field Ω, complement, fin. union

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

(iii) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

σ-field: field & coun. union

(iv) $A_1, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

probability measure on field.

(i) $0 \le P(A) \le 1, \forall A \in \mathcal{F}$.

(ii) $P(\emptyset) = 0$, $P(\Omega) = 1$.

(iii) $A_1, A_2, \cdots$: disjoint in $\mathcal{F}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

$\Rightarrow P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$.

countable additive.

Thm 3.1

A prob. mea. on a field has a unique extension to the generated σ-field.

$P^*(A) := \inf \sum_n P(A_n)$.

$\mathcal{M} := \{A \subseteq \Omega \mid P^*(E) = P^*(EA) + P^*(EA^c), \forall E \subseteq \Omega\}$

L0: 4 properties of P^*

L1 $\mathcal{M}$: field.

L2

$A_1, A_2, \cdots \in \mathcal{M}$, disjoint.

$E \subseteq \Omega$

$\Rightarrow P^*(E(\bigcup_n A_n)) = \sum_{n=1}^{\infty} P^*(EA_n)$

L3

$\mathcal{M}$: σ-field and

$P^*|_{\mathcal{M}}$: countably additive.

L4

$P^* = \inf\{\cdots\}$

Then $\mathcal{F}_0 \subseteq \mathcal{M}$.

L5

$P^* = \inf\{\cdots\}$

Then $P^*(A) = P(A), \forall A \in \mathcal{F}_0$

Rmk:

① L0 ⇒ L1 + L2 + L3

② Explicit formula of P^* ⇒ L4 + L5

③ L3 + L4 + L5 ⇒ existence.

π-system. fin. intersect'n.

$A, B \in \mathcal{P} \Rightarrow AB \in \mathcal{P}$.

λ-system. Ω, com, disj. coun. union

(λ_1) $\Omega \in \mathcal{L}$.

(λ_2) $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$.

(λ_3) $A_1, \cdots \in \mathcal{L}$ disjoint $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$.

Rmk:

σ-field ⇒ λ-system.

But not vice versa.

L6

$\mathcal{A}$: both π-system and λ-system

$\Rightarrow \mathcal{A}$: σ-field.

Thm 3.2 (Dynkin's π-λ thm.)

$\mathcal{P}$: π-system. $\mathcal{L}$: λ-system.

$\mathcal{P} \subseteq \mathcal{L}$.

$\Rightarrow \sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Thm 3.3 (Uniqueness)

P_1, P_2 : prob. mea. on $\sigma(\mathcal{P})$,

$\mathcal{P}$: π-system.

$P_1 = P_2$ on $\mathcal{P}$.

$\Rightarrow P_1 = P_2$ on $\sigma(\mathcal{P})$.

Thm 3.4 (Halmos's M.C.T.)

$\mathcal{F}_0$: field. $\mathcal{M}$: monotone class.

$\mathcal{F}_0 \subseteq \mathcal{M}$.

$\Rightarrow \sigma(\mathcal{F}_0) \subseteq \mathcal{M}$.

$\lim A_n = A$ if

$\underline{\lim} A_n = A = \overline{\lim} A_n$.

Thm 4.1

(i) $P(\underline{\lim} A_n) \le \underline{\lim} P(A_n)$

$\le \overline{\lim} P(A_n) \le P(\overline{\lim} A_n)$.

(ii) $A_n \to A \Rightarrow P(A_n) \to P(A)$.

indep.

$P(A_{k_1} A_{k_2} \cdots A_{k_j}) = P(A_{k_1}) \cdots P(A_{k_j})$,

$\forall\ 1 \le k_1 < \cdots < k_j \le n$.

Rmk:

Pairwise indep. $\not\Rightarrow$ indep.

Thm 4.2

$\mathcal{A}_1, \cdots, \mathcal{A}_n$: indep. π-sys.

$\Rightarrow \sigma(\mathcal{A}_1), \cdots, \sigma(\mathcal{A}_n)$: indep.

Cor 1

$\mathcal{A}_\theta, \theta \in \Theta$: indep. π-sys.

$\Rightarrow \sigma(\mathcal{A}_\theta), \theta \in \Theta$: indep.

Cor 2

$A_{11}\ A_{12} \cdots$

$A_{21}\ A_{22} \cdots$: indep.

$\vdots \quad \vdots$

$\mathcal{F}_i := \sigma$(ith row)

Then

$\mathcal{F}_1, \mathcal{F}_2, \cdots$: indep.

Rmk: (by Thm 4.2)

$A_1, A_2, \cdots, A_n$: indep.

$\Rightarrow P(A_1^c \cdots A_k^c A_{k+1} \cdots A_n)$

$= P(A_1^c) \cdots P(A_k^c) P(A_k) \cdots P(A_n)$.

Thm 4.3 (1st Borel-Cantelli lemma)

$\sum_n P(A_n) < \infty$

$\Rightarrow P(\overline{\lim} A_n) = 0$.

Thm 4.4 (2nd Borel-Cantelli)

$\{A_n\}$: indep.

$\sum_n P(A_n) = \infty$.

$\Rightarrow P(\overline{\lim} A_n) = 1$.

Def $(\Omega, \mathcal{F}, P)$: prob. sp.

$A_1, A_2, \cdots \in \mathcal{F}$.

$\mathcal{T} := \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \cdots)$, called **tail σ-field** of the seq.

Elements in $\mathcal{T}$ are called **tail elements**.

Rmk:

$\underline{\lim} A_n, \overline{\lim} A_n \in \mathcal{T}$.

Thm 4.5 (Kolmogorov's 0-1 law)

$A_1, \cdots, A_n, \cdots$: indep.

$A \in \mathcal{T}$, the tail σ-field.

Then $P(A) = 0$ or 1.

$\mathcal{R}^k$ $:= \sigma(\prod_{i=1}^{k} (a_i, b_i])$

(= Borel σ-field on $\mathbb{R}^k$).

$\mathcal{A} \cap \Omega_0$

$:= \{A \cap \Omega_0 \mid A \in \mathcal{A}\}$.

Thm 10.1 $\Omega_0 \subseteq \Omega$.

(i) $\mathcal{F}$: σ-field on Ω.

$\Rightarrow \mathcal{F} \cap \Omega_0$: σ-field on Ω_0.

(ii) $\mathcal{F} = \sigma(\mathcal{A})$.

$\Rightarrow \mathcal{F} \cap \Omega_0 = \sigma(\mathcal{A} \cap \Omega_0)$.

i.e. $\sigma(\mathcal{A}) \cap \Omega_0 = \sigma(\mathcal{A} \cap \Omega_0)$.

Rmk:

$\Omega_0 \in \mathcal{F}$

$\Rightarrow \mathcal{F} \cap \Omega_0 = \{A \in \mathcal{F} \mid A \subseteq \Omega_0\}$

Thm 10.2

μ : mea. on a field $\mathcal{F}$.

Then

(i) cont. from below.

(ii) " " above.

(iii) countable subadd.

(iv) μ : σ-finite on $\mathcal{F}$.

$\Rightarrow \mathcal{F}$ cannot contain uncountably many collect'n of disjoint $\mathcal{F}$-sets w/ positive μ-measure.

measure on a field.

(i) $\mu(A) \in [0, \infty], \forall A \in \mathcal{F}$.

(ii) $\mu(\emptyset) = 0$.

(iii) $A_1, A_2, \cdots \in \mathcal{F}$ disjoint.

$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

$\Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

σ-finite on $\mathcal{A}$ $\subseteq \mathcal{F}$ if

$\exists\ \mathcal{A}$-sets $A_1, A_2, \cdots$ s.t.

$\mu(A_n) < \infty, \forall n$ and

$\bigcup_{n=1}^{\infty} A_n = \Omega$.

Thm 10.3 (Gen. of Thm 3.3)

μ_1, μ_2 : mea. on $\sigma(\mathcal{P})$

$\mathcal{P}$: π-sys.

In particular, prob. mea.

μ_1, μ_2 : σ-finite on $\mathcal{P}$.

$\mu_1 = \mu_2$ on $\mathcal{P}$.

Then $\mu_1 = \mu_2$ on $\sigma(\mathcal{P})$.

outer measure.

(i) $\mu^*(A) \in [0, \infty], \forall A \subseteq \Omega$.

(ii) $\mu^*(\emptyset) = 0$.

(iii) μ^* : monotone.

(iv) μ^* : countable subadd.

μ^*-mea / **$\mathcal{M}(\mu^*)$**

Thm 11.1

μ^* : outer measure.

Then

$\mathcal{M}(\mu^*)$: σ-field, and

$\mu^*|_{\mathcal{M}(\mu^*)}$: measure.

Thm 11.2

A mea. on a field has an extension to the generated σ-field.

semiring

(i) $\emptyset \in \mathcal{A}$.

(ii) $A, B \in \mathcal{A} \Rightarrow AB \in \mathcal{A}$.

(iii) $A, B \in \mathcal{A}$ w/ $A \subseteq B$
$\Rightarrow \exists$ disjoint $\mathcal{A}$-sets $C_1, C_2, \cdots, C_n$ s.t. $B \setminus A = \bigcup_{k=1}^{n} C_k$.

Thm 11.3

μ: set fun. on $\mathcal{A}$

$\mathcal{A}$: semiring

$\mu(\emptyset) = 0$

μ : finitely add. and countably subadd.

$\mu(A) \in [0, \infty]$, $\forall A \in \mathcal{A}$.

$\Rightarrow \mu$ extends to a measure on $\sigma(\mathcal{A})$.

Rmk : (X)

(1) Thm 11.3 $\Rightarrow$ Thm 11.2

(2) If μ : σ-finite on $\mathcal{A}$, then, by Thm 10.3, the extension is unique.

(3) Why semiring ?
More general than field.
Ex:
$\mathcal{A} := \{(a,b] \mid (a,b] \subseteq \mathbb{R}\}$.
(or $\{\prod_{k=1}^{n}(a_k, b_k] \mid (a_k, b_k] \subseteq \mathbb{R}, \forall k\}$.)

L1 $A \in \mathcal{A}$

$A_1, A_2, \cdots, A_n \in \mathcal{A}$, $\mathcal{A}$: semi-ring.

$\Rightarrow \exists$ disjoint $C_1, \cdots, C_m \in \mathcal{A}$ s.t. $AA_1^c \cdots A_n^c = \bigcup_{k=1}^{m} C_k$.

(General ver. of condit'n (iii)).

Thm 11.4 (Approximat'n thm.)

$\mathcal{A}$: semiring

μ : measure on $\mathcal{F} = \sigma(\mathcal{A})$.

μ : σ-finite on $\mathcal{A}$

Then, for $B \in \mathcal{F}$ and $\varepsilon > 0$,

(i)
$\exists$ countably many disjoint $A_1, A_2, \cdots \in \mathcal{A}$ s.t.
$B \subseteq \bigcup_{k=1}^{\infty} A_k$ and
$\mu(\bigcup_k A_k \setminus B) < \varepsilon$.

(ii)
if $\mu(B) < \infty$, then
$\exists$ finitely many disjoint $A_1, A_2, \cdots, A_n \in \mathcal{A}$ s.t.
$\mu(B \triangle (\bigcup_{k=1}^{n} A_k)) < \varepsilon$.

Cor 1 (X)

μ : finite measure on $\mathcal{F}$

$\mathcal{F} = \sigma(\mathcal{F}_0)$, $\mathcal{F}_0$: field.

$\Rightarrow \forall A \in \mathcal{F}$ and $\varepsilon > 0$,
$\exists B \in \mathcal{F}_0$ s.t.
$\mu(A \triangle B) < \varepsilon$.

Cor 2 (X)

$\mathcal{A}$: semiring

μ_1, μ_2 : mea. on $\mathcal{F} = \sigma(\mathcal{A})$.

$\mu_1 \leq \mu_2$ on $\mathcal{A}$.

μ_1, μ_2 : σ-finite on $\mathcal{A}$

Then $\mu_1 \leq \mu_2$ on $\mathcal{F}$.

L2 (used in S12) (X)

μ : set fun. on $\mathcal{A}$, $\mathcal{A}$: semiring.

$A, A_1, \cdots, A_n \in \mathcal{A}$.

(i) If $\bigcup_{k=1}^{n} A_k \subseteq A$, and $A_1, A_2, \cdots, A_n$: disjoint,
then $\sum_{k=1}^{n} \mu(A_k) \leq \mu(A)$.

$\boxed{\mu : \text{nonnegative, fin-add.}}$

(ii) If $A \subseteq \bigcup_{k=1}^{n} A_k$,
then $\mu(A) \leq \sum_{k=1}^{n} \mu(A_k)$.

Def $A := \prod_{i=1}^{k} (a_i, b_i]$

$x = (x_1, \cdots, x_k)$, a vertex of A.

$\boxed{\mathrm{sgn}_A x} := (-1)^{\#\{i \mid x_i = a_i\}}$, called $\boxed{\text{signum}}$ of x.

$F : \mathbb{R}^k \to \mathbb{R}$

$\boxed{\Delta_A F} := \sum_x (\mathrm{sgn}_A x) \cdot F(x)$, called the $\boxed{\text{difference}}$ of F around vereices of A.

Def $F : \mathbb{R}^k \to \mathbb{R}$.

F : $\boxed{\text{cont. from above}}$ if

$F(x^{(n)}) \to F(x)$, $\forall$

$\forall\ x = (x_1, \cdots, x_k)$

$x^{(n)} = (x_1^{(n)}, \cdots, x_k^{(n)})$

w/ $x_i^{(n)} \downarrow x_i$ as $n \to \infty$.

$\boxed{\text{Thm (12.5)}}$

$F : \mathbb{R}^k \to \mathbb{R}$: cont. from above.

$\Delta_A F \geq 0$, $\forall$ bdd rec A.

$\Rightarrow \exists !$ measure μ on $\mathcal{R}^k$ s.t.

$\mu(A) = \Delta_A F$, $\forall$ bdd rec A.

$\boxed{\text{Rmk:}}$

① μ : finite mea. on $\mathcal{R}^k$.

$F(x) := \mu(\{y = (y_1, \cdots, y_k) \mid y_i \leq x_i, \forall i\})$

$x = (x_1, \cdots, x_k)$.

Then F : cont. from above and

$\Delta_A F = \mu(A)$, $\forall$ bdd rec. A

② $F(x_1, \cdots, x_k) = x_1 \cdots x_k$.

Then μ in (12.5) is Lebesgue mea. in $\mathbb{R}^k$.

Def $T : \Omega \to \Omega'$, $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$.

T : $\boxed{\text{measurable } \mathcal{F}/\mathcal{F}'}$ if

$T^{-1}(A') \in \mathcal{F}$, $\forall A' \in \mathcal{F}'$.

When $(\Omega', \mathcal{F}') = (\mathbb{R}, \mathcal{R}^1)$,

$\boxed{\text{Measurable } \mathcal{F}} :=$ mea. $\mathcal{F}/\mathcal{R}^1$.

$\boxed{\text{Thm (13.1)}}$

$T : \Omega \to \Omega'$, $T' : \Omega' \to \Omega''$.

(i) $T^{-1}(A') \in \mathcal{F}$, $\forall A' \in \mathcal{A}'$, where $\mathcal{A}' \subseteq \mathcal{F}'$ w/ $\sigma(\mathcal{A}') = \mathcal{F}'$.

Then T : mea. $\mathcal{F}/\mathcal{F}'$.

(ii) T : mea. $\mathcal{F}/\mathcal{F}'$, T' : mea. $\mathcal{F}'/\mathcal{F}''$

$\Rightarrow T' \circ T$: mea. $\mathcal{F}/\mathcal{F}''$.

$\boxed{\text{Thm (13.2)}}$

$f : \mathbb{R}^\ell \to \mathbb{R}^k$ cont.

$\Rightarrow f$: Borel.

$\boxed{\text{Thm (13.3)}}$

$g : \mathbb{R}^k \to \mathbb{R}$ mea. (i.e. Borel)

$f_j : \Omega \to \mathbb{R}$ mea. $\mathcal{F}$, $j = 1, \cdots, k$.

$\Rightarrow g(f_1, \cdots, f_k) : \Omega \to \mathbb{R}$ mea. $\mathcal{F}$.

$\boxed{\text{Thm (13.4)}}$

$f_1, f_2, \cdots$ real fun. mea. $\mathcal{F}$. Then

(i) $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim_n} f_n$, $\underline{\lim_n} f_n$: mea. $\mathcal{F}$.

(ii) If $\lim_{n\to\infty} f_n$ exists, then the limit is mea. $\mathcal{F}$.

(iii) $\{\omega \mid \lim_{n\to\infty} f_n(\omega) \text{ exists}\} \in \mathcal{F}$.

(iv) For f mea. $\mathcal{F}$, the set $\{\omega \mid f_n(\omega) \to f(\omega)\} \in \mathcal{F}$.

$\boxed{\text{Thm (13.5)}}$ (*)

f : real mea. $\mathcal{F}$, $f \geq 0$.

Then $\exists$ simple mea. $\mathcal{F}$ f_n s.t. $0 \leq f_n \uparrow f$.

Def $T : \Omega \to \Omega'$, mea. $\mathcal{F}/\mathcal{F}'$.

μ : mea. on $(\Omega, \mathcal{F})$.

Define $\boxed{\mu T^{-1} : \mathcal{F}' \to [0, \infty]}$ by

$\mu T^{-1}(A') := \mu(T^{-1}(A'))$, $\forall A' \in \mathcal{F}'$.

$\boxed{\text{Prop.}}$

(1) μT^{-1} : mea. on $\mathcal{F}'$.

(2) μ : finite $\Rightarrow \mu T^{-1}$: finite.

(3) μ : prob. mea. $\Rightarrow \mu T^{-1}$: prob. mea.

$\boxed{\text{Rmk:}}$

μ : σ-finite $\not\Rightarrow \mu T^{-1}$: σ-finite.

$\boxed{\text{random variable}}$ $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{R}^1)$.

$\boxed{\text{〃 vector}}$ X : 〃 $\to (\mathbb{R}^k, \mathcal{R}^k)$.

$\boxed{\text{〃 element}}$ X : 〃 $\to (\Omega', \mathcal{F}')$.

Def $X : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ r.v.

$(\Omega, \mathcal{F}, P)$: prob. sp.

The $\boxed{\text{distribution}}$ of X is the prob. mea. $\boxed{\mu}$ on $(\mathbb{R}, \mathcal{R}^1)$ defined by $\mu(A) = P(X \in A) = P(X^{-1}(A))$.

$\boxed{F(x)} := \mu((-\infty, x]) = P(X \leq x)$

$\boxed{\text{distribution fun}}$ of X.

$\boxed{\text{Prop}}$

(1) F : right-cont. nondecreasing.

(2) left limit $F(x^-) = P(X < x)$.

(3) F has only countably many discont., which are jumps.

(4) $\lim_{x\to-\infty} F(x) = 0$. $\lim_{x\to\infty} F(x) = 1$.

$\boxed{\text{Thm (14.1)}}$

F : non-decreasing, right-cont.

$\lim_{x\to-\infty} F(x) = 0$. $\lim_{x\to\infty} F(x) = 1$.

Then $\exists$ r.v. X on some prob. sp. s.t. $F(x) = P(X \leq x)$.

$\boxed{\text{Thm (15.1)}}$

(i) $f = \sum_i x_i I_{A_i}$: nonnegative simple.

$\Rightarrow \int f \, d\mu = \sum_i x_i \mu(A_i)$.

(ii) $0 \leq f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$.

(iii) $0 \leq f_n \uparrow f. \Rightarrow 0 \leq \int f_n d\mu \uparrow \int f d\mu$.

(iv) f, g : nonneg. fun.

α, β : 〃 const.

$\Rightarrow \int \alpha f + \beta g \, d\mu = \alpha \int f d\mu + \beta \int g d\mu$.

$\boxed{\text{Thm (15.2)}}$

f, g : nonneg.

(i) $f = 0$ a.e. $\Rightarrow \int f = 0$.

(ii) $\mu(f > 0) > 0. \Rightarrow \int f > 0$.

(iii) $\int f < \infty. \Rightarrow f < \infty$ a.e.

(iv) $f \leq g$ a.e. $\Rightarrow \int f \leq \int g$.

(v) $f = g$ a.e. $\Rightarrow \int f = \int g$.

$\boxed{\text{Thm (16.1)}}$

(i) f, g : integ. $f \leq g$ a.e. $\Rightarrow \int f \leq \int g$.

(ii) … 〃 $\alpha, \beta \in \mathbb{R}. \Rightarrow \int \alpha f + \beta g = \alpha \int f + \beta \int g$.

$\boxed{\text{Rmk:}}$

$|\int f| \leq \int |f|$.

$\boxed{\text{Thm (16.2) (MCT)}}$

$0 \leq f_n \uparrow f$ a.e. $\Rightarrow \int f_n \uparrow \int f$.

$\boxed{\text{Thm (16.3) (Fatou's lemma)}}$

$f_n \geq 0$.

$\Rightarrow \int \underline{\lim_n} f_n \leq \underline{\lim_n} \int f_n$.

$\boxed{\text{Thm (16.4) (LDCT)}}$ imply

$|f_n| \leq g$ a.e. $\forall n$., g : integ.

$f_n \to f$ a.e.

$\Rightarrow \int f_n \to \int f$.

$\boxed{\text{Thm (16.5) (bdd con. thm.)}}$

$\mu(\Omega) < \infty$. f_n : unif. bdd.

$f_n \to f$ a.e.

$\Rightarrow \int f_n \to \int f$.

$\boxed{\text{Thm (16.6)}}$

$f_n \geq 0$.

$\Rightarrow \int \sum_n f_n = \sum_n \int f_n$.

$\boxed{\text{Thm (16.7)}}$

$\sum_n f_n$ con. a.e. $|\sum_{k=1}^{n} f_k| \leq g$ a.e. $\forall n$ w/ g : integ.

$\Rightarrow \sum_n f_n$, f_n : integ., $\int \sum_n f_n = \sum_n \int f_n$.

$\boxed{\text{Thm (16.8)}}$ (Appl. of DCT to conti. diff.).

$\boxed{\text{Thm (16.9)}}$

$A_1, A_2, \cdots$ disjoint in $\mathcal{F}$.

f : either nonneg. or integ.

Then $\int_{\bigcup_n A_n} f = \sum_n \int_{A_n} f$.

$\boxed{\text{Thm (16.10)}}$

(1) f, g : nonneg. $\int_A f = \int_A g$, $\forall A \in \mathcal{F}$.

μ : σ-finite.

Then $f = g$ μ-a.e.

(2) f, g : integ. $\int_A f = \int_A g$, $\forall A \in \mathcal{F}$.

Then $f = g$ μ-a.e.

(3) f, g : integ.

$\int_A f = \int_A g$, $\forall A \in \mathcal{P}$, where

$\mathcal{P}$: π-system w/ $\sigma(\mathcal{P}) = \mathcal{F}$

Ω = countable union of $\mathcal{P}$-sets

Then $f = g$ μ-a.e.

Def

δ : nonneg. μ-mea.

$\nu(A) := \int_A \delta \, d\mu$, $\forall A \in \mathcal{F}$.

Then ν is a mea. and said to $\boxed{\text{have density } \delta \text{ w.r.t. } \mu}$.

$\boxed{\text{Rmk:}}$

① $\nu(A) = 0$, $\forall \mu(A) = 0$.

② ν : finite $\Leftrightarrow \delta$: μ-integ.

③ $\nu(A) = \int_A \delta' d\mu$, $\forall A \in \mathcal{F}$

μ : σ-finite.

$\Rightarrow \delta = \delta'$ μ-a.e.

$\boxed{\text{Thm (16.11) (density theorem)}}$

ν has density δ w.r.t. μ.

Then, for nonneg. or integ. f,

$\int_A f \, d\nu = \int_A f \cdot \delta \, d\mu$, $\forall A \in \mathcal{F}$

Moreover,

f : ν-integ. $\Leftrightarrow f \cdot \delta$: μ-integ.

$\boxed{\text{Thm (16.12) (Scheffe's thm.)}}$

$d\nu_n = \delta_n d\mu$

$d\nu = \delta \, d\mu$.

$\nu_n(\Omega) = \nu(\Omega) < \infty$, $\forall n$.

$\delta_n \to \delta$ μ-a.e.

Then

$\sup_{A \in \mathcal{F}} |\nu(A) - \nu_n(A)|$

$\leq \int_\Omega |\delta - \delta_n| \, d\mu \to 0$ as $n \to \infty$.

<Pf>

$g_n := \delta - \delta_n. \Rightarrow 0 \leq g_n \leq \delta$.

LDCT.

Thm (16.13) (Change of Variables)

$T:(\Omega,\mathcal{F})\to(\Omega',\mathcal{F}')$ mea. $\mathcal{F}/\mathcal{F}'$.

Then $f:(\Omega',\mathcal{F}')\to\mathbb{R}$ integ. w.r.t. μT^{-1}

$\iff f\circ T$: integ. w.r.t. μ.

If f is nonneg. or μT^{-1}-integ., then

$$\int_{T^{-1}(A')} f(T\omega)\,\mu(d\omega) = \int_{A'} f(\omega')\,\mu T^{-1}(d\omega'),\quad \forall A'\in\mathcal{F}'.$$

Motivat'n:

f: integ. $\Rightarrow \lim_{\alpha\to\infty}\int_{[|f|\geq\alpha]}|f|\,d\mu=0$.

Def

$\{f_n\}$: **unif. integ.** if

$$\lim_{\alpha\to\infty}\sup_n\int_{[|f_n|\geq\alpha]}|f_n|\,d\mu=0. \quad (*)$$

Rmk:

① $(*)$ holds and $\mu(\Omega)<\infty$.

$\Rightarrow f_n$: integ., $\forall n$.

② $\{f_n\}$: unif. bdd. $\Rightarrow (*)$ holds. (obvious).

③ $\{f_n\},\{g_n\}$: unif. integ.

$\Rightarrow \{f_n+g_n\}$: unif. integ.

Thm (16.14)

$\mu(\Omega)<\infty$, $f_n\to f$ a.e.

(1) $\{f_n\}$: unif. integ.

$\Rightarrow f$: integ. and $\int f_n\to\int f$.

(2) f_n, f : integ. $\int f_n\to\int f$.

$f_n, f\geq 0$.

$\Rightarrow \{f_n\}$: unif. integ.

Cor

$\mu(\Omega)<\infty$.

f_n, f : integ., $f_n\to f$ a.e.

Then T.F.A.E.

(1) $\{f_n\}$: unif. integ.

(2) $\int|f_n-f|\to 0$.

(3) $\int|f_n|\to\int|f|$.

def & bdd con. thm

tri. ineq.

(16.14) (2)

field: Ω, com, fin. union.

σ-field: field, coun. union.

Thm 3.1

Prob. mea. on a field has a unique extension to gen. σ-field.

π-system: fin. intersection.

λ-sys: Ω, com, disj. coun. union.

Thm 3.2 (Dynkin's π-λ thm.)

$\mathcal{P}$: π-sys. $\mathcal{L}$: λ-sys.

$\mathcal{P}\subseteq\mathcal{L}$. Then $\sigma(\mathcal{P})\subseteq\mathcal{L}$.

$\lim_{n\to\infty}A_n=A$ if $\liminf A_n=A=\limsup A_n$.

Thm 4.1

(i) $P(\liminf A_n)\leq\liminf P(A_n)\leq\limsup P(A_n)\leq P(\limsup A_n)$

(ii) $A_n\to A\Rightarrow P(A_n)\to P(A)$.

Thm 4.2

$\mathcal{A}_1,\cdots,\mathcal{A}_n$: indep. π-sys.

$\Rightarrow \sigma(\mathcal{A}_1),\cdots,\sigma(\mathcal{A}_n)$: indep.

Thm 4.3 (1st Borel-Cantelli)

$\sum_n P(A_n)<\infty\Rightarrow P(\limsup A_n)=0$.

Thm 4.4 (2nd Borel-Cantelli)

$\{A_n\}$: indep., $\sum_n P(A_n)=\infty$.

$\Rightarrow P(\limsup A_n)=1$.

Def $(\Omega,\mathcal{F},P)$: prob. sp. $A_1,A_2,\dots\in\mathcal{F}$.

$\mathcal{T}:=\bigcap_{n=1}^{\infty}\sigma(A_n,A_{n+1},\cdots)$, **tail σ-field**.

tail elements: elements in $\mathcal{T}$.

Rmk: $\liminf A_n,\ \limsup A_n\in\mathcal{T}$.

Thm 4.5 (Kolmogorov's 0-1 law)

$A_1,\cdots$: indep., $A\in\mathcal{T}$.

$\Rightarrow P(A)=0$ or 1.

$\mathcal{A}\cap\Omega_0$ $:=\{A\cap\Omega_0\mid A\in\mathcal{A}\}$.

Thm 10.1 $\Omega_0\subseteq\Omega$.

(i) $\mathcal{F}$: σ-field on Ω.

$\Rightarrow \mathcal{F}\cap\Omega_0$: σ-field on Ω_0.

(ii) $\sigma(\mathcal{A})\cap\Omega_0=\sigma(\mathcal{A}\cap\Omega_0)$.

mea. on a field: $[0,\infty]$, $\phi\mapsto 0$, countable additive.

σ-finite on $\mathcal{A}$ $\subseteq\mathcal{F}$.

Thm 10.2 μ: mea. on a field $\mathcal{F}$.

Then (i) cont. from below/"above"

(ii) count. subadd.

(iv) μ : σ-fin. on $\mathcal{F}$.

$\Rightarrow\mathcal{F}$ can not contain uncount. many disjoint positive μ-mea. set.

Thm 10.3 uniqueness

μ_1,μ_2 : mea. on $\sigma(\mathcal{P})$

$\mathcal{P}$: π-sys. μ_1,μ_2 : σ-fin. on $\mathcal{P}$.

$\mu_1=\mu_2$ on $\mathcal{P}$.

$\Rightarrow\mu_1=\mu_2$ on $\sigma(\mathcal{P})$.

outer measure

(i) $\mu^*(A)\in[0,\infty]$, $\forall A\subseteq\Omega$.

(ii) $\mu^*(\phi)=0$. (iii) μ^* : monotone.

(iv) μ^* : count. subadd.

μ^*-mea. / **$\mathcal{M}(\mu^*)$**.

Thm 11.1 μ^* : outer mea. Then

$\mathcal{M}(\mu^*)$: σ-field, $\mu^*|_{\mathcal{M}(\mu^*)}$: mea.

Thm 11.2

Mea. on a field has an extension to the gen. σ-field.

imply

semiring.

(i) $\phi\in\mathcal{A}$

(ii) $A,B\in\mathcal{A}\Rightarrow AB\in\mathcal{A}$.

(iii) $A,B\in\mathcal{A}$ w/ $A\subseteq B$

$\Rightarrow\exists$ disjoint $C_1,\cdots,C_n\in\mathcal{A}$ s.t.

$B\setminus A=\bigcup_{k=1}^{n}C_k$.

Thm 11.3 (Extension thm.)

μ : set fun. on $\mathcal{A}$, $\mathcal{A}$: semiring

$\mu(\phi)=0$

μ : fin. add., count. subadd.

$\mu(A)\in[0,\infty]$, $\forall A\in\mathcal{A}$.

$\Rightarrow\mu$ extends to a mea. on $\sigma(\mathcal{A})$.

Rmk:

By Thm 10.3, in Thm 11.3, if μ : σ-fin. on $\mathcal{A}$, then the extension is unique.

L1 (Gen. ver. of semiring (iii))

$A,A_1,A_2,\cdots,A_n\in\mathcal{A}$, $\mathcal{A}$: semiring

$\Rightarrow\exists$ disjoint $C_1,\cdots,C_m\in\mathcal{A}$ s.t.

$AA_1^cA_2^c\cdots A_n^c=\bigcup_{k=1}^{m}C_k$.

Thm 11.4. (App. thm).

$\mathcal{A}$: semiring

μ : mea. on $\mathcal{F}=\sigma(\mathcal{A})$

μ : σ-fin. on $\mathcal{A}$.

Then, for $B\in\mathcal{F}$ and $\varepsilon>0$,

(i) $\exists$ disjoint $A_1,\cdots\in\mathcal{A}$ s.t.

$B\subseteq\bigcup_{k=1}^{\infty}A_k$ and $\mu(\bigcup_{k=1}^{\infty}A_k\setminus B)<\varepsilon$.

(ii) if $\mu(B)<\infty$, then $\exists$ fin. many disjoint $A_1,\cdots,A_n\in\mathcal{A}$ s.t.

$\mu(B\,\Delta\,\bigcup_{k=1}^{n}A_k)<\varepsilon$.