Part1a

1 Derivation of 2-body interaction

$$
\begin{aligned}
V &= \frac{1}{2}\sum_{p\sigma_p,q\sigma_q,r\sigma_r,s\sigma_s} < p\sigma_p, q\sigma_q|v|r\sigma_r, s\sigma_s > a^\dagger_{p\sigma_p}a^\dagger_{q\sigma_q}a_{s\sigma_s}a_{r\sigma_r} \\
&= \frac{1}{2}\sum_{p\sigma_p,r} < p\sigma_p, p-\sigma_p|v|r\sigma_p, r-\sigma_p > a^\dagger_{p\sigma_p}a^\dagger_{p-\sigma_p}a_{r-\sigma_p}a_{r\sigma_p} \\
&= \frac{1}{2}\sum_{p\sigma_p,r}(-g)a^\dagger_{p\sigma_p}a^\dagger_{p-\sigma_p}a_{r-\sigma_p}a_{r\sigma_p} \\
&= \frac{1}{2}\sum_{p,r}(-g)\left[\left(a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+}\right)+\left(a^\dagger_{p-}a^\dagger_{p+}a_{r+}a_{r-}\right)\right] \\
&= \frac{1}{2}\sum_{p,r}(-g)\left[\left(a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+}\right)+(-1)^2\left(a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+}\right)\right] \\
&= -g\sum_{p,r}a^\dagger_{p+}a^\dagger_{p-}a_{r-}a_{r+} \\
&\equiv -g\sum_{p,r}P^+_pP^-_r
\end{aligned}
$$

2 Useful relation

$$
\begin{aligned}
[A, BC] &= B[A, C] + [A, C]B \\
\{A, BC\} &= B\{A, C\} - \{A, C\}B
\end{aligned}
$$

then

$$
[AB, CD] = A\{B, C\}D - \{A, C\}BD + CA\{B, D\} - C\{A, D\}B
$$

3 Commutation relation: H_0

$$
\begin{aligned}
[S_z, H_0] &= \sum_{p\sigma_p} \frac{\sigma_p}{2} \sum_{q\sigma_q} (q-1)[a^\dagger_{p\sigma_p} a_{p\sigma_p}, a^\dagger_{q\sigma_q} a_{q\sigma_q}] \\
&= \sum_{p\sigma_p} \frac{\sigma_p}{2} \sum_{q\sigma_q} (q-1)\delta_{p,q}\delta_{\sigma_p,\sigma_q} \left(a^\dagger_{p\sigma_p} a_{q\sigma_q} - a^\dagger_{q\sigma_q} a_{p\sigma_p} \right) \\
&= \sum_{p\sigma_p} \frac{\sigma_p}{2} (p-1) \left(a^\dagger_{p\sigma_p} a_{p\sigma_p} - a^\dagger_{p\sigma_p} a_{p\sigma_p} \right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
[S_+, H_0] &= \sum_{p\sigma_p} (p-1)[a^\dagger_{p+} a_{p-}, a^\dagger_{p\sigma_p} a_{p\sigma_p}] \\
&= \sum_{p\sigma_p} (p-1)[-\delta_{\sigma_p +} a^\dagger_{p\sigma_p} a_{p-} + \delta_{\sigma_p -} a^\dagger_{p+} a_{p\sigma_p}] \\
&= \sum_{p} (p-1)[-a^\dagger_{p+} a_{p-} + a^\dagger_{p+} a_{p-}] \\
&= 0
\end{aligned}
$$

$$
[S_-, H_0] = (-[S_+, H_0])^\dagger = 0
$$

therefore

$$
\begin{aligned}
[S_z, H_0] &= 0 \\
\left[\boldsymbol{S}^2, H_0\right] &= 0
\end{aligned}
$$

4 Commutation relation: V

$$
\begin{aligned}
[P^+_p, S_+] &= [a^\dagger_{p+} a^\dagger_{p-}, a^\dagger_{p+} a_{p-}] \\
&= a^\dagger_{p+} a^\dagger_{p+} \{a^\dagger_{p-}, a_{p-}\} \\
&= 0
\end{aligned}
$$

$$
[P^-_p, S_-] = \left(-[P^+_p, S_+]\right)^\dagger = 0
$$

$$
\begin{aligned}
[P^-_p, S_+] &= [a_{p-} a_{p+}, a^\dagger_{p+} a_{p-}] \\
&= a_{p-} \{a_{p+}, a^\dagger_{p+}\} a_{p-} \\
&= 0
\end{aligned}
$$

$$[P_p^+, S_-] = \left(-[P_p^-, S_+]\right)^\dagger = 0$$

$$\begin{aligned}
[P_p^+, S_z] &= \sum_{\sigma_p} \frac{\sigma_p}{2} [a_{p+}^\dagger a_{p-}^\dagger, a_{p\sigma_p}^\dagger a_{p\sigma_p}] \\
&= \sum_{\sigma_p} \frac{\sigma_p}{2} \left(a_{p\sigma_p}^\dagger a_{p+}^\dagger \delta_{\sigma_p -} - a_{p\sigma_p}^\dagger a_{p-}^\dagger \delta_{\sigma_p +} \right) \\
&= \left(-\frac{1}{2}\right) a_{p-}^\dagger a_{p+}^\dagger - \left(\frac{1}{2}\right) a_{p+}^\dagger a_{p-}^\dagger \\
&= 0
\end{aligned}$$

$$[P_p^-, S_z] = \left(-[P_p^+, S_z]\right)^\dagger = 0$$

therefore

$$\begin{aligned}
[S_z, V] &= 0 \\
\left[\boldsymbol{S}^2, V\right] &= 0
\end{aligned}$$