On the Existence of a Perfect Integer Box

Mark Tarrabain (markt@nerdflat.com)

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Abstract

Constraints on the diagonal dimensions of a rectangular prism with all integer-length edges, integer-length diagonals on each face, and main body or space diagonal also being an integer are discussed, identifying a way to rationally parameterize such a box, and arriving at the previously undiscovered (to the best of this author's knowledge) conclusion that no such box can actually exist.

Introduction

A perfect integer box, or "perfect cuboid" is a rectangular prism whose sides are all integer lengths, whose diagonals on it's 6 faces are each integer lengths, and whose main diagonal through the body of the prism from one corner to the opposite is also an integer.[11]

Does such a box actually exist, and if any do, what are its dimensions?

In spite of the problem's apparent simplicity, finding such a box is actually an unsolved problem[3, 5, 7, 8] in mathematics, which was likely known to Euler in his time[6, 9]. In spite of appearing to be reasonably well known, given the ease with which one can find information about it with no more than a single very obvious Internet search, the problem does not appear to attract quite as much interest in finding a solution as other certain other long-standing problems, some of which have even been solved in relatively recent history, such as Fermat's last theorem[13]. Any insight into why this was so is, as far as this author is aware, mere conjecture.

The remainder of this paper is divided into 3 sections. The first, immediately following this introduction, aquaints the reader with some of the previous work that has been done on this problem. The second section is the main body of the proof, which constructs a couple of rational parameterizations for such a box, and then algebraically shows that no satisfactory rational solutions exists, and in turn no integer solutions will exist either. A primary goal in this proof is to be readily comprehensible so that one can with only a modest effort and easily verify its conclusions with the greatest effort being spent solving the non-linear systems of equations presented in this proof. No further reading is required to understand this proof beyond famiarity with the Pythagorean theorem, and solving systems of polynomial equations. This author personally recommends the applicable Wikipedia articles [14, 15] for readers unfamiliar with these topics. None of the polynomials used by this proof are higher than degree 4, so exact solutions can readily be found. The various solutions this author found to the systems of equations used by this proof were discovered using Maple CAS. The final section is the Appendix which proves lemmas utilized by this proof.

Previous Work

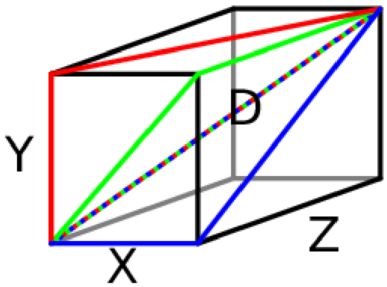
Several papers have been written on this particular subject, and it is not this authors intent to recap all of them to any great detail, but to only briefly summarize a couple of them here. The information presented in these past works is not really needed to understand this particular proof, but is presented here to give the reader some historical context to the nature of this problem, and offer some insight into previous attempts at trying to solve it.

Some of the previously published resources on the subject[7, 12] only go so far as to describe what kind of algebraic constraints would apply to the dimensions of a a perfect integer box, should one ever be discovered, without attempting to draw conclusions about whether or not one exists.

Attempts to even exhaustively search for any perfect integer boxes within very large finite ranges have been previously made[2, 4, 5] but none have produced any conclusive results[6, 9]. Such searches did reveal that should a perfect cuboid exist, at least one of its dimensions would be at least on the order of trillions, if not actually much larger.

Skepticism that such a box even exists is not unheard of[1, 2, 5], and while many have claimed to have discovered a proof of such a box's non-existence, all are either trivially refuted or else are complex enough that they have not yet been independently verified.

Description of Problem



Let us begin by considering the diagram of such a box illustrated here, with dimensions X, Y, and Z, and a main diagonal length of D. Also highlighted on the diagram, but unlabelled, are the face diagonals, which we shall call P, for the diagonal on the face bounded by X and Y, Q for the diagonal on the face bounded by X and Z, and R for the diagonal on the face bounded by Y and Z. Given the requirements for the perfect integer box, we know that the following set of equations will be satisfied in the domain of integers:

$$\begin{aligned} X^2 + Y^2 &= P^2 \\ X^2 + Z^2 &= Q^2 \\ Y^2 + Z^2 &= R^2 \\ X^2 + R^2 &= D^2 \\ Y^2 + Q^2 &= D^2 \\ P^2 + Z^2 &= D^2 \end{aligned} \tag{1}$$

From [14], we know that any primitive all-integer pythagorean triple is of the form:

$$\left(m^2 - n^2, 2mn, m^2 + n^2\right), \tag{2}$$

we can generalize this to apply to any right angle triangle with integer-length sides as follows, per lemma 1 to

$$\left(m' - n', 2\sqrt{m'n'}, m' + n'\right), \tag{3}$$

where the product of m' and n' is square.

Since (3) shows that there at least one of the short edges in a right angle triangle must be even, at least one of the side lengths of any face of a perfect integer box must be even. If only one edge of a box were even, then there would exist a face that had two odd side lengths, and since all pythagerean triples have at least one even number, all possible perfect integer boxes have at least two even side lengths.

We could therefore suppose that $X = 2\sqrt{A_1A_2}$, $Y = 2\sqrt{B_1B_2}$, and $P = 2\sqrt{C_1C_2}$, we can therefore rewrite$X^2 + Y^2 = P^2$ as

$$4A_1A_2 + 4B_1B_2 = 4C_1C_2 \tag{4}$$

considering that $A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = D$, we can rewrite this as:

$$4A_a\left(D - A_a\right) + 4B_b\left(D - B_b\right) = 4C_c\left(D - C_c\right), \tag{5}$$

for arbitrary a, b, c equaling either 1 or 2, which can be rearranged and simplified as

$$A_a^2 + B_b^2 - C_c^2 = D\left(A_a + B_b - C_c\right). \tag{6}$$

We know that C_c cannot be equal to $A_a + B_b$ because c can equal either 1 or 2, and that would mean that that would require that C_1 and C_2 to be equal. Since one of the sides, Z in the above box, would have a length of $C_1 - C_2$, per (3), that would make the Z dimension of the box 0, and degenerate.

Thus, we can safely conclude that under ordinary circumstances, if a perfect integer box exists, then:

$$D = \frac{A_a^2 + B_b^2 - C_c^2}{A_a + B_b - C_c}. \tag{7}$$

Let us say that we orient the box such that we pick a face with all even length sides to be the front of the box. If the integer box has all even length sides, however, then per lemma 2 we are unrestricted in which face we can pick to be the front, so we could then actually find two additional orienations of the perfect integer boxes where:

$$\begin{array}{c} D = \frac{A_a'^2 + B_b'^2 - C_c'^2}{A_a' + B_b' - C_c'} \\ D = \frac{A_a''^2 + B_b''^2 - C_c''^2}{A_a'' + B_b'' - C_c''} \end{array} , \tag{8}$$

with the exact same body dimensions, and with edges reassigned so that:

$$\begin{array}{c} 4A_a(D - A_a) = 4B_b'(D - B_b') \\ 4B_b(D - B_b) = (D - C_c')^2 \\ (D - C_c)^2 = 4A_a'(D - A_a') \end{array} \tag{9}$$

$$\begin{array}{c} 4A_a'(D - A_a') = 4B_b''(D - B_b'') \\ 4B_b'(D - B_b') = (D - C_c'')^2 \\ (D - C_c')^2 = 4A_a''(D - A_a'') \end{array} \tag{10}$$

$$\begin{array}{c} 4A_a''(D - A_a'') = 4B_b(D - B_b) \\ 4B_b''(D - B_b'') = (D - C_c)^2 \\ (D - C_c'')^2 = 4A_a(D - A_a) \end{array} \tag{11}$$

We note that from (9), we find that

$$\begin{array}{c} A_a' = \frac{D}{2} \pm \frac{\sqrt{2DC_c - C_c^2}}{2} \\ B_b' = \frac{D}{2} \pm \left(\frac{D}{2} - A_a\right) \\ C_c' = D \pm 2\sqrt{B_b D - B_b^2} \end{array} , \tag{12}$$

and from (11), we find that

$$\begin{array}{c} A_a'' = \frac{D}{2} \pm \left(\frac{D}{2} - B_b\right) \\ B_b'' = \frac{D}{2} \pm \frac{\sqrt{2DC_c - C_c^2}}{2} \\ C_c'' = D \pm 2\sqrt{A_a(D - A_a)} \end{array} . \tag{13}$$

But if we consider the combination of (7) and (8) as a single system of equations,

$$\begin{array}{ll} DA_a + DB_b - DC_c - A_a^2 - B_b^2 + C_c^2 & = 0 \\ DA_a' + DB_b' - DC_c' - A_a'^2 - B_b'^2 + C_c'^2 & = 0 \\ DA_a'' + DB_b'' - DC_c'' - A_a''^2 - B_b''^2 + C_c''^2 & = 0 \end{array} , \tag{14}$$

and we rewrite (14) using the equivalences from (12) and (13), and after multiplying the left and right sides of the latter two by 4 to get rid of awkward fractions, we get:

$$\begin{array}{ll} DA_a + DB_b - DC_c - A_a^2 - B_b^2 + C_c^2 & = 0 \\ D^2 - 4A_a^2 - 16B_b^2 + C_c^2 \pm 8\sqrt{(D - B_b)\, B_b}\, D + (4A_a + 16B_b - 2C_c)\, D & = 0 \\ D^2 - 16A_a^2 - 4B_b^2 + C_c^2 \pm 8\sqrt{(D - A_a)\, A_a}\, D + (16A_a + 4B_b - 2C_c)\, D & = 0 \end{array} . \tag{15}$$

Solving forA_a, B_b and C_c, we can find that every solution takes the form of one of the following:

$$\begin{array}{c} A_a = \frac{\left(9 \pm \sqrt{65}\right) D}{18}, B_b = \frac{\left(9 \pm \sqrt{65}\right) D}{18}, C_c = \frac{D}{9} \\ A_a = \frac{\left(9 \pm \sqrt{17}\right) D}{18}, B_b = \frac{\left(9 \pm \sqrt{65}\right) D}{18}, C_c = \frac{5D}{9} \\ A_a = \frac{\left(9 \pm \sqrt{65}\right) D}{18}, B_b = \frac{\left(9 \pm \sqrt{17}\right) D}{18}, C_c = \frac{5D}{9} \\ A_a = \frac{\left(3 \pm \sqrt{5}\right) D}{6}, B_b = \frac{\left(3 \pm \sqrt{5}\right) D}{6}, C_c = \frac{D}{3} \\ A_a = \frac{D}{2} \pm \frac{D}{2}, B_b = \frac{D}{2} \pm \frac{D}{2}, C_c = \frac{D}{2} \pm \frac{D}{2} \end{array} \tag{16}$$

We can then observe that where D is assumed to be an integer, (14) can only have rational results for all unknowns in the system when one or more of A_a, B_b, or C_c is either 0 or else equal to D, all of which would specify a degenerate box with a 0 length side. Therefore, no perfect cuboid exists. $\square$

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Appendix

Lemma 1. *Any all-integer pythagorean triple may be parameterized by* $\left(m'-n', 2\sqrt{m'n'}, m'+n'\right)$.

Proof. Since we know that Euclid's formula for pythagrean triples $\left(m^2-n^2, 2mn, m^2+n^2\right)$ generates every possible primitive pythagorean triple (among others), we can multiply each element of the tuple by some integer factor k to generate every possible pythagorean triple. We can then distribute k across the triple such that:

$$m' = km^2$$
$$n' = kn^2$$

And since $\left(k\left(m^2-n^2\right)\right)^2 + (k2mn)^2 = \left(k\left(m^2+n^2\right)\right)^2$, then $(m'-n')^2 + \left(2\sqrt{m'n'}\right)^2 = (m'+n')^2$, which describes the same triangle. Since both m' and n' are both equal to k multiplied by a square number, the square root of their product must also be an integer.

Lemma 2. *For any right angle triangle with all even-length sides, we can always express either of the shorter edges either as twice the square root of the product of two integers, and so be representable as* $2\sqrt{m'n'}$ *where the product of m' and n' are square, and the other shorter side can always be expressed as the difference of the same integers.*

Proof. If we have a right triangle with all integer-length even sides whose shorter two edges can be described as $2\sqrt{mn}$ and $m - n$, and we solve for a triangle exactly twice as long on every edge, we either have:

$$4\sqrt{mn} = 2\sqrt{m'n'}$$
$$2(m - n) = m' - n'$$

Solving for m' and n' we get

$$m' = 2m$$
$$n' = 2n$$

Or if we consider the proposed alternative parameterization:

$$4\sqrt{mn} = m' - n'$$
$$2(m - n) = 2\sqrt{m'n'}$$

Then, we get

$$m' = m + n + 2\sqrt{mn}$$
$$n' = m + n - 2\sqrt{mn}$$

The product of mn here is known to be square per lemma 1, so n' and m' are integers. Therefore, for all-even triples, it does not matter which shorter side we want to parameterize as twice the square root of the product of two integers and which we want to parameterize as a difference of two integers.