ass4

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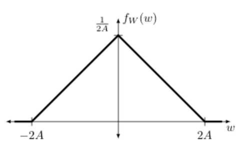
ass4

Carleton University
Department of Systems and Computer Engineering
Digital Communications
Assignment #4
Due on Wednesday, April 10, 2023

1. Suppose the input to the decision device at the receiver is modelled as

$$R = \begin{cases} 0 + W, & \text{if a 0 was transmitted} \\ 2A + W, & \text{if a 1 was transmitted} \end{cases},$$

where the noise variable, W, is characterized by the probability density function shown below:



$$f_W(w) = \begin{cases} \frac{1}{4A^2}(2A - |w|), & |w| \le 2A \\ 0, & |w| > 2A \end{cases}$$

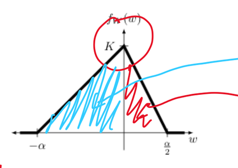
Suppose that 0's and 1's are equally likely to be transmitted. Find:

(a) The decision rule which leads to the minimum probability of error.
(b) The resultant probability of error.

2. Consider a binary antipodal communication system in which 0's and 1's are equally likely to be transmitted. Suppose the input to the decision device at the receiver is modelled as

$$R = \begin{cases} \sqrt{\mathcal{E}_b} + W, & \text{if a 0 was transmitted} \\ -\sqrt{\mathcal{E}_b} + W, & \text{if a 1 was transmitted} \end{cases},$$

where the noise variable, W, is characterized by the probability density function (pdf) shown below:



$$f_W(w) = \begin{cases} 0, & w < -\alpha \\ \frac{K}{\alpha}(w + \alpha), & -\alpha \le w < 0 \\ \frac{K}{\alpha}(-2w + \alpha), & 0 \le w < \alpha/2 \\ 0, & w \ge \alpha/2 \end{cases}$$

$-\frac{5}{3}\sqrt{\mathcal{E}_b} \le w \le 0.$

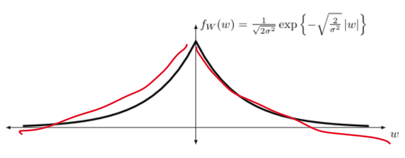
(a) Express K in terms of α so that $f_W(w)$ is a valid pdf.
(b) Sketch the likelihood functions of R given that 0 and 1 were sent. For the sketch, you may assume that $\alpha = \frac{5}{3}\sqrt{\mathcal{E}_b}$.
(c) If $\frac{4}{3}\sqrt{\mathcal{E}_b} \le \alpha \le 2\sqrt{\mathcal{E}_b}$, what rule should the decision device use to minimize the probability of a bit error? Express the rule in the simplest form possible.
(d) If $\alpha = \frac{5}{3}\sqrt{\mathcal{E}_b}$, what is the probability of a bit error if this decision rule is used?

3. Consider a binary communication system in which 0's and 1's are equally likely to be transmitted. Suppose the input to the decision device at the receiver is modelled as

$$R = \begin{cases} -\sqrt{\mathcal{E}_b} + W, & \text{if a 0 was transmitted} \\ \sqrt{\mathcal{E}_b} + W, & \text{if a 1 was transmitted} \end{cases},$$

where the noise variable, W, is characterized by the Laplacian pdf shown below.

(a) Derive the optimal decision rule to minimize the probability of an error. Express the rule in the simplest form possible.
(b) What is the probability of a bit error if this decision rule is used?
(c) What value for the ratio $\mathcal{E}_b/\sigma^2$ in decibels is required to achieve a probability of error of 10^{-5}?



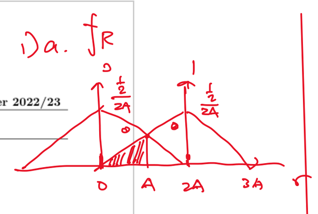
4. Suppose the linear (n, k) block code defined by the generator matrix

$$\underline{\underline{G}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

is used for error detection. Transmission is carried out over a binary symmetric channel with cross-over probability p.

(a) What is n, k, and the rate of this code?
(b) In a table, show which code words are transmitted for each of the possible messages.
(c) Find the minimum distance of the code.
(d) What is the random-error detection capability of this code?
(e) Give the parity check matrix for the code.
(f) What is the syndrome if $\underline{r} = 101010$ is received.
(g) Find P_C, the probability that a code word is received correctly.
(h) Find P_ε, the probability that a code word is received incorrectly, and no errors are detected.
(i) Find P_d, the probability that an error is detected.

1) a. f_R



R_1: $\frac{\frac{1}{4A}}{2A} r = \frac{1}{8A^2} r.$

$-\frac{\frac{1}{4A}}{2A} x + \frac{1}{4A}$

$-\frac{1}{8A^2} x + \frac{1}{4A}$

$R_0: -\frac{1}{8A^2} r + \frac{1}{2A}$

$R_0 = R_1$

$-\frac{1}{8A^2} r + \frac{1}{4A} = \frac{1}{8A^2} r.$

$-\frac{1}{8A^2} r - \frac{1}{8A^2} r = -\frac{1}{4A}$

$-\frac{1}{4A^2} r = -\frac{1}{4A}$

$r = \frac{1}{4A} \times 4A^2$

$r = A$

if $r \ge A$, $\hat{m} = 0$

if $r < A$, $\hat{m} = 1$

b. $P_{\varepsilon|0} = \int_0^A \frac{1}{8A^2} r \, dr.$

$= \frac{A^2}{2} \times \frac{1}{8A^2}$

$= \frac{1}{16}$

$P_{\varepsilon|1} = P_{\varepsilon|0} = \frac{1}{16}$

$P_{\varepsilon|\text{Avg}} = \frac{1}{2}\left[P_{\varepsilon|0} + P_{\varepsilon|1}\right]$

$= \frac{1}{2} \times \frac{2}{16}$

$= \frac{1}{16}$

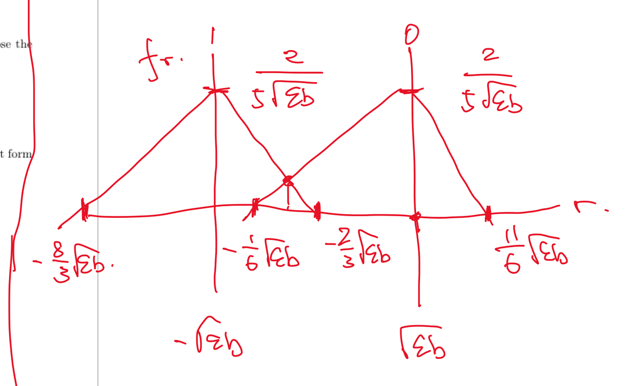
2) a. $\int_{-a}^{0} \frac{k}{a}(w+a)\,dw + \int_0^{\frac{a}{2}} \frac{k}{a}(-2w+a)\,dw = 1$

$\left(a + \frac{a}{2}\right)\frac{k}{2} = 1$

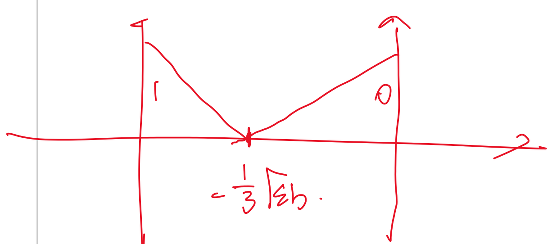
$\frac{3}{2} a k = 2$

$k = \frac{4}{3a}$

b. $a = \frac{5}{3}\sqrt{\varepsilon_b}$. $k = \frac{4}{5\sqrt{\varepsilon_b}}$



c. if $a = \frac{4}{3}\sqrt{\varepsilon_b}$

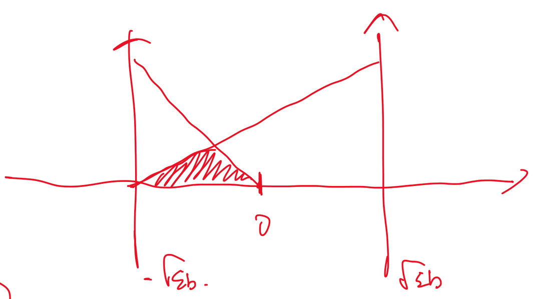


$\sqrt{\varepsilon_b} - \frac{4}{3}\sqrt{\varepsilon_b} = -\frac{1}{3}\sqrt{\varepsilon_b}.$

$-\sqrt{\varepsilon_b} + \frac{2}{3}\sqrt{\varepsilon_b} = -\frac{1}{3}\sqrt{\varepsilon_b}.$

$\rightarrow$ no error.

if $a = 2\sqrt{\varepsilon_b}$



$\sqrt{\varepsilon_b} - 2\sqrt{\varepsilon_b} = -\sqrt{\varepsilon_b}.$

$-\sqrt{\varepsilon_b} + \sqrt{\varepsilon_b} = 0.$

$\rightarrow$ max error.

to minimize bit error should choose $a = \frac{4}{3}\sqrt{\varepsilon_b}$.

if $r > -\frac{1}{3}\sqrt{\varepsilon_b}$, $\hat{m} = 0$

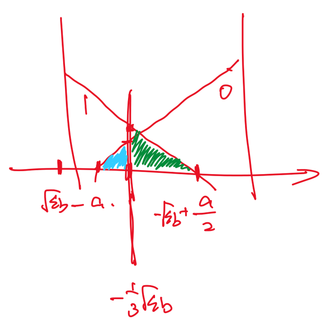
if $r < -\frac{1}{3}\sqrt{\varepsilon_b}$, $\hat{m} = 1$

d. $a = \frac{5}{3}\sqrt{\varepsilon_b}$.

decision rule.

$r > -\frac{1}{3}\sqrt{\varepsilon_b}$, $\hat{m} = 0$

$r < -\frac{1}{3}\sqrt{\varepsilon_b}$, $\hat{m} = 1$



$P_{\varepsilon|0} = \Pr\{\hat{m} = 1 \mid 0 \text{ sent}\}.$

$= \Pr\{r < -\frac{1}{3}\sqrt{\varepsilon_b}\}$

$= \Pr\{\sqrt{\varepsilon_b} + w < -\frac{1}{3}\sqrt{\varepsilon_b}\}.$

$= \Pr\{w < -\frac{4}{3}\sqrt{\varepsilon_b}\}.$

$= \int_{-\frac{5}{3}\sqrt{\varepsilon_b}}^{-\frac{4}{3}\sqrt{\varepsilon_b}} f_W(w)\,dw.$

$= \int_{-\frac{5}{3}\sqrt{\varepsilon_b}}^{-\frac{4}{3}\sqrt{\varepsilon_b}} \frac{k}{a}(w+a)\,dw.$

$k = \frac{4}{3a}$ $a = \frac{5}{3}\sqrt{\varepsilon_b}.$

$\Rightarrow k = \frac{4}{\cancel{3} \times \frac{5}{\cancel{3}}\sqrt{\varepsilon_b}} = \frac{4}{5\sqrt{\varepsilon_b}}$

$\frac{k}{a} = \frac{4}{5\sqrt{\varepsilon_b}} \times \frac{3}{5\sqrt{\varepsilon_b}} = \frac{12}{25\varepsilon_b}.$

$\frac{a}{2} = \frac{1}{2} \times \frac{5}{3}\sqrt{\varepsilon_b}.$

$= \frac{5}{6}\sqrt{\varepsilon_b}.$

$= \int_{-\frac{5}{3}\sqrt{\varepsilon_b}}^{-\frac{4}{3}\sqrt{\varepsilon_b}} \frac{12}{25\varepsilon_b}\left(w + \frac{5}{3}\sqrt{\varepsilon_b}\right)$

$= \frac{12}{25\varepsilon_b} \int_{-\frac{5}{3}\sqrt{\varepsilon_b}}^{-\frac{4}{3}\sqrt{\varepsilon_b}} w + \frac{5}{3}\sqrt{\varepsilon_b}\,dw.$

$= \frac{12}{25\varepsilon_b}\left[\frac{w^2}{2} + \frac{5}{3}\sqrt{\varepsilon_b}\,w\right]_{-\frac{5}{3}\sqrt{\varepsilon_b}}^{-\frac{4}{3}\sqrt{\varepsilon_b}}$

$= \frac{12}{25\varepsilon_b}\left(\left[\frac{\left(-\frac{4}{3}\sqrt{\varepsilon_b}\right)^2}{2} + \frac{5}{3}\sqrt{\varepsilon_b} \times -\frac{4}{3}\sqrt{\varepsilon_b}\right] - \left[\frac{\left(-\frac{5}{3}\sqrt{\varepsilon_b}\right)^2}{2} + \frac{5}{3}\sqrt{\varepsilon_b} \times -\frac{5}{3}\sqrt{\varepsilon_b}\right]\right)$

$= \frac{12}{25\varepsilon_b}\left(\frac{8}{9}\varepsilon_b - \frac{20}{9}\varepsilon_b - \left(\frac{25}{18}\varepsilon_b - \frac{25}{9}\varepsilon_b\right)\right)$

$= \frac{12}{25\cancel{\varepsilon_b}} \times \frac{1}{18}\cancel{\varepsilon_b}.$

$P_{\varepsilon|0} = \frac{2}{75}$

$P_{\varepsilon|1} = \Pr\{\hat{m} = 0 \mid 1 \text{ sent}\}.$

$= \Pr\{r > -\frac{1}{3}\sqrt{\varepsilon_b}\}$

$= \Pr\{-\sqrt{\varepsilon_b} + w > -\frac{1}{3}\sqrt{\varepsilon_b}\}.$ $W + \sqrt{\varepsilon_b}.$

$= \Pr\{w > -\frac{1}{3}\sqrt{\varepsilon_b} + \sqrt{\varepsilon_b}\}$

$= \Pr\{w > \frac{2}{3}\sqrt{\varepsilon_b}\}.$

$= \int_{\frac{2}{3}\sqrt{\varepsilon_b}}^{\frac{5}{6}\sqrt{\varepsilon_b}} \frac{k}{a}(-2w + a)\,dw.$

$= \int_{\frac{2}{3}\sqrt{\varepsilon_b}}^{\frac{5}{6}\sqrt{\varepsilon_b}} \frac{12}{25\varepsilon_b}\left(-2w + \frac{5}{3}\sqrt{\varepsilon_b}\right)dw.$

$= \frac{12}{25\varepsilon_b} \int_{\frac{2}{3}\sqrt{\varepsilon_b}}^{\frac{5}{6}\sqrt{\varepsilon_b}} -2w + \frac{5}{3}\sqrt{\varepsilon_b}\,dw.$

$= \frac{12}{25\varepsilon_b}\left[\frac{-\cancel{2}w^2}{\cancel{2}} + \frac{5}{3}\sqrt{\varepsilon_b}\,w\right]_{\frac{2}{3}\sqrt{\varepsilon_b}}^{\frac{5}{6}\sqrt{\varepsilon_b}}$

$= \frac{12}{25\varepsilon_b}\left[-\left(\frac{5}{6}\sqrt{\varepsilon_b}\right)^2 + \frac{5}{3}\sqrt{\varepsilon_b}\left(\frac{5}{6}\sqrt{\varepsilon_b}\right) - \left(-\left(\frac{2}{3}\sqrt{\varepsilon_b}\right)^2 + \frac{5}{3}\sqrt{\varepsilon_b}\left(\frac{2}{3}\sqrt{\varepsilon_b}\right)\right)\right]$

$= \frac{12}{25\varepsilon_b}\left[-\frac{25}{36}\varepsilon_b + \frac{25}{18}\varepsilon_b - \left(-\frac{4}{9}\varepsilon_b + \frac{10}{9}\varepsilon_b\right)\right]$

$= \frac{12}{25\cancel{\varepsilon_b}} \times \frac{1}{36}\cancel{\varepsilon_b}$

$P_{\varepsilon|1} = \frac{1}{75}$

4. (a) $n = 6$, $k = 3$.

$R = \frac{k}{n} = \frac{1}{2}$

(b)

m	c	
000	000000	
001	001011	3
010	010110	3
011	011101	4
100	100101	3
101	101110	4
110	110011	4
111	111000	3.

(c) $d_{min} = 3.$

(d) REDC $= d_{min} - 1 = 3 - 1 = 2.$

(e) $H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$

(f)

$S = rH^T$

$= [101010]\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [000].$

(g) $p = Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right) = Q\left(\sqrt{\frac{2 R \varepsilon_b}{N_0}}\right) = Q\left(\sqrt{2 \times \frac{3}{6} \times \frac{\varepsilon_{nb}}{N_0}}\right)$

$P_c = (1-p)^6$

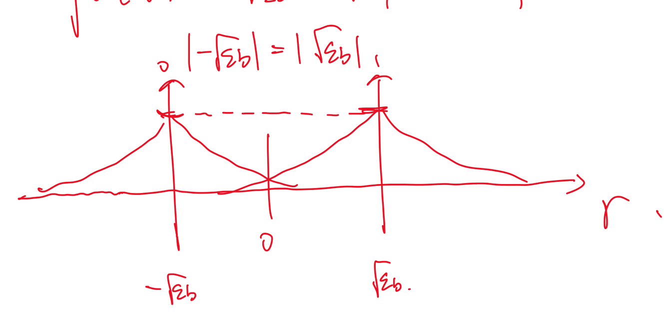
(h) $P_\varepsilon = 4p^3(1-p)^3 + 3p^4(1-p)^2$

(i) $P_d = 1 - P_c - P_\varepsilon.$

3.

a) $R = \begin{cases} -\sqrt{\varepsilon_b} + W, & 0 \text{ sent} \\ \sqrt{\varepsilon_b} + W, & 1 \text{ sent.} \end{cases}$

$f_W(w) = \frac{1}{\sqrt{2\sigma^2}} \exp\left\{-\sqrt{\frac{2}{\sigma^2}}|w|\right\}.$



if $R > 0$, $\hat{m} = 1$ $e^0 = 1$

if $R < 0$, $\hat{m} = 0$.

b) $P_{\varepsilon|0} = \Pr\{\hat{m} = 1, 0 \text{ sent}\}.$

$= \Pr\{R_1 > 0\}.$

$= \Pr\{\sqrt{\varepsilon_b} + W > 0\}.$

$= \Pr\{W > -\sqrt{\varepsilon_b}\}.$

$= \int_{-\sqrt{\varepsilon_b}}^{\infty} \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\sqrt{\frac{2}{\sigma^2}}|w|\right)dw.$

$= \frac{1}{\sqrt{2\sigma^2}}\left[-\sqrt{\frac{\sigma^2}{2}} \exp\left(-\sqrt{\frac{2}{\sigma^2}}|w|\right)\right]_{-\sqrt{\varepsilon_b}}^{\infty}$ $e^0 = 1$

$= -\frac{1}{\sqrt{2\cancel{\sigma^2}}} \times \frac{\sqrt{\cancel{\sigma^2}}}{\sqrt{2}} \times \left(e^{-\infty} - e^{-\sqrt{\frac{2}{\sigma^2}} \times -\sqrt{\varepsilon_b}}\right)$

$\downarrow$ 0

$= \frac{1}{2} e^{\sqrt{\frac{2\varepsilon_b}{\sigma^2}}}$

$P_{\varepsilon|0} = P_{\varepsilon|1} = \frac{1}{2} e^{\sqrt{\frac{2\varepsilon_b}{\sigma^2}}}$

c) $P_\varepsilon \le 10^{-5}$

$\frac{1}{2} e^{\sqrt{\frac{2\varepsilon_b}{\sigma^2}}} \le 10^{-5}$

$e^{\sqrt{\frac{2\varepsilon_b}{\sigma^2}}} \ge 2 \times 10^{-5}$

$\frac{\varepsilon_b}{\sigma^2} \ge \left(\frac{\ln(2 \times 10^{-5})}{2}\right)^2$

$\frac{\varepsilon_b}{\sigma^2} \ge 58.53$. (linear)

$\frac{\varepsilon_b}{\sigma^2}$ dB $= 10\log_{10}(58.53)$

$= 17.67$