On the Existence of a Perfect Integer Box

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Abstract

Constraints on the diagonal dimensions of a rectangular prism with all integer-length edges, integer-length diagonals on each face, and main body or space diagonal also being an integer are discussed, and arriving at the previously undiscovered (to the best of this author's knowledge) conclusion that no such box can actually exist.

Introduction

A perfect integer box, or "perfect cuboid" is a rectangular prism whose sides are all integer lengths, whose diagonals on it's 6 faces are each integer lengths, and whose main diagonal through the body of the prism from one corner to the opposite is also an integer.[11]

Does such a box actually exist, and if any do, what are its dimensions?

In spite of the problem's apparent simplicity, finding such a box is actually an unsolved problem[3, 5, 7, 8] in mathematics, which was likely known to Euler in his time. In spite of appearing to be reasonably well known, given the ease with which one can find information about it with no more than a single very obvious Internet search, the problem does not appear to attract quite as much interest in finding a solution as other certain other long-standing problems, some of which have even been solved in relatively recent history, such as Fermat's last theorem[13]. Any insight into why this was so is, as far as this author is aware, mere conjecture.

The remainder of this paper is divided into 3 sections. The first, immediately following this introduction, aquaints the reader with some of the previous work that has been done on this problem. The second section is the main body of the proof, which constructs a couple of rational parameterizations for such a box, and then algebraically shows that no satisfactory rational solutions exists, and in turn no integer solutions will exist either. A primary goal in this proof is to be readily comprehensible so that one can with only a modest effort and easily verify its conclusions with the greatest effort being spent solving the non-linear systems of equations presented in this proof. No further reading is required to understand this proof beyond famiarity with the Pythagorean theorem. This author personally recommends the applicable Wikipedia article[14] for readers unfamiliar with this topic The final section of this paper is the Appendix which proves lemmas utilized by this proof.

Previous Work

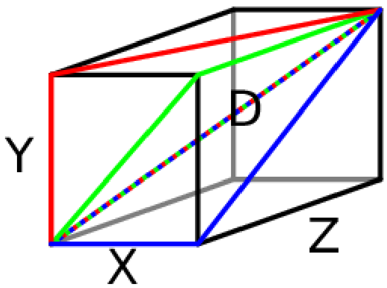
Several papers have been written on this particular subject, and it is not this authors intent to recap all of them to any great detail, but to only briefly summarize a couple of them here. The information presented in these past works is not really needed to understand this particular proof, but is presented here to give the reader some historical context to the nature of this problem, and offer some insight into previous attempts at trying to solve it.

Some of the previously published resources on the subject[7, 12] only go so far as to describe what kind of algebraic constraints would apply to the dimensions of a perfect integer box, should one ever be discovered, without attempting to draw conclusions about whether or not one exists.

Attempts to even exhaustively search for any perfect integer boxes within very large finite ranges have been previously made[2, 4, 5] but none have produced any conclusive results[6, 9]. Such searches did reveal that should a perfect cuboid exist, at least one of its dimensions would be at least on the order of trillions, if not actually much larger.

Skepticism that such a box even exists is not unheard of[1, 2, 5], and while many have claimed to have discovered a proof of such a box's non-existence, all are either trivially refuted or else are complex enough that they have not yet been independently verified.

Description of Problem



Let us begin by considering the diagram of such a box illustrated here, with dimensions X, Y, and Z, and a main diagonal length of D. Also highlighted on the diagram, but unlabelled, are the face diagonals, which we shall call P, for the diagonal on the face bounded by X and Y, Q for the diagonal on the face bounded by X and Z, and R for the diagonal on the face bounded by Y and Z. Given the requirements for the perfect integer box, we know that the following set of equations will be satisfied in the domain of integers:

$$\begin{aligned} X^2 + Y^2 &= P^2 \\ X^2 + Z^2 &= Q^2 \\ Y^2 + Z^2 &= R^2 \\ X^2 + R^2 &= D^2 \\ Y^2 + Q^2 &= D^2 \\ P^2 + Z^2 &= D^2 \end{aligned} \tag{1}$$

From [14], we know that any primitive all-integer pythagorean triple is of the form:

$$\left(m^2 - n^2, 2mn, m^2 + n^2\right), \tag{2}$$

we can generalize this to apply to any right angle triangle with integer-length sides as follows, per lemma 1 to

$$\left(m' - n', 2\sqrt{m'n'}, m' + n'\right), \tag{3}$$

where the product of m' and n' is square.

Since (3) shows that there at least one of the short edges in a right angle triangle must be even, at least one of the side lengths of any face of a perfect integer box must be even. If only one edge of a box were even, then there would exist a face that had two odd side lengths, and since all pythagerean triples have at least one even number, all possible perfect integer boxes have at least two even side lengths. Since a perfect integer box has at least two even side lengths. it follows that the smallest possible perfect integer box must have exactly one odd-length side, and in turn an odd body diagonal.

We could therefore suppose that $X = 2\sqrt{A_1A_2}$, $Y = 2\sqrt{B_1B_2}$, and $P = 2\sqrt{C_1C_2}$, we can therefore rewrite$X^2+Y^2 = P^2$ as

$$4A_1A_2 + 4B_1B_2 = 4C_1C_2 \tag{4}$$

considering that $A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = D$, we can rewrite this as:

$$4A_a\left(D - A_a\right) + 4B_b\left(D - B_b\right) = 4C_c\left(D - C_c\right), \tag{5}$$

for arbitrary a, b, c equaling either 1 or 2, which can be rearranged and simplified as

$$A_a^2 + B_b^2 - C_c^2 = D\left(A_a + B_b - C_c\right). \tag{6}$$

Let us now try to find suitable values for a perfect integer box exactly twice as large as the smallest, whose body diagonal shall designate now as $2D$. Since per lemma 2, we know that either shorter side may be parameterized as twice the square root of the product of two integers, so we could find values for A'_a, B'_b, and C'_c where

$${A'_a}^2 + {B'_b}^2 - {C'_c}^2 = 2D\left(A'_a + B'_b - C'_c\right), \tag{7}$$

and per lemma 3, we know that if this perfect integer box exists, we can find solutions where each of $A'_{a'}$, $B'_{b'}$, and $C'_{c'}$ are all odd. If we add $2D^2$ to both sides, (7) can be rearanged to the following sum of squares equation:

$$\left(A_a' - D\right)^2 + \left(B_b' - D\right)^2 = \left(C_c' - D\right)^2 + D^2. \tag{8}$$

Per lemma 5, all integer solutions to equations of the form $x^2 + y^2 = z^2 + w^2$can be written as

$$\left(mM + nN\right)^2 + \left(mN - nM\right)^2 = \left(mN + nM\right)^2 + \left(mM - nN\right)^2, \tag{9}$$

where m, n, M, and N are integers.

So, we get the following parameterization for (8),

$$\begin{array}{rcl} A_a' - D & = & mM + nN \\ B_b' - D & = & mN - nM \\ C_c' - D & = & mN + nM \\ D & = & mM - nN \end{array}. \tag{10}$$

Substituting the value for D from the above shows us that

$$\begin{array}{rcl} A_a' & = & 2mM \\ B_b' & = & (m-n)(M+N) \\ C_c' & = & (m+n)(M+N) \end{array}. \tag{11}$$

We can then observe, however, that this requires $A_{a'}'$ to be even, but lemma 3 establishes that it must be odd when the smaller triangle has an odd hypotenuse. Since no odd value can be found for $A_{a'}'$ to parameterize a perfect integer box twice as large as the smallest, no perfect integer box exists at all. □

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Appendix

Lemma 1. *Any all-integer pythagorean triple may be parameterized by* $\left(m'-n', 2\sqrt{m'n'}, m'+n'\right)$.

Proof. Since we know that Euclid's formula for pythagrean triples $\left(m^2-n^2, 2mn, m^2+n^2\right)$[14] generates every possible primitive pythagorean triple (among others), we can multiply each element of the tuple by some integer factor k to generate every possible pythagorean triple. We can then distribute k across the triple such that:

$$\begin{array}{c} m' = km^2 \\ n' = kn^2 \end{array}$$

And since $\left(k\left(m^2-n^2\right)\right)^2 + (k2mn)^2 = \left(k\left(m^2+n^2\right)\right)^2$, then $(m'-n')^2 + \left(2\sqrt{m'n'}\right)^2 = (m'+n')^2$, which describes the same triangle. Since both m' and n' are both equal to k multiplied by a square number, the square root of their product must also be an integer.

Lemma 2. *For any right triangle with all even length sides, we can always express either of the shorter edges as twice the square root of the product of two integers, and so be representable as* $2\sqrt{m'n'}$ *where the product of* m' *and* n' *are square, and the other shorter side can always be expressed as the difference of the same integers.*

Proof. If we have a right triangle with all integer length even sides whose shorter two edges can be described as $2\sqrt{mn}$and $m-n$, and we solve for a triangle whose edges are twice as long, we either have:

$$\begin{array}{c} 4\sqrt{mn} = 2\sqrt{m'n'} \\ 2(m-n) = m'-n' \end{array},$$

and solving for m' and n' we get:

$$\begin{array}{c} m' = 2m \\ n' = 2n \end{array}.$$

Or if we consider the proposed alternative parameterization,

$$\begin{array}{c} 4\sqrt{mn} = m'-n' \\ 2(m-n) = 2\sqrt{m'n'} \end{array},$$

then we get:

$$\begin{array}{c} m' = m+n+2\sqrt{mn} \\ n' = m+n-2\sqrt{mn} \end{array}.$$

Since the product of mn here is known to be square per lemma 1, so m' and n' are integers. Therefore, for all-even triples, it does not matter which shorter side we want to parameterize as twice the square root of the product of two integers, and which we want to parameterize as a difference of two integers.

Lemma 3. *For a right angle triangle with integer length sides and an odd length hyptonuse, parameterized by* $(m-n, 2\sqrt{mn}, m+n)$ *,we can parameterize a similar triangle with twice the length of hyptonuse as* $\left(2\sqrt{m'n'}, m'-n', m'+n'\right)$ *where* n' *and* m' *are both odd.*

Proof. Per lemma 2, if we have a right triangle with all integer length even sides whose shorter two edges can be described as $2\sqrt{mn}$and $m-n$, and we solve for a triangle whose edges are twice as long, we can say that

$$\begin{array}{c} m' = m+n+2\sqrt{mn} \\ n' = m+n-2\sqrt{mn} \end{array}.$$

But if the hypotenuse of the smaller triangle is odd, and is equal to $m+n$, since the product of mn is square, that means that with this parameterization, both m' and n' are odd.

Lemma 4. *You can identify the parities, or evenness or oddness, of two otherwise unknown square integers simply by examining their sum.*

Proof. Note that any odd integer represented as $2p + 1$ when squared is $4p^2 + 4p + 1$, and the square of any even integer represented by $2q$ when squared is $4q^2$. This means that the sum of any two square numbers must be one of the following:

- an odd number, which can only happen when adding an odd and an even square,
- equal to exactly 2 more than a multiple of 4, which can only happen when adding two odd squares, or
- equal to a multiple of 4, which can only happen when adding two even squares.

Lemma 5. *The equation $x^2 + y^2 = z^2 + w^2$ can be parameterized and rewritten as $(mM + nN)^2 + (mN - nM)^2 = (mN + nM)^2 + (mM - nN)^2$.*

Proof. Consider the equation

$$x^2 + y^2 = z^2 + w^2.$$

Per lemma 4, the parity of values on the right each have matching parities on the left. We shall say that x and w share parity, and that y and z also share parity.

Since x and w share parity, we can say that $x = mM + nN$ and $w = mM - nN$ for some integers m, n, M, and N, and since y and z share parity, we can say that $y = mN - c$ and $z = mN + c$ for some integer, c.

Substituting this into the equation yields

$$m^2M^2 + 2mMnN + n^2N^2 + m^2N^2 - 2mNc + c^2 = m^2N^2 + 2mNc + c^2 + m^2M^2 - 2mMnN + n^2N^2.$$

This requires $c = nM$, so

$$m^2M^2 + 2mMnN + n^2N^2 + m^2N^2 - 2mNnM + n^2M^2 = m^2N^2 + 2mNnM + n^2M^2 + m^2M^2 - 2mMnN + n^2N^2,$$

which simplifies to

$$(mM + nN)^2 + (mN - nM)^2 = (mN + nM)^2 + (mM - nN)^2.$$