On the Existence of a Perfect Integer Box

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Abstract

Constraints on the diagonal dimensions of a rectangular prism with all integer-length edges, integer-length diagonals on each face, and main body or space diagonal also being an integer are discussed, identifying a way to rationally parameterize such a box, and arriving at the previously undiscovered (to the best of this author's knowledge) conclusion that no such box can actually exist.

Introduction

A perfect integer box, or "perfect cuboid" is a rectangular prism whose sides are all integer lengths, whose diagonals on it's 6 faces are each integer lengths, and whose main diagonal through the body of the prism from one corner to the opposite is also an integer.[11]

Does such a box actually exist, and if any do, what are its dimensions?

In spite of the problem's apparent simplicity, finding such a box is actually an unsolved problem[3, 5, 7, 8] in mathematics, which was likely known to Euler in his time[6, 9]. In spite of appearing to be reasonably well known, given the ease with which one can find information about it with no more than a single very obvious Internet search, the problem does not appear to attract quite as much interest in finding a solution as other certain other long-standing problems, some of which have even been solved in relatively recent history, such as Fermat's last theorem[13]. Any insight into why this was so is, as far as this author is aware, mere conjecture.

The remainder of this paper is divided into 2 main sections. The first, immediately following this introduction, aquaints the reader with some of the previous work that has been done on this problem. The second section is the main body of the proof, which constructs a couple of rational parameterizations for such a box, and then algebraically shows that no satisfactory rational solutions exists, and in turn no integer solutions will exist either. A primary goal in this proof is to be readily comprehensible so that one can with only a modicum of effort, easily verify its conclusions with the greatest effort being spent solving the non-linear systems of equations presented in this proof. No further reading is required to understand this proof beyond famiarity with the Pythagorean theorem, and solving systems of polynomial equations. This author personally recommends the applicable Wikipedia articles [14, 15] for readers unfamiliar with these topics. None of the polynomials in this proof are higher than degree 4, so exact solutions can readily be found. The various solutions this author found to the systems of equations used by this proof were discovered using Maple CAS.

Previous Work

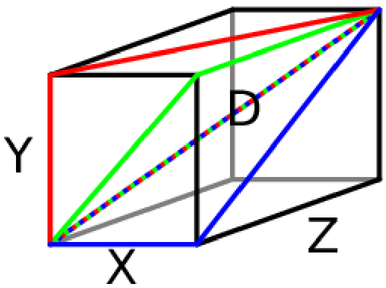
Several papers have been written on this particular subject, and it is not this authors intent to recap all of them to any great detail, but to only briefly summarize a couple of them here. The information presented in these past works is not really needed to understand this particular proof, but is presented here to give the reader some historical context to the nature of this problem, and offer some insight into previous attempts at trying to solve it.

Some of the previously published resources on the subject[7, 12] only go so far as to describe what kind of algebraic constraints would apply to the dimensions of a a perfect integer box, should one ever be discovered, without attempting to draw conclusions about whether or not one exists.

Attempts to even exhaustively search for any perfect integer boxes within very large finite ranges have been previously made[2, 4, 5] but none have produced any conclusive results[6, 9]. Such searches did reveal that should a perfect cuboid exist, at least one of its dimensions would be at least on the order of trillions, if not actually much larger.

Skepticism that such a box even exists is not unheard of[1, 2, 5], and while many have claimed to have discovered a proof of such a box's non-existence, all are either trivially refuted or else are complex enough that they have not yet been independently verified.

Description of Problem



Let us begin by considering the diagram of such a box illustrated here, with dimensions X, Y, and Z, and a main diagonal length of D. Also highlighted on the diagram, but unlabelled, are the face diagonals, which we shall call P, for the diagonal on the face bounded by X and Y, Q for the diagonal on the face bounded by X and Z, and R for the diagonal on the face bounded by Y and Z. Given the requirements for the perfect integer box, we know that the following set of equations will be satisfied in the domain of integers:

$$\begin{array}{c} X^2 + Y^2 = P^2 \\ X^2 + Z^2 = Q^2 \\ Y^2 + Z^2 = R^2 \\ X^2 + R^2 = D^2 \\ Y^2 + Q^2 = D^2 \\ P^2 + Z^2 = D^2 \end{array} \tag{1}$$

From [14], we know that any primitive all-integer pythagorean triple is of the form:

$$\left(m^2 - n^2, 2mn, m^2 + n^2\right), \tag{2}$$

we can generalize this to apply to any right angle triangle with integer-length sides as follows:

$$\left(m - n, 2\sqrt{mn}, m + n\right), \tag{3}$$

where the product of m and n is square.

Since (3) shows that there at least one of the short edges in a right angle triangle must be even, at least one of the side lengths of any face of a perfect integer box must be even. If only one edge of a box were even, then there would exist a face that had two odd side lengths, and since all pythagerean triples have at least one even number, all possible perfect integer boxes have at least two even side lengths.

We could therefore suppose that $X = 2\sqrt{X_1X_2}$, $Y = 2\sqrt{Y_1Y_2}$, and $P = 2\sqrt{Z_1Z_2}$, we can therefore rewrite$X^2+Y^2 = P^2$ as

$$4X_1X_2 + 4Y_1Y_2 = 4Z_1Z_2 \tag{4}$$

considering that $X_1 + X_2 = Y_1 + Y_2 = Z_1 + Z_2 = D$, we can rewrite this as:

$$4X_a\left(D - X_a\right) + 4Y_b\left(D - Y_b\right) = 4Z_c\left(D - Z_c\right) \tag{5}$$

For arbitrary a, b, c equaling either 1 or 2, which can be rearranged and simplified as

$$X_a^2 + Y_b^2 - Z_c^2 = D\left(X_a + Y_b - Z_c\right), \tag{6}$$

and thus:

$$D = \frac{X_a^2 + Y_b^2 - Z_c^2}{X_a + Y_b - Z_c} \tag{7}$$

It is worth noting that if we take an integer box with all even side lengths, then we can find two more perfect integer boxes where:

$$\begin{array}{c} D = \frac{X_a'^2+Y_b'^2-Z_c'^2}{X_a'+Y_b'-Z_c'} \\ D = \frac{X_a''^2+Y_b''^2-Z_c''^2}{X_a''+Y_b''-Z_c''} \end{array} \tag{8}$$

With the exact same body diagonal, and faces reassigned so that:

$$
\begin{gathered}
X_a(D-X_a) = Y'_b(D-Y'_b) \\
4Y_b(D-Y_b) = (D-Z'_c)^2 \\
(D-Z_c)^2 = 4X'_a(D-X'_a) \\
X'_a(D-X'_a) = Y''_b(D-Y''_b) \\
4Y'_b(D-Y'_b) = (D-Z''_c)^2 \\
(D-Z'_c)^2 = 4X''_a(D-X''_a) \\
X''_a(D-X''_a) = Y_b(D-Y_b) \\
4Y''_b(D-Y''_b) = (D-Z_c)^2 \\
(D-Z''_c)^2 = 4X_a(D-X_a)
\end{gathered}
\tag{9}
$$

We can combine (7), (8), and (9) into a single system of equations to get the following set of equations:

$$
\begin{aligned}
DX_a - DY'_b - X_a^2 + Y_b'^2 &= 0 \\
DX'_a - DY''_b - X_a'^2 + Y_b''^2 &= 0 \\
DX''_a - DY_b - X_a''^2 + Y_b^2 &= 0 \\
D^2 - 4DX_a - 2DZ''_c + 4X_a^2 + Z_c''^2 &= 0 \\
D^2 - 4DX'_a - 2DZ_c + 4X_a'^2 + Z_c^2 &= 0 \\
D^2 - 4DX''_a - 2DZ'_c + 4X_a''^2 + Z_c'^2 &= 0 \\
D^2 - 4DY_b - 2DZ'_c + 4Y_b^2 + Z_c'^2 &= 0 \\
D^2 - 4DY'_b - 2DZ''_c + 4Y_b'^2 + Z_c''^2 &= 0 \\
D^2 - 4DY''_b - 2DZ_c + 4Y_b''^2 + Z_c^2 &= 0 \\
DX_a + DY_b - DZ_c - X_a^2 - Y_b^2 + Z_c^2 &= 0 \\
DX'_a + DY'_b - DZ'_c - X_a'^2 - Y_b'^2 + Z_c'^2 &= 0 \\
DX''_a + DY''_b - DZ''_c - X_a''^2 - Y_b''^2 + Z_c''^2 &= 0
\end{aligned}
\tag{10}
$$

We can observe that:

$$
\begin{aligned}
X''_a &= \tfrac{D}{2} \pm \left(\tfrac{D}{2} - Y_b\right) \\
Y''_b &= \tfrac{D}{2} \pm \tfrac{\sqrt{2DZ_c - Z_c^2}}{2} \\
Z''_c &= \tfrac{D}{2} \pm \tfrac{\sqrt{2DX_a - X_a^2}}{2}
\end{aligned}
\tag{11}
$$

We can thus rewrite the remaining equations from (10) that are not made into tautologies from the above equivalences as a set of equations without X''_a, Y''_b, or Z''_b, and we then have:

$$
\begin{aligned}
(X_a - Y'_b)\left(-X_a + D - Y'_b\right) &= 0 \\
DX_a - DY'_b - X_a^2 + Y_b'^2 &= 0 \\
D^2 - 4DX'_a - 2DZ_c + 4X_a'^2 + Z_c^2 &= 0 \\
D^2 - 4DY_b - 2DZ'_c + 4Y_b^2 + Z_c'^2 &= 0 \\
DX_a + DY_b - DZ_c - X_a^2 - Y_b^2 + Z_c^2 &= 0 \\
DX'_a + DY'_b - DZ'_c - X_a'^2 - Y_b'^2 + Z_c'^2 &= 0 \\
-4D^3X_a + 20D^2X_a^2 + 8D^2X_aX'_a + 8D^2X_aY_b + D^2X_a'^2 + 2D^2X'_aY_b & \\
+D^2Y_b^2 - 32DX_a^3 - 8DX_a^2X'_a - 8DX_a^2Y_b - 8DX_aX_a'^2 - 8DX_aY_b^2 & \\
-2DX_a'^3 - 2DX_a'^2Y_b - 2DX'_aY_b^2 - 2DY_b^3 + 16X_a^4 + 8X_a^2X_a'^2 + & \\
8X_a^2Y_b^2 + X_a'^4 + 2X_a'^2Y_b^2 + Y_b^4 &= 0
\end{aligned}
\tag{12}
$$

And reducing further, we can discover that:

$$
\begin{aligned}
X'_a &= \tfrac{D}{2} \pm \tfrac{\sqrt{2DZ_c - Z_c^2}}{2} \\
Y'_b &= \tfrac{D}{2} \pm \left(\tfrac{D}{2} - X_a\right) \\
Z'_c &= \tfrac{3D^2 - 4DX_a - 16DY_b + 2DZ_c + 4X_a^2 + 16Y_b^2 - Z_c^2}{4D}
\end{aligned}
\tag{13}
$$

Rewriting the remaining equations without X'_a, Y'_b or Z'_c, we are left with the following three equations:

$$
\begin{aligned}
DX_a + DY_b - DZ_c - X_a^2 - Y_b^2 + Z_c^2 &= 0 \\
-D^4 - 8D^3X_a + 32D^3Y_b + 4D^3Z_c - 8D^2X_a^2 - 128D^2X_aY_b + & \\
16D^2X_aZ_c - 288D^2Y_b^2 + 64D^2Y_bZ_c - 6D^2Z_c^2 + 32DX_a^3 + 128DX_a^2Y_b - & \\
16DX_a^2Z_c + 128DX_aY_b^2 - 8DX_aZ_c^2 + 512DY_b^3 - 64DY_b^2Z_c - & \\
32DY_bZ_c^2 + 4DZ_c^3 - 16X_a^4 - 128X_a^2Y_b^2 + 8X_a^2Z_c^2 - 256Y_b^4 + & \\
32Y_b^2Z_c^2 - Z_c^4 &= 0 \\
-D^4 + 32D^3X_a - 8D^3Y_b + 4D^3Z_c - 288D^2X_a^2 - 128D^2X_aY_b + & \\
64D^2X_aZ_c - 8D^2Y_b^2 + 16D^2Y_bZ_c - 6D^2Z_c^2 + 512DX_a^3 + 128DX_a^2Y_b & \\
-64DX_a^2Z_c + 128DX_aY_b^2 - 32DX_aZ_c^2 + 32DY_b^3 - 16DY_b^2Z_c - & \\
8DY_bZ_c^2 + 4DZ_c^3 - 256X_a^4 - 128X_a^2Y_b^2 + 32X_a^2Z_c^2 - 16Y_b^4 + & \\
8Y_b^2Z_c^2 - Z_c^4 &= 0
\end{aligned}
\tag{14}
$$

All of the solutions for X_a, Y_b and Z_c are of the form of one of the following:

$$
\begin{gathered}
X_a = \frac{\left(\frac{9}{2} \pm \frac{\sqrt{65}}{2}\right)D}{9}, Y_b = \frac{\left(\frac{9}{2} \pm \frac{\sqrt{65}}{2}\right)D}{9}, Z_c = \frac{\left(\left(\frac{9}{2} \pm \frac{\sqrt{65}}{2}\right)^2 \pm \frac{9\sqrt{65}}{2} - \frac{79}{2}\right)D}{27} \\
X_a = \frac{\left(\frac{9}{2} \pm \frac{\sqrt{65}}{2}\right)D}{9}, Y_b = \frac{\left(\frac{9}{2} \pm \frac{\sqrt{17}}{2}\right)D}{9}, Z_c = \frac{\left(\left(\frac{9}{2} \pm \frac{\sqrt{17}}{2}\right)^2 \pm \frac{\sqrt{17}}{2} - \frac{79}{2}\right)D}{27} \\
X_a = \frac{\left(\frac{9}{2} \pm \frac{\sqrt{17}}{2}\right)D}{9}, Y_b = \frac{\left(\frac{9}{2} \pm \frac{\sqrt{65}}{2}\right)D}{9}, Z_c = \frac{5D}{9} \\
X_a = \frac{\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right)D}{3}, Y_b = \frac{\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right)D}{3}, Z_c = \frac{D}{3} \\
X_a = 0, Y_b = 0, Z_c = D \\
X_a = 0, Y_b = D, Z_c = D \\
X_a = D, Y_b = 0, Z_c = D \\
X_a = D, Y_b = D, Z_c = D
\end{gathered}
\tag{15}
$$

We can then observe that where D is assumed to be an integer, (10) can only have rational results for all unknowns in the system when one or more of X_a, Y_b, or Z_c is either 0 or else equal to D, all of which would specify a degenerate box with a 0 length side. Therefore, no perfect cuboid exists. □

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