MATH523 , Homework 5
Due November 16, 2017 in class

~~1.~~ [8 pts] *Triple recursion formula:* Let $\mathcal{C}[a,b]$ be equipped with the inner product

I think this is just computations.

$$\langle f, g\rangle = \int_a^b f(x)g(x)\omega(x)dx,$$

with weight $\omega(x) > 0$ for $a < x < b$.

~~(a)~~ Show that

$$p_0(x) = 1, \quad p_j(x) = x^j - \sum_{k=0}^{j-1} \frac{\langle x^j, p_k\rangle}{\langle p_k, p_k\rangle} p_k(x)$$

are orthogonal monic polynomials that form a basis of Π_n.

~~(b)~~ Show that there is only one monic polynomial orthogonal to Π_{j-1}, namely p_j, for all $j \geq 1$.

~~(c)~~ Show the following relations for $j \geq 2$:

$$\langle xp_{j-1}, p_{j-2}\rangle = \langle p_{j-1}, p_{j-1}\rangle, \quad \langle xp_{j-1}, p_k\rangle = 0, \quad k < j-2$$

~~(d)~~ Deduce the following triple recursion formula:

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= xp_0(x) - \frac{\langle xp_0, p_0\rangle}{\langle p_0, p_0\rangle} p_0(x) \\ p_j(x) &= \left(x - \frac{\langle xp_{j-1}, p_{j-1}\rangle}{\langle p_{j-1}, p_{j-1}\rangle}\right) p_{j-1}(x) - \frac{\langle p_{j-1}, p_{j-1}\rangle}{\langle p_{j-2}, p_{j-2}\rangle} p_{j-2}(x), \quad j \geq 2. \end{aligned}$$

This formula is better than the one in (a) above to evaluate $\{p_j\}$.

~~2.~~ [8 pts] *Legendre Polynomials:* I think this is also just computations.

~~(a)~~ Use the recursive formula above to show that that the first four (normalized) Legendre polynomials on $[-1, 1]$ are given by:

$$p_0(x) = 1/\sqrt{2}, \quad p_1(x) = x\sqrt{3/2}, \quad p_2(x) = (x^2 - 1/3)\sqrt{45/8}, p_3(x) = (x^3 - 3x/5)\sqrt{175/8}.$$

~~(b)~~ The Rodrigues formula is given by

$$p_n(x) = \frac{1}{2^n n!} D^n[(x^2-1)^n],$$

where D^n denotes the n-th derivative. Use the appropriate binomial formula to show that

$$p_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k},$$

where $[n/2]$ is the integer part of $n/2$. This is another description for Legendre polynomials.

3. [8 pts] Consider a periodic function $f \in \mathcal{C}[0, 2\pi]$ and let us define $\{\hat{f}_k\}_{|k|\leq N}$ to be the discrete Fourier coefficients of $\{f_j\}_{j=0,\ldots,2N}$ sampled at $\{x_j = jh\}_{j=0,1,\ldots,2N}$ where $(2N+1)h = 2\pi$. Now let us subsample f at every P-grid points

$$g_j = f_{jP}, \quad j = 0, 1, \ldots, 2M,$$

so that we only have $2M+1 = (2N+1)/P$. Show that the discrete Fourier coefficients of $\{g_j\}_{j=0,\ldots,2M}$ can be expressed as follows:

$$\hat{g}_\ell = \sum_{k \in \mathcal{A}(\ell)} \hat{f}_k,$$

where $\mathcal{A}(\ell) = \{k : k = \ell + (2M+1)q, q \in \mathbb{Z}, |k| \leq N\}$ is the set of modes that are aliased to wave number ℓ.

Just computation

~~4.~~ [8 pts] If $f(x)$ is a 2π-periodic function, then so are the functions $g(x) = f(\alpha x)$ and $h(x) = f(x - \beta)$, where $\alpha \neq 0$ is an integer number and β is any number. What is the relationship between the Fourier coefficients of $f(x)$ and $g(x)$? What is the relationship between the Fourier coefficients of $f(x)$ and $h(x)$?

5. [8 pts] Let $f(x)$ be the 2π-periodic function $f(x)$ whose values on $[0, 2\pi)$ are given by

$$f(x) = \begin{cases} (x/\pi)^2 - x/\pi, & 0 \leq x \leq \pi, \\ (x-\pi)/\pi - ((x-\pi)/\pi)^2, & \pi \leq x \leq 2\pi \end{cases}$$

   ~~(a)~~ Show that the absolute value of the continuous coefficient $|\hat{f}_k|$ decays like k^{-3} as $k \to \infty$. I think it is just computation.

   (b) Use FFT (with a different number of points) to verify that the discrete coefficient $|\hat{f}_{h,k}|$ also decays slower than k^{-3} as $N \to \infty$. (Note: h denotes the mesh size such that $Nh = 2\pi$ where N is total number of discrete points).

6. [10 pts] Write a program for discrete polynomial least squares approximation of a function f defined on $[-1, 1]$, using the inner product

$$\langle u, v \rangle = \frac{2}{N+1} \sum_{j=0}^{N} u(x_j) v(x_j), \qquad x_j = -1 + \frac{2j}{N}, \quad j = 0, 1, \ldots, N.$$

Follow these steps.

in the inner product defined above

   (a) The recurrence coefficients for the appropriate (monic) orthogonal polynomials $\{\varphi_k\}$ are known explicitly:

$$\varphi_{k+1}(x) = (x - \alpha_k)\varphi_k(x) - \beta_k \varphi_{k-1}(x), \quad k = 0, 1, 2, \ldots,$$

$$\varphi_{-1}(x) = 0, \quad \varphi_0(x) = 1,$$

$$\alpha_k = 0, \quad k = 0, 1, 2, \ldots, N; \qquad \beta_0 = 2,$$

$$\beta_k = \left(1 + \frac{1}{N}\right)^2 \left(1 - \left(\frac{k}{N+1}\right)^2\right) \left(4 - \frac{1}{k^2}\right)^{-1}, \quad k = 1, 2, \ldots, N.$$

   (You do not have to prove this). Define $\gamma_k = \|\varphi_k\|^2 = \langle \varphi_k, \varphi_k \rangle$.

(b) Using the recurrence formula given in (a), generate an array φ of dimension $(N+1, N+1)$ containing $\varphi_k(x_j)$, $k = 0, 1, \ldots, N$; $j = 0, 1, \ldots, N$. (here k is the row index and j is the column index). Define $\mu_k = \max\limits_{0 \le j \le N} |\varphi_k(x_j)|$, $k = \cancel{1,} \ldots, N$.

this is 0

Output $(N = 10)$:

k	β_k	γ_k	μ_k
0			
1			
$\vdots$			
N			

(c) With $\hat{p}_n(x) = \sum\limits_{k=0}^{n} \hat{c}_k \varphi_k(x)$, $n = 0, 1, \ldots, N$, denoting the least squares approximation of degree $\le n$ to the function f on $[-1, 1]$, define

$$\|e_n\|_2 = \|\hat{p}_n - f\|_2 = \langle \hat{p}_n - f, \hat{p}_n - f \rangle^{1/2},$$

$$\|e_n\|_\infty = \max_{0 \le j \le N} |\hat{p}_n(x_j) - f(x_j)|.$$

Using the array φ generated in part (b), compute $\hat{c}_n$, $\|e_n\|_2$, $\|e_n\|_\infty$, $n = 0, 1, \ldots, N$, for the following four functions:

$$f(x) = e^{-x}, \quad f(x) = \ln(x+2), \quad f(x) = \sqrt{1+x}, \quad f(x) = |x|.$$

Be sure you compute $||e_n||_2$ as accurate as possible. Comment on your results.
Output $(N = 10)$, for each f:

n	$\hat{c}_n$	$\|\|e_n\|\|_2$	$\|\|e_n\|\|_\infty$
0			
1			
$\vdots$			
N			

Plot f and compare it to several of its least squares approximations (e.g., $\hat{p}_2$, $\hat{p}_4$, and $\hat{p}_{10}$).