Part1a

1 Derivation of 2-body interaction

$$
\begin{aligned}
V &= \frac{1}{2}\sum_{p\sigma_p,q\sigma_q,r\sigma_r,s\sigma_s} < p\sigma_p, q\sigma_q|v|r\sigma_r, s\sigma_s > a^\dagger_{p\sigma_p}a^\dagger_{q\sigma_q}a_{s\sigma_s}a_{r\sigma_r} \\
&= \frac{1}{2}\sum_{p\sigma_p,r} < p\sigma_p, p-\sigma_p|v|r\sigma_p, r-\sigma_p > a^\dagger_{p\sigma_p}a^\dagger_{p-\sigma_p}a_{r-\sigma_p}a_{r\sigma_p} \\
&= \frac{1}{2}\sum_{p\sigma_p,r}(-g)a^\dagger_{p\sigma_p}a^\dagger_{p-\sigma_p}a_{r-\sigma_p}a_{r\sigma_p} \\
&= \frac{1}{2}\sum_{p,r}(-g)\left(\left(a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+}\right)+\left(a^\dagger_{p-}a^\dagger_{p+}a_{r+}a_{r-}\right)\right) \\
&= \frac{1}{2}\sum_{p,r}(-g)\left(\left(a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+}\right)+(-1)^2\left(a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+}\right)\right) \\
&= -g\sum_{p,r}a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+} \\
&\equiv -g\sum_{p,r}P^+_pP^-_r.
\end{aligned}
$$

2 Commutation relation

2.1 useful relation

$$
\begin{aligned}
[A,BC] &= B[A,C]+[A,C]B, \\
\{A,BC\} &= B\{A,C\}-\{A,C\}B,
\end{aligned}
$$

then

$$
[AB,CD] = A\{B,C\}D-\{A,C\}BD+CA\{B,D\}-C\{A,D\}B.
$$

2.2 S_z vs H_0

By making use of the relation in sec.2.1, we get

$$\begin{aligned}
[S_z, H_0] &= \sum_{p\sigma_p} \frac{\sigma_p}{2} \sum_{q\sigma_q} (q-1)[a^\dagger_{p\sigma_p} a_{p\sigma_p}, a^\dagger_{q\sigma_q} a_{q\sigma_q}] \\
&= \sum_{p\sigma_p} \frac{\sigma_p}{2} \sum_{q\sigma_q} (q-1)\delta_{p,q}\delta_{\sigma_p,\sigma_q} \left(a^\dagger_{p\sigma_p} a_{q\sigma_q} - a^\dagger_{q\sigma_q} a_{p\sigma_p} \right) \\
&= \sum_{p\sigma_p} \frac{\sigma_p}{2} (p-1) \left(a^\dagger_{p\sigma_p} a_{p\sigma_p} - a^\dagger_{p\sigma_p} a_{p\sigma_p} \right) \\
&= 0.
\end{aligned}$$

2.3 $\boldsymbol{S}^2$ vs H_0

By making use of the relation in sec.2.1, we get

$$\begin{aligned}
[S_+, H_0] &= \sum_{p\sigma_p} (p-1)[a^\dagger_{p+} a_{p-}, a^\dagger_{p\sigma_p} a_{p\sigma_p}] \\
&= \sum_{p\sigma_p} (p-1) \left(-\delta_{\sigma_p +} a^\dagger_{p\sigma_p} a_{p-} + \delta_{\sigma_p -} a^\dagger_{p+} a_{p\sigma_p} \right) \\
&= \sum_{p} (p-1) \left(-a^\dagger_{p+} a_{p-} + a^\dagger_{p+} a_{p-} \right) \\
&= 0,
\end{aligned}$$

$$[S_-, H_0] = (-[S_+, H_0])^\dagger = 0,$$

therefore we get

$$\begin{aligned}
\left[\boldsymbol{S}^2, H_0\right] &= [S_z^2 + \frac{1}{2}(S_+S_- + S_-S_+), H_0] \\
&= [S_z, H_0]S_z + S_z[S_z, H_0] + \frac{1}{2}\left([S_+, H_0]S_- + S_+[S_-, H_0] + [S_-, H_0]S_+ + S_-[S_+, H_0]\right) \\
&= 0.
\end{aligned}$$

2.4 S_z vs V

By making use of the relation in sec.2.1, we get

$$\begin{aligned}
[P_p^+, S_z] &= \sum_{\sigma_p} \frac{\sigma_p}{2} [a^\dagger_{p+} a^\dagger_{p-}, a^\dagger_{p\sigma_p} a_{p\sigma_p}] \\
&= \sum_{\sigma_p} \frac{\sigma_p}{2} \left(a^\dagger_{p\sigma_p} a^\dagger_{p+} \delta_{\sigma_p -} - a^\dagger_{p\sigma_p} a^\dagger_{p-} \delta_{\sigma_p +} \right) \\
&= \left(-\frac{1}{2} \right) a^\dagger_{p-} a^\dagger_{p+} - \left(\frac{1}{2} \right) a^\dagger_{p+} a^\dagger_{p-} \\
&= 0,
\end{aligned}$$

$$[P_p^-, S_z] = \left(-[P_p^+, S_z]\right)^\dagger = 0,$$

therefore we get

$$\begin{aligned}
[S_z, V] &= -g\sum_{p,q}[S_z, P_p^+ P_q^-] \\
&= -g\sum_{p,q}\left([S_z, P_p^+]P_q^- + P_p^+[S_z, P_q^-]\right) \\
&= 0.
\end{aligned}$$

2.5 $\boldsymbol{S}^2$ vs V

By making use of the relation in sec.2.1, we get

$$\begin{aligned}
[P_p^+, S_+] &= [a_{p+}^\dagger a_{p-}^\dagger, a_{p+}^\dagger a_{p-}] \\
&= a_{p+}^\dagger a_{p+}^\dagger \{a_{p-}^\dagger, a_{p-}\} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
[P_p^-, S_+] &= \Big(a_{p-}a_{p+}, a_{p+}^\dagger a_{p-}\Big) \\
&= a_{p-}\{a_{p+}, a_{p+}^\dagger\}a_{p-} \\
&= 0,
\end{aligned}$$

then

$$\begin{aligned}
[S_+, V] &= -g\sum_{p,q}[S_+, P_p^+ P_q^-] \\
&= -g\sum_{p,q}\left([S_+, P_p^+]P_q^- + P_p^+[S_+, P_q^-]\right) \\
&= 0,
\end{aligned}$$

$$[S_-, V] = (-[S_+, V])^\dagger = 0.$$

Therefore we get

$$\begin{aligned}
\left[\boldsymbol{S}^2, V\right] &= [S_z^2 + \frac{1}{2}(S_+S_- + S_-S_+), V] \\
&= [S_z, V]S_z + S_z[S_z, V] + \frac{1}{2}\left([S_+, V]S_- + S_+[S_-, V] + [S_-, V]S_+ + S_-[S_+, V]\right) \\
&= 0.
\end{aligned}$$