ECE 1508: Applied Deep Learning

Chapter 1: Preliminaries

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Machine Learning (ML)

Goodfellow et al. *informally* define ML as *". . . a form of applied statistics with increased emphasis on the use of computers to statistically estimate complicated functions. . ."*

Though good, this definition is still unclear! Let's put it in simple words

For most problems, there are two approaches to get to a solution

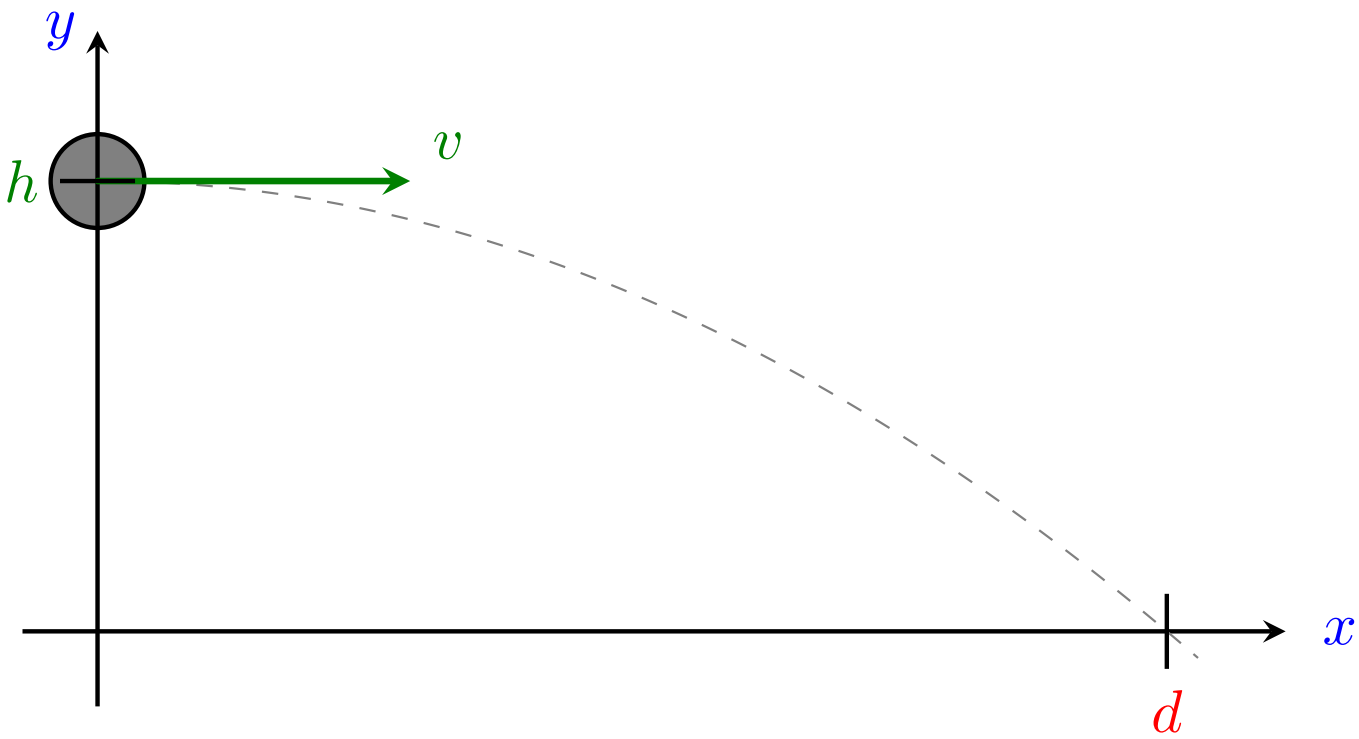
- through analytic derivation
- using data-driven algorithms

ML develops efficient data-driven algorithms for *complicated problems* whose analysis is either *infeasible* or *too complicated*

Let's try a dummy example!

Motion of a Projectile

A projectile, located at height h, is initiated by horizontal velocity v. We are asked to determine horizontal distance d at which *it hits the ground*



Motion of a Projectile: *Analytic Solution*

You may remember the analytic solution from high school or general physics

At time t, the location of the projectile (x, y) is

$$x = vt \qquad\qquad y = -\frac{g}{2}t^2 + h$$

with $g \approx 9.8$ m/s^2 being the gravitational acceleration. When it hits the ground, we have $y = 0$; thus, we can find the hitting time as

$$t_0 = \sqrt{\frac{2h}{g}} = \gamma\sqrt{h}$$

At $t = t_0$, we have $x = d = vt_0$. Defining $\gamma = \sqrt{2/g} \approx 0.45$, we have

$$d = \gamma v\sqrt{h}$$

Motion of a Projectile: *Analytic Solution*

The projectile hits the ground at the horizontal distance

$$d = \gamma v \sqrt{h}$$

To derive this analytic result, we have used two facts

- We knew Newton's laws that describe the motion
- We could solve the equations for d analytically

Well! This is not the case in all problems! In fact, there might be

- no analytic law known that describes the relations, or
- a complicated law whose analysis is computationally infeasible

Let's now look at the ML approach!

Motion of a Projectile: *ML Approach*

A computer scientist could well have forgotten general physics

We conduct the experiment for I different times: in try $i = 1, \ldots, I$

- we initiate with different velocity v_i and height h_i
- we measure the horizontal distance d_i

We then assume that d_i and $[v_i, h_i]$ are related via a *pre-defined function*; for instance, we assume d_i and $[v_i, h_i]$ are related as

$$d_i = w_0 v_i + w_1 h_i$$

We then try to find the optimal values

$$w_0 = w_0^\star \text{ and } w_1 = w_1^\star$$

such that *pre-defined function closely matches our experimental results*

Motion of a Projectile: *ML Approach*

How can we find $w_0^\star$ and $w_1^\star$?

The function should match our experimental data, i.e., *for any i*

$$d_i \overset{!}{=} w_0^\star v_i + w_1^\star h_i \rightsquigarrow (d_i - w_0^\star v_i - w_1^\star h_i)^2 \overset{!}{=} 0$$

By $\overset{!}{=}$, we mean that we intend to have this identity holding

Having this identity *for any i* is equivalent to write

$$\sum_{i=1}^{I} (d_i - w_0^\star v_i - w_1^\star h_i)^2 \overset{!}{=} 0$$

But, such $w_0^\star$ and $w_1^\star$ do not necessarily exist if $I \geqslant 3$!

Don't worry if you don't see it right away! You'll see it in an assignment!

Motion of a Projectile: *ML Approach*

How can we find $w_0^\star$ *and* $w_1^\star$*?* We find $w_0^\star$ and $w_1^\star$ such that

$$\sum_{i=1}^{I} \left(d_i - w_0^\star v_i - w_1^\star h_i\right)^2$$

is as small as possible

Let us define the *loss* $\mathcal{L}\left(w_0, w_1\right)$ as

$$\mathcal{L}\left(w_0, w_1\right) = \sum_{i=1}^{I} \left(d_i - w_0 v_i - w_1 h_i\right)^2$$

We then find $w_0^\star$ *and* $w_1^\star$ *that minimize the loss*

$$\left(w_0^\star, w_1^\star\right) = \underset{w_0, w_1}{\operatorname{argmin}}\, \mathcal{L}\left(w_0, w_1\right)$$

Motion of a Projectile: *ML Approach*

For new given v and h, we find the horizontal distance as

$$d = w_0^\star v + w_1^\star h$$

We could intuitively say that

- *The ML-derived distance is not as accurate as the analytic one*
  - ↳ *The assumed relation between d and $[v, h]$ is not exact*
- *The ML approach gets better as we increase the number of trial I*

In the first assignment, we will program this dummy example!

Motion of a Projectile: *ML Approach*

The ML approach has three main components:

1. *Dataset:* For our example, we collected dataset $\mathbb{D}$

$$\mathbb{D} = \{([v_i, h_i], d_i) : i = 1, \ldots, I\}$$

2. *Model:* We assumed d and $[v, h]$ are related via a *linear* model

$$d = w_0 v + w_1 h$$

3. *Loss:* We evaluated the loss of our model for given w_0 and w_1 as

$$\mathcal{L}(w_0, w_1) = \sum_{i=1}^{I} (d_i - w_0 v_i - w_1 h_i)^2$$

We now take a deeper look into each component

Recap: Vectors and Matrices

We're going to use frequently linear algebra! So, let's recall some basics

$\boldsymbol{x} \in \mathbb{R}^N$ is an N-dimensional *column*-vector with N *real* entries, i.e.,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \rightsquigarrow \boldsymbol{x}^\mathsf{T} = \begin{bmatrix} x_1, \ldots, x_N \end{bmatrix}$$

If we want to make a a *row*-vector, we transpose it, i.e., use $\boldsymbol{x}^\mathsf{T}$

Notation

We show vectors with **bold-face** small letters and drop *column*/*row*

- A vector is by default a *column*-vector
- If we need a *row*-vector, we transpose its *column* version

Recap: Vectors and Matrices

Matrix $\mathbf{A} \in \mathbb{R}^{N \times M}$ can be seen as

- either as the *collection of M column-vectors of dimension N*

$$\mathbf{A} = \left[\mathbf{a}_1, \ldots, \mathbf{a}_M\right]$$

with $\mathbf{a}_m \in \mathbb{R}^N$ for $m = 1, \ldots, M$

- or as the *collection of N row-vectors of dimension M*

$$\mathbf{A} = \begin{bmatrix} \bar{\mathbf{a}}_1^\mathsf{T} \\ \vdots \\ \bar{\mathbf{a}}_N^\mathsf{T} \end{bmatrix}$$

with $\bar{\mathbf{a}}_n \in \mathbb{R}^M$ for $n = 1, \ldots, N$

Recap: Vectors and Matrices

Notation

We show matrices with **bold-face** capital letters

Two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ of *same dimension* N are inner-multiplied as

$$\boldsymbol{x}^\mathsf{T}\boldsymbol{y} = \boldsymbol{y}^\mathsf{T}\boldsymbol{x} = \sum_{n=1}^{N} x_n y_n$$

They can further outer-multiplied as

$$\boldsymbol{x}\boldsymbol{y}^\mathsf{T} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \begin{bmatrix} y_1, \ldots, y_N \end{bmatrix} = \begin{bmatrix} x_1 y_1 & \ldots & x_1 y_N \\ \vdots & \ldots & \vdots \\ x_N y_1 & \ldots & x_N y_N \end{bmatrix} = \left(\boldsymbol{y}\boldsymbol{x}^\mathsf{T}\right)^\mathsf{T}$$

Recap: Vectors and Matrices

Multiplying matrix $\mathbf{A} \in \mathbb{R}^{N \times M}$ with $\boldsymbol{x} \in \mathbb{R}^{M}$ can be seen as

- either as the *linear combination of column-vectors in* $\mathbf{A}$

$$\mathbf{A}\boldsymbol{x} = \left[\mathbf{a}_1, \ldots, \mathbf{a}_M\right] \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix} = \sum_{m=1}^{M} x_m \mathbf{a}_m \in \mathbb{R}^N$$

- or as the *collection of inner-products of row-vectors with* $\boldsymbol{x}$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} \bar{\mathbf{a}}_1^\mathsf{T} \\ \vdots \\ \bar{\mathbf{a}}_N^\mathsf{T} \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \bar{\mathbf{a}}_1^\mathsf{T}\boldsymbol{x} \\ \vdots \\ \bar{\mathbf{a}}_N^\mathsf{T}\boldsymbol{x} \end{bmatrix} \in \mathbb{R}^N$$

ML Components: *Dataset*

Dataset is the collection of data-points; in our example, the dataset was

$$\mathbb{D} = \{([v_i, h_i], d_i) : i = 1, \ldots, I\}$$

which has I data-points $([v_i, h_i], d_i)$ for $i = 1, \ldots, I$

> This is an example of a labeled dataset: a dataset whose data-points contain both the inputs and their corresponding labels (outputs)

More general, a labeled dataset with I data-points is

$$\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$$

- $\boldsymbol{x}_i \in \mathbb{R}^N$ is an input vector with N entries
- y_i is the label of $\boldsymbol{x}_i$ that is a scalar

ML Components: *Supervised Learning*

When the dataset is labeled, our *task* is clear:

- We observe I inputs to a function with their outputs (labels)
- We try to find out (learn) this function

Our task is a *learning task*, because we are trying to

learn the unknown relation between inputs and labels

It describes a *supervised learning problem*, since

a supervisor has filled our dataset with labels

In other words, for the sample inputs in our hand, we know the outputs

In this course, we are mainly focused on supervised learning

ML Components: *Unsupervised Learning*

+ *But can a dataset be unlabeled?*
– Yes! This is the case in *unsupervised learning* in which we are to learn features of a data-point $\boldsymbol{x}$ by investigating a set of its samples

In this case, the dataset is of the form

$$\mathbb{D} = \{\boldsymbol{x}_i : i = 1, \ldots, I\}$$

which is unlabeled. Our *learning task* is further *unsupervised*

Let's make it crystal clear via an example!

ML Components: *Example of Unsupervised Learning*

An N-dimensional data-point $\boldsymbol{x}$ is coming from a natural process, e.g., it contains the pixel values of an image taken from body. It is known that $\boldsymbol{x}$ is a linear combination of $Q \ll N$ *principal vectors*: any $\boldsymbol{x}$ is

$$\boldsymbol{x} = \sum_{q=1}^{Q} a_q \boldsymbol{v}_q = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_Q] \begin{bmatrix} a_1 \\ \vdots \\ a_Q \end{bmatrix} = \mathbf{V}\mathbf{a}$$

with $\mathbf{a} \in \mathbb{R}^Q$ and $\mathbf{V} \in \mathbb{R}^{N \times Q}$. *Nevertheless, we do not know the principle vectors, i.e., matrix $\mathbf{V}$ is unknown to us.*

Our learning task is to

find out what $\mathbf{V}$ is by investigating I samples of $\boldsymbol{x}$

ML Components: *Example of Unsupervised Learning*

In this problem, our dataset is of the form

$$\mathbb{D} = \{\boldsymbol{x}_i : i = 1, \ldots, I\}$$

which is unlabled. We know that these samples are of the form

$$\boldsymbol{x}_i = \mathbf{V}\mathbf{a}_i$$

but we know *neither* $\mathbf{V}$ *nor* $\mathbf{a}_i$. This is an unsupervised learning problem

This is the well-known problem of *dimensionality reduction*

- *we get a large dimensional vector* $\boldsymbol{x}$
- *we derive a feature out of it, i.e.,* $\mathbf{a}$*, which is of lower dimention*

The classical solution is given by Principal Component Analysis (PCA)

ML Components: *Unsupervised* ⇝ *Supervised*

Supervised and unsupervised are not only divisions in terms of dataset:

- We may deal with a semi-supervised learning task
  - ↳ *Dataset contains both labeled and unlabeled data-points*
- We may deal with a reinforcement learning task
  - ↳ *An agent is to learn a set of actions each relying on the others*
  - ↳ *Agent's dataset grows through interactions with the environment*
  - ↳ *Best example is the design of a machine that learns to play chess*

As mentioned, the main focus of this course is on supervised learning

We will also discuss unsupervised later in the course

Reinforcement learning is beyond the scope of this course

There is a separate ECE course on this topic next

Stay tuned for that!

ML Components: *Model*

Now, we know what supervised learning and labeled dataset are

Let's assume we are given by the labeled dataset

$$\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$$

and are to learn the function that relates an input $\boldsymbol{x}$ to its label y

For this task we need to consider a Model

Model is a *parameterized* function that is used to describe the relation between the input $\boldsymbol{x}$ and its label y

In our dummy example, the model was linear

$$\text{label} = d = w_0 v + w_1 h = [w_0, w_1] \begin{bmatrix} v \\ h \end{bmatrix} = \mathbf{w}^\mathsf{T} \boldsymbol{x}$$

Recap: Linear versus Affine Function

Linear Function

Scalar $y \in \mathbb{R}$ is a linear function of vector $\boldsymbol{x} \in \mathbb{R}^N$ if

$$y = \mathbf{w}^\mathsf{T} \boldsymbol{x}$$

for some constant vector $\mathbf{w} \in \mathbb{R}^N$

Affine Function

Scalar $y \in \mathbb{R}$ is an affine function of vector $\boldsymbol{x} \in \mathbb{R}^N$ if

$$y = \mathbf{w}^\mathsf{T} \boldsymbol{x} + b$$

for some constant vector $\mathbf{w} \in \mathbb{R}^N$ and scalar $b \neq 0$

Recap: Linear versus Affine Function

Key difference: If $\boldsymbol{x} = \mathbf{0}_N$ is the vector of all zeros

- Linear function returns zero: *it passes through the origin*
- Affine function returns non-zero: *it does not pass through the origin*

We can simply extend the definition to a vector-valued functions

Let $\mathbf{A} \in \mathbb{R}^{M \times N}$ and $\mathbf{b} \in \mathbb{R}^{M}$; then,

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x}$$

is a linear vector-valued function of $\boldsymbol{x}$ and

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x} + \mathbf{b}$$

is an affine vector-valued function

ML Components: *Model*

In our dummy example, we considered a linear model, but

- we could have considered an affine model

$$\text{label} = b + \mathbf{w}^{\mathsf{T}}\boldsymbol{x}$$

- or a polynomial model

$$\text{label} = b + \mathbf{w}_1^{\mathsf{T}}\boldsymbol{x} + \mathbf{w}_2^{\mathsf{T}}\boldsymbol{x}^2 + \ldots + \mathbf{w}_P^{\mathsf{T}}\boldsymbol{x}^P$$

Notation

By $f(\boldsymbol{x})$ we refer to entry-wise function operation

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \rightsquigarrow \boldsymbol{x}^p = \begin{bmatrix} {x_1}^p \\ \vdots \\ {x_N}^p \end{bmatrix} \text{ or } \sqrt{\boldsymbol{x}} = \begin{bmatrix} \sqrt{x_1} \\ \vdots \\ \sqrt{x_N} \end{bmatrix}$$

ML Components: *Model*

In our dummy example, we considered a linear model, but

- we could have considered even a wired model

$$\text{label} = \left(\mathbf{w}_1^{\mathsf{T}} \boldsymbol{x}\right) \left(\mathbf{w}_2^{\mathsf{T}} \sqrt{\boldsymbol{x}}\right)$$

Models are always parameterized meaning that

they contain some parameters that are to be tuned

Example: $\mathbf{w}$ in linear model or $\mathbf{w}_p$'s and b in polynomial model

Model parameters are of two types:

- *Hyperparameters*
- *Learnable parameters*

ML Components: *Model*

Hyperparameters

Parameters that are required to specify the model *explicitly*

Best example is the order P in the polynomial model: *we need to know* P *in order to write down the model explicitly*

If we know $P = 2$*; then, we know that our model is*

$$\textit{label} = b + \mathbf{w}_1^\mathsf{T}\boldsymbol{x} + \mathbf{w}_2^\mathsf{T}\boldsymbol{x}^2$$

Hyperparameters are specified prior to the start of learning process

+ *How can we tune them?*
– We will discuss it in detail in this course! For the moment, assume that *they are given to us by some genie*

ML Components: *Model*

Learnable Parameters

Once the hyperparameters are set, we have a model with some parameters that are to be learned, such that the model fits our dataset

In our polynomial example after we set $P = 2$, we get the model

$$\text{label} = b + \mathbf{w}_1^\mathsf{T} \boldsymbol{x} + \mathbf{w}_2^\mathsf{T} \boldsymbol{x}^2$$

Now we need to *learn* $\mathbf{w}_1$, $\mathbf{w}_2$, *and* b: *let our dataset be*

$$\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$$

we need to find (learn) values $\mathbf{w}_1^\star$, $\mathbf{w}_2^\star$, *and* $b^\star$ *such that for all* i

$$y_i \approx b^\star + \mathbf{w}_1^{\star\mathsf{T}} \boldsymbol{x}_i + \mathbf{w}_2^{\star\mathsf{T}} \boldsymbol{x}_i{}^2$$

ML Components: *Model*

Learnable Parameters

Once the hyperparameters are set, we have a model with some parameters that are to be learned, such that the model fits our dataset

In our polynomial example with $P = 2$: we learn $\mathbf{w}_1^\star$, $\mathbf{w}_2^\star$, and $b^\star$ such that

$$y_i \approx b^\star + \mathbf{w}_1^{\star\mathsf{T}} \boldsymbol{x}_i + \mathbf{w}_2^{\star\mathsf{T}} {\boldsymbol{x}_i}^2$$

+ *But what does this* $\approx$ *mean?*
– This approximation needs to be quantified!

This is what the loss function does for us

+ *How do we learn the learnable parameters?*
– We answer this after we understand the loss function

ML Components: *Loss*

Loss Function

Loss function quantifies *the difference between the output of the model and the true label*

Back to our polynomial example with dataset $\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$

- We set the hyperparameter P to $P = 2$
- We now set the learnable parameters to

$$\mathbf{w}_1 = \mathbf{w}_1^{(0)} \qquad \mathbf{w}_2 = \mathbf{w}_2^{(0)} \qquad b = b^{(0)}$$

If we give the data-point $\boldsymbol{x}_i$ *to this model as input, it returns*

$$z_i = b^{(0)} + \mathbf{w}_1^{(0)} \boldsymbol{x}_i + \mathbf{w}_2^{(0)} {\boldsymbol{x}_i}^2$$

ML Components: *Loss*

Back to our polynomial example with dataset $\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$

$$z_i = b^{(0)} + \mathbf{w}_1^{(0)} \boldsymbol{x}_i + \mathbf{w}_2^{(0)} {\boldsymbol{x}_i}^2$$

Loss determines the difference between z_i and true label y_i in $\mathbb{D}$

$$\mathcal{L}\left(z_i, y_i\right) = \ell_i \in \mathbb{R}$$

Let's see few examples of loss function:

We can calculate the *squared error*

$$\mathcal{L}\left(z_i, y_i\right) = \left(z_i - y_i\right)^2$$

which intuitively determines the energy of the difference

ML Components: *Loss*

Back to our polynomial example with dataset $\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$

$$z_i = b^{(0)} + \mathbf{w}_1^{(0)} \boldsymbol{x}_i + \mathbf{w}_2^{(0)} {\boldsymbol{x}_i}^2$$

Loss determines the difference between z_i and true label y_i in $\mathbb{D}$

$$\mathcal{L}\left(z_i, y_i\right) = \ell_i \in \mathbb{R}$$

Let's see few examples of loss function:

We can calculate the *error indicator*

$$\mathcal{L}\left(z_i, y_i\right) = \mathbb{1}\left(z_i \neq y_i\right) = \begin{cases} 1 & z_i \neq y_i \\ 0 & z_i = y_i \end{cases}$$

which indicates the occurrence of error

ML Components: *A Quick Wrap-up*

Any ML problem has three components:

1. *Dataset* which is the collection of samples

supervised learning $\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$

2. *Model* that describes the relation between the input and label

$\mathbf{h}$: Hyperparameters $z = f_{\mathbf{h}}(\boldsymbol{x}|\mathbf{w})$ $\mathbf{w}$: Learnable Parameters

3. *Loss* quantifies the difference between the *model's output* and *true label*

$$\mathcal{L}(z_i, y_i)$$

What's Next?

Now, we know the components of an ML problem!

+ *How can we solve the problem? Or, speaking in the language of ML people, how can we address the learning task?*
- Recall that the learning task is to tune the learnable parameters ($\mathbf{w}$ in the last slide): once we tune them, the problem is over

We approximate the lable of a new input $\boldsymbol{x}_{new}$ *as* $y_{new} = f_{\mathbf{h}}\left(\boldsymbol{x}_{new} \middle| \mathbf{w}\right)$

The process of tuning the learnable parameters is called

training of the model

+ *How do we do the training?*
- We see it very shortly, but first we need a more serious example!

Classification

Classification is a supervised learning problem in which

labels belong to a discrete set: $y_i \in \{c_1, \ldots, c_J\}$

the label y_i represents the class to which the input $\boldsymbol{x}_i$ belongs

Best example is image classification:

- The dataset contains some images and their labels

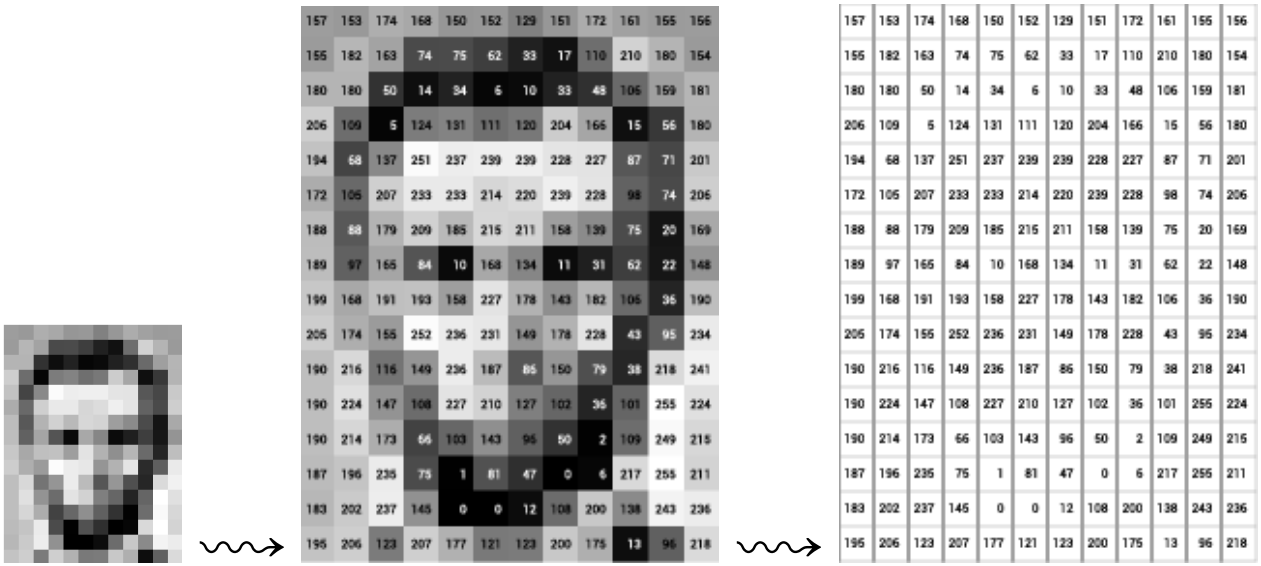
$$\mathbb{D} = \{(\text{image } 1, \text{dog}), (\text{image } 2, \text{cat}), \ldots, (\text{image } I, \text{cat})\}$$

- The images are either *dog* or *cat*

We can simply convert $\mathbb{D}$ *into a numerical dataset*

Classification: *Image Classification*

Any image is nothing but *a collection of pixel values*



We will work with images a lot in this course, so we will learn the meaning of pixel values very well. For now, the important thing is that

for each image, we can make a vector containing its pixel values

Thus, image i in $\mathbb{D}$ can be shown by a pixel vector $\boldsymbol{x}_i$

Classification: *Image Classification*

The labels can further be marked by integer numbers as classes

In our example, we have *two classes of dog and cat*: so, we could say

label of dog images $= 0$ and *label of cat images* $= 1$

So, a collection of I images like

$$\mathbb{D} = \{(\text{image } 1, \text{dog}), (\text{image } 2, \text{cat}), \ldots, (\text{image } I, \text{cat})\}$$

can be represented by the numerical dataset

$$\mathbb{D} = \{(\boldsymbol{x}_1, 0), (\boldsymbol{x}_2, 1), \ldots, (\boldsymbol{x}_I, 1)\}$$

pixel vector of image 1

it's an image of a dog

Classification: *Binary Classification*

We now start with the basic case of binary classification

Binary Classification

A classification problem with only two classes, i.e., $y_i \in \{0, 1\}$

Our example was a binary classification with dog:0 and cat:1

Binary classification is very fundamental, since

- *it is one of the very first problems investigated in ML*
- *other classification problems are reduced into a series of binary classifications: say we want to classify an image as dog, cat or car*
  - *Binary Classification 1: Is it class 0: dog or class 1: {cat, car}?*
    - ↳ *If class 0 ⇝ classification ended*
    - ↳ *If class 1 ⇝ Binary Classification 2: Is it class 0: cat or class 1: car?*

Let's build the main components of this ML problem

Binary Classification: *Dataset*

Dataset is similar to what we had in our dog or cat example

$$\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$$

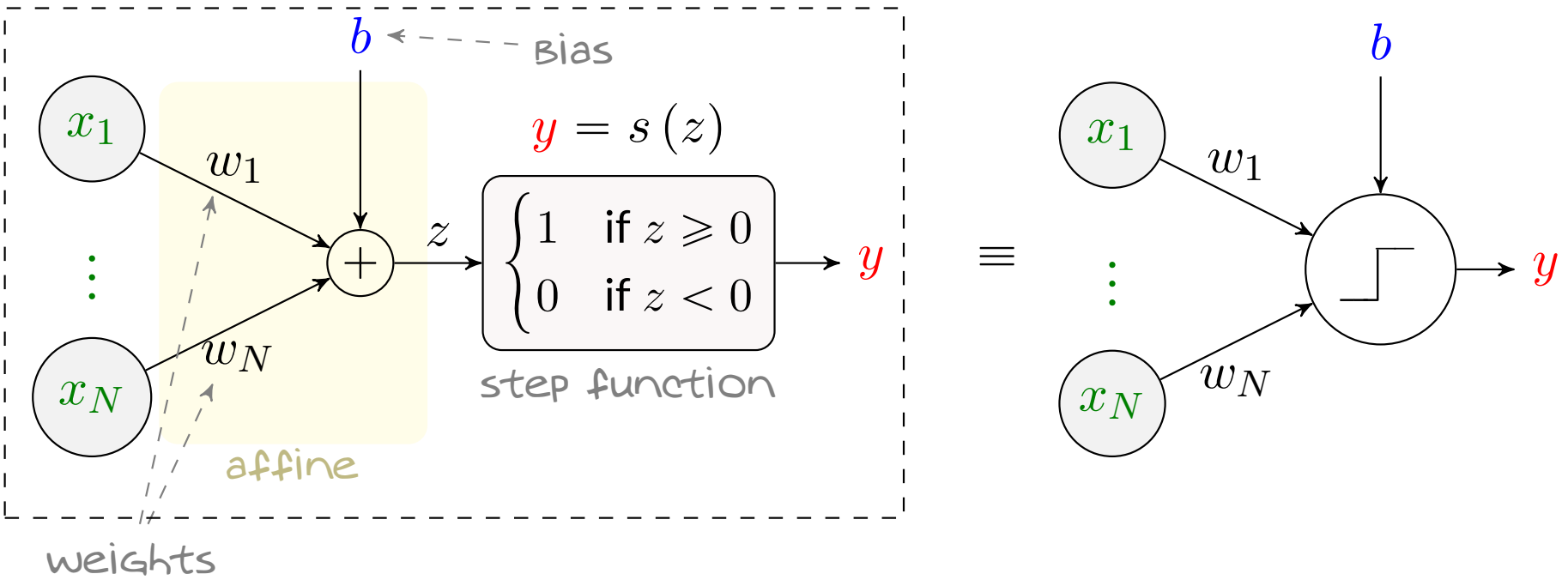
- $\boldsymbol{x}_i \in \mathbb{R}^N$ is a real-valued N-dimensional vector
  - *For instance it contains the pixel values of an image with N pixels*
- $y_i \in \{0, 1\}$ is a binary label

Binary Classification: *Model*

+ *What should our model be?*
- The model gets N real numbers and returns a binary number

Classification models are also called *Classifiers*

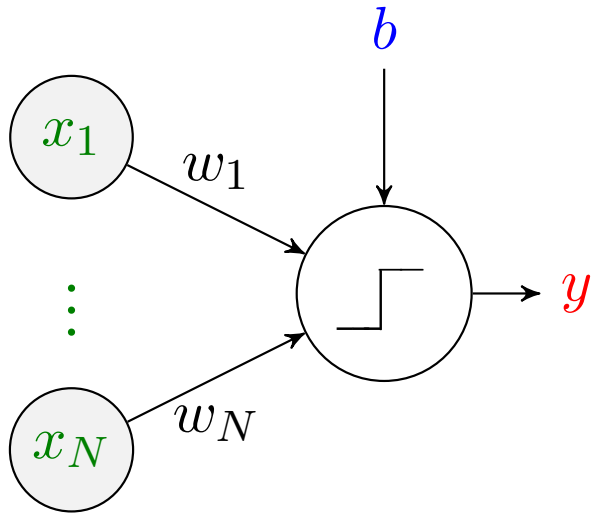
Let's start with perceptron who is the father of neural networks



Binary Classification: *Perceptron*

Perceptron is a *linear classifier*

- *it determines an linear transform of* $\boldsymbol{x}$
- *it classifies using the sign of this transform*

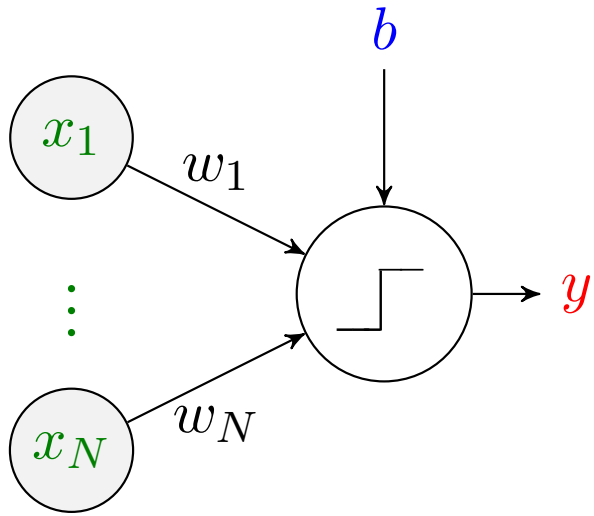


Mathematically, the perceptron is described by

$$y = s(\sum_{n=1}^{N} w_n x_n + b) = s(\underbrace{[w_1, \ldots, w_N]}_{\mathbf{w}^\mathsf{T}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}}_{\boldsymbol{x}} + b) = s(\mathbf{w}^\mathsf{T}\boldsymbol{x} + b)$$

Binary Classification: *Perceptron*

$$y = s(\mathbf{w}^\mathsf{T}\boldsymbol{x} + b)$$

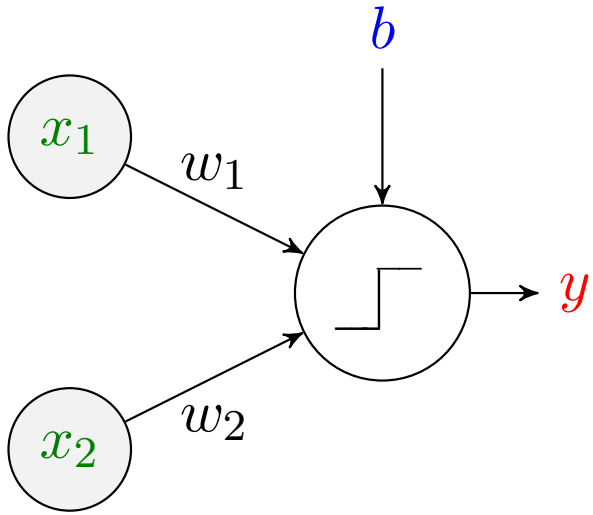


- \+ *Is there any hyperparameter in this model?*
- – We could look at step function as a hyperparameter: *we could have chosen another function to map the affine transform to a binary label*
- \+ *Is there any learnable parameter in this model?*
- – Of course! $\mathbf{w}$ and b are the learnable parameters

Perceptron: *Geometrical Interpretation*

Perceptron geometrically realizes a linear division of $\mathbb{R}^N$

To see this, let's focus on the two-dimensional case, i.e., $N = 2$



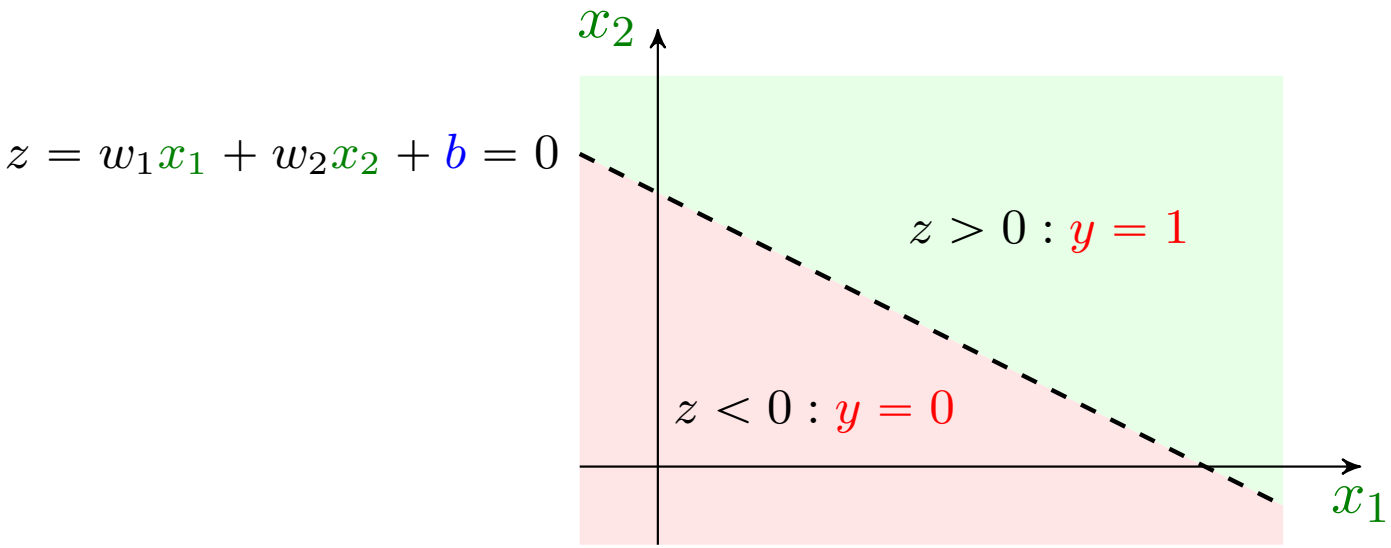
$$y = s(\mathbf{w}^\mathsf{T}\boldsymbol{x} + b) = s(w_1x_1 + w_2x_2 + b) = \begin{cases} 1 & \text{if } w_1x_1 + w_2x_2 + b \geqslant 0 \\ 0 & \text{if } w_1x_1 + w_2x_2 + b < 0 \end{cases}$$

Perceptron: *Geometrical Interpretation*

Let's go to the two-dimensional space with one dimension denoting x_1 and the other x_2 and plot the line

$$z = w_1 x_1 + w_2 x_2 + b = 0$$

for some $w_1, w_2 > 0$ and $b < 0$

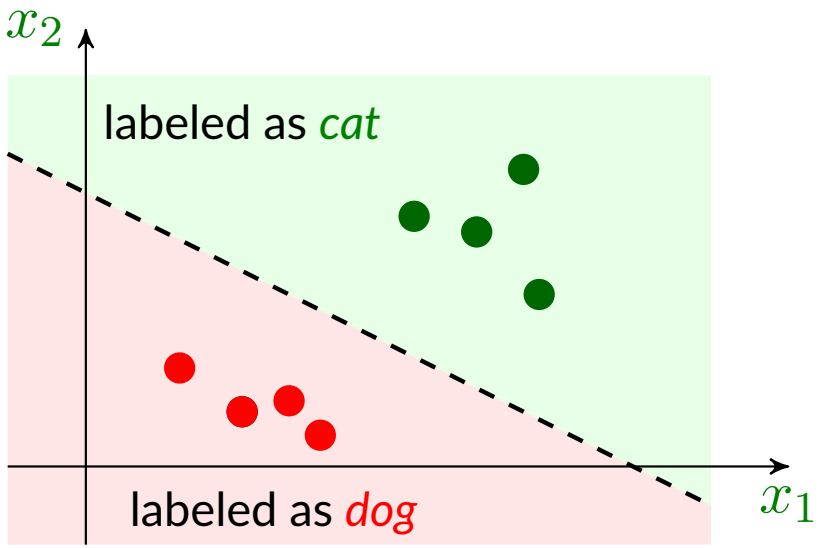


Perceptron draws a line, and then classifies every point above it with label 1 and every point below it with label 0

Perceptron: *Geometrical Interpretation*

So, if we have a perceptron with learnable parameters $w_1, w_2 > 0$, *and* $b < 0$

$$z = w_1 x_1 + w_2 x_2 + b = 0$$



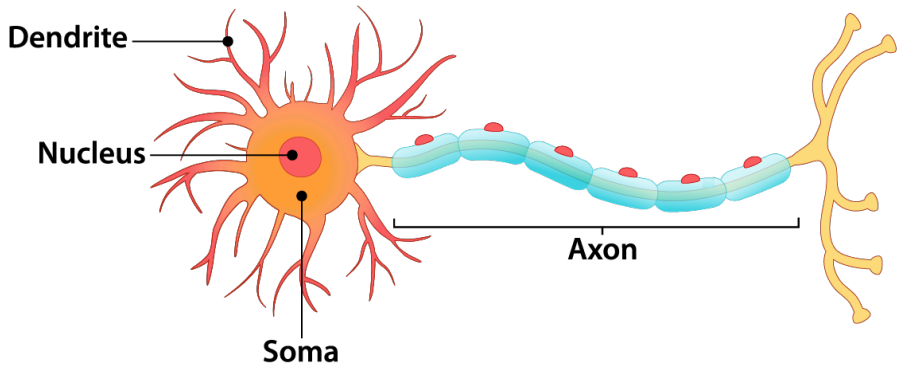
For instance in the example of image classification

- *Every • represents a data-point* $\equiv$ *pixel vector: we should think of a two-pixel image* ☺
- *Perceptron gets the values of the two pixels and predicts whether it is an image of a dog or a cat*

In realistic problems, this viewpoint extends to large-dimensional space

Perceptron: *A Bit of History*

- Perceptron machine was implemented by Frank Rosenblatt in 1957
- He wrote a paper in 1958 illustrating the machine and its algorithm
- The original idea is however older than that
  - ↳ *It was proposed in 1943 by Warren McCulloch and Walter Pitts*
  - ↳ *They introduced it to abstractly describe biological neurons*
  - ↳ *This was why perceptron is also called McCulloch-Pitts neuron*



- Perceptron was a breakthrough in the long-time ongoing attempt to understand the functionality of brain; however,

for us, perceptron is simply a linear classification model

Components of Binary Classification

Back to binary classification: we have the first two components

1. A dataset with N-dimensional inputs $\boldsymbol{x}_i$ and binary labels $y_i \in \{0, 1\}$

$$\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$$

2. Perceptron as the model

$$y = s(\mathbf{w}^\mathsf{T}\boldsymbol{x} + b)$$

We are now looking for the third component, i.e., the loss function

Binary Classification: *Loss*

In general, we can use any loss function

- *squared error*
- *Kullback-Leibler divergence* ⇜ we are going to learn it soon
- error indicator
- ...

Let's use the one initially used in Rosenblatt's machine, i.e., error indicator[1]

For two binary variables y and $\hat{y}$, the loss function is

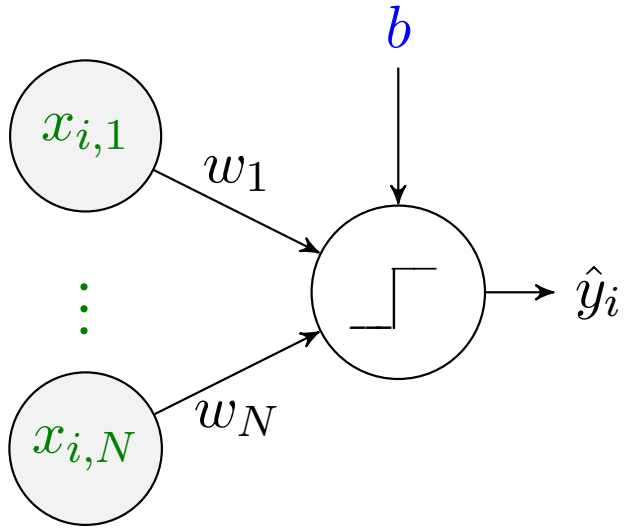
$$\mathcal{L}(\hat{y}, y) = \mathbb{1}(\hat{y} \neq y) = \begin{cases} 1 & \hat{y} \neq y \\ 0 & \hat{y} = y \end{cases}$$

[1] We will realize later that this is not a good choice

Binary Classification: *Loss*

How does this loss look like when we classify via perceptron?

Let $(\boldsymbol{x}_i, y_i)$ be a data-point in dataset: we give $\boldsymbol{x}_i$ to the perceptron



and get $\hat{y}_i$. The loss says whether we got the same label (0) or not (1)

$$\mathcal{L}\left(\hat{y}_i, y_i\right) = \mathbb{1}\left(\hat{y}_i \neq y_i\right) = \begin{cases} 1 & \hat{y}_i \neq y_i \\ 0 & \hat{y}_i = y_i \end{cases}$$

Training a Model

Up to now, we have learned the main components of ML problems and seen a classical example, i.e., classification

---

We now want to understand how we can tune the learnable parameters of a model for our problem.

As mentioned, tuning the learnable parameters of a model is called

training of the model

Before starting with training, let's review some basics of probability theory

Recap: Probability Theory

Discrete Random Variables

Discrete random variable x is described by *probability mass function* $P(x)$

$$\Pr\{x = x_0\} = P(x_0)$$

Assuming that $x \in \mathbb{X}$, we have

$$\sum_{x \in \mathbb{X}} P(x) = 1$$

Continuous Random Variables

Continuous random variable x is described by *probability density function* $P(x)$

$$\Pr\{a < x \leqslant b\} = \int_a^b P(x)\mathrm{d}x$$

We have in this case

$$\int_{-\infty}^{+\infty} P(x)\mathrm{d}x = 1$$

When we talk about a general random variable, we call $P(x)$ the distribution

Recap: Probability Theory

A random vector $\boldsymbol{x}$ is a vector of random variables and its distribution

$$P(\boldsymbol{x}) = P(x_1, \ldots, x_N)$$

is the *joint distribution of the entries*

Assume $\boldsymbol{x} \in \mathbb{R}^N$ is a vector of random variables, its *expectation* is

Discrete $\mathbb{E}\{\boldsymbol{x}\} = \sum_{\boldsymbol{x} \in \mathbb{X}^N} \boldsymbol{x} P(\boldsymbol{x})$ Continuous $\mathbb{E}\{\boldsymbol{x}\} = \int \boldsymbol{x} P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$

and we can extend the definition to any function of $\boldsymbol{x}$, i.e.,

$$\mathbb{E}\{f(\boldsymbol{x})\} = \int f(\boldsymbol{x}) P(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

Recap: Law of Large Numbers

Consider a random sequence $x_1, \ldots, x_I$

we call it independent and identically distributed (i.i.d.) with $x \sim P(x)$

if x_i's are generated *independently* all with the same distribution $P(x)$

Law of Large Numbers

Assume $x_1, \ldots, x_I$ is an i.i.d. sequence with $x \sim P(x)$; then, we have[2]

$$\frac{1}{I} \sum x_i \to \mathbb{E}\{x\}$$

as $I \to \infty$

In simple words: if we *arithmetically average* too many samples of a random process, we get a value very close to its *expectation*

[2] Of course under some conditions which we assume holding

Training a Model

Let us now use the probability theory to derive a meaningful approach of model training: for sake of simplicity, let's assume that we have dataset

$$\mathbb{D} = \{(x_i, y_i) : i = 1, \ldots, I\}$$

with *scalar (one-dimensional) inputs*, i.e., x_i's are real numbers

---

Let us further denote our model as

$$y = f\left(x|\mathbf{w}\right)$$

where in this notation

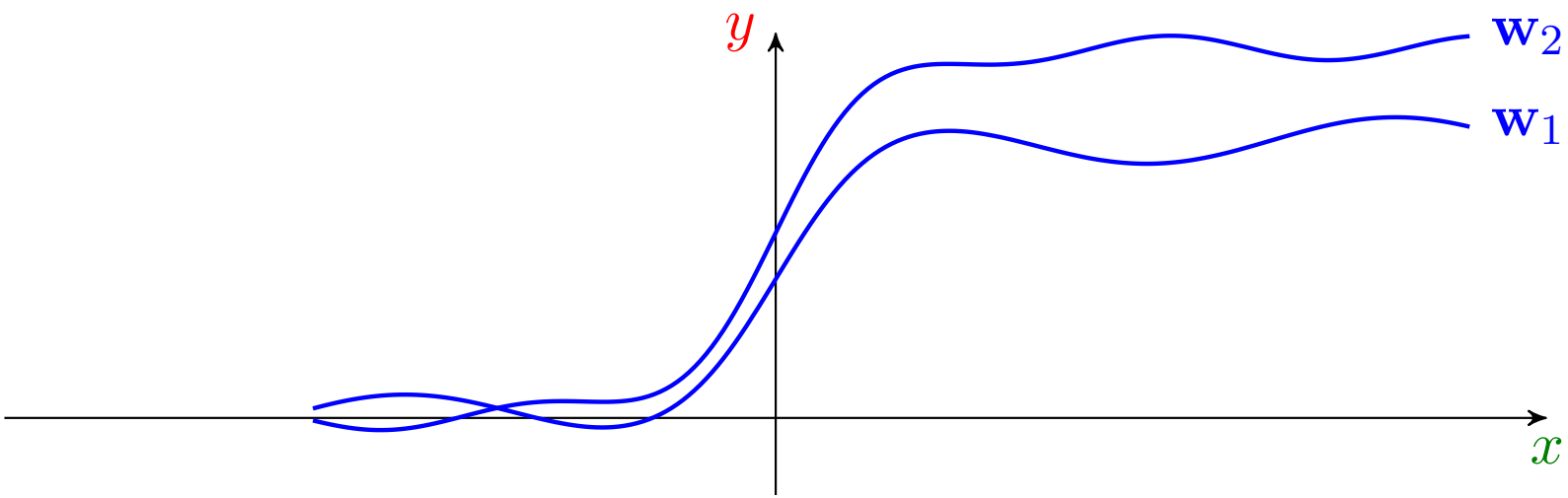
- $\mathbf{w}$ contains all learnable parameters that are to be learned in training
- we drop the hyperparameters, since *they are yet fixed by genie*

Training a Model

The *one-dimensional* assumption helps us visualize the model

$$y = f\left(x|\mathbf{w}\right)$$

With scalar input, we can visualize the model as



As learnable parameters change, the model represents different functions

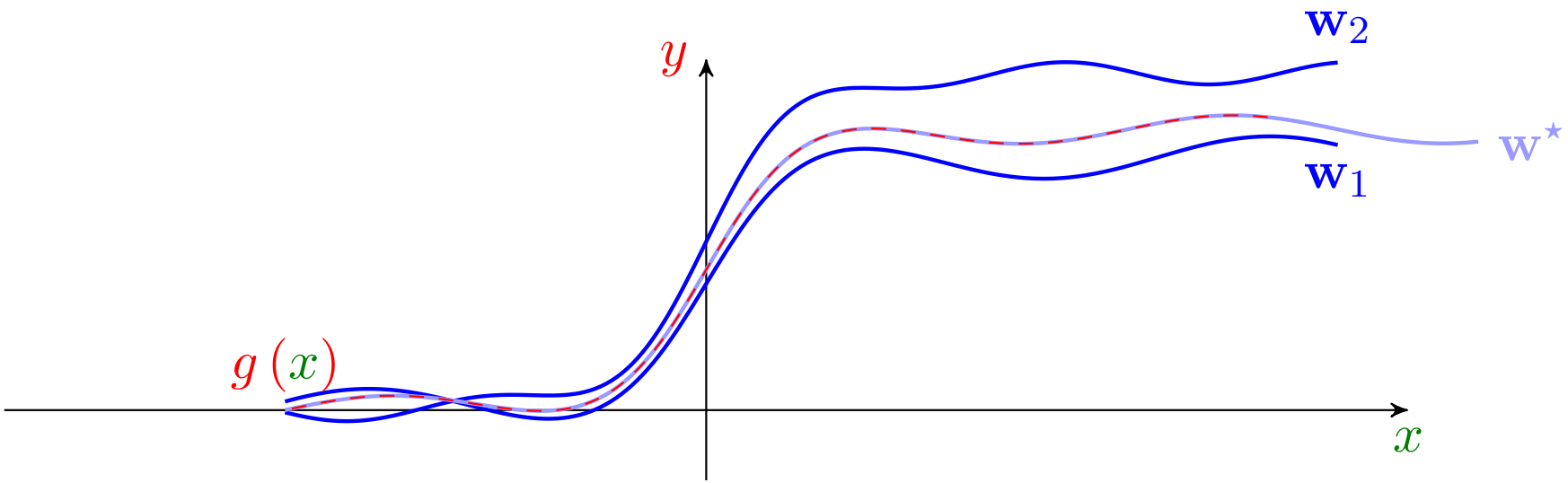
Training a Model

In reality, the labels and inputs are related through a function $g(\cdot)$

$$y = g(x) \rightsquigarrow \text{function } g(\cdot) \text{ is however unknown to us}$$

In fact, our whole learning task is to learn it from the dataset!

+ *What if we knew function* $g(\cdot)$*? How would have we trained our model?*
– Well! We would have tuned $\mathbf{w}$ until the model matches $g(\cdot)$



Training a Model

We could represent such $\mathbf{w}^\star$ using the loss

Point-wise Loss

Let $\mathcal{L}$ be loss function: the model with learnable parameter $\mathbf{w}$ gives us the label $\hat{y}_0 = f\left(x_0|\mathbf{w}\right)$ for input x_0

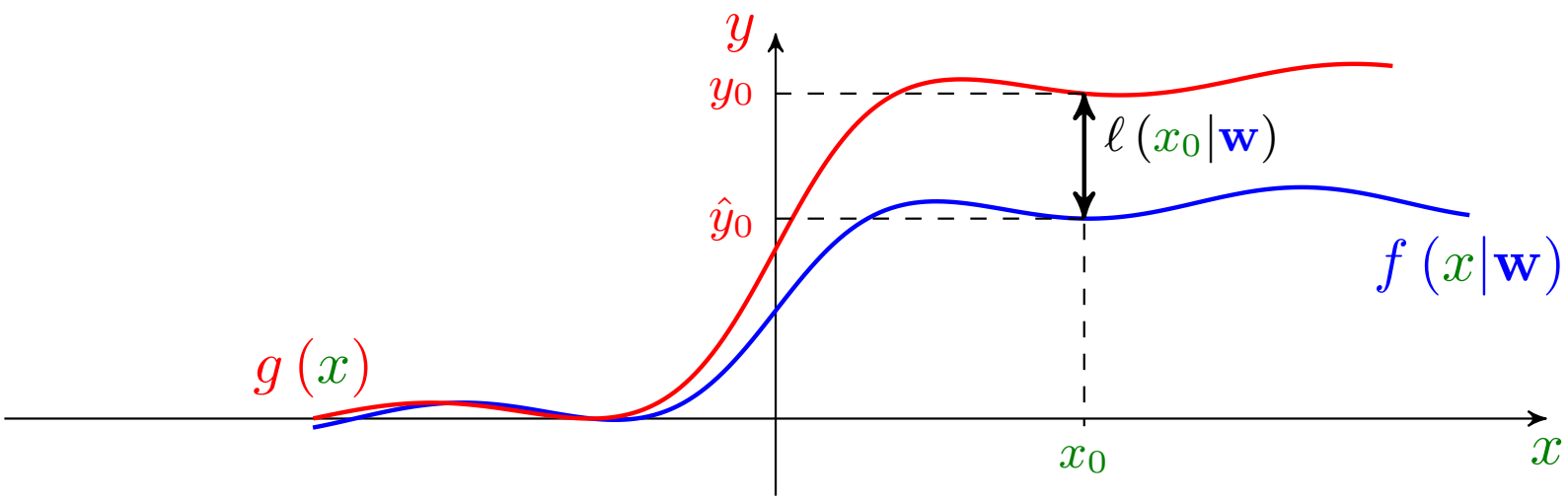
- *It may be different from the true label $y_0 = g\left(x_0\right)$*
- *The loss between the two is given by*

$$\ell\left(x_0|\mathbf{w}\right) = \mathcal{L}\left(\hat{y}_0, y_0\right) = \mathcal{L}\left(f\left(x_0|\mathbf{w}\right), g\left(x_0\right)\right)$$

Training a Model: *Point-wise Loss*

$$\ell\left(x_0|\mathbf{w}\right) = \mathcal{L}\left(\hat{y}_0, y_0\right) = \mathcal{L}\left(f\left(x_0|\mathbf{w}\right), g\left(x_0\right)\right)$$

Let's visualize this loss



Training a Model: *Risk*

We would like our model to recover the true label as closely as possible: so, the best option at point x_0 is to find the choice of $\mathbf{w}$ that minimizes $\ell\left(x_0|\mathbf{w}\right)$

+ *But x_0 is a single point! What about other inputs?!*
– *Right! We should learn $\mathbf{w}$ for any x_0*
+ *How can we do it?*
– *We treat x_0 as a random variable with some distribution $P\left(x_0\right)$, and minimize the expexted loss often called risk*

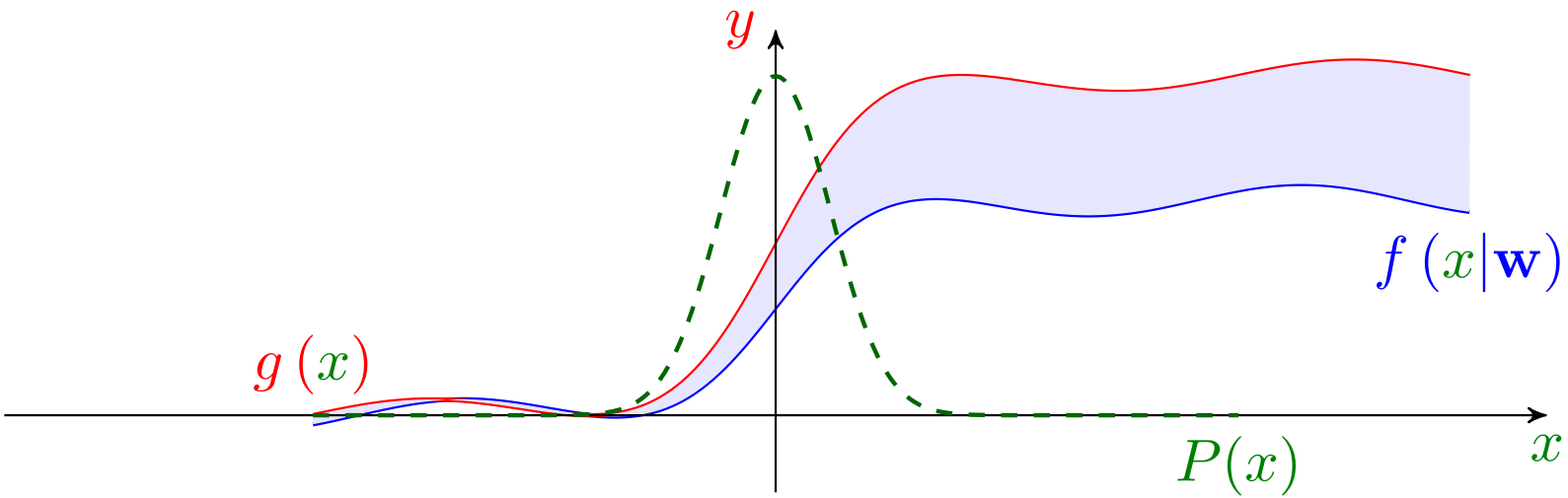
Risk

For given learnable parameter $\mathbf{w}$, the risk is defined as

$$R\left(\mathbf{w}\right) = \mathbb{E}\left\{\ell\left(x_0|\mathbf{w}\right)\right\} = \int \ell\left(x_0|\mathbf{w}\right) P\left(x_0\right) \mathrm{d}x_0$$

Training a Model: *Risk Minimization*

Let's visualize the risk



Risk Minimization

Ideally, the training is formulated as the minimization of risk

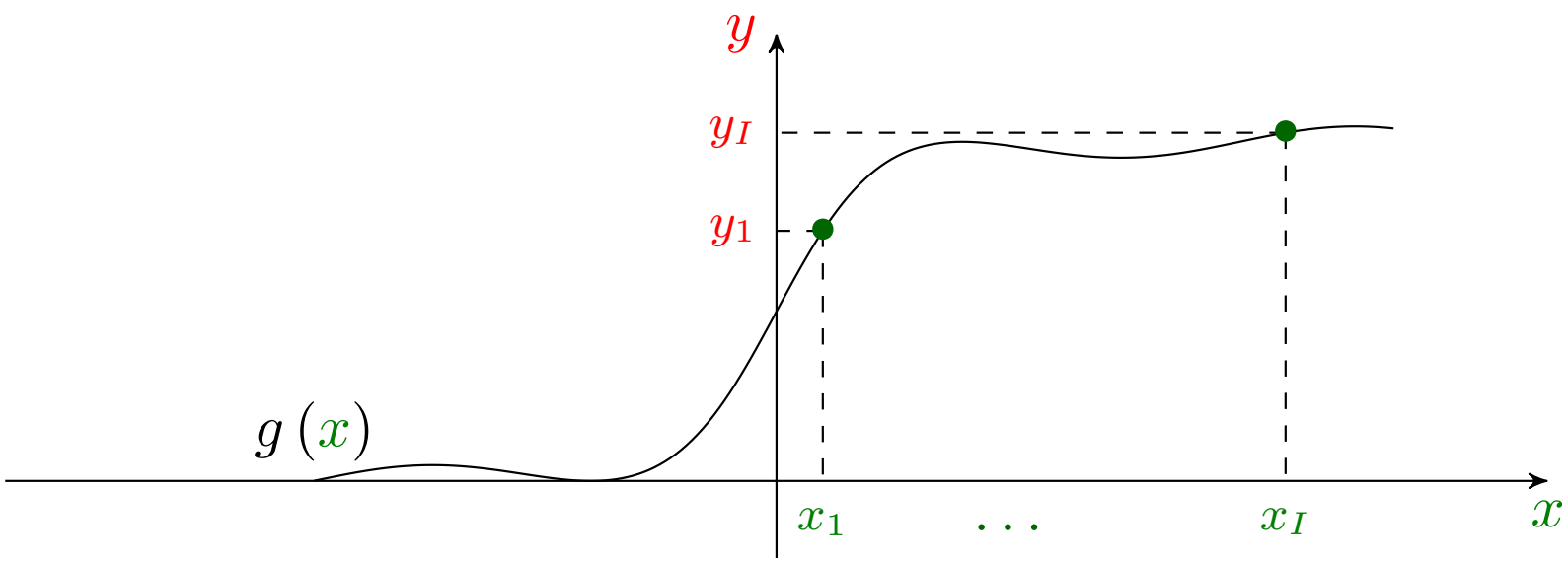
$$\mathbf{w}^{\star} = \underset{\mathbf{w}}{\operatorname{argmin}}\, R\left(\mathbf{w}\right)$$

Training a Model: *Empirical Risk*

+ *Bravo! But, the training seems infeasible, since we have neither the true function $g(\cdot)$, nor the distribution $P(x)$!*
– Right! But, we could handle this approximately using the LLN

Let's look at what we have: *the dataset which contains samples of $g(\cdot)$*

$$\mathbb{D} = \{(x_i, y_i) : i = 1, \ldots, I\}$$



Training a Model: *Empirical Risk*

We can determine point-wise losses for points in the dataset

$$\ell\left(x_i|\mathbf{w}\right) = \mathcal{L}\left(f\left(x_i|\mathbf{w}\right), g\left(x_i\right)\right) = \mathcal{L}\left(f\left(x_i|\mathbf{w}\right), y_i\right)$$

Now, assume that data-points are *drawn from i.i.d. unknown* $P\left(x\right)$

LLN suggests that average of point-wise losses converges to the risk when we have a large enough number of data-points, i.e.

$$\frac{1}{I}\sum_{i=1}^{I}\ell\left(x_i|\mathbf{w}\right) \rightarrow \mathbb{E}\left\{\ell\left(x|\mathbf{w}\right)\right\} = R\left(\mathbf{w}\right)$$

We call this arithmetic average the empirical risk

Empirical risk is the best estimate of risk that we get from our dataset

Training a Model: *Empirical Risk Minimization*

Empirical Risk

Let $\mathbf{w}$ *includes all learnable parameters of the model, and the dataset be*

$$\mathbb{D} = \{(\boldsymbol{x}_i, y_i) : i = 1, \ldots, I\}$$

for loss function $\mathcal{L}$*, the empirical risk is defined as*

$$\hat{R}(\mathbf{w}) = \frac{1}{I} \sum_{i=1}^{I} \mathcal{L}\left(f\left(\boldsymbol{x}_i | \mathbf{w}\right), y_i\right)$$

Training a Model: *Empirical Risk Minimization*

The training is performed by minimizing the empirical risk

Empirical Risk Minimization

To train a model on the dataset, we minimize the empirical risk computed from the dataset

$$\mathbf{w}^{\star} = \underset{\mathbf{w}}{\operatorname{argmin}}\, \hat{R}\left(\mathbf{w}\right) \qquad \text{(Training)}$$

$$= \underset{\mathbf{w}}{\operatorname{argmin}}\, \frac{1}{I} \sum_{i=1}^{I} \mathcal{L}\left(f\left(\boldsymbol{x}_i | \mathbf{w}\right), y_i\right)$$

Wrap-Up: *Universal Scheme for ML*

So, we know pretty much all the theory for supervised learning:

1. We build the main three components:
   1. Dataset
   2. Model
   3. Loss Function
2. We determine the empirical risk of the model using loss function
3. We train the model by minimizing the empirical risk

We call this universal approach ML 1-2-3!

The theory is pretty short; however,

how to execute each step of ML 1-2-3 is a pretty long story

that we are going to learn in the remaining of this course!