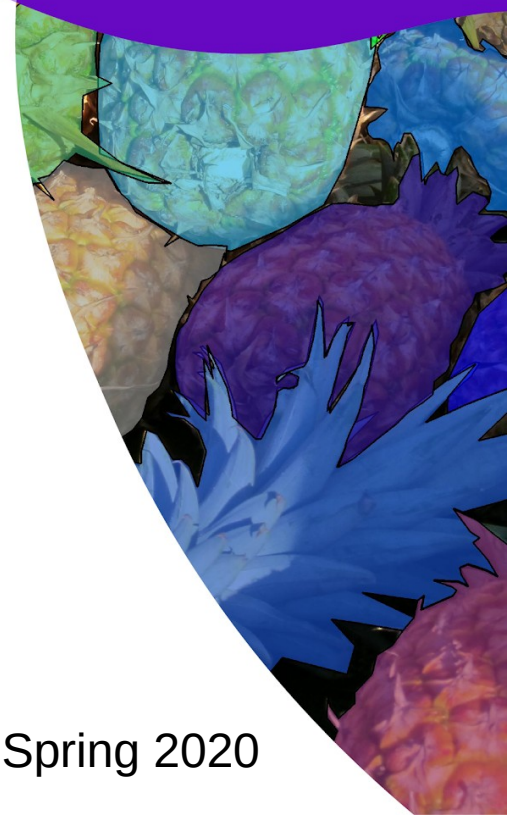
NEW YORK UNIVERSITY

Energy-Based Models (part 2)

http://bit.ly/DLSP20

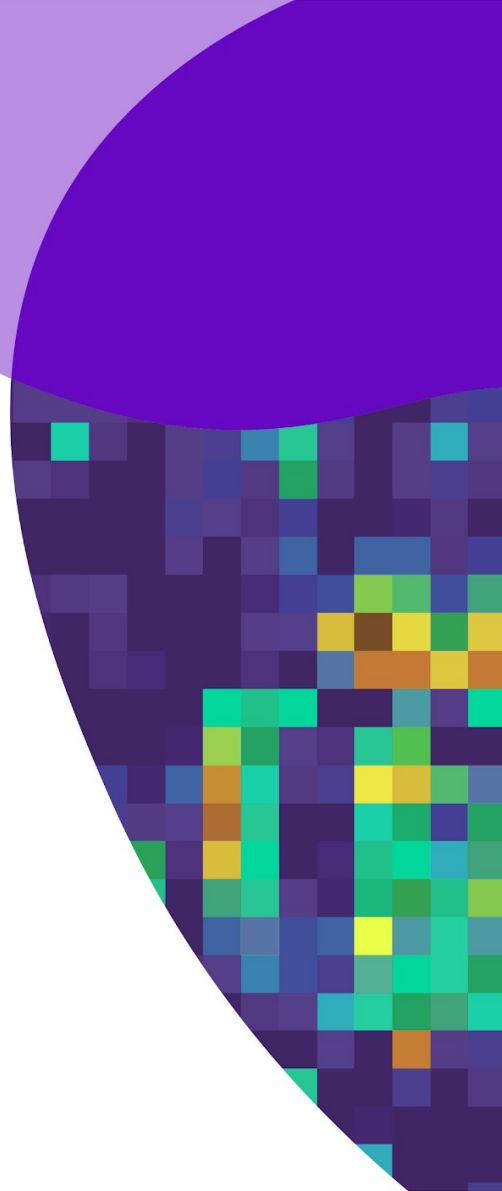
Yann LeCun
NYU - Courant Institute & Center for Data Science
Facebook AI Research
http://yann.lecun.com
TAs: Alfredo Canziani, Mark Goldstein



FACEBOOK AI

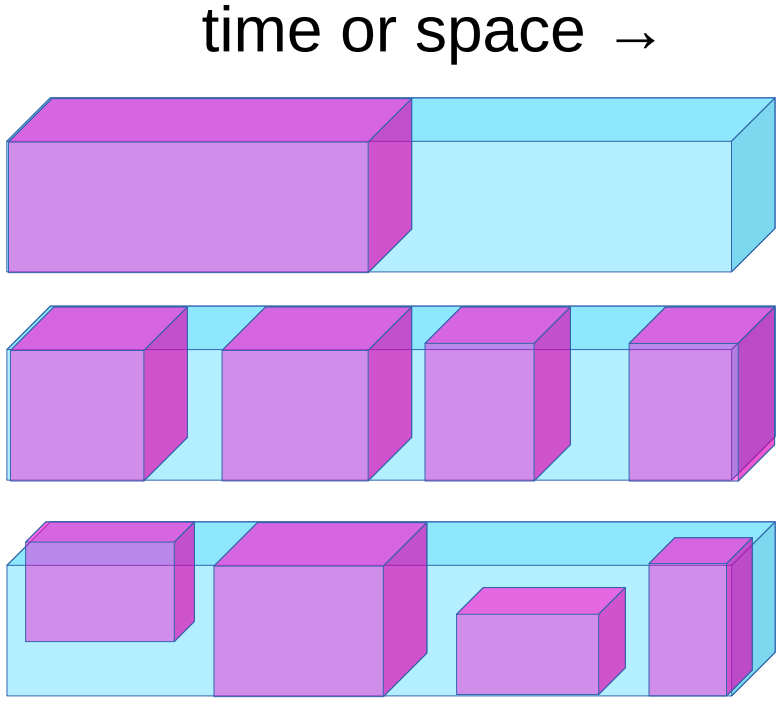
Self-Supervised Learning

Predict everything
from everything else



Self-Supervised Learning = Filling in the Blanks

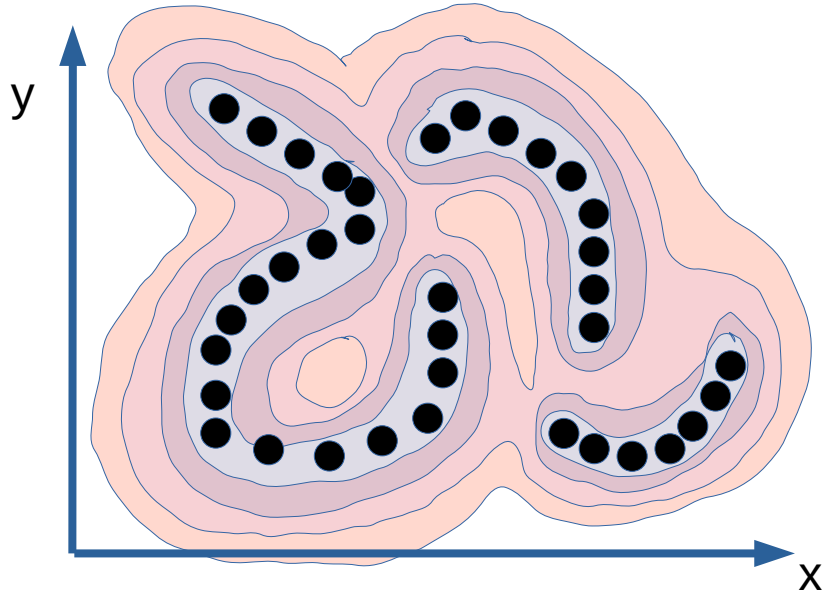
- **Predict any part of the input from any other part.**
- **Predict the future from the past.**
- **Predict the masked from the visible.**
- **Predict the any occluded part from all available parts.**



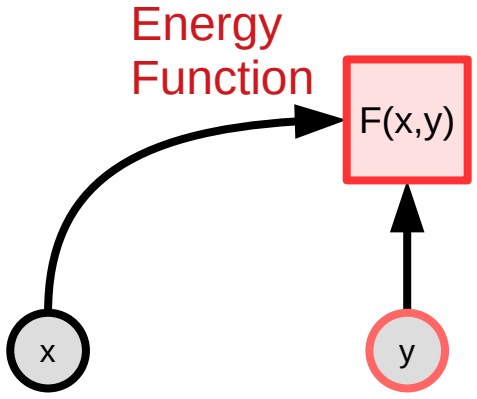
- **Pretend there is a part of the input you don't know and predict that.**
- **Reconstruction = SSL when any part could be known or unknown**

Energy-Based Model

- **Energy function that captures the x,y dependencies:**
  - Low energy near the data points. Higher energy everywhere else.
  - If y is continuous, F should be smooth and differentiable, so we can use gradient-based inference algorithms.

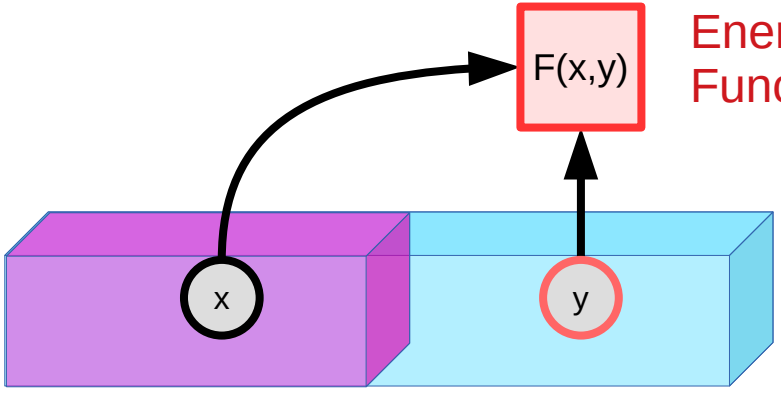


$$\check{y} = \text{argmin}_y F(x, y)$$

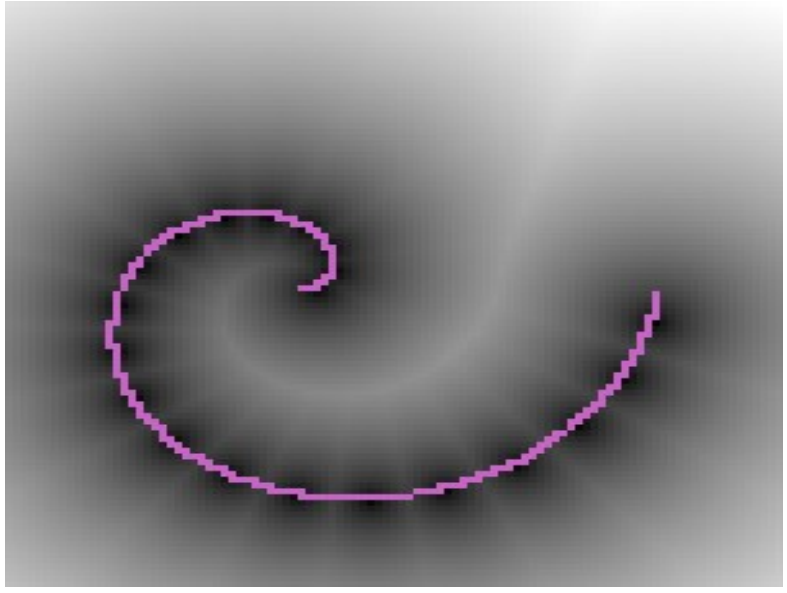


Energy-Based Model

- **Scalar-valued energy function: F(x,y)**
  - measures the compatibility between x and y
  - Low energy: y is good prediction from x
  - High energy: y is bad prediction from x
  - Inference: $\check{y} = argmin_y F(x, y)$



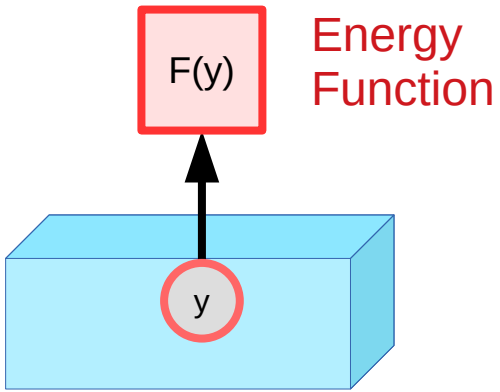
Dark = low energy (good)
Bright = high energy (bad)
Purple = data manifold



[Figure from M-A Ranzato's PhD thesis]

Energy-Based Model: unconditional version

- **Scalar-valued energy function: F(y)**
  - measures the compatibility between the components of y
  - If we don't know in advance which part of y is known and which part is unknown
  - Example: auto-encoders, generative models (energy = -log likelihood)



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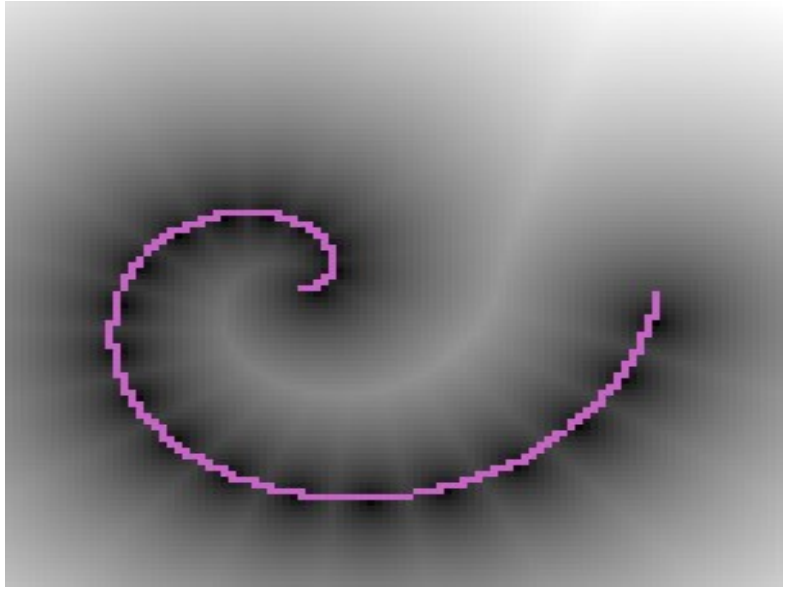


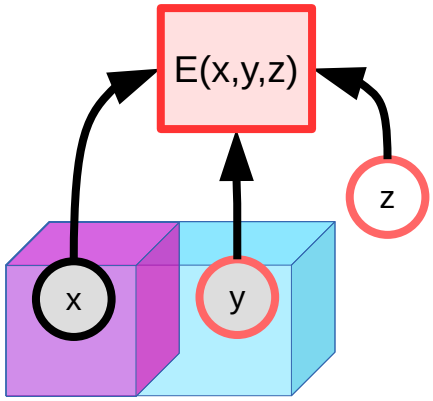
Latent-Variable EBM

- **Allowing multiple predictions through a latent variable**

- **Conditional:**

$$F(x,y)=min_z E(x,y,z)$$

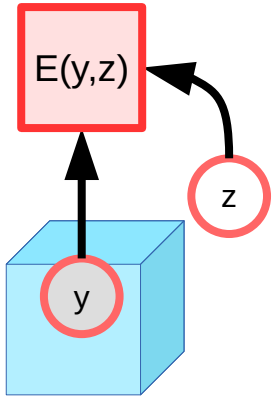
$$F(x,y)=-\frac{1}{\beta}\log[\int_z \exp(-\beta E(x,y,z))]$$



- **Unconditional**

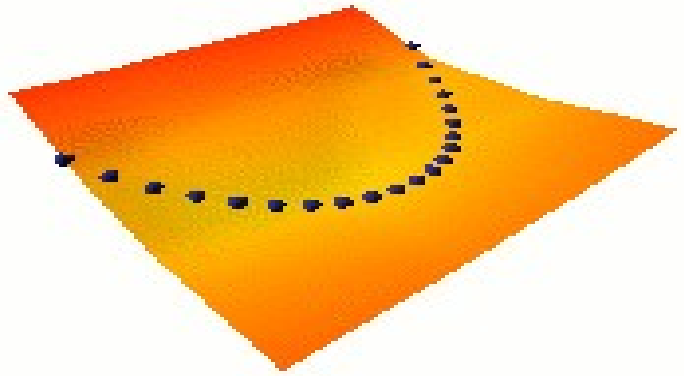
$$F(y)=min_z E(y,z)$$

$$F(y)=-\frac{1}{\beta}\log[\int_z \exp(-\beta E(y,z))]$$

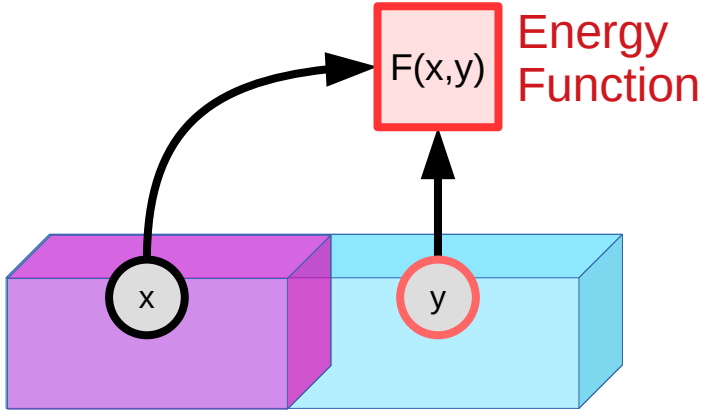


Training an Energy-Based Model

- **Parameterize F(x,y)**
- **Get training data (x[i], y[i])**
- **Shape F(x,y) so that:**
  - F(x[i], y[i]) is strictly smaller than F(x[i], y) for all y different from y[i]
  - Keep F smooth (probabilistic methods break that!)



- **Two classes of learning methods:**
  - 1. **Contrastive methods:** push down on F(x[i], y[i]), push up on other points F(x[i], y')
  - 2. **Architectural Methods:** build F(x,y) so that the volume of low energy regions is limited or minimized through regularization



Seven Strategies to Shape the Energy Function

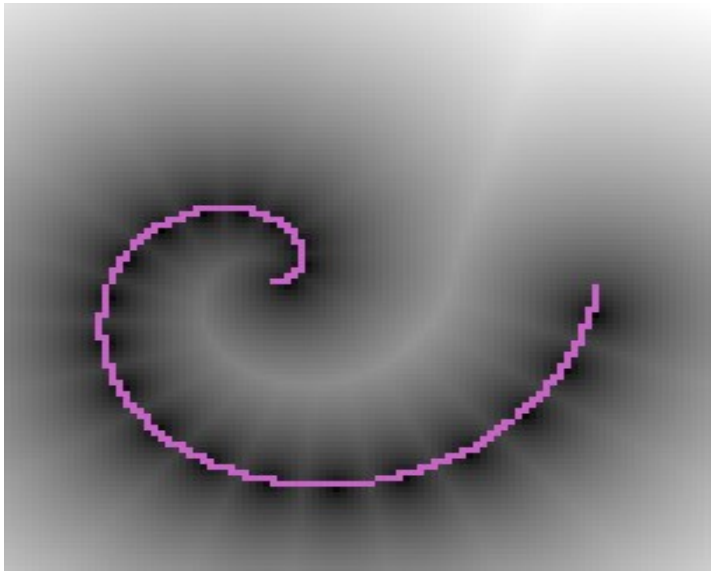
- **Contrastive: [they all are different ways to pick which points to push up]**
  - C1: push down of the energy of data points, push up everywhere else: Max likelihood (needs tractable partition function or variational approximation)
  - C2: push down of the energy of data points, push up on chosen locations: max likelihood with MC/MMC/HMC, Contrastive divergence, Metric learning, Ratio Matching, Noise Contrastive Estimation, Min Probability Flow, adversarial generator/GANs
  - C3: train a function that maps points off the data manifold to points on the data manifold: denoising auto-encoder, masked auto-encoder (e.g. BERT)
- **Architectural: [they all are different ways to limit the information capacity of the code]**
  - A1: build the machine so that the volume of low energy stuff is bounded: PCA, K-means, Gaussian Mixture Model, Square ICA…
  - A2: use a regularization term that measures the volume of space that has low energy: Sparse coding, sparse auto-encoder, LISTA, Variational auto-encoders
  - A3: F(x,y) = C(y, G(x,y)), make G(x,y) as "constant" as possible with respect to y: Contracting auto-encoder, saturating auto-encoder
  - A4: minimize the gradient and maximize the curvature around data points: score matching

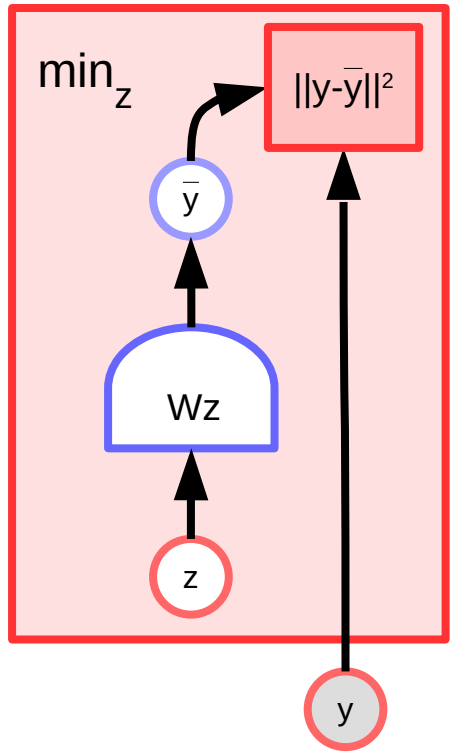
Latent-Variable EBM example: K-means

- **Decoder is linear, z is a 1-hot vector (discrete)**
- **Energy function:** $E(y,z)=\|y-Wz\|^2 \quad z \in 1\,hot$
- **Inference by exhaustive search**

$$F(y)=min_z E(y,z)$$

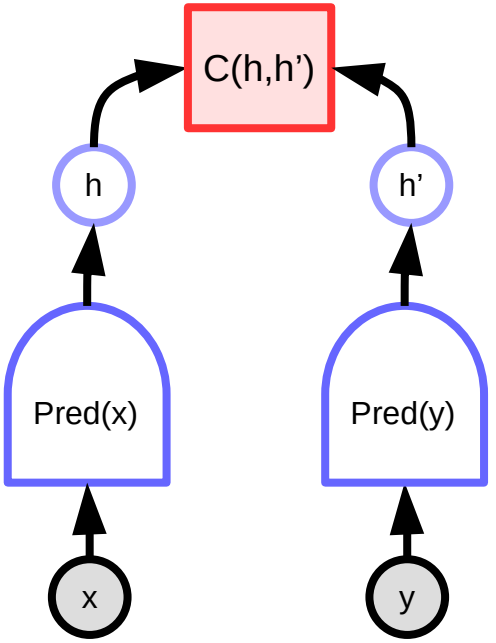
- **Volume of low-energy regions limited by number of prototypes k**





Contrastive Embedding

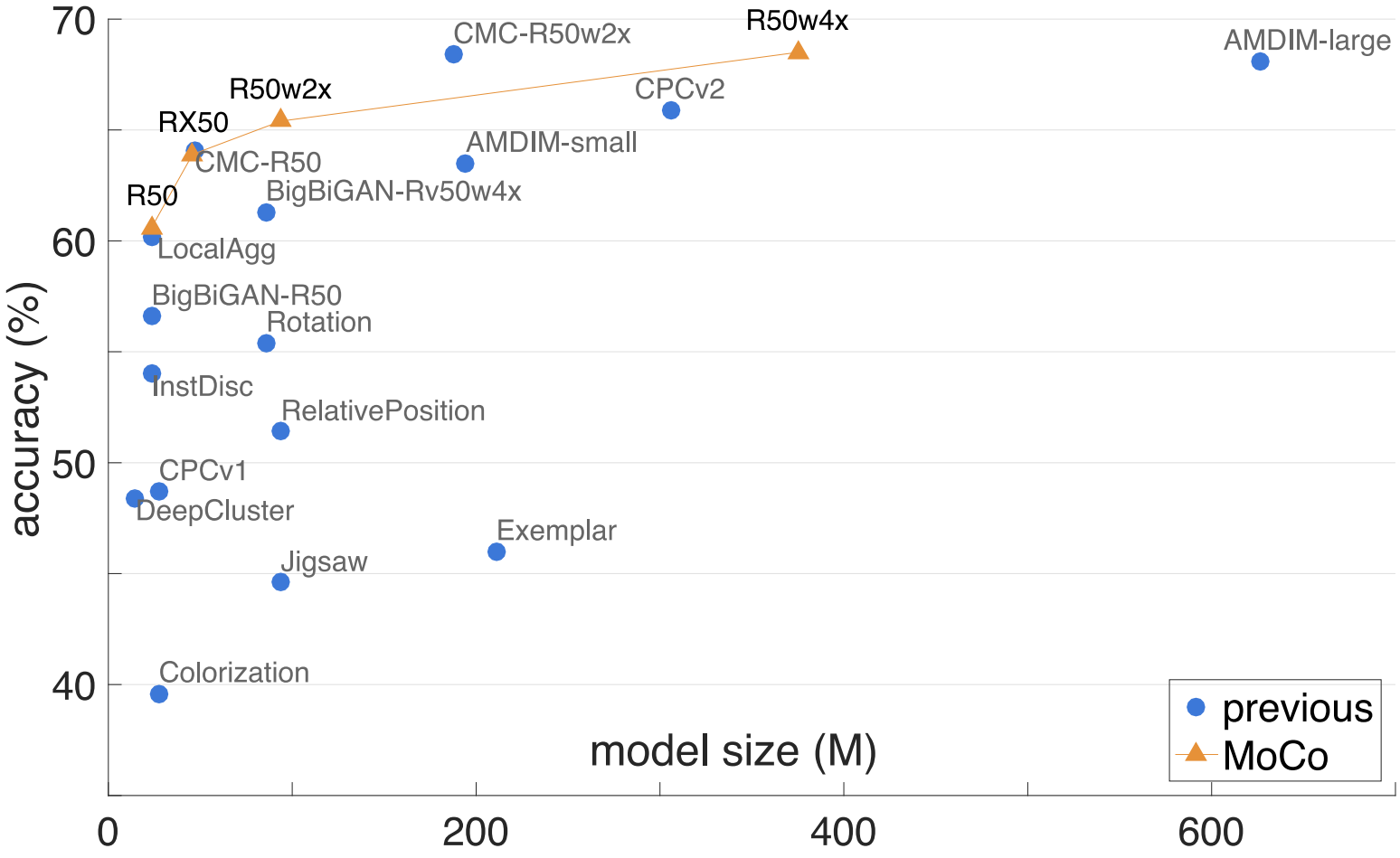
- **Distance measured in feature space**
- **Multiple “predictions” through feature invariance**
- **Siamese nets, metric learning** [YLC NIPS’93,CVPR’05,CVPR’06]
- **Advantage: no pixel-level reconstruction**
- **Difficulty: hard negative mining**
- **Successful examples for images:**
  - DeepFace [Taigman et al. CVPR’14]
  - PIRL [Misra & van der Maaten Arxiv:1912.01991]
  - MoCo [He et al. Arxiv:1911.05722]
  - SimCLR [Chen et al. Arxiv:2002.05709]
- **Video / Audio**
  - Temporal proximity [Taylor CVPR’11]
  - Slow feature [Goroshin NIPS’15]



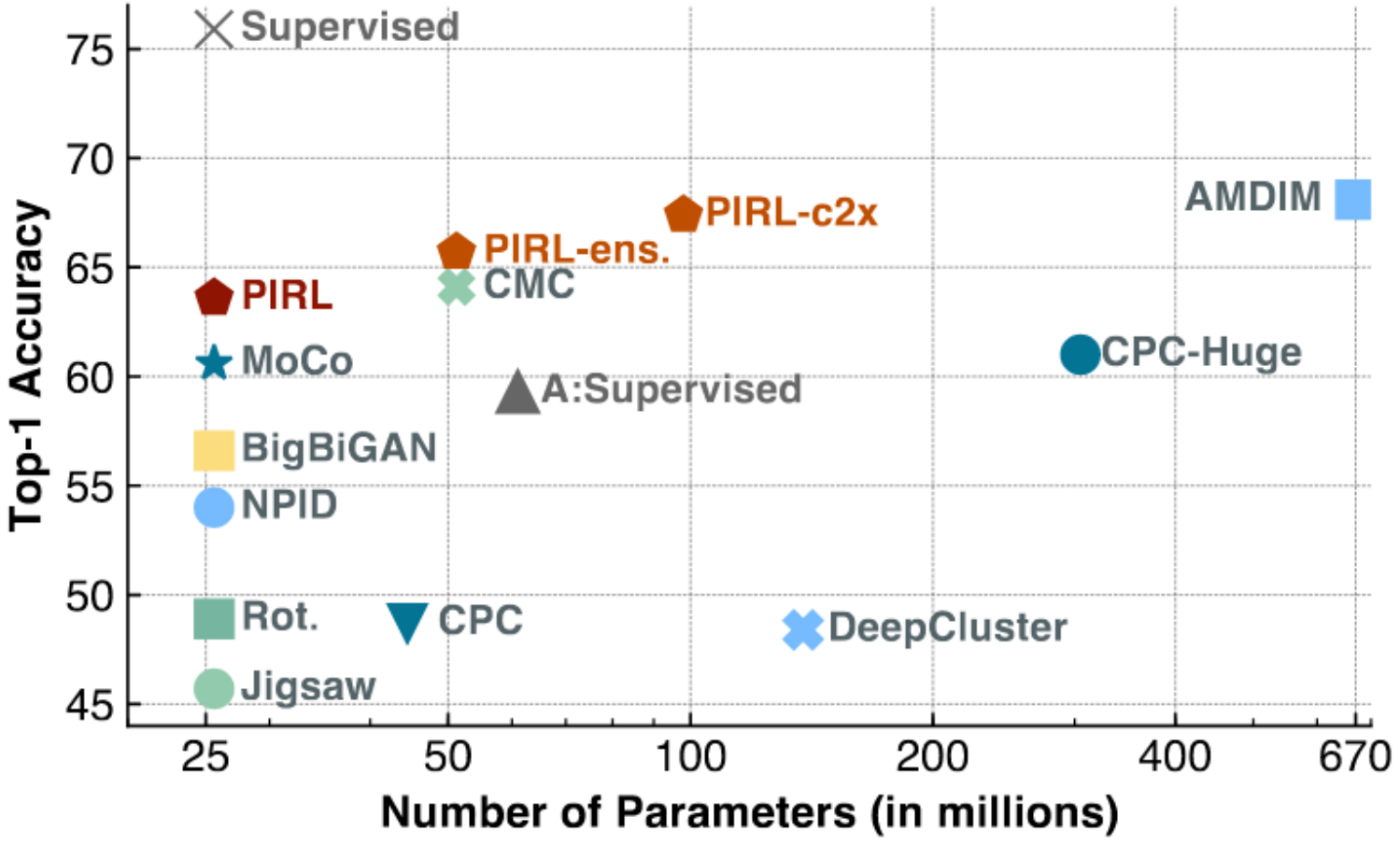
Positive pair:
Make F small

Negative pair:
Make F large

MoCo on ImageNet [He et al. Arxiv:1911.05722]



PIRL on ImageNet [Misra & van der Maaten Arxiv:1912.01991]



PIRL objective function: NCE

- **NCE = noise contrastive estimator**
  - s(u,v) = cosine similarity measure between ConvNet outputs
  - f(u), g(v) different “heads” of the convnets to compute representations

$$h(\mathbf{v_I}, \mathbf{v_{I^t}}) = \frac{\exp\left(\frac{s(\mathbf{v_I}, \mathbf{v_{I^t}})}{\tau}\right)}{\exp\left(\frac{s(\mathbf{v_I}, \mathbf{v_{I^t}})}{\tau}\right) + \sum_{\mathbf{I'} \in \mathcal{D}_N} \exp\left(\frac{s(\mathbf{v_{I^t}}, \mathbf{v_{I'}})}{\tau}\right)}$$

$$L_{\text{NCE}}\left(\mathbf{I}, \mathbf{I}^t\right) = -\log\left[h\left(f(\mathbf{v_I}), g(\mathbf{v_{I^t}})\right)\right] - \sum_{\mathbf{I'} \in \mathcal{D}_N} \log\left[1 - h\left(g(\mathbf{v}_\mathbf{I}^t), f(\mathbf{v_{I'}})\right)\right]$$

PIRL Results on ImageNet

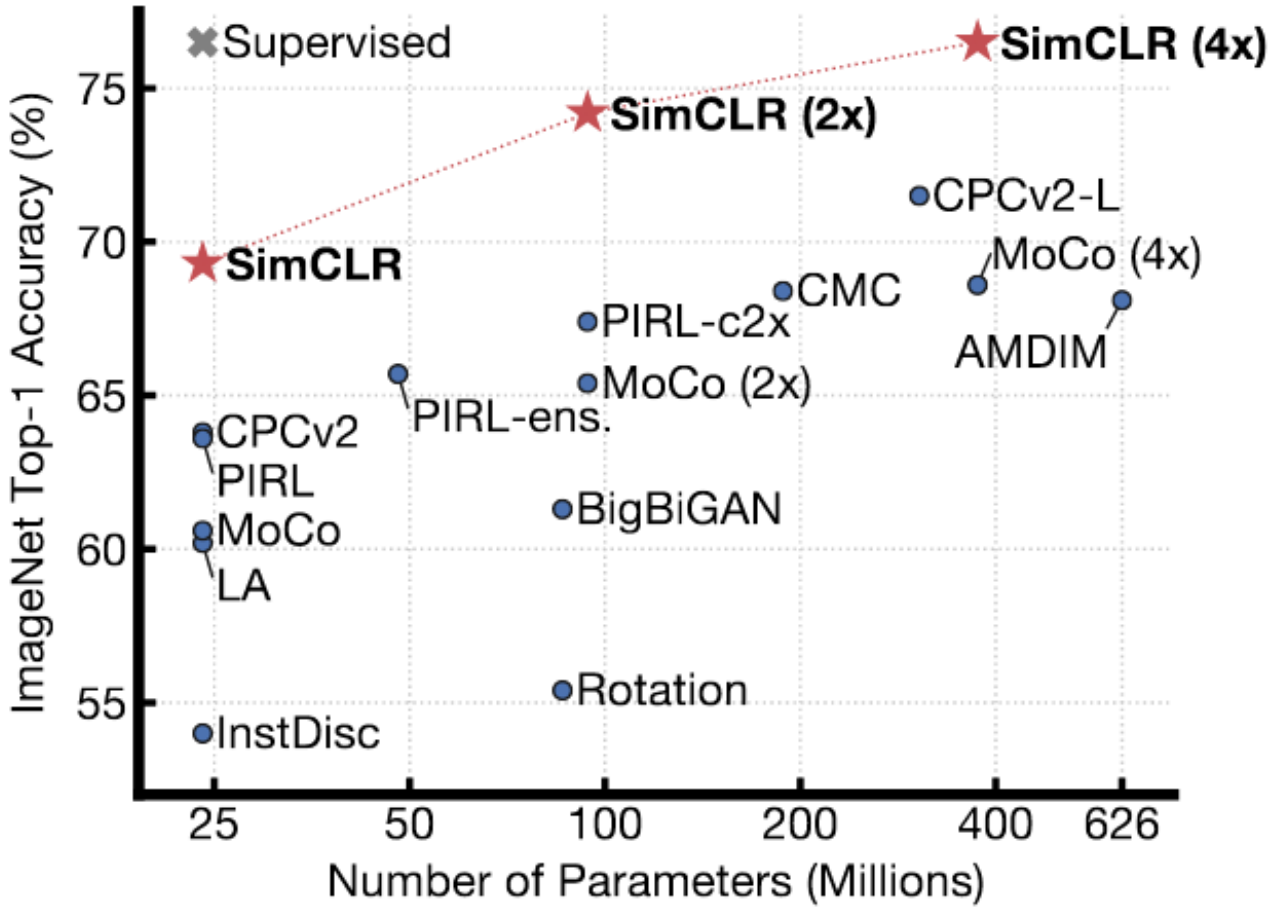
- **Single-crop top-5 accuracy**
- **Contrastive training on full training set.**
- **Supervised fine tuning on 1% and 10% of training samples**

Method	Data fraction → Backbone	1% Top-5 Accuracy	10% Top-5 Accuracy
Random initialization [72]	R-50	22.0	59.0
NPID [72]	R-50	39.2	77.4
Jigsaw [19]	R-50	45.3	79.3
NPID++ [72]	R-50	52.6	81.5
VAT + Ent Min. [20, 45]	R-50v2	47.0	83.4
S^4L Exemplar [75]	R-50v2	47.0	83.7
S^4L Rotation [75]	R-50v2	53.4	**83.8**
PIRL (**ours**)	R-50	**57.2**	**83.8**
Colorization [36]	R-152	29.8	62.0
CPC-Largest [26]	R-170 and R-11	64.0	84.9

Table 3: Semi-supervised learning on ImageNet. Single-crop top-5 accuracy on the ImageNet validation set of self-supervised models that are finetuned on 1% and 10% of the ImageNet training data, following [72]. All numbers except for Jigsaw, NPID++ and PIRL are adopted from the corresponding papers. Best performance is **boldfaced**.

SimCLR [Chen et al. Arxiv:2002.05709]

- **Sophisticated corruption / data augmentation methods**
- **Massive amount of training (on TPU)**
  - Shows the limits of scalability of contrastive methods



SimCLR [Chen et al. Arxiv:2002.05709]

- **ImageNet accuracy with supervised training on 1% and 10% of training data**
  - After SimCLR contrastive pretraining.

Method	Architecture	Label fraction (Top 5) 1%	10%
Methods using other label-propagation:			
Pseudo-label	ResNet50	51.6	82.4
VAT+Entropy Min.	ResNet50	47.0	83.4
UDA (w. RandAug)	ResNet50	-	88.5
FixMatch (w. RandAug)	ResNet50	-	89.1
S4L (Rot+VAT+En. M.)	ResNet50 (4×)	-	91.2
Methods using representation learning only:			
InstDisc	ResNet50	39.2	77.4
BigBiGAN	RevNet-50 (4×)	55.2	78.8
PIRL	ResNet-50	57.2	83.8
CPC v2	ResNet-161(∗)	77.9	91.2
SimCLR (ours)	ResNet-50	75.5	87.8
SimCLR (ours)	ResNet-50 (2×)	83.0	91.2
SimCLR (ours)	ResNet-50 (4×)	**85.8**	**92.6**

Table 7. ImageNet accuracy of models trained with few labels.

SimCLR [Chen et al. Arxiv:2002.05709]

- **Training a linear classifier on the pre-trained features**

Method	Architecture	Param.	Top 1	Top 5
Methods using ResNet-50:				
Local Agg.	ResNet-50	24	60.2	-
MoCo	ResNet-50	24	60.6	-
PIRL	ResNet-50	24	63.6	-
CPC v2	ResNet-50	24	63.8	85.3
SimCLR (ours)	ResNet-50	24	**69.3**	**89.0**
Methods using other architectures:				
Rotation	RevNet-50 (4×)	86	55.4	-
BigBiGAN	RevNet-50 (4×)	86	61.3	81.9
AMDIM	Custom-ResNet	626	68.1	-
CMC	ResNet-50 (2×)	188	68.4	88.2
MoCo	ResNet-50 (4×)	375	68.6	-
CPC v2	ResNet-161 (∗)	305	71.5	90.1
SimCLR (ours)	ResNet-50 (2×)	94	74.2	92.0
SimCLR (ours)	ResNet-50 (4×)	375	**76.5**	**93.2**

Table 6. ImageNet accuracies of linear classifiers trained on representations learned with different self-supervised methods.

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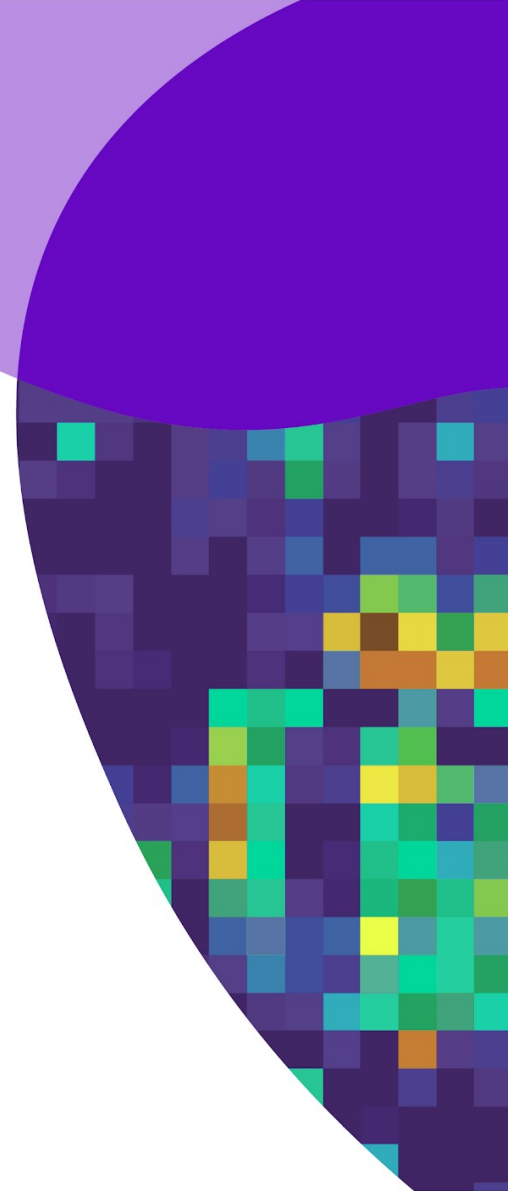
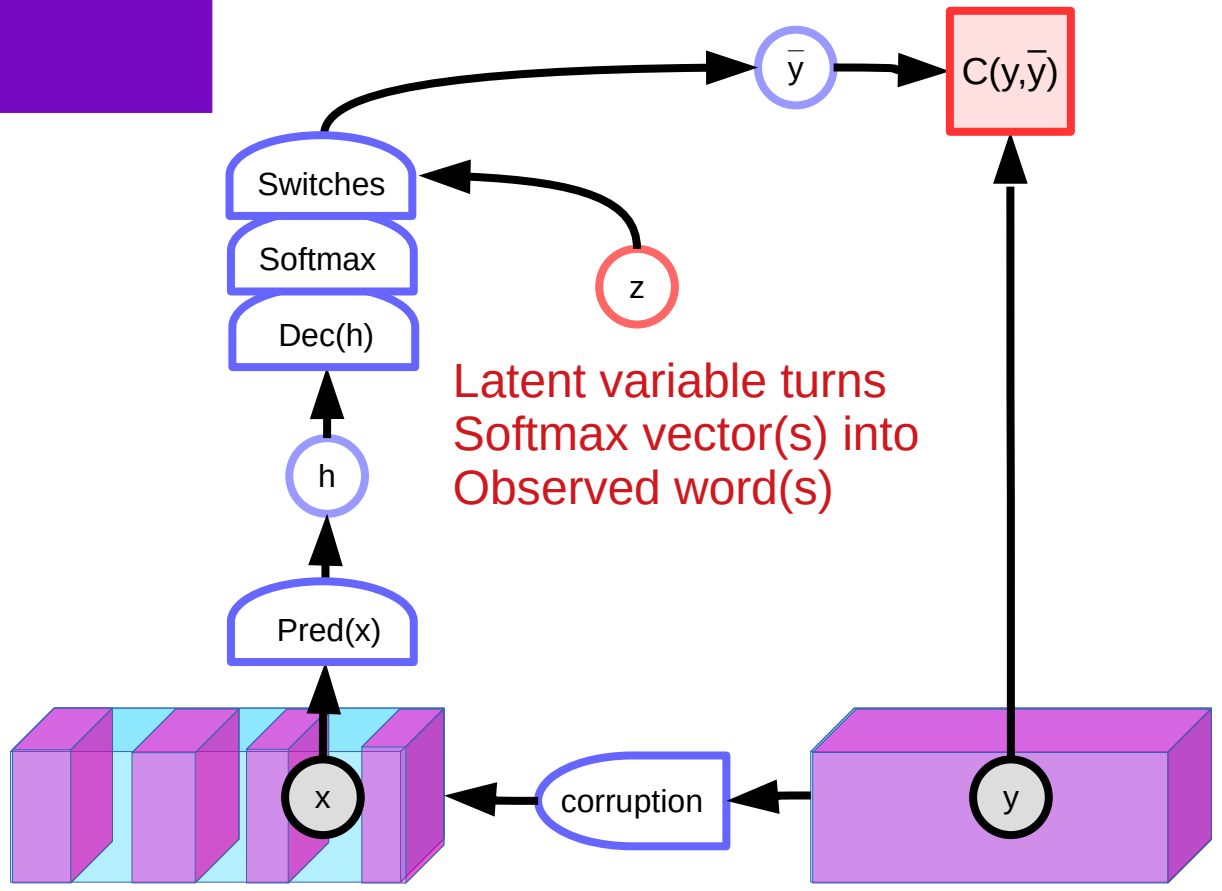
Denoising Auto-Encoder

Contrastive Self-Supervised Learning Method



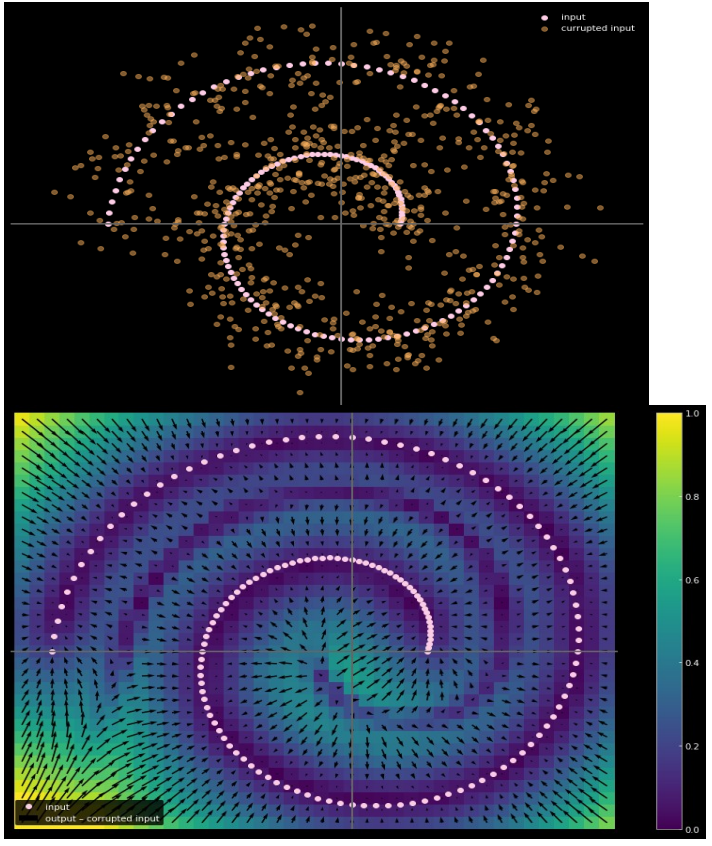
Denoising AE: discrete

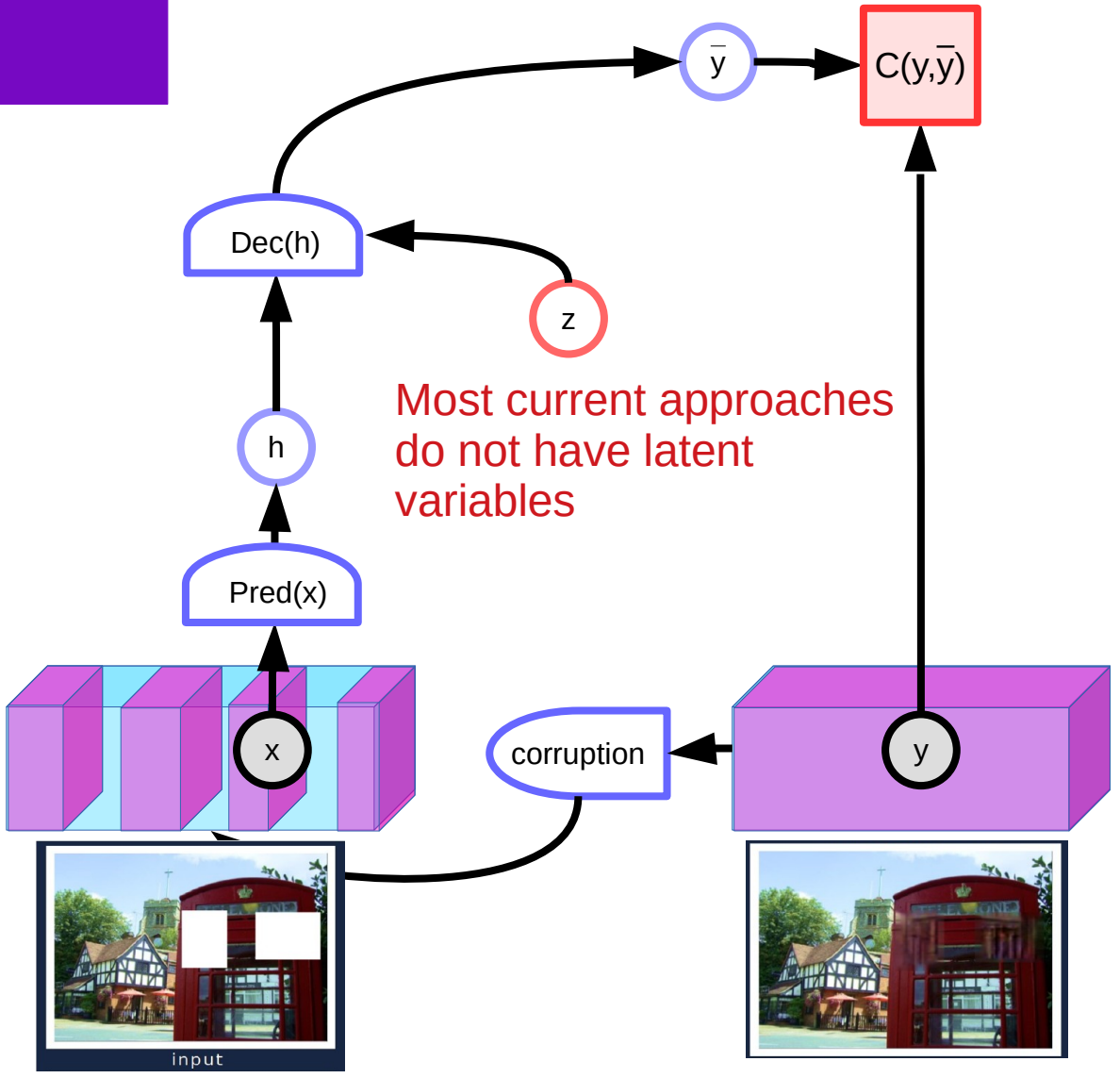
- **[Vincent et al. JMLR 2008]**
- **Masked Auto-Encoder**
  - [BERT et al.]
- **Issues:**
  - latent variables are in output space
  - No abstract LV to control the output
  - How to cover the space of corruptions?



Denoising AE: continuous

- **Image inpainting** [Pathak 17]
- Latent variables? GAN?





Other Contrastive Methods

- **contrastive divergence, Ratio Matching, Noise Contrastive Estimation, Minimum Probability Flow**
- **Contrastive divergence: basic idea**
  - Pick a training sample, lower the energy at that point
  - From the sample, move down in the energy surface with noise
  - Stop after a while
  - Push up on the energy of the point where we stopped
  - This creates grooves in the energy surface around data manifolds
  - CD can be applied to any energy function (not just RBMs)
- **Persistent CD: use a bunch of "particles" and remember their positions**
  - Make them roll down the energy surface with noise
  - Push up on the energy wherever they are
  - Faster than CD
- **RBM** $E(Y,Z) = -Z^T WY$ $\qquad E(Y) = -\log \sum_z e^{Z^T WY}$

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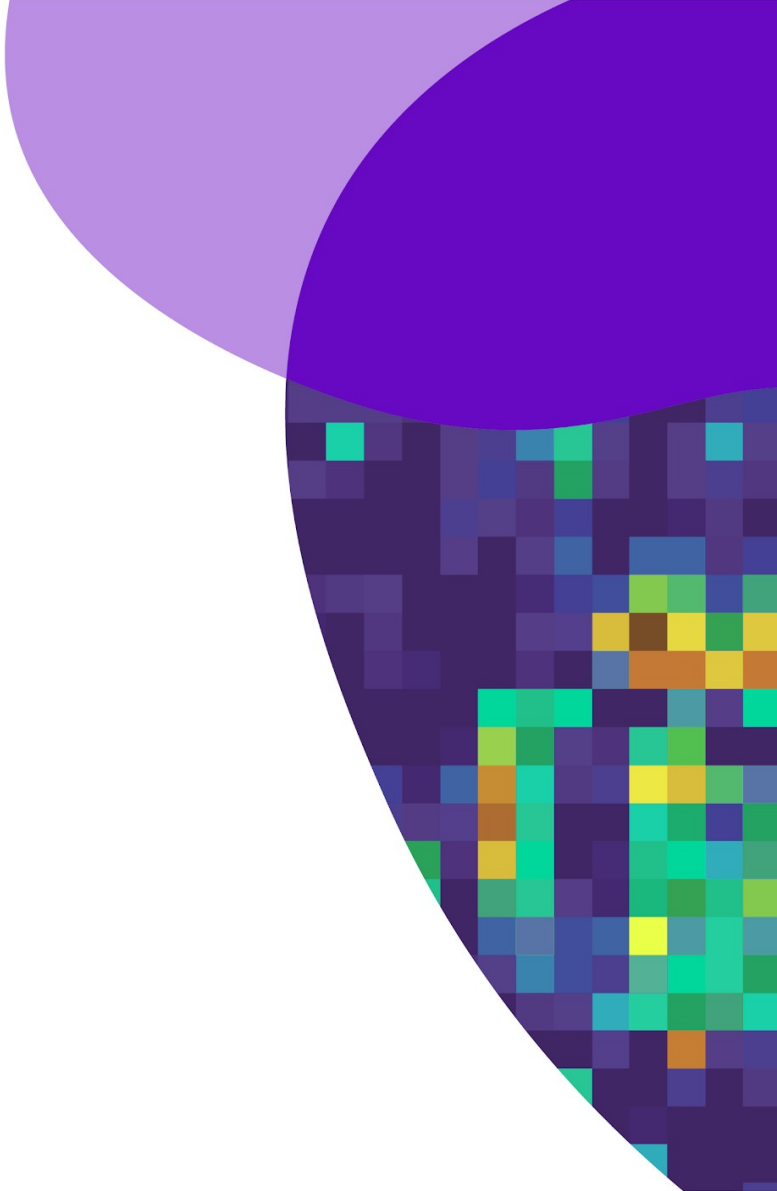
Regularized Latent-Variable EBM



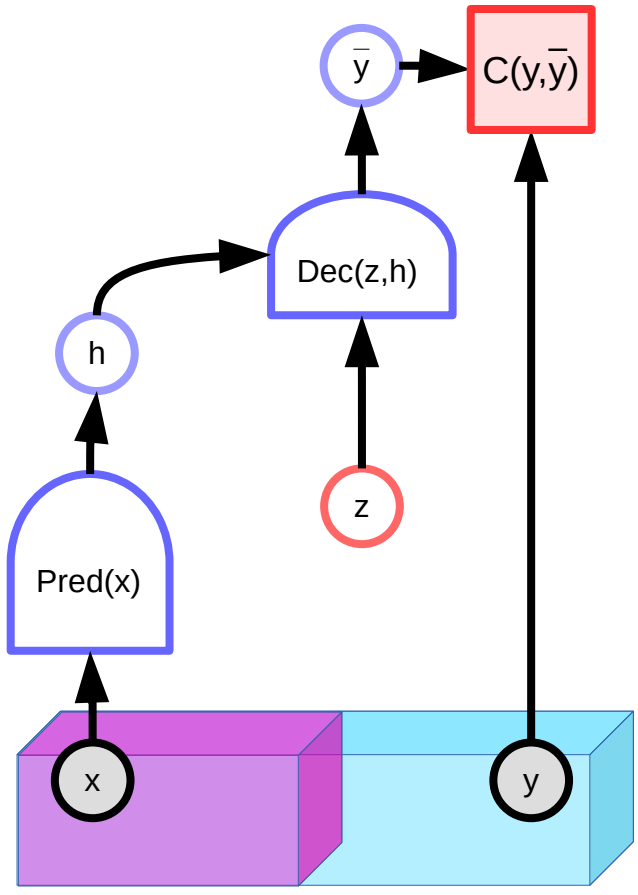
Architectural SSL methods
(non constrastive)

Prediction with Latent Variables

- **If the Latent has too much capacity...**
  - e.g. if it has the same dimension as y
- **... then the entire y space could be perfectly reconstructed**

$$E(x, y, z) = C(y, \mathrm{Dec}(\mathrm{Pred}(x), z))$$

- **For every y, there is always a z that will reconstruct it perfectly**
  - The energy function would be zero everywhere
  - This is no a good model….
- **Solution: limiting the information capacity of the latent variable z.**

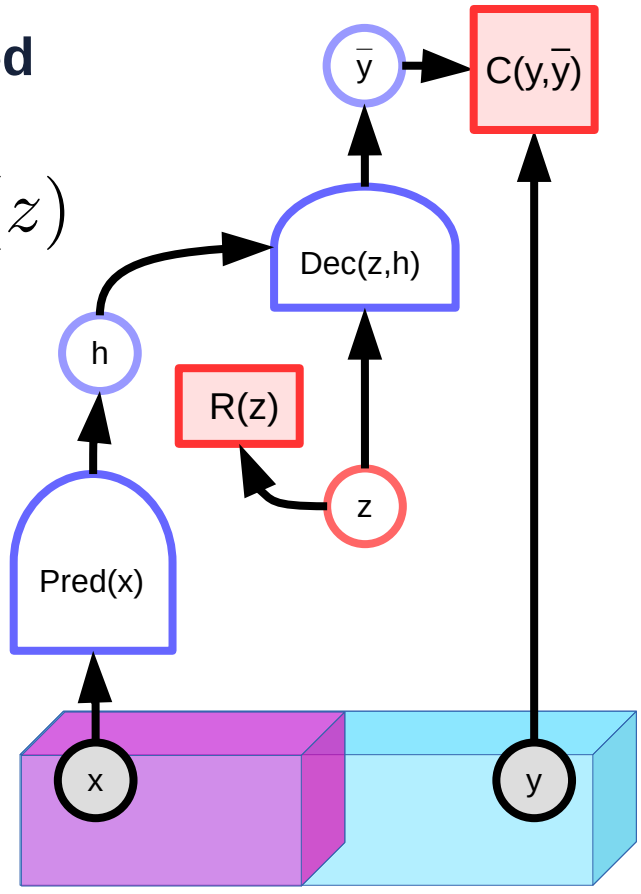


Regularized Latent Variable EBM

- **Regularizer R(z) limits the information capacity of z**
- **Without regularization, every y may be reconstructed exactly (flat energy surface)**

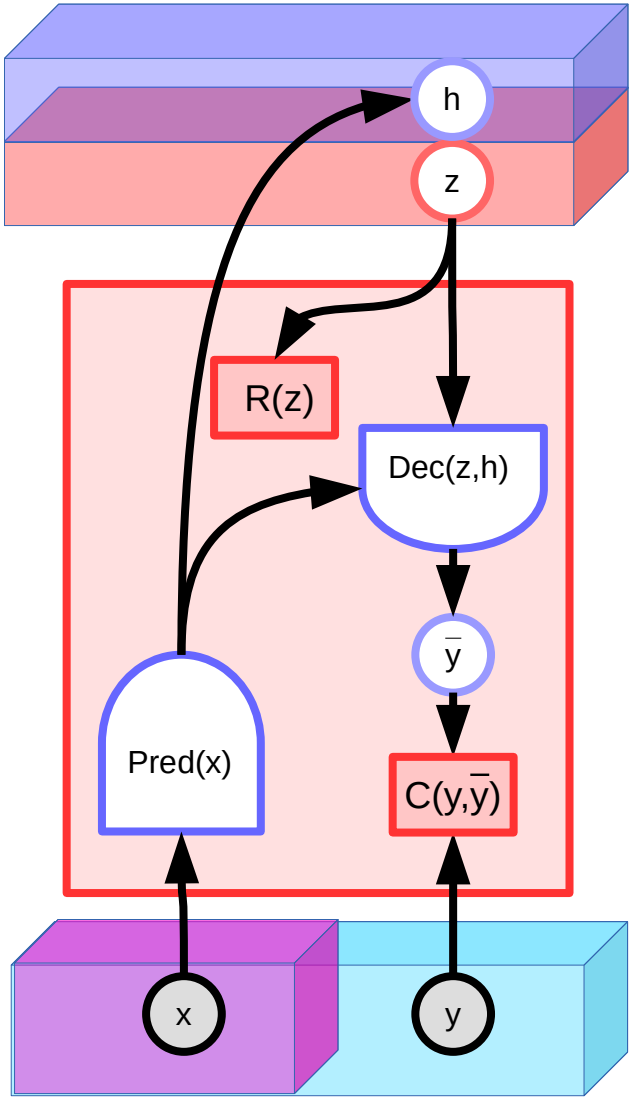
$$E(x, y, z) = C(y, \mathrm{Dec}(\mathrm{Pred}(x), z)) + \lambda R(z)$$

- **Examples of R(z):**
  - Effective dimension
  - Quantization / discretization
  - L0 norm (# of non-0 components)
  - L1 norm with decoder normalization
  - Maximize lateral inhibition / competition
  - Add noise to z while limiting its L2 norm (VAE)
  - <your_information_throttling_method_goes_here>



Sequence → Abstract Features

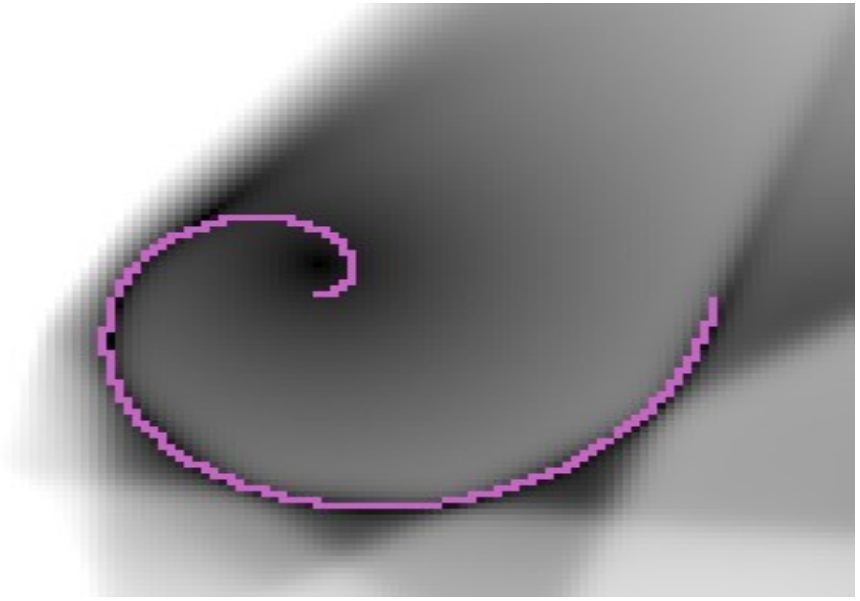
- **Regularized LV EBM is passed over a sequence (e.g. a video, audio, text)**
- **The sequence of corresponding h and z is collected**
  - It contains all the information about the input sequence
  - h contains the information in x that is useful to predict y
  - z contains the complementary information, not present in x or h.
- **Several such SSL modules can be stacked to learn hierarchical representations of sequences**

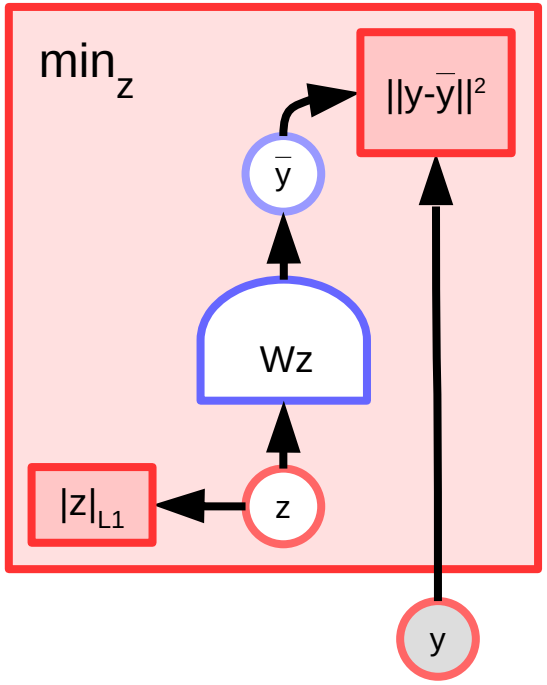


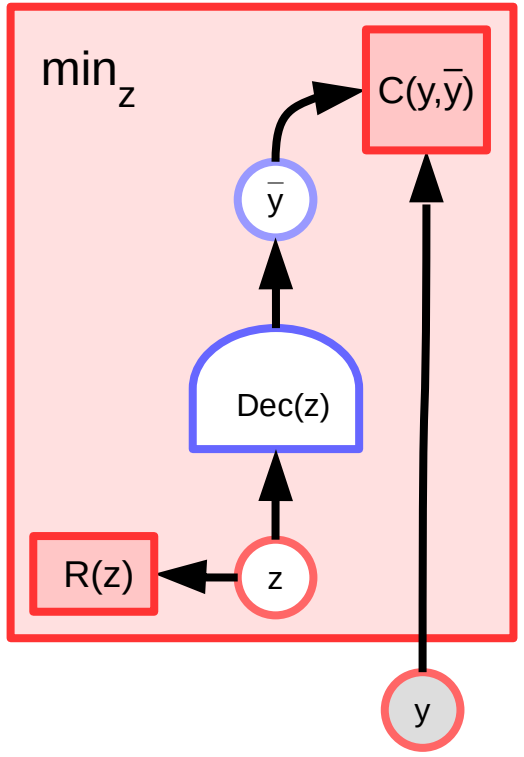
Unconditional Regularized Latent Variable EBM

- **Unconditional form. Reconstruction. No x, no predictor.**
- **Example: sparse modeling**
  - Linear decoder
  - L1 regularizer on Z

$$E(y,z) = \|y - Wz\|^2 + \lambda |z|$$





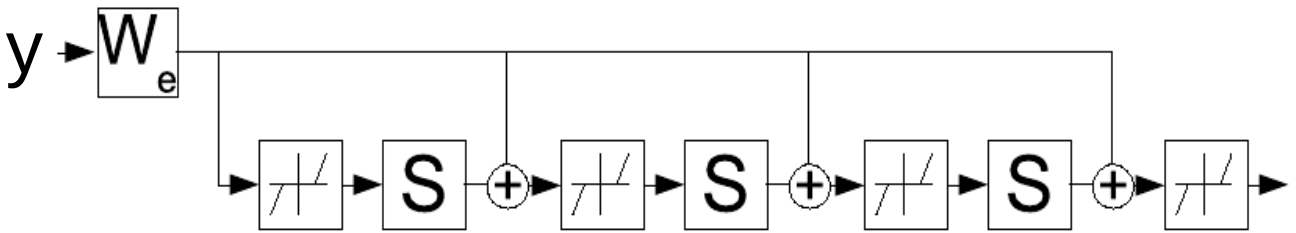


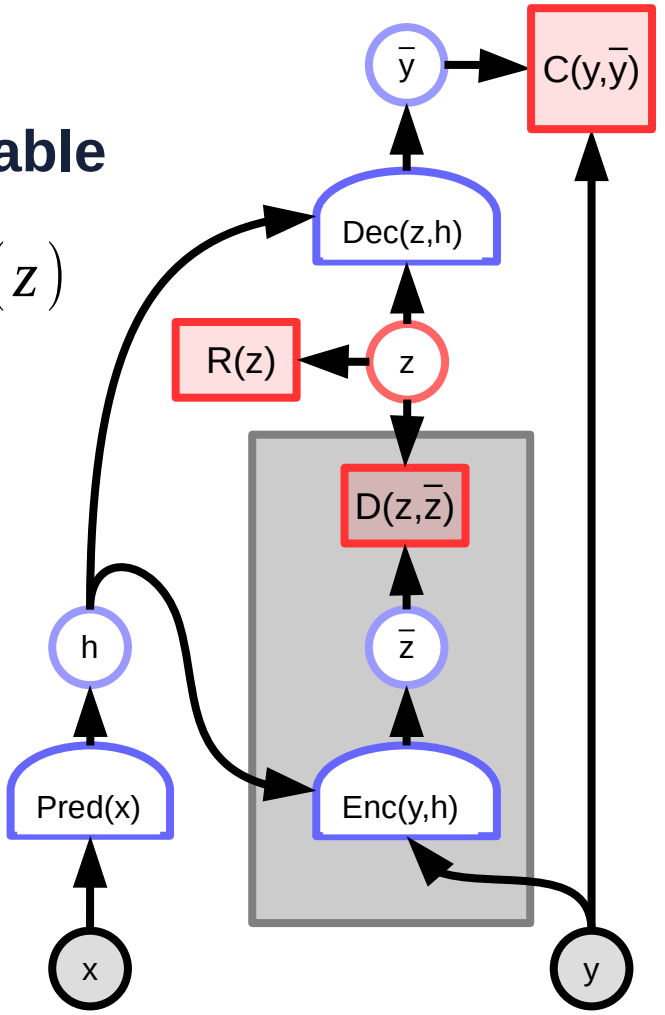
LatVar inference is expensive!

- **Let's train an encoder to predict the latent variable**

$$E(x,y,z)=C(y,Dec(z,h))+D(z,Enc(x,y))+\lambda R(z)$$

- **Predictive Sparse Modeling**
  - R(z) = L1 norm of z
  - Dec(z,h) gain must be bounded (clipped weights)
  - Sparse Auto-Encoder
  - LISTA [Gregor ICML 2010]

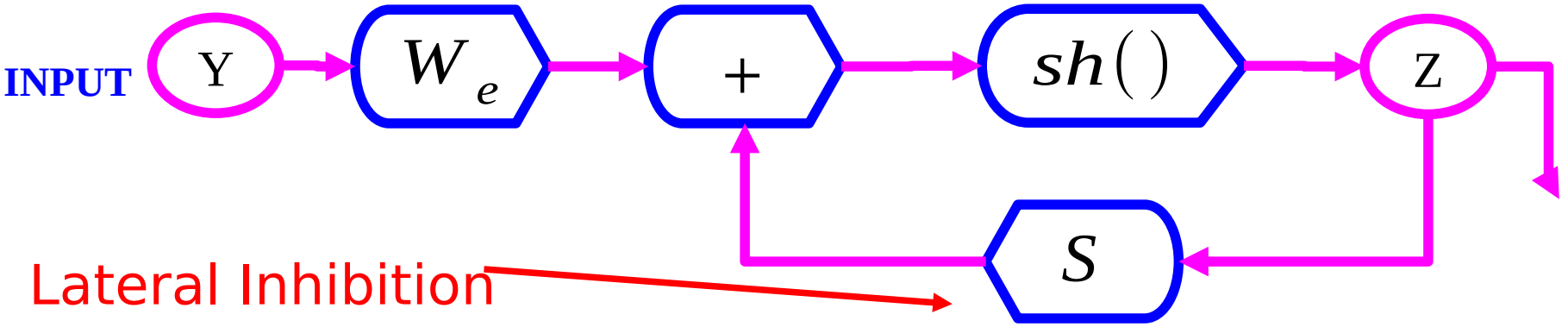




Giving the "right" structure to the encoder

- **ISTA/FISTA: iterative algorithm that converges to optimal sparse code**

[Gregor & LeCun, ICML 2010], [Bronstein et al. ICML 2012], [Rolfe & LeCun ICLR 2013]

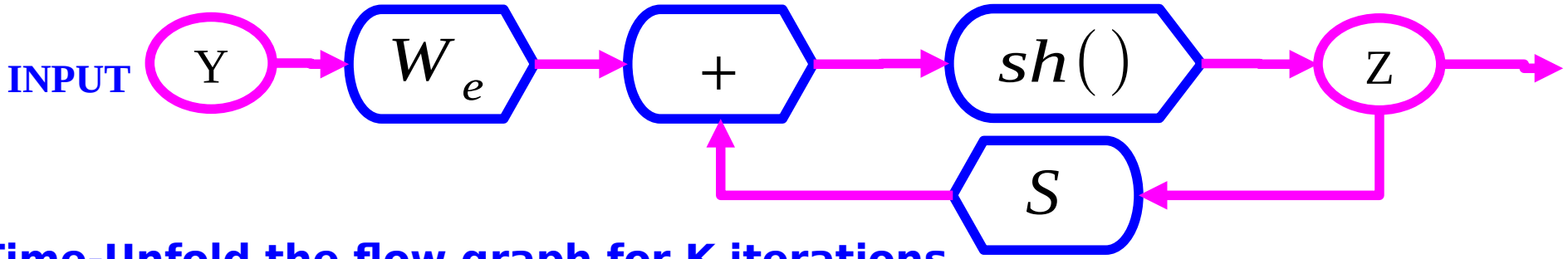


$$Z(t+1) = \text{Shrinkage}_{\lambda/L}\left[Z(t) - \frac{1}{L}W_d^T(W_d Z(t) - Y)\right]$$

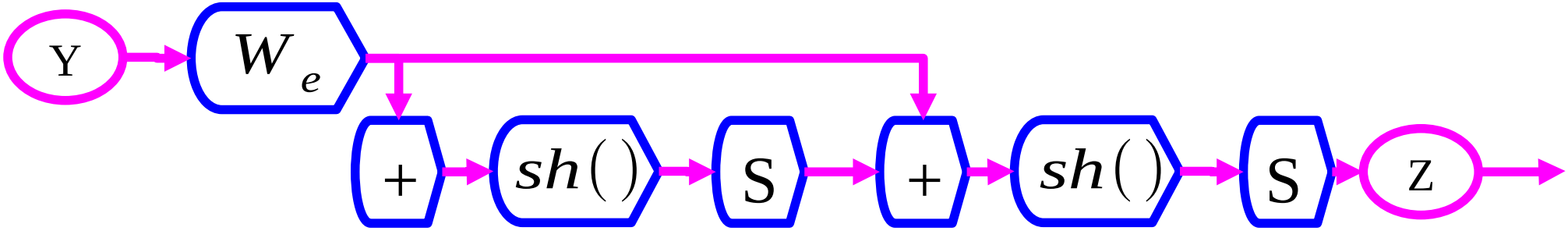
$$Z(t+1) = \text{Shrinkage}_{\lambda/L}\left[W_e^T Y + SZ(t)\right]; \quad W_e = \frac{1}{L}W_d; \quad S = I - \frac{1}{L}W_d^T W_d$$

LISTA: Train We and S matrices to give a good approximation quickly

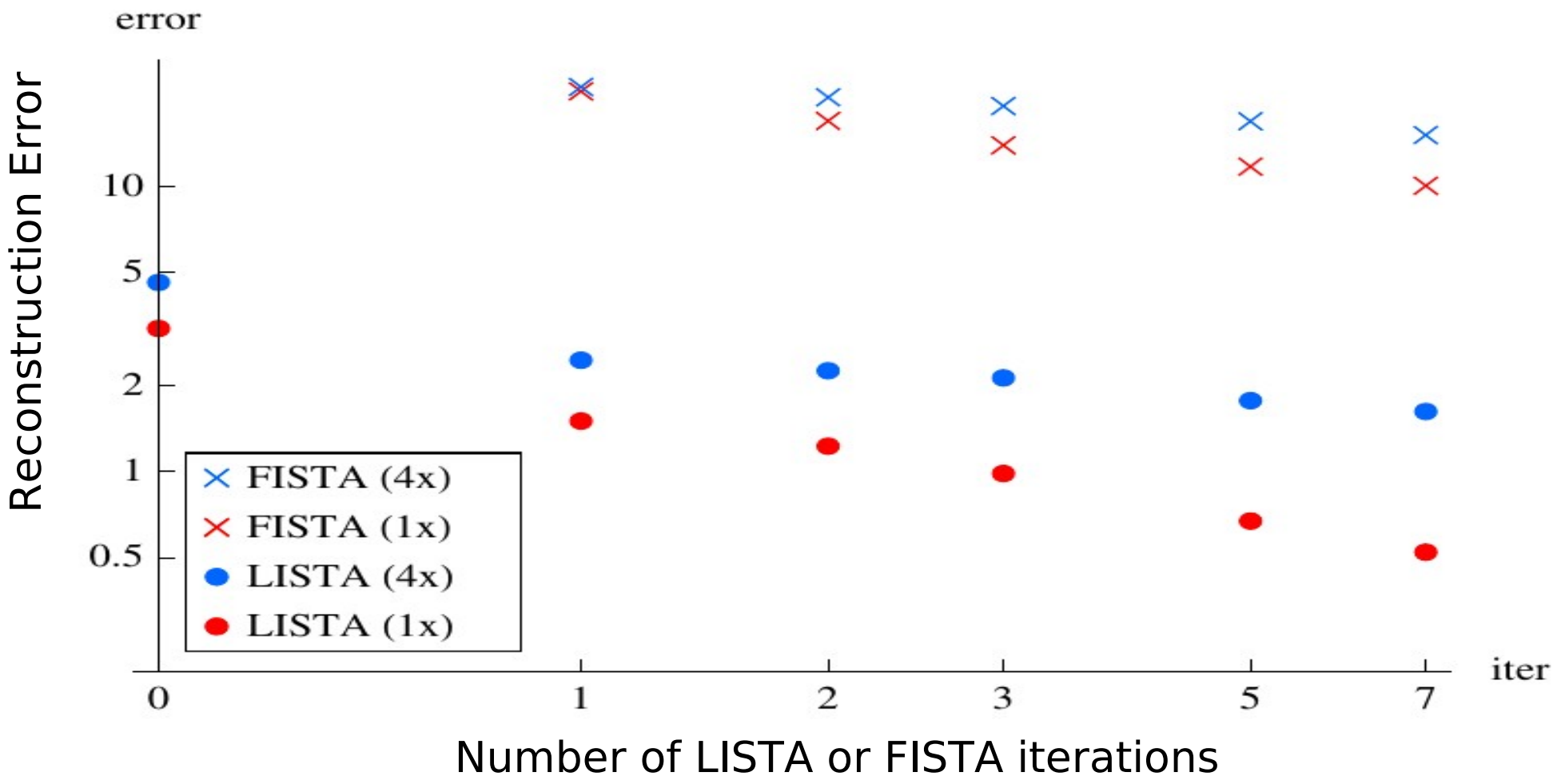
- **Think of the FISTA flow graph as a recurrent neural net where We and S are trainable parameters**



- **Time-Unfold the flow graph for K iterations**
- **Learn the We and S matrices with "backprop-through-time"**
- **Get the best approximate solution within K iterations**

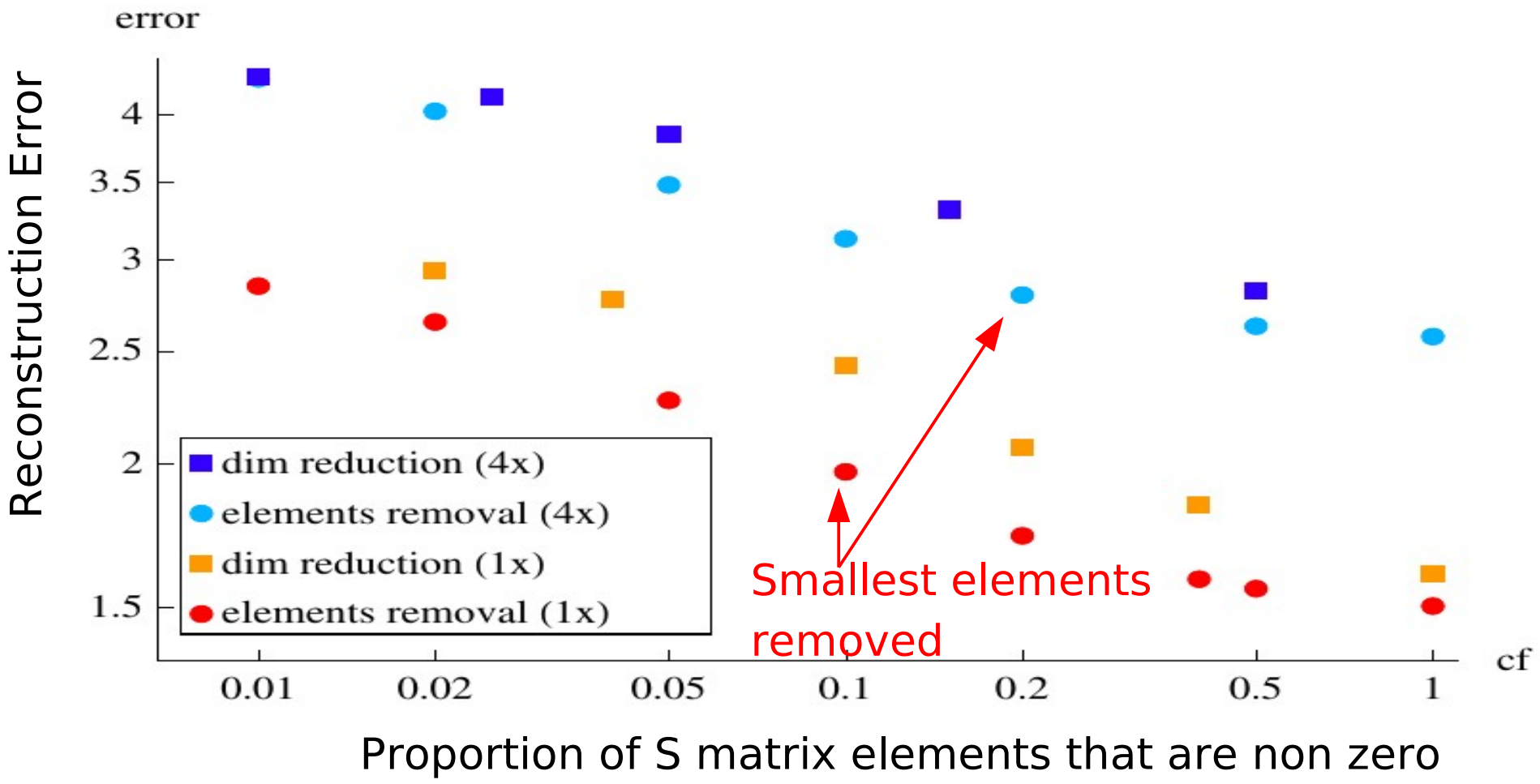


Learning ISTA (LISTA) vs ISTA/FISTA

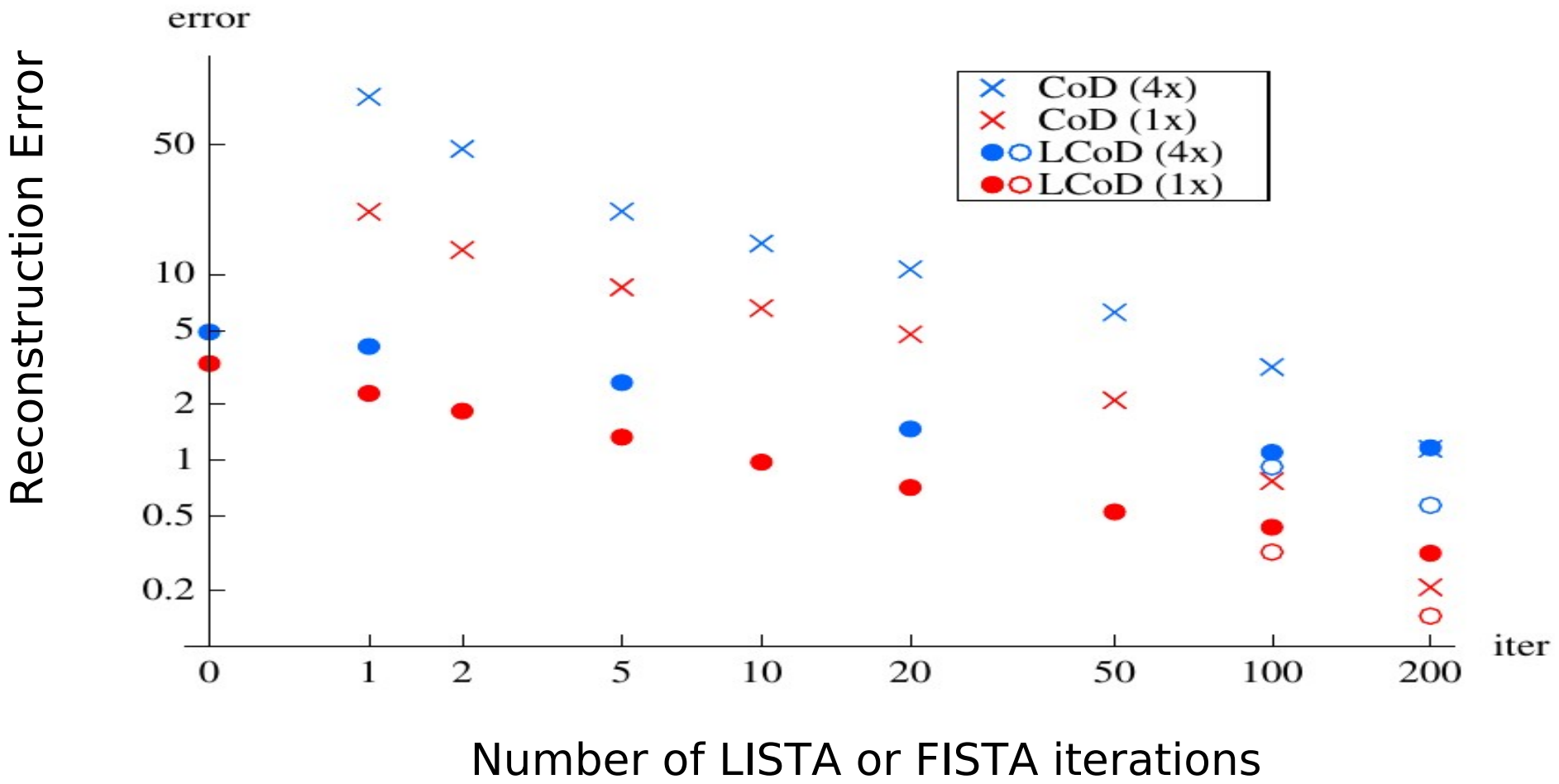


LISTA with partial mutual inhibition matrix

Y. LeCun

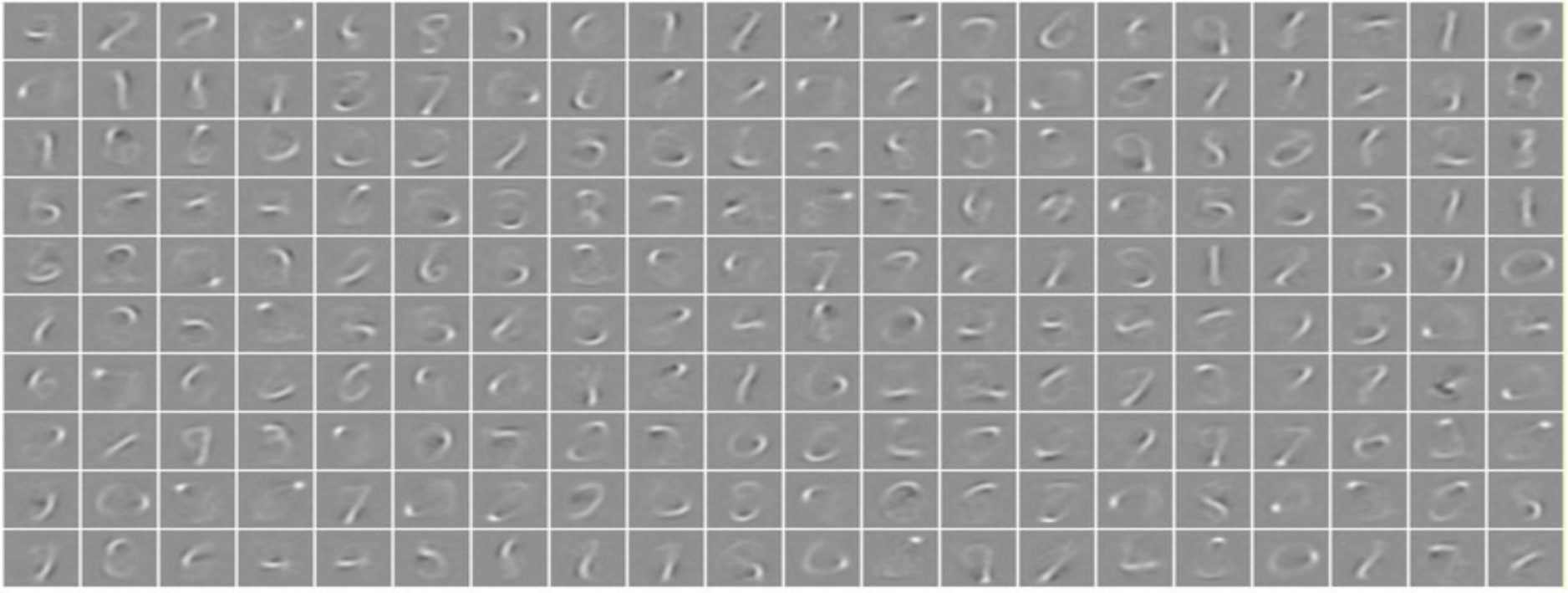


Learning Coordinate Descent (LcoD): faster than LISTA



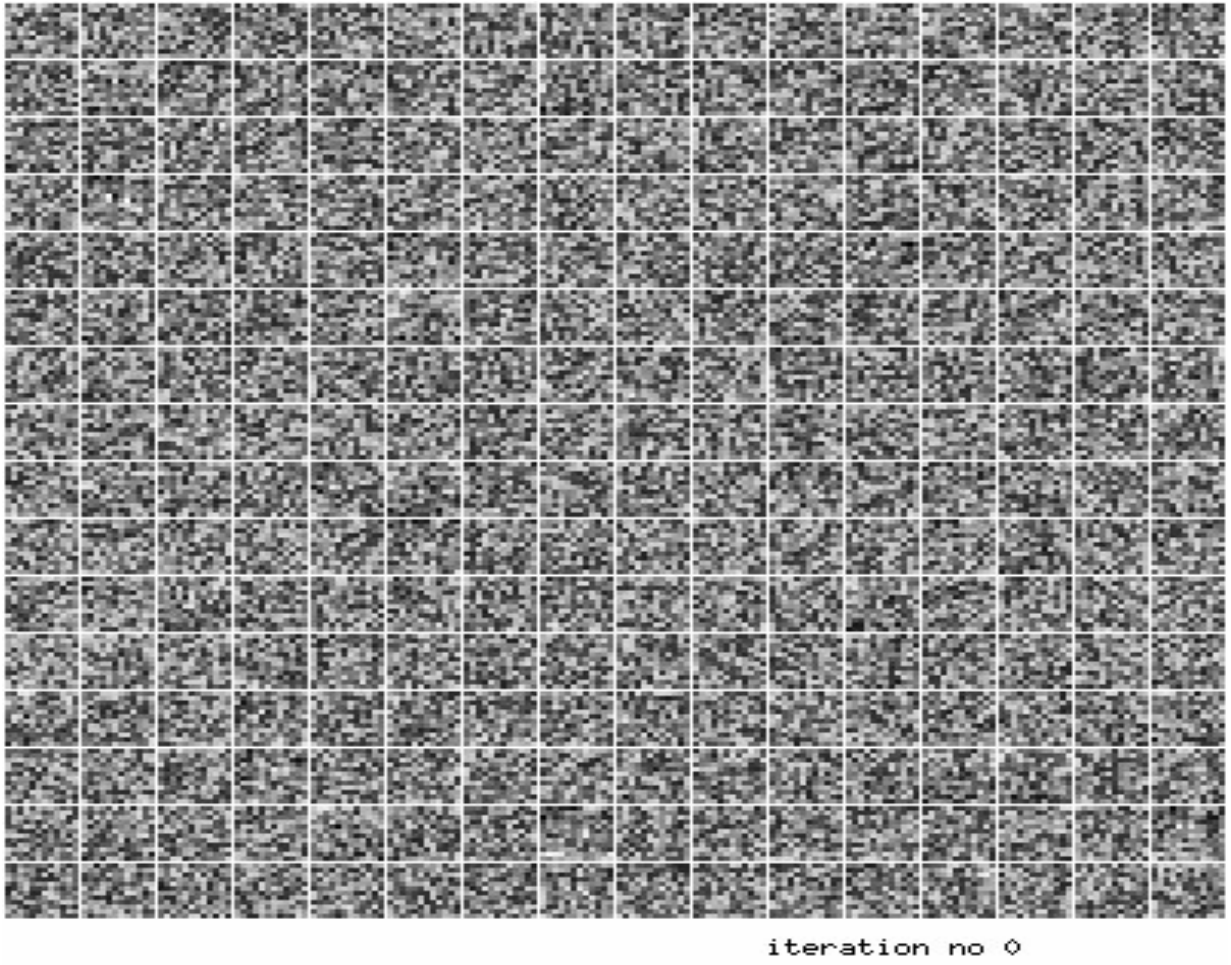
Sparse AE on handwritten digits (MNIST)

- **256 basis functionsBasis functions (columns of decoder matrix) are digit parts**
- **All digits are a linear combination of a small number of these**

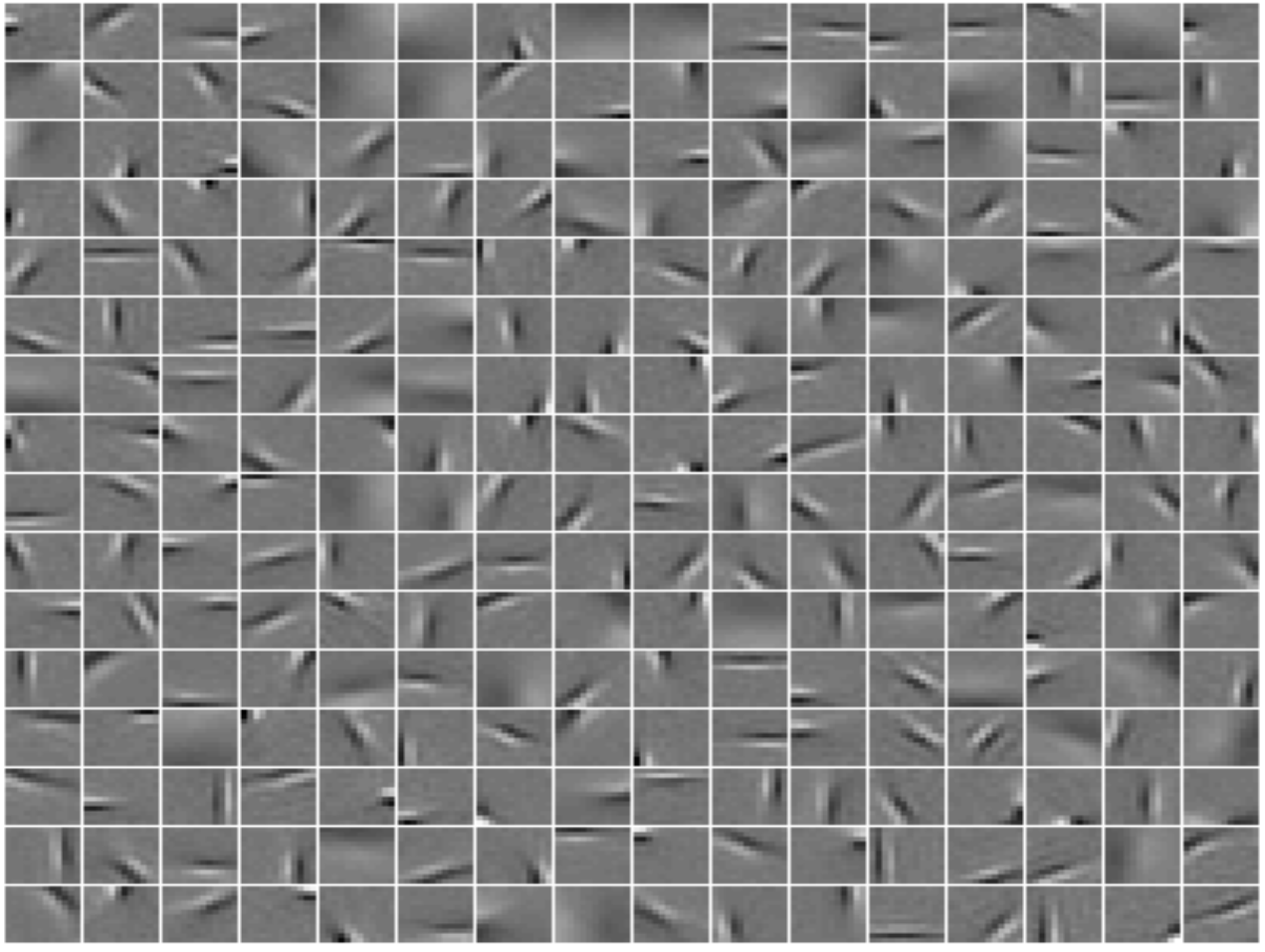


Predictive Sparse Decomposition (PSD): Training

- **Training on natural images patches.**
  - 12X12
  - 256 basis functions
  - [Ranzato 2007]

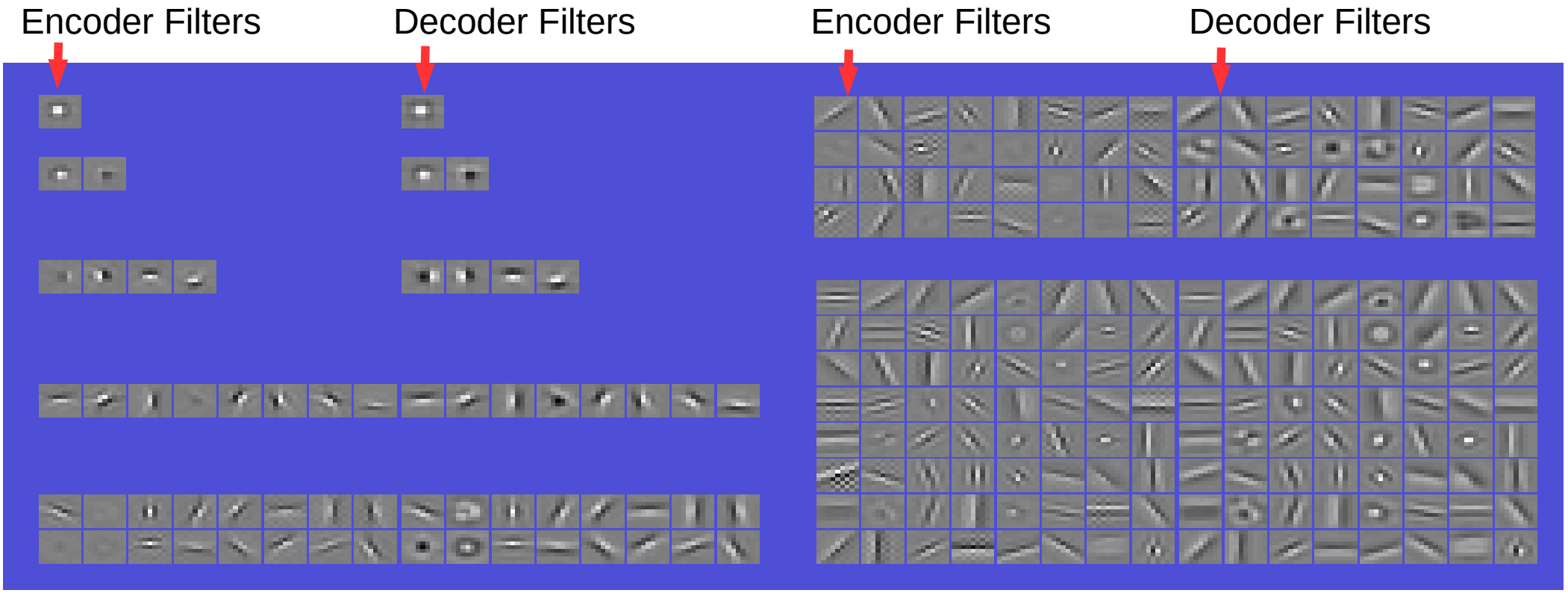


Learned Features: V1-like receptive fields



Convolutional Sparse Auto-Encoder on Natural Images

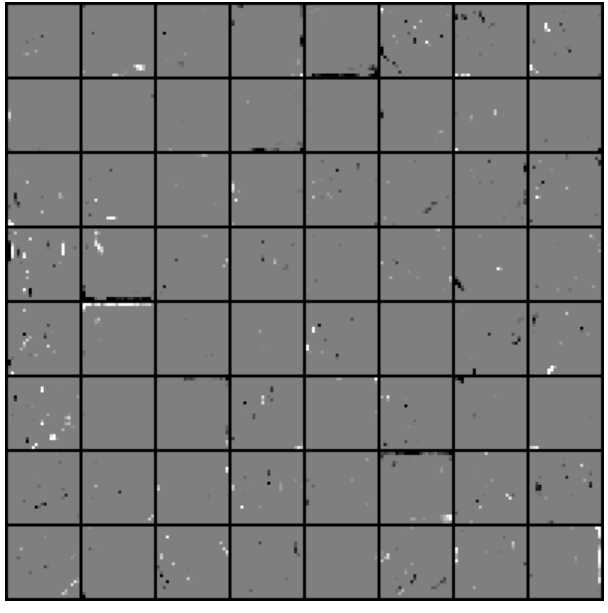
- **Filters and Basis Functions obtained. Linear decoder (conv)**
  - with 1, 2, 4, 8, 16, 32, and 64 filters [Kavukcuoglu NIPS 2010]



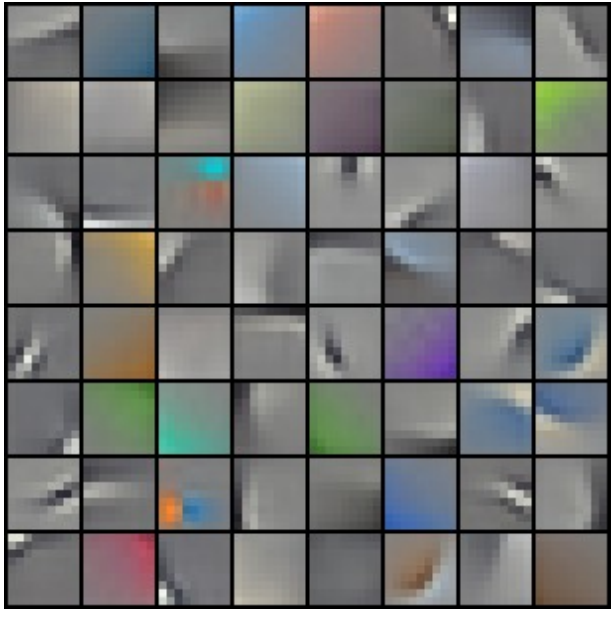
Convolutional Sparse Auto-Encoder on Natural Images

- **Trained on CIFAR 10 (32x32 color images)**
- **Architecture: Linear decoder, LISTA recurrent encoder**
- **Pytorch implementation (talk to Jure Zbontar)**

sparse codes (z) from encoder



9x9 decoder kernels



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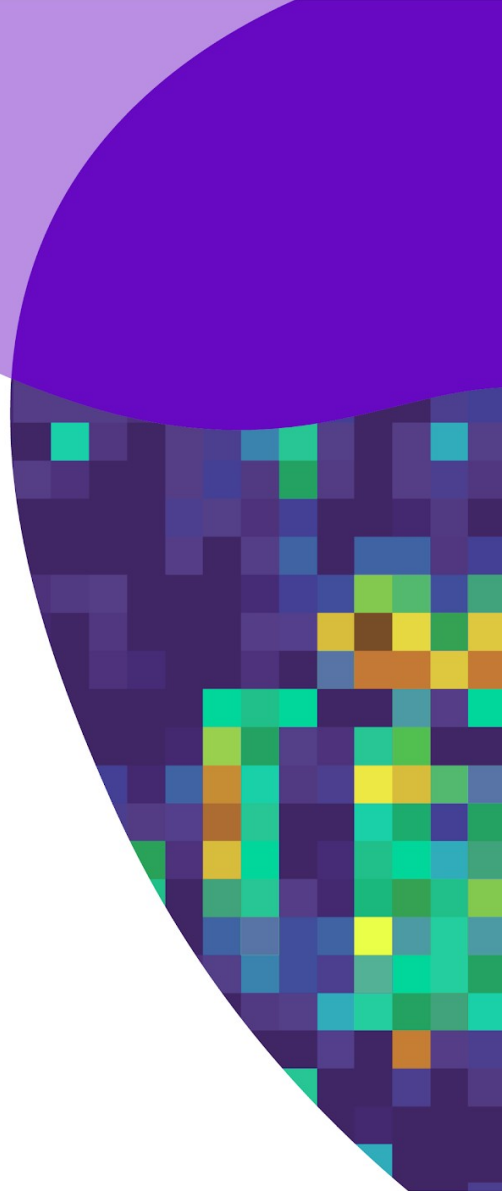
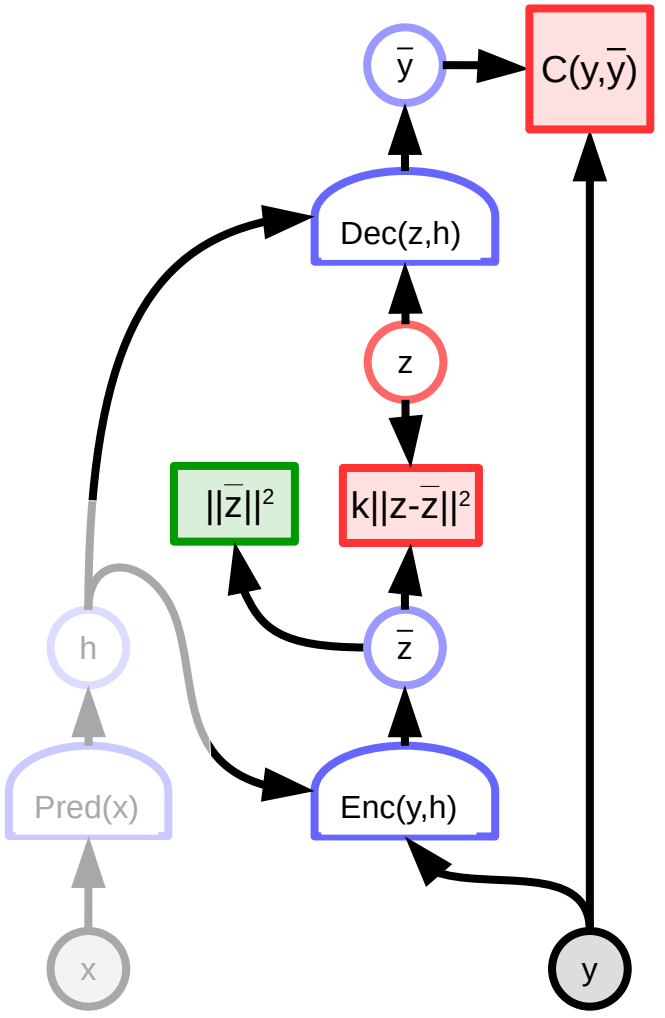
Variational AE

Architectural SSL methods
(non constrastive)



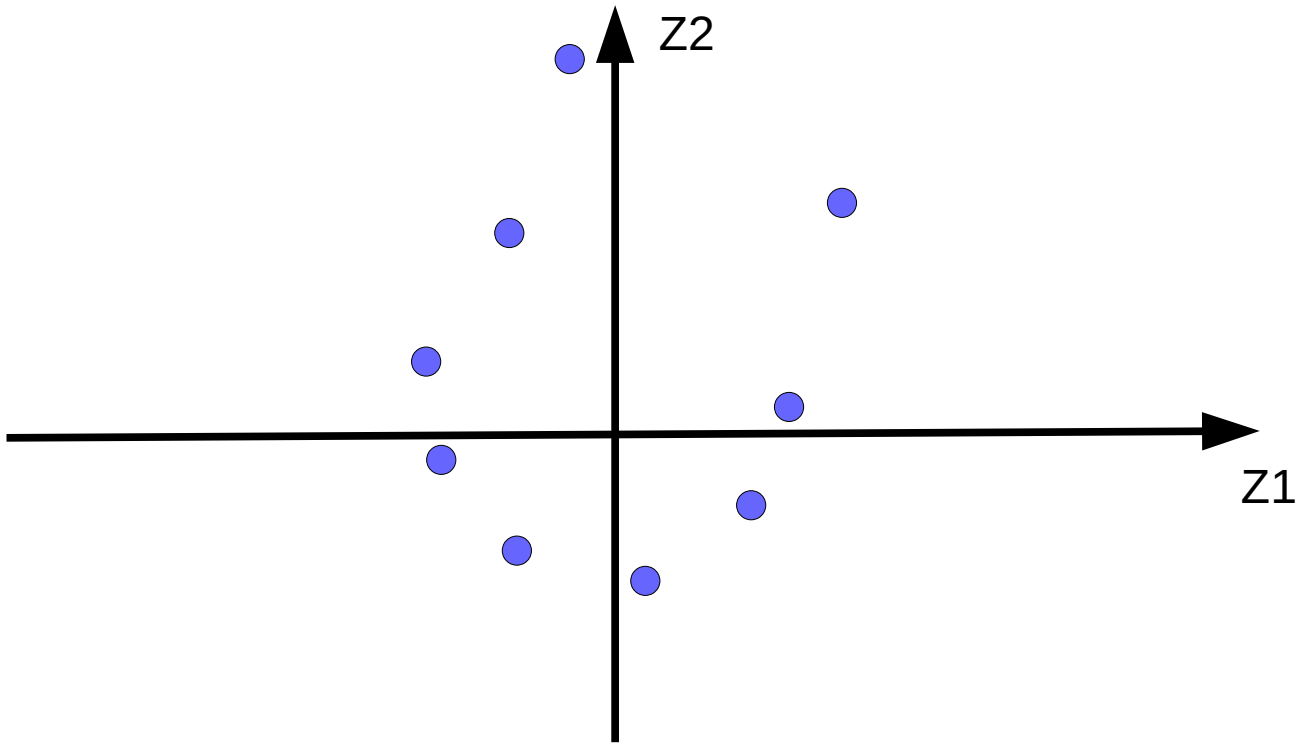
Variational Auto-Encoder

- **Limiting the information capacity of the code by adding Gaussian noise**
- **The energy term $k||z-\bar{z}||^2$ is seen as the log of a prior from which to sample z**
- **The encoder output is regularized to have a mean and a variance close to zero.**



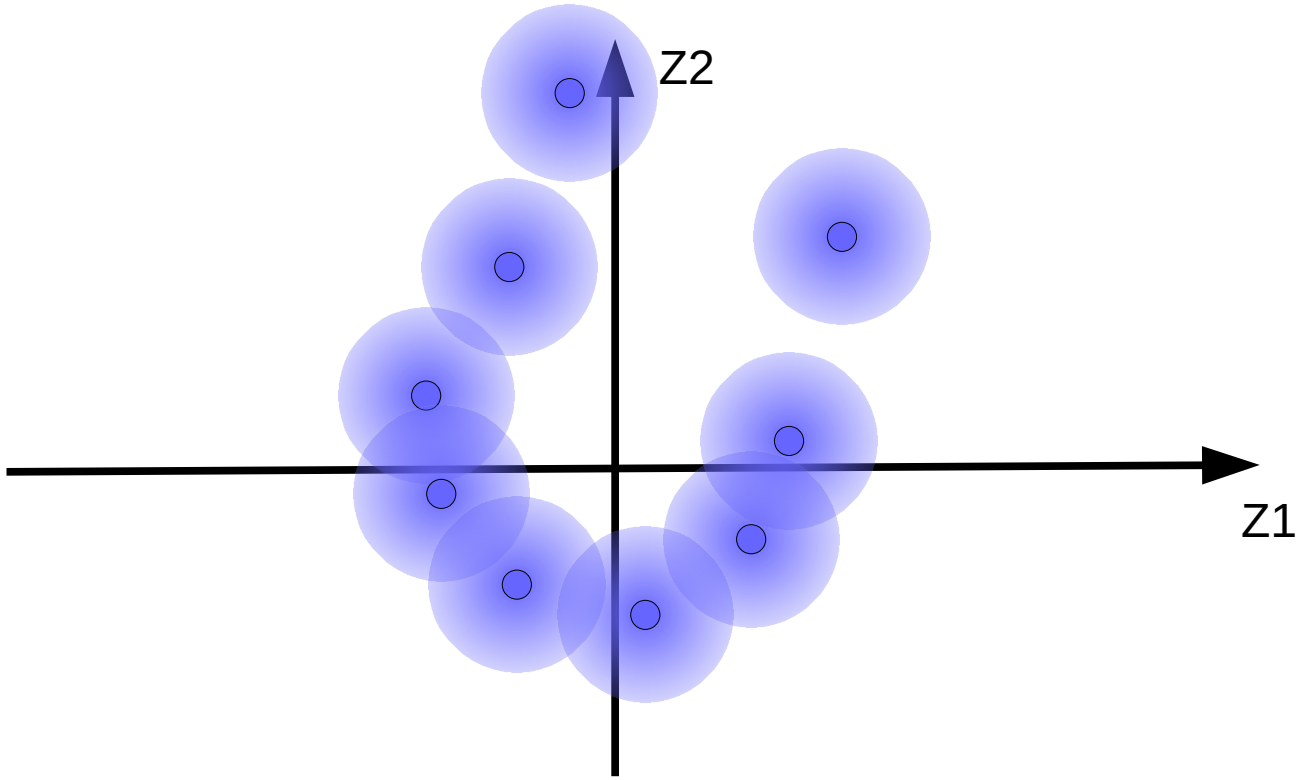
Variational Auto-Encoder

- **Code vectors for training samples**



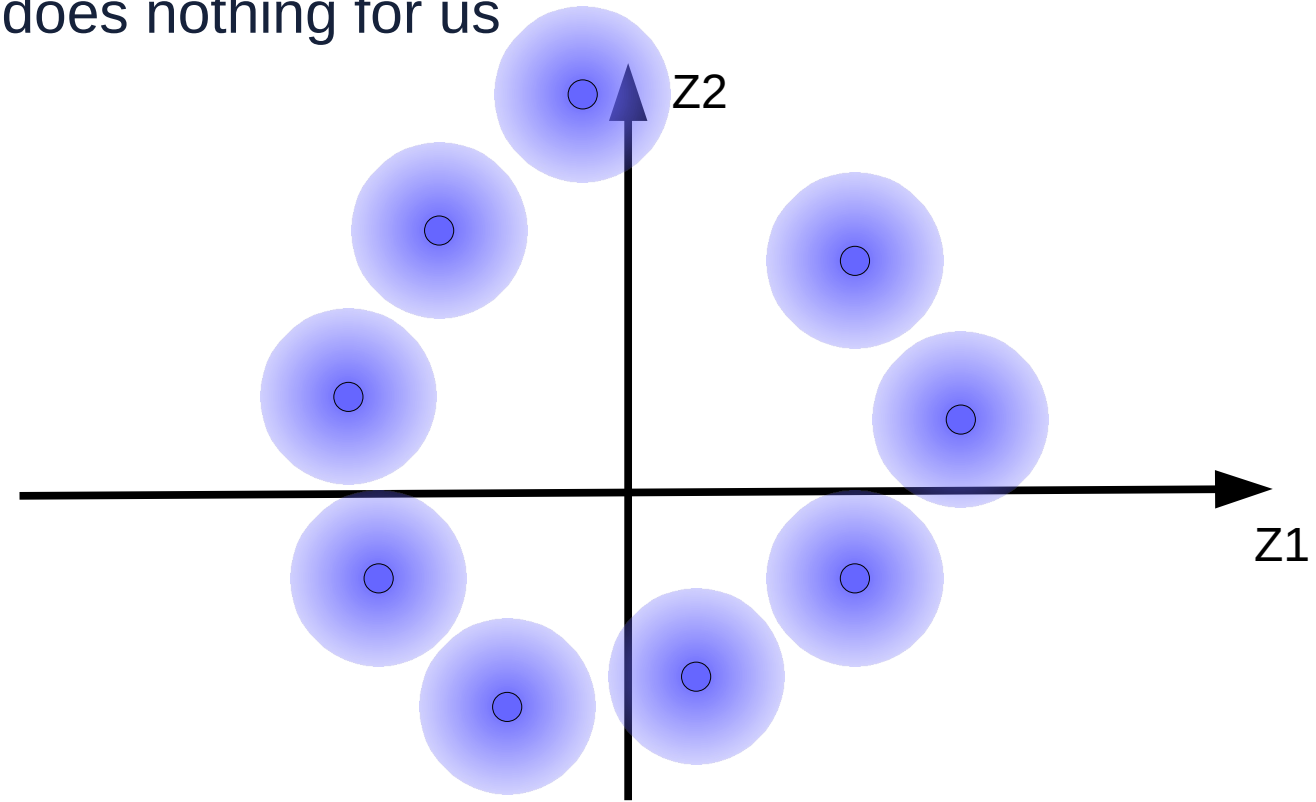
Variational Auto-Encoder

- **Code vectors for training sample with Gaussian noise**
  - Some fuzzy balls overlap, causing bad reconstructions



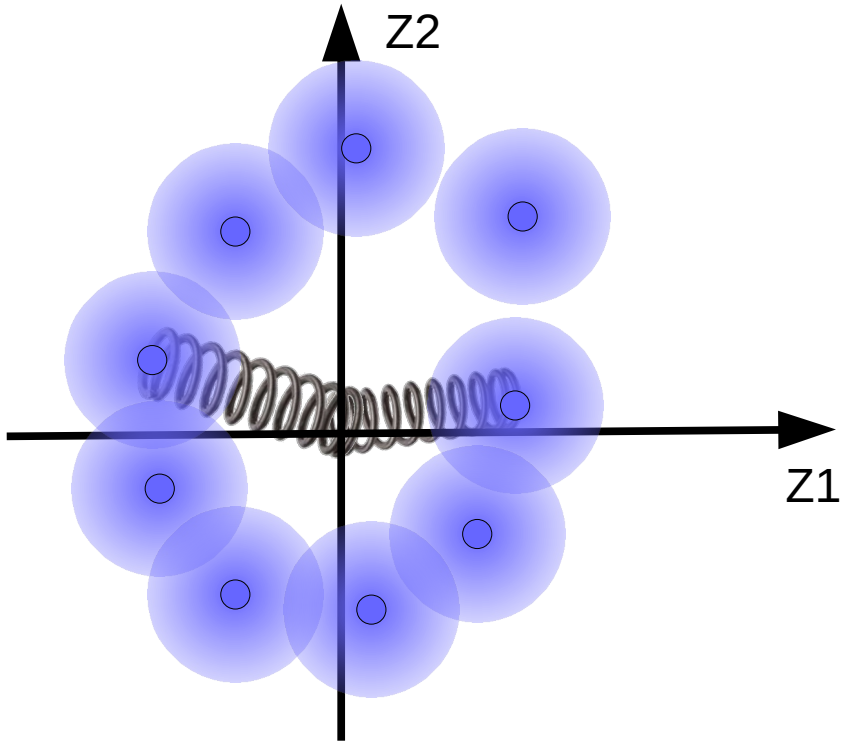
Variational Auto-Encoder

- **The code vectors want to move away from each other to minimize reconstruction error**
  - But that does nothing for us



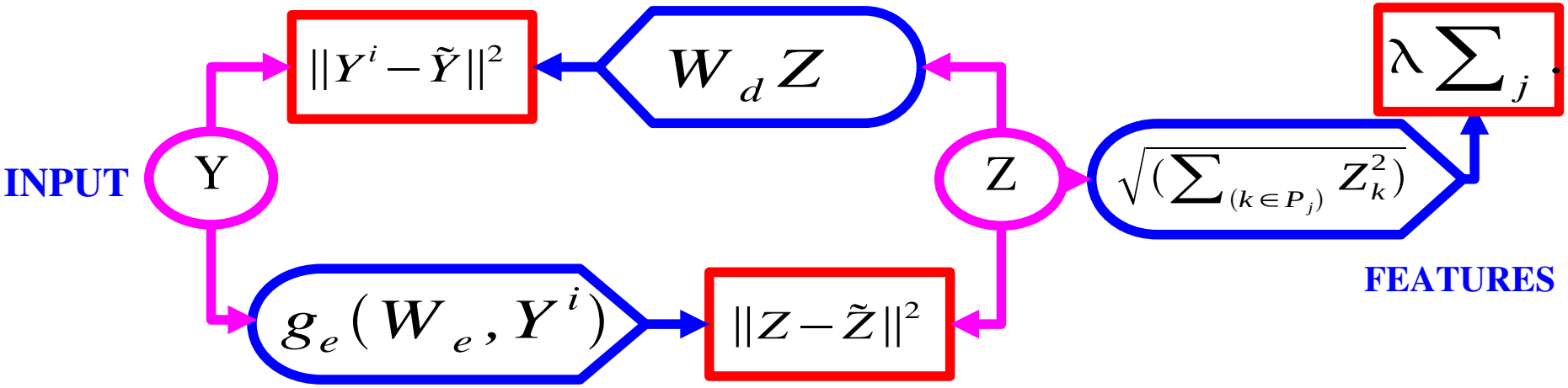
Variational Auto-Encoder

- **Attach the balls to the center with a sping, so they don't fly away**
  - Minimize the square distances of the balls to the origin
- **Center the balls around the origin**
  - Make the center of mass zero
- **Make the sizes of the balls close to 1 in each dimension**
  - Through a so-called KL term



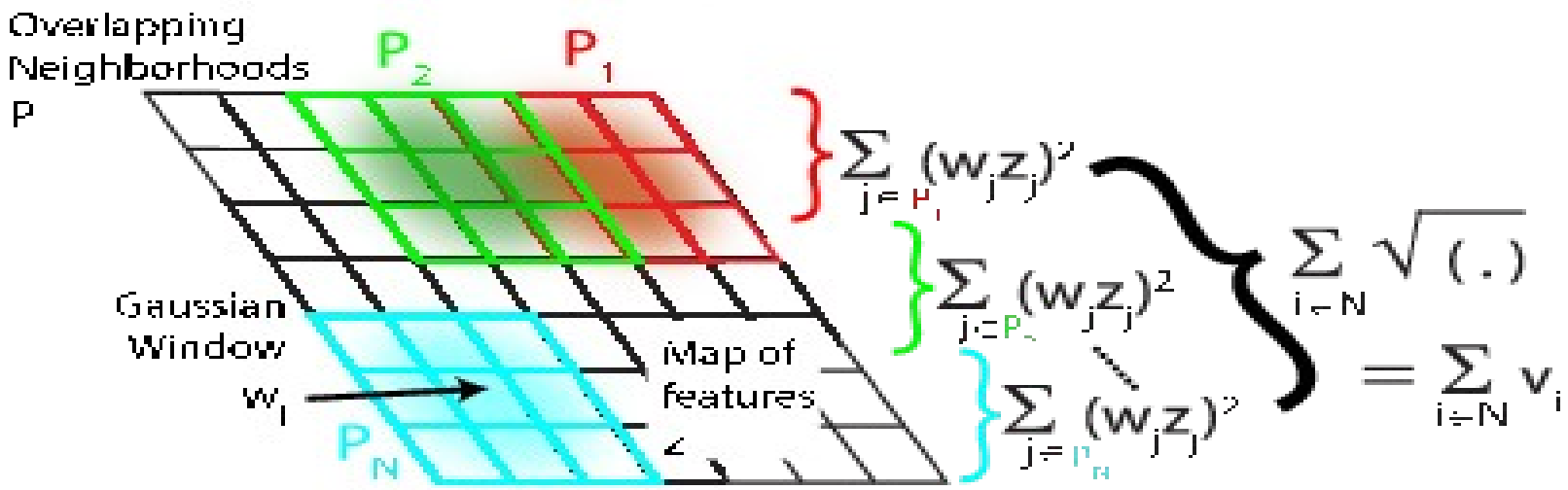
Group Sparsity [Kavukcuoglu et al. CVPR 2009]

- **Unsupervised PSD ignores the spatial pooling step.**
- **Could we devise a similar method that learns the pooling layer as well?**
- **Idea [Hyvarinen & Hoyer 2001]: group sparsity on pools of features**
  - Minimum number of pools must be non-zero
  - Number of features that are on within a pool doesn't matter
  - Polls tend to regroup similar features



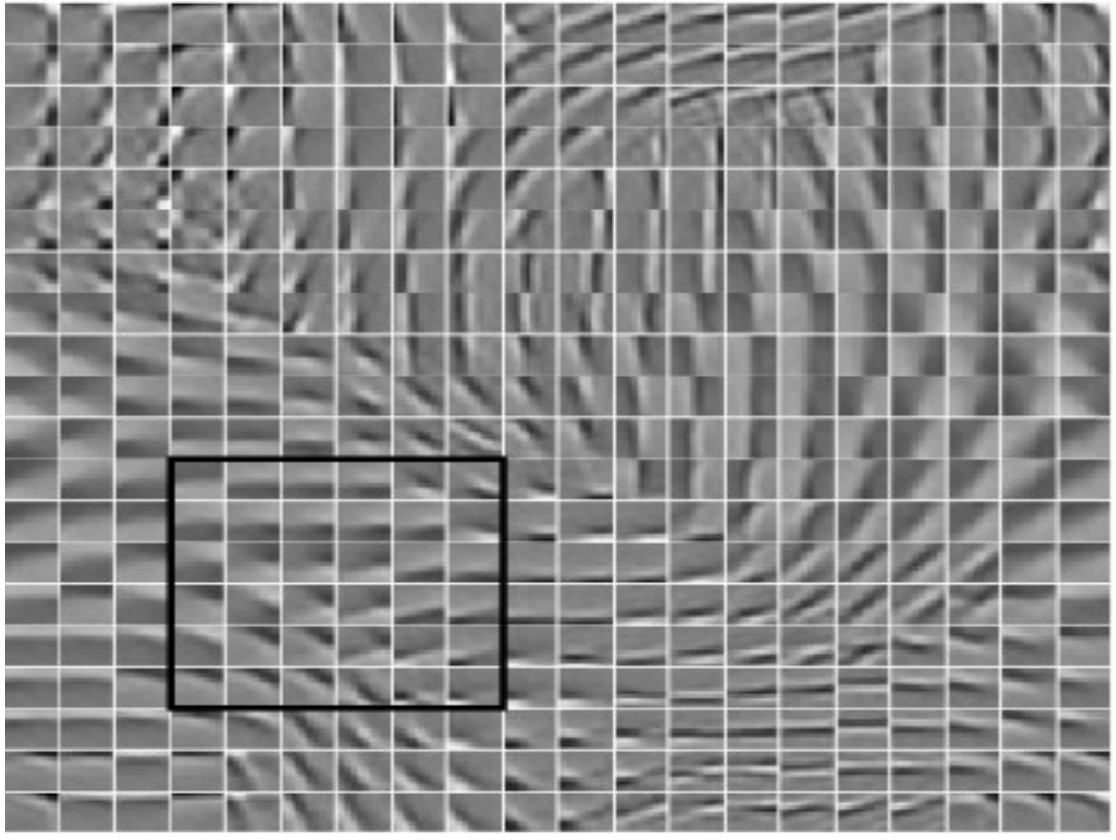
Why just pool over space? Why not over orientation?

- **Using an idea from Hyvarinen: topographic square pooling (subspace ICA)**
  - 1. Apply filters on a patch (with suitable non-linearity)
  - 2. Arrange filter outputs on a 2D plane
  - 3. square filter outputs
  - 4. minimize sqrt of sum of blocks of squared filter outputs



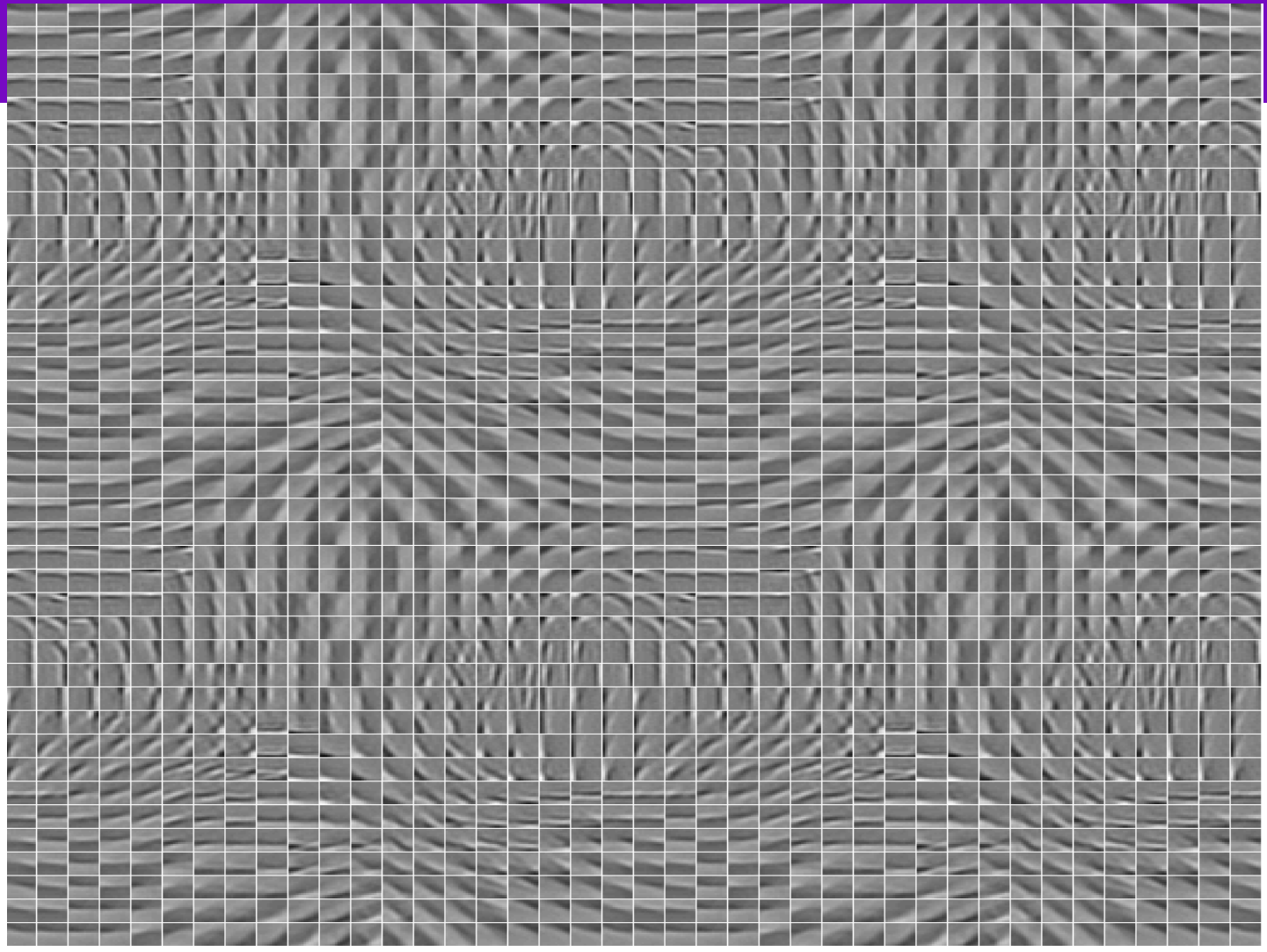
Why just pool over space? Why not over orientation?

- **The filters arrange themselves spontaneously so that similar filters enter the same pool.**
- **The pooling units can be seen as complex cells**
- **They are invariant to local transformations of the input**
  - For some it's translations, for others rotations, or other transformations.



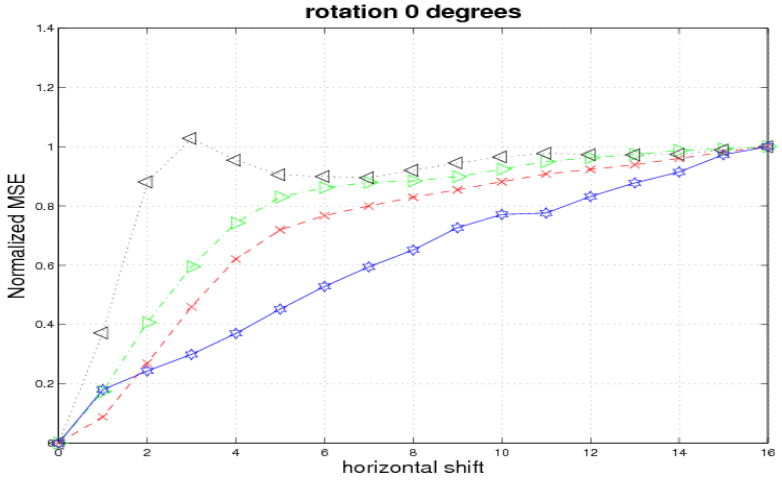
Pinwheels!

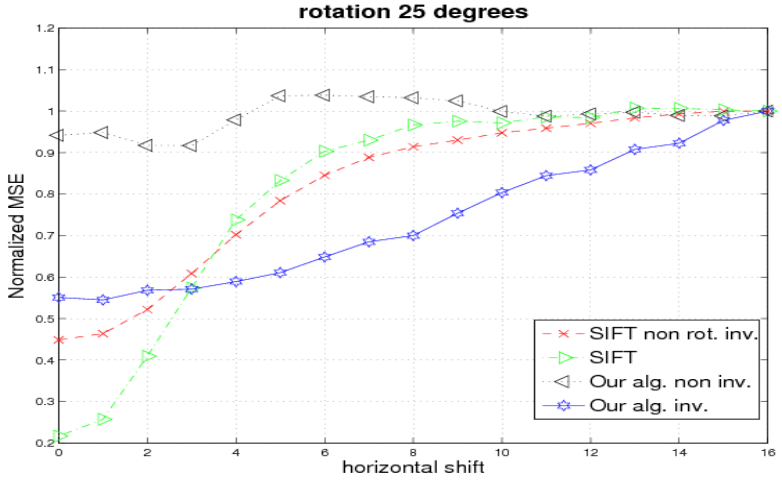
- **Does that look pinwheely to you?**



Invariance Properties Compared to SIFT

- **Measure distance between feature vectors (128 dimensions) of 16x16 patches from natural images**
  - Left: normalized distance as a function of translation
  - Right: normalized distance as a function of translation when one patch is rotated 25 degrees.





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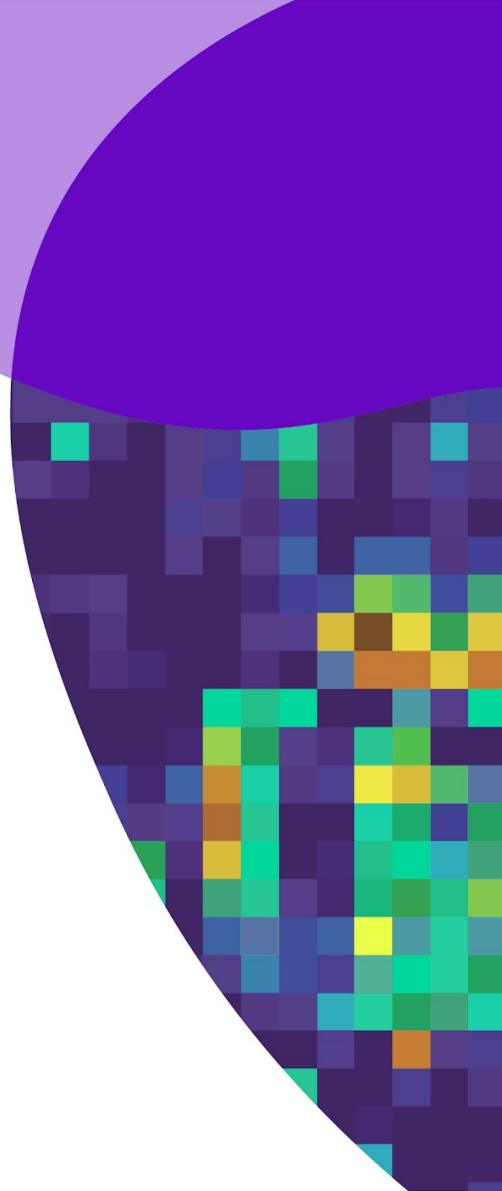
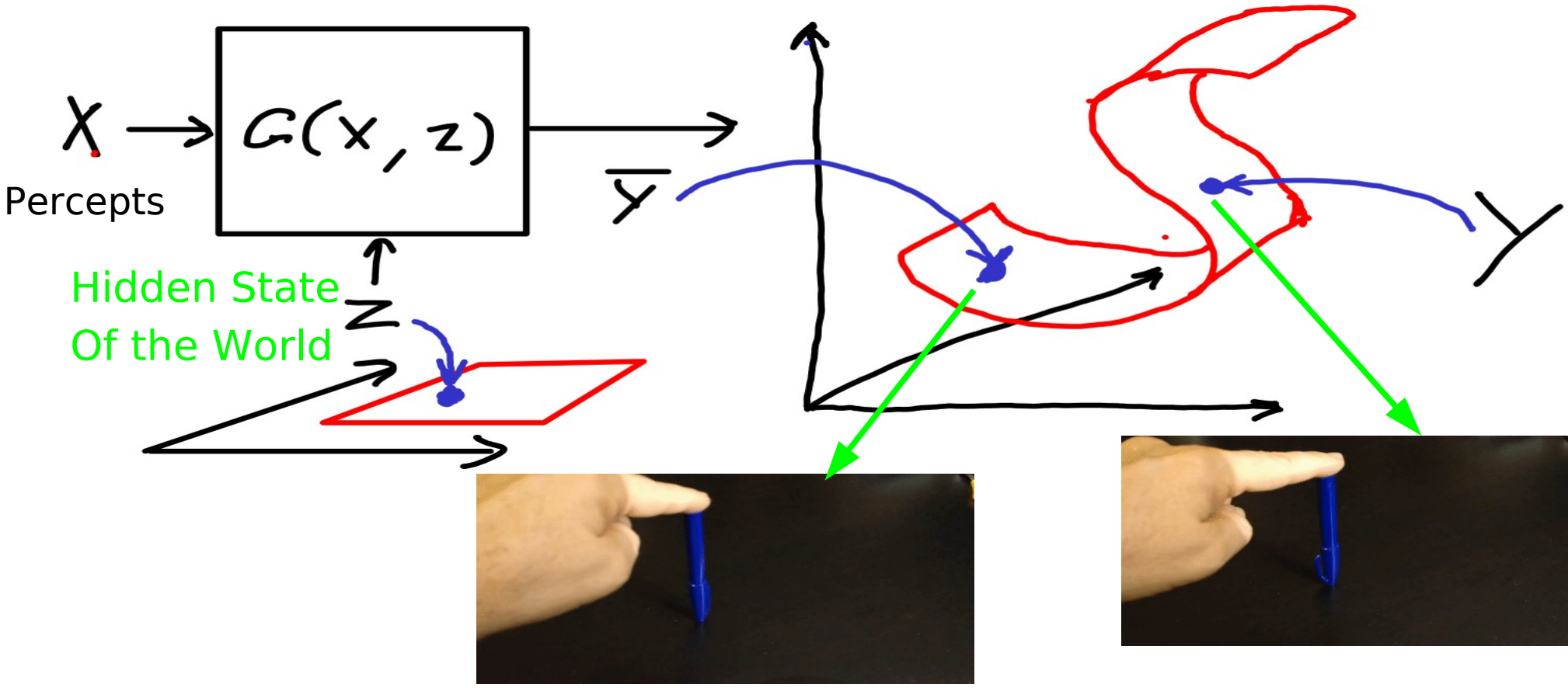
Generative Adversarial Networks



Contrastive method in which a generator network produces the contrastive samples

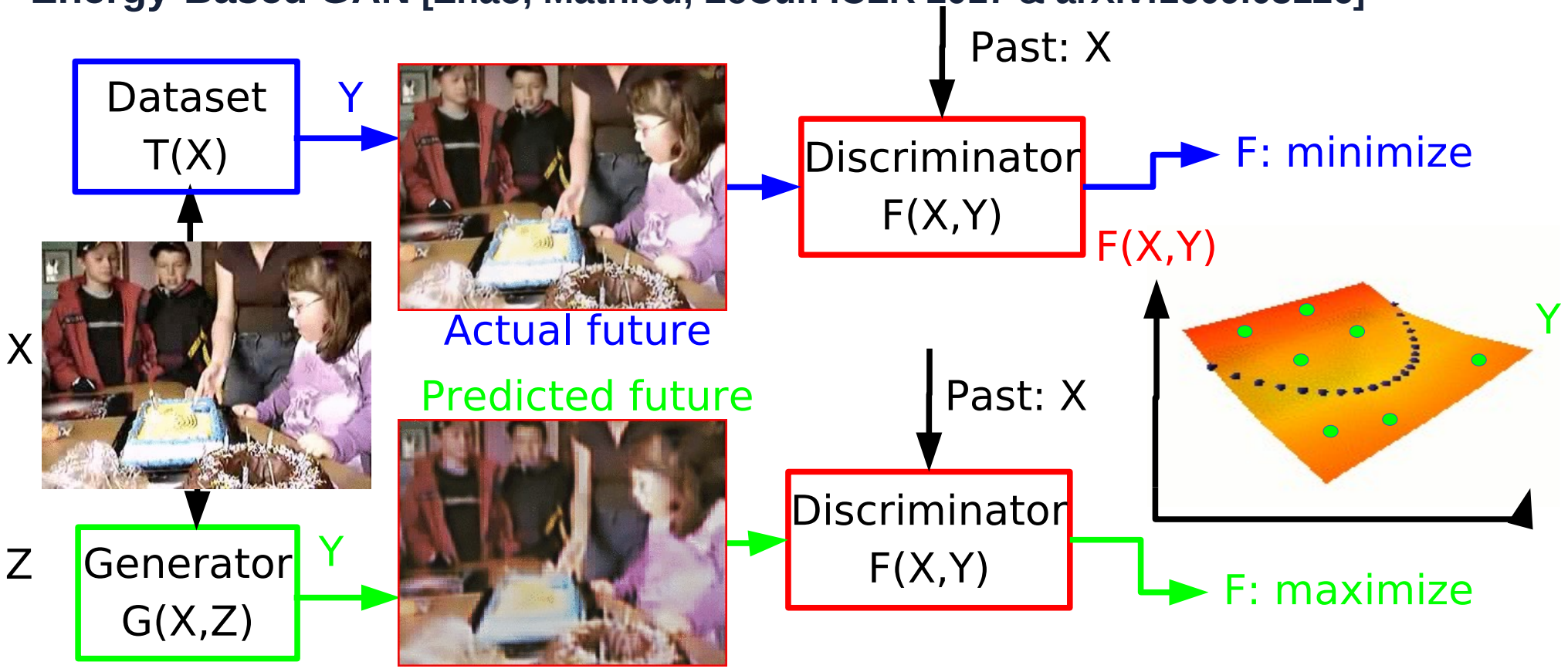
The Hard Part: Prediction Under Uncertainty

- **Invariant prediction: The training samples are merely representatives of a whole set of possible outputs (e.g. a manifold of outputs).**



Adversarial Training: the key to prediction under uncertainty?

- **Generative Adversarial Networks (GAN) [Goodfellow et al. NIPS 2014],**
- **Energy-Based GAN [Zhao, Mathieu, LeCun ICLR 2017 & arXiv:1609.03126]**



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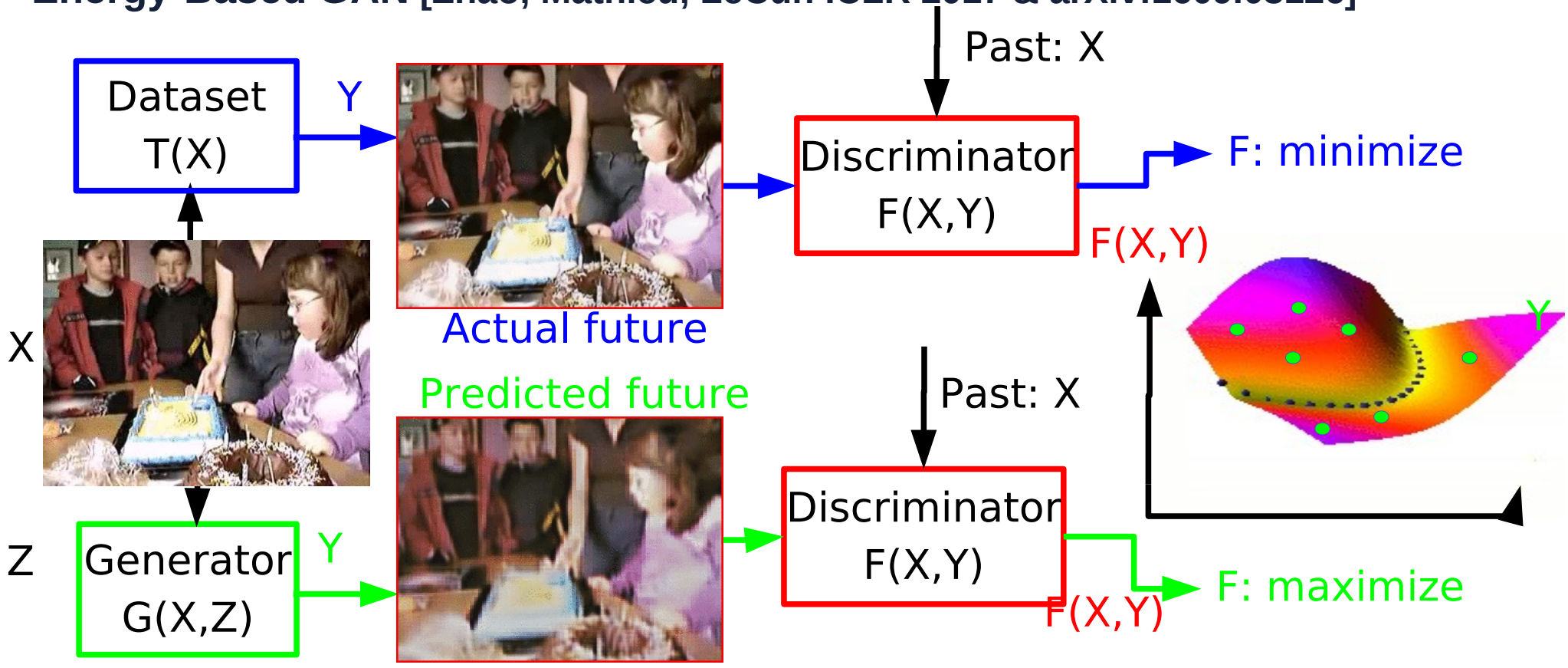

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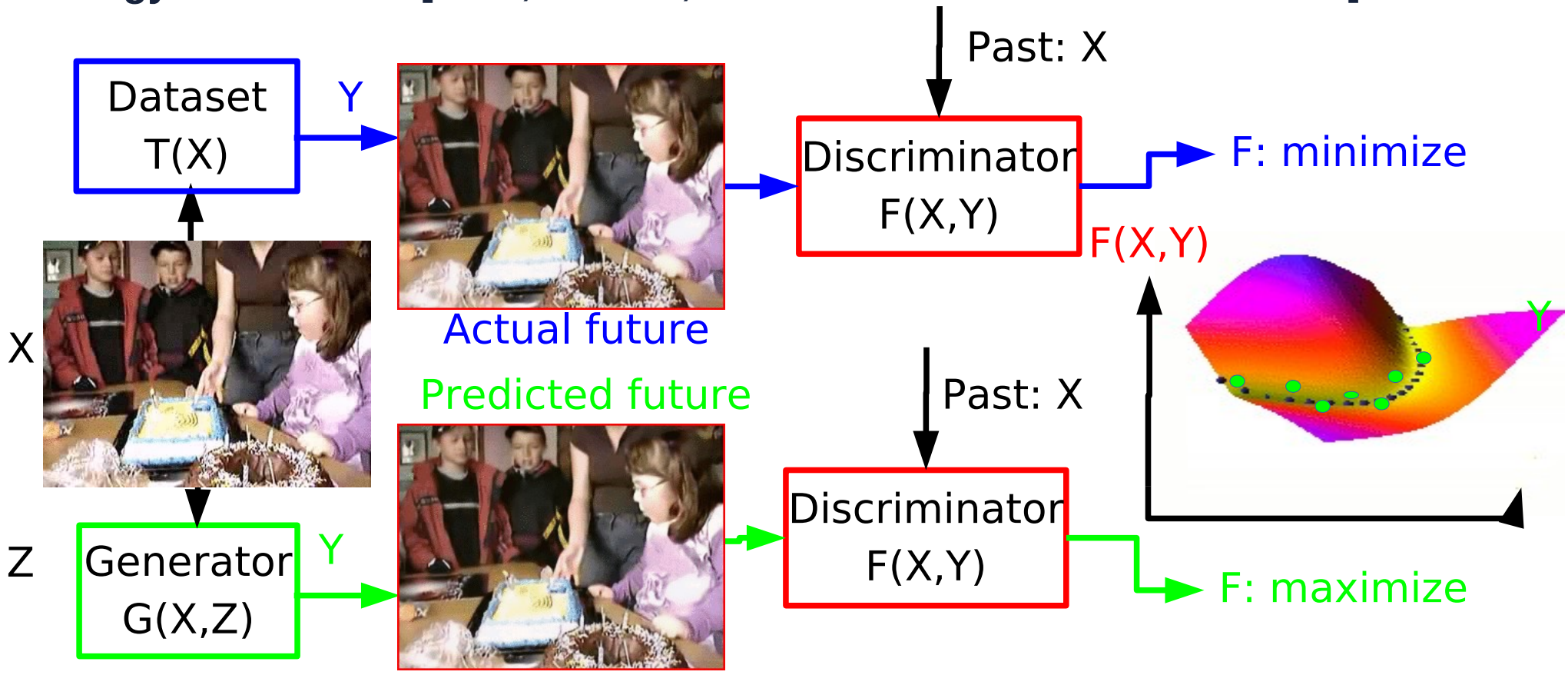
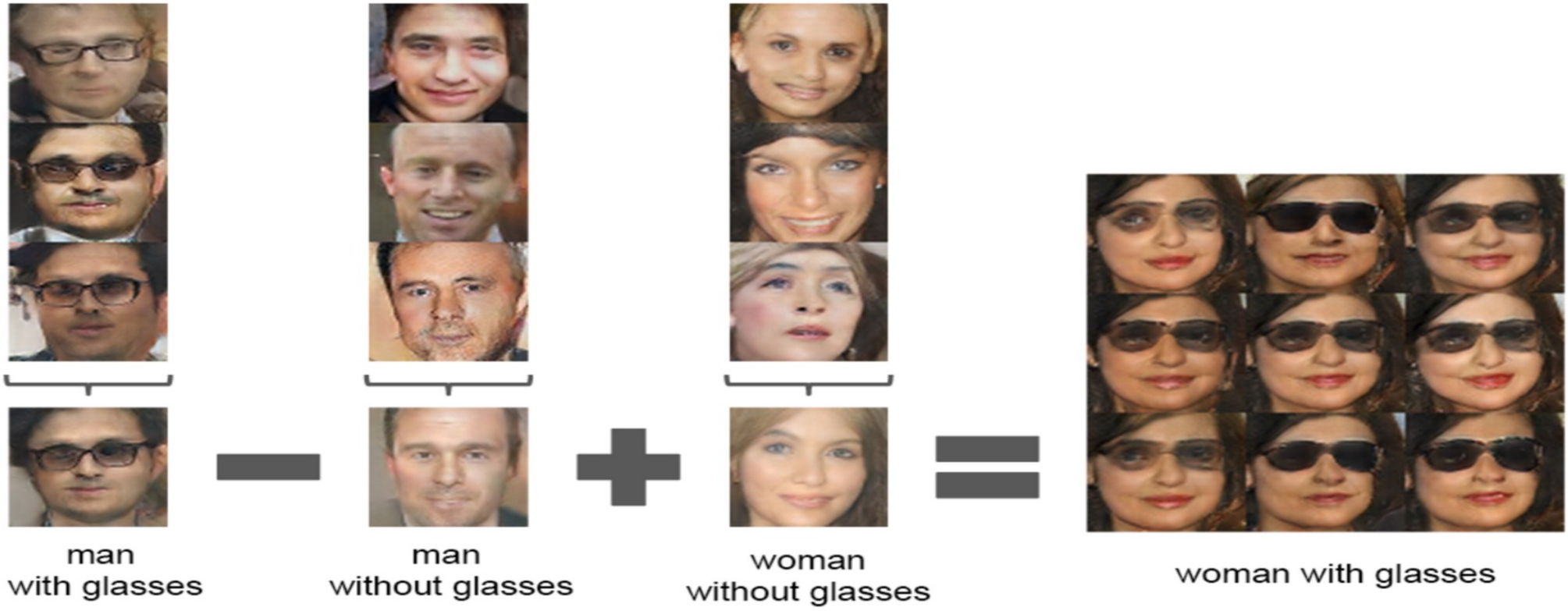


Face Algebra (in DCGAN space)

- **DCGAN: adversarial training to generate images.**
  - [Radford, Metz, Chintala 2015]



Faces “invented” by a GAN (Generative Adversarial Network)

- **Random vector → Generator Network → output image** [Goodfellow NIPS 2014]
  [Karras et al. ICLR 2018] (from NVIDIA)

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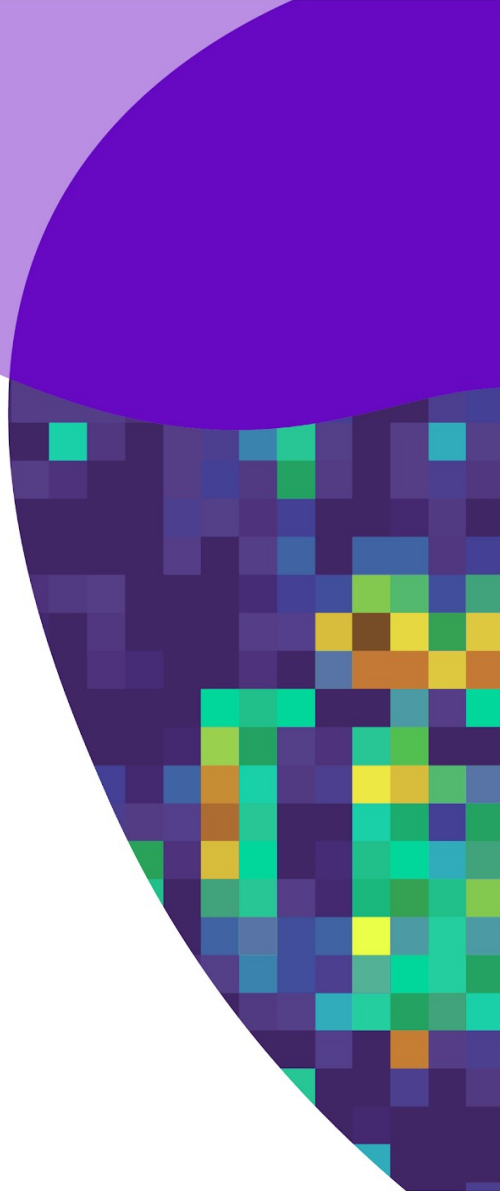
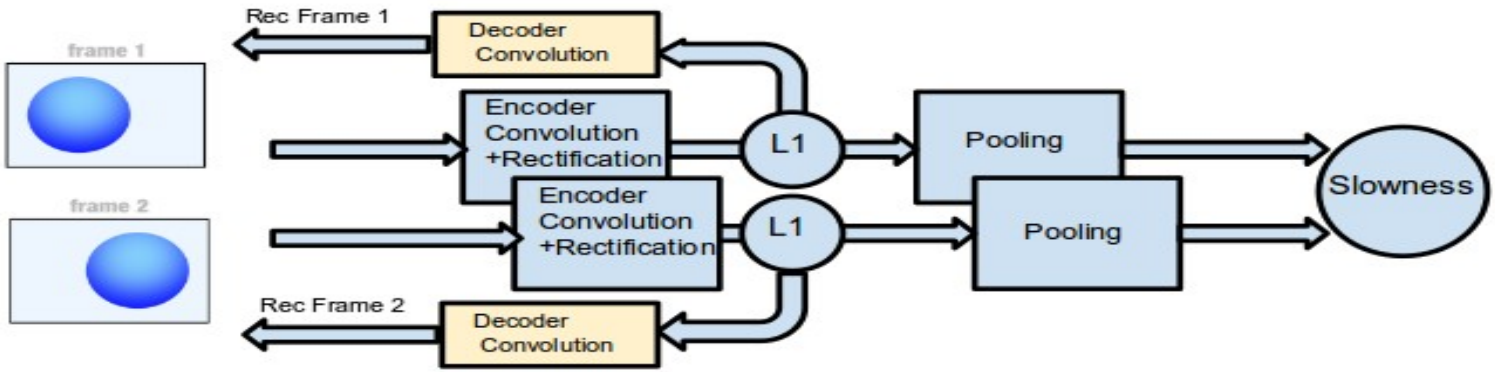
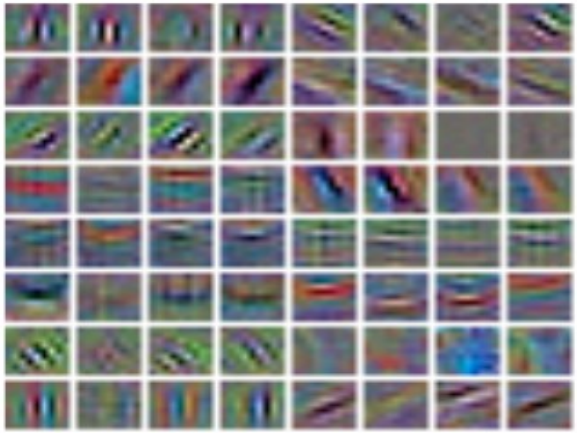
Other Latent representation regularization

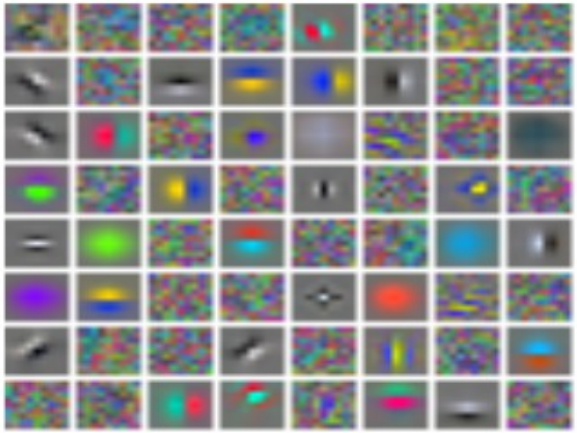
Using temporal constancy

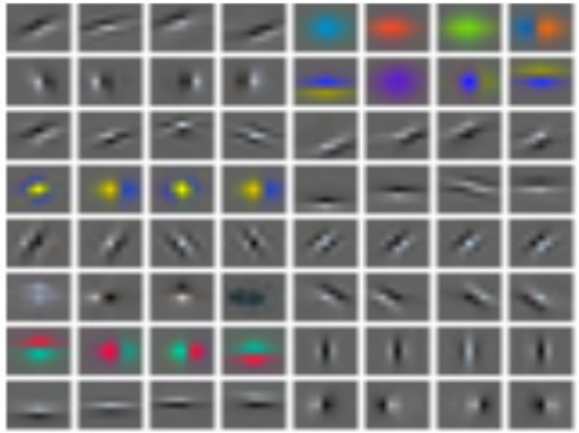


Sparse Auto-Encoder with “Slow Feature” Penalty









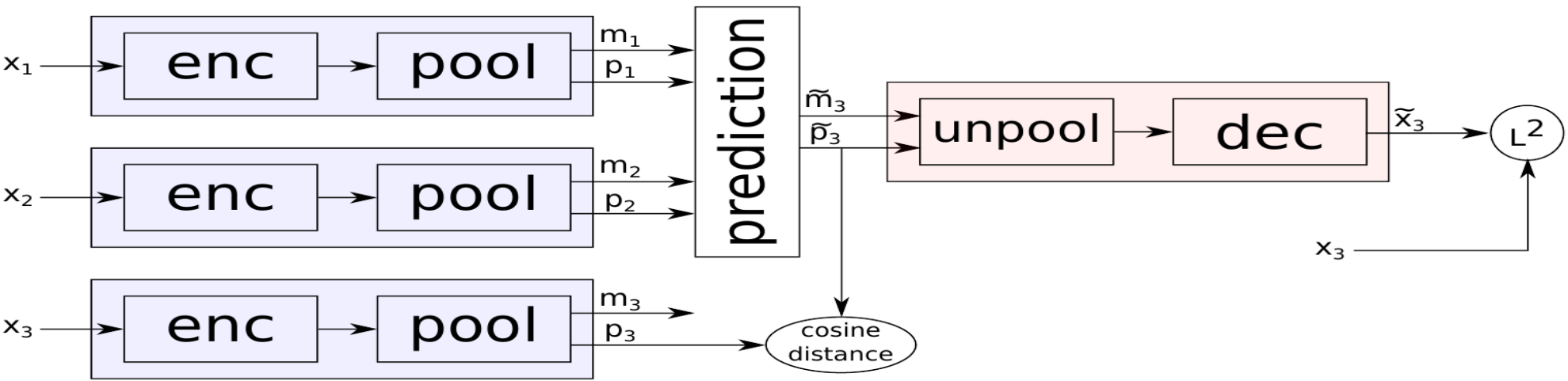
- Supervised filters CIFAR10
- sparse conv. auto-encoder
- slow & sparse conv. auto-encoder Trained on YouTube videos
- [Goroshin et al. ICCV 2015, Arxiv:1412.6056]

Unsupervised Learning by Prediction and Linearization

[Goroshin, Mathieu, LeCun NIPS 2015, arXiv:1506.03011]

- **Trained on 3 successive frames of video**
- **Maximize the colinearity between successive changes of feature vectors**
- **So that "natural" changes stay in linear manifold in feature space.**

$$L = \frac{1}{2}\|G_W(\mathbf{a}\begin{bmatrix} z^t & z^{t-1}\end{bmatrix}^T) - x^{t+1}\|_2^2 - \lambda \frac{(z^t - z^{t-1})^T(z^{t+1} - z^t)}{\|z^t - z^{t-1}\|\,\|z^{t+1} - z^t\|}$$



Unsupervised Learning by Prediction and Linearization

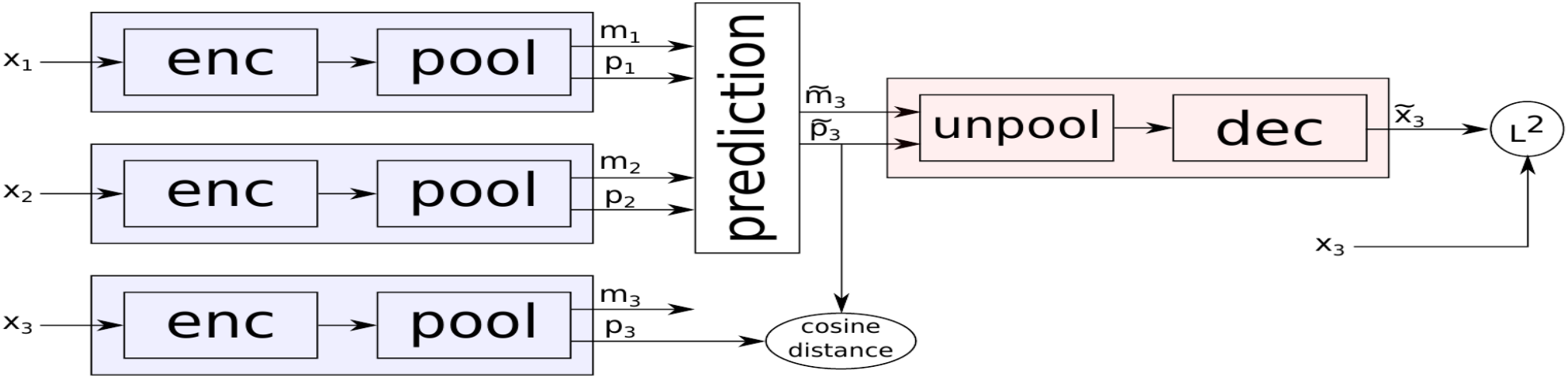
$$L = \frac{1}{2}\|G_W(\mathbf{a}\,[z^t \quad z^{t-1}]^T) - x^{t+1}\|_2^2 - \lambda \frac{(z^t - z^{t-1})^T(z^{t+1} - z^t)}{\|z^t - z^{t-1}\|\,\|z^{t+1} - z^t\|}$$

- **Magnitude**
  - (soft max)

$$m_k = \sum_{N_k} z(f,x,y)\frac{e^{\beta z(f,x,y)}}{\sum_{N_k} e^{\beta z(f',x',y')}} \approx \max_{N_k} z(f,x,y)$$

- **Phase**
  - (soft argmax)

$$\mathbf{p}_k = \sum_{N_k} \begin{bmatrix} f \\ x \\ y \end{bmatrix} \frac{e^{\beta z(f,x,y)}}{\sum_{N_k} e^{\beta z(f',x',y')}} \approx \arg\max_{N_k} z(f,x,y)$$

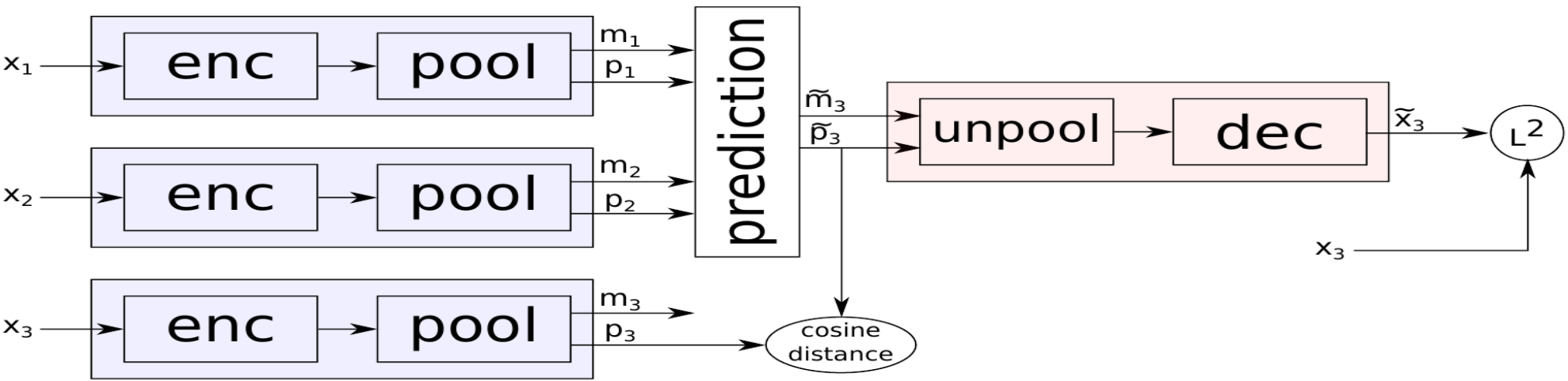


Unsupervised Learning by Prediction and Linearization

[Goroshin, Mathieu, LeCun NIPS 2015, arXiv:1506.03011]

- **Version 2:**
  - Make sure the “what”/magnitude changes slowly
  - Make sure the “where”/phase changes linearly

$$m^{t+1} = \frac{m^t + m^{t-1}}{2}$$
$$\mathbf{p}^{t+1} = 2\mathbf{p}^t - \mathbf{p}^{t-1}$$



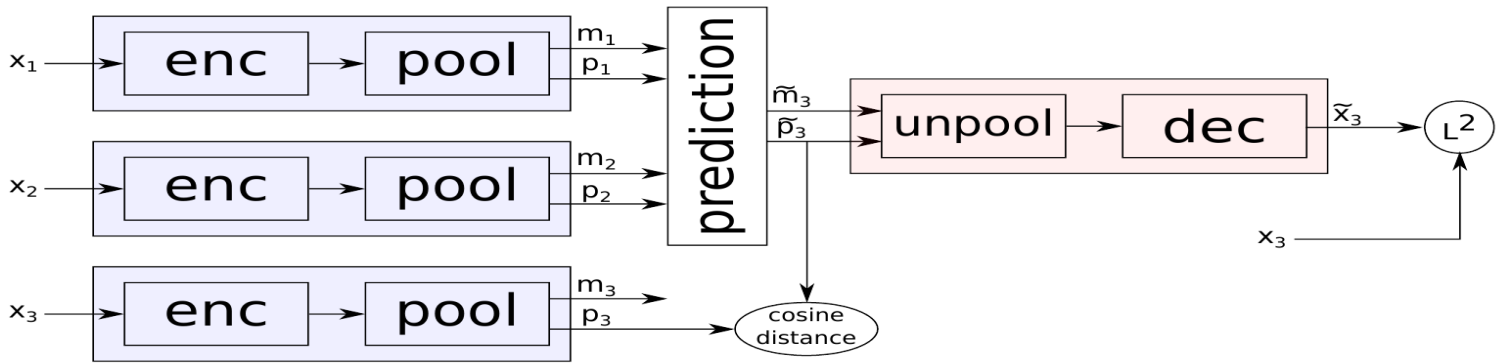
Unsupervised Learning by Prediction and Linearization

[Goroshin, Mathieu, LeCun arXiv:1506.03011]

- **Version 3: the world is unpredictable**
  - Add latent variable to compensate for that.
  - Minimize over them during training

$$\hat{z}_\delta^{t+1} = z^t + (W_1 \delta) \odot \mathbf{a} \begin{bmatrix} z^t & z^{t-1} \end{bmatrix}^T$$

$$L = \min_\delta \| G_W(\hat{z}_\delta^{t+1}) - x^{t+1} \|_2^2 - \lambda \frac{(z^t - z^{t-1})^T (z^{t+1} - z^t)}{\| z^t - z^{t-1} \| \| z^{t+1} - z^t \|}$$



Algorithm (with latent variables)

Algorithm 1 Minibatch stochastic gradient descent training for prediction with uncertainty. The number of δ-gradient descent steps (k) is treated as a hyper-parameter.

for number of training epochs **do**
  Sample a mini-batch of temporal triplets $\{x^{t-1}, x^t, x^{t+1}\}$
  Set $\delta_0 = 0$
  Forward propagate x^{t-1}, x^t through the network and obtain the codes z^{t-1}, z^t and the prediction $\hat{x}_0^{t+1}$
  for i =1 to k **do**
    Compute the L^2 prediction error
    Back propagate the error through the decoder to compute the gradient $\frac{\partial L}{\partial \delta^{i-1}}$
    Update $\delta_i = \delta_{i-1} - \alpha \frac{\partial L}{\partial \delta^{i-1}}$
    Compute $\hat{z}_{\delta_i}^{t+1} = z^t + (W_1 \delta_i) \odot \mathbf{a} \left[z^t \quad z^{t-1} \right]^T$
    Compute $\hat{x}_i^{t+1} = G_W(z_{\delta^i}^{t+1})$
  end for
  Back propagate the full encoder/predictor loss from Equation 7 using δ_k, and update the weight matrices W and W_1
end for

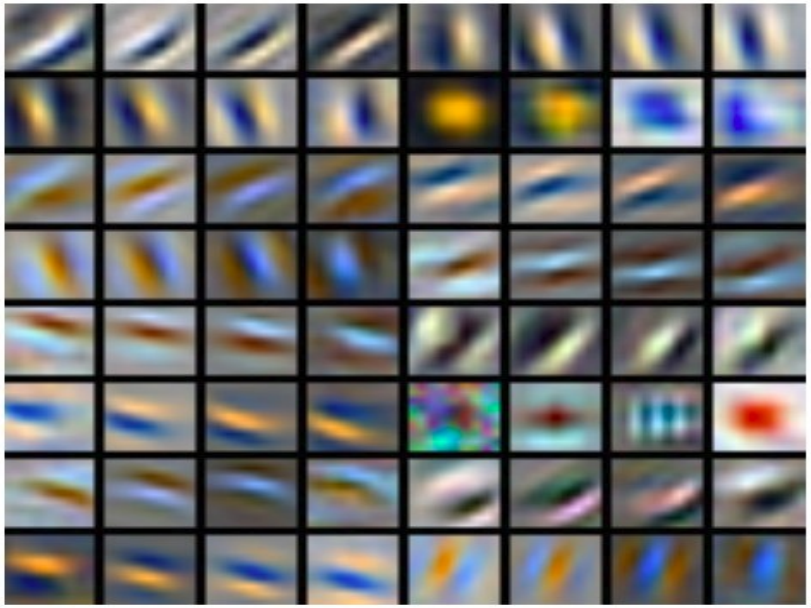
$$\hat{z}_{\delta}^{t+1} = z^t + (W_1 \delta) \odot \mathbf{a} \left[z^t \quad z^{t-1} \right]^T$$

$$L = \min_{\delta} \| G_W(\hat{z}_{\delta}^{t+1}) - x^{t+1} \|_2^2 - \lambda \frac{(z^t - z^{t-1})^T (z^{t+1} - z^t)}{\| z^t - z^{t-1} \| \| z^{t+1} - z^t \|}$$

Architectures

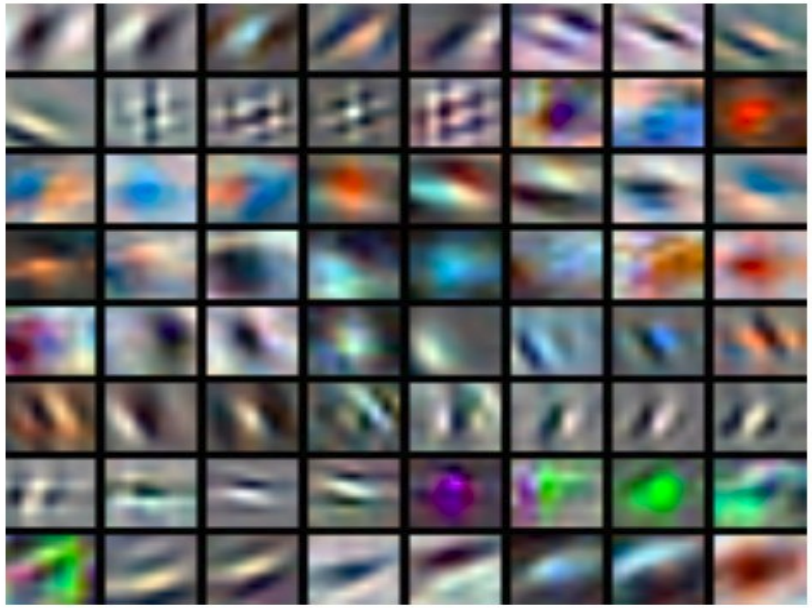
	Encoder	Prediction	Decoder
Shallow Architecture 1	Conv+ReLU $64 \times 9 \times 9$ Phase Pool 4	Average Mag. Linear Extrap. Phase	Conv $64 \times 9 \times 9$
Shallow Architecture 2	Conv+ReLU $64 \times 9 \times 9$ Phase Pool 4 stride 2	Average Mag. Linear Extrap. Phase	Conv $64 \times 9 \times 9$
Deep Architecture 1	Conv+ReLU $16 \times 9 \times 9$ Conv+ReLU $32 \times 9 \times 9$ FC+ReLU 8192×4096	None	FC+ReLU $\mathbf{8192} \times 8192$ Reshape $32 \times 16 \times 16$ SpatialPadding 8×8 Conv+ReLU $16 \times 9 \times 9$ SpatialPadding 8×8 Conv $1 \times 9 \times 9$
Deep Architecture 2	Conv+ReLU $16 \times 9 \times 9$ Conv+ReLU $32 \times 9 \times 9$ FC+ReLU 8192×4096	Linear Extrapolation	FC+ReLU 4096×8192 Reshape $32 \times 16 \times 16$ SpatialPadding 8×8 Conv+ReLU $16 \times 9 \times 9$ SpatialPadding 8×8 Conv $1 \times 9 \times 9$
Deep Architecture 3	Conv+ReLU $16 \times 9 \times 9$ Conv+ReLU $32 \times 9 \times 9$ FC+ReLU 8192×4096 Reshape $64 \times 8 \times 8$ Phase Pool 8×8	Average Mag. Linear Extrap. Phase	Unpool 8×8 FC+ReLU 4096×8192 Reshape $32 \times 16 \times 16$ SpatialPadding 8×8 Conv+ReLU $16 \times 9 \times 9$ SpatialPadding 8×8 Conv $1 \times 9 \times 9$

Filters: pooling over groups of 4 filters.



(a) Shallow Architecture 1

Non-overlapping pooling
4x4x4 (feat,x,y)

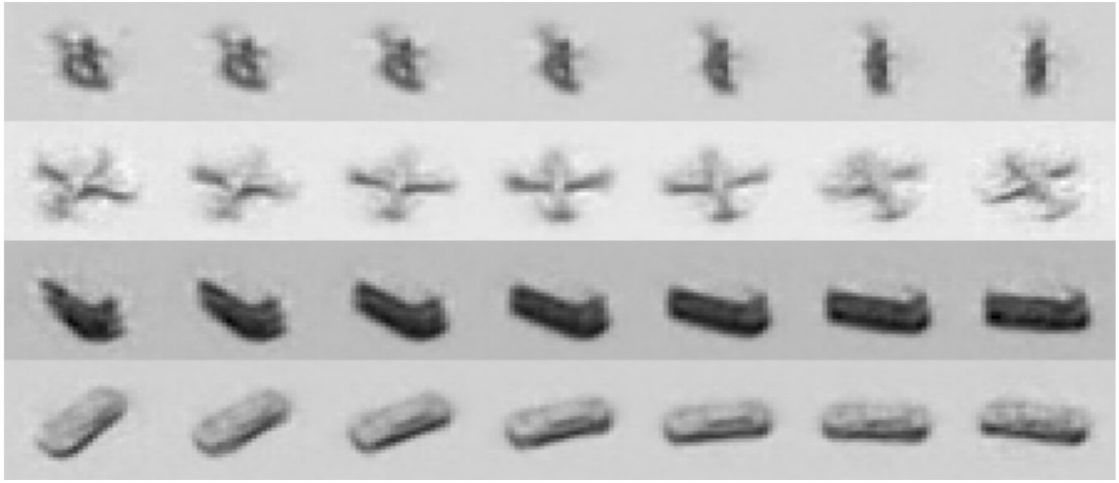


(b) Shallow Architecture 2

Overlapping pooling
4x4x4 (feat,x,y)
Stride 2 over features

Image interpolation in feature space: deterministic world

- **No phase pooling**
- **No curvature regularization**



- **With phase pooling**
- **With curvature regularization**

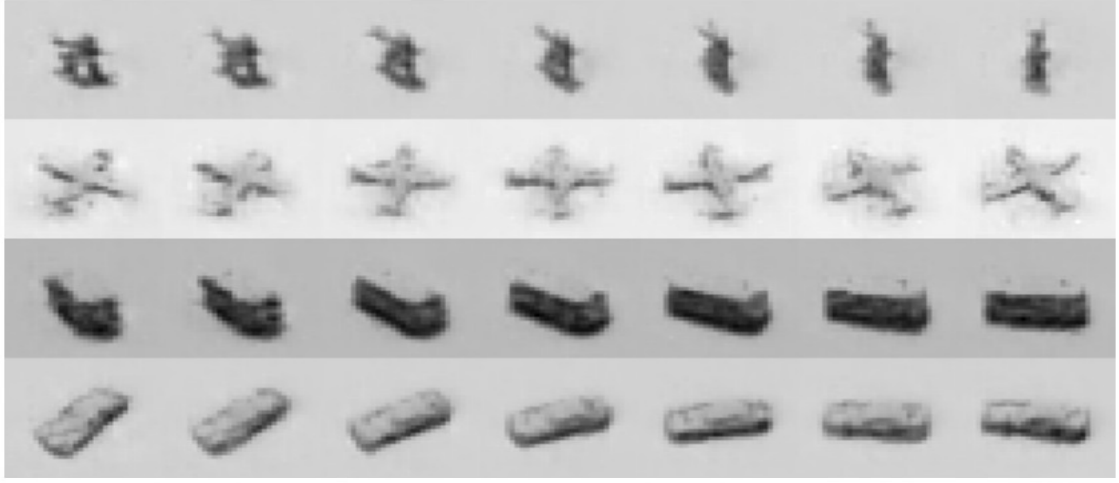


Image interpolation in feature space: deterministic world

- **Image interpolation with the basic model (siamese net)**

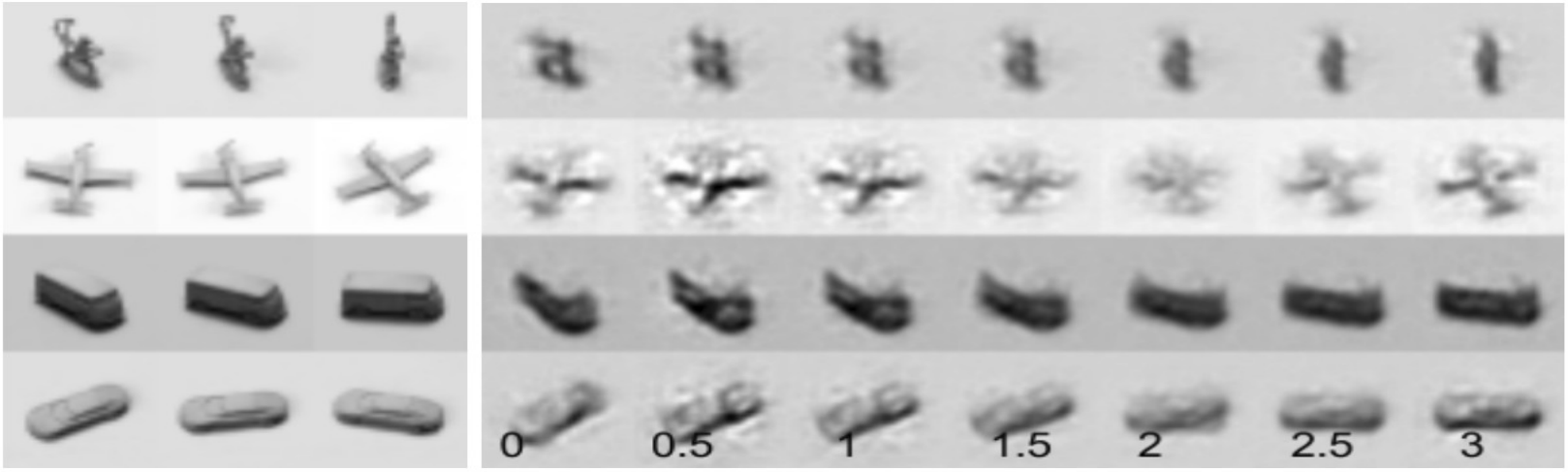
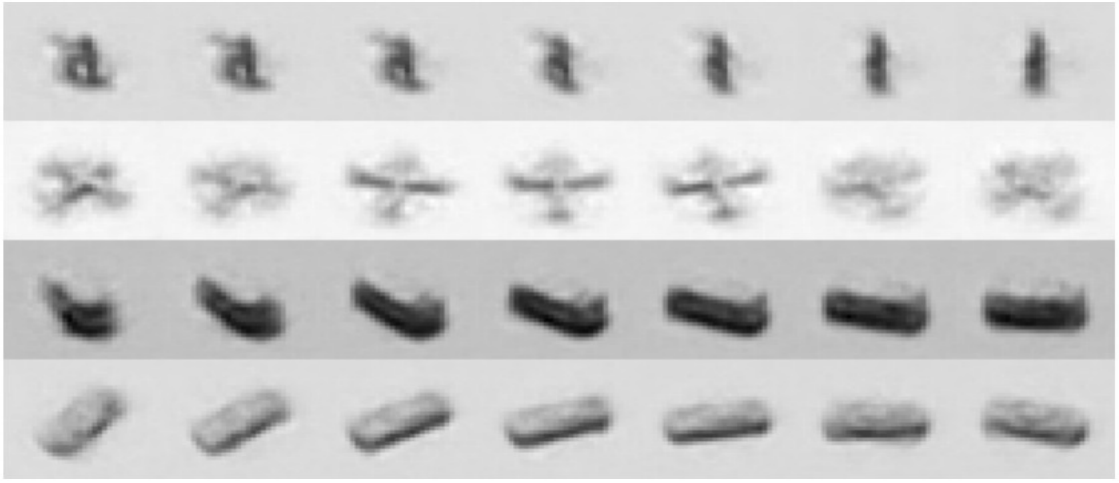


Image interpolation in feature space: unpredictable world

- **Phase interpolation**
- **No latent variable**



- **Phase interpolation**
- **With latent variable**