

On the Existence of a Perfect Integer Box

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Abstract

Constraints on the diagonal dimensions of a rectangular prism with all integer-length edges, integer-length diagonals on each face, and main body or space diagonal also being an integer are discussed, and arriving at the previously undiscovered (to the best of this author's knowledge) conclusion that no such box can actually exist.

Introduction

A perfect integer box, or “perfect cuboid” is a rectangular prism whose sides are all integer lengths, whose diagonals on its 6 faces are each integer lengths, and whose main diagonal through the body of the prism from one corner to the opposite is also an integer.[11]

Does such a box actually exist, and if any do, what are its dimensions?

In spite of the problem's apparent simplicity, finding such a box is actually an unsolved problem[3, 5, 7, 8] in mathematics, which was likely known to Euler in his time. In spite of appearing to be reasonably well known, given the ease with which one can find information about it with no more than a single very obvious Internet search, the problem does not appear to attract quite as much interest in finding a solution as other certain other long-standing problems, some of which have even been solved in relatively recent history, such as Fermat's last theorem[13]. Any insight into why this was so is, as far as this author is aware, mere conjecture.

The remainder of this paper is divided into 3 sections. The first, immediately following this introduction, acquaints the reader with some of the previous work that has been done on this problem. The second section is the main body of the proof, which constructs a couple of rational parameterizations for such a box, and then algebraically shows that no satisfactory rational solutions exists, and in turn no integer solutions will exist either. A primary goal in this proof is to be readily comprehensible so that one can with only a modest effort understand it and easily verify its conclusions. No further reading is required to understand this proof beyond familiarity with the Pythagorean theorem and familiarity with solving systems of equations. This author personally recommends the applicable Wikipedia articles[14, 15] for readers unfamiliar with these topics. None of the polynomials used by this proof are higher than degree 4, so exact solutions can readily be found. The various solutions this author found to the systems of equations used by this proof were discovered using Maple CAS, and solutions verified on www.wolframalpha.com. The final section of this paper is the Appendix which proves lemmas utilized by this proof.

Previous Work

Several papers have been written on this particular subject, and it is not this authors intent to recap all of them to any great detail, but to only briefly summarize a couple of them here. The information presented in these past works is not really needed to understand this particular proof, but is presented here to give the reader some historical context to the nature of this problem, and offer some insight into previous attempts at trying to solve it.

Some of the previously published resources on the subject[7, 12] only go so far as to describe what kind of algebraic constraints would apply to the dimensions of a perfect integer box, should one ever be discovered, without attempting to draw conclusions about whether or not one exists.

Attempts to even exhaustively search for any perfect integer boxes within very large finite ranges have been previously made[2, 4, 5] but none have produced any conclusive results[6, 9]. Such searches did reveal that should a perfect cuboid exist, at least one of its dimensions would be at least on the order of trillions, if not actually much larger.

Skepticism that such a box even exists is not unheard of[1, 2, 5], and while many have claimed to have discovered a proof of such a box's non-existence, all are either trivially refuted or else are complex enough that they have not yet been independently verified.

Description of Problem



Let us begin by considering the diagram of such a box illustrated here, with dimensions X , Y , and Z , and a main diagonal length of D . Also highlighted on the diagram, but unlabelled, are the face diagonals, which we shall call P , for the diagonal on the face bounded by X and Y , Q for the diagonal on the face bounded by X and Z , and R for the diagonal on the face bounded by Y and Z . Given the requirements for the perfect integer box, we know that the following set of equations will be satisfied in the domain of integers:

$$\begin{aligned} X^2 + Y^2 &= P^2 \\ X^2 + Z^2 &= Q^2 \\ Y^2 + Z^2 &= R^2 \\ X^2 + R^2 &= D^2 \\ Y^2 + Q^2 &= D^2 \\ P^2 + Z^2 &= D^2 \end{aligned} \tag{1}$$

From [14], we know that any primitive all-integer pythagorean triple is of the form:

$$(m^2 - n^2, 2mn, m^2 + n^2), \tag{2}$$

we can generalize this to apply to any right angle triangle with integer-length sides as follows, per lemma 1 to

$$(m' - n', 2\sqrt{m'n'}, m' + n'), \tag{3}$$

where the product of m' and n' is square.

Since (3) shows that there at least one of the short edges in a right angle triangle must be even, at least one of the side lengths of any face of a perfect integer box must be even. If only one edge of a box were even, then there would exist a face that had two odd side lengths, and since all pythagorean triples have at least one even number, all possible perfect integer boxes have at least two even side lengths. Since a perfect integer box has at least two even side lengths, it follows that the smallest possible perfect integer box must have exactly one odd-length side, and in turn an odd body diagonal.

We could therefore suppose that $X = 2\sqrt{A_1A_2}$, $Y = 2\sqrt{B_1B_2}$, and $P = 2\sqrt{C_1C_2}$, we can therefore rewrite $X^2 + Y^2 = P^2$ as

$$4A_1A_2 + 4B_1B_2 = 4C_1C_2 \tag{4}$$

considering that $A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = D$, we can rewrite this as:

$$4A_a(D - A_a) + 4B_b(D - B_b) = 4C_c(D - C_c), \tag{5}$$

for arbitrary a, b, c equaling either 1 or 2, which can be rearranged and simplified as

$$A_a^2 + B_b^2 - C_c^2 = D(A_a + B_b - C_c). \tag{6}$$

Let us now try to find suitable values for a perfect integer box exactly twice as large as the smallest, whose body diagonal shall designate now as $2D$, so we could find values for A'_a , B'_b , and C'_c where

$$A_a'^2 + B_b'^2 - C_c'^2 = 2D(A'_a + B'_b - C'_c), \tag{7}$$

If we add $2D^2$ to both sides, (7) can be rearranged to the following sum of squares equation:

$$(A'_a - D)^2 + (B'_b - D)^2 = (C'_c - D)^2 + D^2. \tag{8}$$

Per lemma 3, all integer solutions to equations of the form $x^2 + y^2 = z^2 + w^2$ can be written as

$$(mM + nN)^2 + (mN - nM)^2 = (mN + nM)^2 + (mM - nN)^2, \quad (9)$$

where m, n, M , and N are integers.

So, we get the following parameterization for (8),

$$\begin{aligned} A'_a - D &= mM + nN \\ B'_b - D &= mN - nM \\ C'_c - D &= mN + nM \\ D &= mM - nN \end{aligned} \quad (10)$$

Substituting the value for D from the above shows us that

$$\begin{aligned} A'_a &= 2mM \\ B'_b &= (m - n)(M + N) \\ C'_c &= (m + n)(M + N) \\ D &= mM - nN \end{aligned} \quad (11)$$

If, for C'_c , we say that $m + n = f$, and $M + N = F$, where $fF = C'_c$, we have the following system of equations:

$$\begin{aligned} A'_a &= 2mM \\ B'_b &= (m - n)(M + N) \\ f &= m + n \\ F &= M + N \\ D &= mM - nN \end{aligned} \quad (12)$$

Solving for m, n, M, N , and D we get

$$\begin{aligned} D &= \frac{2FfA'_a + B'^2_b - F^2f^2}{2(Ff + B'_b)} \\ M &= \frac{A'_aF}{Ff + B'_b} \\ N &= \frac{F(Ff - A'_a + B'_b)}{Ff + B'_b} \\ m &= \frac{Ff + B'_b}{2F} \\ n &= \frac{Ff - B'_b}{2F} \end{aligned} \quad (13)$$

And substituting $Ff = C'_c$ we have the following solution for D :

$$D = \frac{2A'_aC'_c + B'^2_b - C'^2_c}{2(C'_c + B'_b)} \quad (14)$$

Substituting this value of D in (7), gives us

$$A'^2_a + B'^2_b - C'^2_c = \frac{2(2A'_aC'_c + B'^2_b - C'^2_c)(A'_a + B'_b - C'_c)}{2C'_c + 2B'_b} \quad (15)$$

Which gives us 3 solutions for C'_c :

$$\begin{aligned} C'_c &= B'_b \\ C'_c &= \frac{A'_a}{2} \\ C'_c &= A'_a - B'_b \end{aligned} \quad (16)$$

Recall, however, that since c can be either 1 or 2, so this means that either $C'_1 = C'_2$, which would result in a zero length side in the integer box, or else C'_1 equals one of the solutions in (16), and C'_2 equals another.

If either C'_1 or C'_2 equals to one of B'_b , for b equalling 1 or 2, however, then that means that the pair (B'_1, B'_2) and pair (C'_1, C'_2) describe identical triangles when parameterized per (3). This would mean that unless there was a 0-length edge on the box, there would be two identical sides on the box that would result in an irrational diagonal on the face with those edges. Therefore if the box is not degenerate, it must be the case that $C'_1 + C'_2 = \frac{A'_a}{2} + A'_a - B'_b$, which should equal $2D$.

Since b can be either 1 or 2, however, and since $B'_2 = 2D - B'_1$ and $B'_1 = 2D - B'_2$, that means that both of the following must be true

$$\begin{aligned}\frac{A'_a}{2} + A'_a - B'_b &= 2D \\ \frac{A'_a}{2} + A'_a - 2D + B'_b &= 2D\end{aligned}\tag{17}$$

This requires that

$$\begin{aligned}A'_a &= 2D \\ B'_b &= D\end{aligned}\tag{18}$$

Which requires that $D=0$, meaning that no perfect integer box exists. □

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Appendix

Lemma 1. Any all-integer pythagorean triple may be parameterized by $(m' - n', 2\sqrt{m'n'}, m' + n')$.

Proof. Since we know that Euclid's formula for pythagorean triples $(m^2 - n^2, 2mn, m^2 + n^2)$ [14] generates every possible primitive pythagorean triple (among others), we can multiply each element of the tuple by some integer factor k to generate every possible pythagorean triple. We can then distribute k across the triple such that:

$$\begin{aligned} m' &= km^2 \\ n' &= kn^2 \end{aligned}$$

And since $(k(m^2 - n^2))^2 + (k2mn)^2 = (k(m^2 + n^2))^2$, then $(m' - n')^2 + (2\sqrt{m'n'})^2 = (m' + n')^2$, which describes the same triangle. Since both m' and n' are both equal to k multiplied by a square number, the square root of their product must also be an integer.

Lemma 2. You can identify the parities, or evenness or oddness, of two otherwise unknown square integers simply by examining their sum.

Proof. Note that any odd integer represented as $2p + 1$ when squared is $4p^2 + 4p + 1$, and the square of any even integer represented by $2q$ when squared is $4q^2$. This means that the sum of any two square numbers must be one of the following:

- an odd number, which can only happen when adding an odd and an even square,
- equal to exactly 2 more than a multiple of 4, which can only happen when adding two odd squares, or
- equal to a multiple of 4, which can only happen when adding two even squares.

Lemma 3. The equation $x^2 + y^2 = z^2 + w^2$ can be parameterized and rewritten as $(mM + nN)^2 + (mN - nM)^2 = (mN + nM)^2 + (mM - nN)^2$.

Proof. Consider the equation

$$x^2 + y^2 = z^2 + w^2.$$

Per lemma 2, the parity of values on the right each have matching parities on the left. We shall say that x and w share parity, and that y and z also share parity.

Since x and w share parity, we can say that $x = mM + nN$ and $w = mM - nN$ for some integers m, n, M , and N , and since y and z share parity, we can say that $y = mN - c$ and $z = mN + c$ for some integer, c .

Substituting this into the equation yields

$$m^2M^2 + 2mMnN + n^2N^2 + m^2N^2 - 2mNc + c^2 = m^2N^2 + 2mNc + c^2 + m^2M^2 - 2mMnN + n^2N^2.$$

This requires $c = nM$, so

$$m^2M^2 + 2mMnN + n^2N^2 + m^2N^2 - 2mNnM + n^2M^2 = m^2N^2 + 2mNnM + n^2M^2 + m^2M^2 - 2mMnN + n^2N^2,$$

which simplifies to

$$(mM + nN)^2 + (mN - nM)^2 = (mN + nM)^2 + (mM - nN)^2.$$