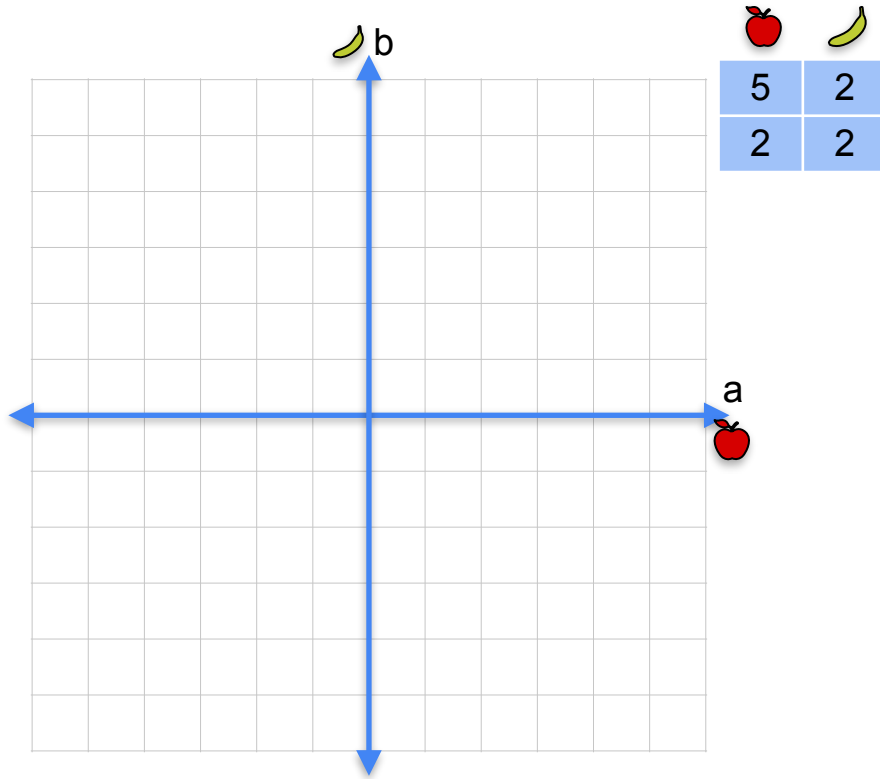
# Matrices as linear transformations

	
5	2
2	2

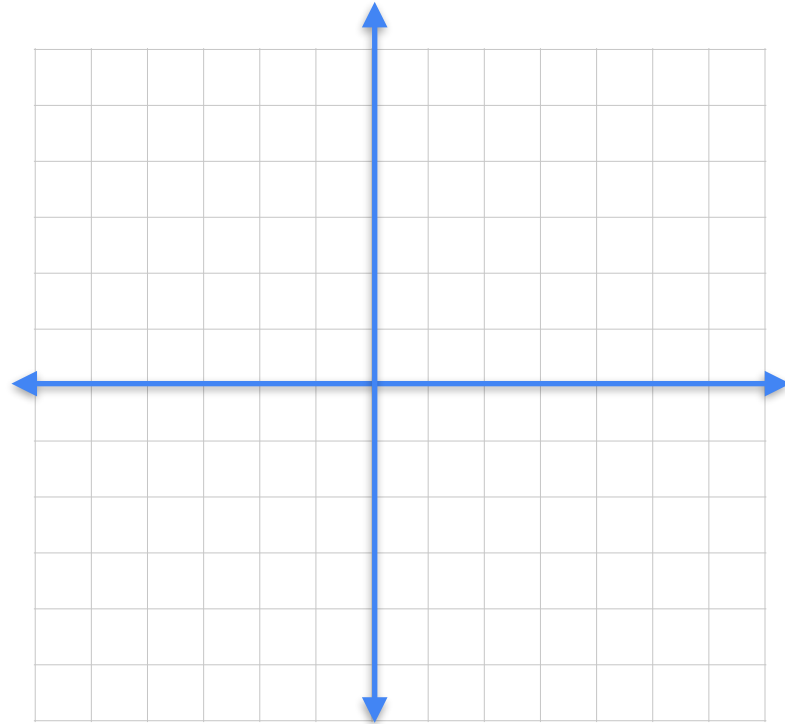
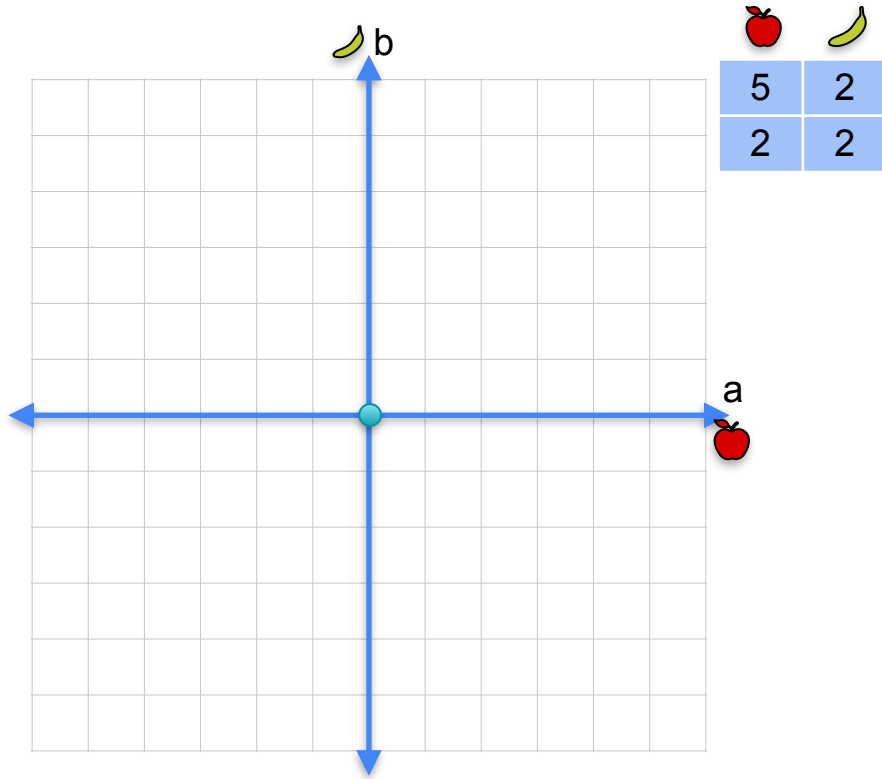
# Matrices as linear transformations



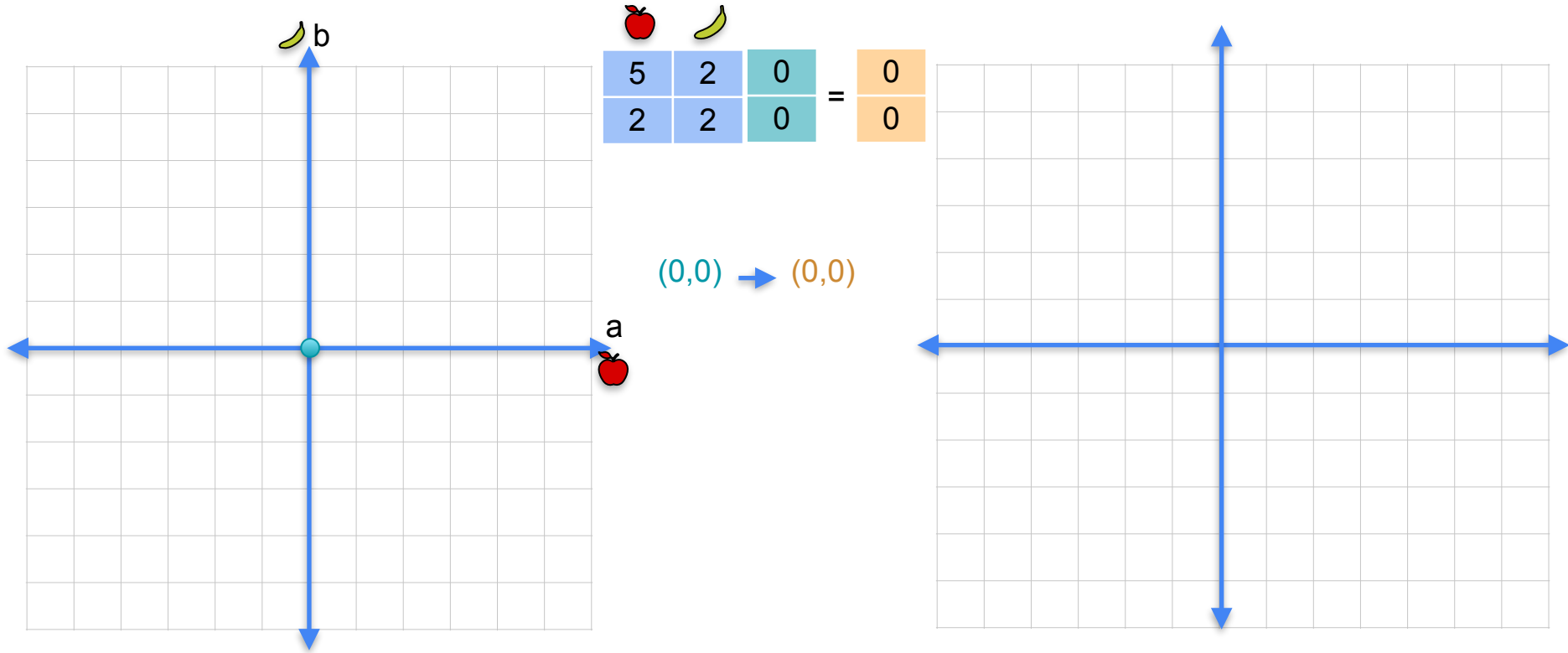
# Matrices as linear transformations



# Matrices as linear transformations

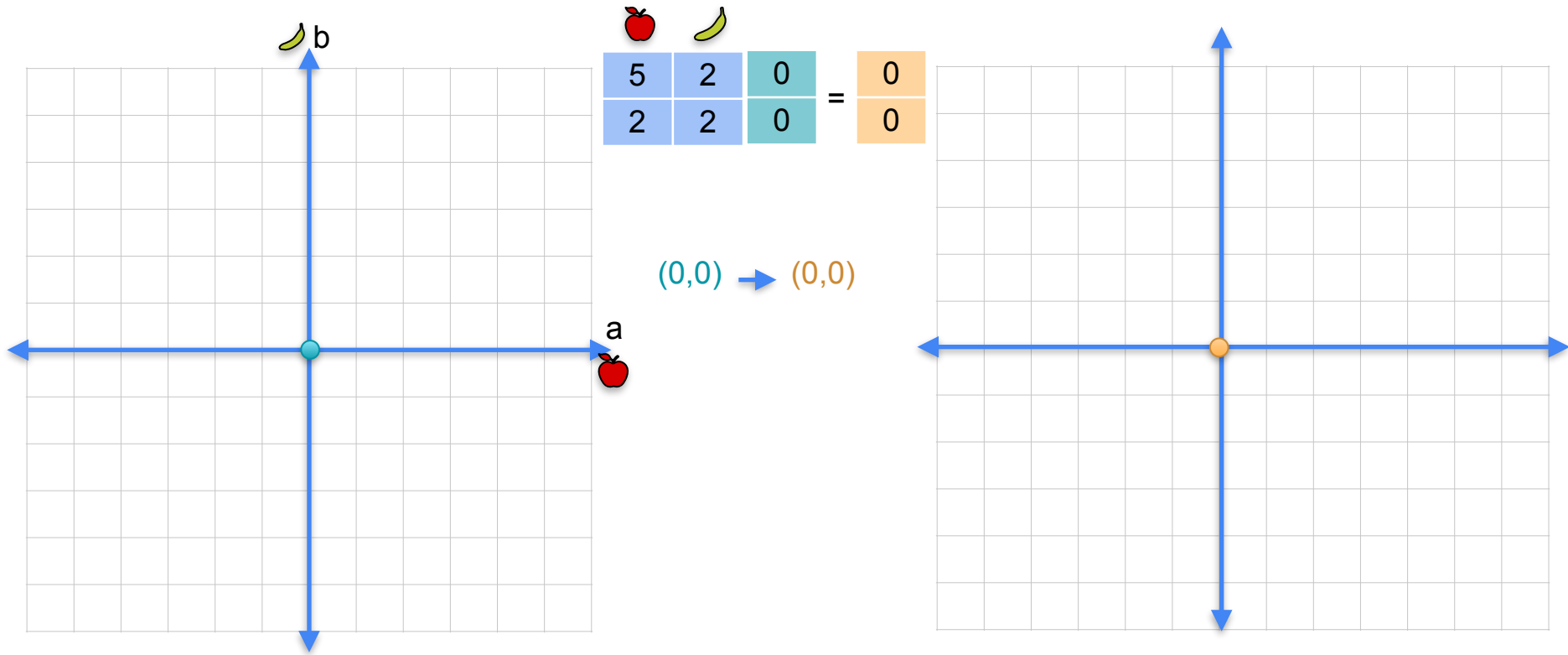


# Matrices as linear transformations

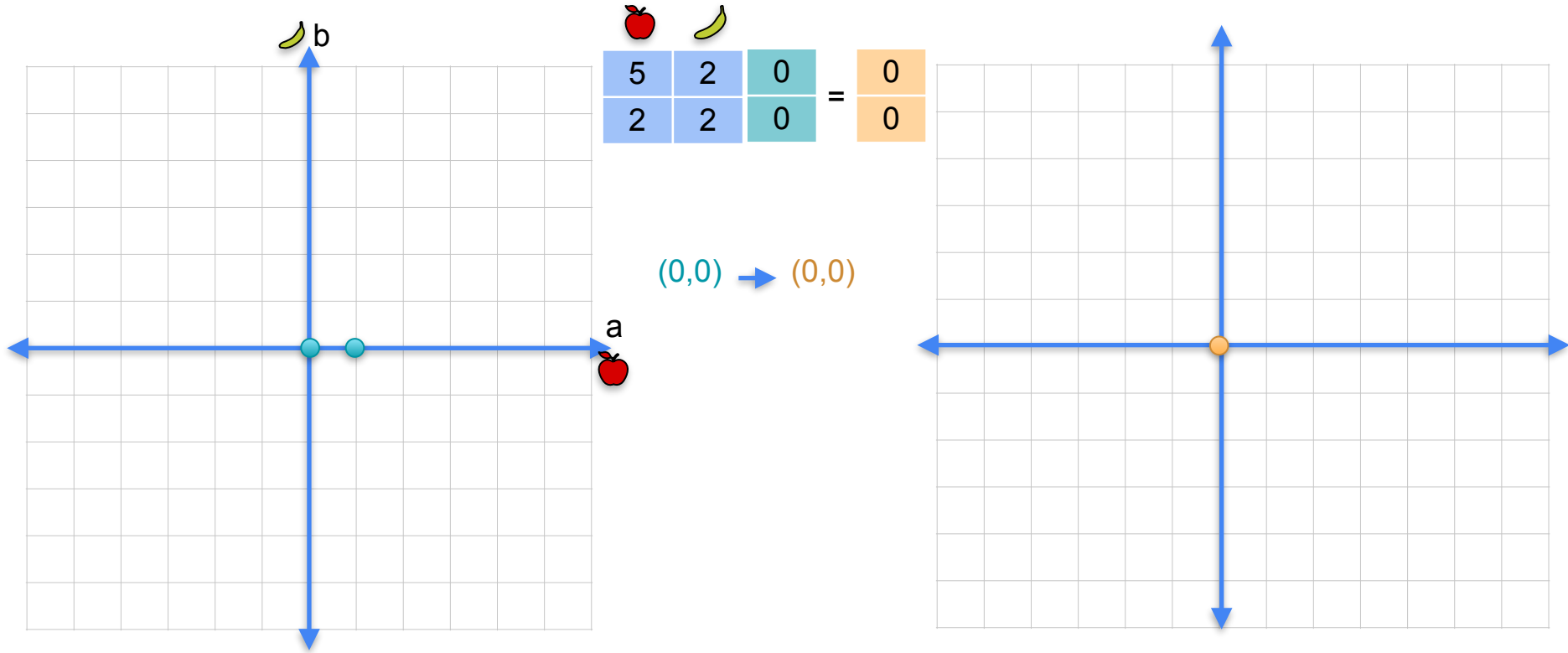




# Matrices as linear transformations



# Matrices as linear transformations



# Matrices as linear transformations

b

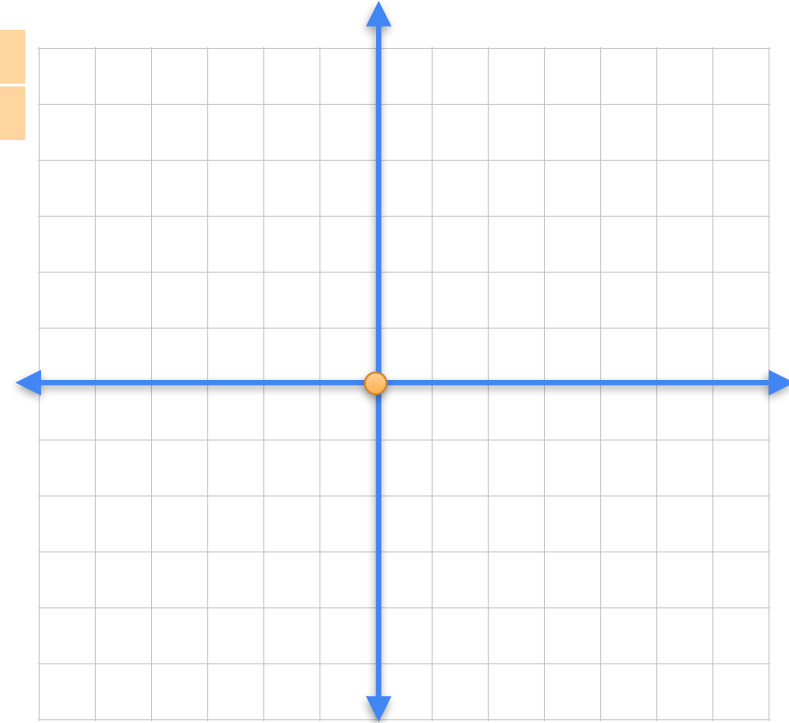


5	2	1	=	5
2	2	0		2

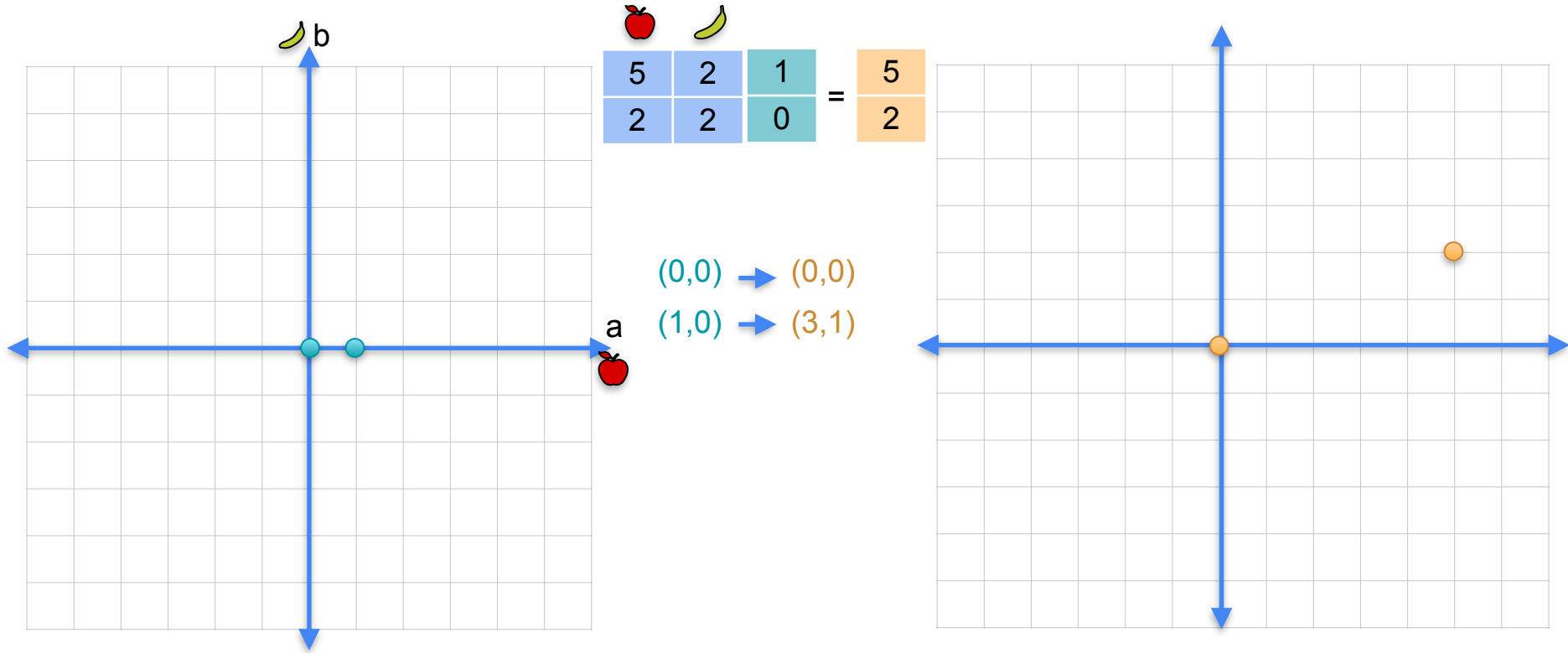
$(0,0) \rightarrow (0,0)$

$(1,0) \rightarrow (3,1)$

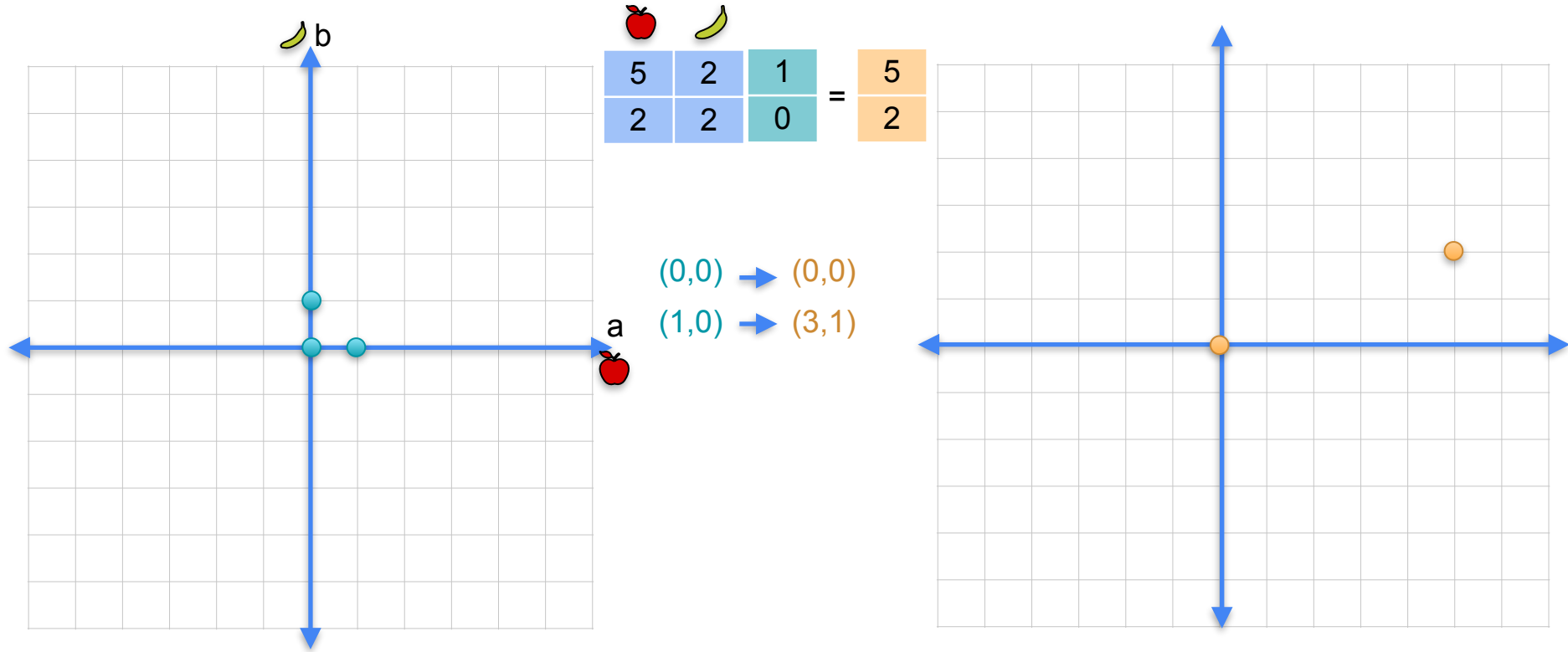
a



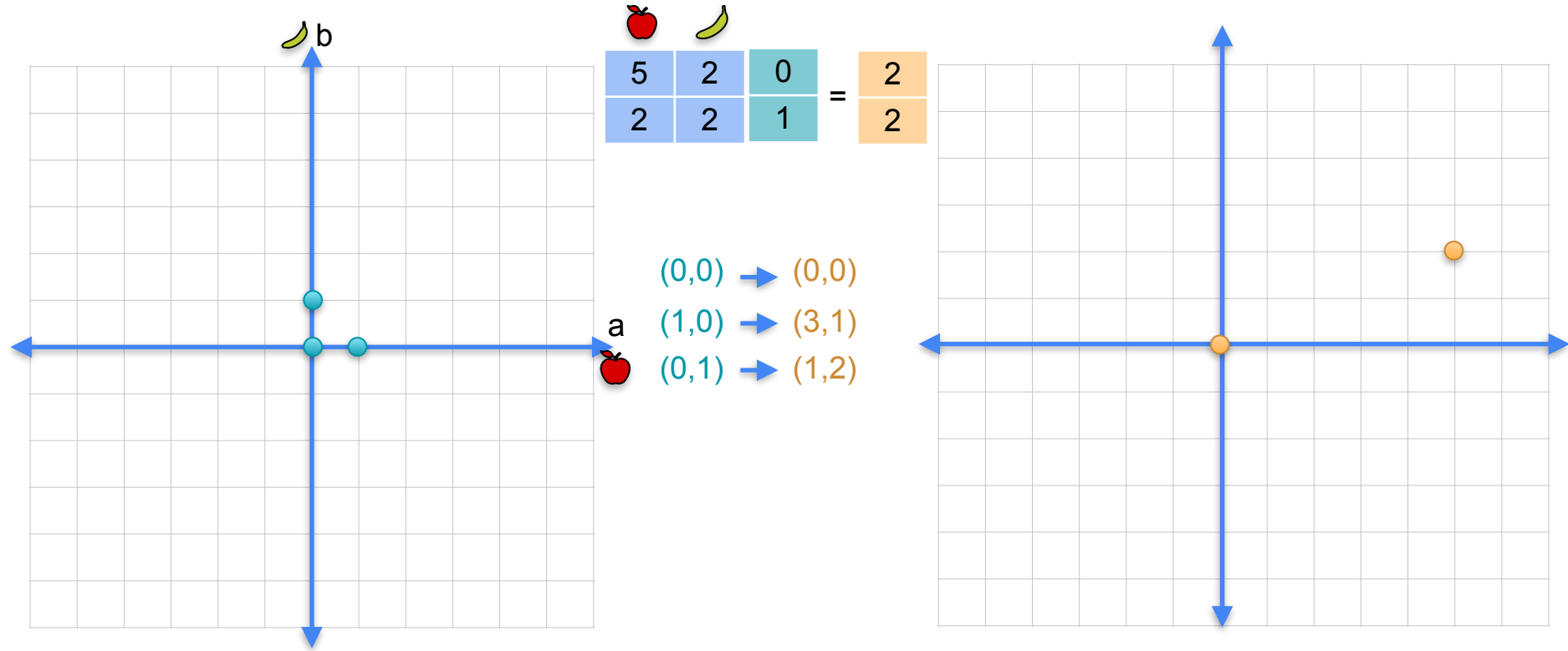
# Matrices as linear transformations



# Matrices as linear transformations



# Matrices as linear transformations



# Matrices as linear transformations

b

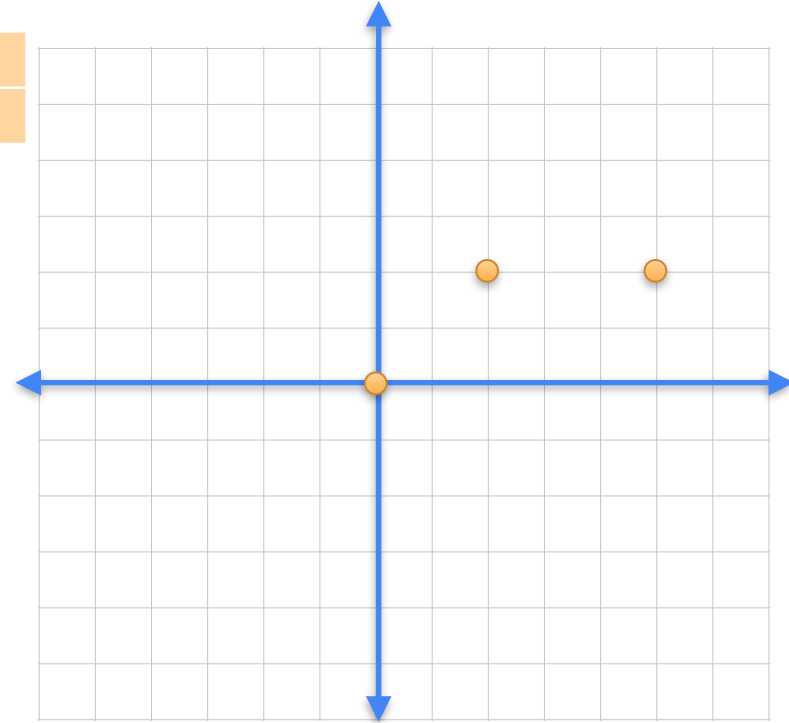


$$\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

a



$$\begin{aligned} (0,0) &\rightarrow (0,0) \\ (1,0) &\rightarrow (3,1) \\ (0,1) &\rightarrow (1,2) \end{aligned}$$



# Matrices as linear transformations

b



5	2	0
2	2	1

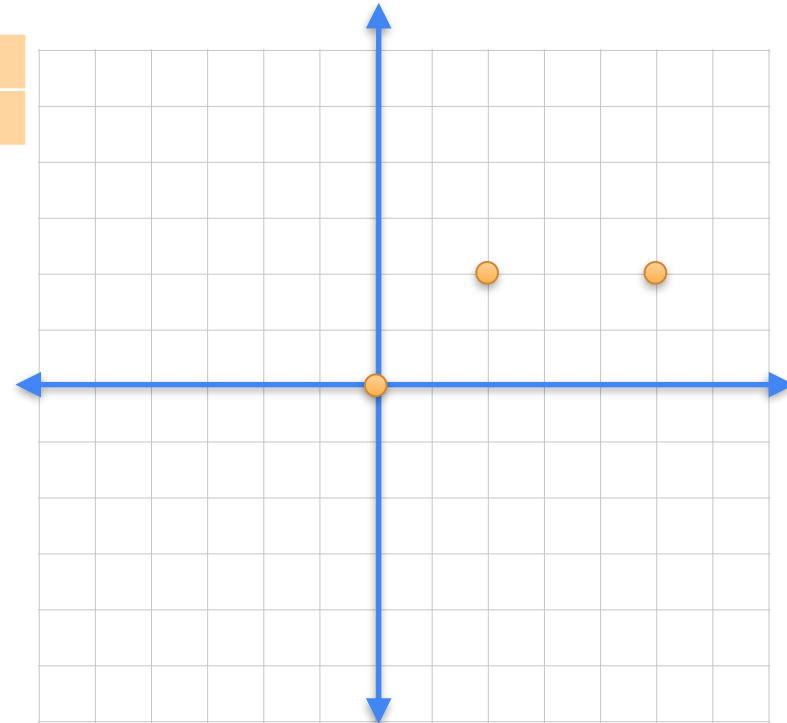
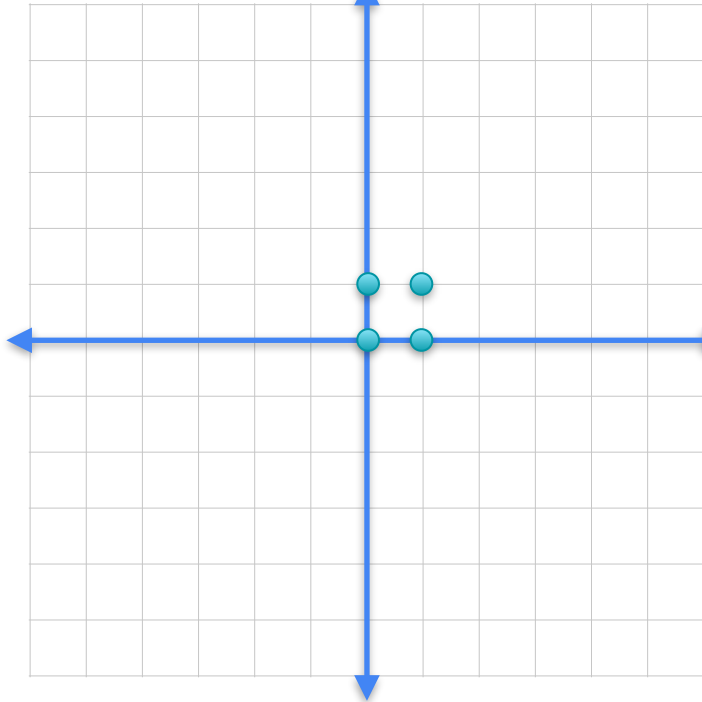
 = 

2
2

a



$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$





# Matrices as linear transformations

b

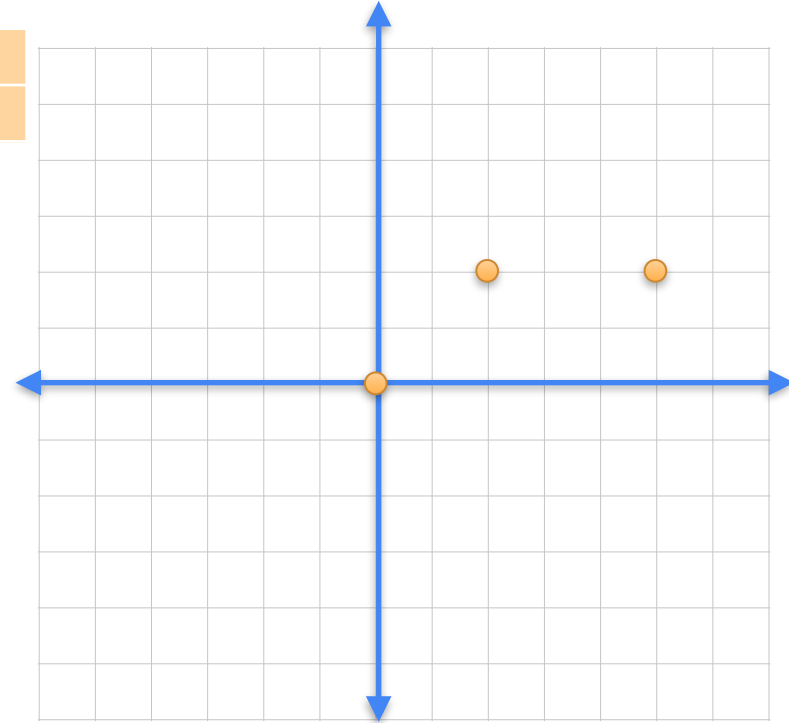
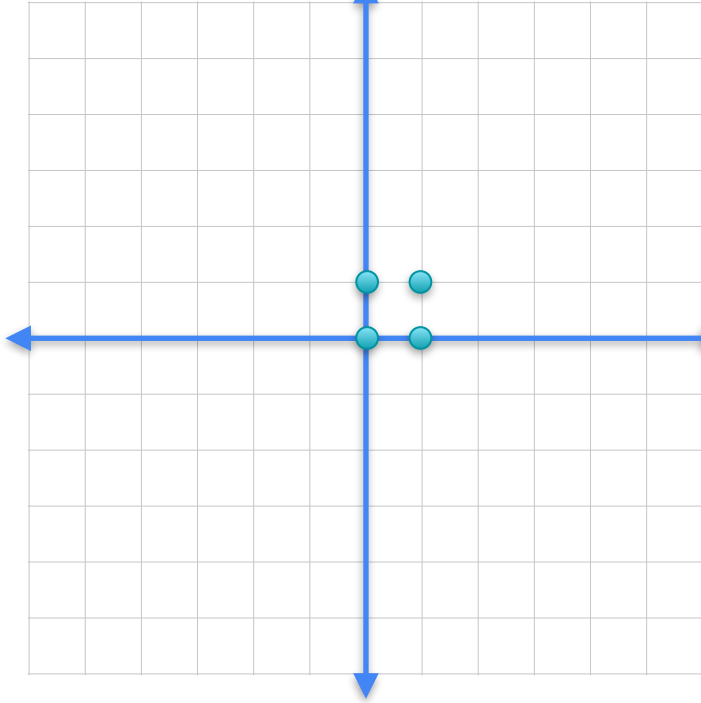


5	2	1	=	7
2	2	1		2

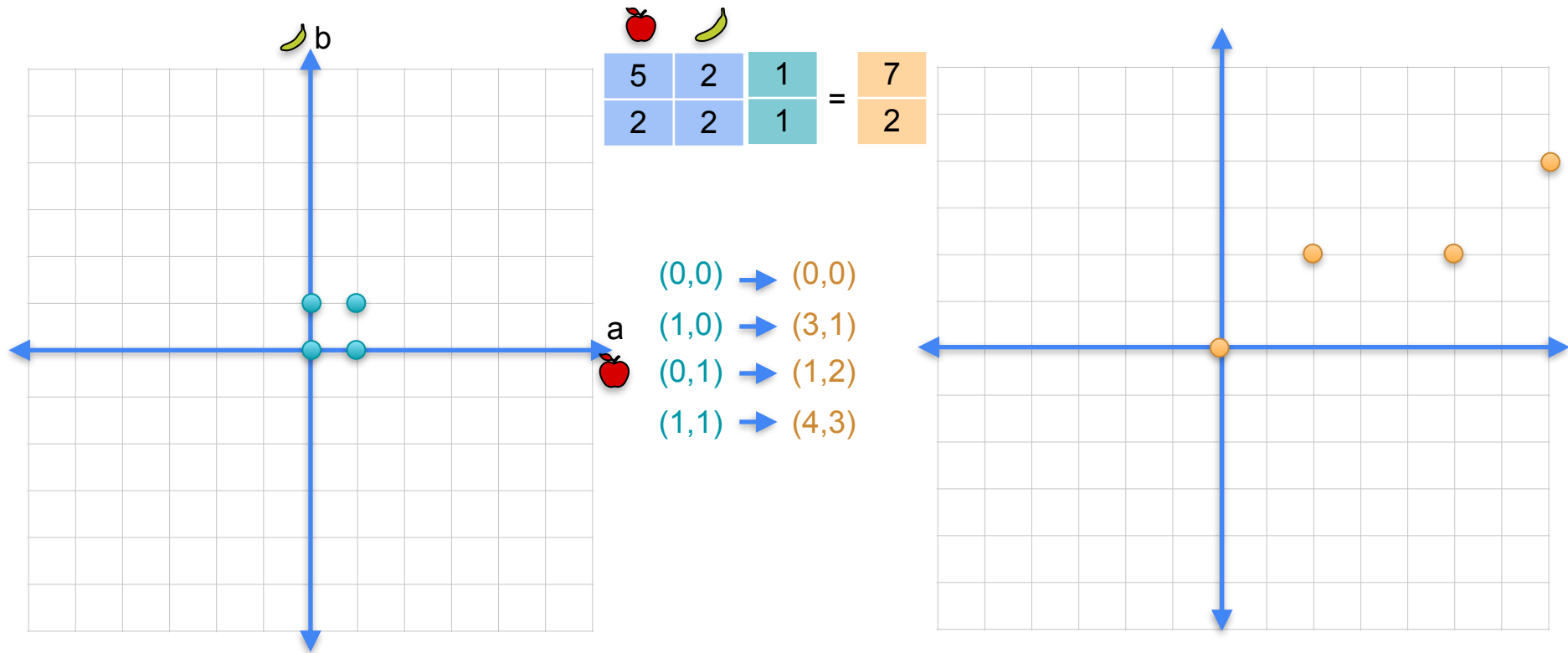
a



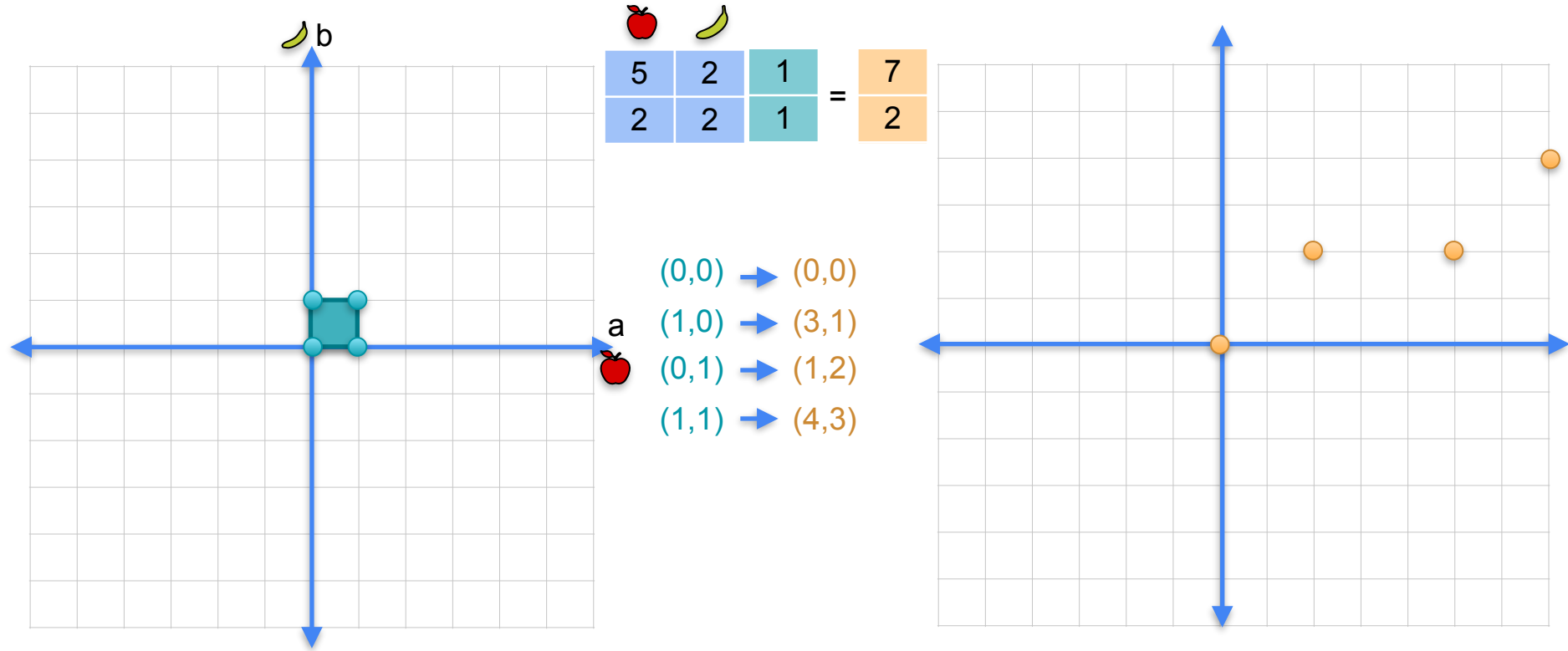
$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



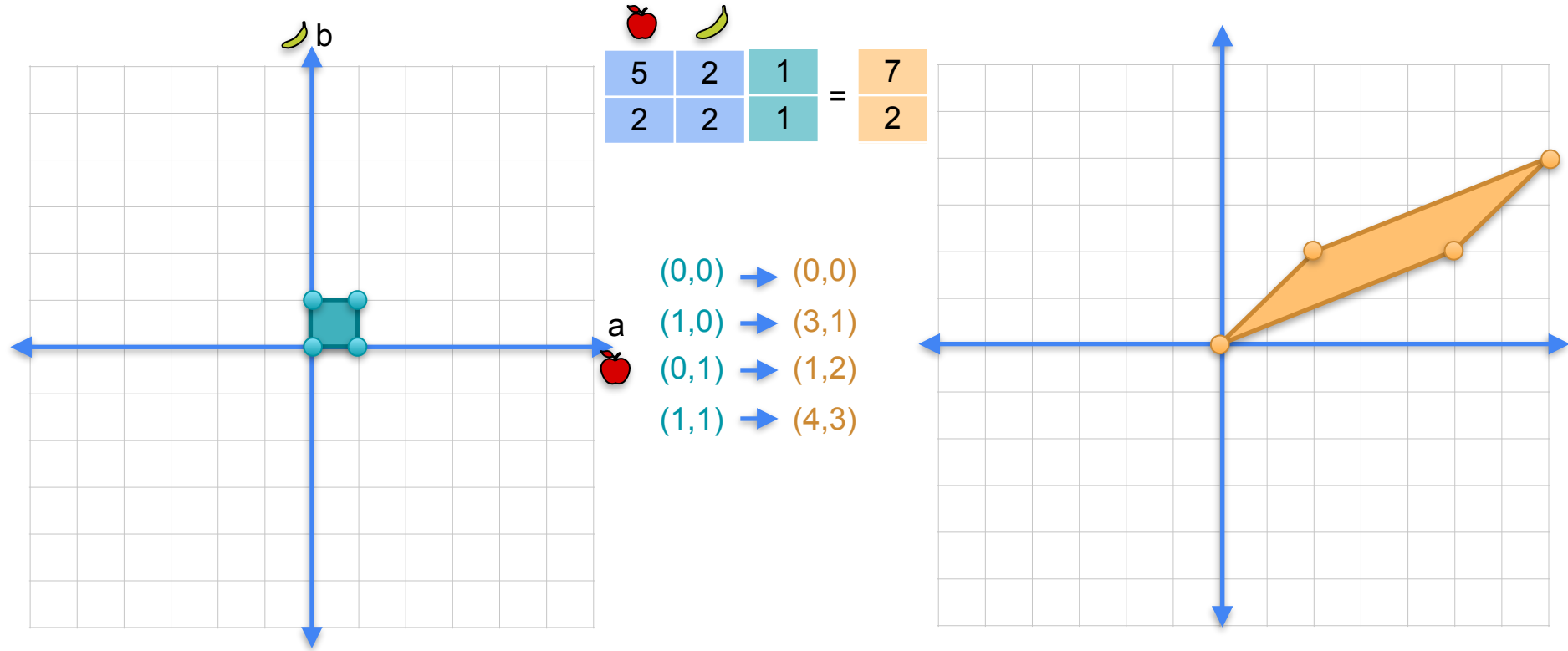
# Matrices as linear transformations





# Matrices as linear transformations



# Matrices as linear transformations

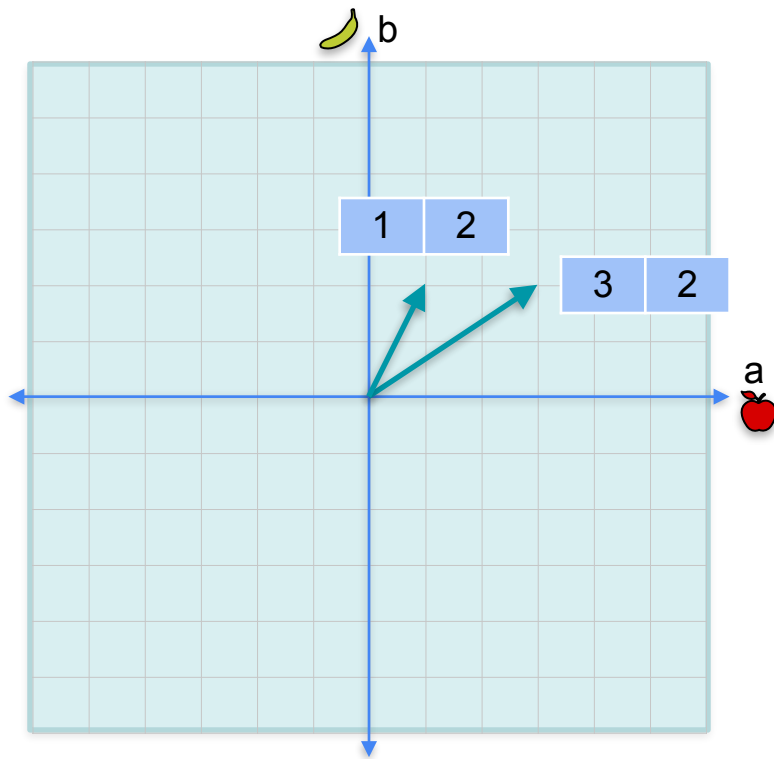


# Row span of a matrix



	
3	2
1	2

Rows

3	2
1	2

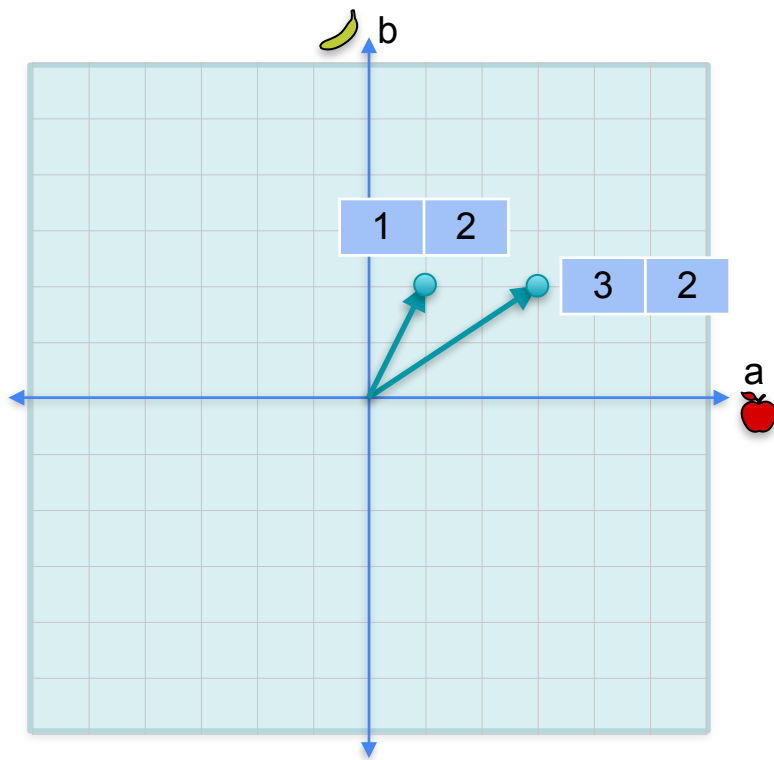


# Row span of a matrix

	
3	2
1	2

Rows

3	2
1	2



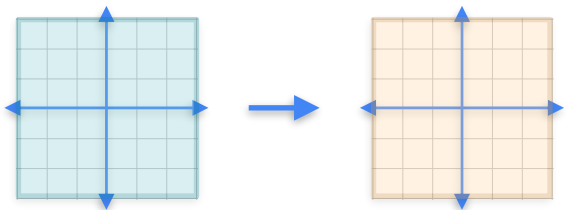
# Span of the rows

Non-singular



3	1
1	2

Rank = 2



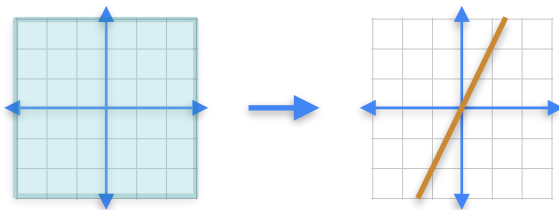
The whole plane

Singular



1	1
2	2

Rank = 1



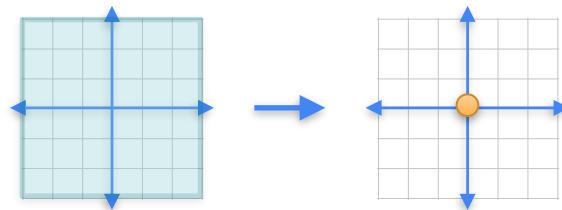
A line

Singular



0	0
0	0

Rank = 0



A point

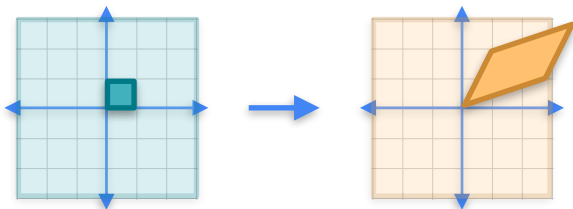
# Basis vectors

Non-singular



3	1
1	2

Rank = 2



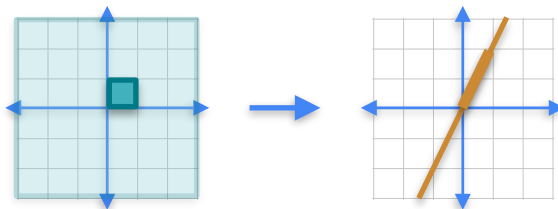
The whole plane

Singular



1	1
2	2

Rank = 1



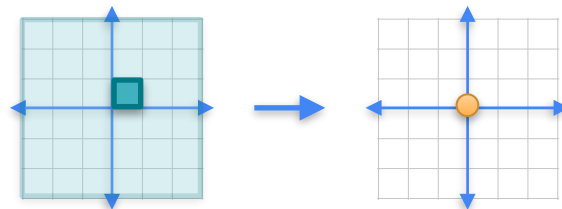
A line

Singular



0	0
0	0

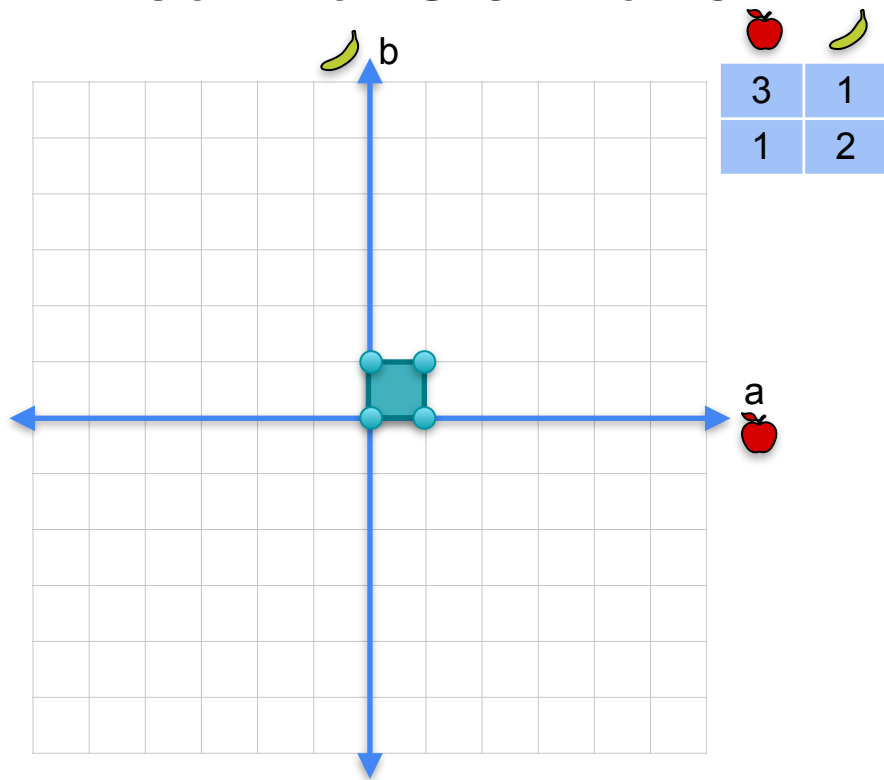
Rank = 0



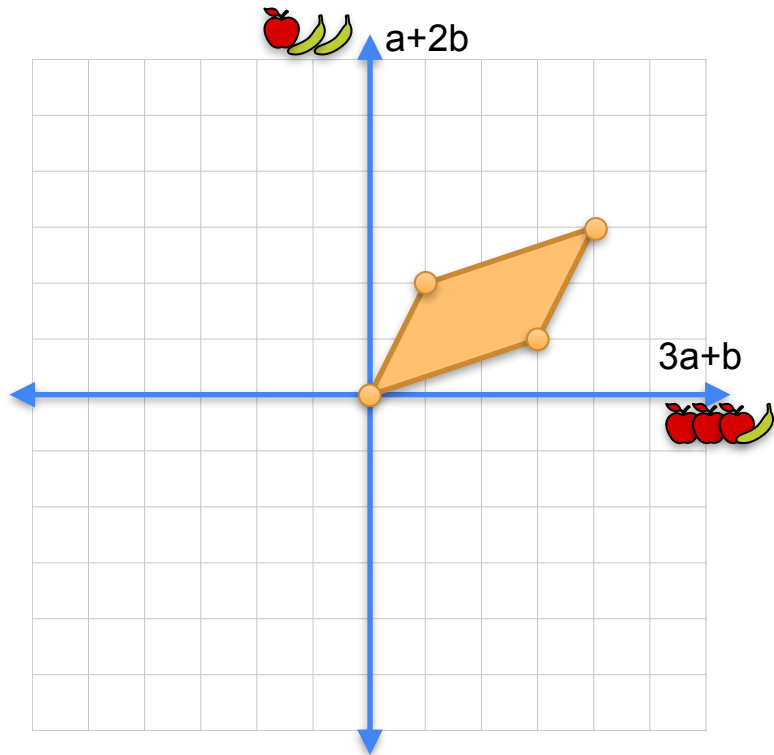
A point



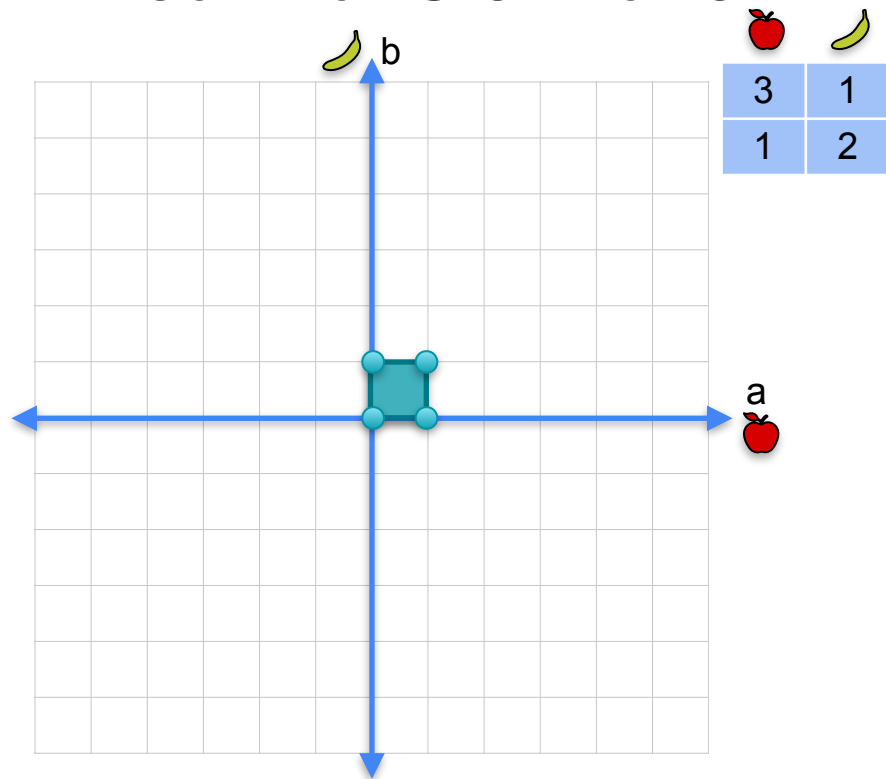
# Linear transformation



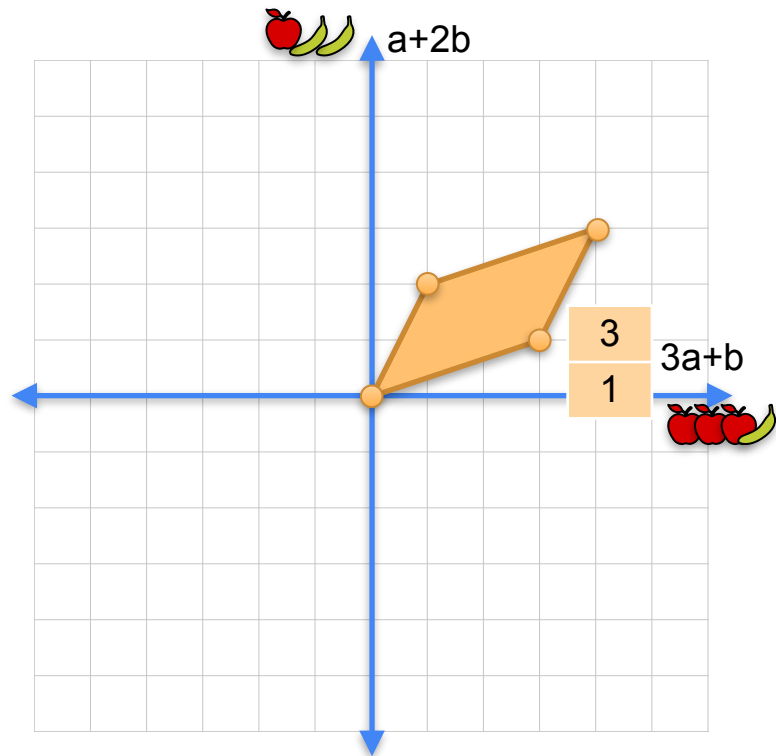
=



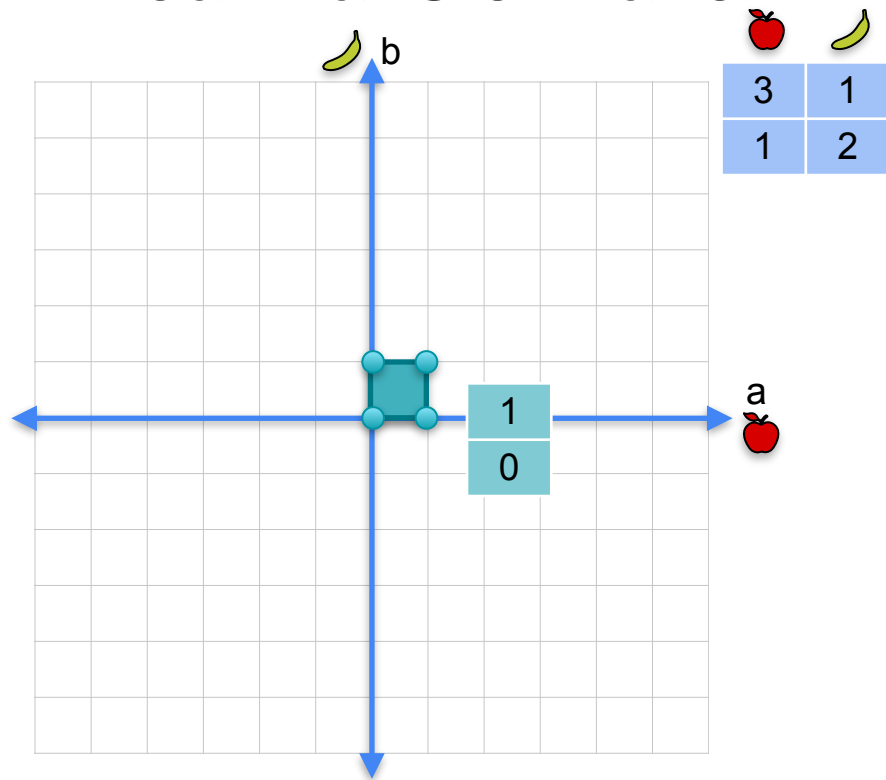
# Linear transformation



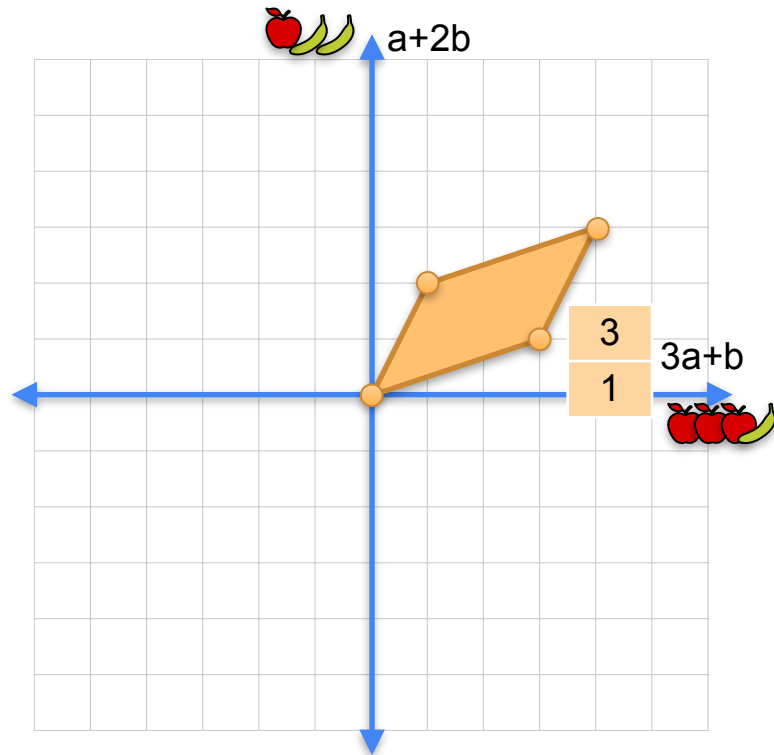
=



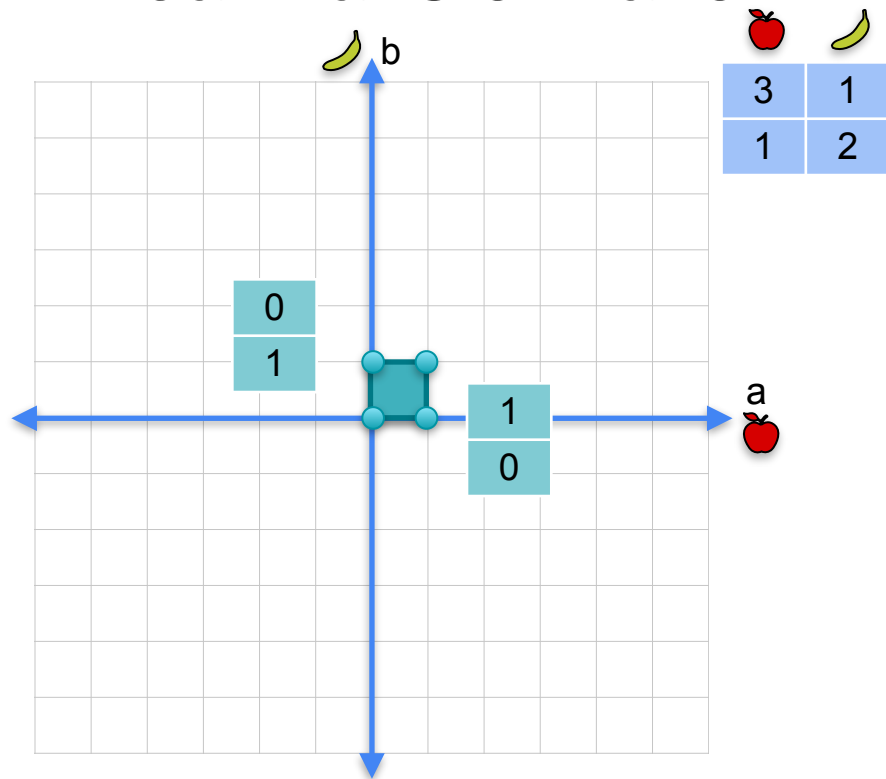
# Linear transformation



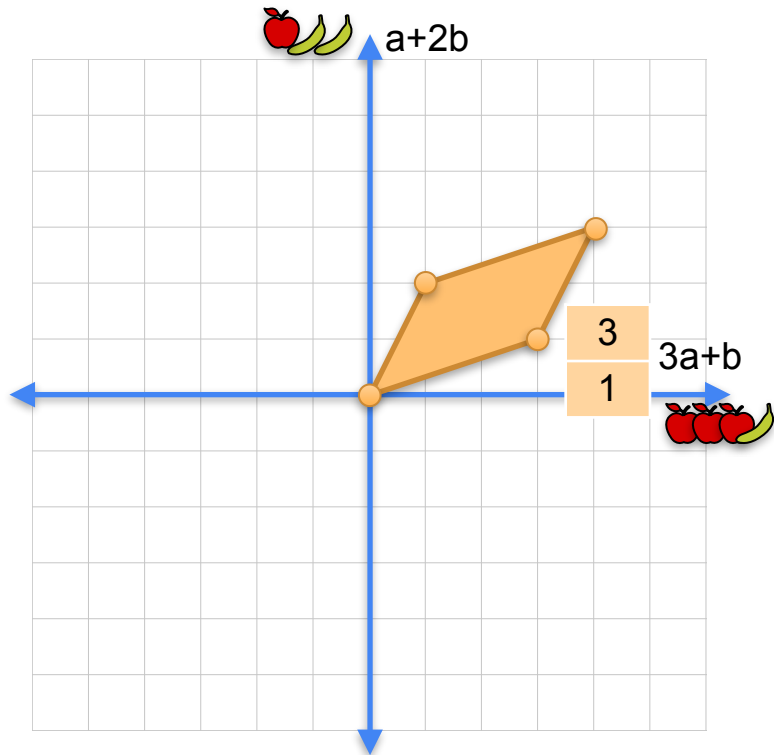
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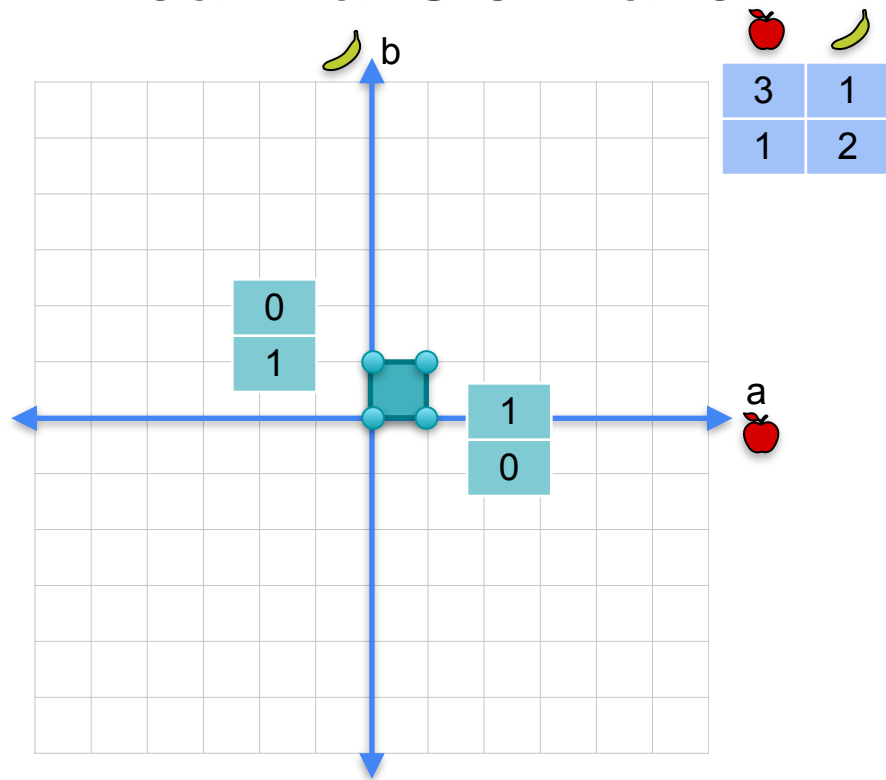
# Linear transformation



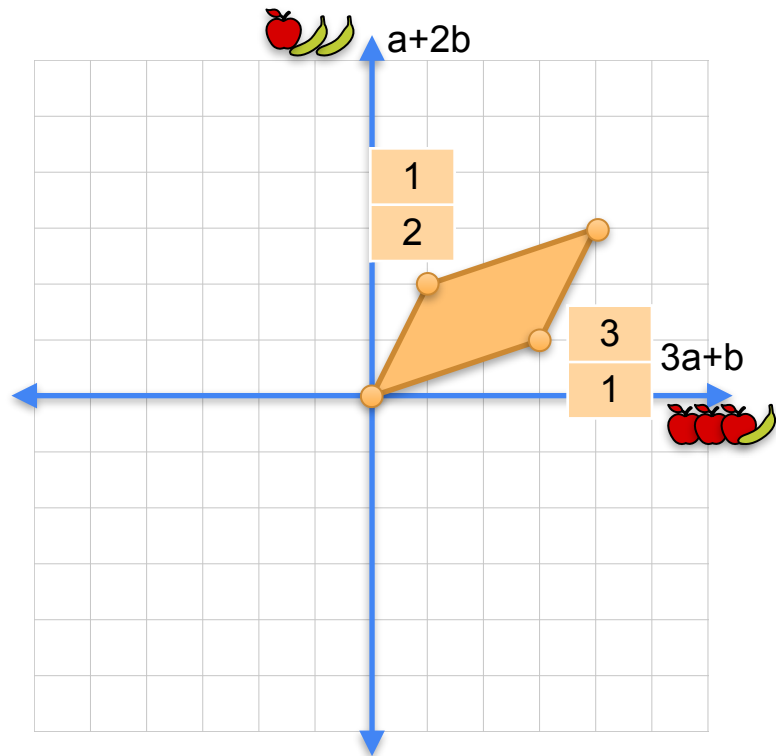
=



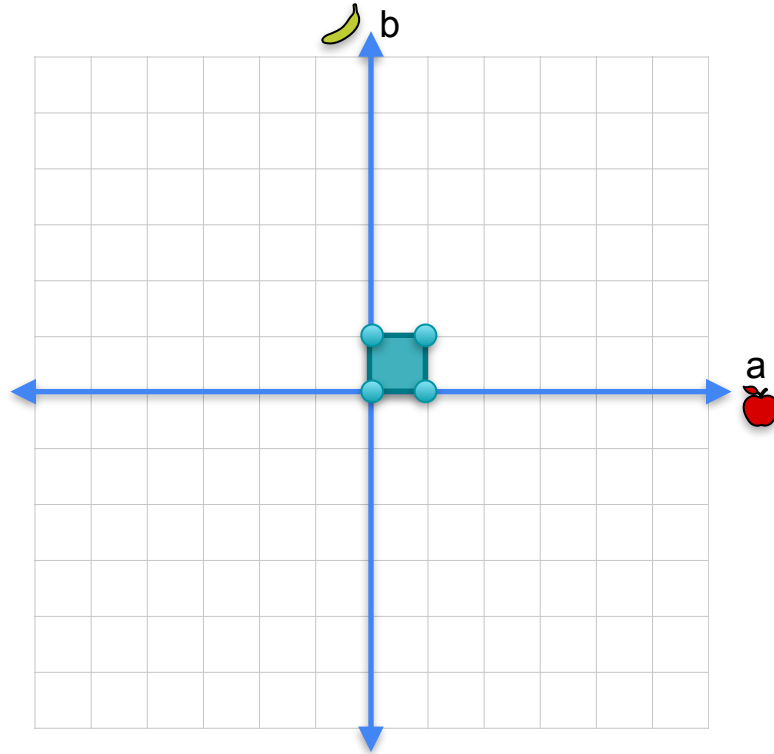
# Linear transformation



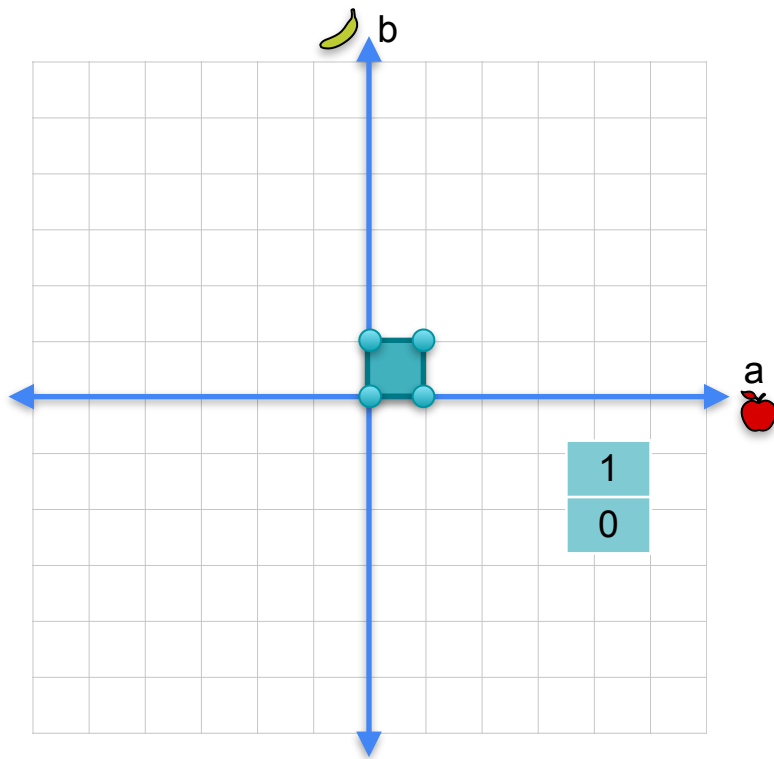
=



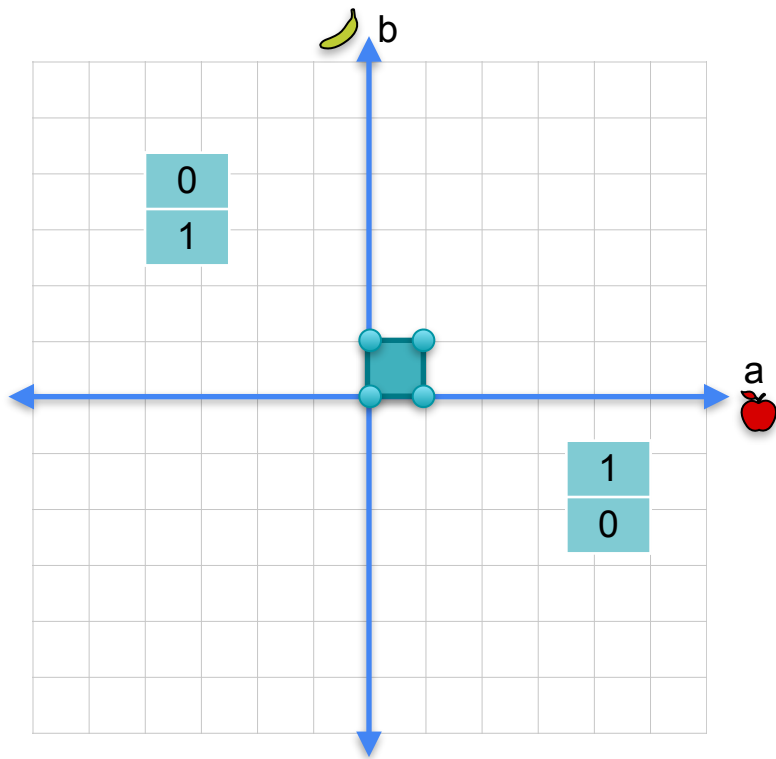
# Linear transformation



# Linear transformation



# Linear transformation





# 3D



DeepLearning.AI

# Math for Machine Learning

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## Linear algebra - Week 4

Vectors

Matrices



Dot product

Matrix multiplication



Linear transformations



# A matrix and its corresponding system of equations

# A matrix and its corresponding system of equations



	
1	1
1	2



# A matrix and its corresponding system of equations



	
1	1
1	2

	
1	1
2	2

# A matrix and its corresponding system of equations





	
1	1
1	2



	
1	1
2	2



	
0	0
0	0



# A matrix and its corresponding system of equations

## System 1

- $a + b = 0$   
 
- $a + 2b = 0$   
 





	
1	1
1	2



	
1	1
2	2

	
0	0
0	0






# A matrix and its corresponding system of equations



## System 1



- $a + b = 0$   
 
- $a + 2b = 0$   
 

	
1	1
1	2

## System 2

- $a + b = 0$   
 
- $2a + 2b = 0$   
  





	
1	1
2	2



	
0	0
0	0









# A matrix and its corresponding system of equations



System 1

- $a + b = 0$   
 
- $a + 2b = 0$   
 

	
1	1
1	2



System 2

- $a + b = 0$   
 
- $2a + 2b = 0$   
   

	
1	1
2	2

System 3



- $0a + 0b = 0$
- $0a + 0b = 0$

	
0	0
0	0

# A matrix and its corresponding system of equations



## System 1

- $a + b = 0$
- $a + 2b = 0$

	
1	1
1	2



## System 2

- $a + b = 0$
- $2a + 2b = 0$

	
1	1
2	2

## System 3

- $0a + 0b = 0$
- $0a + 0b = 0$

	
0	0
0	0

The only two numbers  $a$ ,  
 $b$ , such that

- $a + b = 0$
- and
- $a + 2b = 0$



are:

$a=0$  and  $b=0$

# A matrix and its corresponding system of equations

## System 1

- $a + b = 0$
- $a + 2b = 0$

	
1	1
1	2

The only two numbers  $a$ ,  
 $b$ , such that



- $a + b = 0$
- and
- $a + 2b = 0$

are:

$a=0$  and  $b=0$

## System 2

- $a + b = 0$
- $2a + 2b = 0$

	
1	1
2	2

Any pair  $(x, -x)$  satisfies that



- $a + b = 0$
- and
- $a + 2b = 0$

For example:

$(1, -1)$ ,  $(2, -2)$ ,  $(-8, 8)$ , etc.

## System 3



- $0a + 0b = 0$
- $0a + 0b = 0$

	
0	0
0	0

# A matrix and its corresponding system of equations

## System 1

- $a + b = 0$
- $a + 2b = 0$

	
1	1
1	2

The only two numbers  $a$ ,  $b$ , such that

- $a+b = 0$

and



- $a+2b = 0$

are:

$a=0$  and  $b=0$

## System 2

- $a + b = 0$
- $2a + 2b = 0$

	
1	1
2	2

Any pair  $(x, -x)$  satisfies that

- $a+b = 0$

and



- $a+2b = 0$

For example:

$(1,-1)$ ,  $(2,-2)$ ,  $(-8,8)$ , etc.

## System 3

- $0a + 0b = 0$
- $0a + 0b = 0$

	
0	0
0	0

Any pair of numbers satisfies that

- $0a+0b = 0$

and

- $0a+0b = 0$

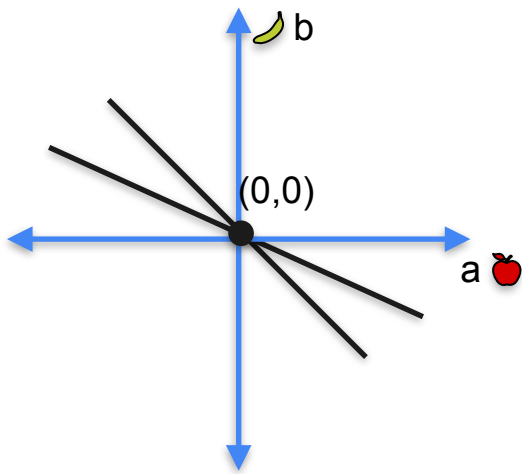
For example:

$(1,2)$ ,  $(3,-9)$ ,  $(-90,8.34)$ , etc.

# The set of solutions of a system of equations

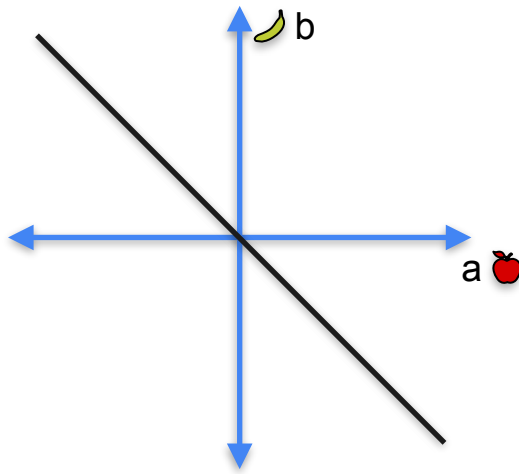
System 1

- $a + b = 0$   
🍏 🍌
- $a + 2b = 0$   
🍏 🍌🍌



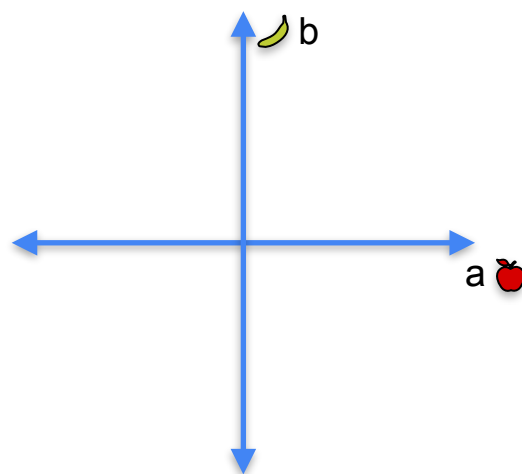
System 2

- $a + b = 0$   
🍏 🍌
- $2a + 2b = 0$   
🍏🍏 🍌🍌



System 3

- $0a + 0b = 0$
- $0a + 0b = 0$



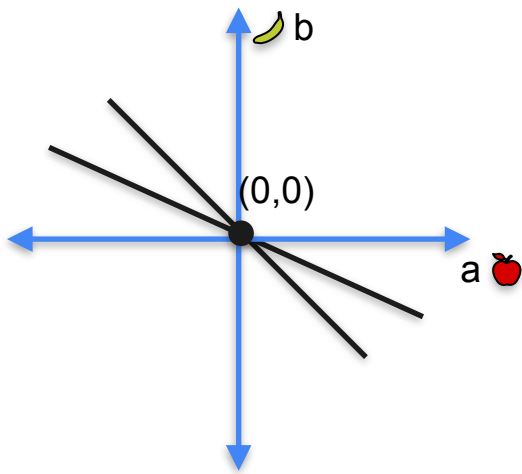
# The set of solutions of a system of equations

System 1

- $a + b = 0$
- $a + 2b = 0$

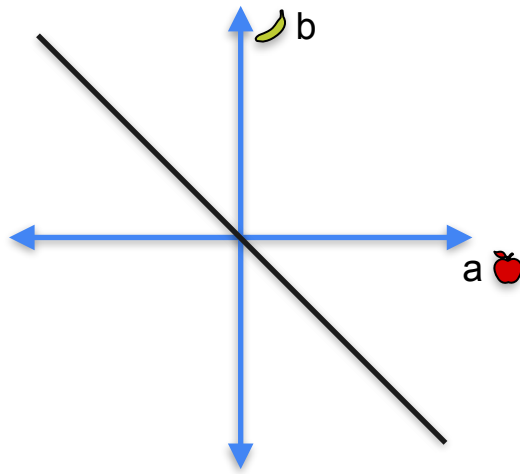
**Solution**

- $a = 0$
- $b = 0$



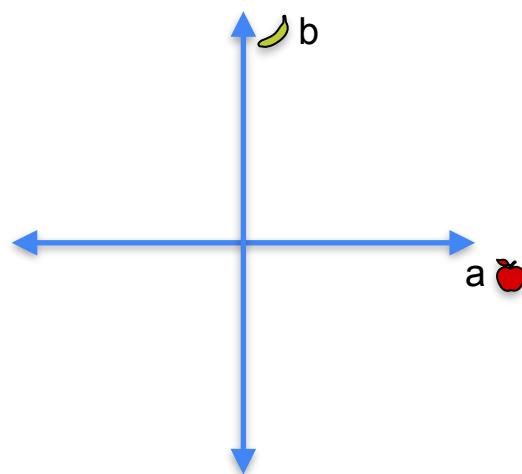
System 2

- $a + b = 0$
- $2a + 2b = 0$



System 3

- $0a + 0b = 0$
- $0a + 0b = 0$



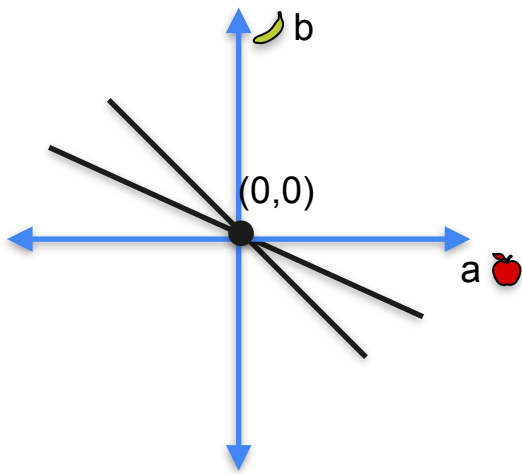
# The set of solutions of a system of equations

System 1

- $a + b = 0$
- $a + 2b = 0$

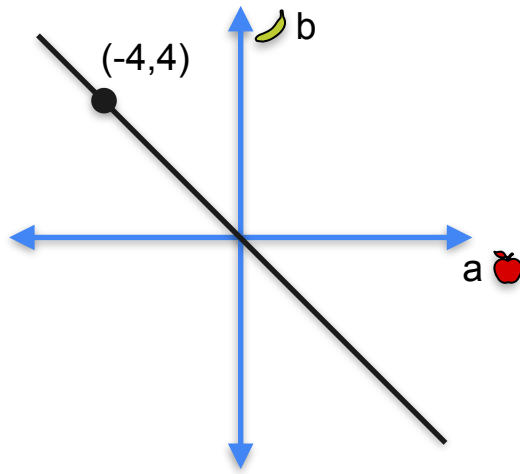
**Solution**

- $a = 0$
- $b = 0$



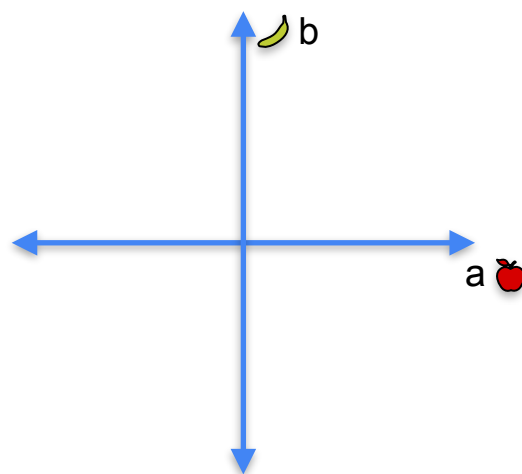
System 2

- $a + b = 0$
- $2a + 2b = 0$



System 3

- $0a + 0b = 0$
- $0a + 0b = 0$



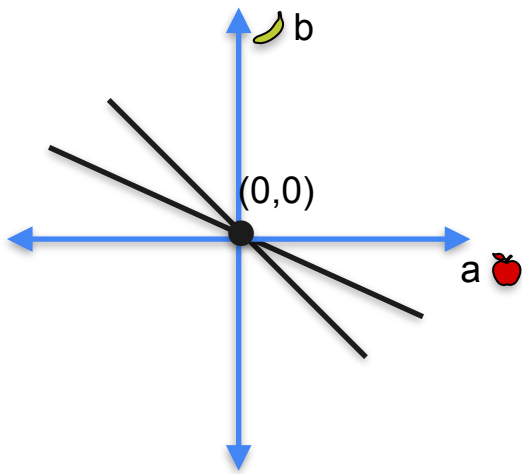
# The set of solutions of a system of equations

System 1

- $a + b = 0$
- $a + 2b = 0$

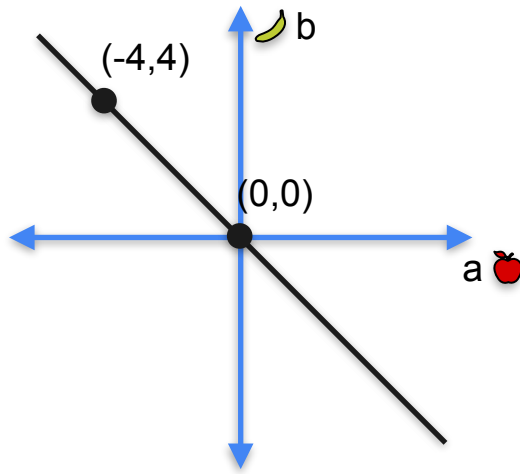
**Solution**

- $a = 0$
- $b = 0$



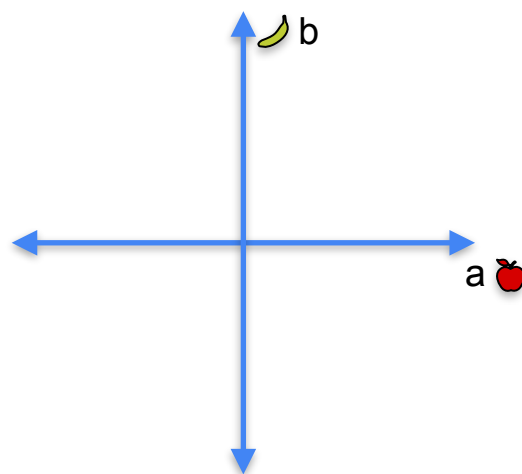
System 2

- $a + b = 0$
- $2a + 2b = 0$



System 3

- $0a + 0b = 0$
- $0a + 0b = 0$





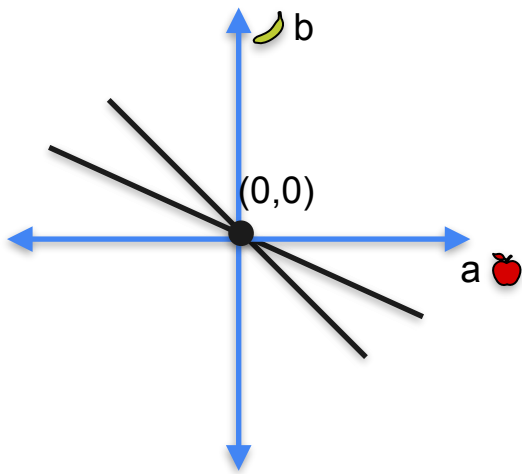
# The set of solutions of a system of equations

System 1

- $a + b = 0$
- $a + 2b = 0$

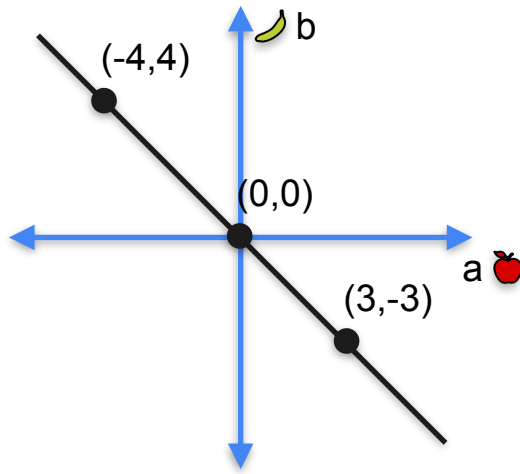
Solution

- $a = 0$
- $b = 0$



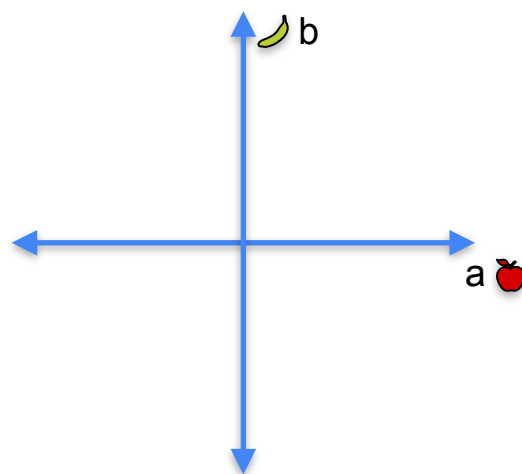
System 2

- $a + b = 0$
- $2a + 2b = 0$



System 3

- $0a + 0b = 0$
- $0a + 0b = 0$



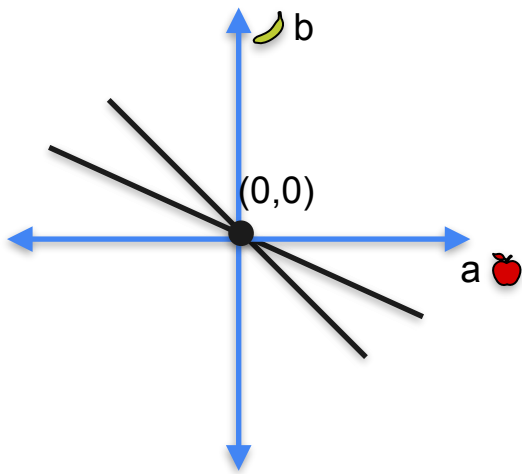
# The set of solutions of a system of equations

## System 1

- $a + b = 0$
- $a + 2b = 0$

### Solution

- $a = 0$
- $b = 0$

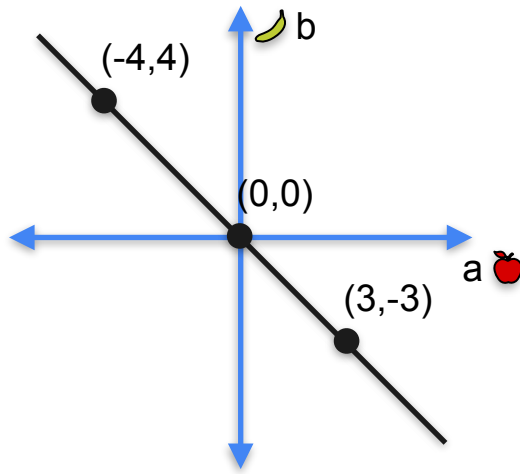


## System 2

- $a + b = 0$
- $2a + 2b = 0$

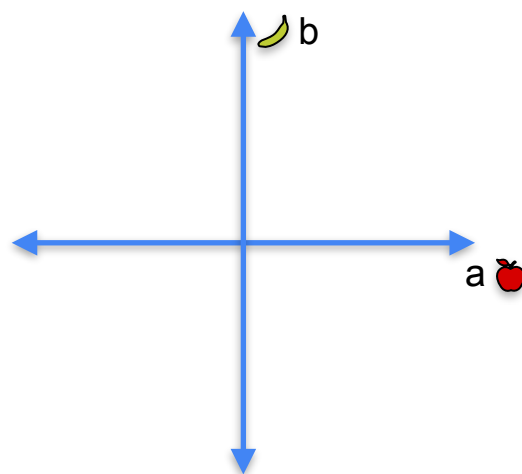
### Solutions

- any  $a$
- $b = -a$



## System 3

- $0a + 0b = 0$
- $0a + 0b = 0$



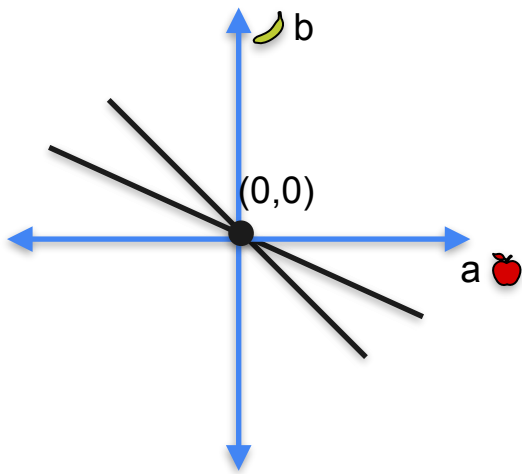
# The set of solutions of a system of equations

System 1

- $a + b = 0$
- $a + 2b = 0$

**Solution**

- $a = 0$
- $b = 0$

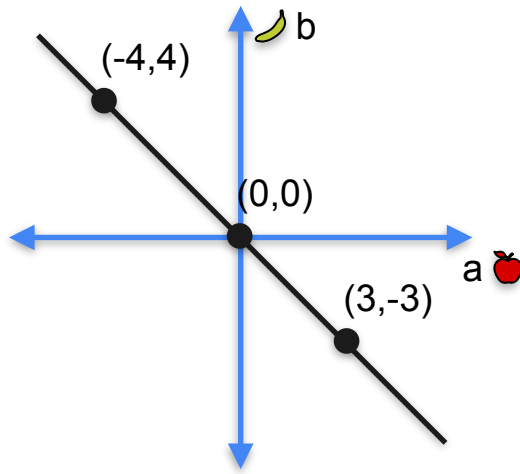


System 2

- $a + b = 0$
- $2a + 2b = 0$

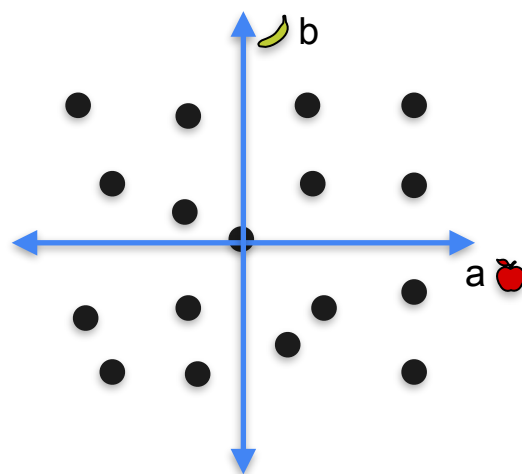
**Solutions**

- any  $a$
- $b = -a$



System 3

- $0a + 0b = 0$
- $0a + 0b = 0$



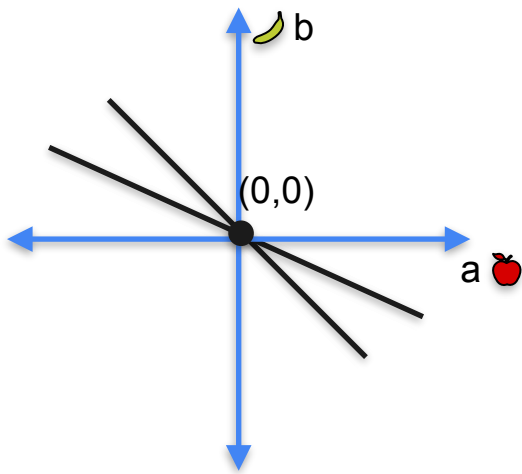
# The set of solutions of a system of equations

System 1

- $a + b = 0$
- $a + 2b = 0$

**Solution**

- $a = 0$
- $b = 0$

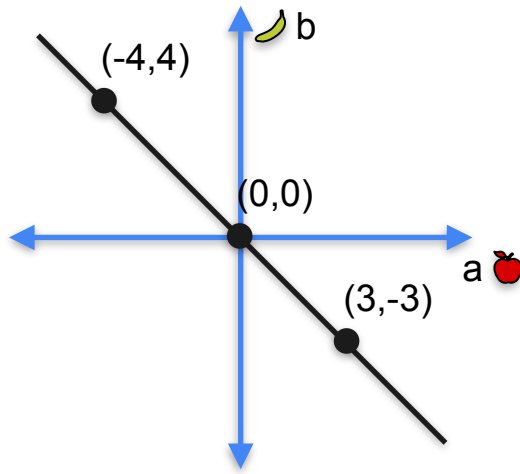


System 2

- $a + b = 0$
- $2a + 2b = 0$

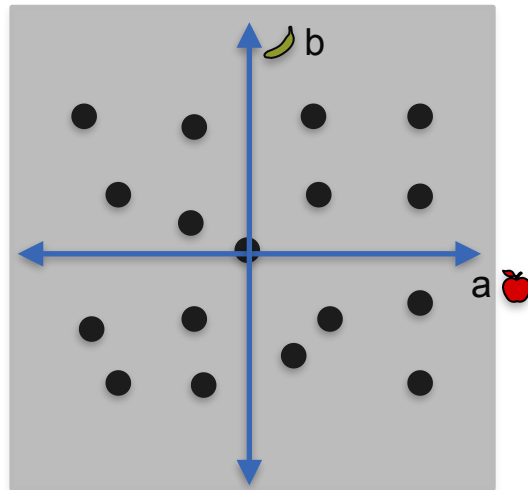
**Solutions**

- any  $a$
- $b = -a$



System 3

- $0a + 0b = 0$
- $0a + 0b = 0$



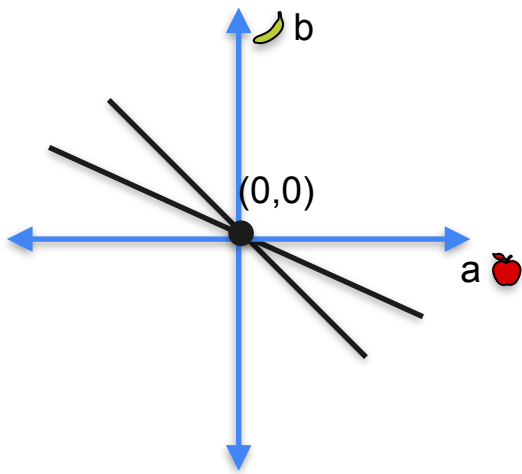
# The set of solutions of a system of equations

System 1

- $a + b = 0$
- $a + 2b = 0$

**Solution**

- $a = 0$
- $b = 0$

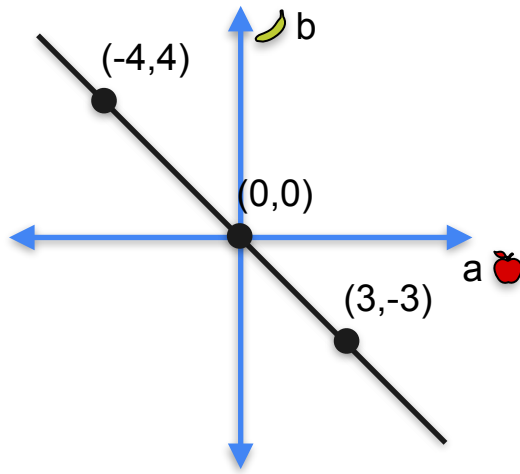


System 2

- $a + b = 0$
- $2a + 2b = 0$

**Solutions**

- any  $a$
- $b = -a$

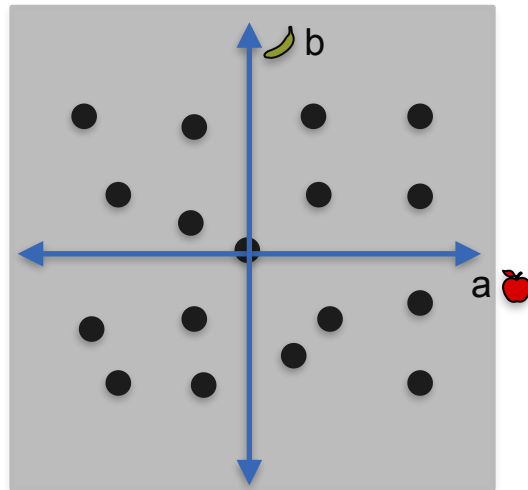


System 3

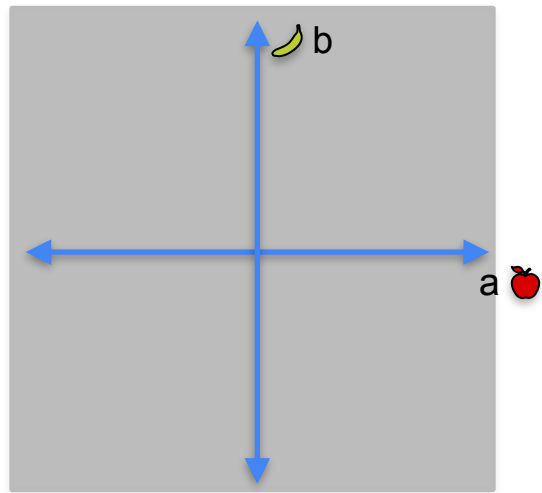
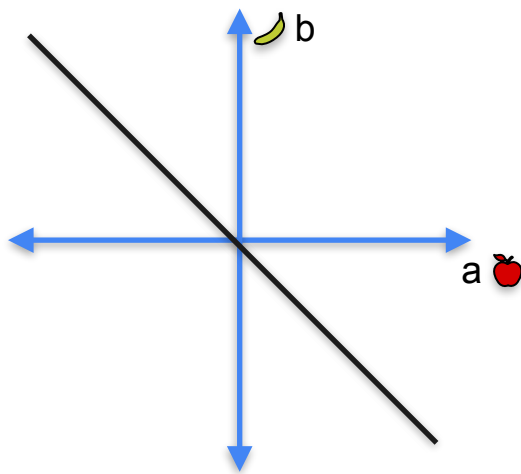
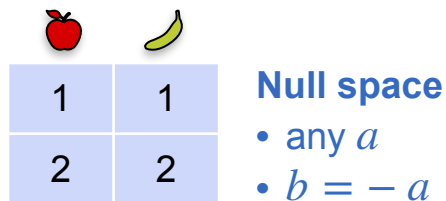
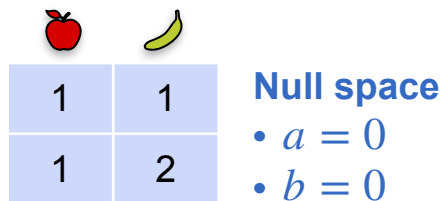
- $0a + 0b = 0$
- $0a + 0b = 0$

**Solutions**



- any  $a$
- any  $b$



# The null space of a matrix



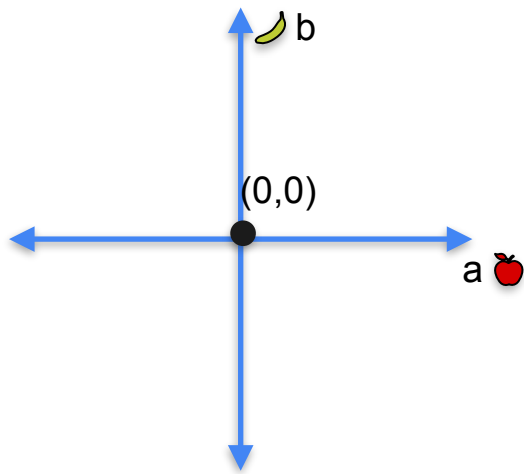
# The null space of a matrix



	
1	1
1	2

**Null space**

- $a = 0$
- $b = 0$

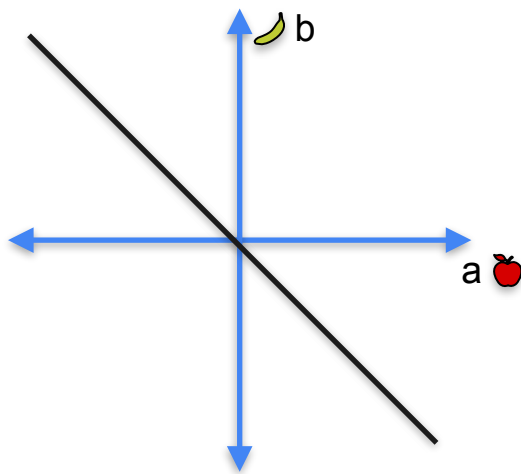
Dimension = 0




	
1	1
2	2

**Null space**

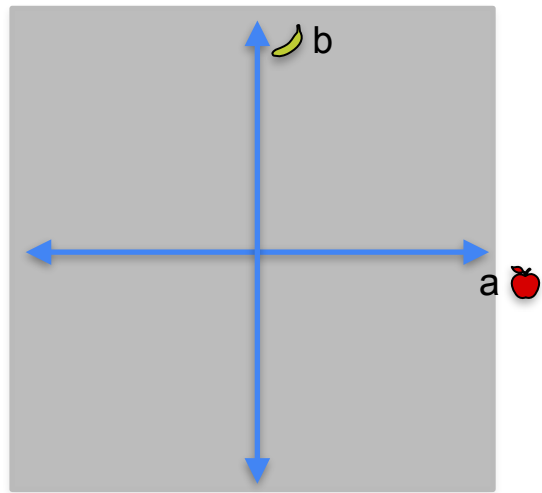
- any  $a$
- $b = -a$





	
0	0
0	0

**Null space**

- any  $a$
- any  $b$



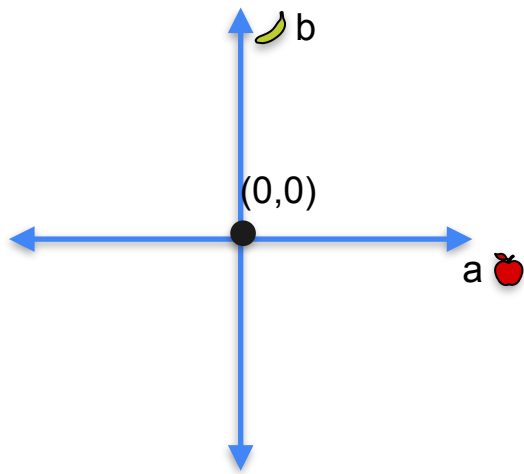
# The null space of a matrix



	
1	1
1	2

**Null space**

- $a = 0$
- $b = 0$

Dimension = 0

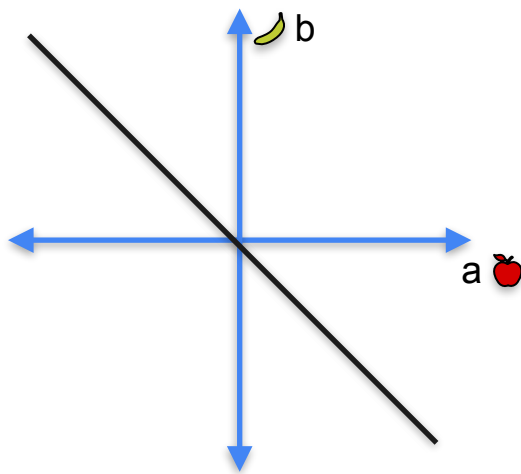



	
1	1
2	2

**Null space**

- any  $a$
- $b = -a$

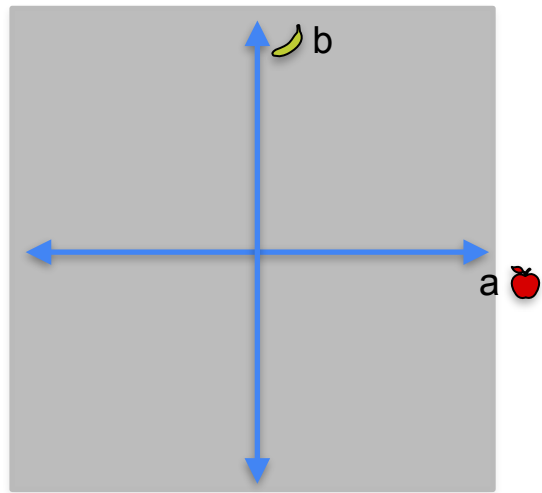
Dimension = 1



	
0	0
0	0

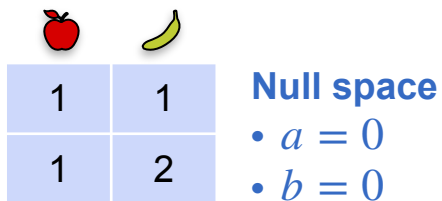
**Null space**

- any  $a$
- any  $b$

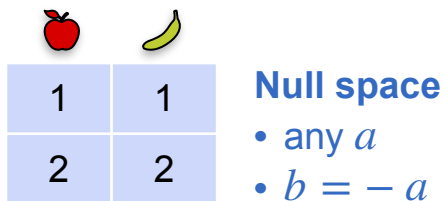
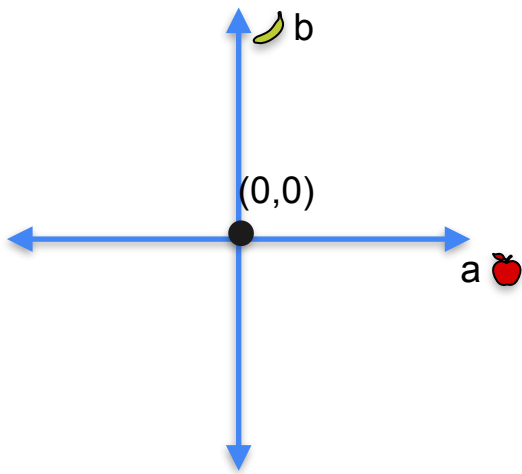




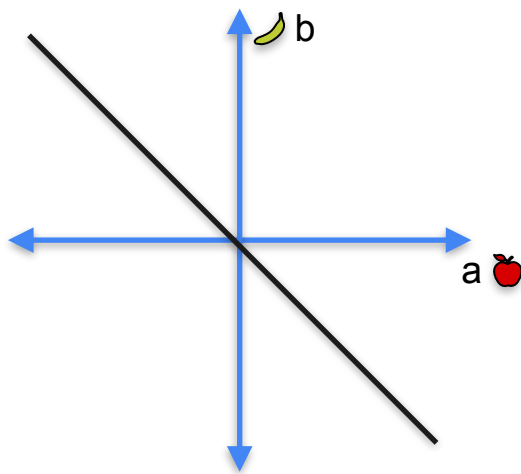
# The null space of a matrix



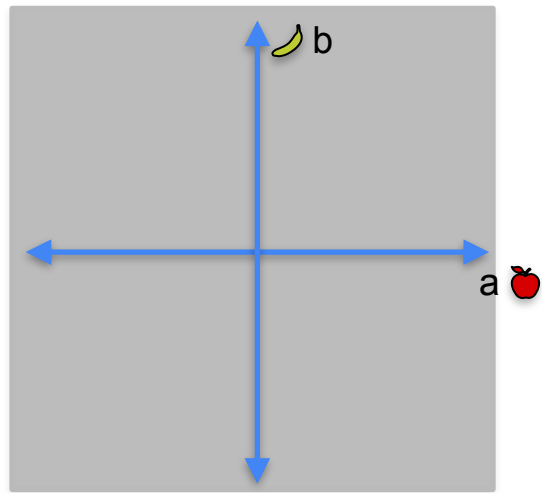
Dimension = 0





Dimension = 1



Dimension = 2



# The null space of a matrix



	
1	1
1	2

**Null space**

- $a = 0$
- $b = 0$

Dimension = 0




	
1	1
2	2

**Null space**

- any  $a$
- $b = -a$

Dimension = 1



	
0	0
0	0



**Null space**

- any  $a$
- any  $b$

Dimension = 2



# The null space of a matrix

	
1	1
1	2



**Null space**

- $a = 0$
- $b = 0$

Dimension = 0



Non-singular


	
1	1
2	2

**Null space**

- any  $a$
- $b = -a$

Dimension = 1



	
0	0
0	0



**Null space**

- any  $a$
- any  $b$

Dimension = 2



# The null space of a matrix

	
1	1
1	2



**Null space**

- $a = 0$
- $b = 0$

Dimension = 0



Non-singular

	
1	1
2	2



**Null space**

- any  $a$
- $b = -a$

Dimension = 1



Singular

	
0	0
0	0



**Null space**

- any  $a$
- any  $b$

Dimension = 2



# The null space of a matrix

	
1	1
1	2



**Null space**

- $a = 0$
- $b = 0$

Dimension = 0



Non-singular

	
1	1
2	2


**Null space**

- any  $a$
- $b = -a$

Dimension = 1



Singular

	
0	0
0	0

**Null space**



- any  $a$
- any  $b$

Dimension = 2



Singular

# The null space of a matrix

	
1	1
1	2



**Null space**

- $a = 0$
- $b = 0$

Dimension = 0



Non-singular

	
1	1
2	2



**Null space**

- any  $a$
- $b = -a$

Dimension = 1



Singular

	
0	0
0	0

**Null space**

- any  $a$
- any  $b$

Dimension = 2



Singular

# More conceptual explanation of the null space

- Elaborate here

# Quiz: Null space of a matrix

**Problem:** Determine the dimension of the null space of the following two matrices

**Matrix 1**

5	1
-1	3

**Matrix 2**

2	-1
-6	3



# Solutions: Null space of a matrix

**Matrix 1:** Notice that this is a non-singular matrix, since the determinant is 16. Therefore, the null space is only the point (0,0). The dimension is 0.

5	1
-1	3

**Matrix 2:** The corresponding system of equation has the equations  $2a - b = 0$  and  $-6a + 3b = 0$ . Some inspection shows that the first equation has the points (1,2), (2,4), (3,6), etc. as solutions. All of them are also solutions to the second equation,  $-6a + 3b = 0$ . Therefore the null space is all the points of the form (x, 2x). The dimension of this null space is 1, and the matrix is singular.

2	-1
-6	3

# Systems of linear equations

# Systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

# Systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

# Systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

# Systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

# Systems of linear equations

## System 1

- $a + b + c = 0$
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- $a + b + 2c = 0$

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- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

1	1	1
1	2	1
1	1	2

# Systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

1	1	1
1	2	1
1	1	2

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

1	1	1
1	1	2
1	1	3

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$



# Systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

1	1	1
1	2	1
1	1	2

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

1	1	1
1	1	2
1	1	3

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

1	1	1
2	2	2
3	3	3

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

# Systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

1	1	1
1	2	1
1	1	2

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

1	1	1
1	1	2
1	1	3

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

1	1	1
2	2	2
3	3	3

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

0	0	0
0	0	0
0	0	0

# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

# Null space for systems of linear equations

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- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

## Solution space



# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

### Solution space



## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space



## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

### Solution space



## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space



## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

### Solution space



## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

### Solution space



## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space



## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

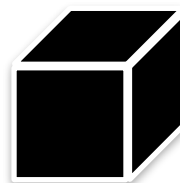
### Solution space



## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

### Solution space



# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

Solution space



Dimension = 0

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

Solution space



## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

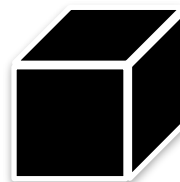
Solution space



## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

Solution space





# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

### Solution space



Dimension = 0

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space



Dimension = 1

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

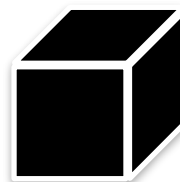
### Solution space



## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

### Solution space



# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

### Solution space



Dimension = 0

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space



Dimension = 1

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

### Solution space

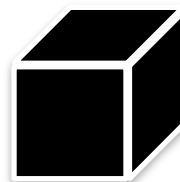


Dimension = 2

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

### Solution space



# Null space for systems of linear equations

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

### Solution space



Dimension = 0

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space



Dimension = 1

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

### Solution space

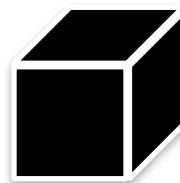


Dimension = 2

## System 4

- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$
- $0a + 0b + 0c = 0$

### Solution space



Dimension = 3

# Null space for matrices

Matrix 1

1	1	1
1	2	1
1	1	2

Null space



Dimension = 0

Matrix 2

1	1	1
1	1	2
1	1	3

Null space



Dimension = 1

Matrix 3

1	1	1
2	2	2
3	3	3

Null space

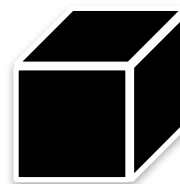


Dimension = 2

Matrix 4

0	0	0
0	0	0
0	0	0

Null space



Dimension = 3

# Quiz: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

1	1	1
1	1	2
0	0	-1

1	1	1
0	2	2
0	0	3

# Solution: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

- $a + c = 0$
- $b = 0$
- $3a + 2b + 3c = 0$

1	1	1
1	1	2
0	0	-1

- $a + b + c = 0$
- $a + b + 2c = 0$
- $c = 0$

1	1	1
0	2	2
0	0	3

- $a + b + c = 0$
- $2b + 2c = 0$
- $3c = 0$

# Solution: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

- $a + c = 0$
- $b = 0$
- $3a + 2b + 3c = 0$

1	1	1
1	1	2
0	0	-1

- $a + b + c = 0$
- $a + b + 2c = 0$
- $c = 0$

1	1	1
0	2	2
0	0	3

- $a + b + c = 0$
- $2b + 2c = 0$
- $3c = 0$

**All points of the form**  
 $(x, 0, -x)$

# Solution: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

- $a + c = 0$
- $b = 0$
- $3a + 2b + 3c = 0$

1	1	1
1	1	2
0	0	-1

- $a + b + c = 0$
- $a + b + 2c = 0$
- $c = 0$

1	1	1
0	2	2
0	0	3

- $a + b + c = 0$
- $2b + 2c = 0$
- $3c = 0$

**All points of the form**

$(x, 0, -x)$

**Dimension = 1**



# Solution: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

- $a + c = 0$
- $b = 0$
- $3a + 2b + 3c = 0$

**All points of the form**  
 $(x, 0, -x)$   
**Dimension = 1**

1	1	1
1	1	2
0	0	-1

- $a + b + c = 0$
- $a + b + 2c = 0$
- $c = 0$

**All points of the form**  
 $(x, -x, 0)$

1	1	1
0	2	2
0	0	3

- $a + b + c = 0$
- $2b + 2c = 0$
- $3c = 0$

# Solution: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

- $a + c = 0$
- $b = 0$
- $3a + 2b + 3c = 0$

**All points of the form**

$(x, 0, -x)$

**Dimension = 1**

1	1	1
1	1	2
0	0	-1

- $a + b + c = 0$
- $a + b + 2c = 0$
- $c = 0$

**All points of the form**

$(x, -x, 0)$

**Dimension = 1**

1	1	1
0	2	2
0	0	3

- $a + b + c = 0$
- $2b + 2c = 0$
- $3c = 0$

# Solution: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

- $a + c = 0$
- $b = 0$
- $3a + 2b + 3c = 0$

**All points of the form**  
 $(x, 0, -x)$   
**Dimension = 1**

1	1	1
1	1	2
0	0	-1

- $a + b + c = 0$
- $a + b + 2c = 0$
- $c = 0$

**All points of the form**  
 $(x, -x, 0)$   
**Dimension = 1**

1	1	1
0	2	2
0	0	3

- $a + b + c = 0$
- $2b + 2c = 0$
- $3c = 0$

**The point**  
 $(0, 0, 0)$

# Solution: Null space

**Problem:** Determine the dimension of the null space for the following matrices.

1	0	1
0	1	0
3	2	3

- $a + c = 0$
- $b = 0$
- $3a + 2b + 3c = 0$

**All points of the form**  
 $(x, 0, -x)$   
**Dimension = 1**

1	1	1
1	1	2
0	0	-1

- $a + b + c = 0$
- $a + b + 2c = 0$
- $c = 0$

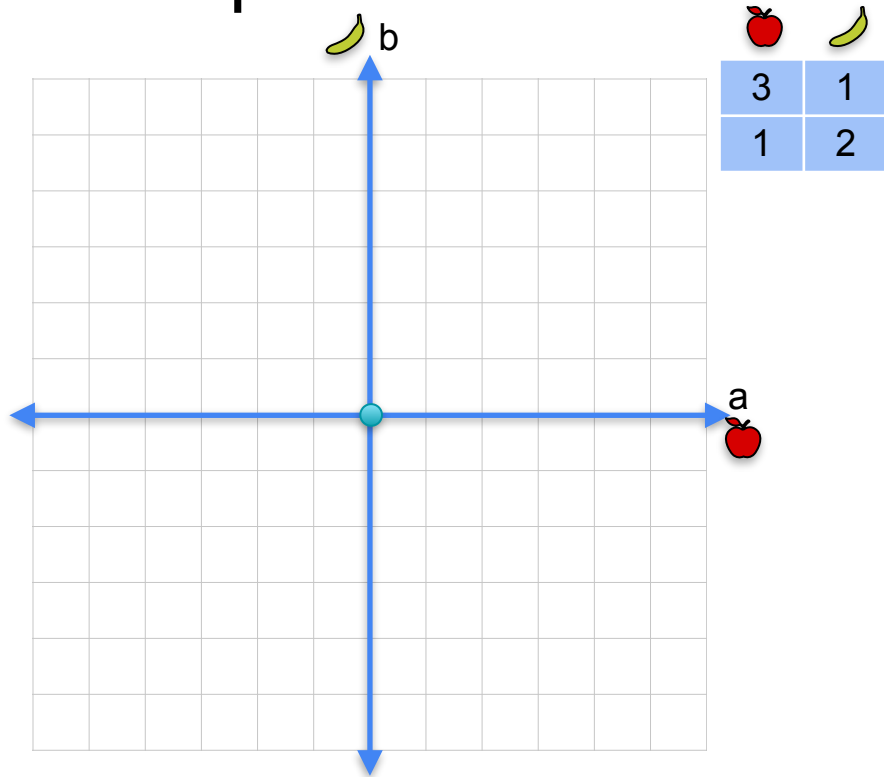
**All points of the form**  
 $(x, -x, 0)$   
**Dimension = 1**

1	1	1
0	2	2
0	0	3

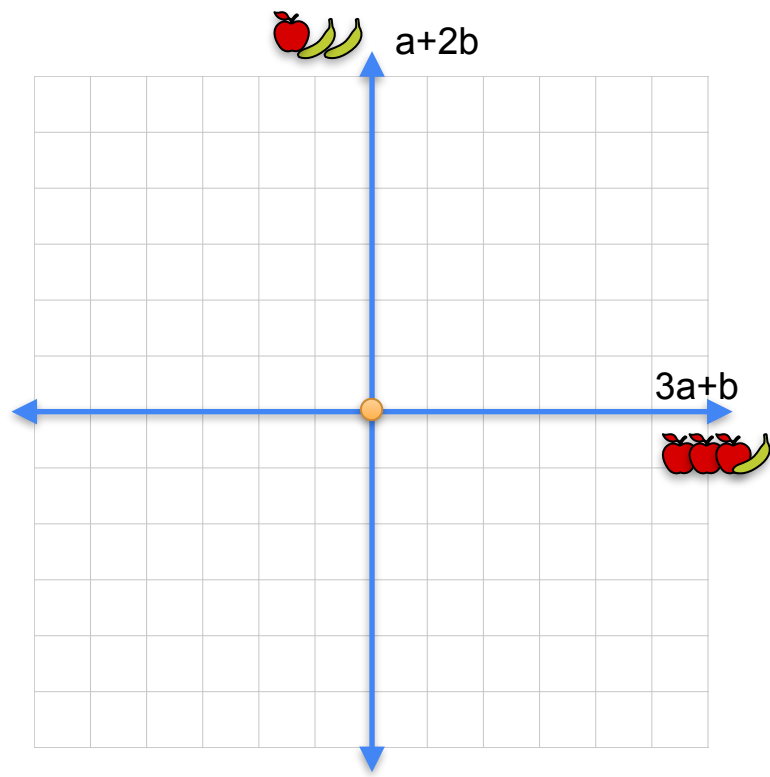
- $a + b + c = 0$
- $2b + 2c = 0$
- $3c = 0$

**The point**  
 $(0, 0, 0)$   
**Dimension = 0**

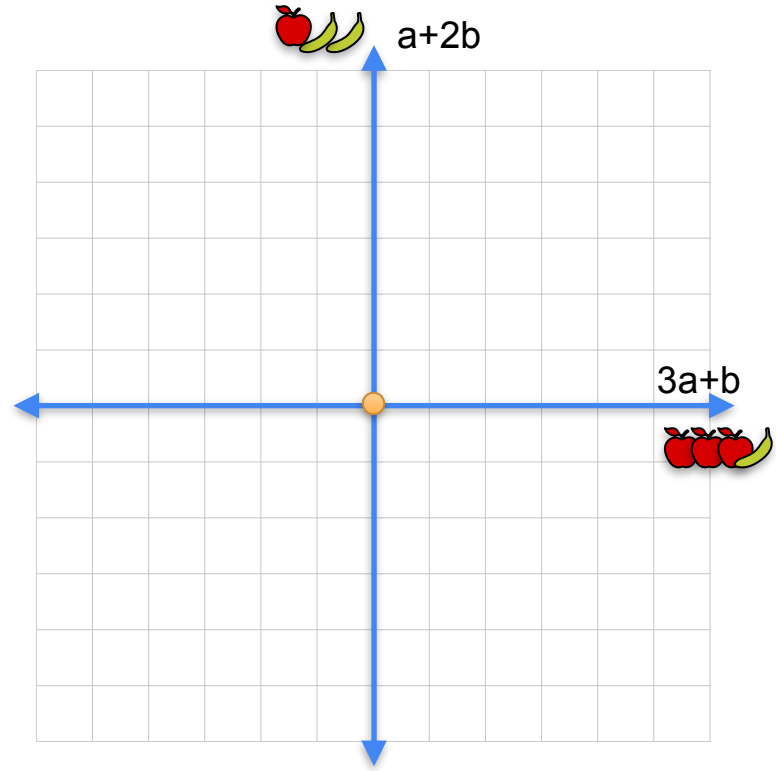
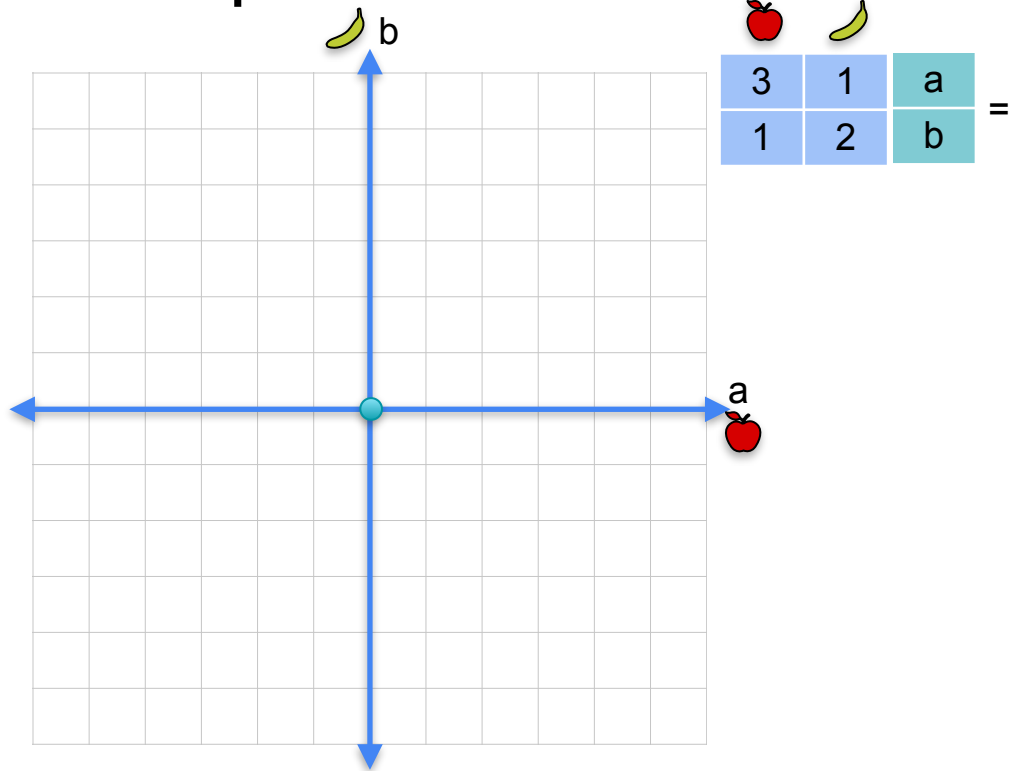
# Null space



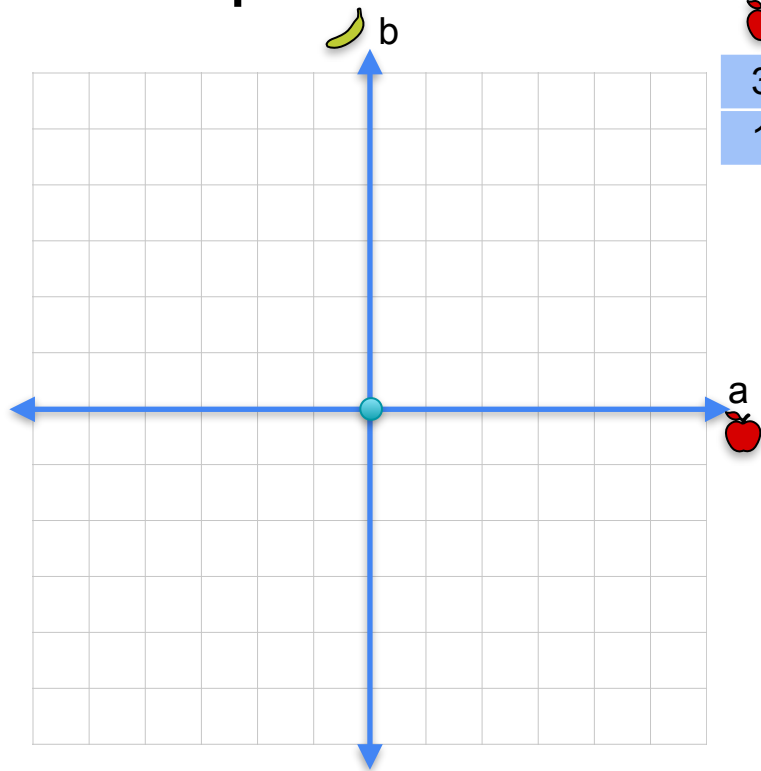
=





# Null space



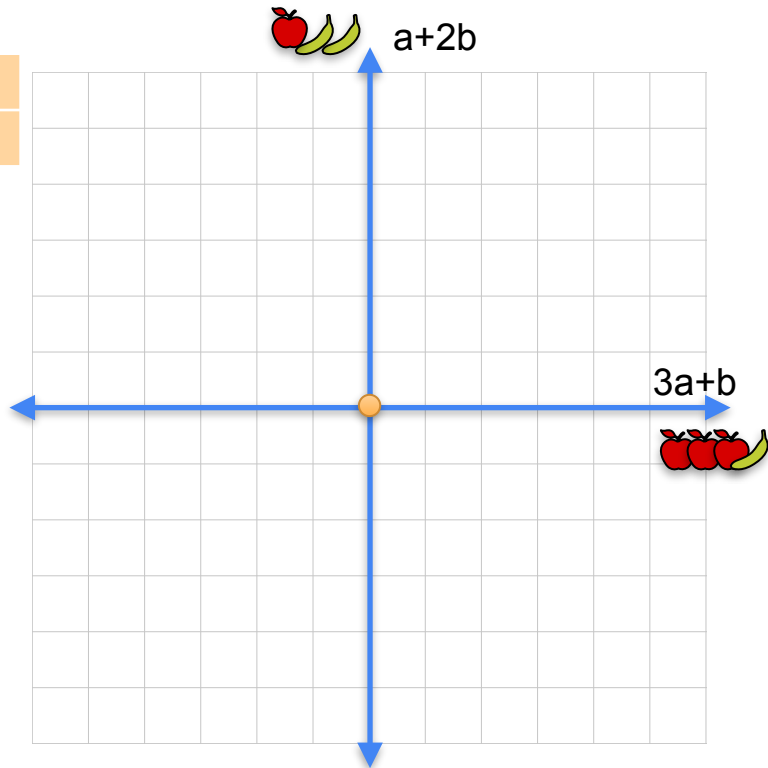
# Null space



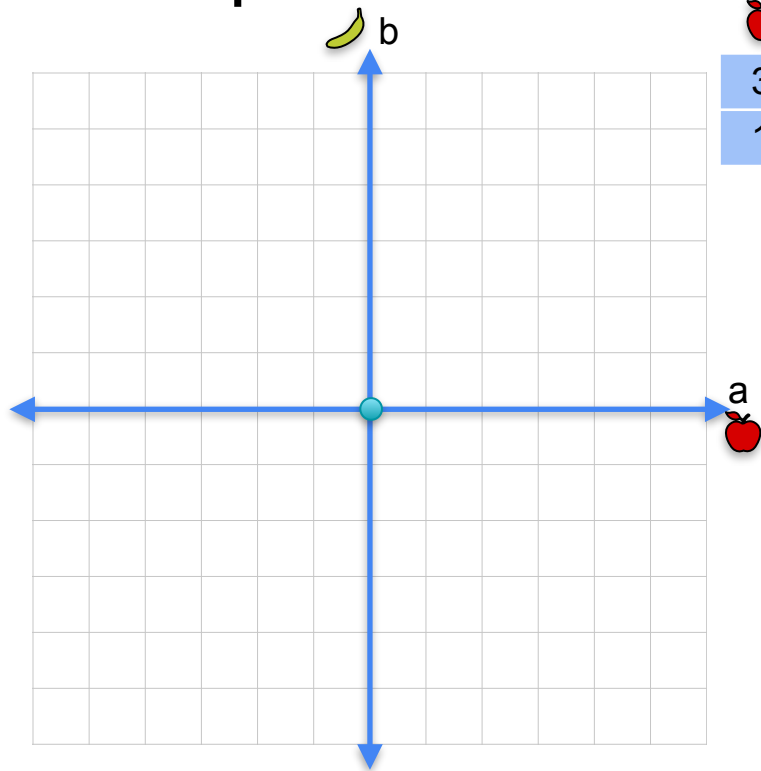
		
3	1	a
1	2	b



 = 

0
0

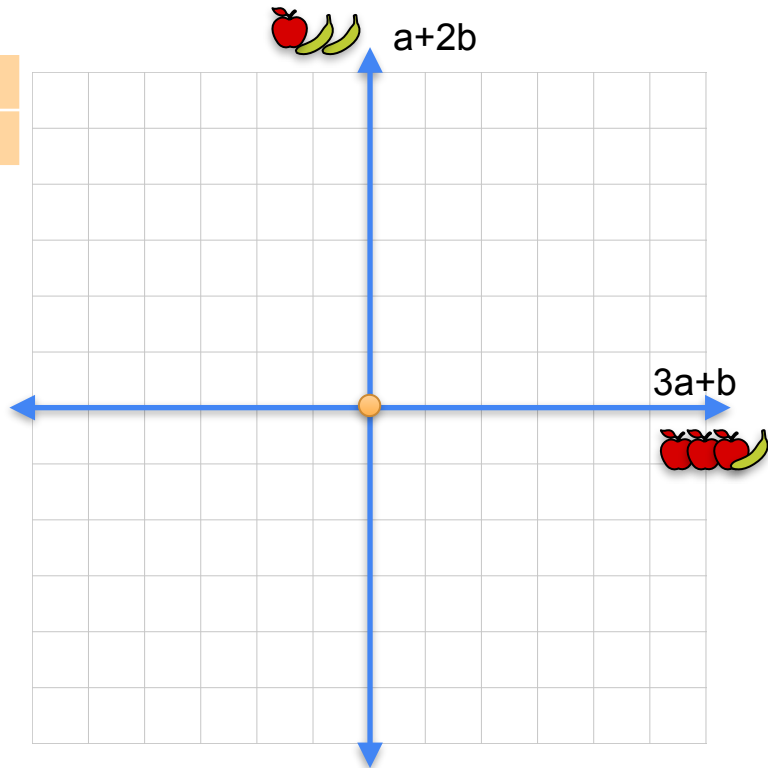


# Null space



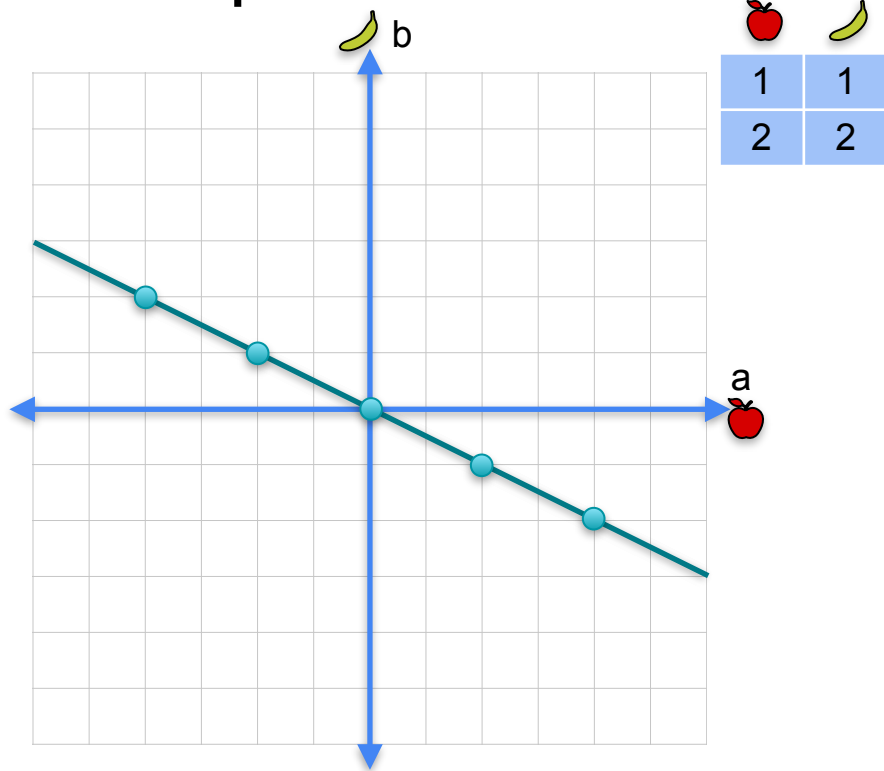
			
3	1	0	=
1	2	0	

0
0

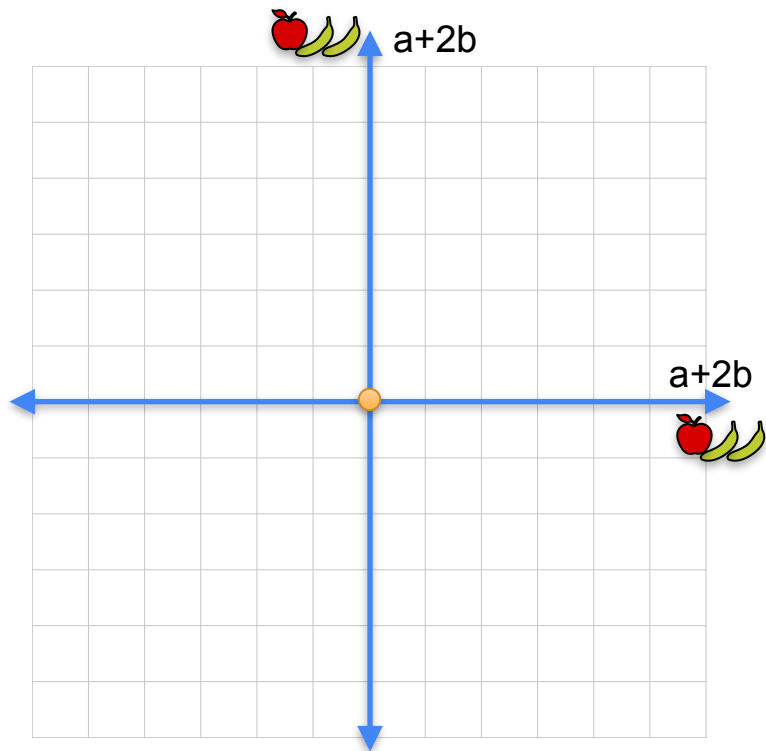




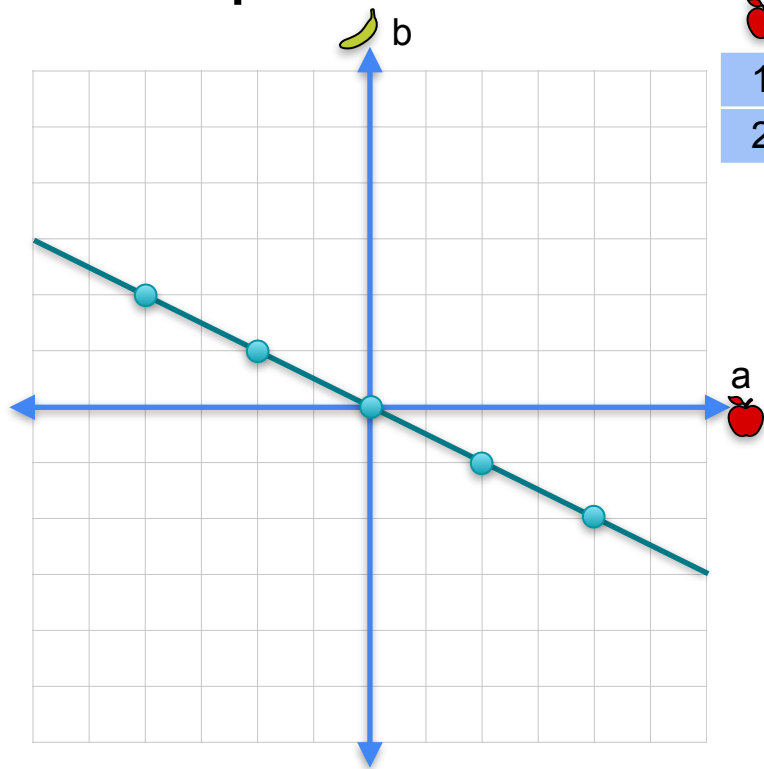
# Null space





=

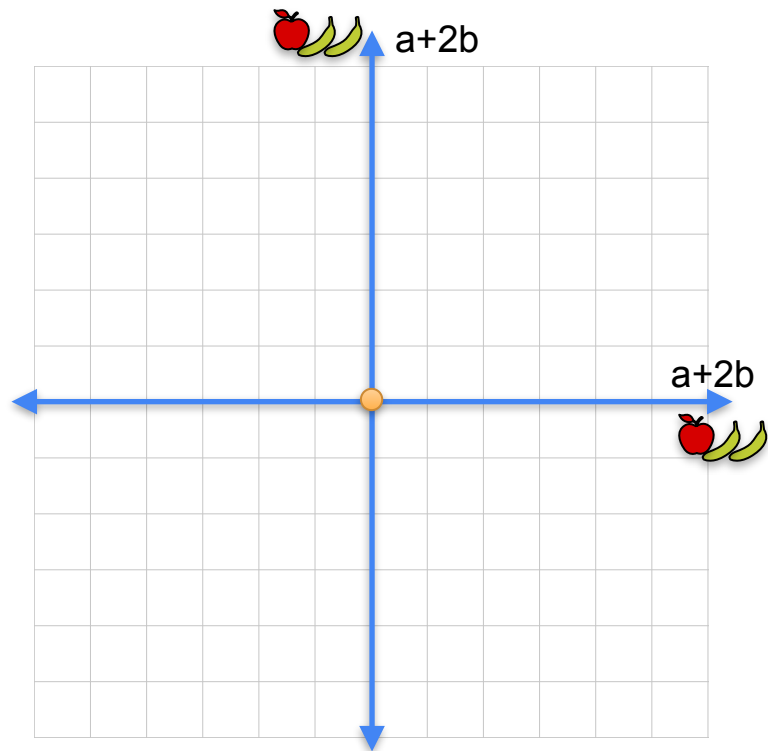


# Null space

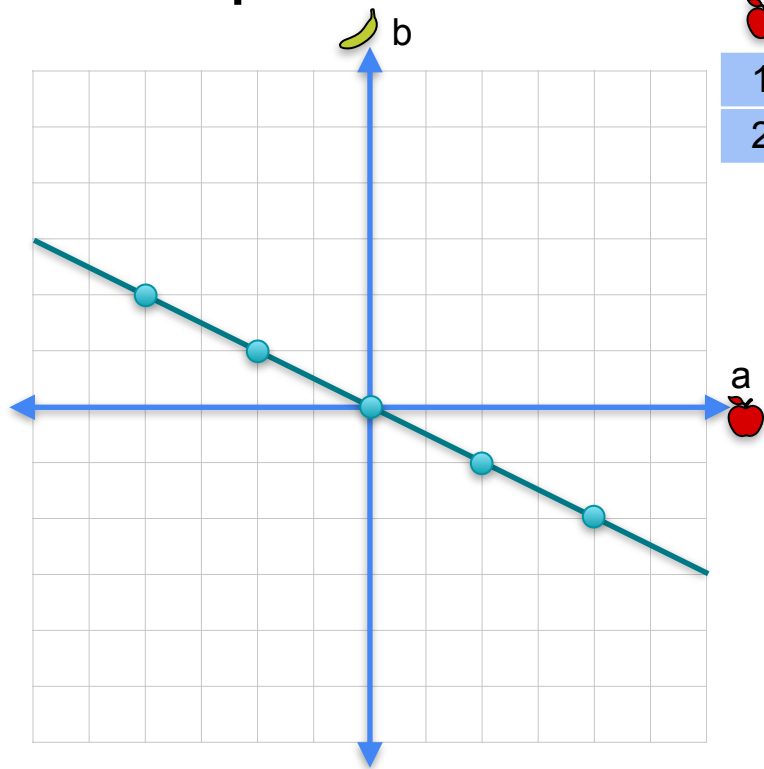




		
1	1	a
2	2	b

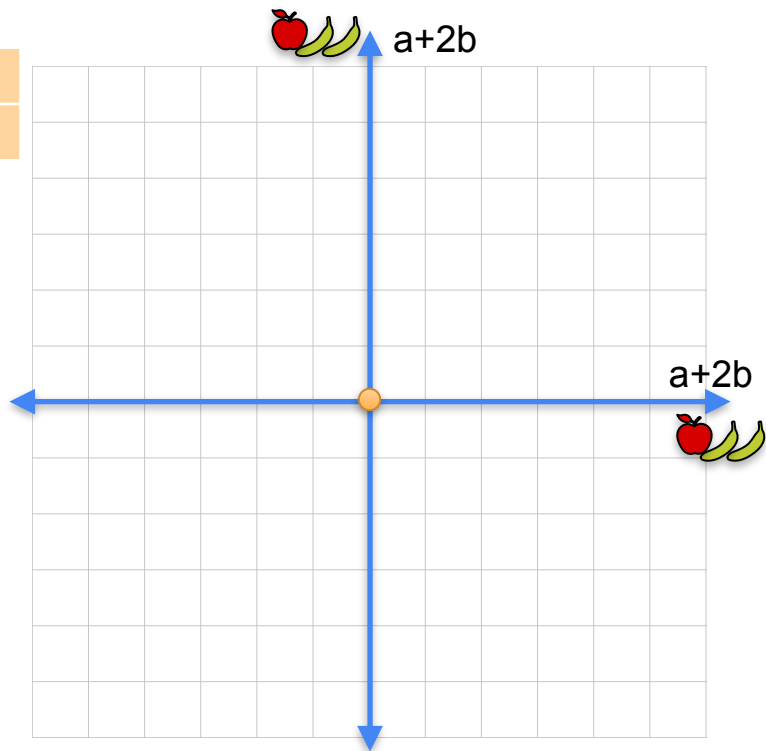
=



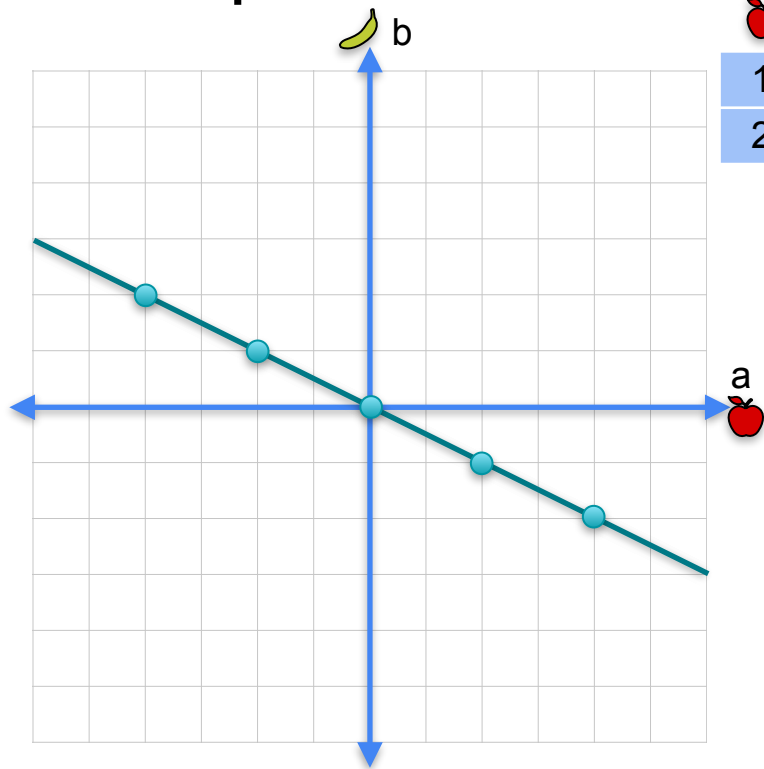
# Null space





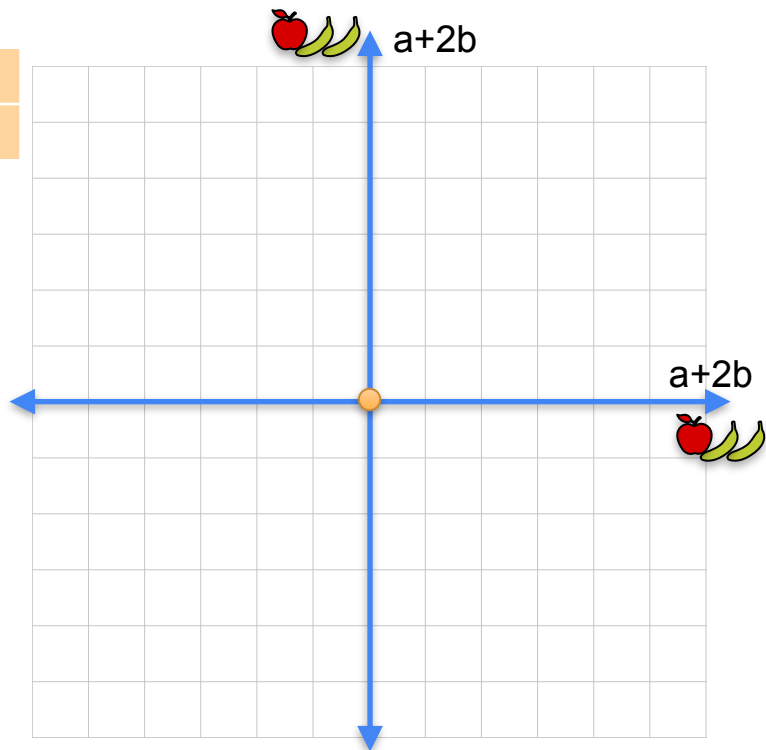
				
1	1	a	=	0
2	2	b		0



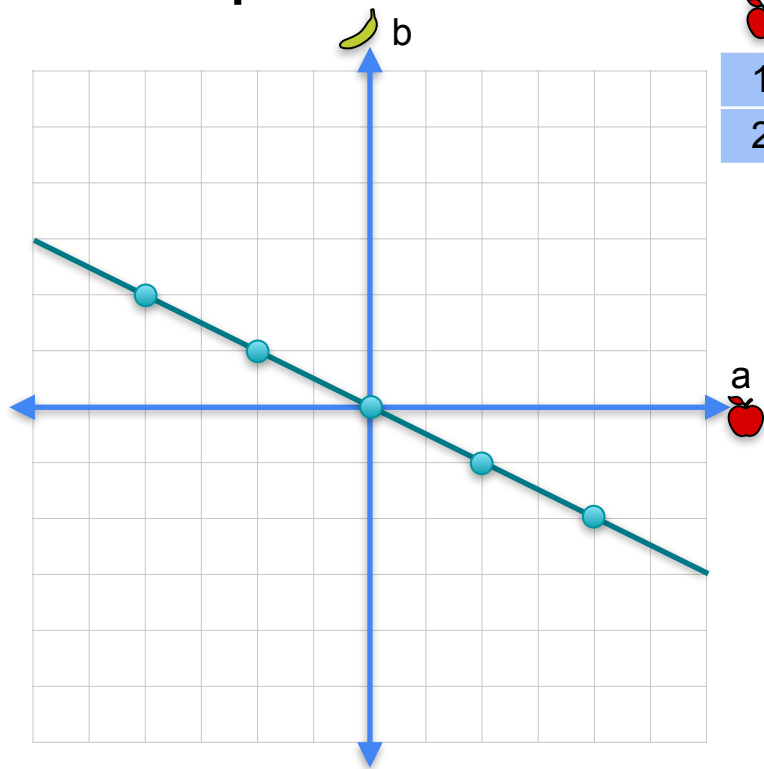
# Null space





				
1	1	0	=	0
2	2	0		0



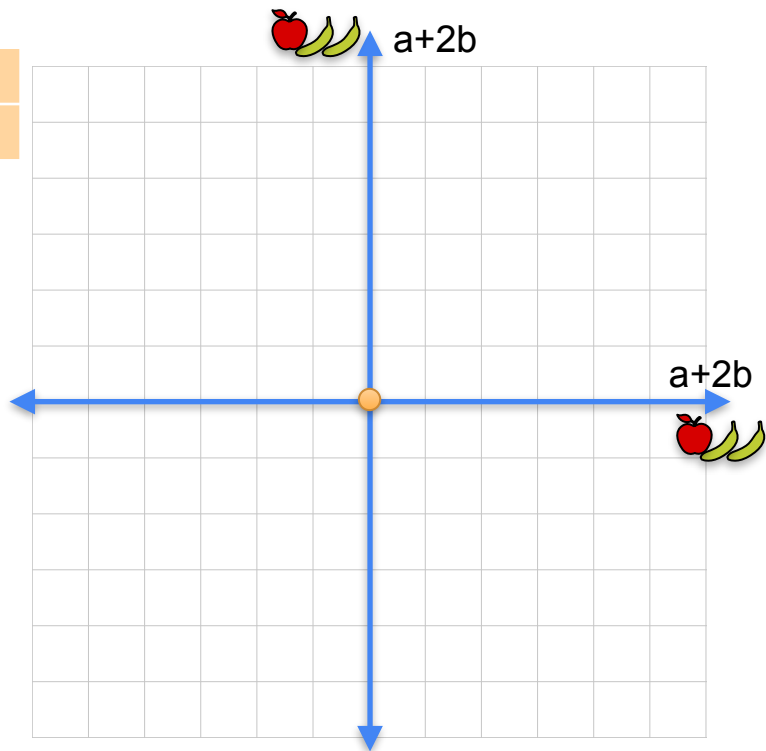
# Null space



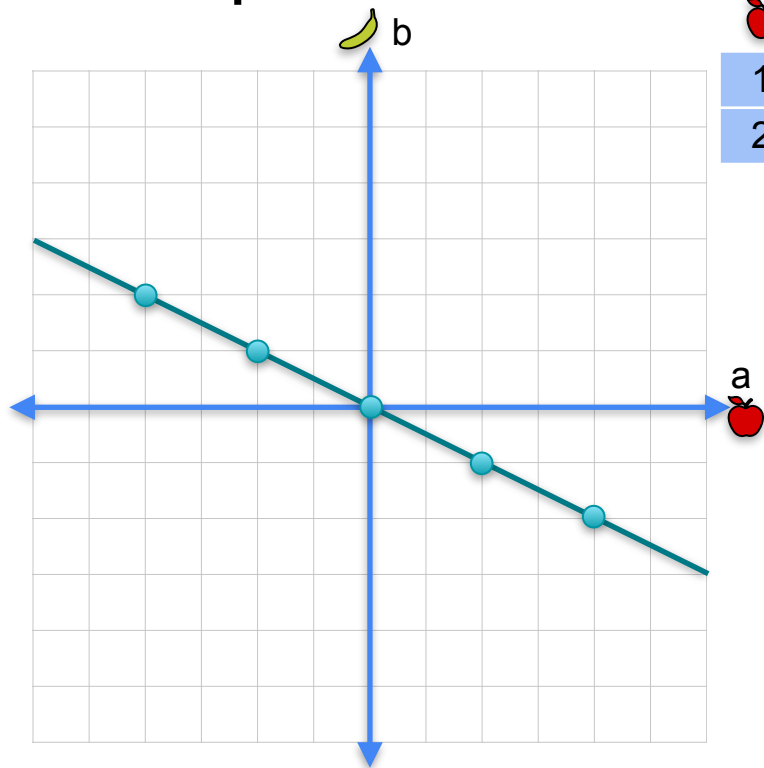
			
1	1	0	=
2	2	0	



0
0

2
-1

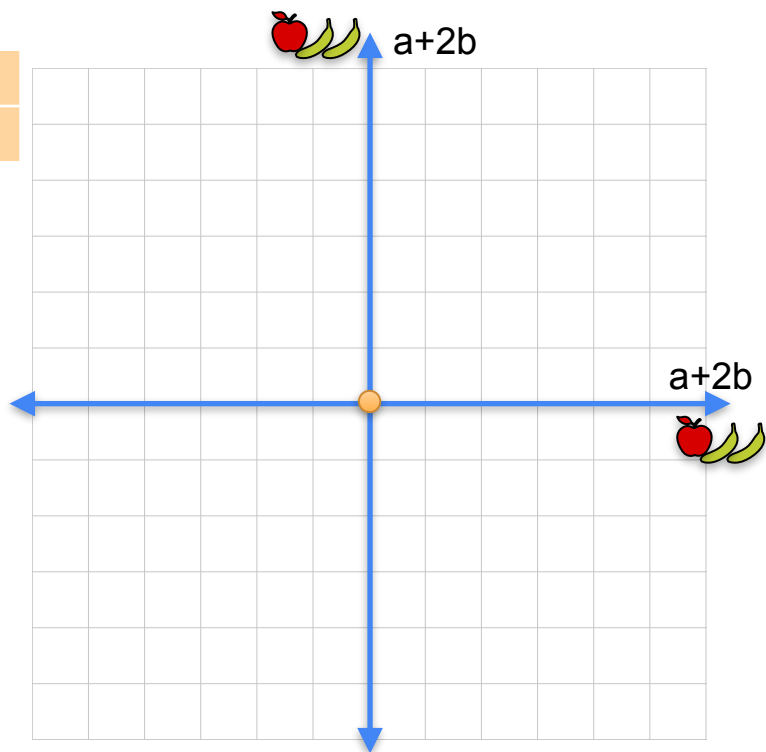


# Null space

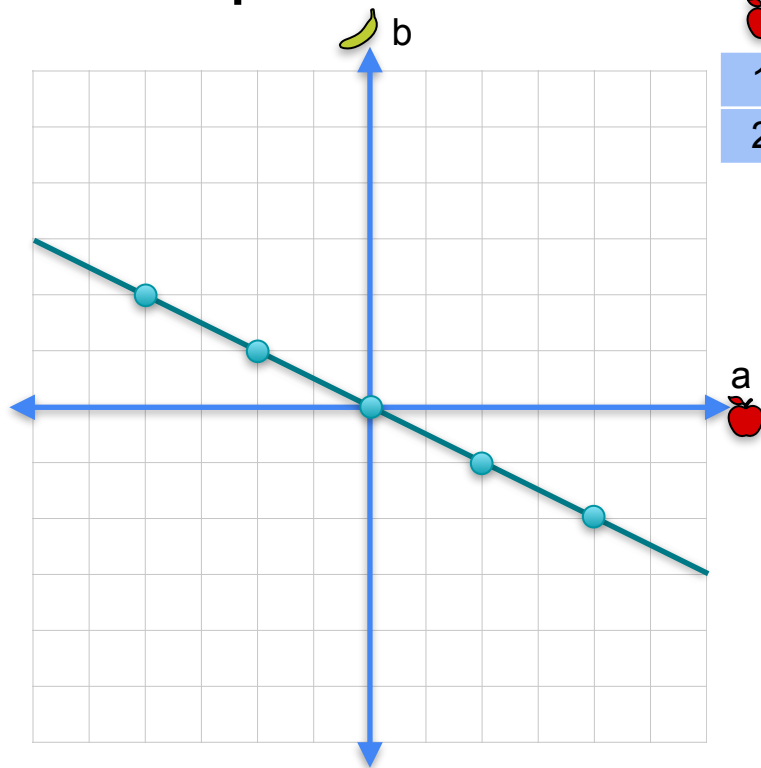


						
1	1	0	=	<table border="1"><tr><td>0</td></tr><tr><td>0</td></tr></table>	0	0
0						
0						
2	2	0				

2	4
-1	-2



# Null space



Apple	Banana	
1	1	0
2	2	0

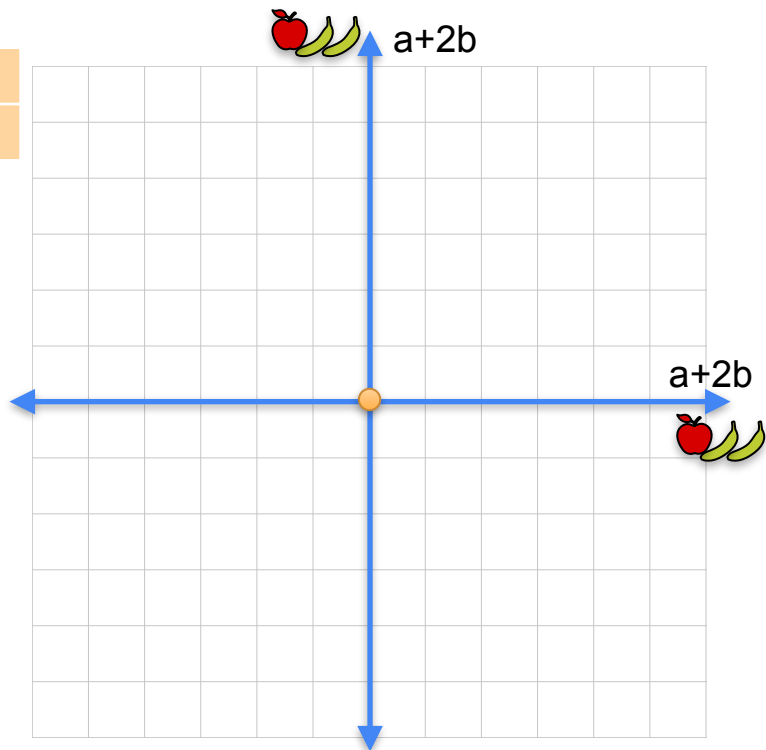
 = 

0
0

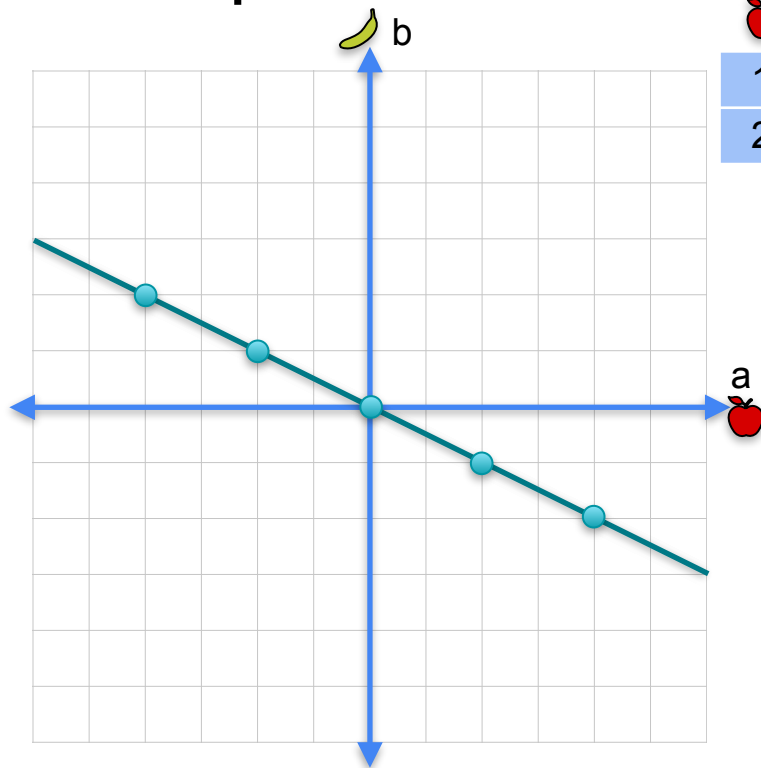
2	4
-1	-2



  

-2
1



# Null space

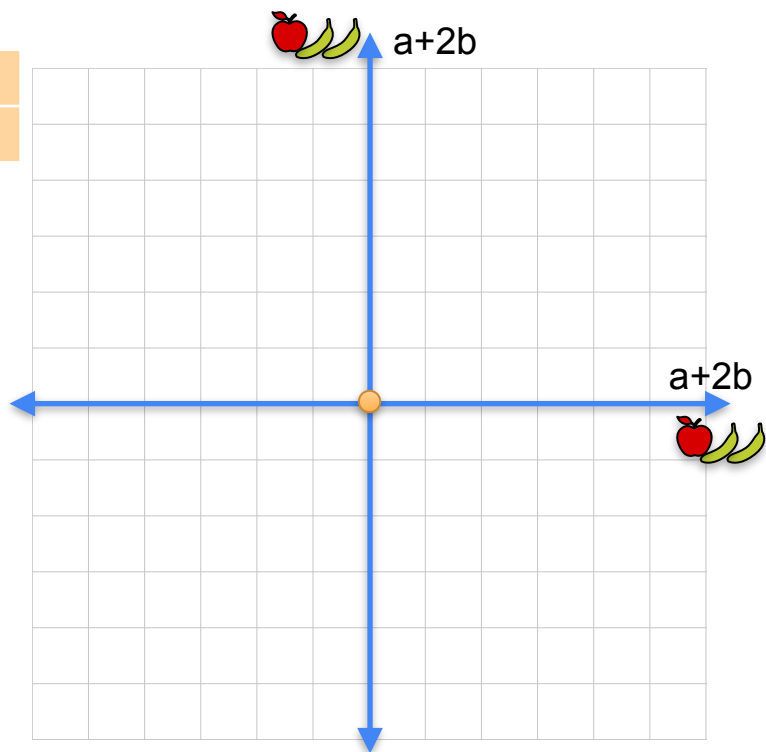


		
1	1	0
2	2	0

 = 

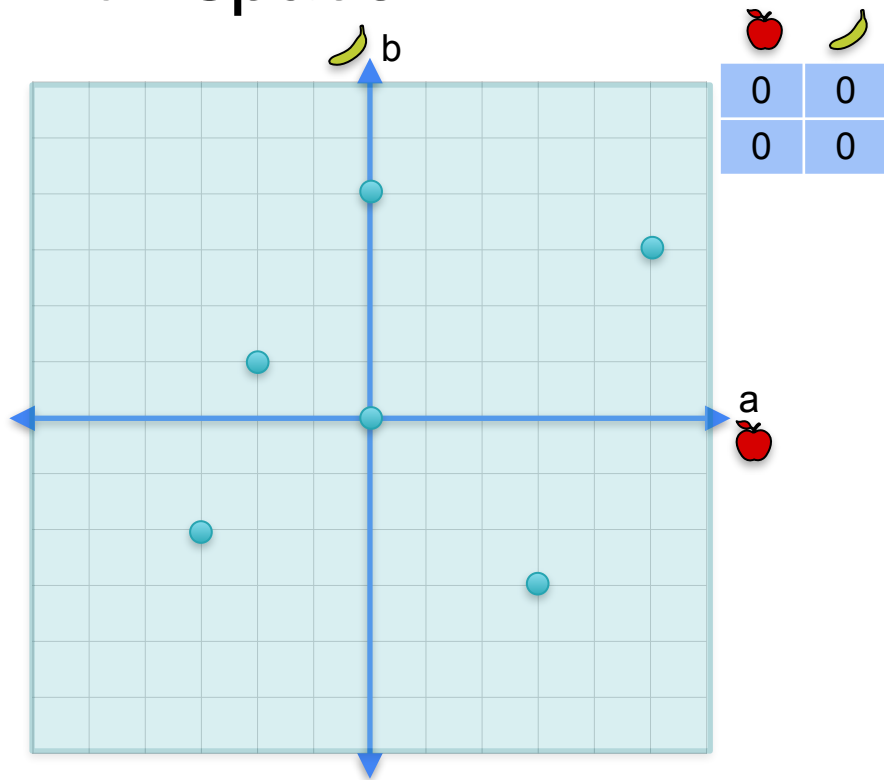
0
0

2	4
-1	-2
-2	-4
1	2

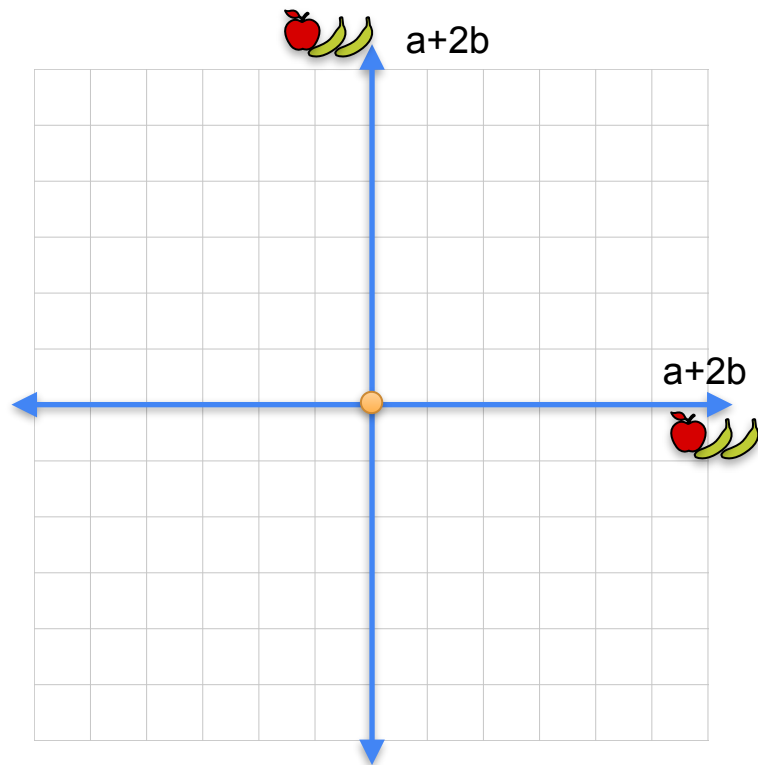




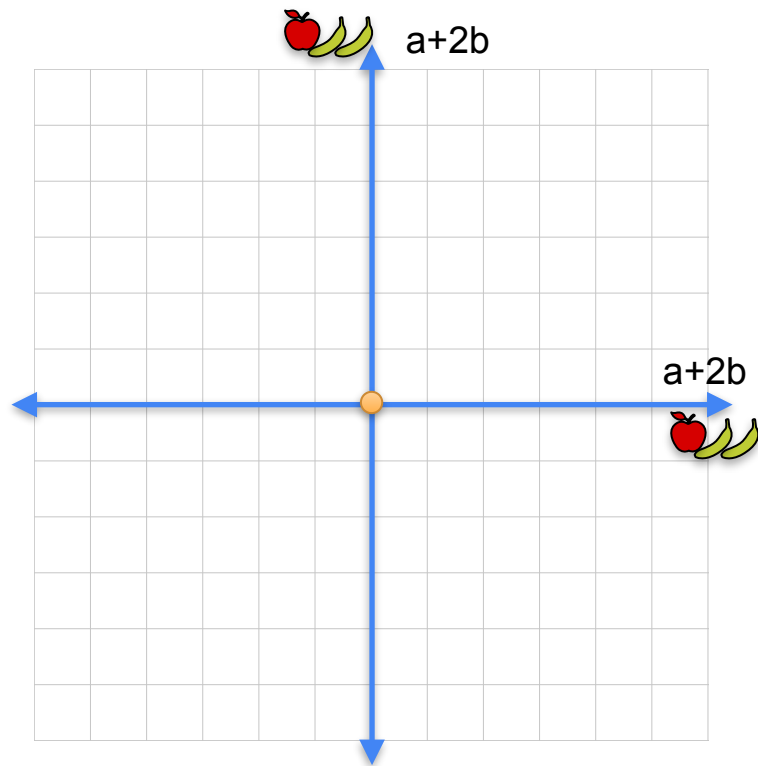
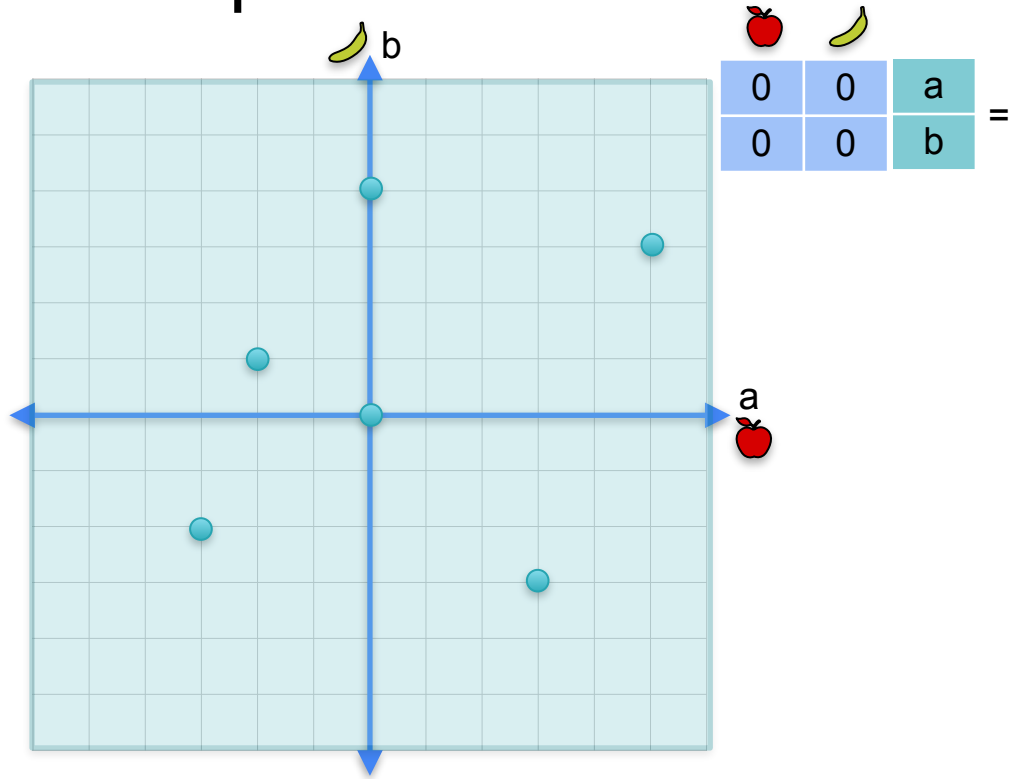
# Null space



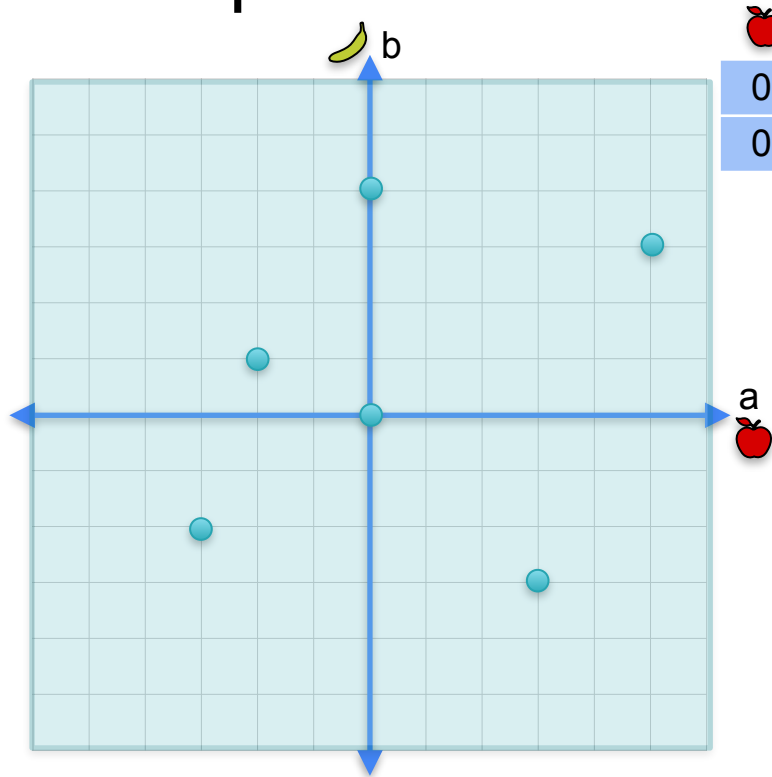
=



# Null space



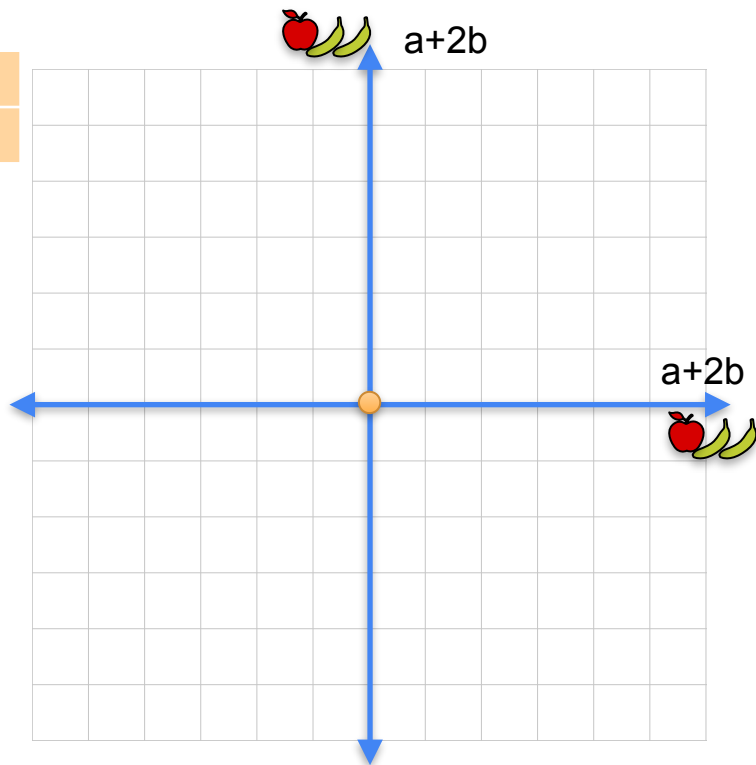
# Null space



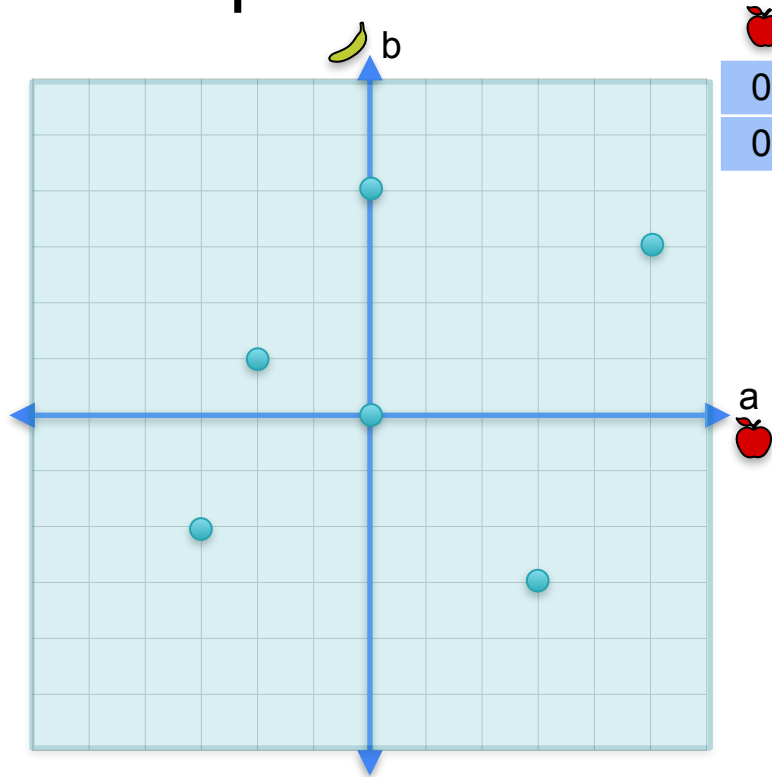
0	0	a
0	0	b



 = 

?
?



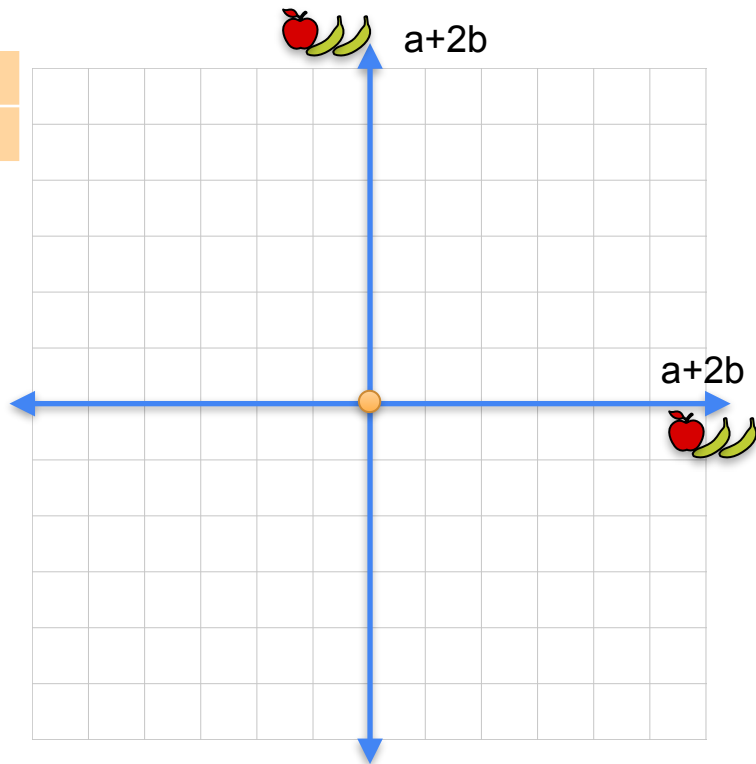
# Null space



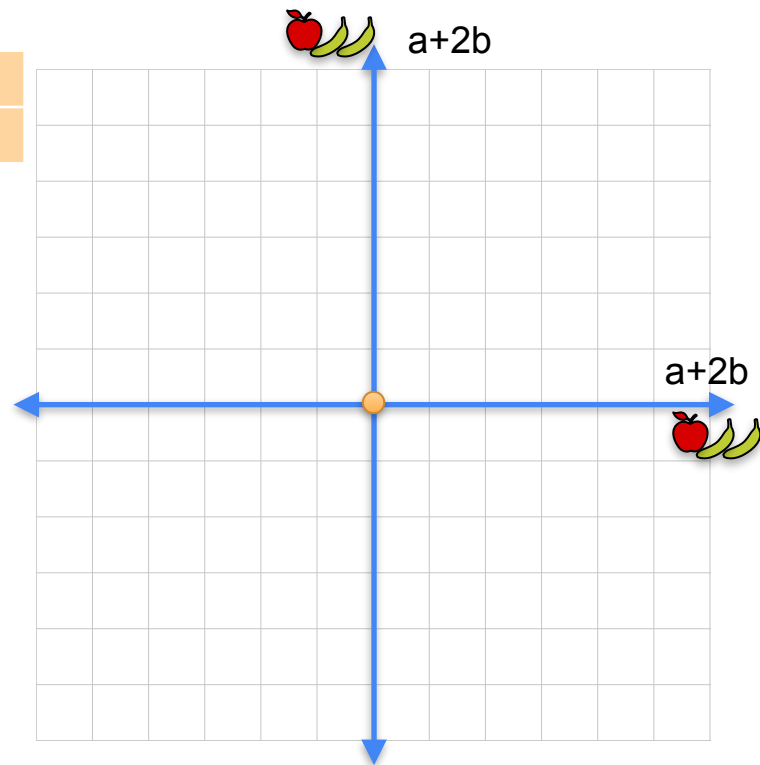
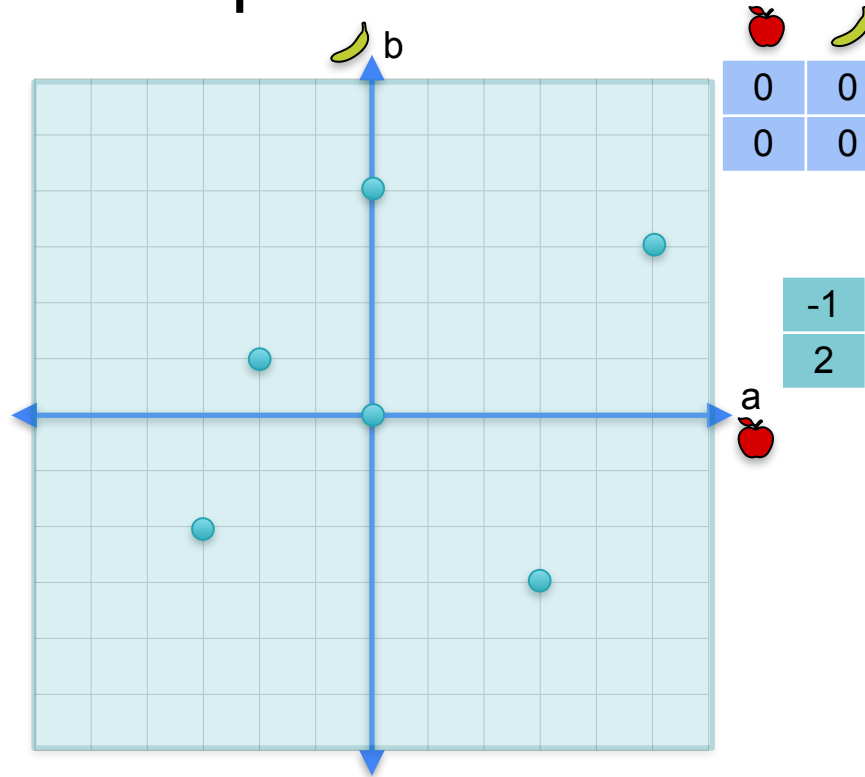
		
0	0	0
0	0	0

 = 

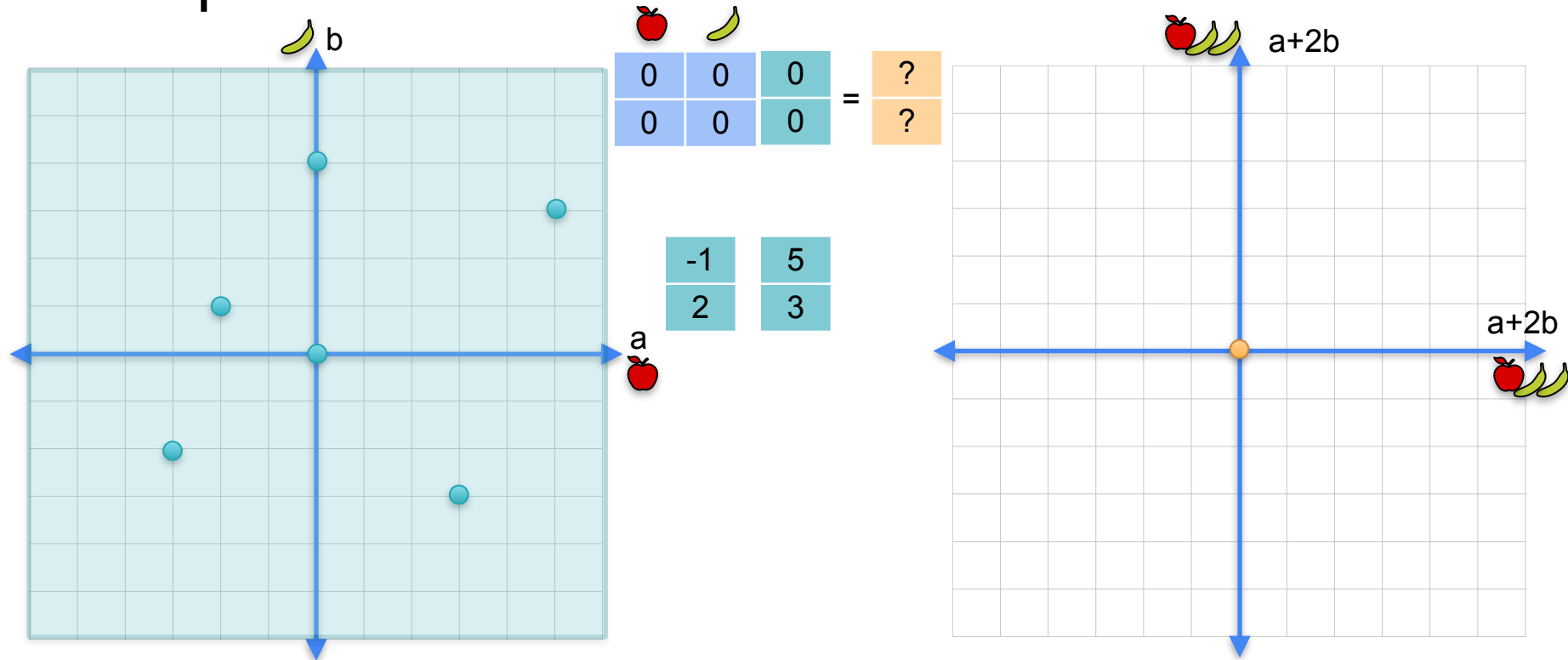
?
?



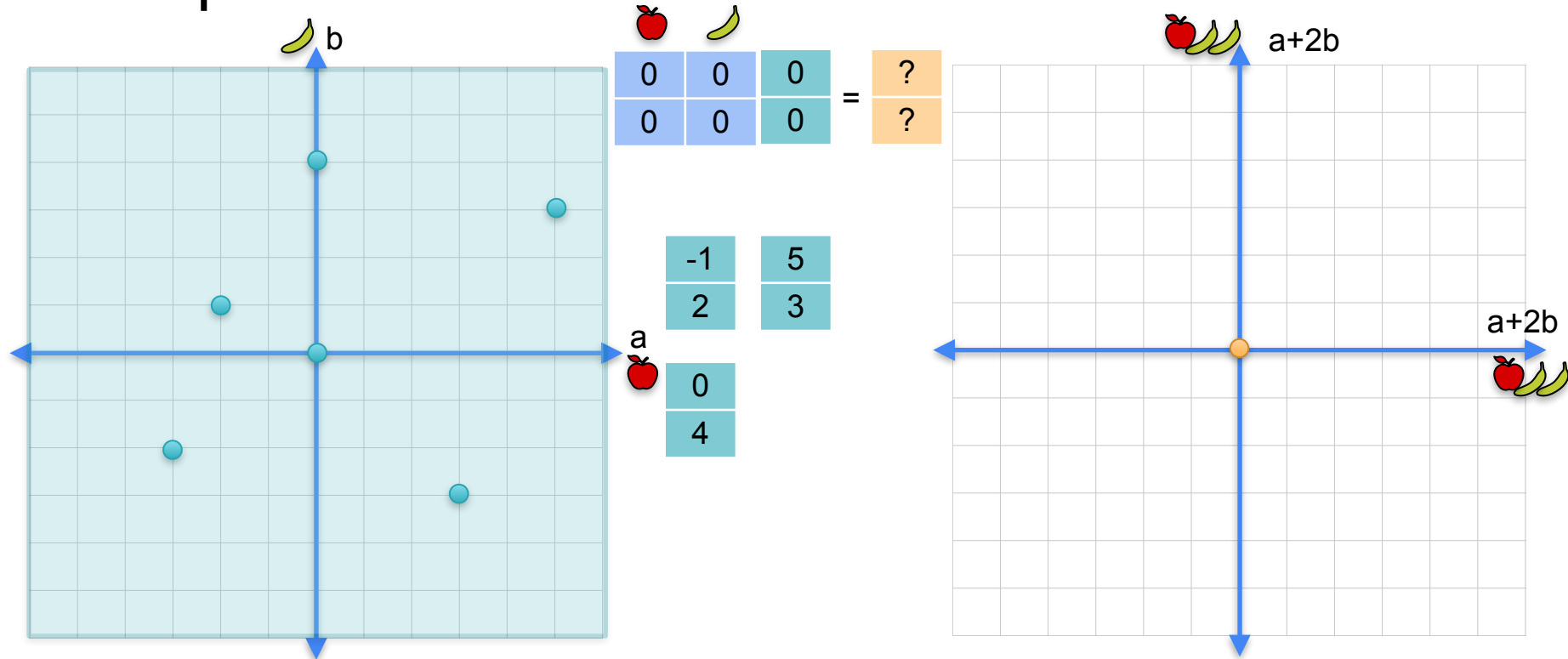
# Null space



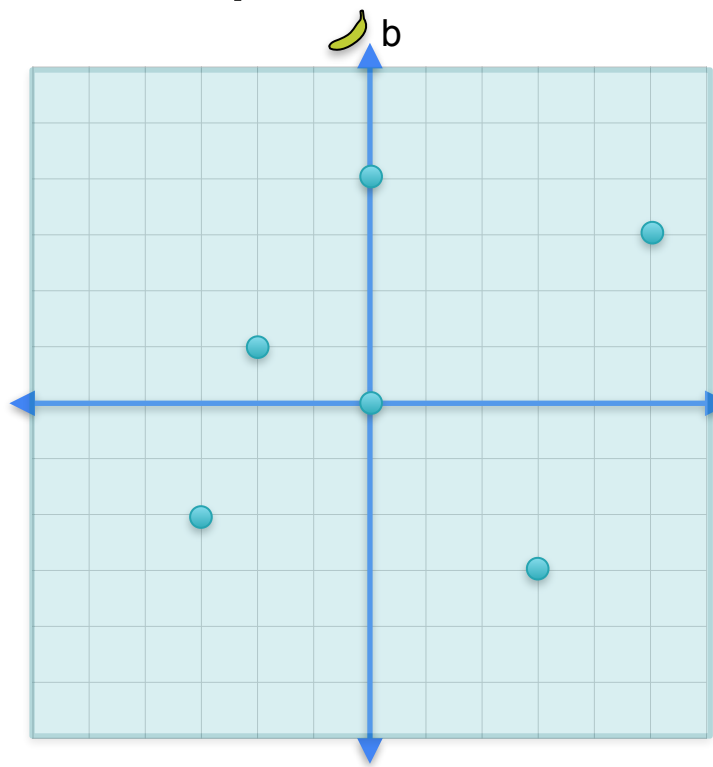
# Null space





# Null space




# Null space

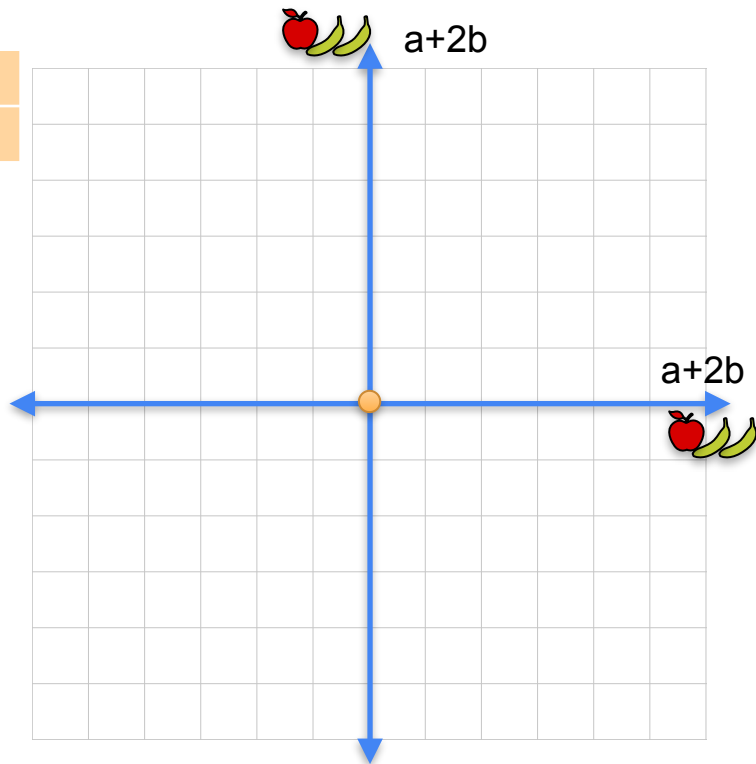


		
0	0	0
0	0	0

 = 

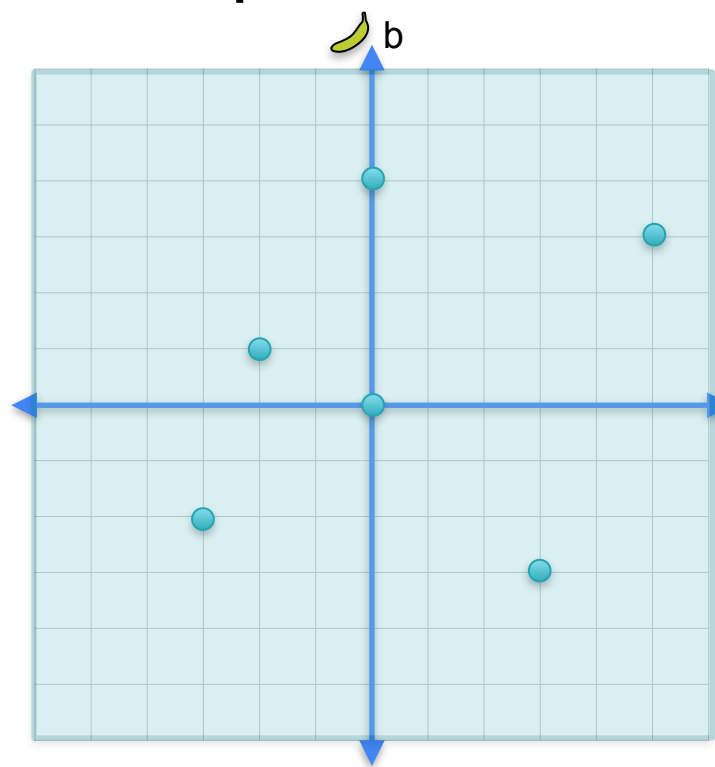
?
?

	
-1	5
2	3
0	-1
4	-3





# Null space



🍏	🍌	
0	0	0
0	0	0

 $=$ 

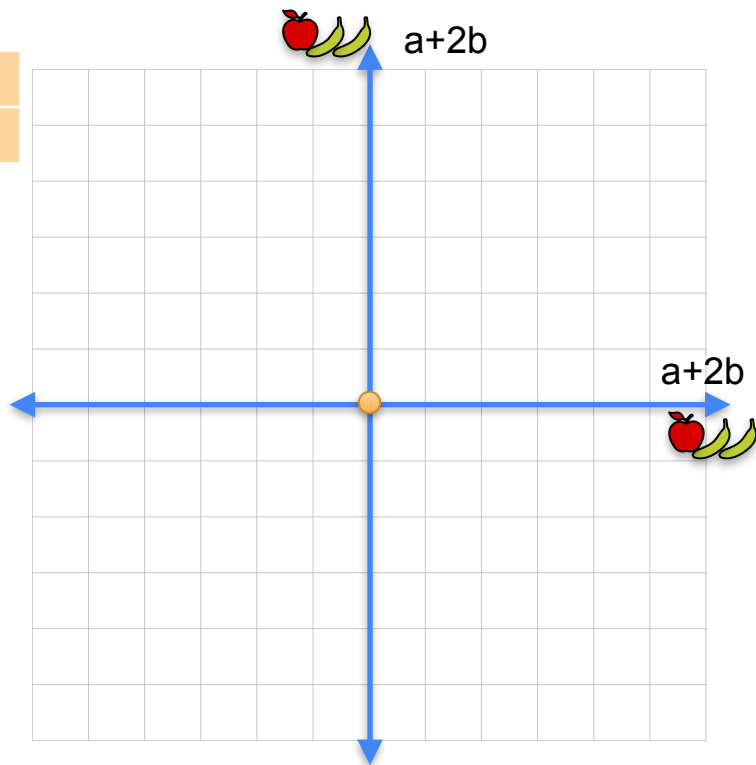
?
?

-1	5
2	3

 $\begin{matrix} \text{🍏} \\ a \end{matrix}$ 



0	-1
4	-3

3
-3

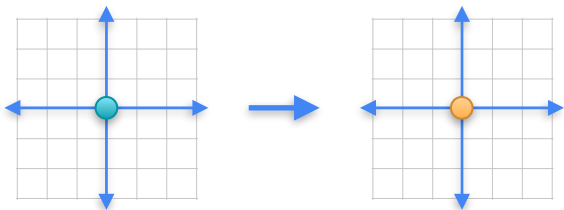


# Null space

Non-singular



	
3	1
1	2

Rank = 2

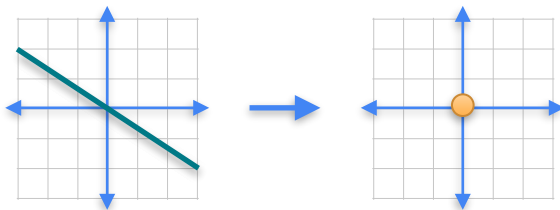


Dimension = 0

Singular



	
1	1
2	2

Rank = 1

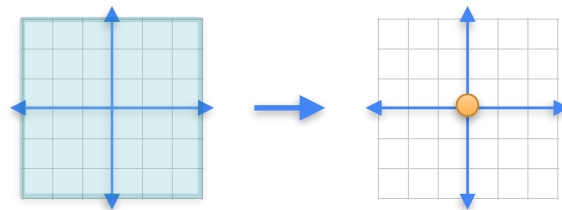


Dimension = 1

Singular

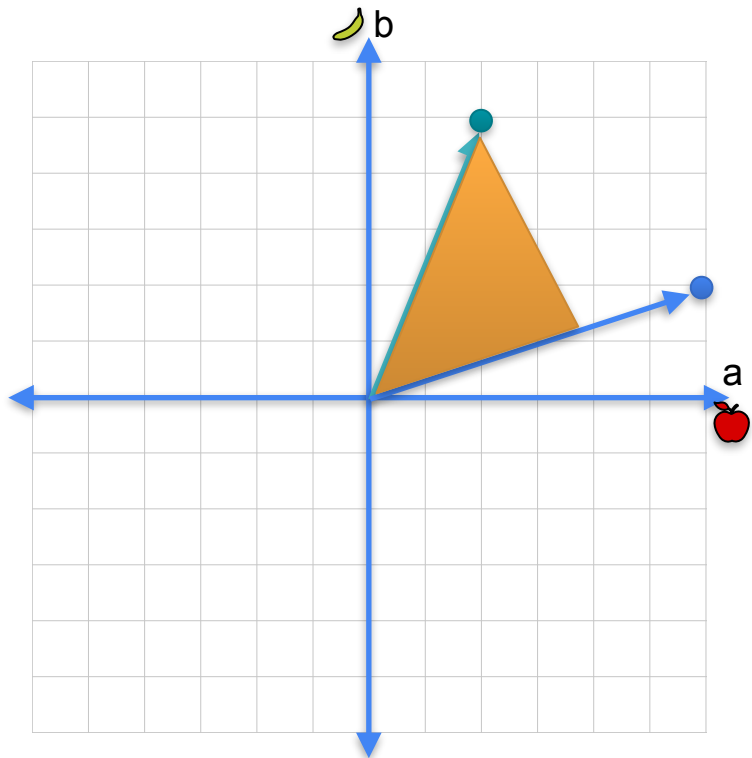
	
0	0
0	0

Rank = 0



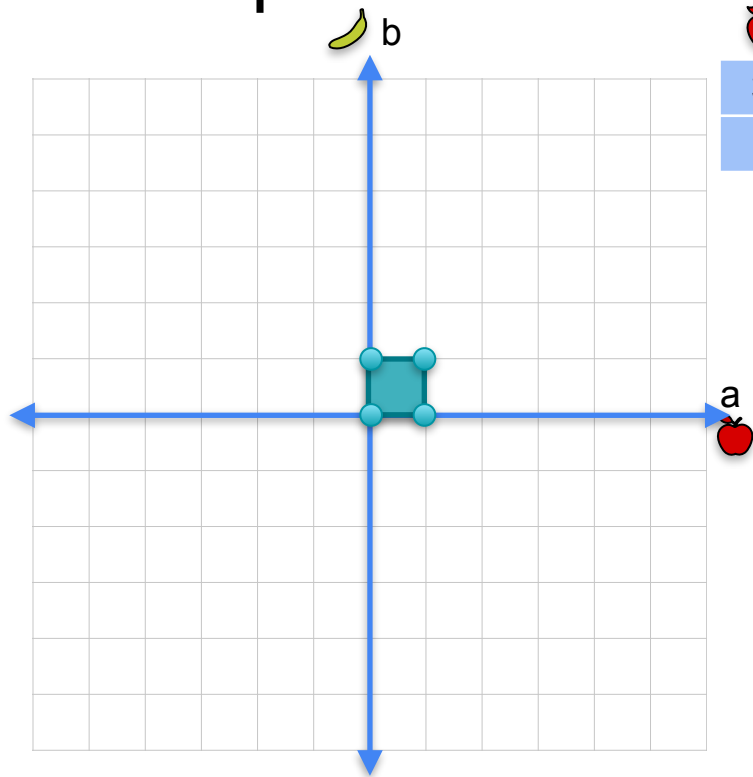
Dimension = 2



# Dot product as an area



$$\begin{matrix} \text{🍏} & \text{🍌} \\ 6 & 2 \end{matrix} \cdot \begin{matrix} \$ \text{🍏} & 2 \\ \$ \text{🍌} & 5 \end{matrix} = \$ 22$$

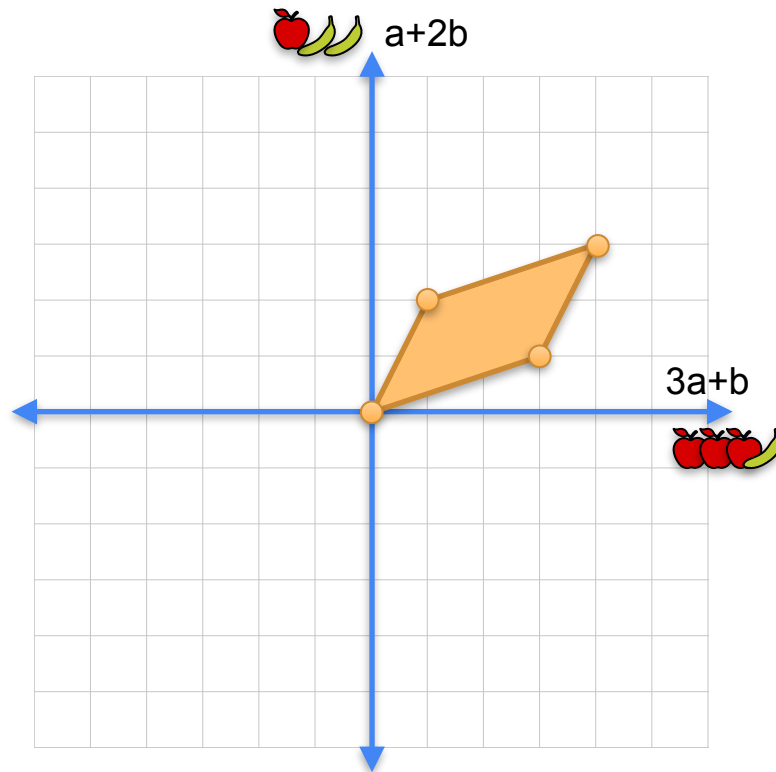
# Row space



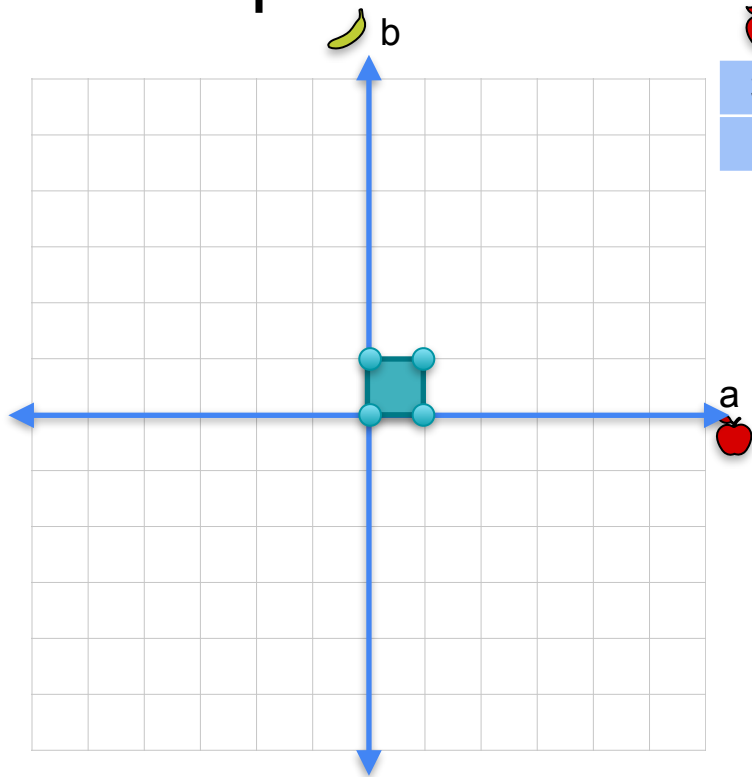
	
3	1
1	2



=

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



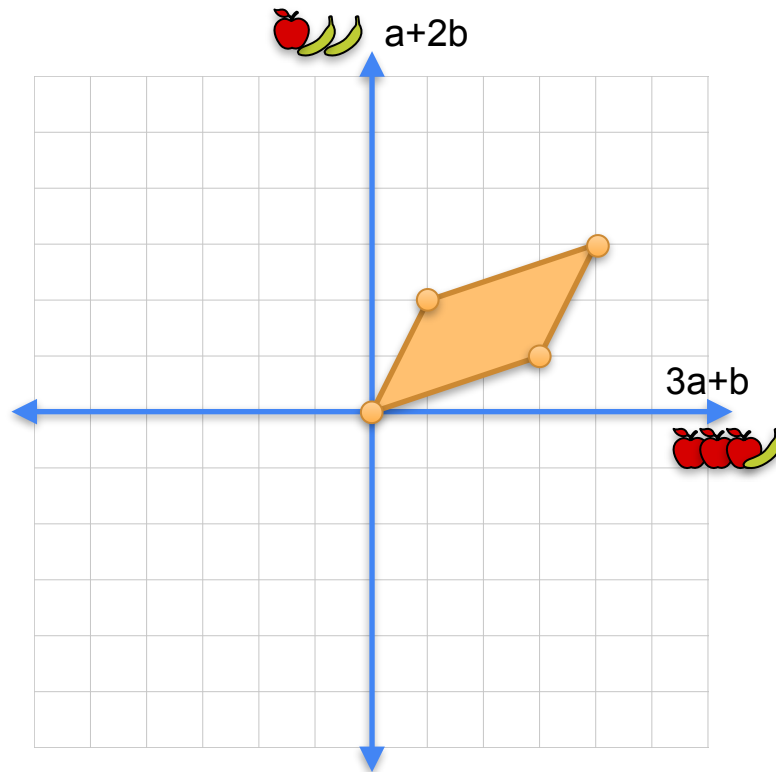
# Row space



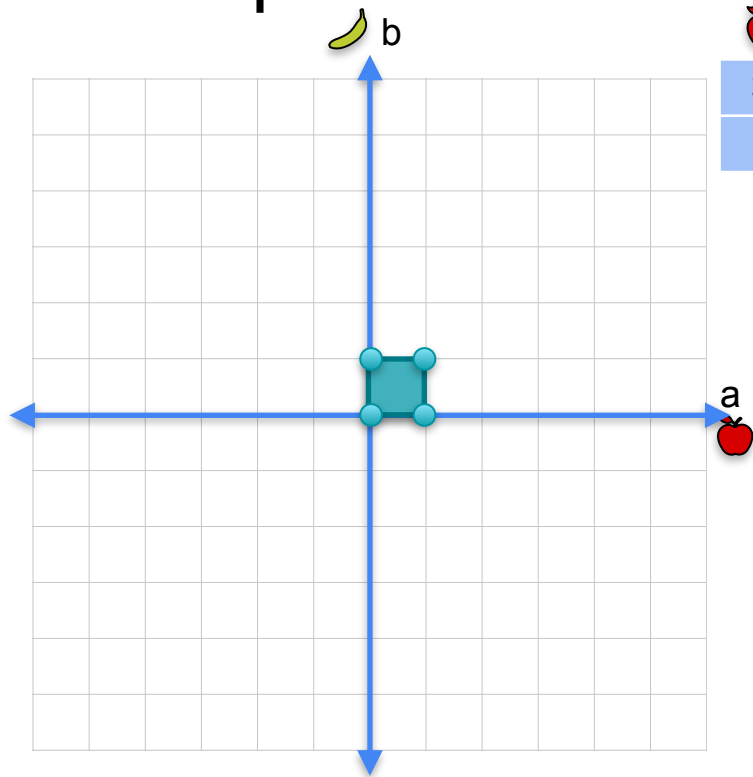
		
3	1	0
1	2	0



 =

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



# Row space

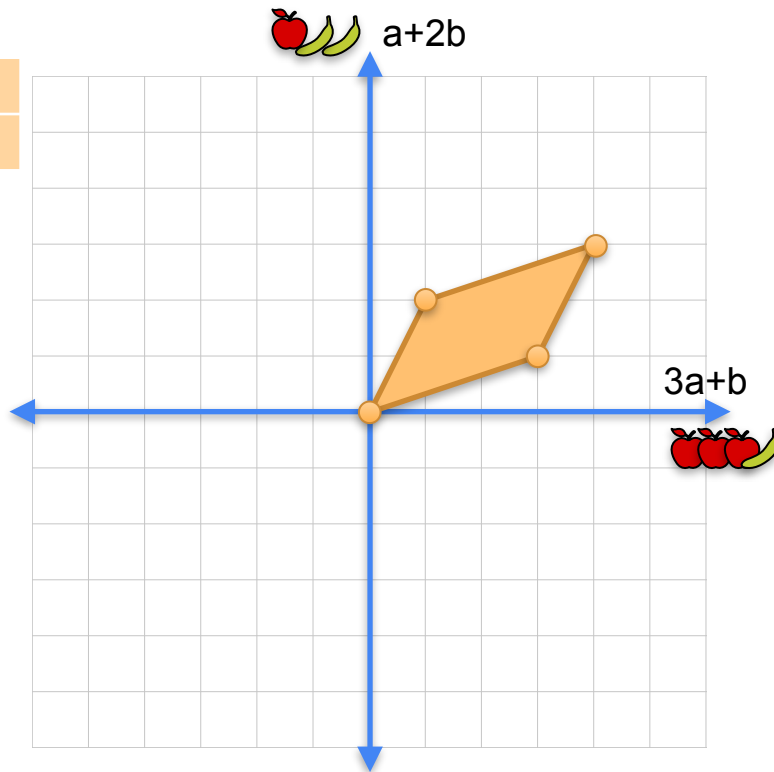


			
3	1	0	0
1	2	0	0

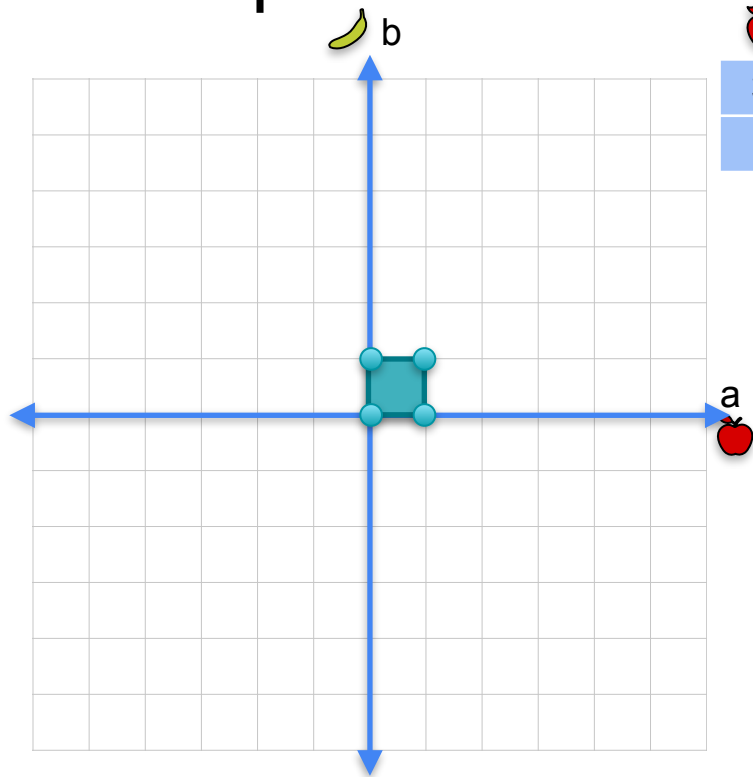
 = 



0
0

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



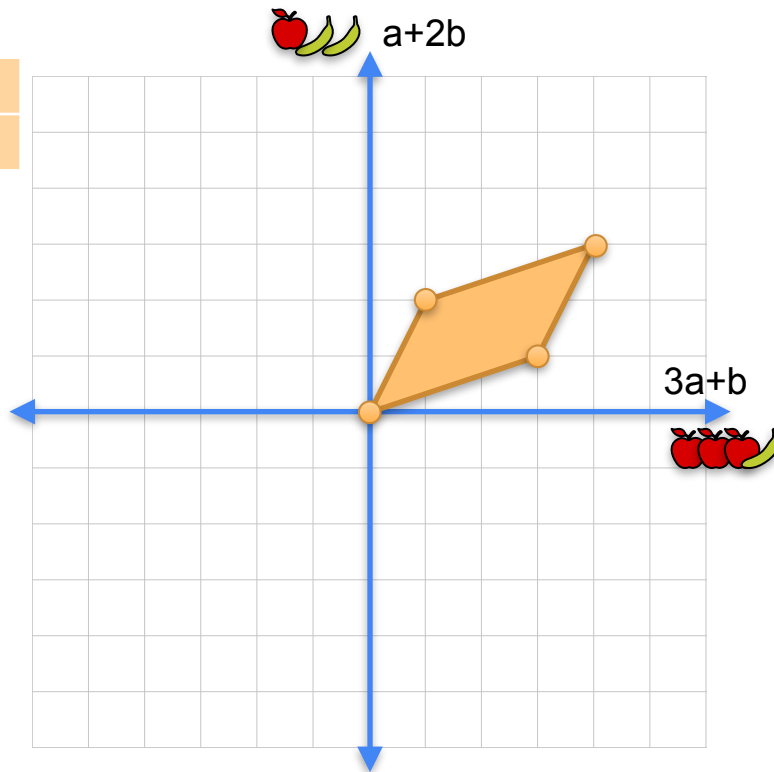
# Row space



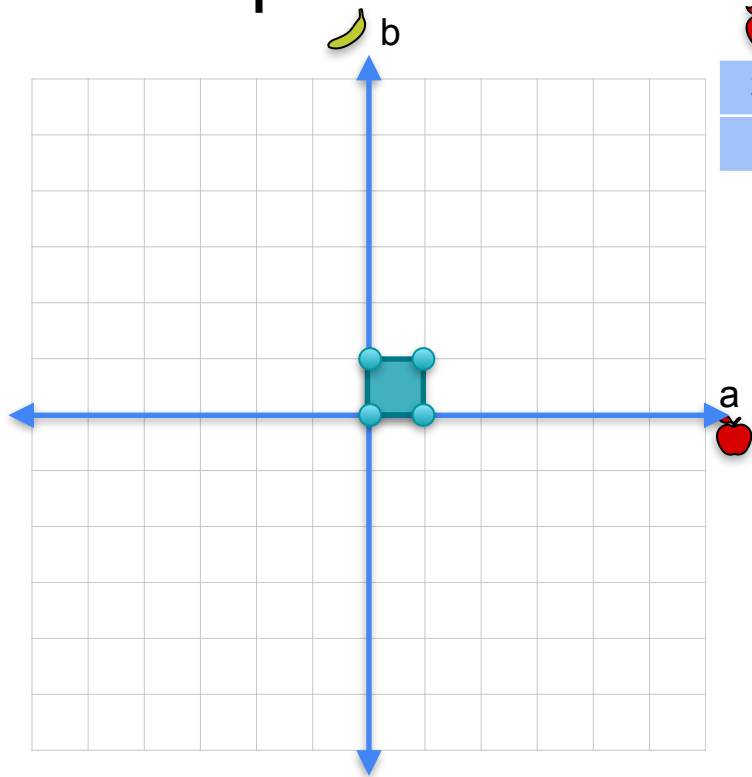
			
3	1	1	0
1	2	0	0



 =

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



# Row space

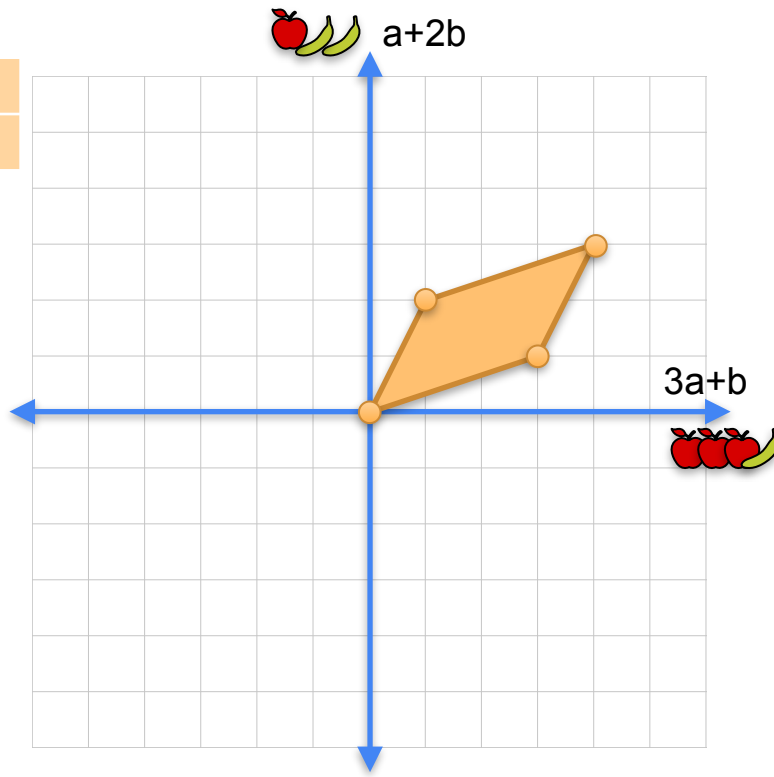


			
3	1	1	3
1	2	0	1

=

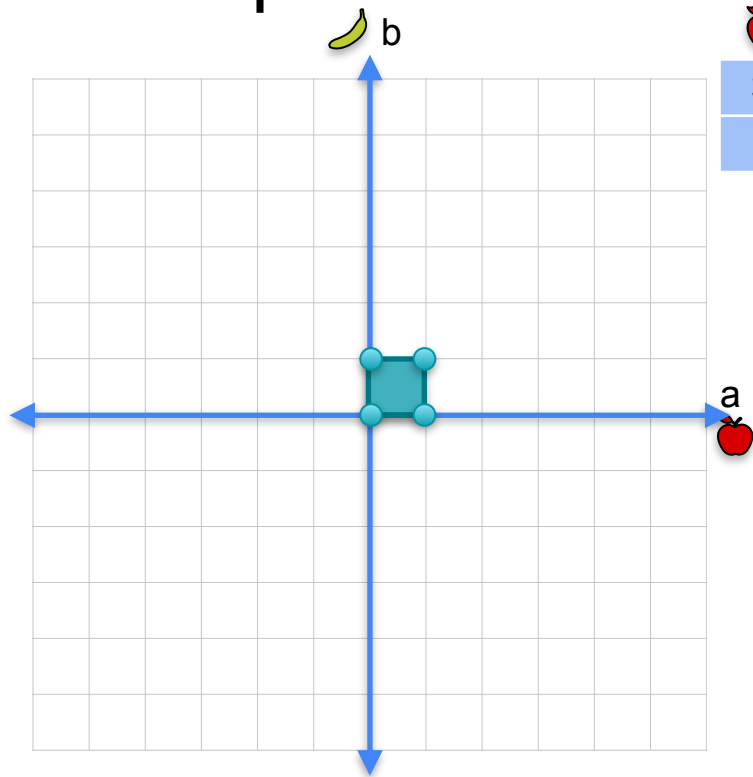
3
1



$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$





# Row space

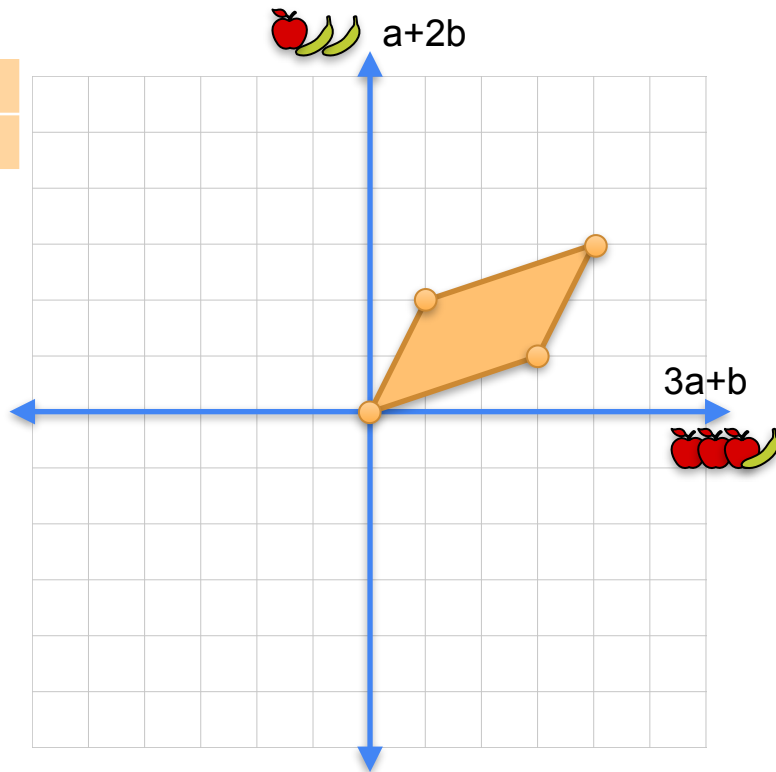


			
3	1	0	3
1	2	1	1

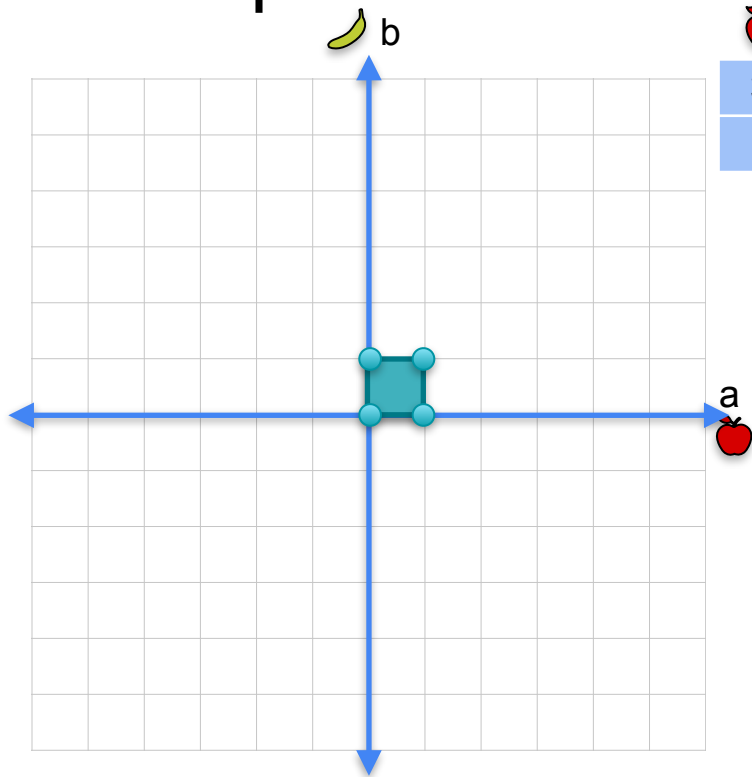
=



3
1

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



# Row space

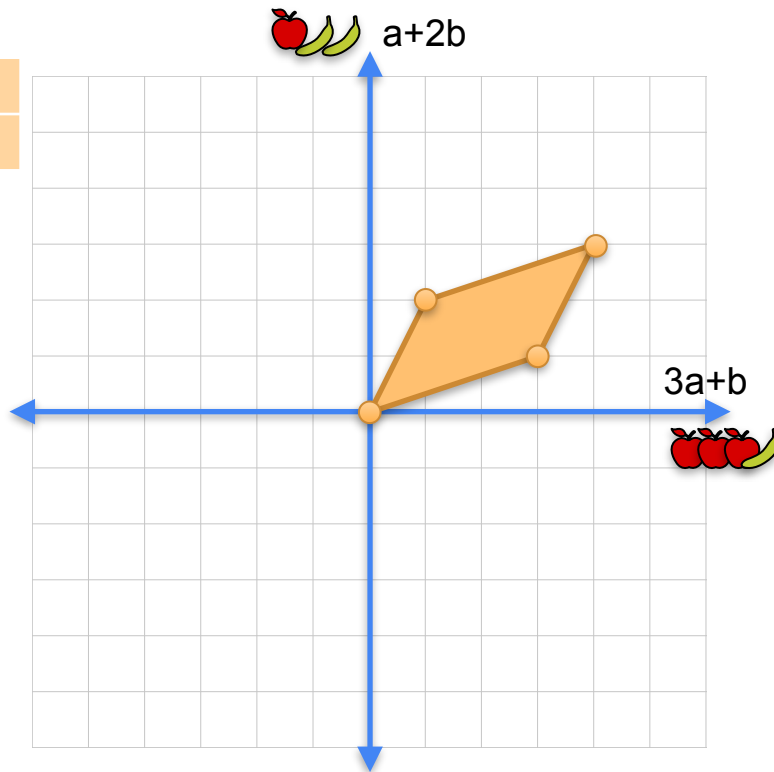


		
3	1	0
1	2	1

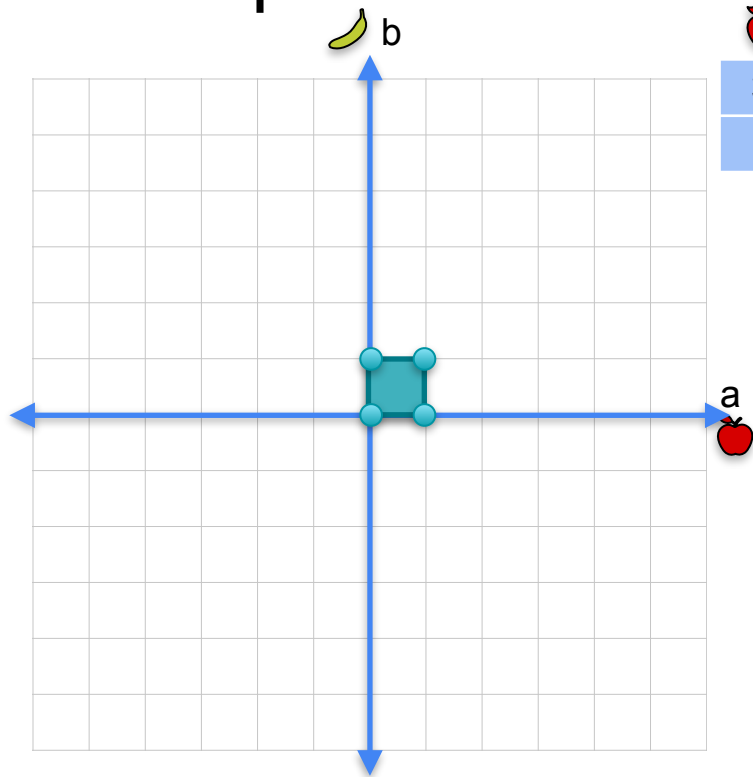
 = 



1
2

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



# Row space

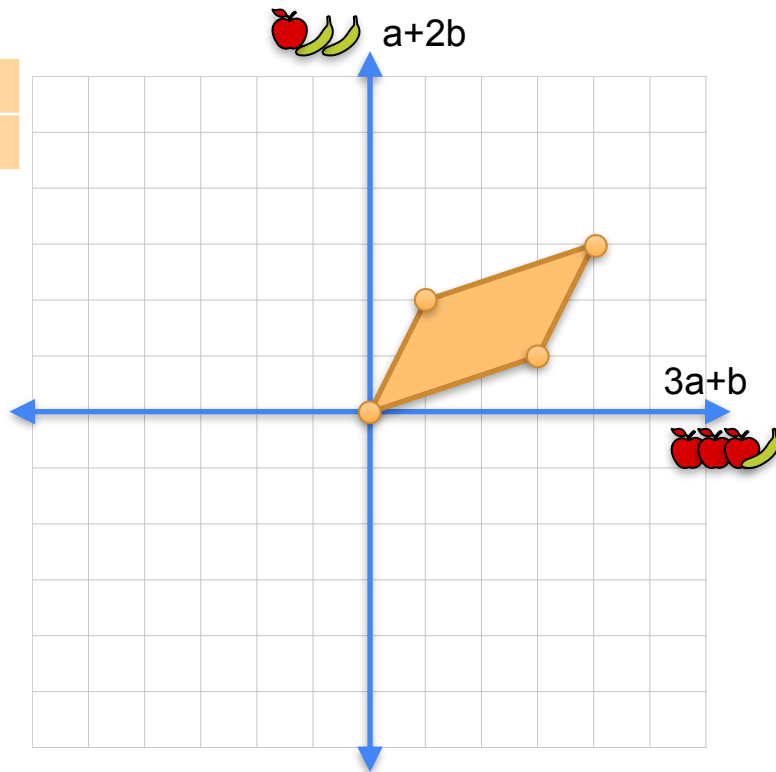


		
3	1	1
1	2	1

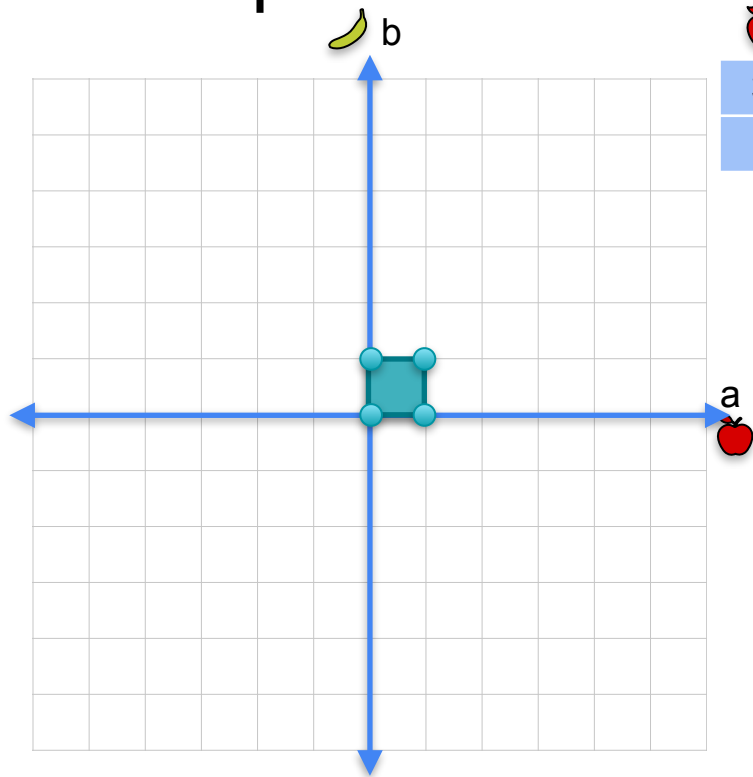
 = 



1
2

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



# Row space

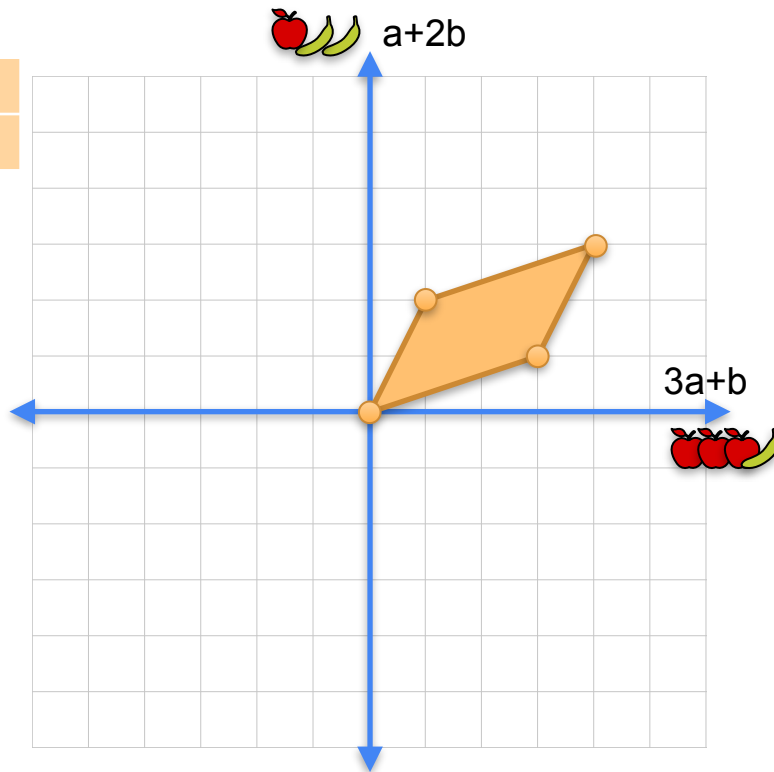


		
3	1	1
1	2	1

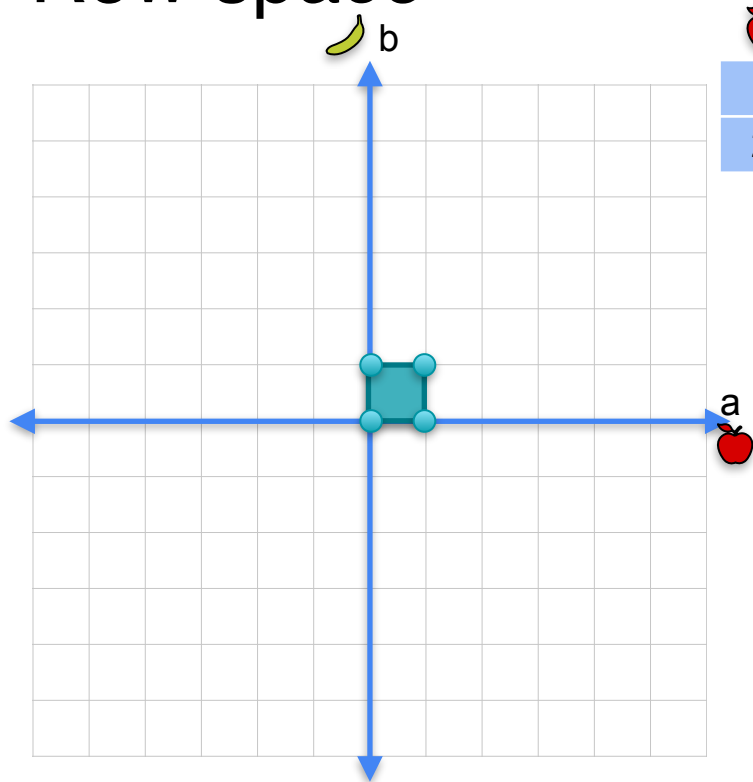
 = 

4
3

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (3,1)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (4,3)$



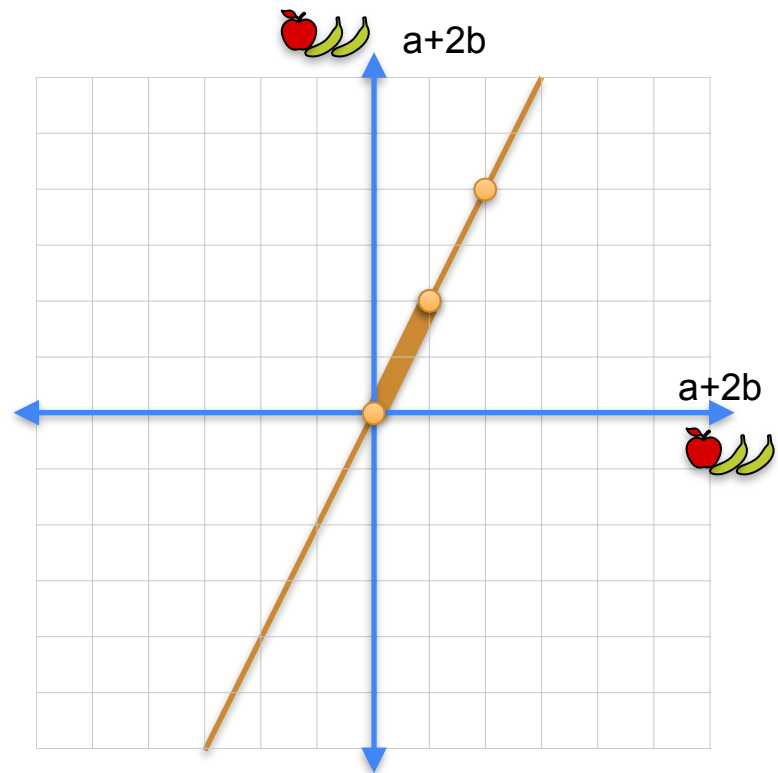
# Row space



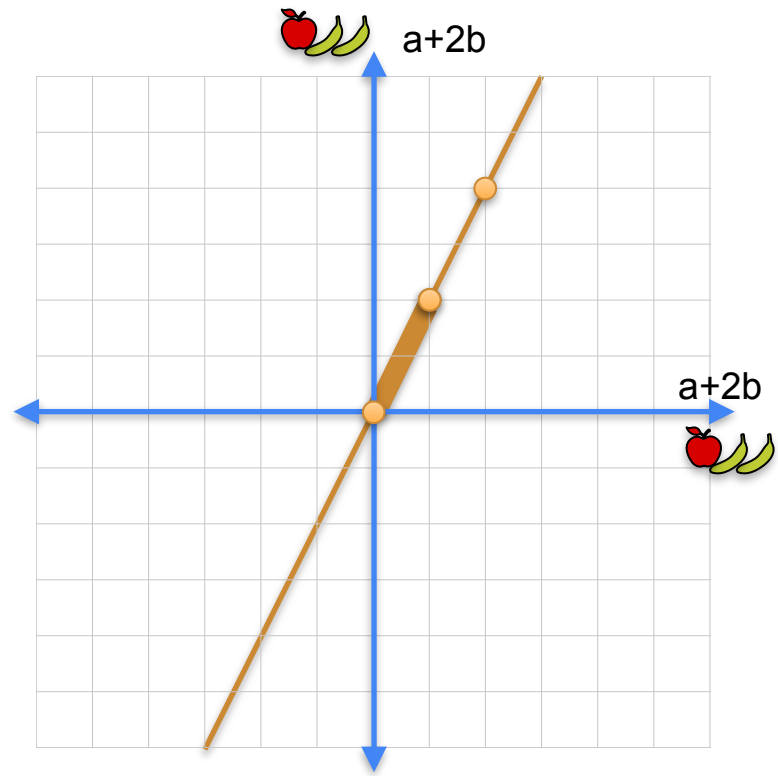
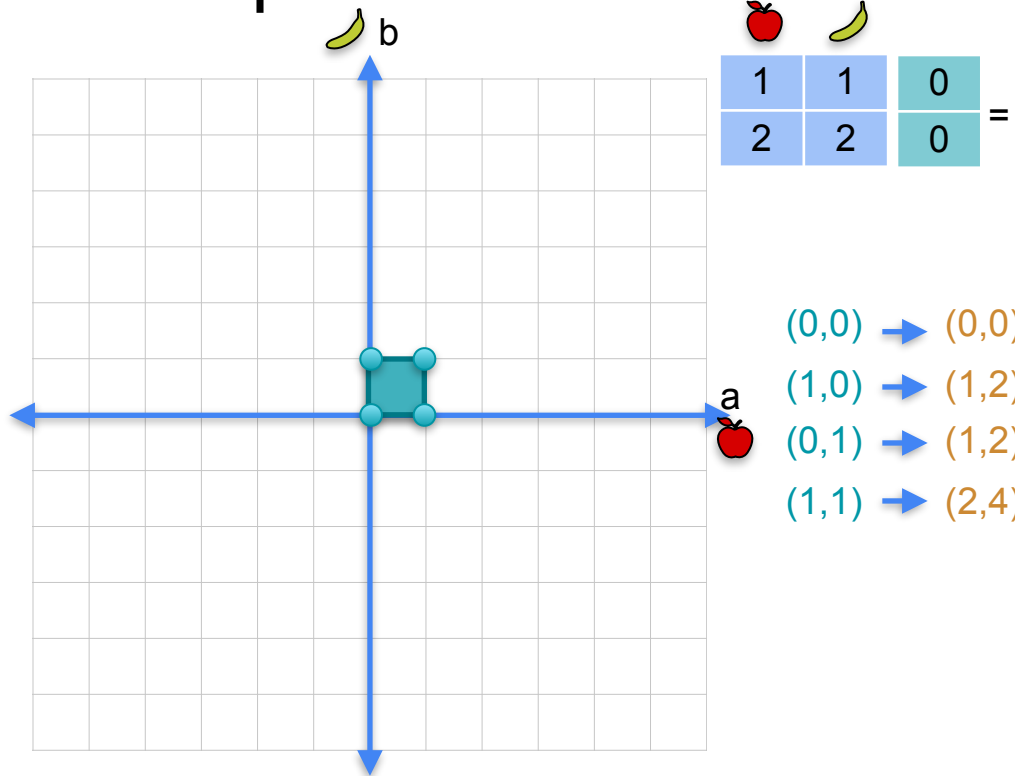
1	1
2	2

=

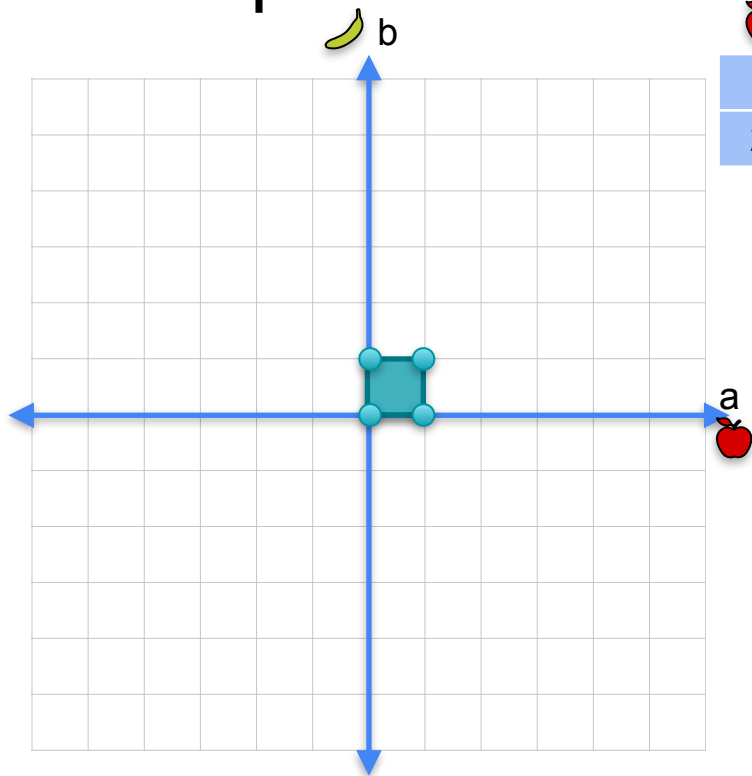
$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$



# Row space

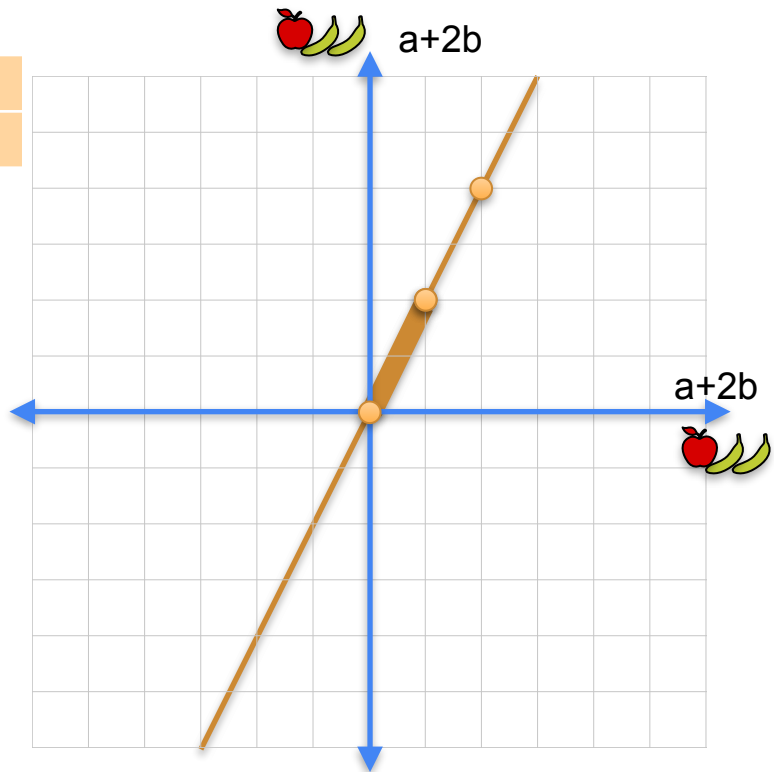


# Row space

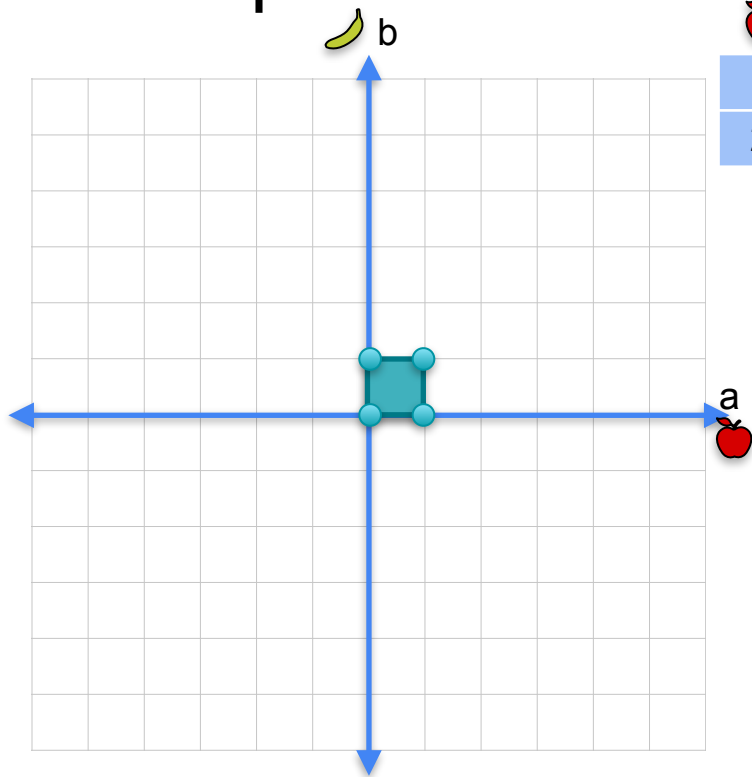


1	1	0	=	0
2	2	0		0

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$

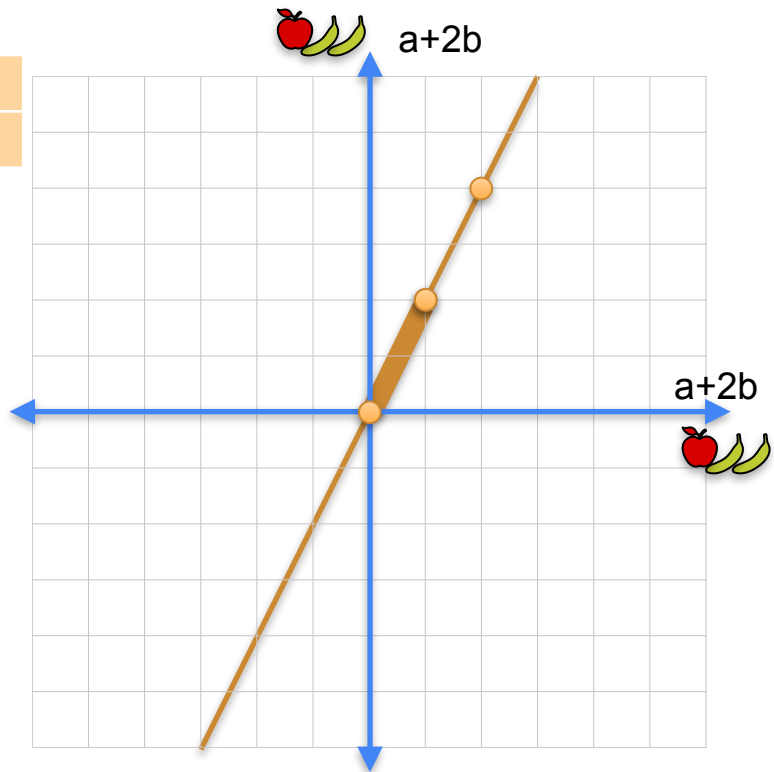


# Row space



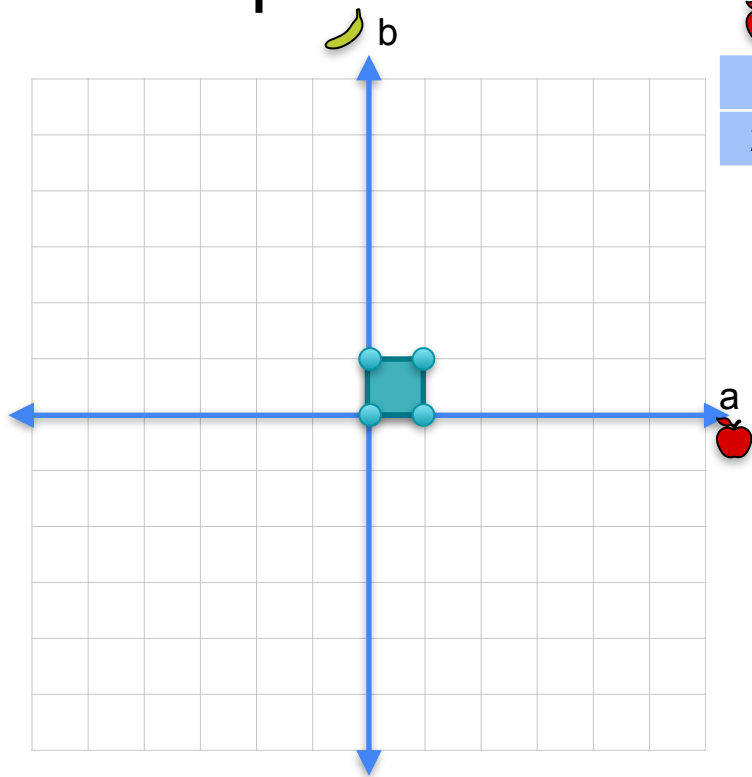
1	1	1	=	0
2	2	0		0



$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$



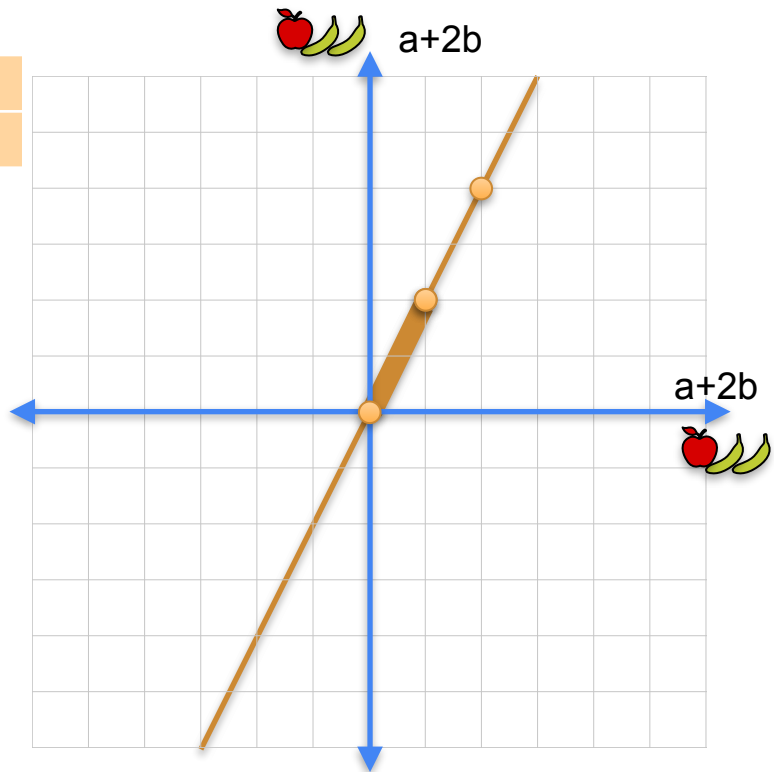


# Row space

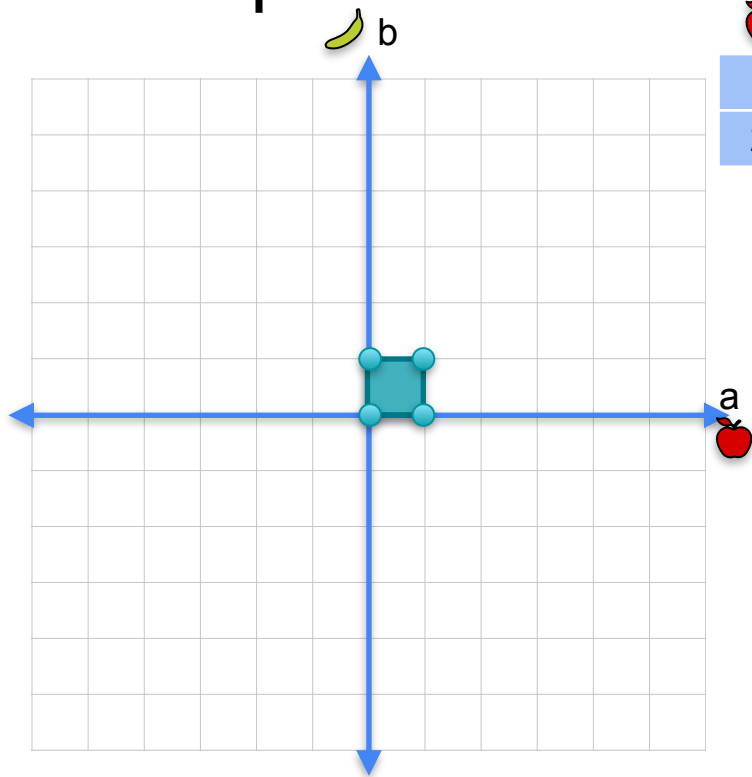




			
1	1	1	3
2	2	0	1

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$



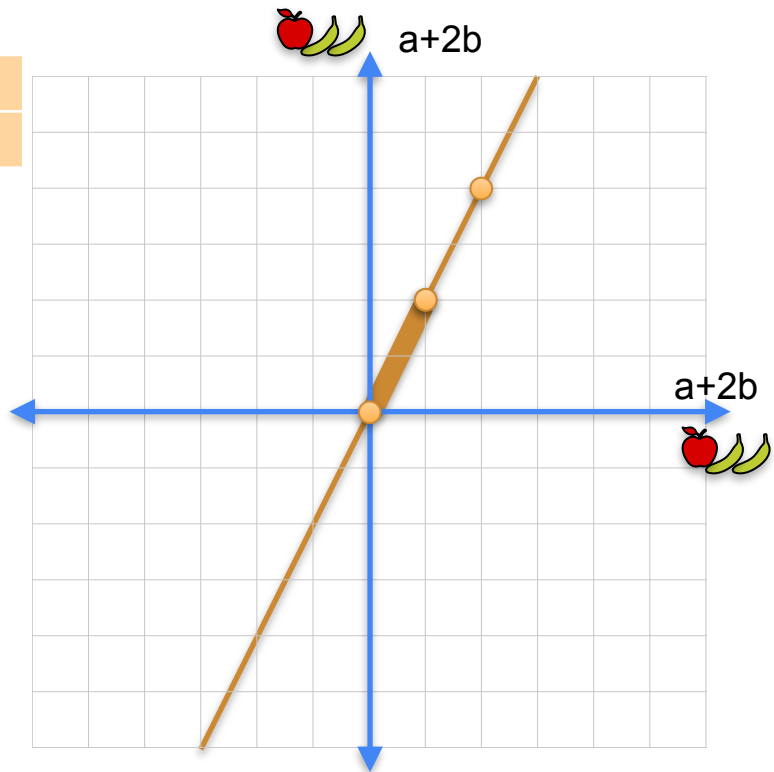
# Row space



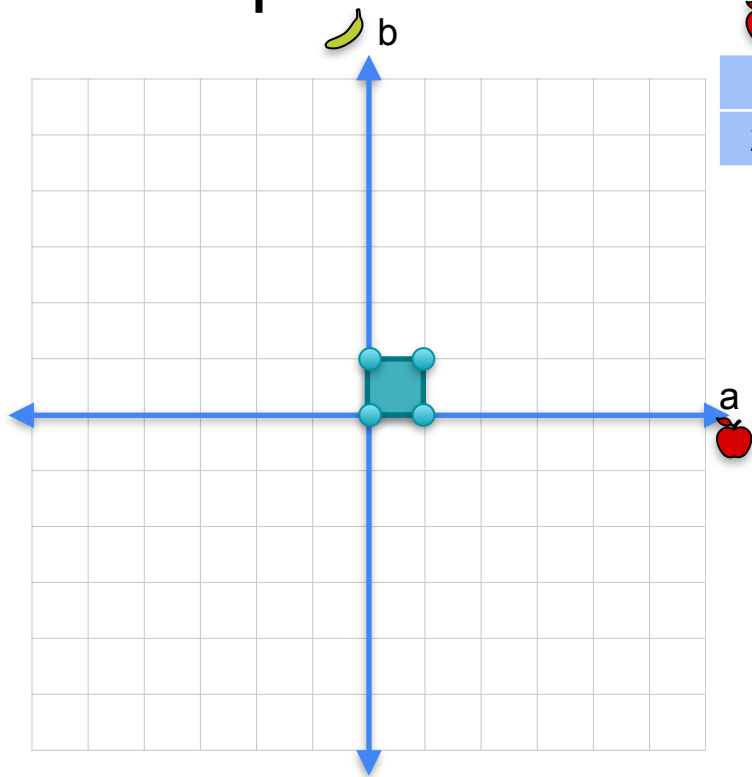
			
1	1	0	3
2	2	1	1



=

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$



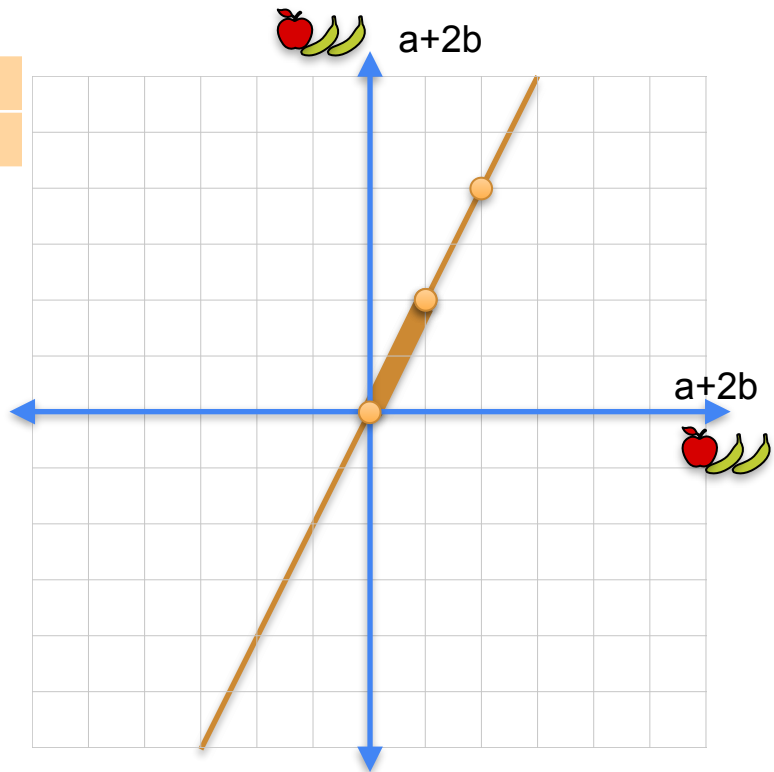
# Row space



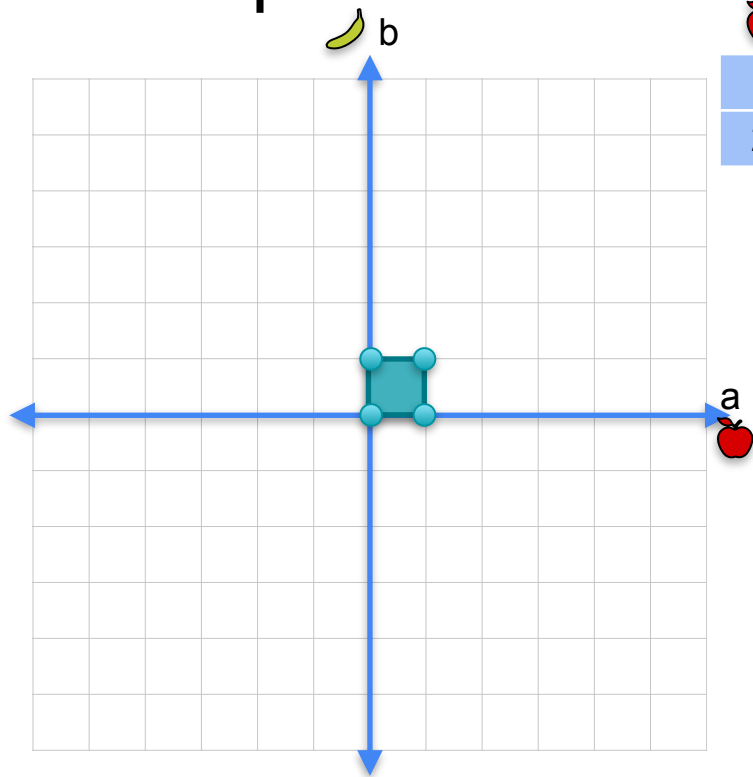
			
1	1	0	1
2	2	1	2



 =

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$



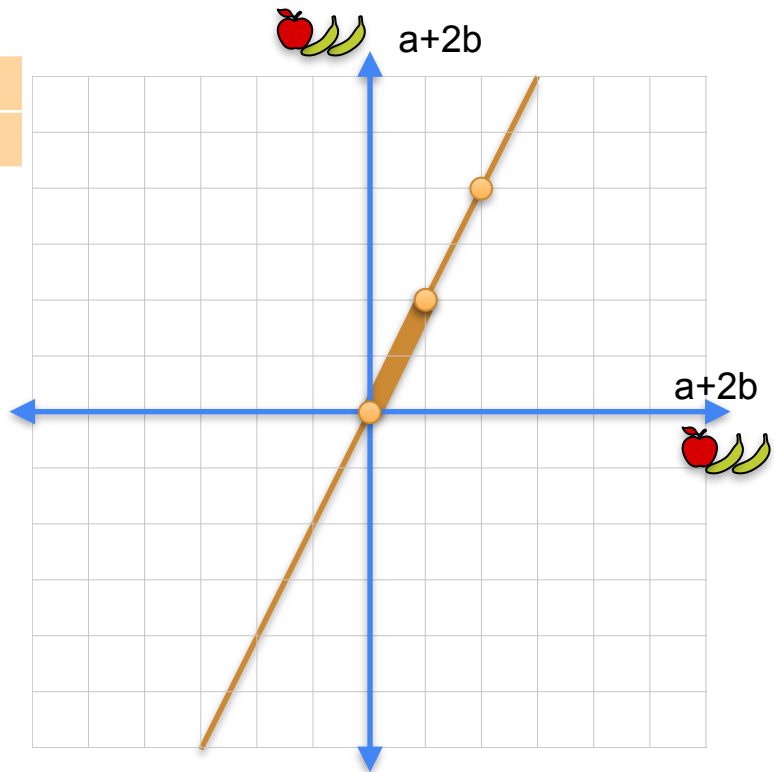
# Row space



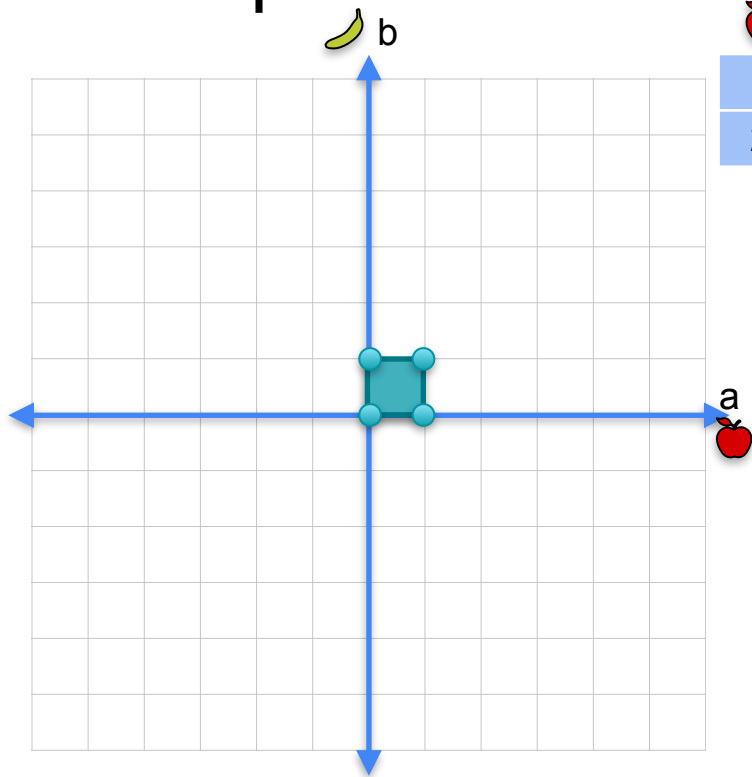
			
1	1	1	1
2	2	1	2

=

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$

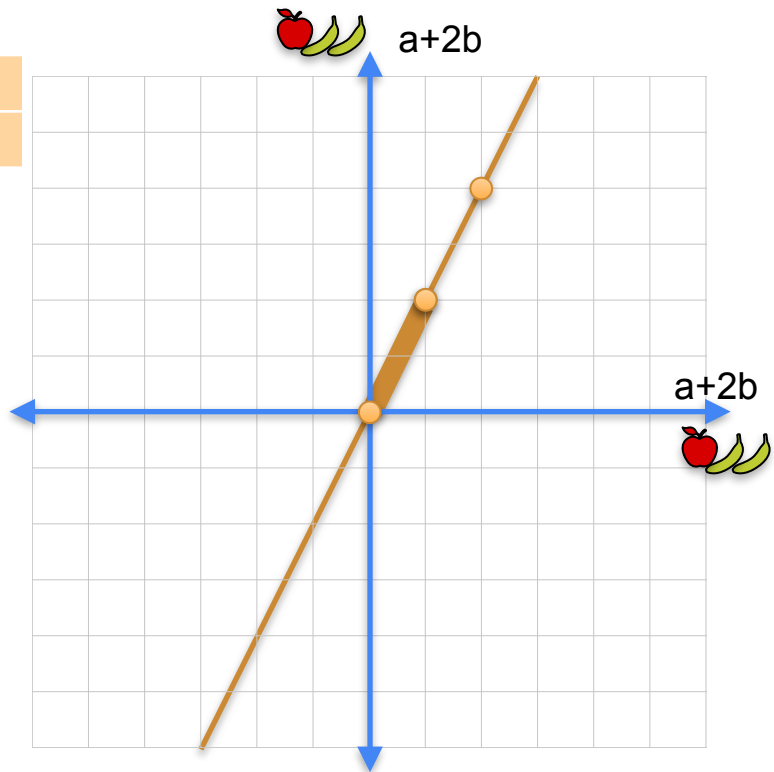


# Row space



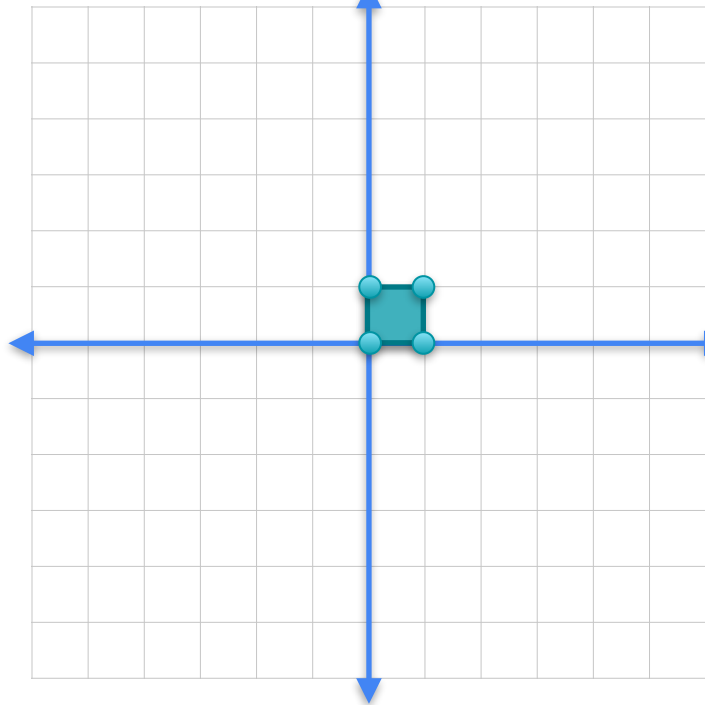
1	1	1	=	4
2	2	1		3



$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (1,2)$   
 $(0,1) \rightarrow (1,2)$   
 $(1,1) \rightarrow (2,4)$



# Row space

 b

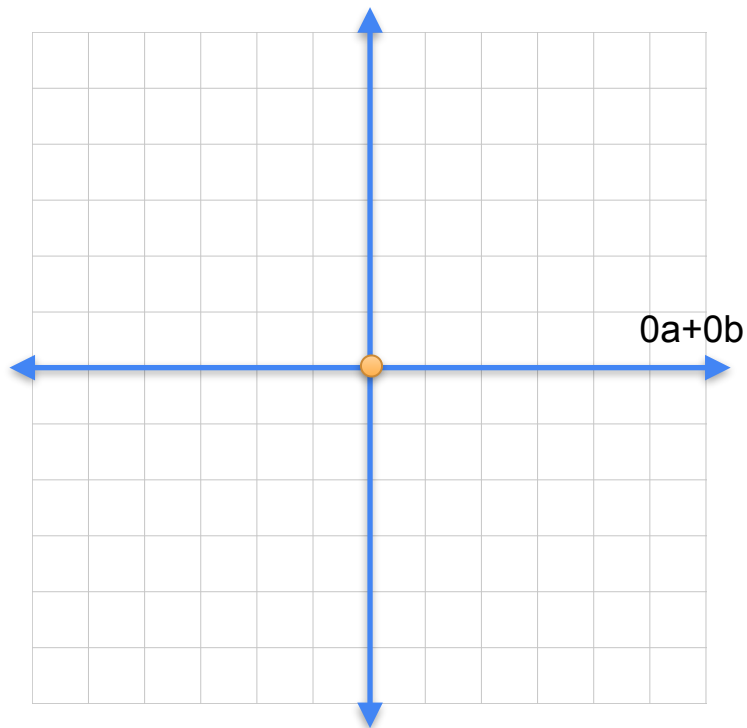


	
0	0
0	0

=

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (0,0)$   
 $(0,1) \rightarrow (0,0)$   
 $(1,1) \rightarrow (0,0)$

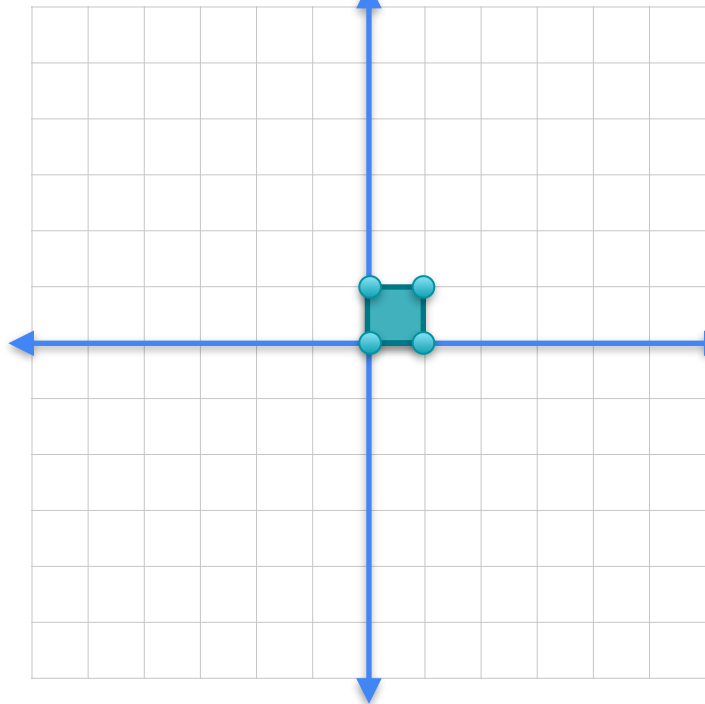
$0a+0b$



# Row space



b



0	0	a
0	0	b

=

(0,0) → (0,0)

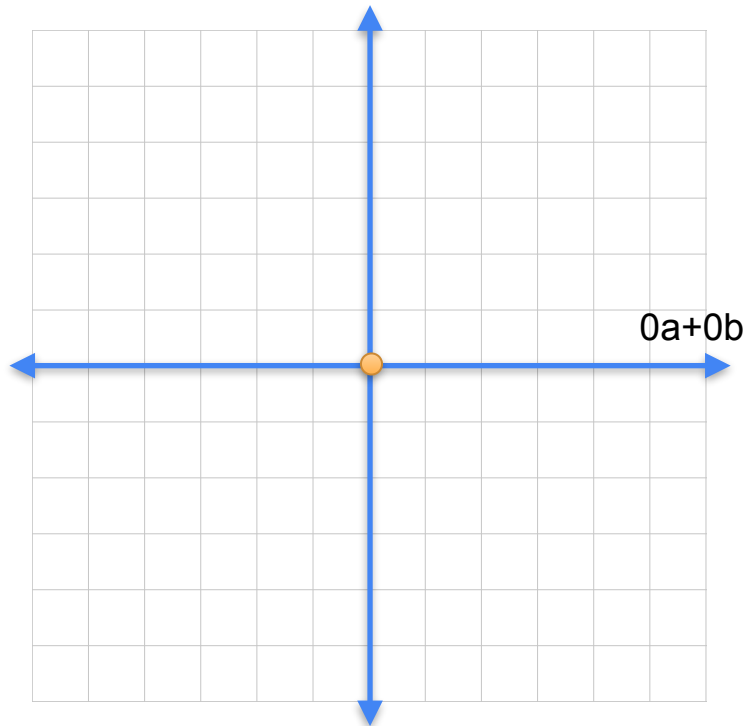
(1,0) → (0,0)

(0,1) → (0,0)

(1,1) → (0,0)



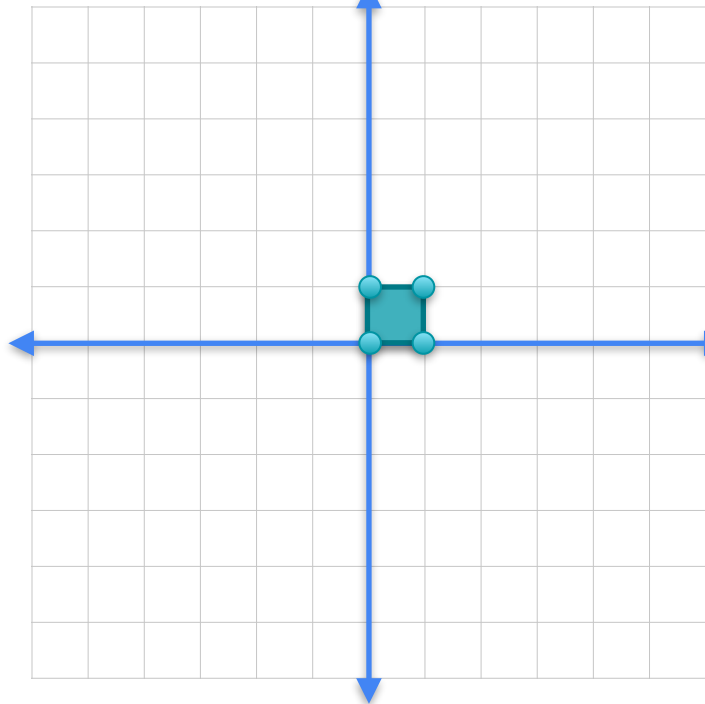
$0a+0b$



# Row space



b



0	0	a	=	0
0	0	b	=	0

$(0,0) \rightarrow (0,0)$

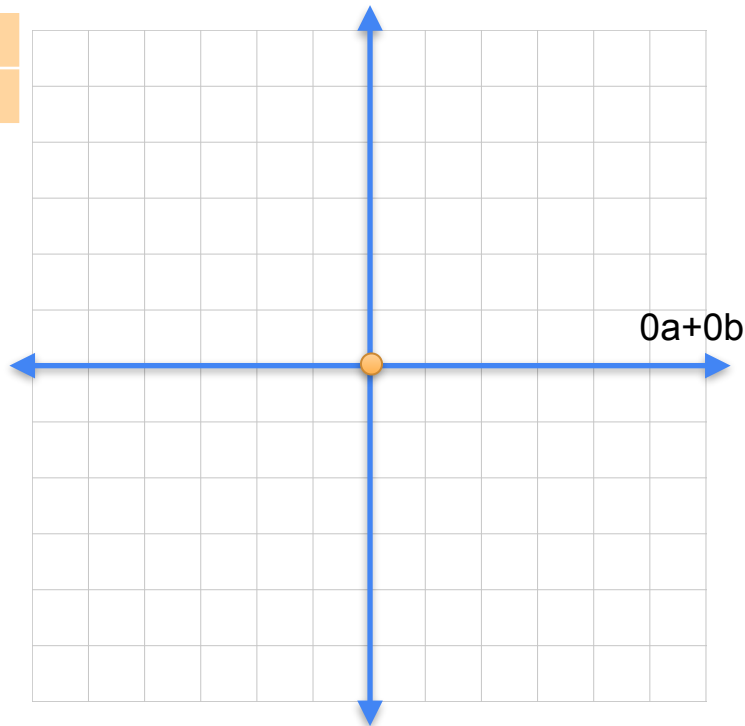
$(1,0) \rightarrow (0,0)$

$(0,1) \rightarrow (0,0)$

$(1,1) \rightarrow (0,0)$





$0a+0b$



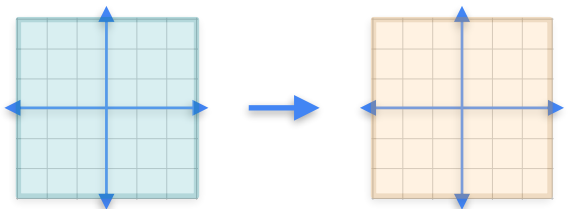


# Row space

Non-singular



	
3	1
1	2

Rank = 2

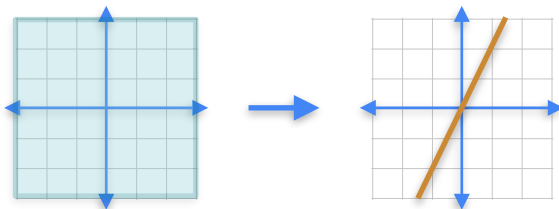


Dimension = 2

Singular



	
1	1
2	2

Rank = 1

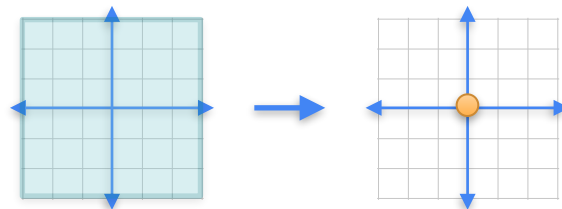


Dimension = 1

Singular

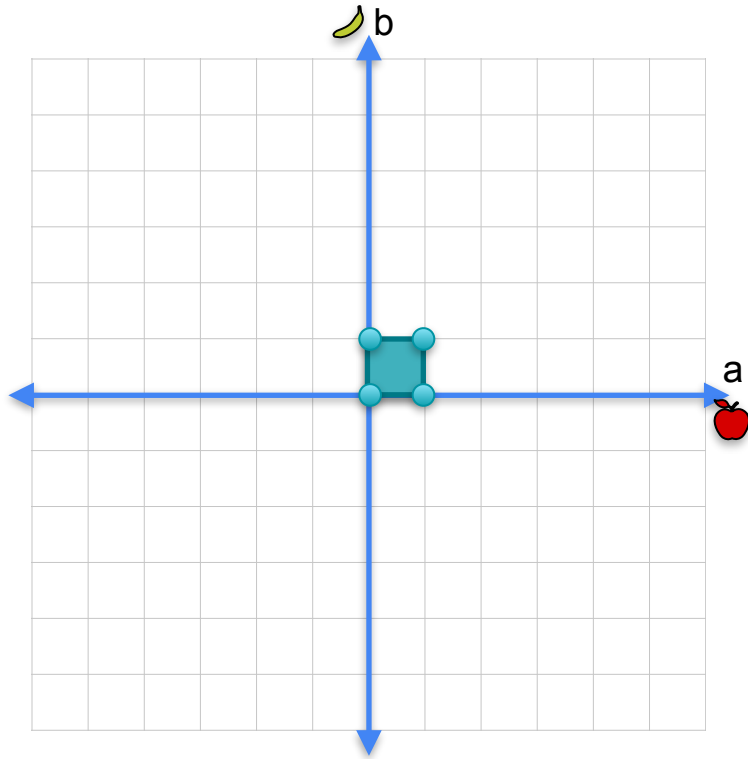
	
0	0
0	0



Rank = 0



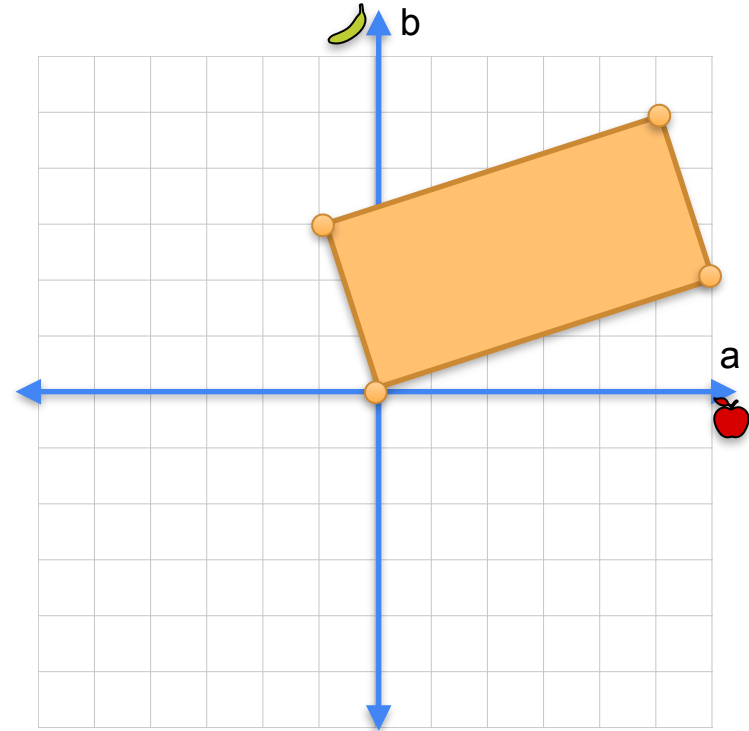
Dimension = 0

# Orthogonal matrix

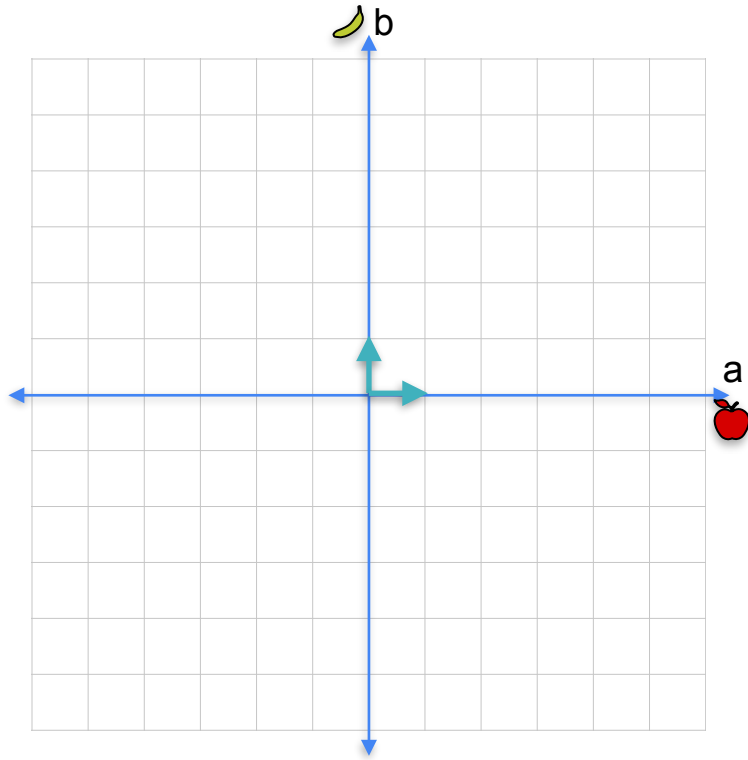




	
6	-1
2	3

$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (6,2)$   
 $(0,1) \rightarrow (-1,3)$   
 $(1,1) \rightarrow (5,5)$

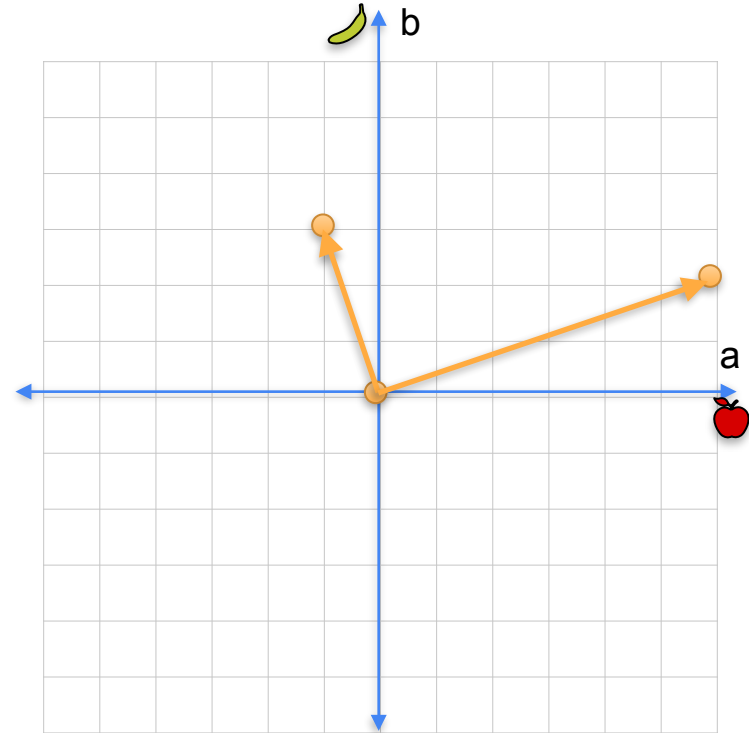


# Orthogonal matrix



	
6	-1
2	3

$(1,0) \rightarrow (6,2)$   
 $(0,1) \rightarrow (-1,3)$



# Orthogonal matrices have orthogonal columns

6	-1
2	3

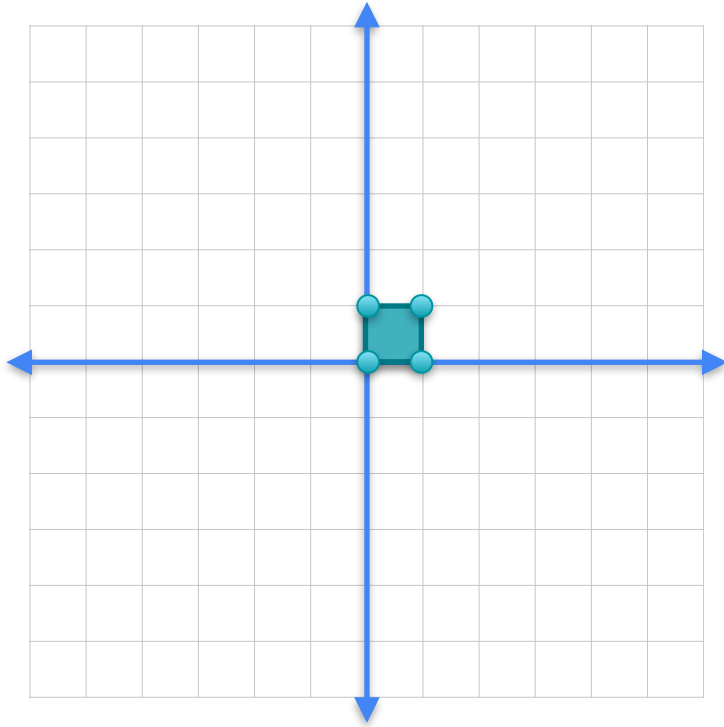
6	-1
2	3

 = 0

6	-1	2	3
---	----	---	---

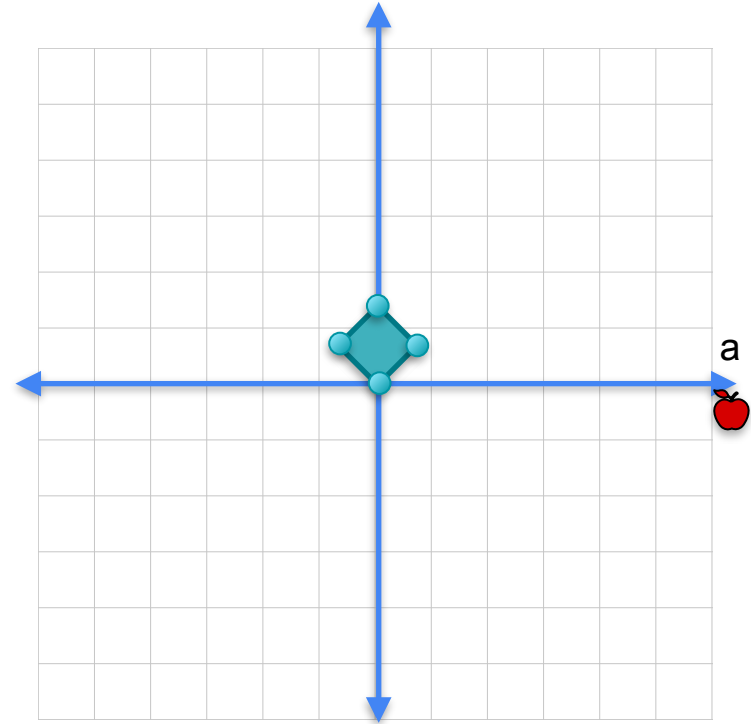
 = 0

# Orthogonal matrix

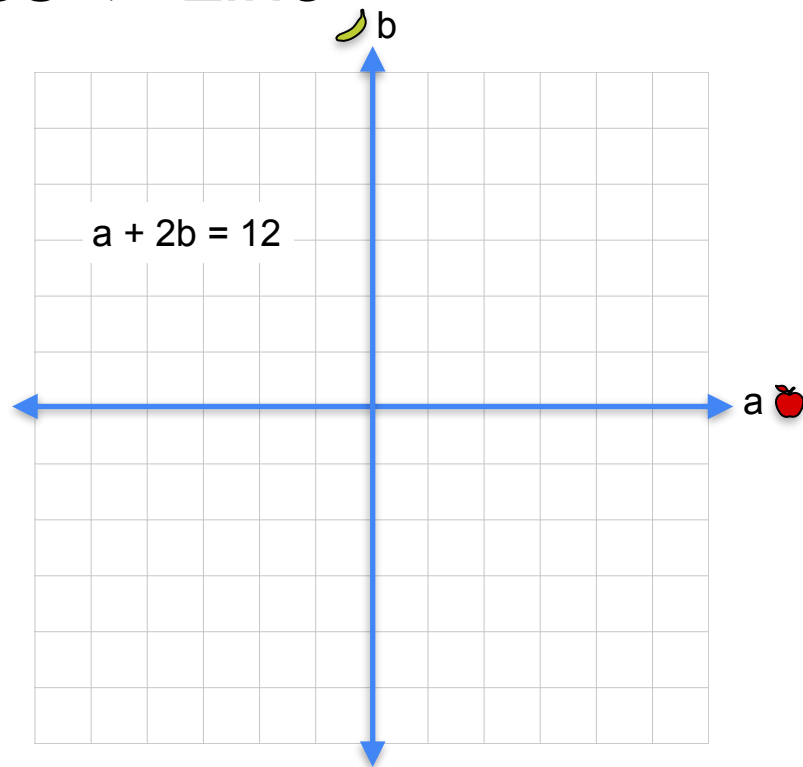
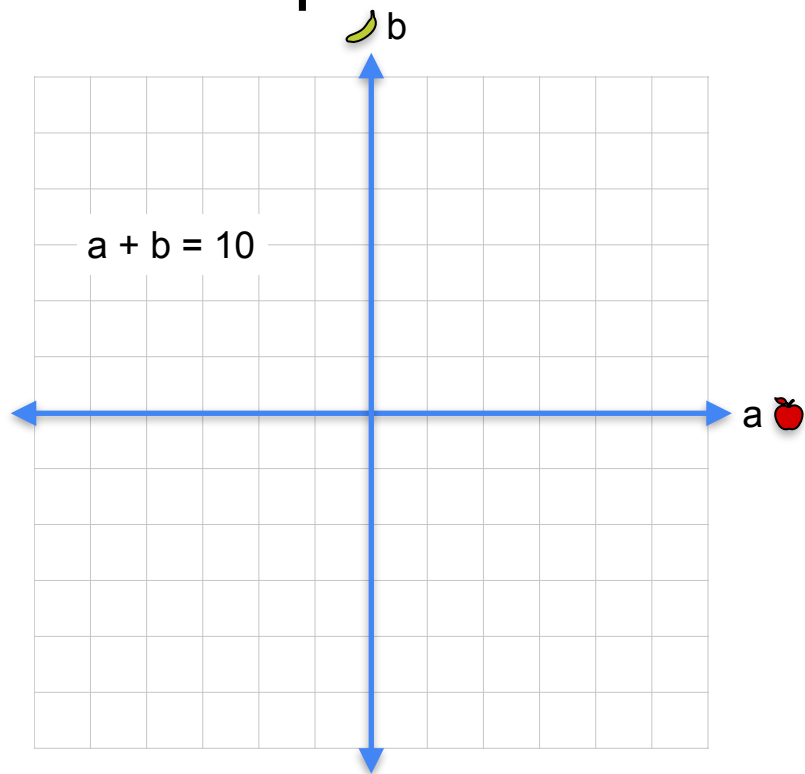


0.7071	0.7071
-0.7071	0.7071

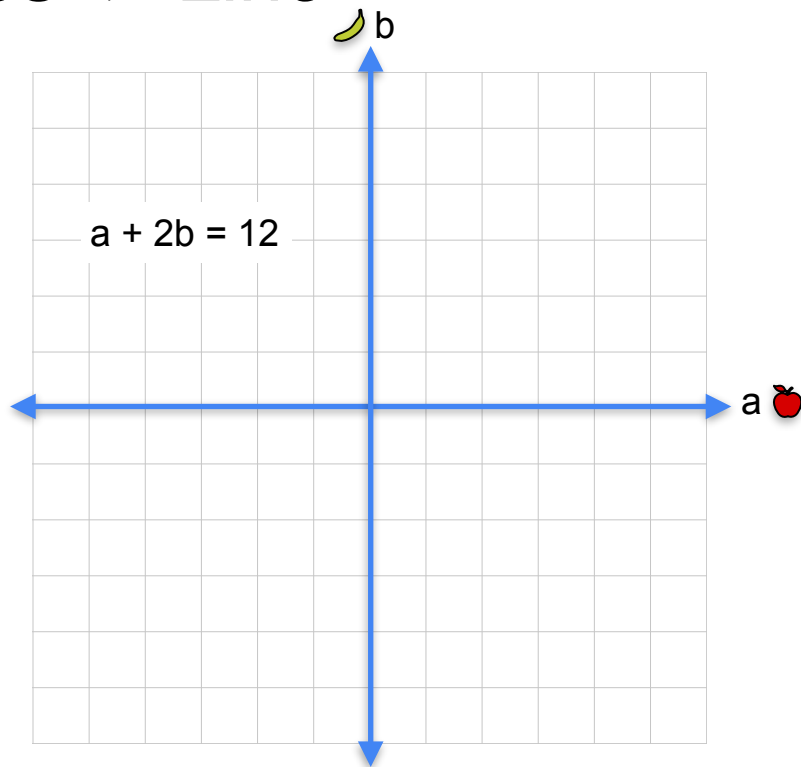
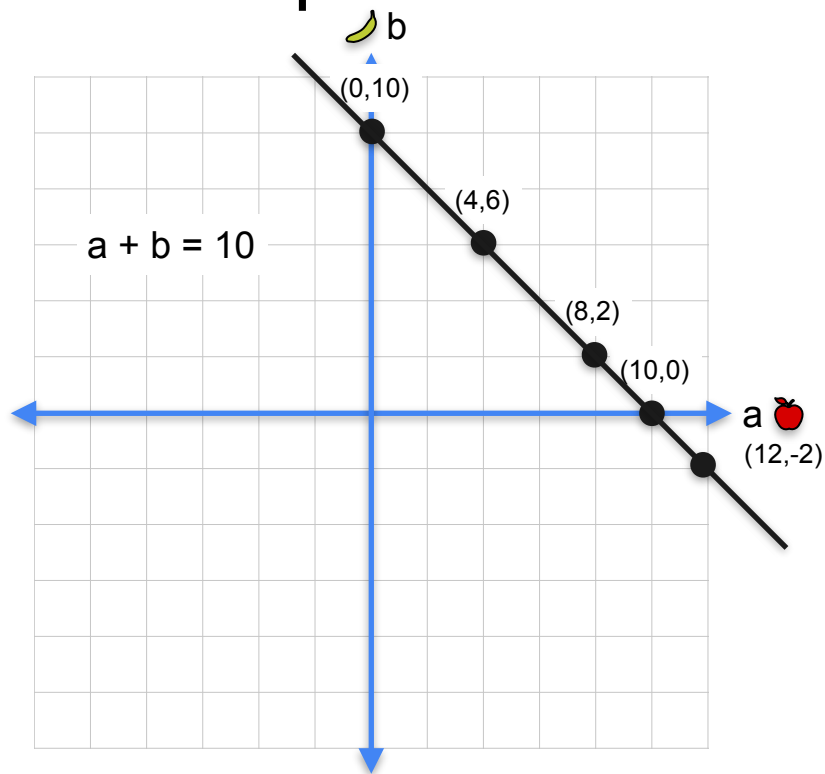
$(0,0) \rightarrow (0,0)$   
 $(1,0) \rightarrow (0.7071, 0.7071)$   
 $(0,1) \rightarrow (-0.7071, 0.7071)$   
 $(1,1) \rightarrow (0, 1.4142)$



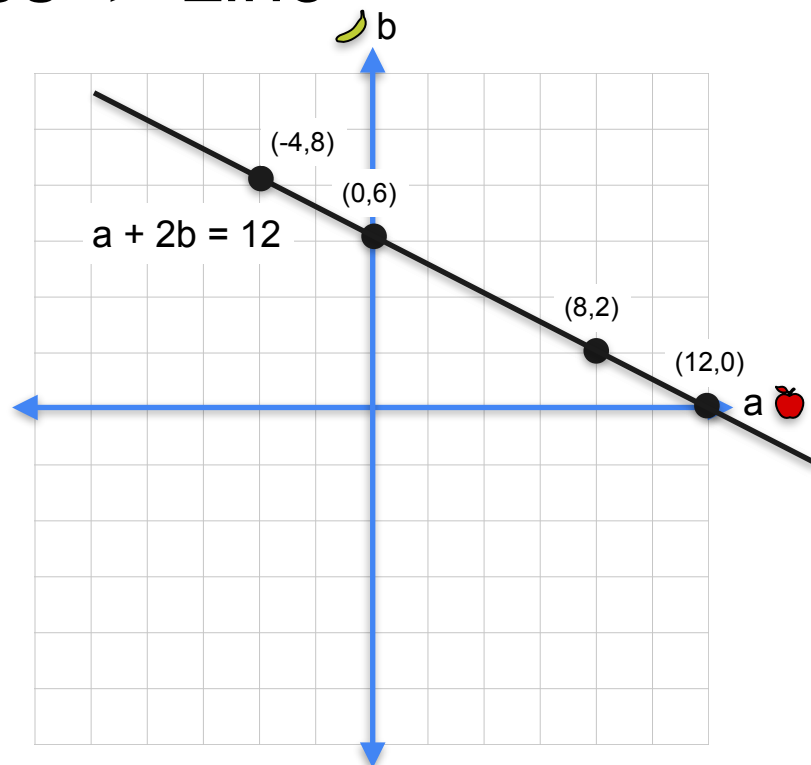
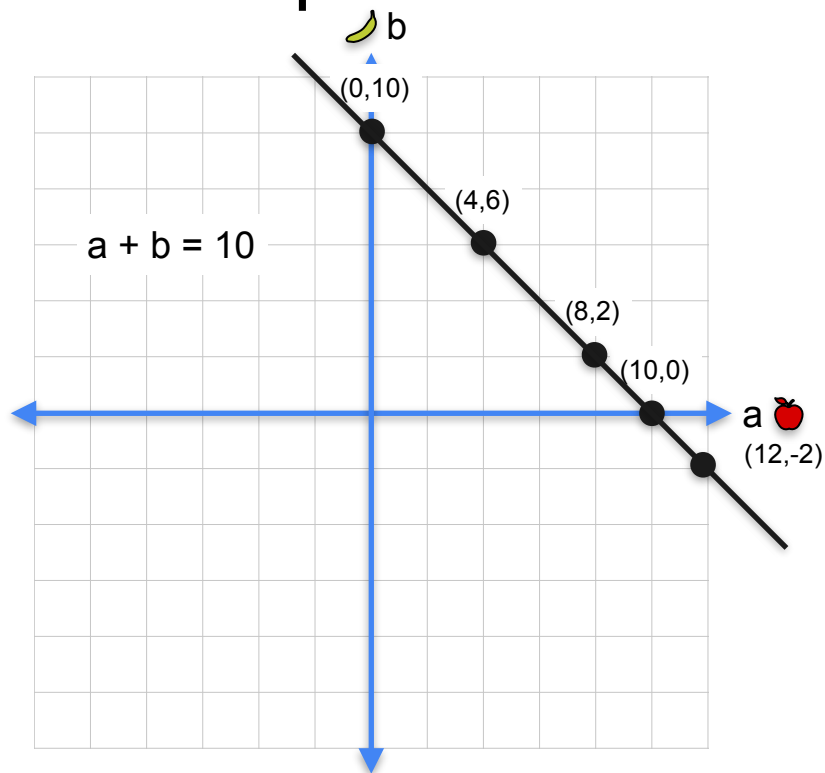
# Linear equation in 2 variables -> Line



# Linear equation in 2 variables -> Line



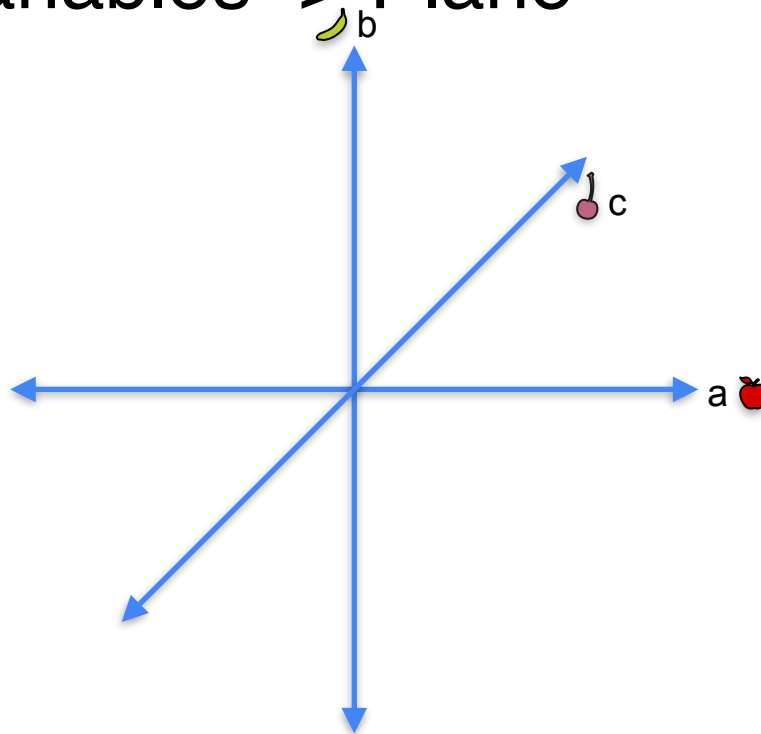
# Linear equation in 2 variables -> Line





# Linear equation in 3 variables -> Plane

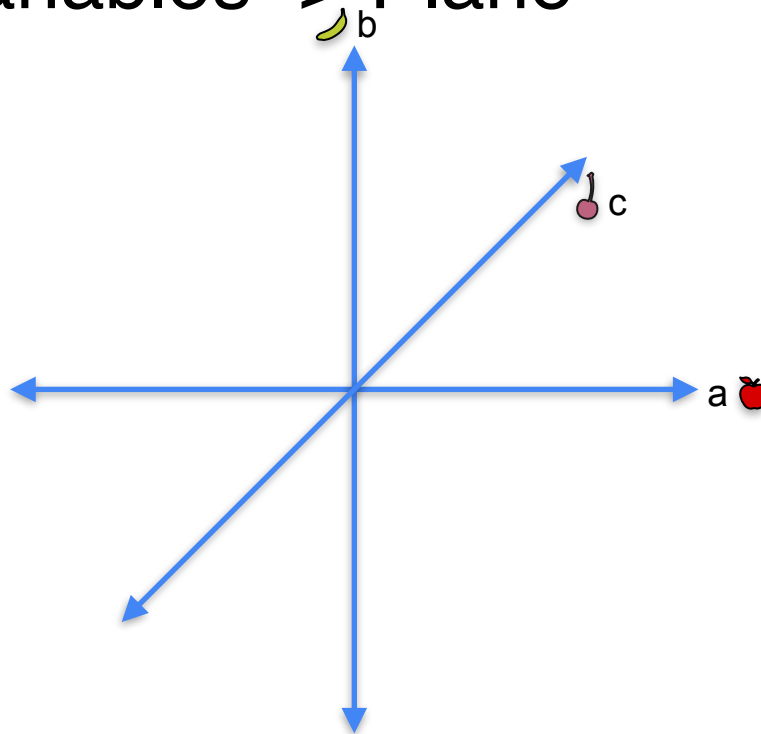
$$a + b + c = 1$$



# Linear equation in 3 variables -> Plane

$$a + b + c = 1$$

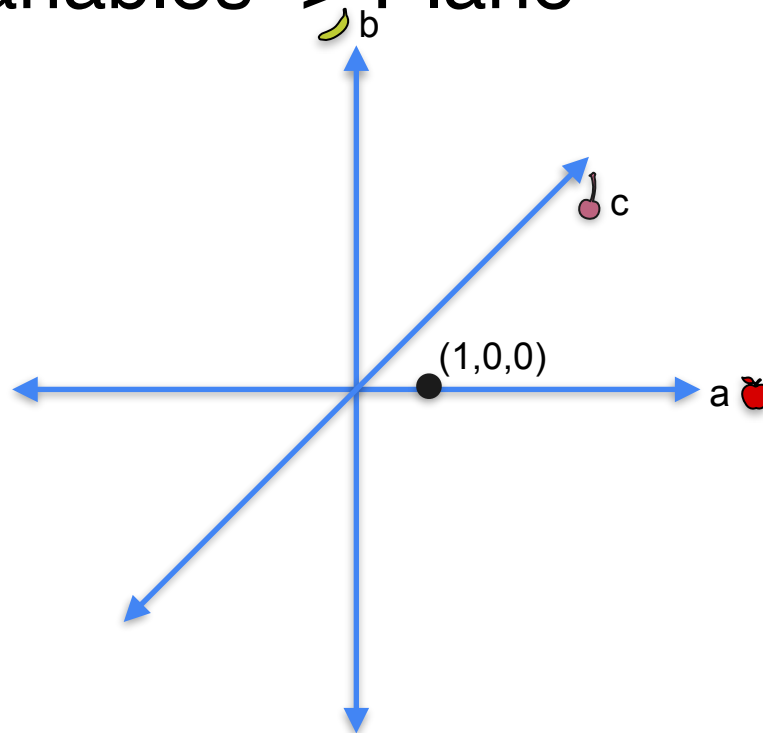
$$1 + 0 + 0 = 1$$



# Linear equation in 3 variables -> Plane

$$a + b + c = 1$$

$$1 + 0 + 0 = 1$$

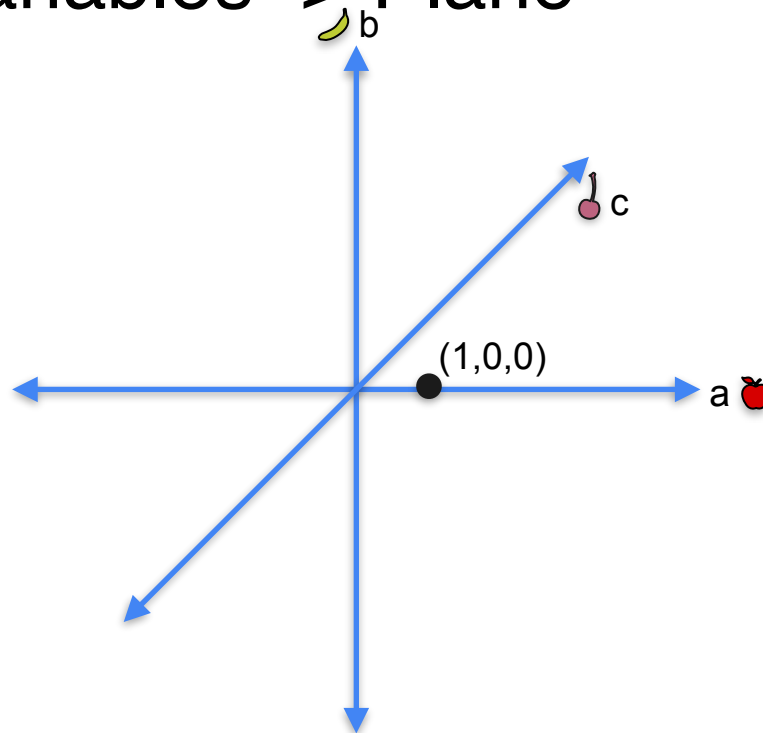


# Linear equation in 3 variables -> Plane

$$a + b + c = 1$$

$$1 + 0 + 0 = 1$$

$$0 + 1 + 0 = 1$$

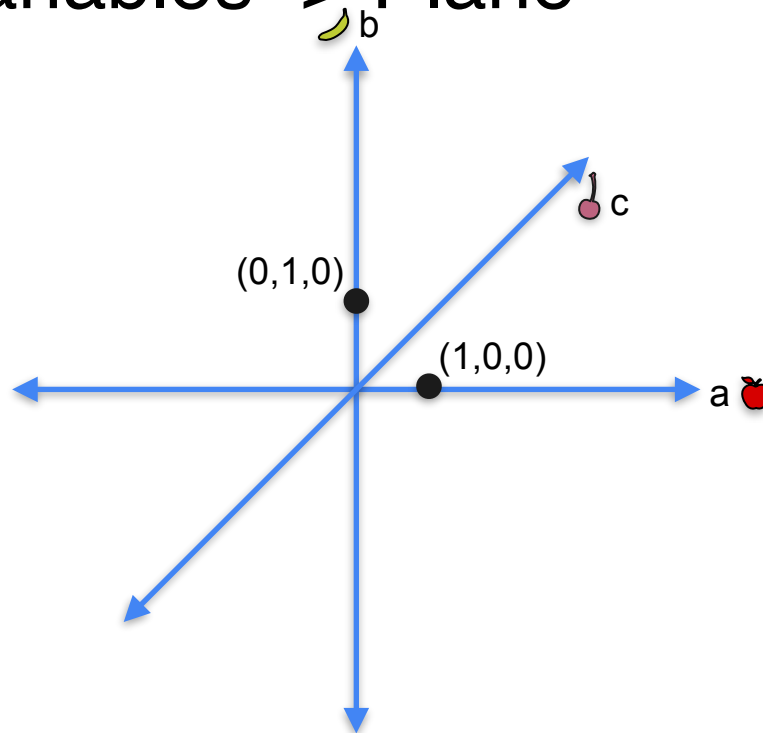


# Linear equation in 3 variables -> Plane

$$a + b + c = 1$$

$$1 + 0 + 0 = 1$$

$$0 + 1 + 0 = 1$$



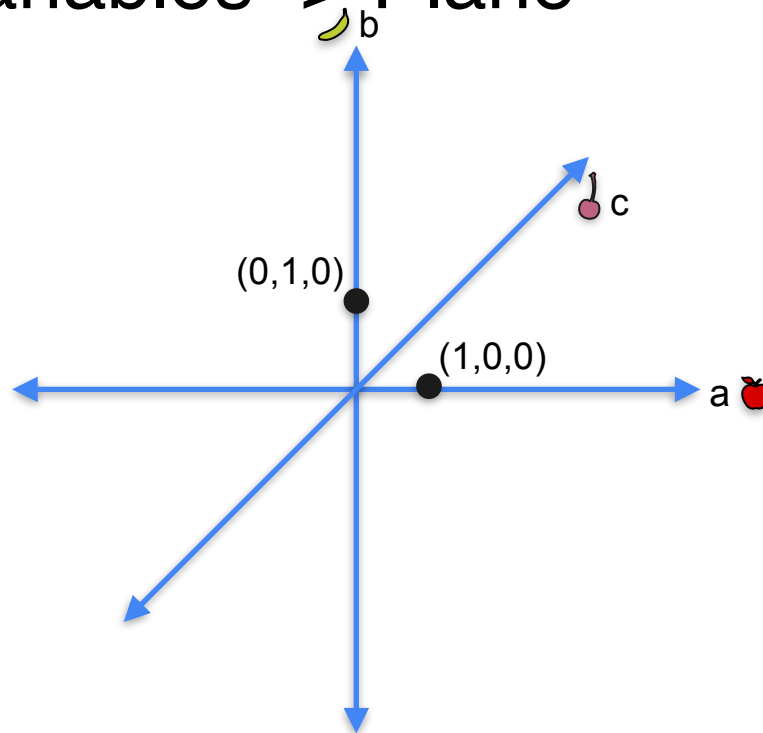
# Linear equation in 3 variables -> Plane

$$a + b + c = 1$$

$$1 + 0 + 0 = 1$$

$$0 + 1 + 0 = 1$$

$$0 + 0 + 1 = 1$$



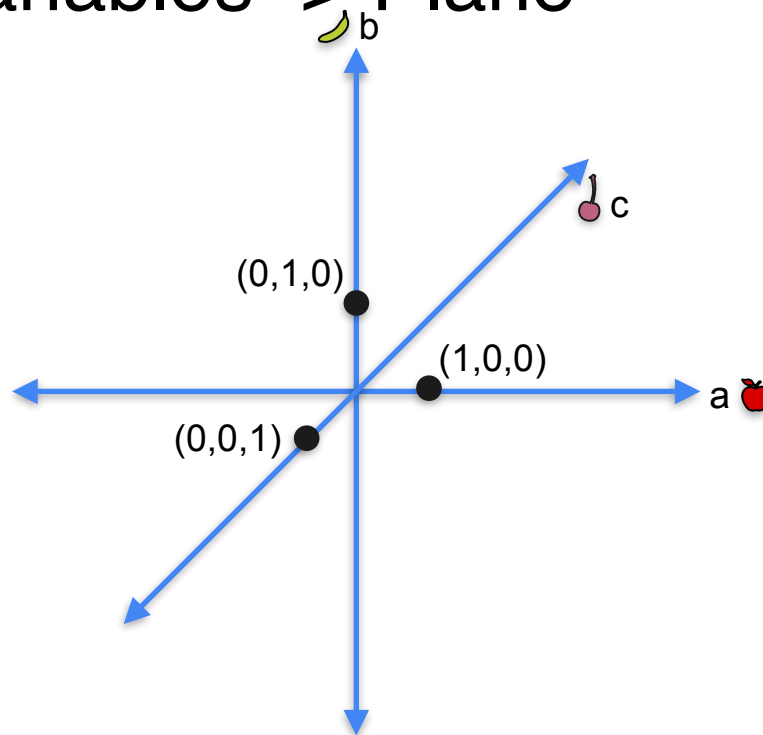
# Linear equation in 3 variables -> Plane

$$a + b + c = 1$$

$$1 + 0 + 0 = 1$$

$$0 + 1 + 0 = 1$$

$$0 + 0 + 1 = 1$$



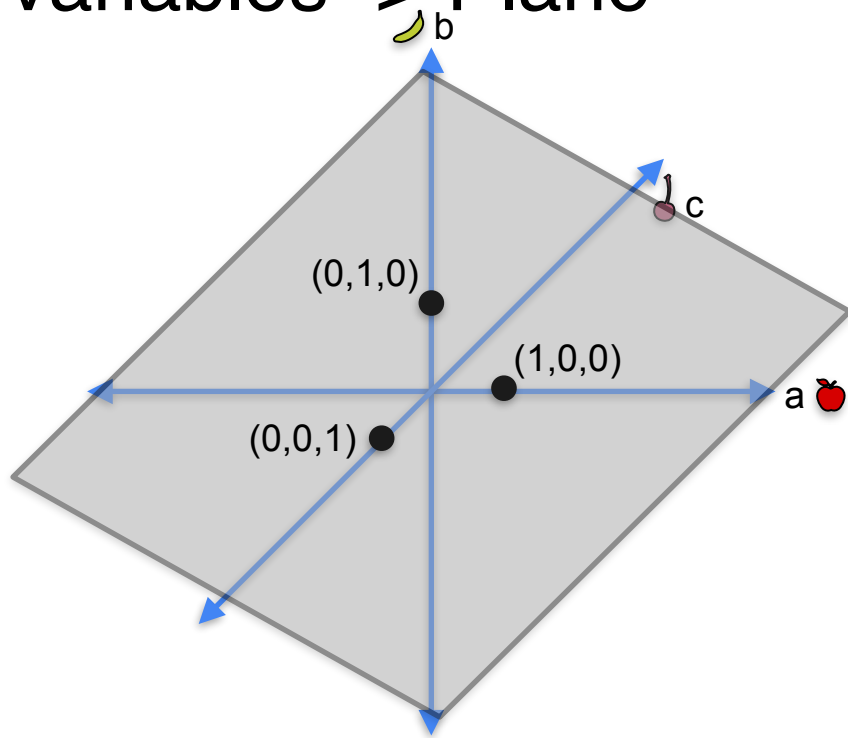
# Linear equation in 3 variables -> Plane

$$a + b + c = 1$$

$$1 + 0 + 0 = 1$$

$$0 + 1 + 0 = 1$$

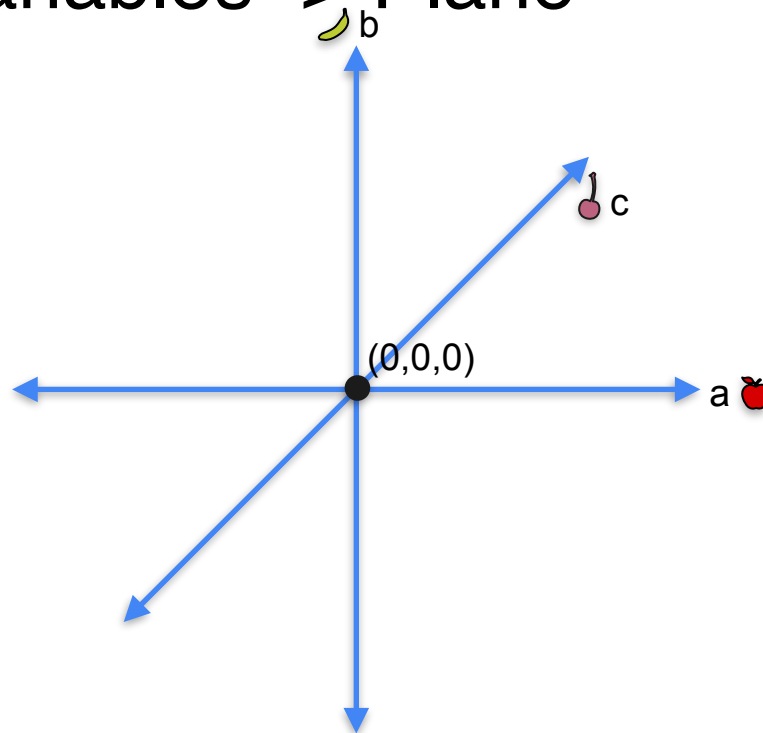
$$0 + 0 + 1 = 1$$





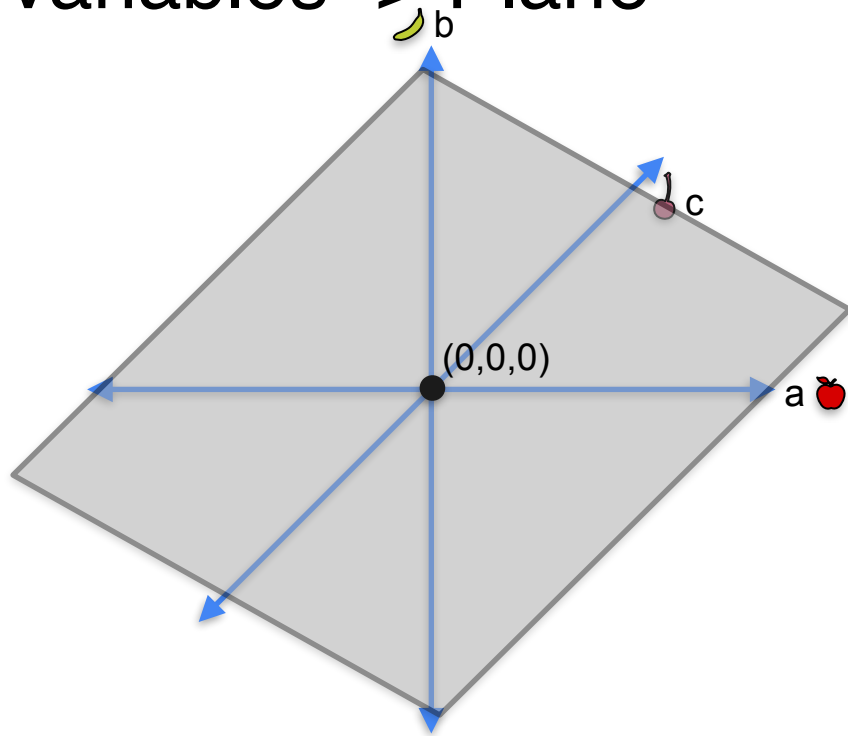
# Linear equation in 3 variables -> Plane

$$3a - 5b + 2c = 0$$



# Linear equation in 3 variables -> Plane

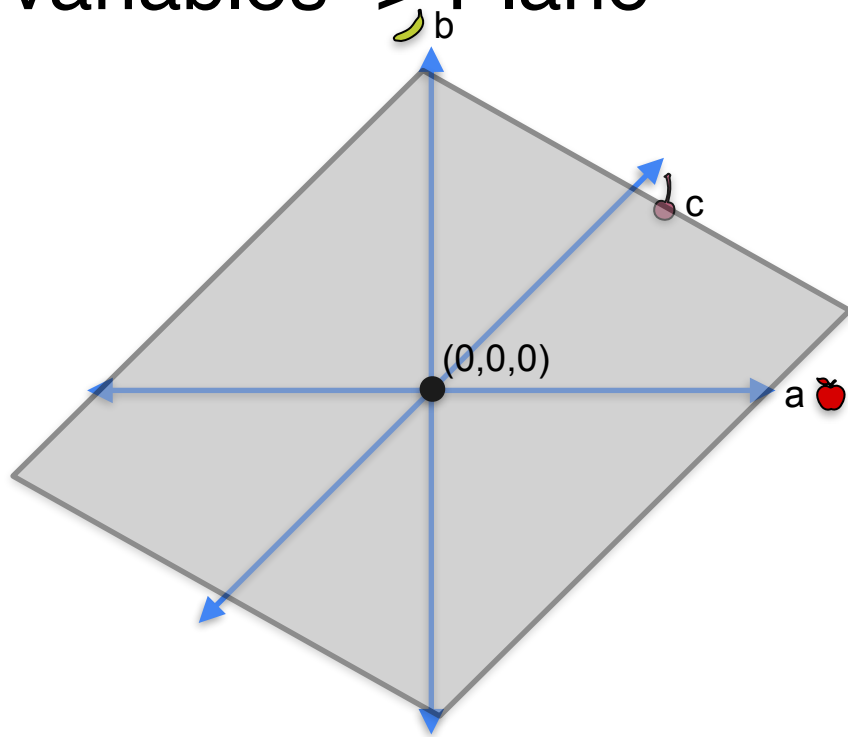
$$3a - 5b + 2c = 0$$



# Linear equation in 3 variables -> Plane

$$3a - 5b + 2c = 0$$

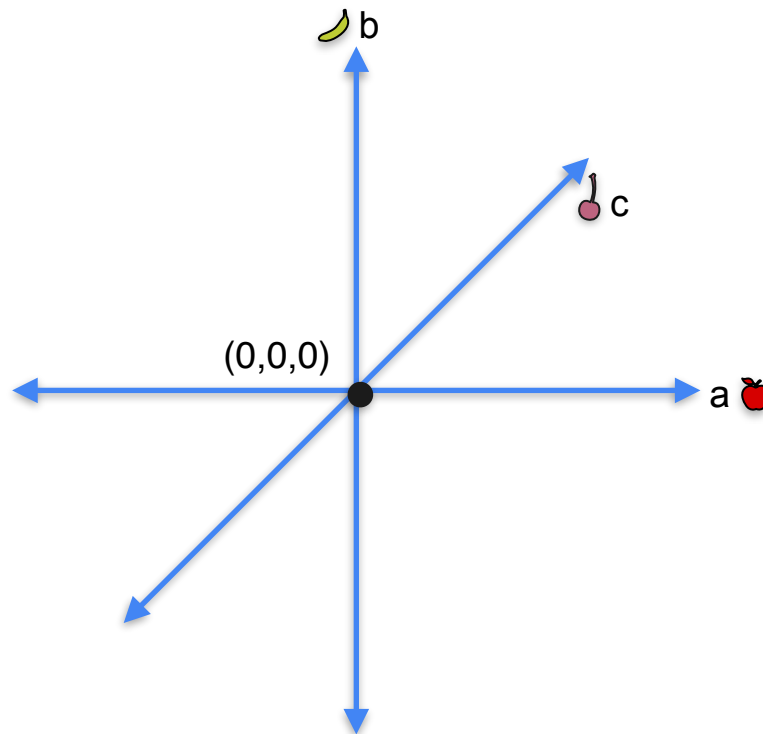
$$3(0) + 5(0) + 2(0) = 0$$



# System 1

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$



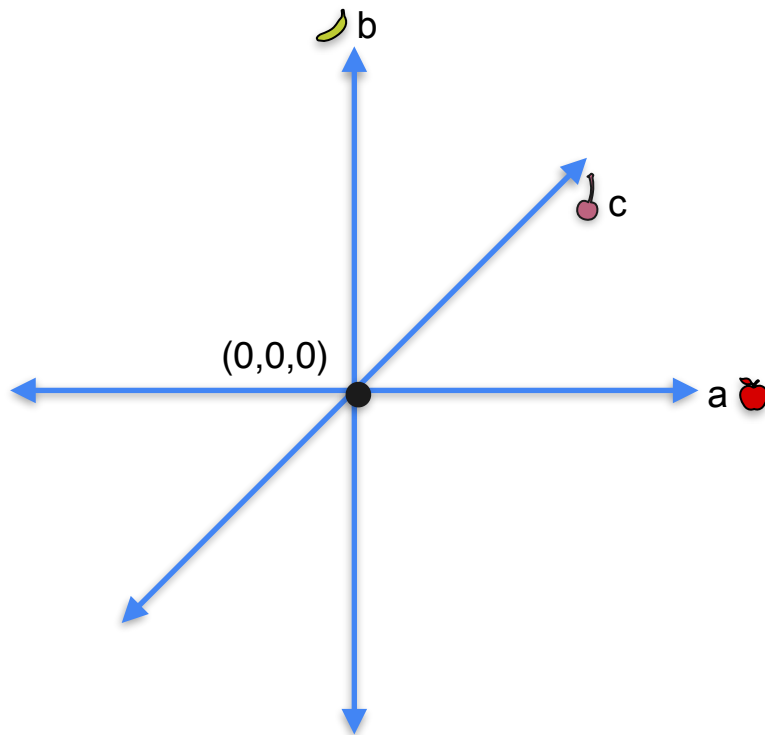
# System 1

## System 1

- $a + b + c = 0$

- $a + 2b + c = 0$

- $a + b + 2c = 0$



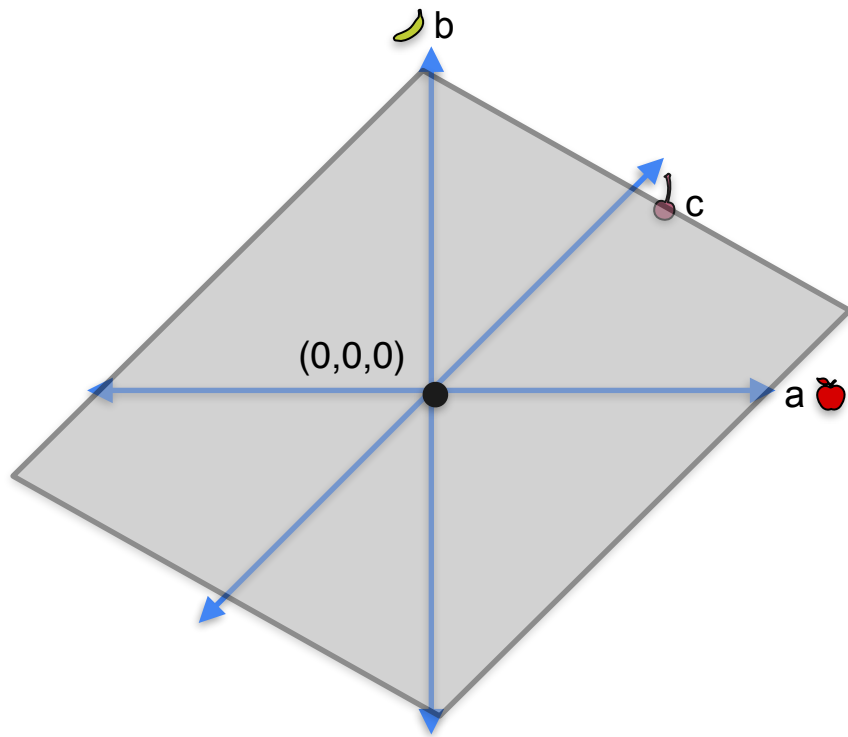
# System 1

## System 1

- $a + b + c = 0$

- $a + 2b + c = 0$

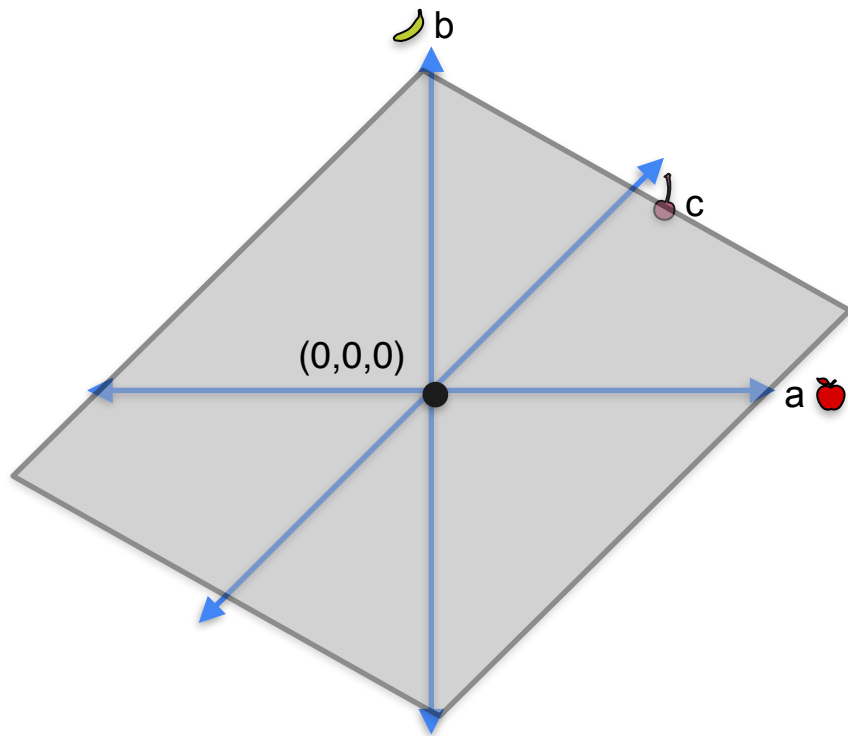
- $a + b + 2c = 0$



# System 1

## System 1

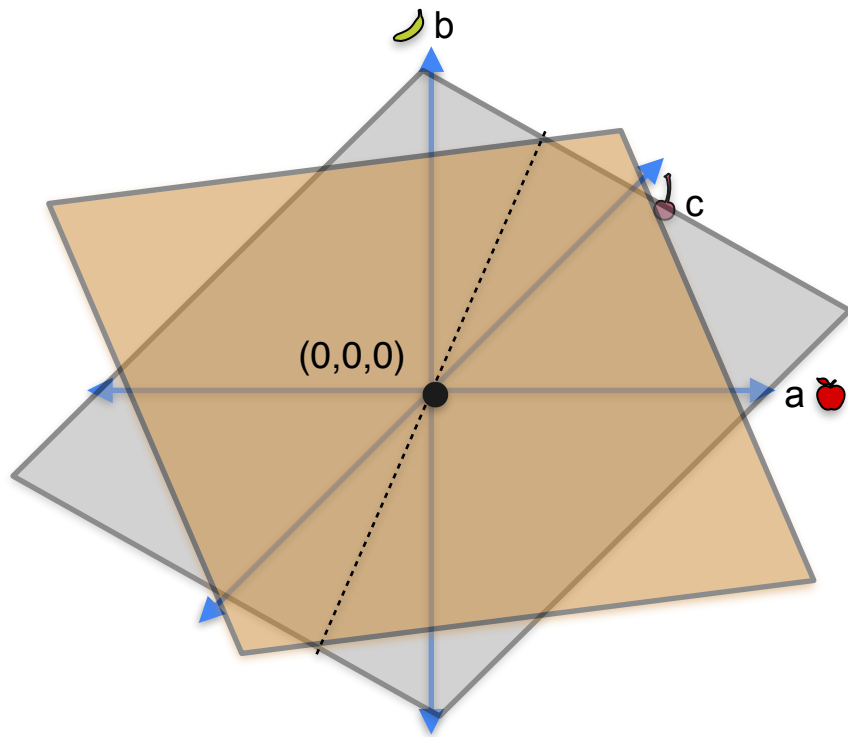
- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$



# System 1

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

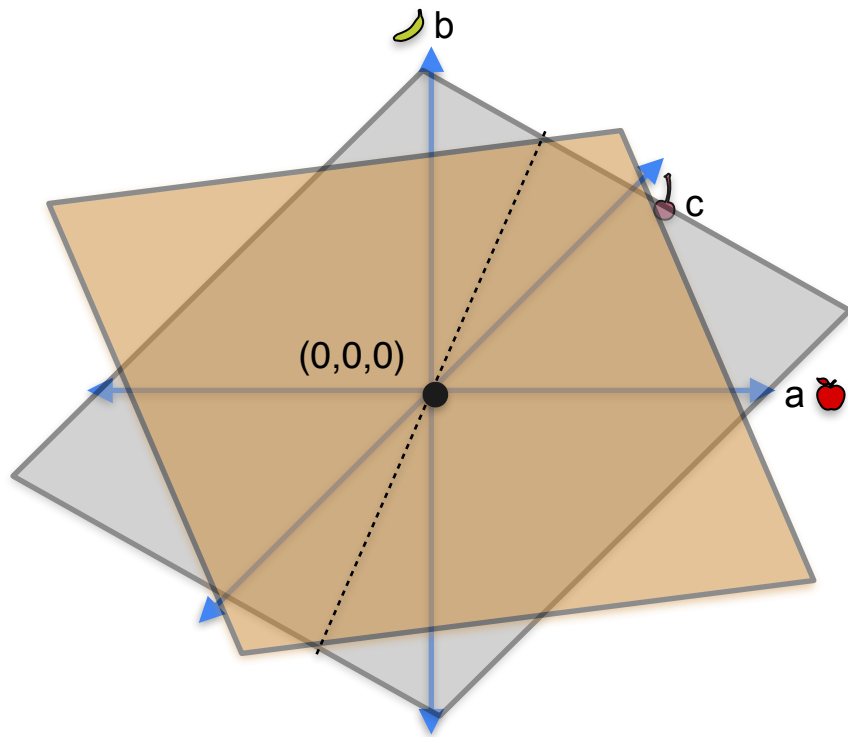




# System 1

## System 1

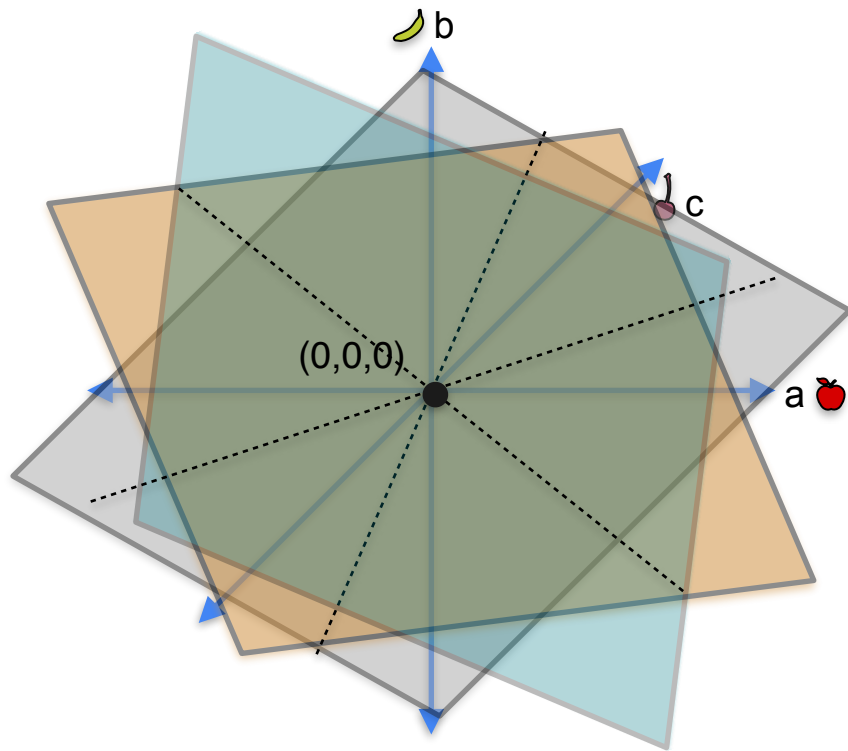
- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$



# System 1

## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$



# System 1

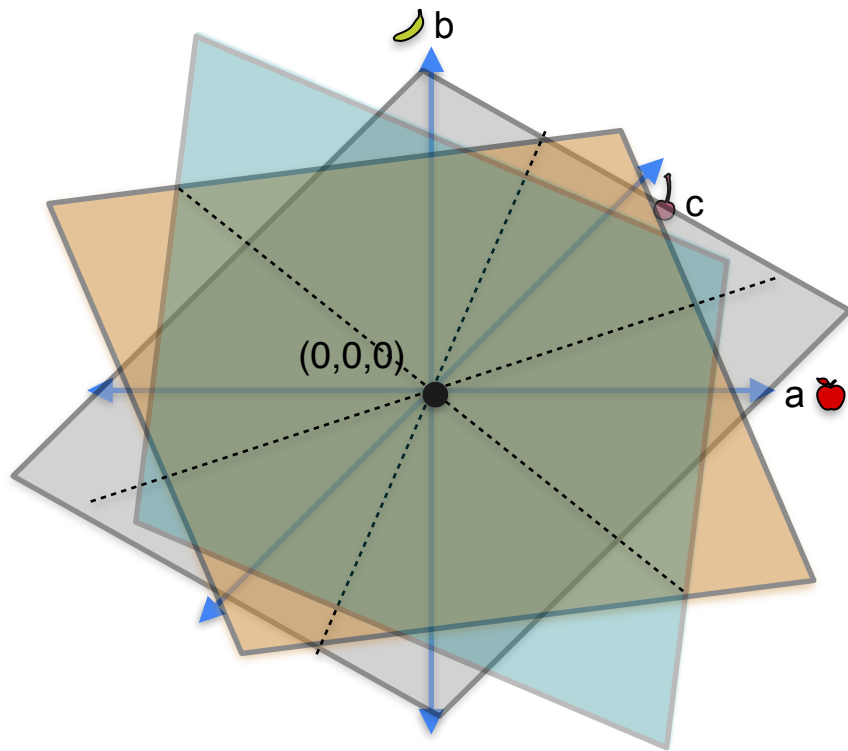
## System 1

- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$



## Solution space

- $a = 0$
- $b = 0$
- $c = 0$



# System 1

## System 1

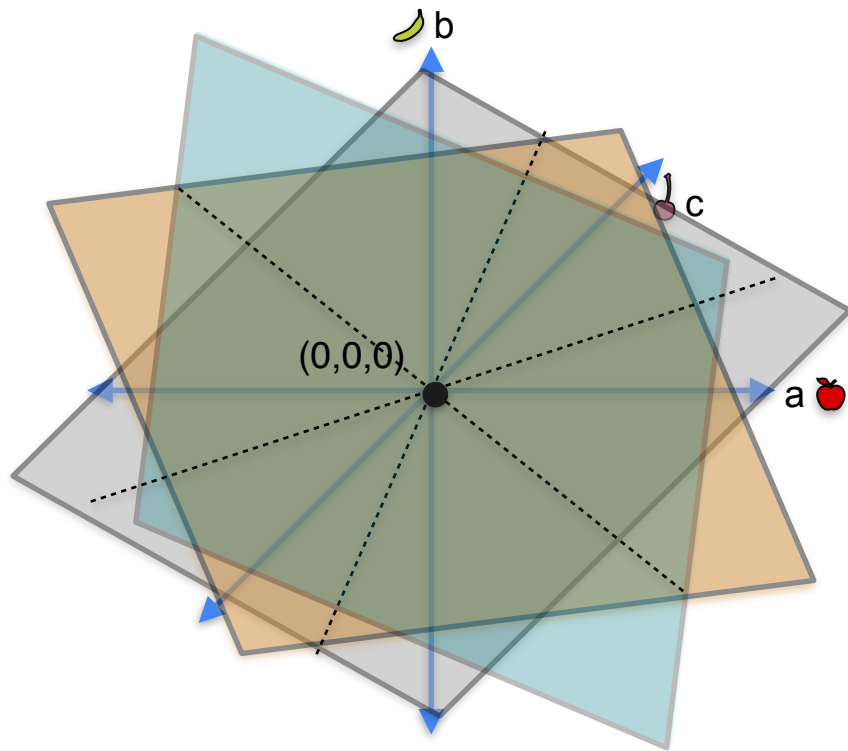
- $a + b + c = 0$
- $a + 2b + c = 0$
- $a + b + 2c = 0$

### Solution space

- $a = 0$
- $b = 0$
- $c = 0$



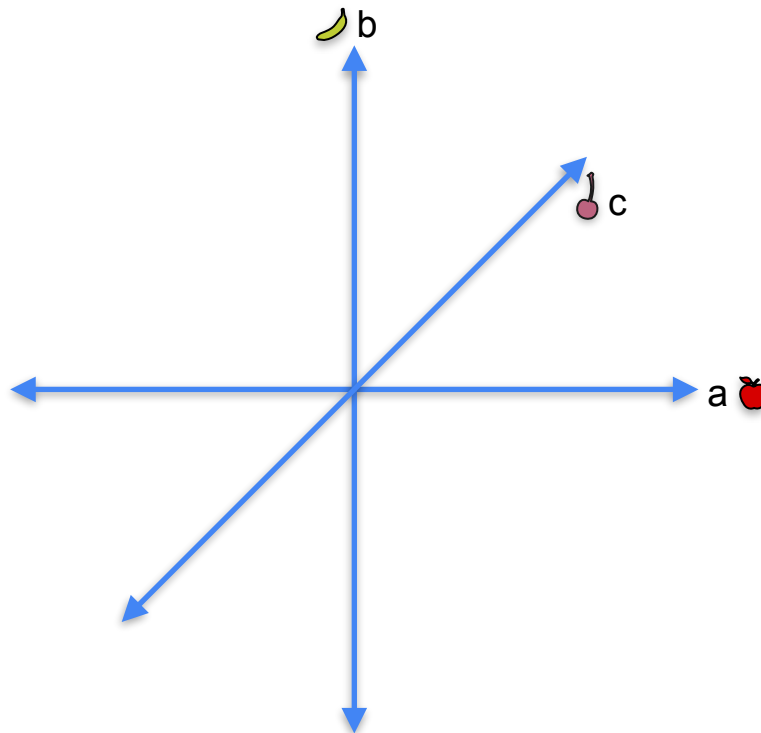
The point  
(0,0,0)



# System 2

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



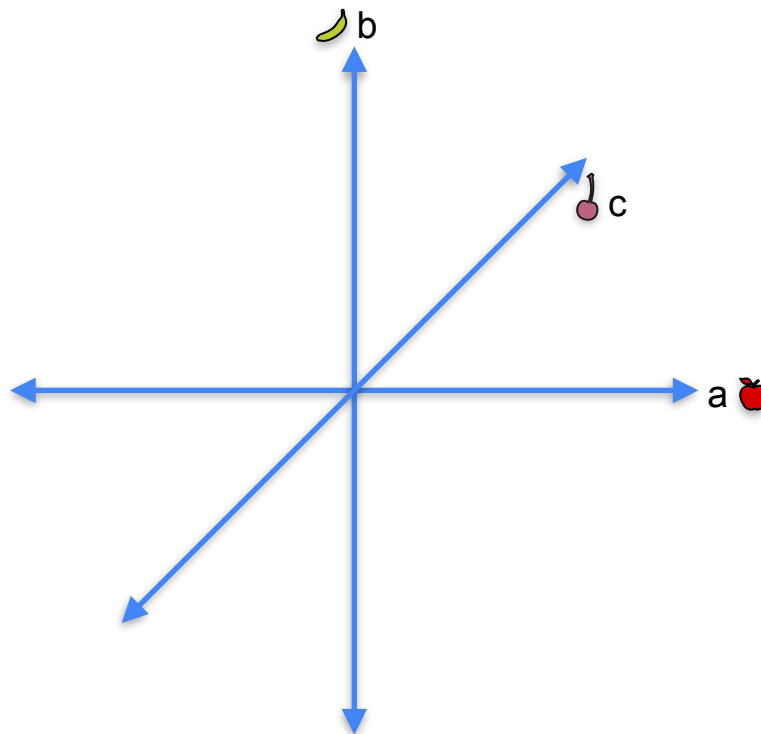
# System 2

## System 2

- $a + b + c = 0$

- $a + b + 2c = 0$

- $a + b + 3c = 0$



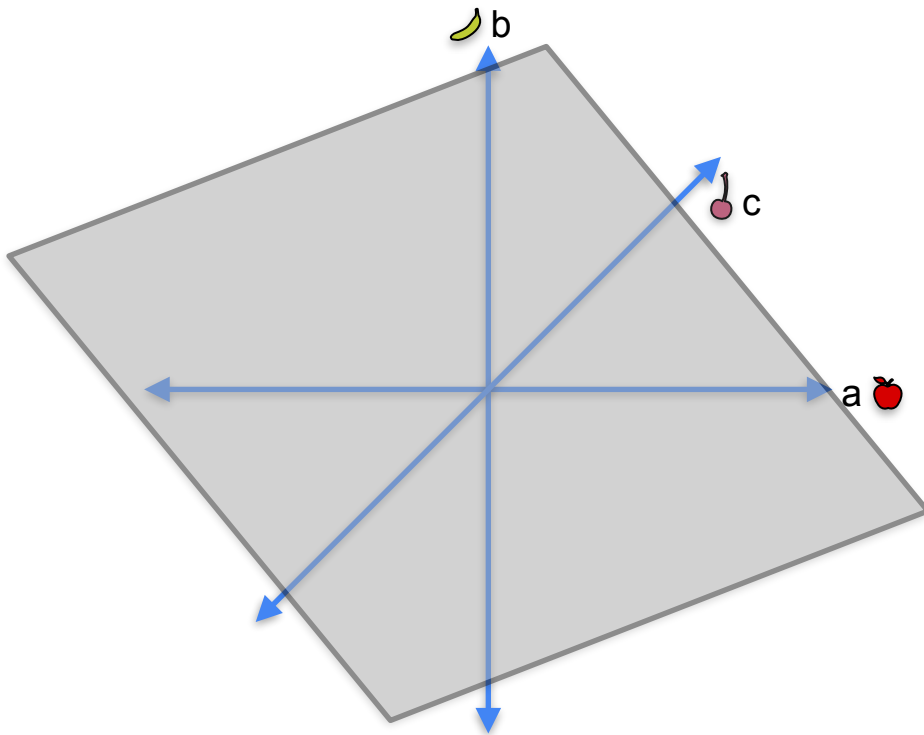
# System 2

## System 2

- $a + b + c = 0$

- $a + b + 2c = 0$

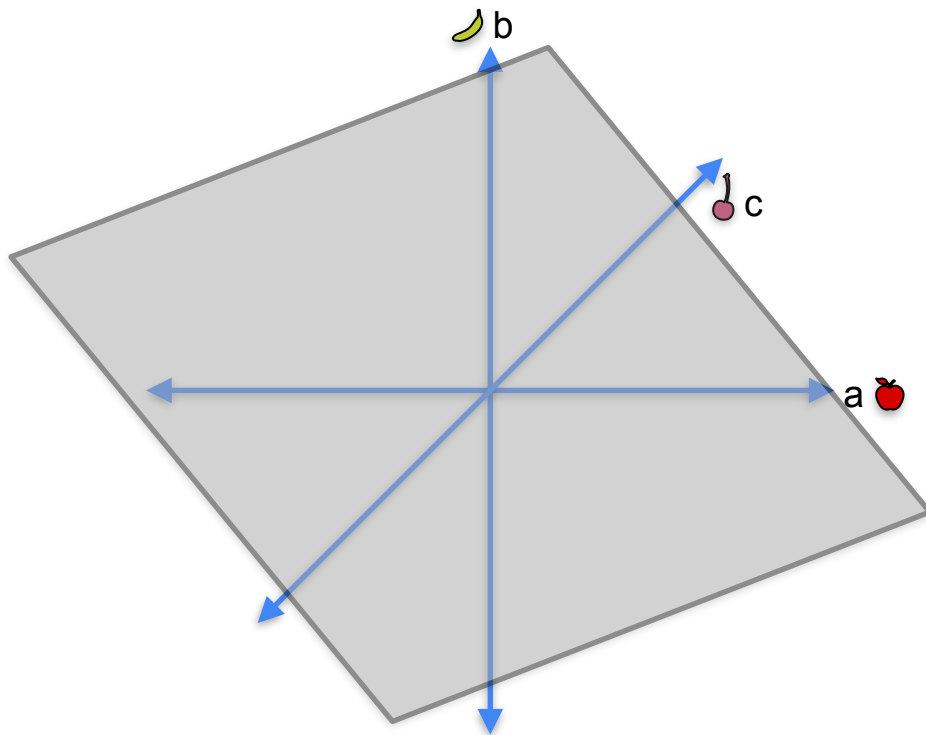
- $a + b + 3c = 0$



# System 2

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

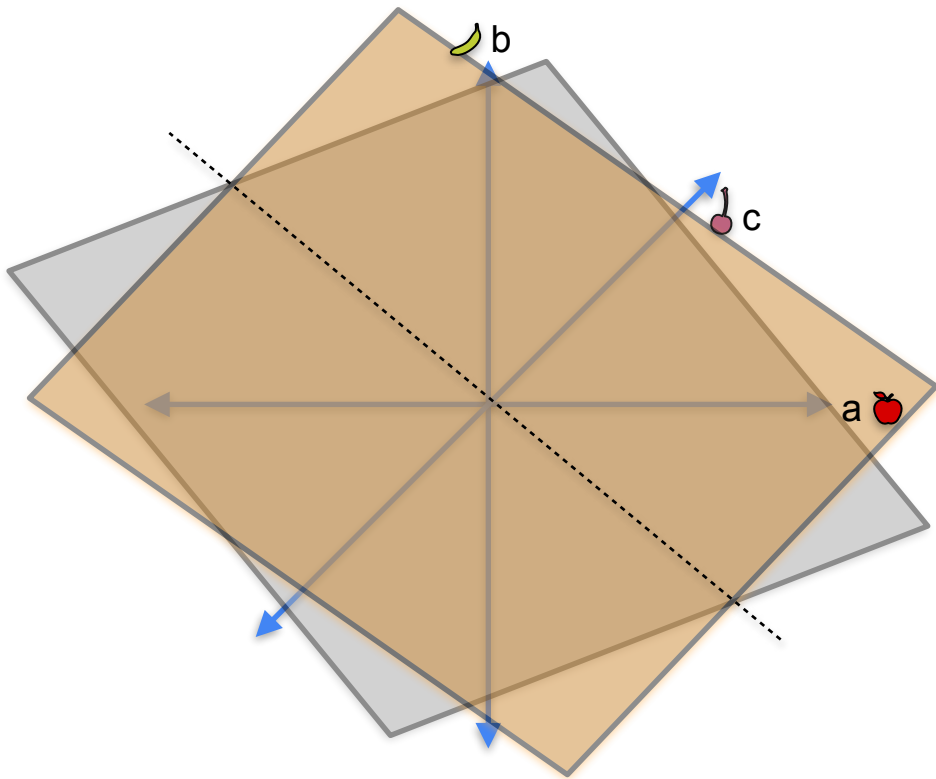




# System 2

## System 2

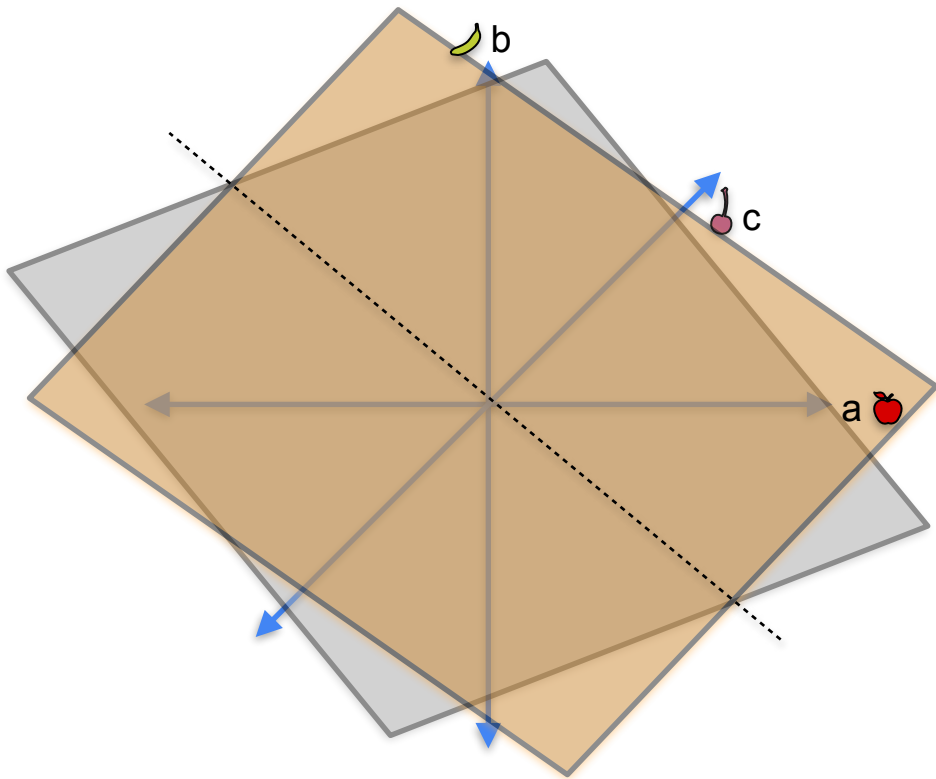
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



# System 2

## System 2

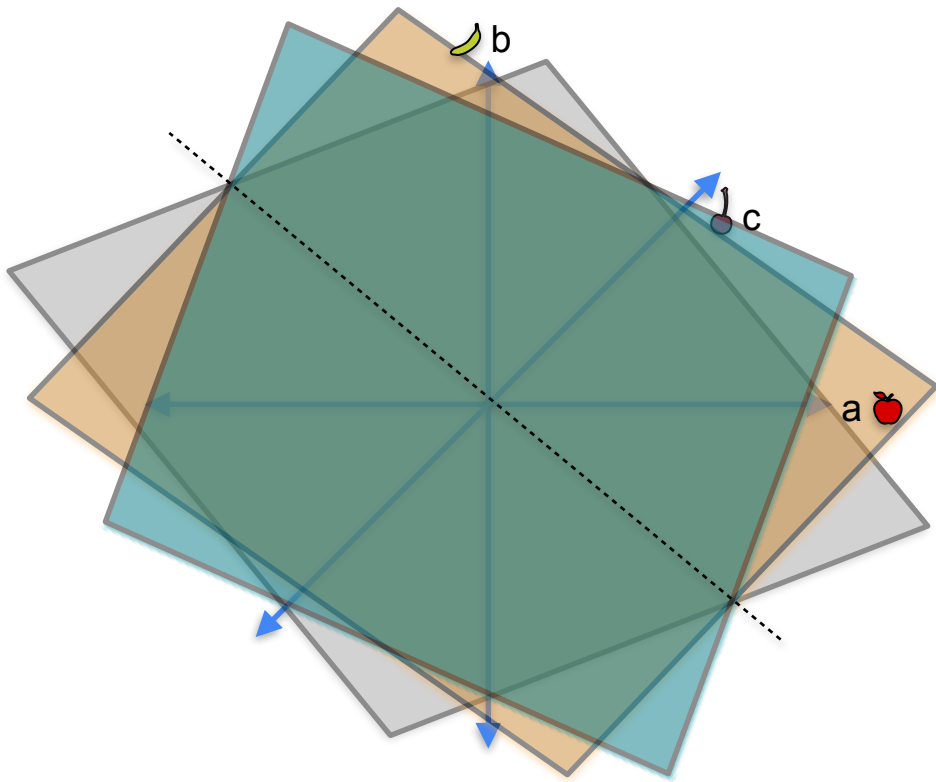
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



# System 2

## System 2

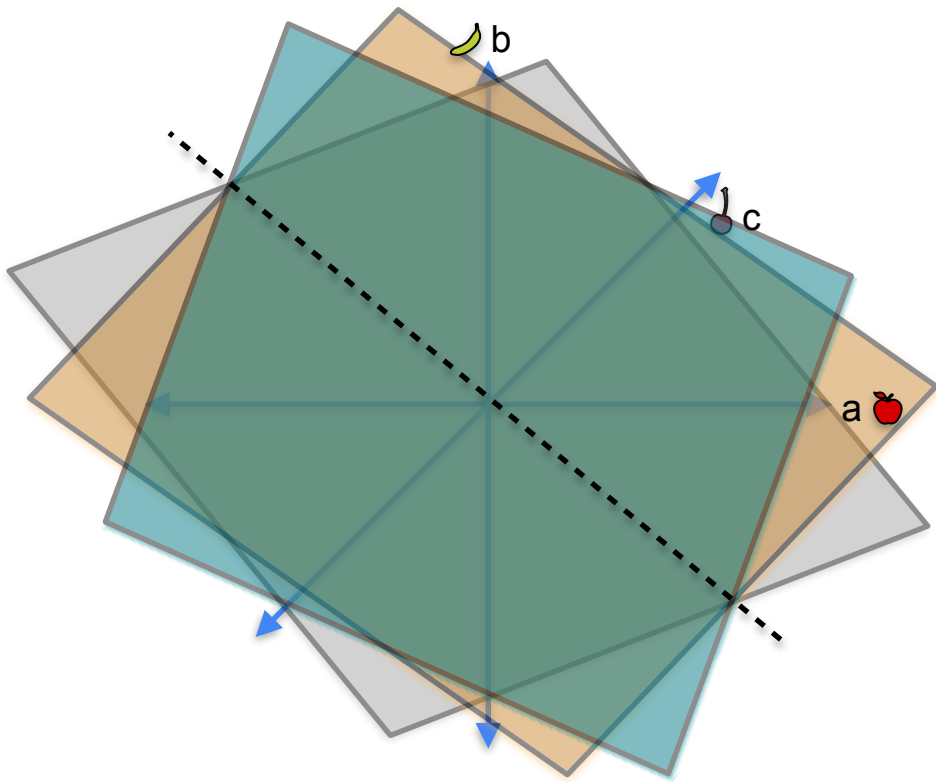
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



# System 2

## System 2

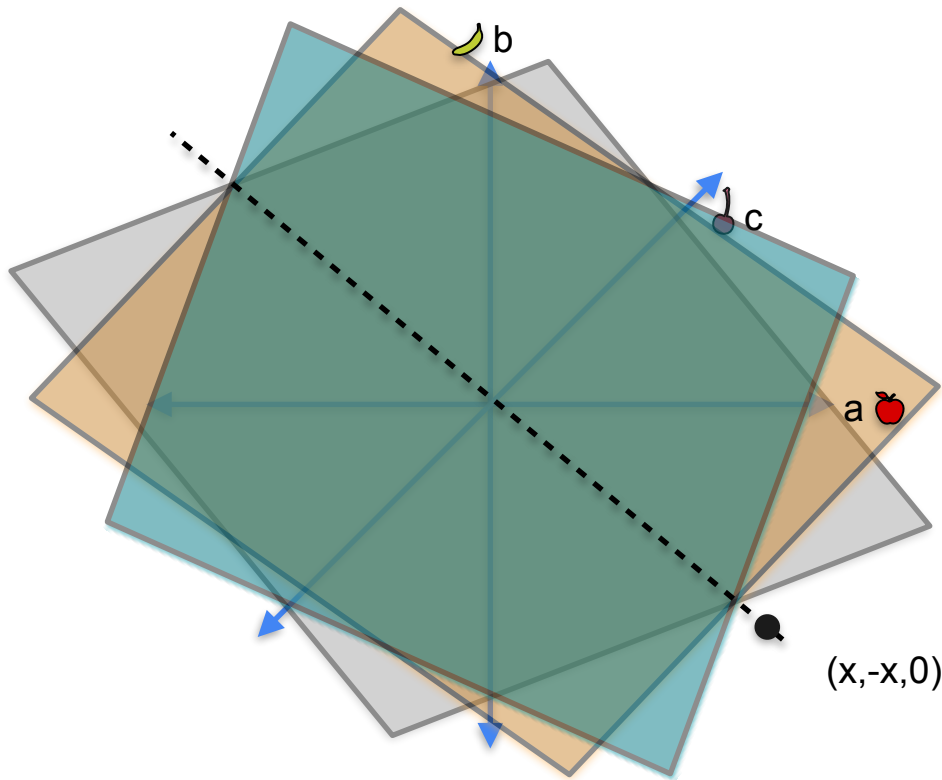
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



# System 2

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$



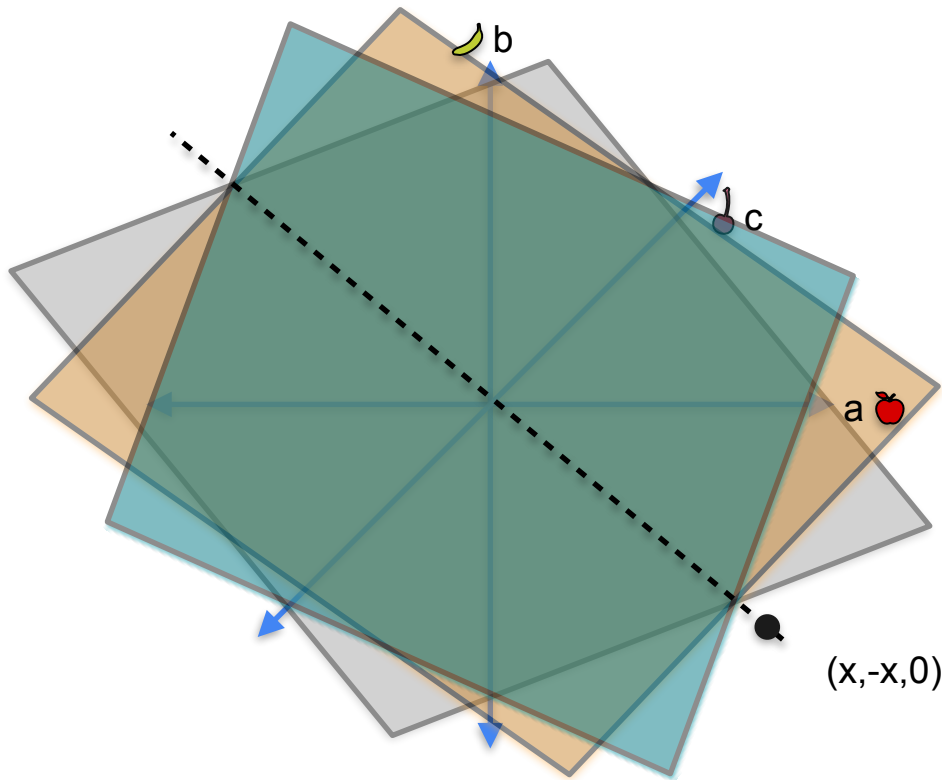
# System 2

## System 2

- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space

- $c = 0$
- $b = -a$



# System 2

## System 2

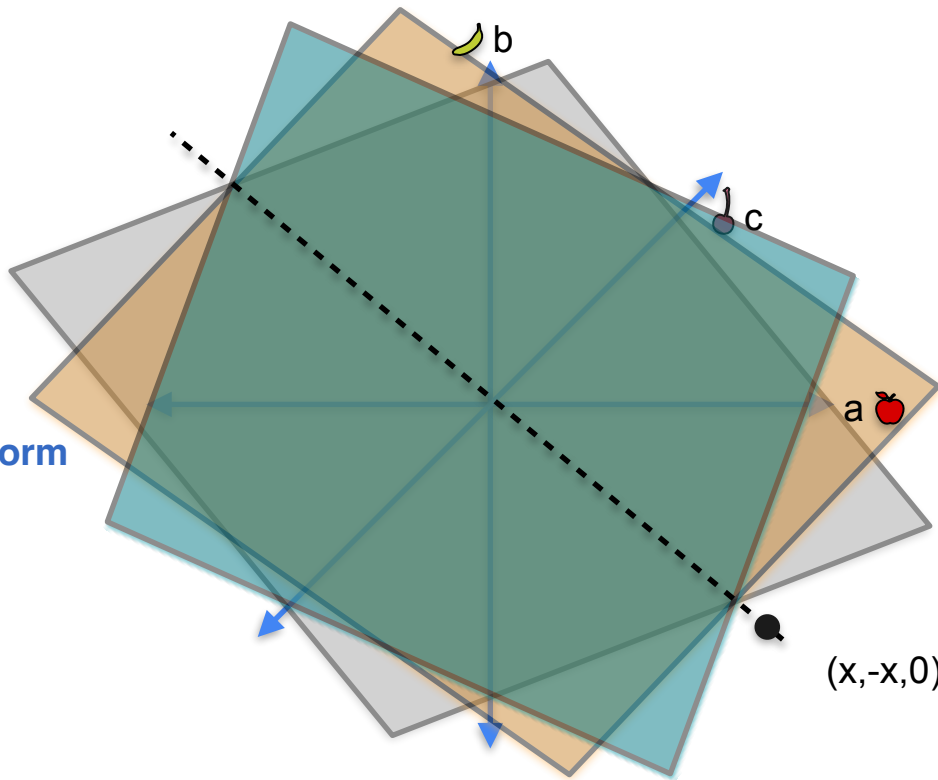
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space

- $c = 0$
- $b = -a$



All points of the form  
 $(x, -x, 0)$



# System 2

## System 2

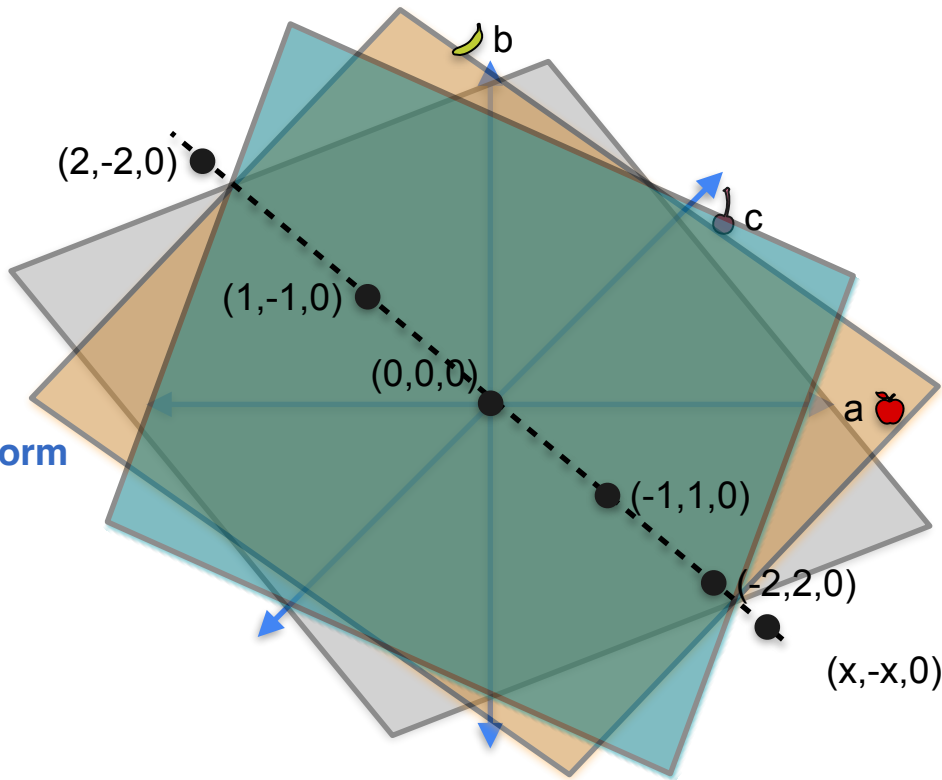
- $a + b + c = 0$
- $a + b + 2c = 0$
- $a + b + 3c = 0$

### Solution space

- $c = 0$
- $b = -a$



All points of the form  
 $(x, -x, 0)$

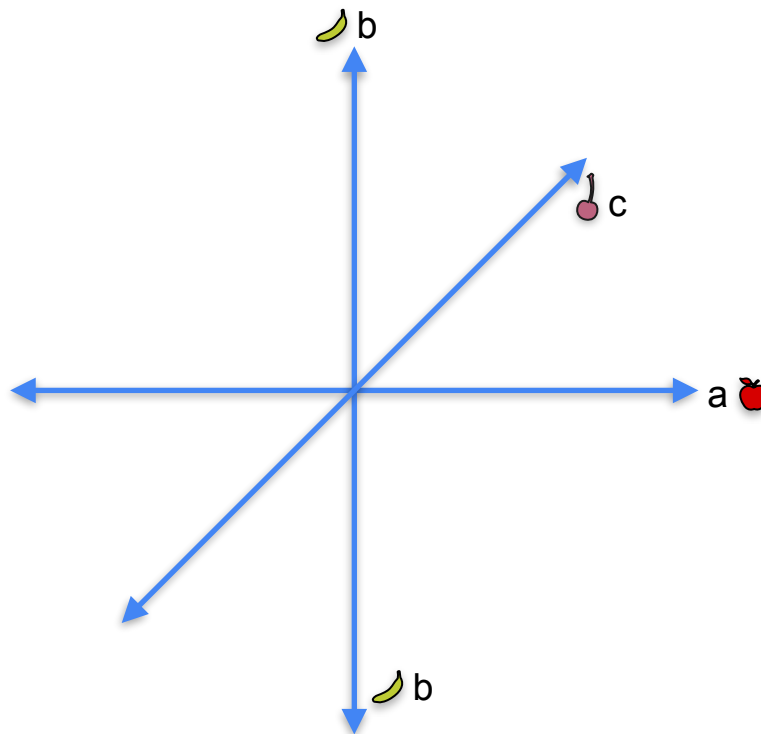




# System 3

## System 3

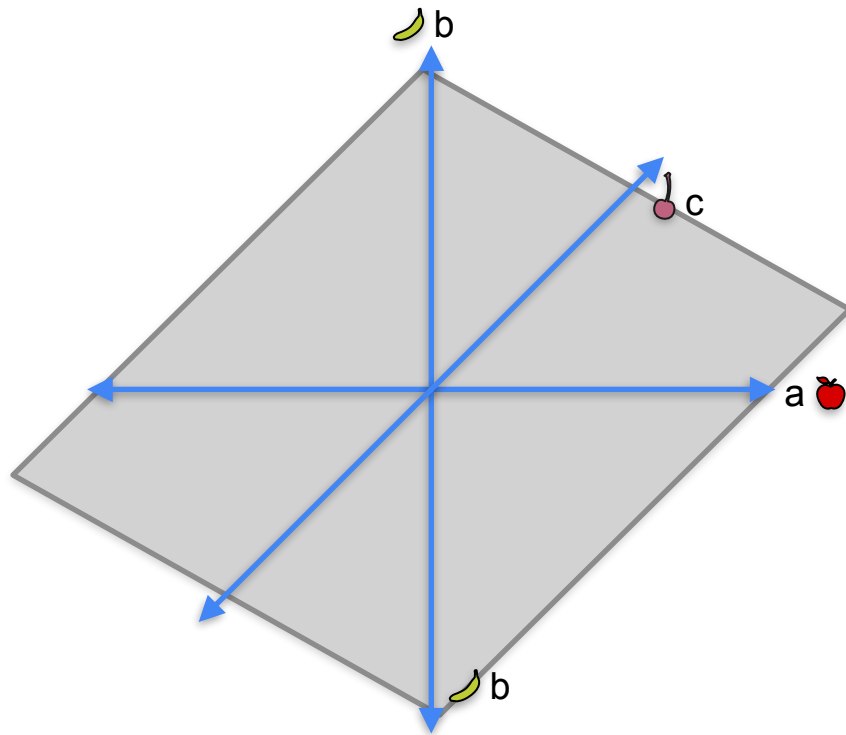
- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$



# System 3

## System 3

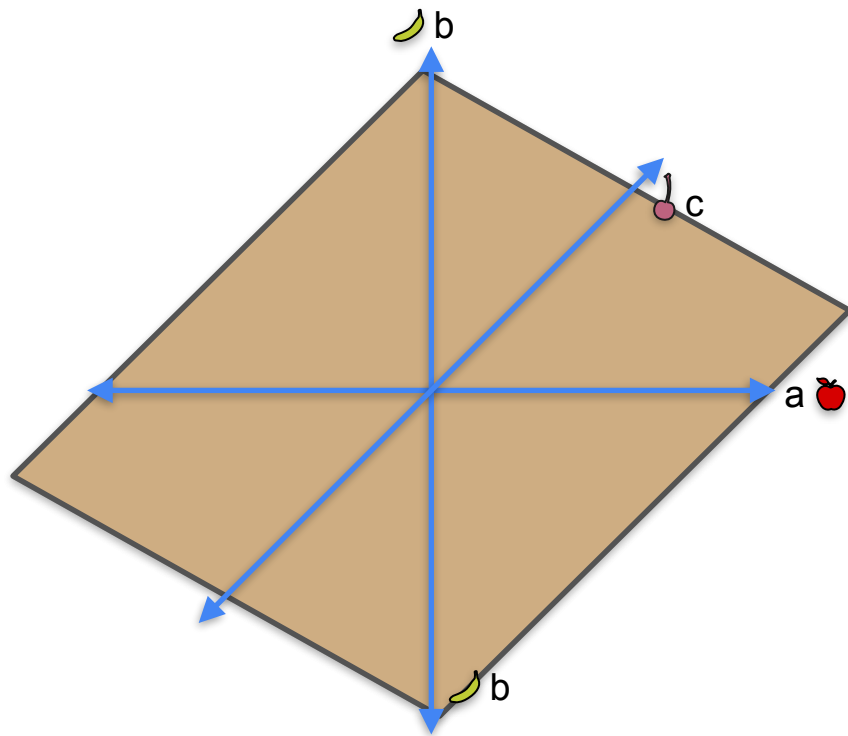
- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$



# System 3

## System 3

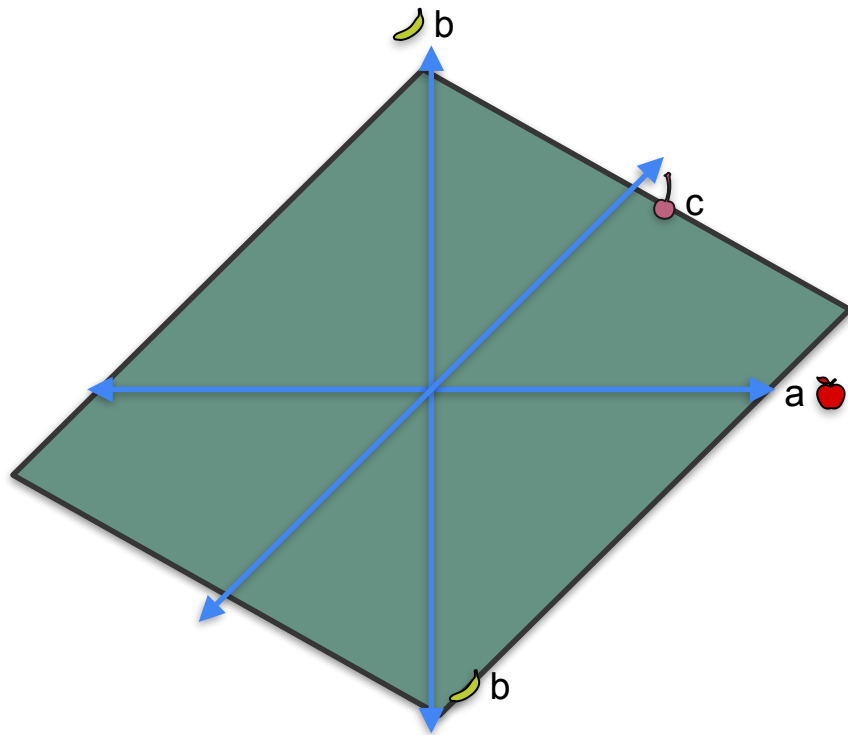
- $a + b + c = 0$
- $2a + 2b + 2c = 0$  ←
- $3a + 3b + 3c = 0$



# System 3

## System 3

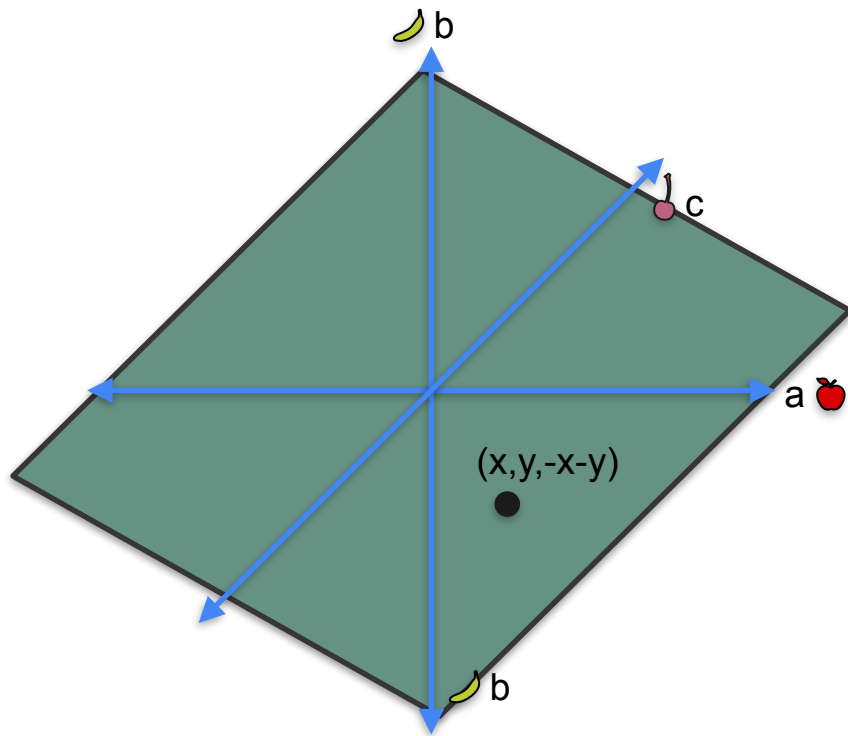
- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$



# System 3

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$  ←



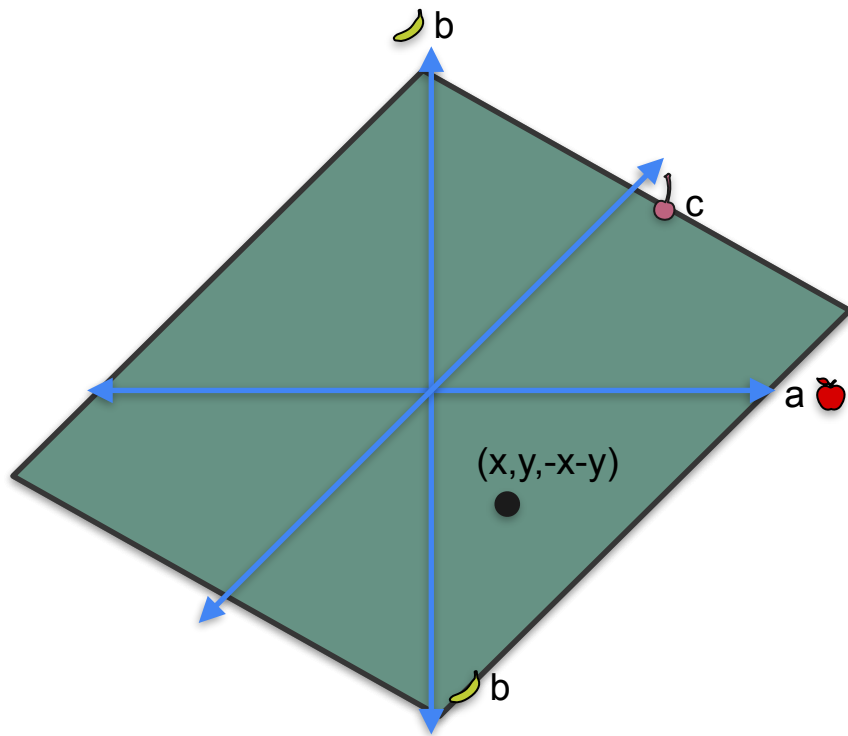
# System 3

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

### Solution space

- $a + b + c = 0$



# System 3

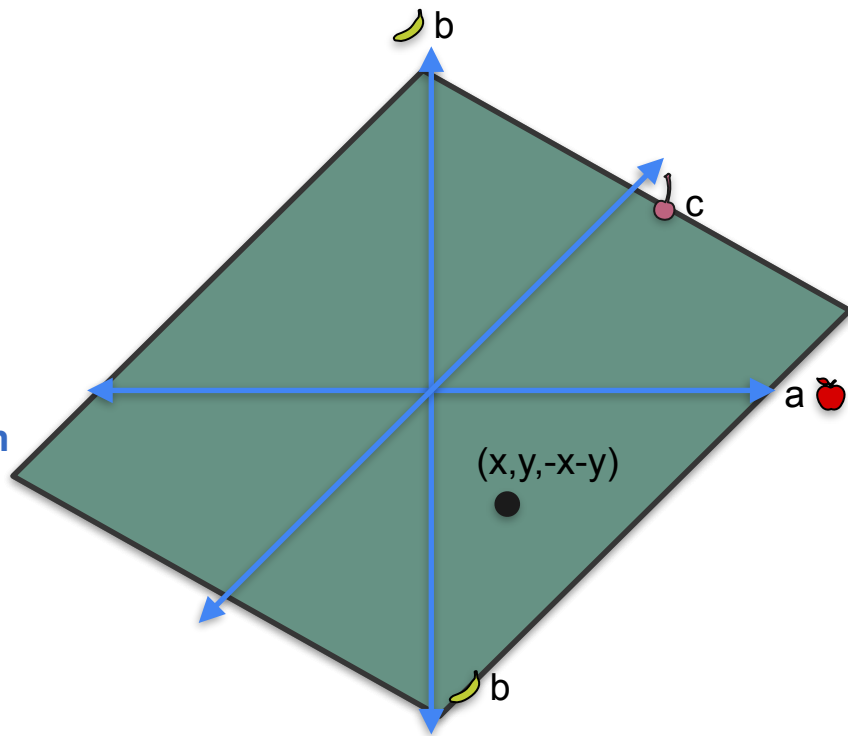
## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

### Solution space

- $a + b + c = 0$

All points of the form  
 $(x, y, -x - y)$



# System 3

## System 3

- $a + b + c = 0$
- $2a + 2b + 2c = 0$
- $3a + 3b + 3c = 0$

### Solution space

- $a + b + c = 0$

All points of the form  
 $(x, y, -x - y)$

