On the Existence of a Perfect Integer Box

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Mar, 2024

Abstract

Constraints on the diagonal dimensions of a rectangular prism with all integer-length edges, integer-length diagonals on each face, and main body or space diagonal also being an integer are discussed, and arriving at the previously undiscovered (to the best of this author's knowledge) conclusion that no such box can actually exist.

Introduction

A perfect integer box, or "perfect cuboid" is a rectangular prism whose sides are all integer lengths, whose diagonals on it's 6 faces are each integer lengths, and whose main diagonal through the body of the prism from one corner to the opposite is also an integer.[11]

Does such a box actually exist, and if any do, what are its dimensions?

In spite of the problem's apparent simplicity, finding such a box is actually an unsolved problem[3, 5, 7, 8] in mathematics, which was likely known to Euler in his time. In spite of appearing to be reasonably well known, given the ease with which one can find information about it with no more than a single very obvious Internet search, the problem does not appear to attract quite as much interest in finding a solution as other certain other long-standing problems, some of which have even been solved in relatively recent history, such as Fermat's last theorem[13]. Any insight into why this was so is, as far as this author is aware, mere conjecture.

The remainder of this paper is divided into 3 sections. The first, immediately following this introduction, aquaints the reader with some of the previous work that has been done on this problem. The second section is the main body of the proof, which constructs a couple of algebraic parameterizations for such a box, and then algebraically shows that no satisfactory integer solutions exists. A primary goal in this proof is to be readily comprehensible so that one can with only a modest effort understand it and easily verify its conclusions. No further reading is required to understand this proof beyond famiarity with the Pythagorean theorem and familiarity with solving systems of equations. This author personally recommends the applicable Wikipedia articles[14, 15] for readers unfamiliar with these topics. Ahtough polynomials as high as degree 8 exist in this proof, higher degrees readily cancel eachother out when evaluated in context of the equation being considered, and we are left with no higher degrees than 2, so exact solutions can readily be found. The various solutions this author found to the systems of equations used by this proof were discovered using Maple CAS, and solutions verified on www.wolframalpha.com. The final section of this paper is the Appendix which proves lemmas utilized by this proof.

Previous Work

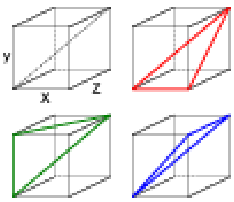
Several papers have been written on this particular subject, and it is not this authors intent to recap all of them to any great detail, but to only briefly summarize a couple of them here. The information presented in these past works is not really needed to understand this particular proof, but is presented here to give the reader some historical context to the nature of this problem, and offer some insight into previous attempts at trying to solve it.

Some of the previously published resources on the subject[7, 12] only go so far as to describe what kind of algebraic constraints would apply to the dimensions of a perfect integer box, should one ever be discovered, without attempting to draw conclusions about whether or not one exists.

Attempts to even exhaustively search for any perfect integer boxes within very large finite ranges have been previously made[2, 4, 5] but none have produced any conclusive results[6, 9]. Such searches did reveal that should a perfect cuboid exist, at least one of its dimensions would be at least on the order of trillions, if not actually much larger.

Skepticism that such a box even exists is not unheard of[1, 2, 5], and while many have claimed to have discovered a proof of such a box's non-existence, all are either trivially refuted or else are complex enough that they have not yet been independently verified.

Description of Problem



Let us begin by considering the diagram of such a box illustrated here, with dimensions X, Y, and Z, and a main diagonal length of D. Also illustrated are embedded right angle triangles within the box that we be utilizing for this proof. We shall label the face diagonals P, for the diagonal on the face bounded by X and Y, Q for the diagonal on the face bounded by X and Z, and R for the diagonal on the face bounded by Y and Z. Given the requirements for the perfect integer box, we know that the following set of equations will be satisfied in the domain of integers:

$$\begin{aligned} X^2 + Y^2 &= P^2 \\ X^2 + Z^2 &= Q^2 \\ Y^2 + Z^2 &= R^2 \\ X^2 + R^2 &= D^2 \\ Y^2 + Q^2 &= D^2 \\ P^2 + Z^2 &= D^2 \end{aligned} \tag{1}$$

From [14], we know that any primitive all-integer pythagorean triple is of the form:

$$\left(m^2 - n^2, 2mn, m^2 + n^2\right), \tag{2}$$

we can generalize this to apply to any right angle triangle with integer-length sides as follows, per lemma 1 to

$$\left(m' - n', 2\sqrt{m'n'}, m' + n'\right), \tag{3}$$

where the product of m' and n' is square.

Since (3) shows that there at least one of the short edges in a right angle triangle must be even, at least one of the side lengths of any face of a perfect integer box must be even. If only one edge of a box were even, then there would exist a face that had two odd side lengths, and since all pythagerean triples have at least one even number, all possible perfect integer boxes have at least two even side lengths. Since a perfect integer box has at least two even side lengths. it follows that the smallest possible perfect integer box must have exactly one odd-length side, and in turn an odd body diagonal.

We could therefore suppose that $X = 2\sqrt{A_1A_2}$, $Y = 2\sqrt{B_1B_2}$, and $P = 2\sqrt{C_1C_2}$, we can therefore rewrite$X^2+Y^2 = P^2$ as

$$4A_1A_2 + 4B_1B_2 = 4C_1C_2 \tag{4}$$

considering that $A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = D$, we can rewrite this as:

$$4A_a\left(D - A_a\right) + 4B_b\left(D - B_b\right) = 4C_c\left(D - C_c\right), \tag{5}$$

for arbitrary a, b, c equaling either 1 or 2, which can be rearranged and simplified as

$$A_a^2 + B_b^2 - C_c^2 = D\left(A_a + B_b - C_c\right). \tag{6}$$

Henceforth, we will A_a, B_b, and C_c as A, B, and C, respectively, knowing that either subscript may be applied, and simplifying further equations: We can thus rearrange (6) as an expression for the main diagonal.

$$D = \frac{A^2 + B^2 - C^2}{A + B - C}. \tag{7}$$

And that this will define a perfect integer box as long as $A\left(D - A\right)$, $B\left(D - B\right)$, and $C\left(D - C\right)$ are all square.

Consider that per lemma (2), we can know we can find rational values a and b such that:

$$\begin{aligned} \frac{X}{D} &= \frac{2a}{1+a^2+b^2} \\ \frac{Y}{D} &= \frac{2b}{1+a^2+b^2} \\ \frac{Z}{D} &= \frac{1-a^2-b^2}{1+a^2+b^2} \end{aligned} \tag{8}$$

This means that the diagonal P on the front face must be equal to $D\frac{2\sqrt{a^2+b^2}}{1+a^2+b^2}$
So that means that

$$\begin{aligned} 4A(D-A) &= \left(D\frac{2a}{1+a^2+b^2}\right)^2 \\ 4B(D-B) &= \left(D\frac{2b}{1+a^2+b^2}\right)^2 \\ 4C(D-C) &= \left(D\frac{2\sqrt{a^2+b^2}}{1+a^2+b^2}\right)^2 \end{aligned} \tag{9}$$

Solving for A, B, and C frrom (9) give us:

$$\begin{aligned} A &= \frac{D\left(a^4+2a^2b^2+b^4+a^2+2b^2+1\right)}{a^4+2a^2b^2+b^4+2a^2+2b^2+1} \\ B &= \frac{D\left(a^4+2a^2b^2+b^4+2a^2+b^2+1\right)}{a^4+2a^2b^2+b^4+2a^2+2b^2+1} \\ C &= \frac{D\left(a^4+2a^2b^2+b^4+a^2+b^2+1\right)}{a^4+2a^2b^2+b^4+2a^2+2b^2+1} \end{aligned} \tag{10}$$

When we substitute this into (7), we discover that:

$$D = \frac{\left(a^8+4\left(b^2+1\right)a^6+6\left(b^2+1\right)^2a^4+2\left(2b^6+6b^4+5b^2+2\right)a^2+\left(b^2+1\right)^4\right)D}{(a^2+b^2+1)^4} \tag{11}$$

Which means (unless D were 0, a degenerate case that we are not considering), we could divide both side by D to get:

$$1 = \frac{a^8+4\left(b^2+1\right)a^6+6\left(b^2+1\right)^2a^4+2\left(2b^6+6b^4+5b^2+2\right)a^2+\left(b^2+1\right)^4}{(a^2+b^2+1)^4} \tag{12}$$

Which means that the denonominator of the right hand side equals the numerator. So starting with:

$$\left(a^2+b^2+1\right)^4 = a^8+4\left(b^2+1\right)a^6+6\left(b^2+1\right)^2a^4+2\left(2b^6+6b^4+5b^2+2\right)a^2+\left(b^2+1\right)^4 \tag{13}$$

Fully expanding both sides and then subtracting the right hand side from both sides gives us:

$$2a^2b^2 = 0 \tag{14}$$

Which means that either a or b are 0, which per (8) would result in a 0-length edge, and therefore the box must be degenerate. Therefore, no perfect integer box exists. □

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Appendix

Lemma 1. *Any all-integer pythagorean triple may be parameterized by* $\left(m'-n', 2\sqrt{m'n'}, m'+n'\right)$.

Proof. Since we know that Euclid's formula for pythagrean triples $\left(m^2-n^2, 2mn, m^2+n^2\right)$[14] generates every possible primitive pythagorean triple (among others), we can multiply each element of the tuple by some integer factor k to generate every possible pythagorean triple. We can then distribute k across the triple such that:

$$m' = km^2$$
$$n' = kn^2$$

And since $\left(k\left(m^2-n^2\right)\right)^2 + (k2mn)^2 = \left(k\left(m^2+n^2\right)\right)^2$, then $(m'-n')^2 + \left(2\sqrt{m'n'}\right)^2 = (m'+n')^2$, which describes the same triangle. Since both m' and n' are both equal to k multiplied by a square number, the square root of their product must also be an integer.

Lemma 2. *The rational points on a unit sphere can be represented as* $\frac{2a}{1+a^2+b^2}$, $\frac{2b}{1+a^2+b^2}$, *and* $\frac{1-a^2-b^2}{1+a^2+b^2}$, *for any rational* a *and* b.

Proof. $\left(\frac{2a}{1+a^2+b^2}\right)^2 + \left(\frac{2b}{1+a^2+b^2}\right)^2 + \left(\frac{1-a^2-b^2}{1+a^2+b^2}\right)^2 = \left(\frac{1+a^2+b^2}{1+a^2+b^2}\right)^2 = 1$