MATH523 , Homework 2

Due September 28, 2017 in class

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1. [10 pts] *Interpolation with Newton's divided difference.*

   ~~(a)~~ Write a MATLAB program that finds the unique polynomial of degree $\leq n$ in Newton form which interpolates data points $(x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$, using divided differences. The input arguments should be two vectors $\vec{x} = (x_1, \ldots, x_{n+1})$ and $\vec{y} = (y_1, \ldots, y_{n+1})$, and the output should be the coefficients (or divided differences) $\vec{c} = (c_1, \ldots, c_{n+1})$. Check your code on the following data (it is useful to do first the calculation at hands):

   $$\vec{x} = (0, 1, 1.5, 2), \quad \vec{y} = (-1, 3, 2, 4).$$

   (b) Perform an operation count for the divided difference algorithm (only division is enough) !!

   ~~(c)~~ Write a MATLAB script which takes $\vec{c}, \vec{x}, z$ and evaluate the polynomial using Horner's rule at z.

   (d) Interpolate the following functions on $[-1, 1]$ with $n+1$ equally spaced points $-1 = x_1 < x_2 < \ldots < x_{n+1} = 1$ for $n = 5, 11, 15$:

   $$f(x) = e^{2x} \sin 3\pi x, \quad f(x) = |x|.$$

   For each function and n, plot the function and the interpolation polynomials with finer mesh as well as the coarse data $y_i = f(x_i)$ in the same figure! Draw

   ?? conclusions about the approximation quality as n increases and relate to the error estimate derived in class under the assumption that $f \in \mathcal{C}^{n+1}[-1, 1]$.

2. [5 pts] *Interpolation error*

   (a) Use the linear interpolation on $[0, 1]$ and a scaling argument to prove the error estimate for linear interpolation at x_1 and $x_2 = x_1 + h$:

   $$|f(x) - p(x)| \leq \frac{\|f"\|_{L^\infty(x_1, x_2)}}{8} h^2.$$

   Use Theorem 8.2 in Q's book

(b) Consider the Bessel function of order zero,

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)dt.$$

How small must h be chosen so that the function above can be "linearly interpolated" with error less than 10^{-6} in absolute value?

3. [10 pts] *Inverse interpolation:* This problem shows how to use interpolation for root finding. We want to find a zero x^* of the function $f(x) = \cos(x) - x$. Note that

$$f(0.6) = 0.22534, \quad f(0.7) = 0.06484, \quad f(0.9) = -0.27839.$$

Let $g(y) = f^{-1}(y)$ be the inverse of $f(x)$. The unknown zero x^* can then be expressed as $x^* = g(0)$.

(a) Construct the quadratic polynomial $p_2(y)$ that interpolates g (not f!) at the given points. Determine the divided difference table using only 5 decimal digits (use the MATLAB function from problem 1.(a)). Determine $p_2(0)$ as an approximation to x^* using the function from Problem 1.(c).

(b) Give a theoretical estimate for error $|p_2(0) - x^*|$. Use the following relation where $y = f(x)$:

$$g'''(y) = 3f"(x)^2 f'(x)^{-5} - f'''(x)f'(x)^{-4}.$$

(c) Given $f(0.8) = -.10329$ and the data points above, determine the polynomial p_3 that interpolates g at the given four points. Compute $|p_2(0) - p_3(0)|$ and regard it as an estimate for the true error.

4. [10 pts] *Chebyshev function*

(a) Write a recursive MATLAB function $t = Chebyshev(n, x)$ which evaluates Chebyshev polynomial $T_n(x)$ of degree n at x; x may be a vector. Plot the function for $n = 0, 1, 2, \ldots, 5$ and discrete points $x = linspace(-1, 1, 100)$. Plot also the extrema and the roots from the analytical formula derived in class.

(b) Repeat problem 1(d) for $f(x) = |x|$ by interpolating on Chebyshev nodes instead of equi-spaced points.

5. [5 pts] Let $\Pi_n[-1, 1]$ denotes the space of polynomials of degree $\leq n$. Define the following function $\langle \cdot, \cdot \rangle : \Pi_n \to \mathbb{R}$:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}}.$$

(a) Show that $(\Pi_n, \langle \cdot, \cdot \rangle)$ is an inner product space.

(b) Show that the Chebyshev polynomials $\{T_j(x)\}_{j=0,\ldots,n}$ form an orthogonal basis of inner product space $(\Pi_n, \langle \cdot, \cdot \rangle)$.

6. [5 pts] *Lebesque constant* is defined to be

$$\lambda_n(x) = \sum_{i=0}^{n} |\ell_i(x)|,$$

where $\ell_i(x)$ are the Lagrange polynomial.

~~(a)~~ Let $x_0, x_1, \ldots, x_n$ be $n+1$ distincts points in $[a, b]$ and $f_i = f(x_i), i = 0, \ldots, n$, for some function f. Let $\tilde{f}_i = f + \epsilon_i$, where $|\epsilon_i| \leq \epsilon$. Use the Lagrange interpolation formula to show that

$$|p_n\tilde{f}(x) - p_n f(x)| \leq \epsilon\lambda_n(x), \quad x \in [a, b].$$

~~(b)~~ For quadratic interpolation at three equally spaced points, show that $\lambda_2(x) \leq 1.25$ for any x between the three points.

~~(c)~~ For $x_0 = 0, x_1 = 1, x_2 = p$ where $p \gg 1$, show that

$$\lambda_2(x) = 1 + c(x-1)(p-x)$$

for some constant c and $1 \leq x \leq p$. Then determine $\max_{1\leq x\leq p} \lambda_2(x)$. How fast does this maximum grow with p?

7. [5 pts] *Pade approximation:* Consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}.$$

on $f(x) = \cos(x)$. Determine the coefficients in r in such a way that

$$f(x) - r(x) = \mathcal{O}(x^8).$$