MATH 597C: Applied Algebraic Topology Fall ~~2013~~ 2015

Instructor : **Vladimir Itskov**; office: McAllister Building 226; email: vladimir.itskov@math.psu.edu
Office hours: Tuesday 11:00-1:00pm or by appointment.

Class time : Monday, Wednesday, Friday 12:20 PM - 1:10 PM, 101 Electrical Eng West.

Texts: We will not follow a single textbook for the entire course, however the following books will be used:

- H. Edelsbrunner & J. Harer. *"Computational Topology"*, ISBN-13: 978-0821849255
- R. Ghrist, *"Elementary Applied Topology"*, ISBN 978-1502880857, Sept. 2014. `https://www.math.upenn.edu/~ghrist/notes.html`

Homework : Homework assignments are an important part of this course.

Grading: Your course grade will be based on 2 components:
- Homework (60%), and
- Final Exam (40%).

Expectations: You are expected to hand in your own write-up of each homework assignment, even if you worked with others. Please feel free to come to my office hours to ask for help if you are stuck on a homework problem. I also expect you will attend class regularly.

Class Webpage: I will use "angel.psu.edu" to post some course-related materials. The correct web address is `https://cms.psu.edu`.

The scope of the Course:

This class will serve as an introduction to the rapidly developing new field of applied mathematics that builds on ideas and techniques from Algebraic Topology. This course will serve as an introduction to combinatorial topology and homological algebra, as well as their applications. The tentative list of topics will be chosen as a proper subset of the following: manifolds, simplicial complexes, Chech complexes, Vietoris-Rips complexes, convex geometry, Euler characteristic, homology, sequences, cohomology, sheaves, topology of point cloud data, persistent homology, topological data analysis.

Students with disabilities: Penn State welcomes students with disabilities into the University's educational programs. Every Penn State campus has an office for students with disabilities. The Office for Disability Services (ODS) Web site provides contact information for every Penn State campus: http://equity.psu.edu/ods/dcl. For further information, please visit the Office for Disability Services Web site: http://equity.psu.edu/ods. In order to receive consideration for reasonable accommodations, you must contact the appropriate disability services office at the campus where you are officially enrolled, participate in an intake interview, and provide documentation: http://equity.psu.edu/ods/doc-guidelines. If the documentation supports your request for reasonable accommodations, your campus' disability services office will provide you with an accommodation letter.

Math MATH 597C, Fall 2015 Homework #1 Due on Friday September 11th, 2015.

Problem 0. Suppose that X is a topological space, and *every* function $f : X \to \mathbb{R}^1$ is continuous. Describe the topology of X. Justify your answer.

Problem 1. (a) Prove that if X is a connected topological space and $f\colon X \to Y$ is continuous, then $f(X) \subseteq Y$ is connected in the induced topology. (b) Prove that path connectedness implies connectedness. (c)* (for your entertainment) Convince yourself that the converse is not true by considering the example of $X = \{(x, y) \in \mathbb{R}^2 \,|\, x = 0 \text{ or } y = \sin\frac{1}{x}\}$.

Problem 2. Provide a proof that the following topological spaces are *not* homeomorphic:

1. The Möbius strip and the cylinder
2. $\mathbb{R}^3$ and S^3
3. S^3 and S^4 (* optional: this is harder without any of the theorems that will be discussed in the future)

Problem 3. Let $X = \cup_{n=1}^{\infty} C_n$ be the union of the circles $C_n = \{(x, y) \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\} \subset \mathbb{R}^2$ equipped with the topology induced from the Euclidean plane. Let Y be the quotient of the real line (with the Euclidean topology) by the equivalence relation

$$x \sim y \qquad \text{if and only if} \quad x, y \in \mathbb{Z}.$$

Prove that X and Y are *not* homeomorphic.

Problem 4. [baby nerve lemma] Prove that a topological space X is connected (here I mean "connected", not "path-connected") if and only if every open covering $\mathcal{U} = \{U_\alpha\}$ has the following property: For each pair of sets U, V in $\mathcal{U}$ there is a sequence of open sets $\{U_0, U_1, \ldots, U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \varnothing$ for all i.

Problem 5.

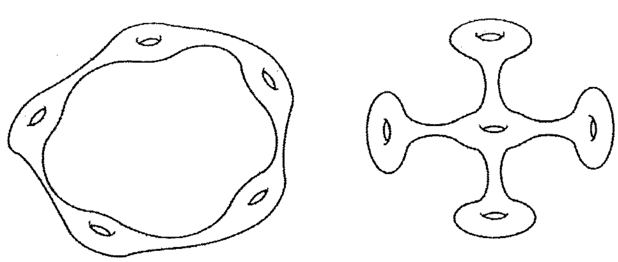

Figure 1: (a) Are the two compact orientable surfaces depicted here homeomorphic? Give a proof of your answer.

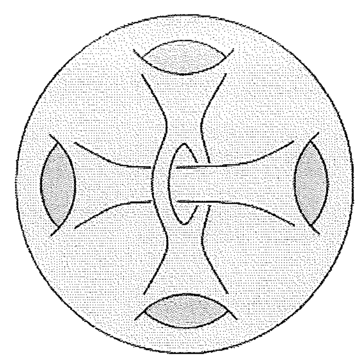

Figure 2: (b) Characterize the depicted surface in terms of genus and orientability.

Problem 6. Construct triangulations of the following spaces: (1) $\mathbb{R}P^2$; (2) S^3; (3) A closed 3-dimensional ball $B^3 \subset \mathbb{R}^3$ with a two-dimensional strip of paper attached along two different non-overlapping one-dimensional intervals on the boundary of that ball. First, draw a picture of this.

Problem 1. Use the Mayer-Vietoris sequence of singular homology to find the homology groups for all n-dimensional spheres S^n with coefficients in an arbitrary Abelian group G. (*Hint:* use induction on $n \geq 0$.)

Problem 2. Compute the homology of $\mathbb{R}^n$ with $q \geq 1$ points removed.

Problem 3. Use the long exact sequence theorem, to formulate and prove the analogous theorem for the Mayer-Vietoris sequence for the case of reduced homology $\tilde{H}_*$.

Problem 4. Given a topological space X, its suspension is defined as

$$SX = (X \times I)/\{(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X\}$$

Prove that $\tilde{H}_n(SX) = \tilde{H}_{n-1}(X)$.

Problem 5. Given two topological spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, the wedge sum is the quotient of the disjoint union $X \sqcup Y$ obtained by "gluing" x_0 and y_0, i.e. $X \vee Y = (X \sqcup Y)/\sim$, where $x \sim y \iff x = x_0$ and $y = y_0$. Assuming that both x_0 and y_0 possess open contractible neighborhoods, prove that

$$\tilde{H}_*(X \vee Y) = \tilde{H}_*(X) \oplus \tilde{H}_*(Y).$$

Problem 6:

Since $H_n(S^n, \mathbb{Z}) = \mathbb{Z}$, there are two choices of generators for $H_n(S^n)$. Each is called an *orientiation* of S^n. Given a choice of orientation, we can think of the top homology as the integers. If $f : S^n \to S^n$ is a continuous map between oriented spheres, then $f_* : \mathbb{Z} = H_n(S^n) \to \mathbb{Z} = H_n(S^n)$ is multiplication by an integer, which we call the *degree* of f. Note that, for maps from an n-sphere to itself, the degree doesn't depend on the orientation[1], but for maps between distinct spheres, you have to pick orientations before you can define a degree.

Problem 6a. Prove that $deg(f \circ g) = deg(f)deg(g)$ if both f and g are maps $S^n \to S^n$, and that $deg(f_0) = deg(f_1)$ if f_0 and f_1 are homotopic.

Problem 6b. Consider the reflection map $R(x_1, x_2) = (-x_1, x_2)$ that maps $\mathbb{R}^2$ to itself, and also maps S^1 to itself. Show that R has degree -1. [*Hint:* Pick an explicit generator α of $H_1(S^1)$ such that $R_\#(\alpha) = -\alpha$.] Now consider reflection on the n-sphere, where $R(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$. Show that R has degree -1.

Problem 6c. Show that the antipodal map $x \mapsto (-x)$ of S^n is homotopic to the identity if, and only if, $n > 0$ is odd. *Hint:* You can start with proving that the antipodal map of the even-dimensional sphere has degree (-1) and later constructing an explicit homotopy of the antipodal map to the identity on any odd-dimensional sphere.

[1] This is because switching the orientation would involve multiplying by (-1) twice.

✓ **Problem 1.** Find a collection of open sets $\{\mathcal{U}_i\}$ in $\mathbb{R}^2$ so that $|\mathcal{N}(\{U_i\})| \simeq S^2$. That is, the geometric realization of the nerve of $\{U_i\}$ is homotopy equivalent to the two-dimensional sphere. *Hint:* you may want to violate the assumptions of the Nerve Lemma ...

Problem 2. Let $S(\mathbb{T}^2)$ denote the suspension of the two-dimensional torus and S^3 denote the three-dimensional sphere. Prove that any cover of $X = S(\mathbb{T}^2) \vee S^3$, by open contractible sets $\{U_i\}_{i=1}^n$ of X, such that each of their non-empty intersection is contractible necessarily has $n \geq 7$ of such open sets.

✓ **Problem 3.** Let M and N be compact connected n-dimensional manifolds. Prove that the Euler characteristic of their connected sum satisfies

$$\chi(M\#N) = \chi(M) + \chi(N) - \chi(S^n).$$

✓ **Problem 4.** Let v be a vector field on the connected sum $M = \mathrm{KL}\#\mathbb{T}^2$ of the Klein bottle and the two-dimensional torus. Assume that this vector field has at most seven zeroes (i.e. fixed points of the appropriate differential equation). What is the maximal number of asymptotically stable fixed points (steady states) that this vector field can possibly have? Provide the proof of your estimate.

✓ **Problem 5.** Let $K_0 \subseteq K$ and $L_0 \subseteq L$ be pairs of simplicial complexes that satisfy $L \subseteq K$ and $L \setminus L_0 = K \setminus K_0$. Prove that they have isomorphic relative homology groups, that is, $H_p(K, K_0) \simeq H_p(L, L_0)$ for all dimensions p.

Math MATH 597C, Fall 2015 **Homework #4** Due on Monday, November 16th, 2015.

Problem 1. Use the definitions of the simplicial cohomology groups to compute the cohomology of the Klein bottle with coefficients in Z and in $\mathbb{Z}_2$.

Problem 2. Formulate and prove Mayer Vietoris long exact sequence theorem for singular cohomology.

Problem 3. Show that if $f\colon S^n \to S^n$ has degree d, then $f^* : H^n(S^n, G) \to H^n(S^n, G)$ is a multiplication by d.

Problem 4.

a) Let $\{U_i\}_{i=1}^{\infty}$ be compact convex sets in R^n such that every $(n+1)$-tuple of these sets have a common point. Prove that $\cap_{i=1}^{\infty} U_i \neq \varnothing$.

b) Give an example of an infinite family $\{U_i\}_{i=1}^{\infty}$ of convex sets (not compact!) in R^n such that every $n+1$ sets have a common point but there are no points common to all the sets U_i.

Math MATH 597C, Fall 2015 **Homework #5** Due on Monday, November 30th, 2015.

Problem 1. Use the existence of a partition of unity on an n-dimensional manifold to prove that M is orientable if and only if there exists a nowhere-vanishing differential form $\omega \in \Omega^n(M)$.

Problem 2. Compute the de Rham cohomology $H^*(S^1)$ and $H^*(T^2)$ using the definition. This includes the algebra structure.

Problem 3. Recall that the support of a function $f\colon M \to \mathbb{R}$ is the closure,

$$\mathrm{supp}(f) = \mathrm{closure}\{x \in M | f(x) \neq 0\}.$$

Define $\Omega_c^*(M)$ to be the differential forms with compact supports and observe that $d : \Omega_c^k(M) \to \Omega_c^{k+1}(M)$ is well-defined. This yields the deRham cohomology with compact supports, denoted as $H_c^*(M)$.

a) Compute $H_c^*(\text{a point})$ and $H_c^*(\mathbb{R})$ (hint: do not hesitate to integrate..).

b) Does the functor $H_c^*(-)$ respect the homotopy invariance? Give a proof of your answer.

c) Explain why pullback is not well-defined on $\Omega_c^*(-)$ for arbitrary continuous maps.

d) Let $U \subset \mathbb{R}^n$ be an open set. Use integration to define the pairing in deRham cohomology

$$H^k(U) \otimes H_c^{n-k}(U) \to \mathbb{R}.$$

Here you can use (without proof) the Stokes theorem that states that for every $\omega \in \Omega^{n-1}(\mathrm{closure}(U))$

$$\int_U d\omega = \int_{\partial U} \omega.$$

Problem 4*. Let N and M be manifolds. Assume that they both have finite good covers. Prove the isomorphism of deRham cohomology:

$$H^*(M \times N) = H^*(M) \otimes H^*(N).$$

Math MATH 597C, Fall 2015 **Homework #6** Due on Monday, December 14th, 2015.

Problem 1. Let X be a topological space, and $\mathcal{F}, \mathcal{G}$ be presheaves over X.

- A presheaf homomorphism $f\colon \mathcal{F} \to \mathcal{G}$ is a collection of group homomorphisms, $f_{|U}\colon \mathcal{F}(U) \to \mathcal{G}(U)$ defined for each open subset $U \subseteq X$, such that $\rho^U_V \circ f_{|U} = f_{|V} \circ \rho^U_V$ for every $V \subseteq U$.
- A fine sheaf over a topological space X is a sheaf with ~~with~~ "partitions of unity". More precisely, a sheaf $\mathcal{F}$ is fine if for any open cover $\{U_i\}_{i\in I}$ of the space X one can find a family of homomorphisms $\phi_i\colon \mathcal{F} \to \mathcal{F}$ from the sheaf to itself with $\sum_{i\in I} \phi_i = \mathrm{id}_{\mathcal{F}(X)}$ such that each homomorphism ϕ_i is zero, when restricted to any open subset V with $U_i \cap V = \varnothing$.

Prove that if $\mathcal{F}$ is a fine sheaf over X, then its sheaf cohomology can be described as $H^0(X,\mathcal{F}) = \mathcal{F}(X)$ and $H^q(X,\mathcal{F}) = 0$ for any $q > 0$.

Hint: you can recycle the idea of the proof of the proposition where we proved the vanishing of the cohomology of the sheaf of differential forms over a manifold. This relied on explicit construction of the homotopy operator, which should be the same as far as the formula is concerned; the meaning of each letter in that formula changes though...

Problem 2. Let X be a topological space. Assume that the topology of X is "so poor", that it has only four nonempty open sets: $X = U \cup V$, U, V, $E = U \cap V$. Denote by $\mathcal{U} = \{U, V\}$ its "natural" cover.

(a) Define <u>a sheaf</u> $\mathcal{F}$ as follows:

(*i*) $\mathcal{F}(U) = \mathbb{R}^m$, $\mathcal{F}(E) = \mathbb{R}^n$, and $\mathcal{F}(V) = 0$;

(*ii*) $\rho^U_E = A$, where $A\colon \mathbb{R}^m \to \mathbb{R}^n$ is an $n \times m$ matrix.

Compute $H^*(\mathcal{U},\mathcal{F})$ in terms of $\ker A$, $\operatorname{im} A$, and $\operatorname{coker} A$. Assuming that $\mathcal{F}$ is a sheaf, what is $\mathcal{F}(X)$?

(b) Define <u>a sheaf</u> $\mathcal{F}$ as follows:

(*i*) $\mathcal{F}(U) = \mathbb{R}^m$, $\mathcal{F}(E) = \mathbb{R}^n$, and $\mathcal{F}(V) = \mathbb{R}^p$ for some positive integers $m, n, p \in \mathbb{N}$.

(*ii*) $\rho^U_E = A$, and $\rho^V_E = B$ where $A\colon \mathbb{R}^m \to \mathbb{R}^n$ and $B\colon \mathbb{R}^p \to \mathbb{R}^n$ are the appropriate linear transformations.

Compute $H^*(\mathcal{U},\mathcal{F})$ in terms of the properties of the linear equation $Ax - By = 0$. Assuming that $\mathcal{F}$ is a sheaf, what is $\mathcal{F}(X)$?

(c) Why are the above two sheaves are (generally) not fine?

Math MATH 597C   **Take-home assignment**   Due at 11:00 am on Friday, Dec. 18th.

Please finish five out of the following six problems.

Problem 1. Compute the Euler characteristic of the following:

(*a*) The two-dimensional surface obtained from the regular octagon by identifying its parallel sides (all oriented clock-wise).

(*b*) The three-dimensional torus $T^3 = S^1 \times S^1 \times S^1$.

Problem 2. For each of the manifolds in the Problem 1, assume that a vector field on that manifold has at most five fixed points (zeros). What is the maximal number of stable fixed points (steady states) of such a vector field?

Problem 3. Use the Alexander duality to prove that the Klein bottle can not be embedded into the three-dimensional Euclidean space without self-intersections.

Problem 4. Consider the spaces $X = S^2 \vee S^1 \vee S^1$ and the two-dimensional torus $Y = T^2 = S^1 \times S^1$. Show that

(*a*) $H_k(X, G) = H_k(Y, G)$ for all k and G,

(*b*)* The spaces X and Y are <u>not</u> homotopy equivalent.

Problem 5. Let $A \subset V \subset X$ be a triple of subspaces. Prove that there is a long exact sequence in relative (singular) homology:

$$\cdots \to H_n(V, A) \to H_n(X, A) \to H_n(X, V) \to H_{n-1}(V, A) \to \cdots$$

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Problem 6. Let $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$ be a collection of N points in the d-dimensional Euclidean space.

(*a*) If $d = 1$, what is the relationship of the Rips and Cech complexes associated with this arrangement of points on a line?

(*b*) If $d = 2$, find a counter-example to the property you found for part (a).

(*c*) Find at least one homological condition to bound the *embedding dimension* d from below, based on

(*i*) The Cech complex.

(*ii*)* The Rips complex.