Recursive Form of the Discrete Fourier Transform for Two-Dimensional Signals

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Abstract. In this paper, recursive fast Fourier transform is presented for two-dimensional signals. When applying to real-time analysis, the computational efficiency is highly improved by integrating a recursive procedure. The recursive procedure highly reduces the number of complex arithmetic operations, and provide detailed spectral analysis for one or two-dimensional signals.
In the first stage, the recursive algorithm is realized for one-dimensional signals. Then, recursive fast Fourier transform is presented for two-dimensional signals. The advantages of the recursive algorithm are presented by giving examples for one and two-dimensional signals.

1 Introduction

In past years, time-frequency signal representation such as the short-time Fourier transform (STFT), the short-time Hartley transform and the Wigner distribution, have received considerable attention as powerful tools for analysing a variety of signals and systems [1,2,3]. In particular, if the frequency content is time varying, as in non-stationary signals, these approaches are very attractive. In applications of spectral analysis such as speech processing and the detection or estimation of a narrow band spectral peak, the spectral content over a small portion of the band or even at arbitrary frequencies is required.

Not long after the key contribution to digital spectral analysis, the development of the fast Fourier transform (FFT) by Cooley and Tukey [4], Halberstein [5] proposed the idea of evaluating the successive DFTs using the recursive computation. Bongiovanni presented the formulated procedure, which allowed a moving size M to be any integer power of 2; nonetheless, M is no greater than the frame size [6]. Several algorithms for efficient computation of moving-frame DFTs have been developed thereafter [7].

2 Recursive Procedures of the Fourier Transform

In the paper, the moving size M is selected as one. The recursive procedure for the succeeding DFT process uses sequence samples from x(1) to x(N-1), which are computed for the previous window. Fig. 1. shows the previous and succeeding windows for recursive procedure.

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The results obtained from one-dimensional signals are extended into the analysis of two-dimensional signals.

2.1 Recursive Procedure for One-Dimensional Signals

Let X(k) denote the N-point DFT of non-stationary long-term sequence x(n). Since the running Fourier spectrum is updated at a rate of one sample point (M=1), X(k) is given by the following equation:

$$X(k) = x(0)e^{\frac{-j2\pi k.0}{N}} + x(1)e^{\frac{-j2\pi k.1}{N}} + \ldots\ldots + x(N-1)e^{\frac{-j2\pi k(N-1)}{N}} . \quad (1)$$

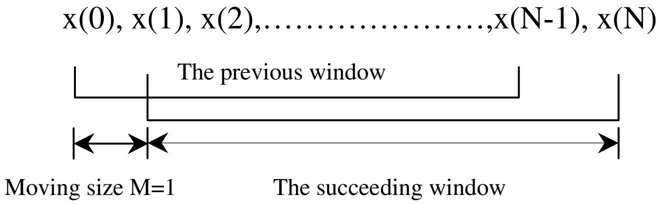


Fig. 1. The previous and succeeding windows for the recursive procedure

The succeeding DFT becomes as follows:

$$X'(k) = x(1+M)e^{\frac{-j2\pi k.0}{N}} + \ldots + x(N+M-1)e^{\frac{-j2\pi k(N-2)}{N}} + x(N+M)e^{\frac{-j2\pi k(N-1)}{N}} . \quad (2)$$

In the analysis, moving size M is selected as 1. Let us subtract the first term of the summation in equation (1) from X(k),

$$X(k) - x(0)e^{\frac{-j2\pi k.0}{N}} = x(1)e^{\frac{-j2\pi k.1}{N}} + \ldots\ldots + x(N-1)e^{\frac{-j2\pi k(N-1)}{N}} \quad (3)$$

and then, multiply equation (3) by $e^{\frac{+j2\pi k.1}{N}}$:

$$e^{\frac{+j2\pi k.1}{N}} \cdot \left(X(k) - x(0)e^{\frac{-j2\pi k.0}{N}} \right) = x(1)e^{\frac{-j2\pi k.0}{N}} + \ldots\ldots + x(N-1)e^{\frac{-j2\pi k(N-2)}{N}} . \quad (4)$$

Let us subtract the last term of $X'(k)$ in equation (2) from $X'(k)$,

$$X'(k) - x(N)e^{\frac{-j2\pi k(N-1)}{N}} = x(1)e^{\frac{-j2\pi k.0}{N}} + \ldots\ldots + x(N-1)e^{\frac{-j2\pi k(N-2)}{N}} \quad (5)$$

and equate the left side of equation (4) to that of equation (5):

$$X'(k) - x(N)e^{\frac{-j2\pi k(N-1)}{N}} = X(k) \cdot e^{\frac{j2\pi k}{N}} - x(0) \cdot e^{\frac{j2\pi k}{N}} ; \qquad e^{\frac{-j2\pi k(N-1)}{N}} = e^{\frac{+j2\pi k}{N}}$$

$$X'(k) = e^{\frac{j2\pi k}{N}} \cdot \big(X(k) + x(N) - x(0) \big) . \quad (6)$$

There exists a class of algorithms, called the fast Fourier transform (FFT), which requires $O(N \times \log_2 N)$ operations for the succeeding DFT, where one operation is a real multiplication and a real addition. Equation (6) requires one multiplication and two addition operations for a single term of the succeeding DFT (for M=1). The overall computation requires O(N) multiplication and O(2N) addition operations.

2.2 Recursive Procedure for Two-Dimensional Signals

In this paper, two different recursive procedures are formulated depending on the moving direction of a sub-image (for M=1). Fig. 2 shows the sub-image moved along n1 axis. Two square regions (shown by bold borders) represent the $x(n_1,n_2)$ and $x(n_1+1,n_2)$ sub-images, where the sub-image $x(n_1+1,n_2)$ is shown by dark gray.

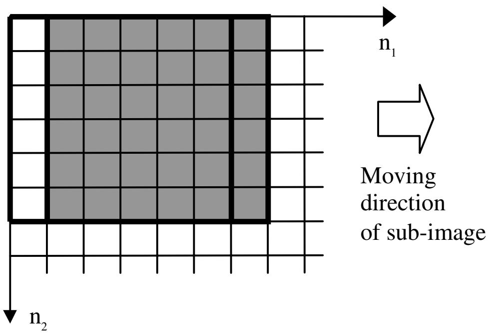


Fig. 2. Sub-image moved along the n_1 axis

The two-dimensional DFT of an $N_1 \times N_2$ sub-image ($x(n_1,n_2)$) is a separable transform defined as follows:

$$X(k_1,k_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2).e^{\frac{-j2\pi k_1 n_1}{N_1}}\,e^{\frac{-j2\pi k_2 n_2}{N_2}} \qquad W_1=e^{\frac{-j2\Pi}{N_1}} \qquad W_2=e^{\frac{-j2\Pi}{N_2}}$$

$$\begin{aligned}X(k_1,k_2)=\ & x(0,0).W_1^{k_1 0}\cdot W_2^{k_2 0}+x(0,1).W_1^{k_1 0}\cdot W_2^{k_2 1}+\ldots\ldots\\ & +x(N_1-1,N_2-1).W_1^{k_1(N_1-1)}\cdot W_2^{k_2(N_2-1)}\end{aligned} \tag{7}$$

The succeeding 2D-DFT becomes

$$X(k_1,k_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1+1,n_2).e^{\frac{-j2\pi k_1 n_1}{N_1}}\,e^{\frac{-j2\pi k_2 n_2}{N_2}}$$

$$\begin{aligned}X'(k_1,k_2)=\ & x(1,0).W_1^{k_1 0}\cdot W_2^{k_2 0}+x(1,1).W_1^{k_1 0}\cdot W_2^{k_2 1}+\ldots\ldots\\ & +x(N_1,N_2-1).W_1^{k_1(N_1-1)}\cdot W_2^{k_2(N_2-1)}\end{aligned} \tag{8}$$

Let us subtract the terms belonging to $x(0, n_2)$ in equation (7) from $X(k_1,k_2)$,

$$X(k_1,k_2)-\sum_{n_2=0}^{N_2-1} x(0,n_2)\cdot W_1^{k_1 0}\cdot W_2^{k_2 n_2}=\sum_{n_1=1}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1,n_2).e^{\frac{-j2\pi k_1 n_1}{N_1}}\,e^{\frac{-j2\pi k_2 n_2}{N_2}} \tag{9}$$

and then, multiply equation (9) by $W_1^{-k_1}$;

$$W_1^{-k_1}\left[\mathrm{X}(\mathrm{k}_1,\mathrm{k}_2)-\sum_{n_2=0}^{N_2-1}x(0,n_2)\cdot W_1^{k_1 0}\cdot W_2^{k_2 n_2}\right]=\sum_{n_1=1}^{N_1-1}\sum_{n_2=0}^{N_2-1}\mathrm{x}(\mathrm{n}_1,\mathrm{n}_2).\mathrm{W}_1^{k_1(n_1-1)}.\mathrm{W}_2^{k_2 n_2} \quad (10)$$

Now, let us subtract the terms belonging to $x(N_1, n_2)$ in equation (8) from $X'(k_1,k_2)$,

$$X'(k_1,k_2)-\sum_{n_2=0}^{N_2-1}x(N_1,n_2)W_1^{k_1(N_1-1)}\cdot W_2^{k_2 n_2}=\sum_{n_1=0}^{N_1-2}\sum_{n_2=0}^{N_2-1}\mathrm{x}(\mathrm{n}_1+1,\mathrm{n}_2).\mathrm{W}_1^{k_1 n_1}.\mathrm{W}_2^{k_2 n_2} \quad (11)$$

and equate the left side of equation (10) to that of equation (11):

$$W_1^{-k_1}\left[\mathrm{X}(\mathrm{k}_1,\mathrm{k}_2)-\sum_{n_2=0}^{N_2-1}x(0,n_2)\cdot W_2^{k_2 n_2}\right]=X'(k_1,k_2)-\sum_{n_2=0}^{N_2-1}x(N_1,n_2)\cdot W_1^{k_1(N_1-1)}\cdot W_2^{k_2 n_2}$$

$$X'(k_1,k_2)=W_1^{-k_1}\left[\mathrm{X}(\mathrm{k}_1,\mathrm{k}_2)-\sum_{n_2=0}^{N_2-1}x(0,n_2)\cdot W_2^{k_2 n_2}+\sum_{n_2=0}^{N_2-1}x(N_1,n_2)\cdot W_2^{k_2 n_2}\right]. \quad (12)$$

For the sub-window moved along n2 axis, recursive procedure is formulated as follows:

$$X'(k_1,k_2)=W_2^{-k_2}\left[\mathrm{X}(\mathrm{k}_1,\mathrm{k}_2)-\sum_{n_1=0}^{N_1-1}x(n_1,0)\cdot W_1^{k_1 n_1}+\sum_{n_1=0}^{N_1-1}x(n_1,N_2)\cdot W_1^{k_1 n_1}\right]. \quad (13)$$

Two-dimensional fast Fourier transform requires $O(N_1 \times N_2 \times Log_2 N_1)$ operations for the succeeding DFT, where one operation is a real multiplication and a real addition.

For the recursive algorithm, firstly, the second and third terms in square brackets of equation (12) are computed and stored for all k_2 values. These processes require $O(2 \times N_2 \times N_2)$ multiplication operations. And then, $W_1^{k_1}$ is multiplied by the term in the square brackets of equation (12). This process requires $O(N_1 \times N_2)$ multiplication operations. Thus, the overall computation requires $O(N_1 \times N_2 + 2 \times N_2 \times N_2)$ multiplication operations.

For N_2=1, equation (12) is defined as follows:

$$X'(k_1,1)=W_1^{-k_1}[\ \mathrm{X}(\mathrm{k}_1,1)-x(0,0)\cdot W_2^{k_2 0}+x(N_1,0)\cdot W_2^{k_2 0}\ ]. \quad (14)$$

For N_1=1, equation (13) is defined as follows:

$$X'(1,k_2)=W_2^{-k_2}[\ \mathrm{X}(1,\mathrm{k}_2)-x(0,0)\cdot W_1^{k_1 0}+x(0,N_2)\cdot W_1^{k_1 0}\ ]. \quad (15)$$

It is observed that equation (6) has a similar form with equations (14) and (15).

3 Computer Simulations

For the analysis of one-dimensional signals, a signal with four different frequency components at four different time intervals is given as an example in order to determine the starting and ending times of the bursts. The signal is sampled at 2048 Hz, and contains 2048 data. The interval of 171 to 190 msec. has a 400 Hz sinusoid, the interval of 195 to 215 msec. has a 200 Hz sinusoid, the interval of 732 to 747 msec. has a 300 Hz sinusoid, and finally the interval 879 to 898 msec. has a 100 Hz sinusoid. In the analysis, a window of 128 samples is formed. Fig. 3 shows the STFS obtained by the recursive procedure. It is observed that the recursive procedure gives more detailed spectral analysis by consuming less computational time.

The results obtained from one-dimensional signals are extended to the analysis of two-dimensional signals. As an example, in texture processing, textures and their boundaries can be determined by analysing the Fourier spectrum. By choosing the moving size M as 1, changes in the Fourier spectrum will be observed in detail.

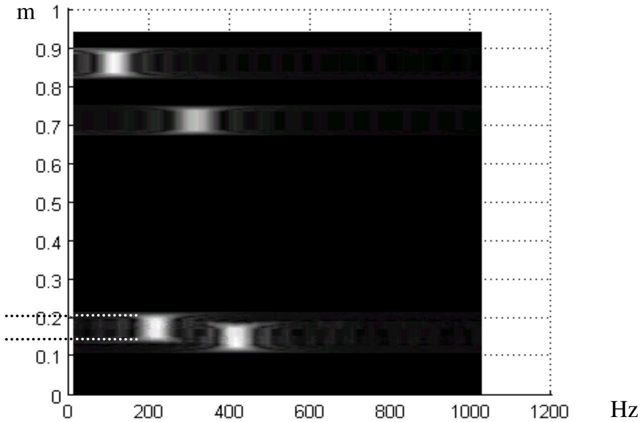


Fig. 3. The STFS obtained by the recursive procedure

As an application to image processing, the ultrasound image of kidney cyst is segmented by using the ISOM network developed in a previous study [8]. The kidney cyst image is shown in Fig. 4.a. The only difference from the previous study is to move the sub-image over the entire image, instead of splitting the entire image into square blocks. Feature vectors are formed by using 2D-DFT of the sub-images. Training set consists of the feature vectors formed by moving the sub-image over the entire ultrasound image.

Threshold value determines the number of classes (tissues) in the ultrasound image. Threshold is selected as 60000 (the same as in the previous study). Size of the sub-image is selected as 4×4 pixels. For 100 sub-images, the consumed times by using recursive process and the standart DFT are 8 and 40 seconds, respectively. Ultrasound image of kidney cyst in Fig. 4.b is segmented into five tissues.

In this study, the recursive 2D-DFT algorithm enables us to analyse the Fourier spectrum in detail. Hence, better classification performance is achieved by using the recursive algorithm.

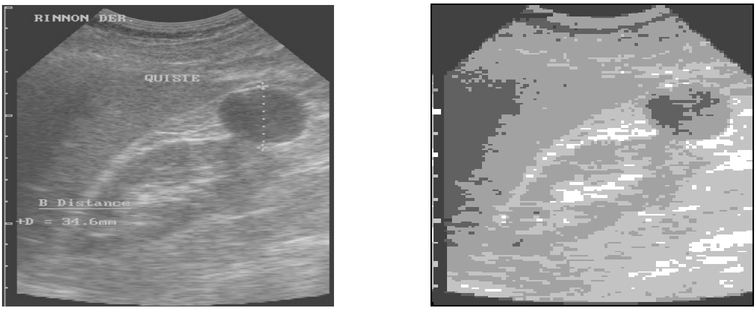
All simulations are performed on Pentium III-450 MHz PC using MATLAB 6.0.

4 Conclusion

In the analysis of one-dimensional signals, the starting and ending times of bursts can be determined easily, assuming that the moving window does not contain the same frequency content at disconnected locations in the window. In Fig. 3, it is observed that the durations of the four frequencies in the Fourier spectrum are longer than the values explained in the section of Computer Simulations. The ending times of bursts are easily obtained as t_2 from Fig. 3. However, the starting time of a burst is found by adding the t_1 value appeared in Fig. 3 to the duration of the moving window. This analysis enables us to find the starting and ending times of bursts in detail.

In the previous study [8], we developed a novel method for the segmentation of ultrasound images. Splitting the entire image into square blocks decreased the performance of the classification process. In this study, by moving the sub-image over the entire image, the boundaries of textures are determined in detail. However, this process increases the overall computational time. By using the recursive algorithm, the consumed times for training and classification processes are decreased.

In this study, the recursive two-dimensional Fourier transform is developed for general-purpose image processing applications. The recursive algorithm can be used as a pre-process for pattern recognition applications or texture processing.



(a) (b)

Fig. 4.(a) Ultrasound image of kidney cyst, (b) segmented image by the ISOM.

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