Technical note in support of the paper:
PASTA

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1 Moments

In this note we provide the theoretical results underpinning the methodology presented in the paper [] submitted for review. We first introduce some preliminary notation and concepts.

1.1 Notation

- A point cloud in $\mathbb{R}^3$ (some results may be generalized to $\mathbb{R}^n$) is described as a finite set with $d \in \mathbb{N}$ elements:
$$X = \{x^{(1)}, x^{(2)}, \dots, x^{(d)}\}, \tag{1}$$
where $x^{(i)} \in \mathbb{R}^3$ for each $i \in \{1, 2, \dots, d\}$.

- We denote the convex hull of a finite set of points X as:
$$H(X) = \left\{ \sum_{i=0}^{d} \lambda_i x^{(i)} \mid \sum_{i=0}^{d} \lambda_i = 1 \text{ and } \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } i \in \{1, 2, \dots, d\} \right\}. \tag{2}$$

- Given a compact set $H \subset \mathbb{R}^n$, and μ the Lebesgue measure on $\mathbb{R}^n$, we denote the volume of H as:
$$|H| = \mu(H) = \int_H \mathrm{d}\mu, \tag{3}$$
and its radius as:
$$\rho(H) = \min_{q \in \mathbb{R}^n} \max_{x \in H} \|x - q\|. \tag{4}$$
Intuitively, ρ is the radius of the smallest ball containing H.

- We define the "centroid", "center of mass", or "first moment" of a compact set $H \subset \mathbb{R}^3$ as:
$$c = \frac{1}{|H|} \int_{x \in E} x \, \mathrm{d}\mu. \tag{5}$$

- We define the "covariance matrix", or "second moment" of H relative to c as:
$$\Sigma = \frac{1}{|H|} \int_{x \in H} (x - c)(x - c)^T \, \mathrm{d}\mu. \tag{6}$$

- Given two compact sets $H_1, H_2 \subset \mathbb{R}^n$ of non-zero measure, we introduce a notion of "overlap" between the sets. This overlap is represented by a number $\delta \in [0, 1]$, and is defined as follows:
$$\delta = \frac{|H_1 \cap H_2|}{\max\{|H_1|, |H_2|\}}, \tag{7}$$
where the volume of the empty set is considered to be zero. Intuitively, $\delta = 0$ denotes that H_1 and H_2 are disjoint, while $\delta = 1$ indicates that $H_1 = H_2$.

1.2 Ideal Reconstruction

We consider the following scenario. First, note that if we have available a compact set H_1, and its rigid transformation $H_2 = p + RH_1$, according to some translation vector $p \in \mathbb{R}^n$ and orthonormal rotation matrix $R \in \mathbb{R}^{n \times n}$, these can be exactly reconstructed (up to a sign ambiguity on R) via knowledge of the first and second moments of H_1 and H_2. The relationship between the moments of H_1 and H_2 is:

$$\begin{aligned}
c_1 &= \frac{\int_{H_1} x \, \mathrm{d}\mu}{\int_{H_1} \mathrm{d}\mu}, & \Sigma_1 &= \frac{\int_{H_1} (x-c_1)(x-c_1)^T \, \mathrm{d}\mu}{\int_{H_1} \mathrm{d}\mu}, \\
c_2 &= \frac{\int_{H_2} x \, \mathrm{d}\mu}{\int_{H_2} \mathrm{d}\mu} & \Sigma_2 &= \frac{\int_{H_2} (x-c_2)(x-c_2)^T \, \mathrm{d}\mu}{\int_{H_2} \mathrm{d}\mu} \\
&= \frac{\int_{H_1} (p + Rz)\det(R) \, \mathrm{d}\mu}{\int_{H_1} \det(R) \, \mathrm{d}\mu} & &= \frac{\int_{H_2} (x - (p + Rc_1))(x - (p + Rc_1))^T \, \mathrm{d}\mu}{\int_{H_2} \mathrm{d}\mu} \\
&= p + R \int_{H_1} z \, \mathrm{d}\mu & &= \frac{\int_{H_1} ((p + Rz) - (p + Rc_1))((p + Rz) - (p + Rc_1))^T \det(R) \, \mathrm{d}\mu}{\int_{H_1} \det(R) \, \mathrm{d}\mu} \\
&= p + Rc_1, & &= \frac{\int_{H_1} R(z - c_1)(z - c_1)^T R^T \, \mathrm{d}\mu}{\int_{H_1} \mathrm{d}\mu} \\
& & &= R\Sigma_1 R^T.
\end{aligned} \tag{8}$$

Where for the computation of both c_2 and Σ_2, we used the change of variables $g(x) = p + Rz$ to rewrite the integrals over H_2 as integrals over H_1.

Then, with $c_1, c_2, \Sigma_1, \Sigma_2$ known and the relationship between the the moments, we need a to find solution in (p, R) to the system of equations:

$$\begin{cases} c_2 = p + Rc_1 \\ \Sigma_2 = R\Sigma_1 R^T. \end{cases} \tag{9}$$

First, observe that the covariance matrices are symmetric, and thus can be rewritten as $\Sigma_1 = V_1 \Lambda_1 V_1^T$, and $\Sigma_2 = V_2 \Lambda_2 V_2^T$, where Λ is a diagonal matrix whose i-th element is the i-th eigenvalue of the corresponding Σ, and V is the matrix whose i-th column is the eigenvector of Σ corresponding to its i-th eigenvalue. In addition, since the diagonalization of a matrix is unique up to a reordering of the eigenvalues, we know that $\Lambda_1 = \Lambda_2$, because:

$$R\Sigma_1 R^T = RV_1\Lambda_1 V_1^T R^T = V_2 \Lambda_1 V_2^T = \Sigma_2, \quad \text{where } V_2 = RV_1. \tag{10}$$

From the above we can compute R as:

$$R = V_2 V_1^T. \tag{11}$$

With R known, the translation vector can also be obtained:

$$p = c_2 - Rc_1 = c_2 - V_2 V_1^T c_1. \tag{12}$$

1.3 Imperfect Reconstruction

In the previous section we have shown how, under ideal conditions, the rigid transformation between two volumes can be obtained from knowledge of their first and second moments. In a real scenario, the sets H_1 and H_2 are obtained from point clouds corresponding to scans of some shape or environment, and if the scans do not sample the same points, in general we won't be able to compute the moments of H_2, but those of some $\tilde{H}_2$ that is the rigid transformation of some "perturbed" $\tilde{H}_1$, where the perturbation is caused by the difference in the sampled points. Following, we provide bounds on the reconstruction of (p, R) when we only have available $\tilde{H}_2 = p + R\tilde{H}_1$, parameterized by how "close" $\tilde{H}_1$ is to H_1.

If we apply the same computations from the previous section to this imperfect scenario, the rotation matrix would be computed as:

$$\hat{R} = \tilde{V}_2 V_1^T = R\tilde{V}_1 V_1^T, \quad \text{where } \tilde{V}_2 = R\tilde{V}_1. \tag{13}$$

If we define $\tilde{V}_1 = EV_1$ for some error rotation matrix[1] E, we arrive at

$$\hat{R} = REV_1V_1^T = RE. \tag{14}$$

Consequently, we can write the distance between $\hat{R}$ and R as:

$$\begin{aligned} \|\hat{R} - R\| &= \|RE - R\| \\ &= \|R(E - I)\| \\ &= \|E - I\|, \end{aligned} \tag{15}$$

or how close the error rotation E is to the identity (the null rotation). Note that the last equality holds because R is norm preserving.

Similarly, under these non-ideal conditions, the estimated translation vector would be:

$$\hat{p} = \tilde{c}_2 - \hat{R}c_1, \tag{16}$$

and if we define $e = \tilde{c}_1 - c_1$, we can bound the difference between the estimated and the real translation vector as:

$$\begin{aligned} \|\hat{p} - p\| &= \|\tilde{c}_2 - \hat{R}c_1 - c_2 + Rc_1\| \\ &= \|p + R\tilde{c}_1 - REc_1 - p - Rc_1 + Rc_1\| \\ &= \|R(c_1 + e) - REc_1\| \\ &= \|R((I - E)c_1 + e)\| \\ &= \|(I - E)c_1 + e\| \\ &\leq \|I - E\|\|c_1\| + \|e\|, \end{aligned} \tag{17}$$

where again the penultimate equation holds because R is norm preserving. In the following sections we provide bounds on $\|e\| = \|\tilde{c}_1 - c_1\|$ and $\|I - E\| = \|I - \tilde{V}_1V_1^T\|$.

1.4 Moment Bounds

The first result describes a bound on how much the first moments of two compact sets H, $\tilde{H}$ can differ depending on the level of overlap δ and radius of the union of the two sets $\rho(H \cup \tilde{H})$.

Lemma 1 (First moment perturbation) *Let $H, \tilde{H}$ be compact subsets of R^3 of non-zero measure, and denote their centroids as $c, \tilde{c}$ respectively. Further, assume that they have an overlap of δ. Then:*

$$\|c - \tilde{c}\| \leq 3(1 - \delta)\rho(H \cup \tilde{H}). \tag{18}$$

Proof: To start, we define the following vector:

$$b = \arg\min_{q \in \mathbb{R}^3} \max_{x \in H \cup \tilde{H}} \|x - q\|. \tag{19}$$

We introduce this quantity to perform some of the bounds later in the proof.

We first rewrite the expression of the difference between the centroids of H and $\tilde{H}$ as follows:

$$\begin{aligned} \|c - \tilde{c}\| &= \|c - b - \tilde{c} + b\| \\ &= \left\| \frac{1}{|H|} \int_{x \in H} x \,\mathrm{d}\mu - b - \frac{1}{|\tilde{H}|} \int_{x \in \tilde{H}} x \,\mathrm{d}\mu + b \right\| \\ &= \left\| \frac{1}{|H|} \int_{x \in H} (x - b) \,\mathrm{d}\mu - \frac{1}{|\tilde{H}|} \int_{x \in \tilde{H}} (x - b) \,\mathrm{d}\mu \right\|, \end{aligned} \tag{20}$$

where the second equality can be obtained by noting that:

$$\frac{1}{|H|} \int_H \mathrm{d}\mu = \frac{1}{|\tilde{H}|} \int_{\tilde{H}} \mathrm{d}\mu = 1. \tag{21}$$

[1] This always exists since V_1 and $\tilde{V}_1$ are orthonormal.

We can now proceed to bound the difference between the centroids. In doing so, we will exploit the fact that the integral over H can be split into a sum of two integrals: one over $H \cap \tilde{H}$, and one over $H \setminus \tilde{H}$, with $\setminus$ denoting the set difference operation. The integral over $\tilde{H}$ can be split in an analogous way, and we can write:

$$\begin{aligned}
\|c - \tilde{c}\| &= \left\| \frac{1}{|H|} \int_{x \in H} (x-b)\,\mathrm{d}\mu - \frac{1}{|\tilde{H}|} \int_{x \in \tilde{H}} (x-b)\,\mathrm{d}\mu \right\| \\
&= \left\| \frac{1}{|H|} \left(\int_{x \in H \cap \tilde{H}} (x-b)\,\mathrm{d}\mu + \int_{x \in H \setminus \tilde{H}} (x-b)\,\mathrm{d}\mu \right) - \frac{1}{|\tilde{H}|} \left(\int_{x \in H \cap \tilde{H}} (x-b)\,\mathrm{d}\mu + \int_{x \in \tilde{H} \setminus H} (x-b)\,\mathrm{d}\mu \right) \right\| \\
&\leq \left\| \frac{1}{|H|} \int_{x \in H \cap \tilde{H}} (x-b)\,\mathrm{d}\mu - \frac{1}{|\tilde{H}|} \int_{x \in H \cap \tilde{H}} (x-b)\,\mathrm{d}\mu \right\| + \left\| \frac{1}{|H|} \int_{x \in H \setminus \tilde{H}} (x-b)\,\mathrm{d}\mu - \frac{1}{|\tilde{H}|} \int_{x \in \tilde{H} \setminus H} (x-b)\,\mathrm{d}\mu \right\| \\
&\leq \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \left\| \int_{x \in H \cap \tilde{H}} (x-b)\,\mathrm{d}\mu \right\| + \frac{1}{|H|} \left\| \int_{x \in H \setminus \tilde{H}} (x-b)\,\mathrm{d}\mu \right\| + \frac{1}{|\tilde{H}|} \left\| \int_{x \in \tilde{H} \setminus \tilde{H}1} (x-b)\,\mathrm{d}\mu \right\| \\
&\leq \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \int_{x \in H \cap \tilde{H}} \|x-b\|\,\mathrm{d}\mu + \frac{1}{|H|} \int_{x \in H \setminus \tilde{H}} \|x-b\|\,\mathrm{d}\mu + \frac{1}{|\tilde{H}|} \int_{x \in \tilde{H} \setminus \tilde{H}1} \|x-b\|\,\mathrm{d}\mu \\
&\leq \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \int_{x \in H \cap \tilde{H}} \rho(H \cup \tilde{H})\,\mathrm{d}\mu + \frac{1}{|H|} \int_{x \in H \setminus \tilde{H}} \rho(H \cup \tilde{H})\,\mathrm{d}\mu + \frac{1}{|\tilde{H}|} \int_{x \in \tilde{H} \setminus \tilde{H}1} \rho(H \cup \tilde{H})\,\mathrm{d}\mu \\
&\leq \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| |H \cap \tilde{H}| \rho(H \cup \tilde{H}) + \frac{|H \setminus \tilde{H}|}{|H|} \rho(H \cup \tilde{H}) + \frac{|\tilde{H} \setminus H|}{|\tilde{H}|} \rho(H \cup \tilde{H}) \\
&\leq (1-\delta)\rho(H \cup \tilde{H}) + (1-\delta)\rho(H \cup \tilde{H}) + (1-\delta)\rho(H \cup \tilde{H}) \\
&= 3(1-\delta)\rho(H \cup \tilde{H}).
\end{aligned} \tag{22}$$

The fourth inequality holds by definition of b: the vector whose maximum distance from any point in H or $\tilde{H}$ is the radius of $H \cup \tilde{H}$. The sixth inequality holds because of the following observations:

- $\left| \frac{|\tilde{H}| - |H|}{|\tilde{H}|} \right| \leq \frac{|\tilde{H} \setminus H|}{|\tilde{H}|} = \frac{|\tilde{H}| - |H \cap \tilde{H}|}{|\tilde{H}|} = 1 - \frac{|H \cap \tilde{H}|}{|\tilde{H}|} \leq 1 - \frac{|H \cap \tilde{H}|}{\max\{|H|, |\tilde{H}|\}} = 1 - \delta$
- $\frac{|H \cap \tilde{H}|}{|H|} \leq 1$
- $\frac{|H \setminus \tilde{H}|}{|H|} = \frac{|H| - |H \cap \tilde{H}|}{|H|} = 1 - \frac{|H \cap \tilde{H}|}{|H|} \leq 1 - \frac{|H \cap \tilde{H}|}{\max\{|H|, |\tilde{H}|\}} = 1 - \delta.$

□

* * *

We can provide a theorem analogous to the previous one for the perturbation of the second moment.

Lemma 2 (Second moment perturbation) *Let $H, \tilde{H}$ be non empty bounded subsets of R^3 and denote their covariance matrices as $\Sigma, \tilde{\Sigma}$ respectively. Further, assume that they have an overlap of δ. Then:*

$$\|\Sigma - \tilde{\Sigma}\| \leq \left(25(1-\delta)^2 + 8(1-\delta)\right) \rho^2(H \cap \tilde{H}). \tag{23}$$

Proof: The norm of the difference between Σ and $\tilde{\Sigma}$ is:

$$\begin{aligned}
\|\Sigma - \tilde{\Sigma}\| &= \left\| \frac{1}{|H|} \int_H (x-c)(x-c)^T \, \mathrm{d}\mu - \frac{1}{|\tilde{H}|} \int_{\tilde{H}} (x-\tilde{c})(x-\tilde{c})^T \, \mathrm{d}\mu \right\| \\
&= \| \frac{1}{|H|} \left(\int_{H\cap\tilde{H}} (x-c)(x-c)^T \, \mathrm{d}\mu + \int_{H\setminus\tilde{H}} (x-c)(x-c)^T \, \mathrm{d}\mu \right) - \\
&\qquad \frac{1}{|\tilde{H}|} \left(\int_{H\cap\tilde{H}} (x-\tilde{c})(x-\tilde{c})^T \, \mathrm{d}\mu + \int_{\tilde{H}\setminus H} (x-\tilde{c})(x-\tilde{c})^T \, \mathrm{d}\mu \right) \| \\
&= \| \int_{H\cap\tilde{H}} \frac{(x-c)(x-c)^T}{|H|} - \frac{(x-\tilde{c})(x-\tilde{c})^T}{|\tilde{H}|} \, \mathrm{d}\mu + \\
&\qquad \frac{1}{|H|} \int_{H\setminus\tilde{H}} (x-c)(x-c)^T \, \mathrm{d}\mu - \frac{1}{|\tilde{H}|} \int_{\tilde{H}\setminus H} (x-\tilde{c})(x-\tilde{c})^T \, \mathrm{d}\mu \| \\
&\le \left\| \int_{H\cap\tilde{H}} \frac{(x-c)(x-c)^T}{|H|} - \frac{(x-\tilde{c})(x-\tilde{c})^T}{|\tilde{H}|} \, \mathrm{d}\mu \right\| + \\
&\qquad \frac{1}{|H|} \left\| \int_{H\setminus\tilde{H}} (x-c)(x-c)^T \, \mathrm{d}\mu \right\| + \frac{1}{|\tilde{H}|} \left\| \int_{\tilde{H}\setminus H} (x-\tilde{c})(x-\tilde{c})^T \, \mathrm{d}\mu \right\| \\
&\le \int_{H\cap\tilde{H}} \left\| \frac{(x-c)(x-c)^T}{|H|} - \frac{(x-\tilde{c})(x-\tilde{c})^T}{|\tilde{H}|} \right\| \mathrm{d}\mu + \\
&\qquad \frac{1}{|H|} \int_{H\setminus\tilde{H}} \left\|(x-c)(x-c)^T\right\| \mathrm{d}\mu + \frac{1}{|\tilde{H}|} \int_{\tilde{H}\setminus H} \left\|(x-\tilde{c})(x-\tilde{c})^T\right\| \mathrm{d}\mu.
\end{aligned} \tag{24}$$

We then proceed to compute bounds for the three separate terms in the last line:

- Defining $e = \tilde{c} - c$ and $\Delta H = |\tilde{H}| - |H|$, we can write:

$$\begin{aligned}
\text{first term} &= \int_{H\cap\tilde{H}} \left\| \frac{(x-c)(x-c)^T}{|H|} - \frac{(x-\tilde{c})(x-\tilde{c})^T}{|\tilde{H}|} \right\| \mathrm{d}\mu \\
&= \int_{H\cap\tilde{H}} \left\| \frac{(x-c)(x-c)^T}{|H|} - \frac{(x-c-e)(x-c-e)^T}{|H| + \Delta H} \right\| \mathrm{d}\mu \\
&= \int_{H\cap\tilde{H}} \left\| \frac{(|H| + \Delta H)(x-c)(x-c)^T - |H|(x-c-e)(x-c-e)^T}{|H||\tilde{H}|} \right\| \mathrm{d}\mu \\
&= \int_{H\cap\tilde{H}} \left\| \frac{\Delta H(x-c)(x-c)^T + |H|((x-c)e^T + e(x-c)^T - ee^T)}{|H||\tilde{H}|} \right\| \mathrm{d}\mu \\
&\le \int_{H\cap\tilde{H}} \left\| \frac{\Delta H(x-c)(x-c)^T}{|H||\tilde{H}|} \right\| + \left\| \frac{(x-c)e^T + e(x-c)^T - ee^T}{|\tilde{H}|} \right\| \mathrm{d}\mu \\
&\le \int_{H\cap\tilde{H}} \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \|(x-c)(x-c)^T\| + \frac{1}{|\tilde{H}|} \left(\|(x-c)e^T\| + \|e(x-c)^T\| + \|ee^T\| \right) \mathrm{d}\mu \\
&= \int_{H\cap\tilde{H}} \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \|x-c\|^2 + \frac{1}{|\tilde{H}|} \left(2\|x-c\|\|e\| + \|e\|^2 \right) \mathrm{d}\mu \\
&\le \int_{H\cap\tilde{H}} \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \|x-c\|^2 + \frac{1}{|\tilde{H}|} \left(2\|x-c\|\|e\| + \|e\|^2 \right) \mathrm{d}\mu \\
&\le |H\cap\tilde{H}| \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \|x-c\|^2 + |H\cap\tilde{H}| \frac{1}{|\tilde{H}|} \left(2\|x-c\|\|e\| + \|e\|^2 \right) \\
&\le (1-\delta)^2 \|x-c\|^2 + (1-\delta) 2\|x-c\|\|e\| + \|e\|^2 \\
&\le (1-\delta)^2 \left(2\rho(H\cup\tilde{H}) \right)^2 + (1-\delta) 2 (2\rho(H\cup\tilde{H}))(3\rho(H\cup\tilde{H})) + (3\rho(H\cup\tilde{H}))^2 \\
&\le 4(1-\delta)^2 \rho^2(H\cup\tilde{H}) + 12(1-\delta)^2 \rho^2(H\cup\tilde{H}) + 9(1-\delta)^2 \rho^2(H\cup\tilde{H}) \\
&= 25(1-\delta)^2 \rho^2(H\cup\tilde{H}).
\end{aligned} \tag{25}$$

The fifth inequality holds because, like for Theorem 1, the following inequalities hold:

$$|H \cap \tilde{H}| \left| \frac{|\tilde{H}| - |H|}{|H||\tilde{H}|} \right| \leq 1, \qquad \frac{|H \cap \tilde{H}|}{|E2|} \leq 1 - \delta \leq 1. \tag{26}$$

The sixth inequality holds because $\|x - c\|$, for $x \in H \cap \tilde{H}$ is less than the diameter of the smallest ball containing $H \cup \tilde{H}$, i.e. $\|x - c\| \leq 2\rho(H \cup \tilde{H})$, and $\|e\| \leq 3(1-\delta)\rho(H \cup \tilde{H})$, from Theorem 1.

- For the second term we have:

$$\begin{aligned}
\text{second term} &= \frac{1}{|H|} \int_{H \setminus \tilde{H}} \left\|(x-c)(x-c)^T\right\| \mathrm{d}\mu \\
&\leq \frac{1}{|H|} \int_{H \setminus \tilde{H}} \|x-c\|^2 \,\mathrm{d}\mu \\
&\leq \frac{1}{|H|} \int_{H \setminus \tilde{H}} 4\rho^2(H \cup \tilde{H}) \,\mathrm{d}\mu \\
&= \frac{|H \setminus \tilde{H}|}{|H|} 4\rho^2(H \cup \tilde{H}) \\
&\leq 4(1-\delta)\rho^2(H \cup \tilde{H}).
\end{aligned} \tag{27}$$

- Analogously to the second term, for the third term we have:

$$\text{third term} = \frac{1}{|\tilde{H}|} \int_{\tilde{H} \setminus H} \left\|(x-\tilde{c})(x-\tilde{c})^T\right\| \mathrm{d}\mu \leq 4(1-\delta)\rho^2(H \cup \tilde{H}). \tag{28}$$

Putting all the bounds together we can then write:

$$\begin{aligned}
\|\mathcal{K}_1 - \mathcal{K}_2\| &\leq 25(1-\delta)^2\rho^2(H \cap \tilde{H}) + 4(1-\delta)\rho^2(H \cap \tilde{H}) + 4(1-\delta)\rho^2(H \cap \tilde{H}) \\
&= \left(25(1-\delta)^2 + 8(1-\delta)\right)\rho^2(H \cap \tilde{H}).
\end{aligned} \tag{29}$$

□

* * *

The error in the estimated rotation matrix depends on how the eigenvectors of $\tilde{\Sigma}$ are perturbed with respect to those of Σ. Therefore, we introduce the following result to connect the bound on $\|\Sigma - \tilde{\Sigma}\|$ to a bound on the eigenvector difference.

Lemma 3 (Perturbation of eigenvectors) *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix, and $B \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Consider the function $M : [0,1] \to \mathbb{R}^{n \times n}$ defined as $M(x) = A + xB$. Suppose that all eigenvalues of A are simple, and $2\|B\| < \min_{i,j} |\lambda_i - \lambda_j|,\ i,j \in \{1,2,\ldots,n\}$, with λ_i denoting the i-th eigenvalue of A. Then, the following holds:*

1. *The eigenvalues of $M(x)$ are simple for all $x \in [0,1]$, and $\min_{i,j} |\lambda_i(x) - \lambda_j(x)| \leq \min_{i,j} |\lambda_i - \lambda_j| - 2\|B\|$.*
2. *The unit-normalized eigenvectors $u_i(x)$ of $M(x)$ are such that, for $x \in [0,1]$ and $i \in \{1,2,\ldots,n\}$:*

$$\|u_i(x) - u_i(0)\| \leq \frac{\|B\|}{\min_{i,j} |\lambda_i - \lambda_j| - 2\|B\|}.$$

Proof: We prove each statement in sequence:

1. By [2], the absolute values of the first term in the taylor expansion centered at x of the i-th simple eigenvalue is:

$$\left| \frac{y_i(x)^T B x_i(x)}{y_i(x)^T x_i(x)} \right| = \left| \frac{y_i(x)^T B x_i(x)}{1} \right| \leq \|y_i(x)\|\|B\|\|x_i(x)\| = \|B\|, \tag{30}$$

where $x_i(x)$ and $y_i(x)$ are the right and left eigenvectors of $A + xB$, normalized so that they have norm 1, and since $A + xB$ is symmetric, $x_i(x) = y_i(x)$. Because of this, $y_i(x)^T x_i(x) = x_i(x)^T x_i(x) = \|x_i(x)\|^2 = 1$.

The absolute value of this coefficient is the absolute value of the derivative $\left|\frac{\partial \lambda_i}{\partial x}(x)\right| = \left|\frac{y_i(x)^T B x_i(x)}{y_i(x)^T x_i(x)}\right| \leq \|B\|$. Then, for $x \in [0,1]$, the maximum change in the i-th eigenvalue is bounded by:

$$\begin{aligned} |\lambda_i(x) - \lambda_i(0)| &= \left|\int_{s=0}^{s=x} \frac{\partial \lambda_i}{\partial x}(s)\, \mathrm{d}s\right| \\ &\leq \int_{s=0}^{s=x} \left\|\frac{\partial \lambda_i}{\partial x}(s)\right\| \mathrm{d}s \\ &\leq \int_{s=0}^{s=x} \|B\|\, \mathrm{d}s \\ &= (x-0)\|B\| \\ &\leq \|B\|, \end{aligned} \tag{31}$$

meaning that the minimum difference between the eigenvalues can change by at most $2\|B\|$:

$$\min_{i,j} |\lambda_i(x) - \lambda_j(x)| \geq \min_{i,j} |\lambda_i - \lambda_j| - 2\|B\|. \tag{32}$$

2. By Corollary 4 from [1], we know that an upper bound for the norm of the derivative of $u_i(x)$ is:

$$\frac{\|B\|}{|\lambda_i(x) - \mu(x)|}, \tag{33}$$

where $\mu(x)$ is the eigenvalue of $M(x)$ closest to $\lambda_i(x)$ and $\mu(x) \neq \lambda_i(x)$. The denominator is lower bounded by the smallest difference in eigenvalues of $M(x)$, so that:

$$\begin{aligned} \left\|\frac{\partial u_i}{\partial x}(x)\right\| &\leq \frac{\|B\|}{|\lambda_i(x) - \mu(x)|} \\ &\leq \frac{\|B\|}{\min_{i,j} |\lambda_i(x) - \lambda_j(x)|} \\ &\leq \frac{\|B\|}{\min_{i,j} |\lambda_i - \lambda_j| - 2\|B\|}. \end{aligned} \tag{34}$$

Then, for $x \in [0,1]$, the i-th eigenvector can change by at most:

$$\begin{aligned} \|u_i(x) - u_i(0)\| &= \left\|\int_{s=0}^{s=x} \frac{\partial u_i}{\partial x}(s)\, \mathrm{d}x\right\| \\ &\leq \int_{s=0}^{s=x} \left\|\frac{\partial u_i}{\partial x}(s)\right\| \mathrm{d}x \\ &\leq \int_{s=0}^{s=x} \frac{\|B\|}{\min_{i,j} |\lambda_i - \lambda_j| - 2\|B\|}\, \mathrm{d}x \\ &= (x-0)\frac{\|B\|}{\min_{i,j} |\lambda_i - \lambda_j| - 2\|B\|} \\ &\leq \frac{\|B\|}{\min_{i,j} |\lambda_i - \lambda_j| - 2\|B\|}. \end{aligned} \tag{35}$$

□

* * *

This final result connects the bound on the difference between normalized eigenvectors to the error rotation matrix E aligning H with $\tilde{H}$. This is expressed by a norm bound describing how close E is to the identity. The result is proven for $n = 3$, but the proof for any n is completely analogous.

Lemma 4 (Rotation between frames) *Let $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be orthonormal bases of vectors in $\mathbb{R}^3$, and assume that for $i \in \{1,2,3\}$:*

$$\|u_i - v_i\| \leq \varepsilon. \tag{36}$$

Then, letting $U = \begin{bmatrix} u_1, u_2, u_3 \end{bmatrix}$ and $V = \begin{bmatrix} v_1, v_2, v_3 \end{bmatrix}$, there exists an orthonormal matrix $R \in \mathbb{R}^{3\times 3}$ such that:

$$V = UR, \qquad \text{and} \qquad \|R - I\| \leq \varepsilon\sqrt{3}. \tag{37}$$

Proof: Since $\{u_1, u_2, u_3\}$ forms a basis of $\mathbb{R}^3$, we can decompose v_i, for $i \in \{1,2,3\}$ as:

$$v_i = \sum_{j=1}^{3} r_{j,i} u_j, \tag{38}$$

for some choice of $r_{i,j} \in \mathbb{R}$. Thus, we can write:

$$\begin{bmatrix} v_1, v_2, v_3 \end{bmatrix} = \begin{bmatrix} u_1, u_2, u_3 \end{bmatrix} \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix}, \tag{39}$$

meaning that:

$$U^{-1}V = U^T V = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} = R, \tag{40}$$

where $U^{-1} = U^T$ because U is orthonormal. Since R is the product of orthonormal matrices, it is also orthonormal, and we can write the bound:

$$\begin{aligned} \|R - I\| &= \left\|U^T V - I\right\| \\ &= \left\|(U - V + V)^T V - I\right\| \\ &= \left\|(U - V)^T V + V^T V - I\right\| \\ &= \left\|(U - V)^T V + I - I\right\| \\ &\le \left\|(U - V)^T\right\| \|V\| \\ &= \|U - V\| \\ &= \max_{x \in \mathbb{R}^3, \|x\|=1} \|(U - V)x\| \\ &= \max_{x \in \mathbb{R}^3, \|x\|=1} \|(u_1 - v_1)x_1 + (u_2 - v_2)x_2 + (u_3 - v_3)x_3\| \\ &\le \max_{x \in \mathbb{R}^3, \|x\|=1} \|u_1 - v_1\| \, |x_1| + \|u_2 - v_2\| \, |x_2| + \|u_3 - v_3\| \, |x_3| \\ &\le \max_{x \in \mathbb{R}^3, \|x\|=1} \varepsilon(|x_1| + |x_2| + |x_3|) \\ &= \varepsilon\sqrt{3}. \end{aligned} \tag{41}$$

□

1.5 Main Result

We are now in a position to state the main result, connecting the overlap between H and $\tilde{H}$ with the error in the position and rotation estimation.

Theorem 1 (Pose Error Bound) *Let $H, \tilde{H}$ be non empty bounded subsets of R^3 with an overlap of δ. Let c be the first moment of H, $\bar{c} = 3(1-\delta)\rho(H \cup \tilde{H})$, and $\bar{\sigma} = \left(25(1-\delta)^2 + 8(1-\delta)\right)\rho^2(H \cap \tilde{H})$. Then, if $\min_{i,j} |\lambda_i - \lambda_j| > 2\bar{\sigma}$, where λ_i is i-th eigenvalue of the second moment of H, the following bounds hold:*

$$\begin{aligned} \|\hat{R} - R\| &\le \sqrt{3}\frac{\bar{\sigma}}{\min_{i,j} |\lambda_i - \lambda_j| - 2\bar{\sigma}} \\ \|\hat{p} - p\| &\le \|c\|\sqrt{3}\frac{\bar{\sigma}}{\min_{i,j} |\lambda_i - \lambda_j| - 2\bar{\sigma}} + \bar{c}. \end{aligned} \tag{42}$$

Proof: From Lemma 1, we know that $\|\tilde{c}_1 - c_1\| \le \bar{c}$, and similarly, from Lemma 2 we know that $\|\tilde{\Sigma} - \Sigma\| \le \bar{\sigma}$. Then, we can write the perturbed covariance as $\tilde{\Sigma} = \Sigma + \Sigma_e$, with $\|\Sigma_e\| \le \bar{\sigma}$.

Applying Lemma 3 with $A = \Sigma$ and $B = \Sigma_e$, we can claim that:

$$\|\tilde{u}_i - u_i\| \le \frac{\|\Sigma_e\|}{\min_{i,j} |\lambda_i - \lambda_j| - 2\|\Sigma_e\|} \le \frac{\bar{\sigma}}{\min_{i,j} |\lambda_i - \lambda_j| - 2\bar{\sigma}}, \tag{43}$$

with $\tilde{u}_i$ the i-th eigenvector of $\tilde{\Sigma}$, u_i the i-th eigenvector of Σ, and λ_i the i-th eigenvector of Σ.

Applying Lemma 4 with $\epsilon = \frac{\bar{\sigma}}{\min_{i,j}|\lambda_i - \lambda_j| - 2\bar{\sigma}}$, we can finally conclude that:

$$\|\hat{R} - R\| = \|E - I\| \leq \varepsilon\sqrt{3} = \frac{\bar{\sigma}}{\min_{i,j}|\lambda_i - \lambda_j| - 2\bar{\sigma}}. \tag{44}$$

As for the translation, we can simply write the bound:

$$\|\hat{p} - p\| \leq \|E - I\|\|c\| + \|\tilde{c} - c\| \leq \|c\|\sqrt{3}\frac{\bar{\sigma}}{\min_{i,j}|\lambda_i - \lambda_j| - 2\bar{\sigma}} + \bar{c}. \tag{45}$$

□

1.6 Point Registration vs Pose Estimation

The theory above has been developed under the setting where we attempt to estimate the translation vector p and rotation matrix R such that $H_2 = p + RH_1$. When we are doing localization, we want to identify the *pose* of the second frame with respect to the first, which results in estimating the "inverse" transformation, or finding the p', R' such that $H_1 = p' + R'H_2$. This involves solving the system of equations:

$$\begin{cases} c_1 = p' + R'c_2 \\ \Sigma_1 = R'\Sigma_2 R'^T. \end{cases} \tag{46}$$

Therefore, the bounds obtained in Theorem 1 are the same as in the case of pose estimation, with the roles of H_1 and H_2 swapped.

2 Fast Simplex Moment Computation

Our method of rototranslation estimation relies on the computation of first and second moments as *integral* quantities over a set, rather than a summation over a set of points. In the case of sets H_1 and H_2 defined as convex hulls of finite sets of points, the resulting sets are *polytopes*, and as such can be partitioned into a set of simplices. The first and second moments can then be computed via weighted summation of the moments of the separate simplices. The first and second moments of a simplex can be easily computed as linear and quadratic functions, respectively, of the vertex coordinates, making the fast and exact computation of them feasible. The proof of these results is provided in the following sections.

2.1 Auxiliary Mathematical Results

We begin by introducing an auxiliary result that is necessary for the following theorem.

Lemma 5 *For all $k, a, b \in \mathbb{N}$ and $x_1, x_2, \dots, x_{k-1} \in \mathbb{R}$ the following holds:*

$$\int_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} x_k^a \left(1 - \sum_{i=1}^{k-1} x_i - x_k\right)^b \mathrm{d}x_k = \begin{cases} \frac{a!b!}{(b+a+1)!}\left(1 - \sum_{i=1}^{k-2} x_i - x_{k-1}\right)^{b+a+1}, & \text{if } k > 1 \\ \frac{a!b!}{(b+a+1)!}, & \text{if } k = 1. \end{cases} \tag{47}$$

Proof: We first prove the case for $k > 1$. The proof follows directly from integration by parts. First, note that we can write:

$$\begin{aligned}
&\int_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} x_k^a \left(1 - \sum_{i=1}^{k-1} x_i - x_k\right)^b \mathrm{d}x_k \\
&\quad = \left[-\frac{x_k^a \left(1-\sum_{i=1}^{k-1} x_i - x_k\right)^{b+1}}{b+1} \right]_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} + \frac{a}{b+1} \int_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} x_k^{a-1} \left(1 - \sum_{i=1}^{k-1} x_i - x_k\right)^{b+1} \mathrm{d}x_k \qquad (48) \\
&\quad = \frac{a}{b+1} \int_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} x_k^{a-1} \left(1 - \sum_{i=1}^{k-1} x_i - x_k\right)^{b+1} \mathrm{d}x_k.
\end{aligned}$$

Then, repeatedly integrating by parts until the exponent on x_k is zero, we arrive at:

$$\begin{aligned}
\int_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} x_k^a \left(1 - \sum_{i=1}^{k-1} x_i - x_k\right)^b \mathrm{d}x_k &= \frac{a(a-1)\dots 1}{(b+1)(b+2)\dots(b+a)} \int_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} \left(1 - \sum_{i=1}^{k-1} x_i - x_k\right)^{b+a} \mathrm{d}x_k \\
&= \frac{a(a-1)\dots 1}{(b+1)(b+2)\dots(b+a)} \left[-\frac{\left(1-\sum_{i=1}^{k-1} x_i - x_k\right)^{b+a+1}}{b+a+1} \right]_{x_k=0}^{1-\sum_{i=1}^{k-1} x_i} \qquad (49) \\
&= \frac{a(a-1)\dots 1}{(b+1)(b+2)\dots(b+a+1)} \left(1 - \sum_{i=1}^{k-1} x_i\right)^{b+a+1} \\
&= \frac{a!b!}{(b+a+1)!} \left(1 - \sum_{i=1}^{k-2} x_i - x_{k-1}\right)^{b+a+1}.
\end{aligned}$$

Following the same computations, we can prove the case for $k = 1$. □

* * *

We now prove a closed form expression for a family of multidimensional integrals. This expression will be used to calculate a set of coefficients used in the computation of the moments of a simplex.

Theorem 2 *For all $k \in \mathbb{N}$, and for all $n_1, n_2, \dots, n_k \in \mathbb{N}$, the following holds:*

$$\int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \cdots \int_{x_k=0}^{1-\sum_{i=0}^{k-1} x_i} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \, \mathrm{d}x_k \, \mathrm{d}x_{k-1} \dots \mathrm{d}x_1 = \frac{\prod_{i=0}^{k} n_i!}{(k + \sum_{i=0}^{k} n_i)!}. \qquad (50)$$

Proof: We first note that we can rewrite the integral as:

$$\int_{x_1=0}^{1} x_1^{n_1} \int_{x_2=0}^{1-x_1} x_2^{n_2} \cdots \int_{x_k=0}^{1-\sum_{i=0}^{k-1} x_i} x_k^{n_k} \, \mathrm{d}x_k \, \mathrm{d}x_{k-1} \dots \mathrm{d}x_1. \qquad (51)$$

Then, we apply Lemma 5 to the inner integral [2] and obtain:

$$\begin{aligned}
&\int_{x_1=0}^{1} x_1^{n_1} \int_{x_2=0}^{1-x_1} x_2^{n_2} \cdots \int_{x_k=0}^{1-\sum_{i=0}^{k-1} x_i} x_k^{n_k} \, \mathrm{d}x_k \, \mathrm{d}x_{k-1} \dots \mathrm{d}x_1 \\
&\quad = \int_{x_1=0}^{1} x_1^{n_1} \int_{x_2=0}^{1-x_1} x_2^{n_2} \cdots \int_{x_k=0}^{1-\sum_{i=0}^{k-2} x_i} x_{k-1}^{n_{k-1}} \frac{n_k!}{(n_k+1)!} \left(1 - \sum_{i=1}^{k-2} x_i - x_{k-1}\right)^{n_k+1} \mathrm{d}x_{k-1} \, \mathrm{d}x_{k-2} \dots \mathrm{d}x_1 \qquad (52) \\
&\quad = \frac{n_k!}{(n_k+1)!} \int_{x_1=0}^{1} x_1^{n_1} \int_{x_2=0}^{1-x_1} x_2^{n_2} \cdots \int_{x_k=0}^{1-\sum_{i=0}^{k-2} x_i} x_{k-1}^{n_{k-1}} \left(1 - \sum_{i=1}^{k-2} x_i - x_{k-1}\right)^{n_k+1} \mathrm{d}x_{k-1} \, \mathrm{d}x_{k-2} \dots \mathrm{d}x_1.
\end{aligned}$$

[2] Note that $x_k^{n_k} = x_k^{n_k} \left(1 - \sum_{i=0}^{k-1} x_i - x_k\right)^0$.

Applying the lemma repeatedly until we exhaust the integrals we arrive at:

$$\begin{aligned}
&\int_{x_1=0}^{1} x_1^{n_1} \int_{x_2=0}^{1-x_1} x_2^{n_2} \cdots \int_{x_k=0}^{1-\sum_{i=0}^{k-1} x_i} x_k^{n_k} \,\mathrm{d}x_k \,\mathrm{d}x_{k-1} \ldots \mathrm{d}x_1 \\
&\quad = \frac{n_k!}{(n_k+1)!} \times \frac{n_{k-1}!(n_k+1)!}{(n_{k-1}+n_k+2)!} \times \cdots \times \frac{n_1!(n_2+\cdots+n_k+(k-1))!}{(n_1+n_2+\cdots+n_k+k)!} \\
&\quad = \frac{n_1!n_2!\ldots n_k!}{(n_1+n_2+\cdots+n_k+k)!} \\
&\quad = \frac{\prod_{i=1}^{k} n_i!}{(k+\sum_{i=1}^{k} n_i)!}.
\end{aligned} \tag{53}$$

□

In addition, to simplify some of the following notation, we define:

Definition 1 *For any $k \in \mathbb{N}$, and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ we define $M_{n_1,n_2,\ldots,n_k}$ as:*

$$M(n_1, n_2, \ldots, n_k) = \frac{\prod_{i=1}^{k} n_i!}{(k+\sum_{i=1}^{k} n_i)!}. \tag{54}$$

2.2 Simplex Moments

We can now proceed with the computation of the moments of a simplex. The computation relies on a change of coordinates that will be used to compute the necessary integrals. Consider a simplex $S \subset \mathbb{R}^n$, determined by $n+1$ vertices $v_0, v_1 \ldots, v_n \in \mathbb{R}^n$. Let us introduce the change of variables $g : \mathbb{R}^n \to \mathbb{R}^n$:

$$\begin{aligned}
g(x) &= g(x_1, x_2, \ldots, x_n) \\
&= v_0 + x_1(v_1 - v_0) + x_2(v_2 - v_0) + \cdots + x_n(v_n - v_0) \\
&= \begin{bmatrix} z_1 & z_2 & \ldots z_n \end{bmatrix}^T \\
&= z.
\end{aligned} \tag{55}$$

Then, defining $V = \begin{bmatrix} v_1 - v_0 & v_2 - v_0 & \ldots & v_n - v_0 \end{bmatrix}$, the Jacobian of g is:

$$\frac{\partial g}{\partial x}(x) = V. \tag{56}$$

Then, any integral over S of some function f can be rewritten as:

$$\begin{aligned}
\int_{z \in S} f(z) \,\mathrm{d}\mu &= \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \cdots \int_{x_n}^{1-\sum_{i=1}^{n-1} x_i} f(v_0 + Vx) \det(V) \,\mathrm{d}x_n \,\mathrm{d}x_{n-1} \ldots \mathrm{d}x_1 \\
&= \det(V) \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \cdots \int_{x_n}^{1-\sum_{i=1}^{n-1} x_i} f(v_0 + Vx) \,\mathrm{d}x_n \,\mathrm{d}x_{n-1} \ldots \mathrm{d}x_1.
\end{aligned} \tag{57}$$

2.2.1 Volume

The volume of a simplex is:

$$\int_{z \in S} \mathrm{d}\mu = \det(V) \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \cdots \int_{x_n}^{1-\sum_{i=1}^{n-1} x_i} \mathrm{d}x_n \,\mathrm{d}x_{n-1} \ldots \mathrm{d}x_1. \tag{58}$$

Then, we can apply Theorem 2:

$$\det(V) \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \cdots \int_{x_n}^{1-\sum_{i=1}^{n-1} x_i} \mathrm{d}x_n \,\mathrm{d}x_{n-1} \ldots \mathrm{d}x_1 = \det(V) M(0, 0, \ldots, 0) = \frac{\prod_{i=1}^{n} 0!}{(n+\sum_{i=1}^{n} 0)!} = \frac{1}{n!}\det(V). \tag{59}$$

2.2.2 First Moment

The first moment of a simplex is:

$$
\begin{aligned}
\frac{\int_{z\in S} z \,\mathrm{d}\mu}{\int_{z\in S} \mathrm{d}\mu} &= \frac{n!}{\det(V)}\det(V)\int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1}\cdots\int_{x_n}^{1-\sum_{i=1}^{n-1}x_i}(v_0+Vx)\,\mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_1\\
&= n!\,(v_0M(0,\ldots,0)+(v_1-v_0)M(1,\ldots,0)+(v_2-v_0)M(0,1,\ldots,0)+\cdots+(v_n-v_0)M(0,\ldots,1))\\
&= n!\left(\frac{1}{n!}v_0+\frac{1}{(n+1)!}\sum_{i=1}^{n}(v_i-v_0)\right)\\
&= v_0+\frac{1}{n+1}\sum_{i=1}^{n}(v_i-v_0)\\
&= \frac{1}{n+1}\sum_{i=0}^{n}v_i.
\end{aligned}
\tag{60}
$$

2.2.3 Second Moment

The second moment of a simplex, relative to a point $q \in \mathbb{R}^n$ is:

$$
\begin{aligned}
\frac{\int_{z\in S}(z-q)(z-q)^T\,\mathrm{d}\mu}{\int_{z\in S}\mathrm{d}\mu} &= \frac{n!}{\det(V)}\det(V)\int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1}\cdots\int_{x_n}^{1-\sum_{i=1}^{n-1}x_i}(v_0+Vx-q)(v_0+Vx-q)^T\,\mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_1\\
&= n!\int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1}\cdots\int_{x_n}^{1-\sum_{i=1}^{n-1}x_i}(v_0+Vx-q)(v_0+Vx-q)^T\,\mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_1\\
&= n!\int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1}\cdots\int_{x_n}^{1-\sum_{i=1}^{n-1}x_i}\begin{bmatrix}v_0-q & V\end{bmatrix}\begin{bmatrix}1\\ x\end{bmatrix}\begin{bmatrix}1 & x^T\end{bmatrix}\begin{bmatrix}(v_0-q)^T\\ V^T\end{bmatrix}\mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_1\\
&= n!\begin{bmatrix}v_0-q & V\end{bmatrix}\int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1}\cdots\int_{x_n}^{1-\sum_{i=1}^{n-1}x_i}\begin{bmatrix}1 & x^T\\ x & xx^T\end{bmatrix}\mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_1\begin{bmatrix}(v_0-q)^T\\ V^T\end{bmatrix}\\
&= \begin{bmatrix}v_0-q & V\end{bmatrix}K\begin{bmatrix}(v_0-q)^T\\ V^T\end{bmatrix},
\end{aligned}
\tag{61}
$$

where the matrix $K \in \mathbb{R}^{(n+1)\times(n+1)}$ is defined as:

$$
K = n!\int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1}\cdots\int_{x_n}^{1-\sum_{i=1}^{n-1}x_i}\begin{bmatrix}1 & x^T\\ x & xx^T\end{bmatrix}\mathrm{d}x_n\,\mathrm{d}x_{n-1}\ldots\mathrm{d}x_1. \tag{62}
$$

Exploiting Theorem 2, the entries of K follow the pattern:

$$
\begin{aligned}
k_{0,0} &= n!M(0,\ldots,0) = n!\frac{1}{n!} = 1\\
k_{i,0} = k_{0,i} &= n!M(0,\ldots,1,\ldots,0) = n!\frac{1}{(n+1)!} = \frac{1}{n+1}, \qquad i>0\\
k_{i,j} &= n!M(0,\ldots,1,\ldots,1,\ldots,0) = n!\frac{1}{(n+2)!} = \frac{1}{(n+1)(n+2)}, \qquad i,j>1,\ i\neq j\\
k_{i,i} &= n!M(0,\ldots,2,\ldots,0) = n!\frac{2}{(n+2)!} = \frac{2}{(n+1)(n+2)}, \qquad i>1.
\end{aligned}
\tag{63}
$$

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[1] Carl D Meyer and Gilbert W Stewart. Derivatives and perturbations of eigenvectors. *SIAM Journal on Numerical Analysis*, 25(3):679–691, 1988.

[2] JH Wilkinson. The algebraic eigenvalue problem. In *Handbook for Automatic Computation, Volume II, Linear Algebra*. Springer-Verlag New York, 1971.