PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, May 11, 2016

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

√1. Find all prime ideals and all maximal ideals in the ring $\mathbb{Z}[x, y]$ which contain the ideal $I := (35, x^2+1, y)$. (The whole ring is not considered a prime ideal.) Justify your answer.

√2. Determine the Galois group G of the polynomial X^4+4 over $\mathbb{Q}$. In particular, determine whether G is abelian and whether it is cyclic.

3. Prove that there is no simple group of order 120.

√4. Find the cardinality of the set of all distinct non-isomorphic $\mathbb{F}_p[x]$-module structures on $\mathbb{F}_p^2$. Justify your answer.

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DATE: May 6, 2016

FROM: Becky Becky

TO: Men-Chu Wu

RE: Qualifying Exams

You signed up to take the Analysis A / algebra B exam(s).

The number of your exam packet is 68. The exams will be given as follows:

Monday, May 9

Analysis – Part A 9:00-11:00 am
Analysis – Part B/C 1:30-3:30 pm

Wednesday, May 11

Algebra–Part A 9:00-11:00 am
Algebra-Part B/C 1:30-3:30 pm

All exams will be held in Room 114 McAllister Building.

If this information is not correct, please let me know as soon as possible.

QE Algebra. Fall 2015

1.

[Sol]

$30 = 2\cdot 3\cdot 5$.

$n_3 \equiv 1 \mod 3$. $n_3 \mid 10$. $\Rightarrow n_3 = 1$ or 10.

$n_5 \equiv 1 \mod 5$. $n_5 \mid 6$. $\Rightarrow n_5 = 1$ or 6.

Note we cannot have both $n_3 = 10$ and $n_5 = 6$. o.w. the number of elements of order 3 would be at least $10\cdot 2 = 20$ and that of order 5 would be at least $6\cdot 4 = 24$; in total, $44 > 30$ $\times$.

Hence, $n_3 = 1$ or $n_5 = 1$.

Let P_3 and P_5 be a Sylow 3-subgp and 5-subgp, resp.

$|P_3| = 3$ and $|P_5| = 5$. $\Rightarrow$ P_3 and P_5 are both cyclic, say $P_3 = \{1, \alpha, \alpha^2\}$ and $P_5 = \{1, \beta, \cdots, \beta^4\}$

case1 $n_3 = 1$.

Then P_3: normal. $\Rightarrow \beta\alpha\beta^{-1} = \alpha$ or α^2.

Suppose $\beta\alpha\beta^{-1} = \alpha^2$. Then $\alpha^4 = \beta\alpha^2\beta^{-1} = \beta^2\alpha\beta^{-2}$. $\Rightarrow \alpha^8 = \beta^2\alpha^2\beta^{-2} = \beta^3\alpha\beta^{-3}$. $\Rightarrow \alpha^{16} = \beta^3\alpha^2\beta^{-3} = \beta^4\alpha\beta^{-4}$.

$\Rightarrow \alpha^{32} = \beta^4\alpha^2\beta^{-4} = \beta^5\alpha\beta^{-5} = \alpha$. $\Rightarrow \alpha^{31} = 1$. $\Rightarrow (\alpha^3)^{10}\alpha = 1$. $\Rightarrow \alpha = 1$. $\times$

Thus, $\beta\alpha\beta^{-1} = \alpha$. i.e. $\beta\alpha = \alpha\beta$.

case2 $n_5 = 1$.

Then P_5: normal. $\Rightarrow \alpha\beta\alpha^{-1} = \beta^k$, some $k \in \{1,2,3,4\}$.

$\Rightarrow \beta^{k^2} = \alpha\beta^k\alpha^{-1} = \alpha^2\beta\alpha^{-2}$. $\Rightarrow \beta^{k^3} = \alpha^2\beta^k\alpha^{-2} = \alpha^3\beta\alpha^{-3} = \beta$. $\Rightarrow \beta^{k^3-1} = 1$. $\Rightarrow 5 \mid k^3 - 1$.

In $\{1,2,3,4\}$, $k=1$ is the only one making $5 \mid k^3-1$. $\Rightarrow \alpha\beta\alpha^{-1} = \beta$. $\Rightarrow \alpha\beta = \beta\alpha$.

In both cases, $\alpha\beta = \beta\alpha$. $\Rightarrow \langle\alpha\beta\rangle = \{\alpha^i\beta^j \mid 0 \le i \le 2,\ 0 \le j \le 4\}$. $\Rightarrow |\langle\alpha\beta\rangle| = 15$.

$\langle\alpha\beta\rangle$ is obviously cyclic and normal since $(G : \langle\alpha\beta\rangle) = 30/15 = 2$. #

2.

[Sol]. $R := \mathbb{Z}[x,y]/(6, (x-2)^2, y^6)$. $I := (6, (x-2)^2, y^6)$

By correspondence theorem, ideals of R correspond to ideals of $\mathbb{Z}[x,y]$ containing $(6, (x-2)^2, y^6)$.

Let P be a prime ideal of R. Then it corresponds to a prime ideal $\widetilde{P}$ of $\mathbb{Z}[x,y]$ containing I.

$\because (x-2)^2 \in \widetilde{P}$ $\therefore x-2 \in \widetilde{P}$. ($\widetilde{P}$: prime). $\because y^6 \in \widetilde{P}$ $\therefore y \in \widetilde{P}$. ($\widetilde{P}$: prime).

$\because 6 = 2\cdot 3 \in \tilde{P}$ $\therefore 2 \in \tilde{P}$ or $3 \in \tilde{P}$.

<u>case1</u> $2 \in \tilde{P}$. $\Rightarrow \tilde{P} \supseteq (2, x-2, y)$.

Note $\mathbb{Z}[x,y]/(2,x-2,y) \cong \mathbb{Z}_2$ is a field. $\Rightarrow (2,x-2,y)$: maximal.

$\Rightarrow \tilde{P} = (2,x-2,y)$. $\Rightarrow P = (2,x-2,y)/I$. #

<u>case2</u> $3 \in \tilde{P}$.

Similarly, $\tilde{P} = (3,x-2,y)$. $\Rightarrow P = (3,x-2,y)/I$. #

In summary, R has 2 prime ideals and both are maximal. #

3.

Consider $M_1 = R/(x-1) \oplus R/(x-2)$ and $M_2 = R/(x) \oplus R/(x-3)$.

They are not R-module iso. by Fundamental theorem.

They are irreducible are easy to prove. #

PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, December 13, 2015

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let G be a group of order 30. Show that G has a normal subgroup that is cyclic of order 15.

2. Count the number of prime ideals in the ring
$$\mathbb{Z}[x,y]/(6,(x-2)^2,y^6).$$
How many of these prime ideals are maximal? (The whole ring is not considered a prime ideal.)

3. An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Let $R = \mathbb{Q}[x]$. Construct two non-isomorphic irreducible R-modules whose underlying abelian group is $\mathbb{Q} \times \mathbb{Q}$.

4. Construct an extension field K of $\mathbb{Q}$ such that $K/\mathbb{Q}$ is Galois and the Galois group of K over $\mathbb{Q}$ is cyclic of order 5.

PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, May 11, 2015

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

✓ 1. Let p be a prime number. Classify, up to isomorphism, all groups of order $2p$.

✓ 2. Let $R = \mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}$. $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as abelian gp. $\mathbb{Z}_4$ as abelian gp.

   ✓ (a) Compute the factor rings $R/(2)$ and $R/(2i)$. Are these rings isomorphic? No.

   ✓ (b) Is R a principal ideal domain? No. ($\because$ $R/(2i)$: not field). or $2i$: irr. not prime. $2i \mid 2(-2+2i)$, but not either of them.

   Justify your answers.

✓ 3. An element m of an R-module M is called a *torsion element* if $rm = 0$ for some non-zero element $r \in R$. Let R be a principal ideal domain. Let F be the fraction field of R. Let M_1 and M_2 be finitely generated R-modules such that

$$M_1 \otimes_R F \cong M_2 \otimes_R F.$$

   ✓ (a) Prove that if both M_1 and M_2 have no torsion elements then $M_1 \cong M_2$.

   ✓ (b) Give an explicit example which shows that the conclusion in (a) is false if M_1 or M_2 have torsion elements. $R = \mathbb{Z}$, $F = \mathbb{Q}$, $M_1 = \mathbb{Z}$, $M_2 = \mathbb{Z} \oplus \mathbb{Z}_2$.

✓ 4. Let $f(x) = x^8 - 1$. Find the Galois group of $f(x)$ over each of the following fields:

   (a) The rational field $\mathbb{Q}$. $C_2 \times C_2$.

   (b) The field $\mathbb{Q}(i)$. C_2

   (c) The field $\mathbb{F}_3$ of three elements. C_2

PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, August 21, 2014

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let M be the cokernel of the mapping from $\mathbb{Z}^2$ to $\mathbb{Z}^3$ given by the matrix

$$\begin{bmatrix} 2 & 8 \\ 4 & 10 \\ 6 & 12 \end{bmatrix}$$

How many $\mathbb{Z}$-module homomorphisms are there from M to $\mathbb{Z}/3\mathbb{Z}$? 9

2. Let $K = \mathbb{Q}\left(\frac{-1+\sqrt{-3}}{2}\right)$. Give an example of two non-isomorphic field extensions L_1 and L_2 of K such that $\mathrm{Gal}(L_1/K) \cong \mathrm{Gal}(L_2/K) \cong \mathbb{Z}/3\mathbb{Z}$. Justify your claims.

3. Let G be a finite group of order p^n, where p is prime and $n \geq 1$. Suppose G acts on a finite set S. Let S' be the subset of S consisting of elements fixed by G:

$$S' = \{x \in S \mid gx = x \text{ for all } g \in G\}.$$

Prove that the order of S' is congruent to the order of S modulo p.

4. Let $F[x]$ be the ring of polynomials with coefficients in an algebraically closed field F. Describe, with justification,

   (i) All ideals of $F[x] \times F[x]$.

   (ii) All prime ideals of $F[x] \times F[x]$.

PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, May 12, 2014

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Suppose that $R = \mathbf{Z}[i]$ and that M and N are finitely generated R-modules. Suppose further that $P = \langle 1 + i \rangle$.

   (i) Show that P is a prime ideal.

   (ii) Suppose that $M \oplus R/P \oplus P$ is isomorphic to $N \oplus R/P \oplus P$. Prove that M and N are isomorphic.

2. Find all the prime ideals in $R = \mathbf{Z}[i][x]/\langle 1 + i, x^2 + 2 \rangle$.

3. Prove that there is no simple group of order 96.

4. Suppose that $f \in \mathbb{Q}[x]$ is an irreducible polynomial and that $\alpha, \beta \in \mathbb{C}$ are roots of f. Suppose that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ is such that $K/\mathbb{Q}$ is a finite Galois extension. Show that $\mathbb{Q}[\alpha] \cap K$ is isomorphic to $\mathbb{Q}[\beta] \cap K$.

   Hint: We know there is an isomorphism $\sigma : \mathbb{Q}[\alpha] \to \mathbb{Q}[\beta]$ sending α to β. Show that σ map extends to an automorphism of some larger field that sends K to K.

PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, August 20, 2013

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let R denote the ring $\mathbb{Q}[x]$ and let N denote the R-module $R/\langle x^2+1\rangle$. Further suppose that M and M' are finitely generated R-modules such that

$$M \oplus N \cong M' \oplus N,$$

in other words, $M \oplus N$ and $M' \oplus N$ are isomorphic as R-modules. Prove that $M \cong M'$ as R-modules.

2. Find an ideal I in the ring $A = \mathbb{Z}[x]$ such that A/I has exactly three prime ideals. Identify the ideals and justify your assertion.

3. Show there is no simple group of order 108.

4. Suppose that $\alpha \in \mathbb{C}$ with $\alpha^n \in \mathbb{Q}$ such that $\mathbb{Q}[\alpha] \supseteq \mathbb{Q}$ is Galois. Further suppose that F is the field containing $\mathbb{Q}$ generated by all the roots of unity in $\mathbb{Q}[\alpha]$. Show that $\mathrm{Gal}(\mathbb{Q}[\alpha] : F)$ is a cyclic group.

PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, May 6, 2013

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

✓ 1. Show that a group G of order 36 cannot be simple.

✓ 2. Let R be a principal ideal domain, and let I, J be two nonzero ideals of R. Show that $IJ = I \cap J$ if and only if $I + J = R$.

✓ 3. Let K be a field, and let L be an extension field of K. Let $u \in L$, and assume that the minimal polynomial of u over K is $x^n - a$ for some $a \in K$. Let $n = md$ for positive integers m, d.

   (a) Show that $[K(u^m) : K] = d$.

   (b) What is the minimal polynomial of u^m over K?

✓ 4. Let E be a splitting field of $x^{35} - 1$ over $\mathbb{F}_8$. Determine the cardinality of E and make a diagram showing all subfields of E and the inclusions between them.

$|E| = 8^4$

$$\begin{array}{c} E \\ | \Big) 2 \\ M \\ | \Big) 2 \\ \mathbb{F}_8 \end{array}$$

PhD QUALIFYING EXAMINATION IN ALGEBRA
Part B, August 23, 2012

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $R = \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients and $I = \langle 3, x^2 + 3x + 5 \rangle$ the ideal in R generated by 3 and $x^2 + 3x + 5$. Find all the prime ideals that contain I. $\langle 3, x+1 \rangle, \langle 3, x-1 \rangle$

2. Show that the alternating group A_6 has no subgroup of order $72 = 6 \cdot 4 \cdot 3$. You may use the fact that A_6 is simple.

3. Let F be the splitting field of $f = x^4 - 11$ over $\mathbb{Q}$. Show that $G = \mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to D_4 (the dihedral group of order $8 = 4 \cdot 2$).

4. Let $R = \mathbb{F}_2[t]$ and M be an R-module generated by elements a, b, c subject to the relations:

$$a + tb + (t^2 + t + 1)c = 0, \quad (t + 1)b + (t^2 + t)c = 0.$$

Write M as a direct sum of cyclic R-modules.

PhD QUALIFYING EXAMINATION IN ALGEBRA
Paper B, May 9th, 2012

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

√ 1. Let S_i be a family of subgroups in some group G. Suppose that an element $x_i \in G$ is chosen for each i, and we form the intersection of all the cosets x_iS_i. Prove that this intersection is either empty or a coset of the intersection of all the S_i.

√ 2. Suppose that we have an abelian group given by generators x, y, z and relations

$$3x + 2y + 4z = 0, \quad 2x - y + 10z = 0, \quad 5x + 3y - 2z = 0.$$

Find a product of cyclic groups isomorphic to this group.

√ 3. Suppose that an integral domain D has a field F inside it, and D is a finite-dimensional vector space over F. Prove that D is actually a field itself.

4. Suppose E/K is a Galois extension with an abelian Galois group. Prove that all fields intermediate between E and K are Galois extensions of K.

Should I consider infinite Galois for this problem?