3/11 **§6.** SKIP 3.

4. $\boxed{P_n(k)} \triangleq \begin{cases} C_n^k\, p^k q^{n-k} & , k=0,1,\cdots,n \\ 0 & , k=n+1,\cdots . \end{cases}$

Poisson's Thm.

$p(n) \to 0$ as $n \to \infty$, $np(n) \to \lambda$, as $n \to \infty$, where $\lambda > 0$.

$\Rightarrow$ for each $k = 0, 1, 2, 3, \cdots$, we have

$$P_n(k) \to \pi_k \text{ as } n \to \infty, \text{ where } \pi_k = \frac{\lambda^k e^{-\lambda}}{k!}, k=0,1,\cdots.$$

<Pf>

$np(n) \to \lambda \Leftrightarrow n(p(n) - \frac{1}{n}\lambda) \to 0 \Leftrightarrow \dfrac{p(n) - \lambda/n}{1/n} \to 0 \Leftrightarrow p(n) - \lambda/n = o(1/n) \Leftrightarrow p(n) = \lambda/n + o(1/n).$

Thus $P_n(k) = C_n^k\, p(n)^k (1-p(n))^k$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}\left[\frac{\lambda}{n} + o\left(\frac{1}{n}\right)\right]^k \left[1 - \frac{\lambda}{n} - o\left(\frac{1}{n}\right)\right]^{n-k}$$

$$= \frac{1}{k!}\underbrace{\left\{n(n-1)\cdots(n-k+1)\left[\frac{\lambda}{n} + o\left(\frac{1}{n}\right)\right]^k\right\}}_{*_1}\underbrace{\left\{\left[1 - \frac{\lambda}{n} - o\left(\frac{1}{n}\right)\right]^{n-k}\right\}}_{*_2}.$$

$*_1 = \dfrac{n(n-1)\cdots(n-k+1)}{n^k}\cdot[\lambda + o(1)]^k \to \lambda^k$, as $n \to \infty$.

$*_2 = \left[1 - \dfrac{\lambda + o(1)}{n}\right]^{n-k} \to e^{-\lambda}$, as $n \to \infty$.

Hence, $P_n(k) \to \dfrac{\lambda^k e^{-\lambda}}{k!}$, as $n \to \infty$, for $k = 0, 1, 2, \cdots$. (#)

Rmk:

For a concrete and reasonable reason of the assumption of Poisson's Theorem, see P23 in $\boxed{\text{Meester's book}}$.

Def

The set of numbers $\{\pi_k \mid k = 0, 1, 2, \cdots\}$ is called the $\boxed{\text{Poisson probability distribution.}}$

SKIP 5.

6.

Def The function $\boxed{\Phi(x)} = \displaystyle\int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt$ is called the $\boxed{\text{normal}}$ or $\boxed{\text{Gaussian}}$ $\boxed{\text{distribution}}$ on the real line w/ density $\boxed{\varphi(x)} = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$.

Rmk: This function shows up in De Moivre-Laplace integral thm and is also important in other places.

SKIP 7. and Jump to § 8. (i.e. also SKIP §7.)

§8. Conditional Probabilities and Mathematical Expectations with Respect to Decompositions.

1.

Def $(\Omega, \mathcal{A}, P)$: probability space.

$\mathscr{D} = \{D_1, \cdots, D_k\}$: decomposition of Ω (i.e. $D_i \in \mathcal{A}, \forall i$. $D_i \neq \phi, \forall i$ and $\Omega = D_1 + \cdots + D_k$). + means disjoint union.

$P(D_i) > 0, \ \forall i = 1, \cdots, k$.

A : event from $\mathcal{A}$.

Define a random variable π on Ω by

$$\pi(\omega) = \sum_{i=1}^{k} P(A|D_i) I_{D_i}(\omega) \quad \cdots (1).$$

i.e. on D_i, it takes the value $P(A|D_i)$, the conditional probability of A given D_i.

This random variable is usually denoted by $\boxed{P(A|\mathscr{D})}$ and called the conditional probability of the event A with respect to the decomposition $\mathscr{D}$.

or $P(A|\mathscr{D})(\omega)$ to remind that it's a random variable.

Prop

(a) $P(A+B|\mathscr{D}) = P(A|\mathscr{D}) + P(B|\mathscr{D})$. $\cdots$ (2).

(b) $P(A|\Omega) = P(A)$, the constant random variable, where Ω is itself the trivial decomposition $\{\Omega\}$. $\cdots$ (3).

(c) (formula for total probability)

$$EP(A|\mathscr{D}) = P(A). \cdots (4) \quad \left(\text{Original form} : \sum_{i=1}^{k} P(A|D_i)P(D_i) = P(A)\right)$$

Def Given a random variable η on Ω. Assume $\text{im}(\eta) = \{y_1, \cdots, y_k\}$.

Then $\eta = \sum_{j=1}^{k} y_j I_{D_j}$, where $D_j = \{\eta = y_j\}$.

The decomposition $\boxed{\mathscr{D}_\eta} = \{D_1, \cdots, D_k\}$ is called the decomposition induced by η.

The random variable $P(A|\mathscr{D}_\eta)$, for an event A, will be denoted by $\boxed{P(A|\eta)}$ or $\boxed{P(A|\eta)(\omega)}$ to emphasize its being a random variable, and called the conditional probability of A with respect to the random variable η.

For a collection $\eta_1, \cdots, \eta_m$ of random variables, we can form the decomposition by $D_{y_1, y_2, \cdots, y_m} = \{\eta_1 = y_1, \eta_2 = y_2, \cdots, \eta_m = y_m\}$, where $y_i \in \text{im}(\eta_i)$, and similarly define $\boxed{P(A|\eta_1, \cdots, \eta_m)}$, called the conditional probability of A w.r.t. $\eta_1, \cdots, \eta_m$.

Exercise: ξ, η : indep. identically distributed random variables w/ values 0, 1 and probability p, q.
Compute $P(\xi + \eta = k|\eta)$, for $k = 0, 1, 2$.