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1 PPT in Standard Form

We will convert the SDP given by Consentino for PPT measurements into more standard form.

Let $\mathbb{H}^n$ be the space of $n \times n$ Hermitian matrices The standard primal SDP problem is:

$$\begin{aligned} &\min_{X \in \mathbb{H}^n} \langle C, X \rangle_{\mathbb{H}^n} \\ \text{subject to: } &\langle A_k, X \rangle_{\mathbb{H}^n} = b_k, \quad k = 1, \ldots, m \\ &X \succcurlyeq 0 \end{aligned} \tag{1}$$

while the dual SDP is

$$\begin{aligned} &\max_{y \in \mathbb{C}^m} \langle b, y \rangle_{\mathbb{C}^m} \\ \text{subject to: } &C \succcurlyeq \sum_{j=1}^m y_j A_j \end{aligned} \tag{2}$$

The problem of finding the optimal PPT measurement for distinguishing states ρ_j in $\mathbb{H}^n$ that each appear with probability p_j is

$$\begin{aligned} \max : &\sum_{j=1}^k p_j \langle P_j, \rho_j \rangle \\ \text{subject to: } &\sum_{j=1}^k P_j = \mathbb{I}_{\mathcal{A}} \otimes \mathbb{I}_{\mathcal{B}}, \\ &P_1, \ldots, P_k \in PPT(\mathcal{A} : \mathcal{B}) \end{aligned} \tag{3}$$

Notice that the condition that $P_j \in PPT(\mathcal{A} : \mathcal{B})$ is the same as the condition

$$\begin{aligned} T_{\mathcal{A}}(P_j) &\succcurlyeq 0 \\ P_j &\succcurlyeq 0 \end{aligned} \tag{4}$$

where $T_{\mathcal{A}}$ is the partial transpose operation. Let's consider the matrices

$$\begin{aligned} \rho_T &= \frac{1}{2} \bigoplus_{j=1}^k p_j T_{\mathcal{A}}(\rho_j) \oplus \bigoplus_{j=1}^k p_j \rho_j \\ P_T &= \bigoplus_{j=1}^k T_{\mathcal{A}}(P_j) \oplus \bigoplus_{j=1}^k P_j \end{aligned} \tag{5}$$

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Then

$$\langle \rho_T, P_T \rangle = \sum_{j=1}^{k} p_j \langle P_j, \rho_j \rangle. \tag{6}$$

because

$$T_{\mathcal{A}}^{\dagger} = T_{\mathcal{A}}. \tag{7}$$

Let

$$\begin{aligned} A_{i,j}^1 &= \bigoplus \alpha_{i,j} \oplus \bigoplus \mathbb{O}^n \\ A_{i,j}^2 &= \bigoplus \mathbb{O}^n \oplus \bigoplus \alpha_{i,j} \end{aligned} \tag{8}$$

where $\alpha_{i,j}$ is an $n \times n$ matrix such that $\alpha_{i,j}[x,y]\delta_{ix}\delta_{jy}$ and $\mathbb{O}^n$ is the $n \times n$ all 0's matrix. Then we have

$$\langle A_{i,j}^1, P_T \rangle = \delta_{i,j} \text{ for } i,j \in [n] \leftrightarrow \langle A_{i,j}^2, P_T \rangle = \delta_{i,j} \text{ for } i,j \in [n] \leftrightarrow \sum_{j=1}^{k} P_j = \mathbb{I}_{\mathcal{A}} \otimes \mathbb{I}_{\mathcal{B}} \tag{9}$$

We also need a condition to make the i^{th} block of the first half correspond to the i^{th} block of the second half (i.e. $T_{\mathcal{A}}(P_j) \sim P_j$). So we have

$$\langle B_{i,j}^l, P_T \rangle = 0 \tag{10}$$

where $B_{i,j}^l = 1$ on the (i,j) element of the l^{th} block of the first half, and $B_{i,j}^l = -1$ on the corresponding (after partial transpose of (i,j)) element of the l^{th} block of the second half. Therefore, we can rewrite the PPT SDP in more standard form as

$$\begin{aligned} \text{min} \quad & \langle -\rho_T, P_T \rangle \\ \text{subject to:} \quad & \langle A_{i,j}^1, P_T \rangle = \delta_{i,j} \\ & \langle A_{i,j}^1, P_T \rangle = \delta_{i,j} \\ & \langle B_{i,j}^l, P_T \rangle = 0 \\ & P_T \succcurlyeq 0 \end{aligned} \tag{11}$$

This gives us the dual form

$$\begin{aligned} \text{max} \quad & \sum \delta_{i,j} y_{i,j}^1 + \sum \delta_{i,j} y_{i,j}^2 \\ \text{subject to:} \quad & -\rho_T \succcurlyeq \sum y_{i,j}^1 A_{i,j}^1 + \sum y_{i,j}^2 A_{i,j}^2 + \sum y_{l,i,j} B_{i,j}^l \end{aligned} \tag{12}$$

Let's define matrices

$$\begin{aligned} H^1 &= \sum -y_{i,j}^1 \alpha_{i,j} \\ H^2 &= \sum -y_{i,j}^2 \alpha_{i,j} \\ Y^l &= \sum y_{l,i,j} \alpha_{i,j} \end{aligned} \tag{13}$$

Looking at the block structure of ρ_T, we see we can rewrite the dual form as

$$\begin{aligned} \max \quad & -\operatorname{tr} H^1 - \operatorname{tr} H^2 \\ \text{subject to:} \quad & -\frac{1}{2} p_j T_{\mathcal{A}}(\rho_j) \succcurlyeq -H^1 + Y^l \quad \forall j \\ & -\frac{1}{2} p_j \rho_j \succcurlyeq -H^2 - T_{\mathcal{A}}(Y^l) \quad \forall j \end{aligned} \tag{14}$$

[**skimmel:**Not sure about the following:] We can combine the two constraints and set $Y^l = 0$ since it doesn't affect the objective function. Getting rid of the negatives, we have

$$\begin{aligned} \min \quad & \operatorname{tr} H \\ \text{subject to:} \quad & H \succcurlyeq p_j T_{\mathcal{A}}(\rho_j) \quad \forall j \end{aligned} \tag{15}$$

For the case of $p_i = 1/k$, we have

$$\begin{aligned} \min \quad & \frac{1}{k} \operatorname{tr} H \\ \text{subject to:} \quad & H \succcurlyeq T_{\mathcal{A}}(\rho_j) \quad \forall j \end{aligned} \tag{16}$$

References