**Problem 1.8.** Let $P$ be a probability measure on a field $\mathcal{F}_0$ and for every subset $A$ of $\Omega$, let $P^*(A)$ be the outer measure defined in class. Let $\tilde{P}$ be the extension of $P$ to $\sigma(\mathcal{F}_0)$. Show that

$$P^*(A) = \inf\{\tilde{P}(B) : \ A \subseteq B, \ B \in \mathcal{F}\}.$$

**Problem 1.9.** Show that a $\lambda$-system can be equivalently defined by these three conditions:

1. $\Omega \in \mathcal{L}$;
2. If $A \in \mathcal{L}$, $B \in \mathcal{L}$, and $A \subseteq B$, then $BA^c \in \mathcal{L}$;
3. If $A_1, A_2, \ldots$ are a disjoint sequence of members of $\mathcal{L}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{L}$.

## 2 HW2: due 9/23/2016

√ **Problem 2.1.** Let $\Omega$ be the unit square $(0,1] \times (0,1]$, and let

$$\mathcal{F} = \{A \times (0,1] : \ A \in \mathcal{B}\},$$

where $\mathcal{B}$ is the Borel $\sigma$-field on $(0,1]$. For any member $A \times (0,1]$ of $\mathcal{F}$, define

$$P(A \times (0,1]) = \lambda(A),$$

where $\lambda$ is the Lebesgue measure on $\mathcal{B}$. Show that $\mathcal{F}$ is a $\sigma$-field and $P$ is a probability on $\mathcal{F}$.

√ **Problem 2.2.** Prove the following statements.

√1. A $\lambda$-system satisfies the following conditions

($\lambda_4$) $A, B \in \mathcal{L}$ and $A \cap B = \varnothing$ imply $A \cup B \in \mathcal{L}$;

($\lambda_5$) $A_1, A_2, \ldots \in \mathcal{L}$ and $A_n \uparrow A$ imply $A \in \mathcal{L}$;

($\lambda_6$) $A_1, A_2, \ldots \in \mathcal{L}$ and $A_n \downarrow A$ imply $A \in \mathcal{L}$.

√2. $\mathcal{L}$ is a $\lambda$-system if and only if it satisfies ($\lambda_1$), ($\lambda_2'$) and ($\lambda_5$). Recall that ($\lambda_2'$) means

$$A, B \in \mathcal{L} \text{ and } A \subseteq B \text{ imply } BA^c \in \mathcal{L}.$$

√ **Problem 2.3.** Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of sets. Prove that

$$I_{\limsup_n A_n} = \limsup_n (I_{A_n}), \quad I_{\liminf_n A_n} = \liminf_n (I_{A_n}).$$

$\boxed{\text{characteristic fun?}}$

(Recall that, for a sequence of numbers $a_n$, $\limsup_n a_n$ is defined to be $\lim_n \sup_{k \geq n} a_k$; $\liminf_n a_n$ is defined to be $\lim_n \inf_{k \geq n} a_k$).

√ **Problem 2.4.** Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of subsets of $\Omega$. Let

$$B_n = \cap_{k=n}^{\infty} A_k, \quad C_n = \cup_{k=n}^{\infty} A_k.$$

Show that

$$B_n \uparrow \liminf_n A_n, \quad C_n \downarrow \limsup_n A_n.$$

**Problem 2.5.** (a) Prove that

$$
\begin{aligned}
(\limsup_n A_n) \cap (\limsup_n B_n) &\supseteq \limsup_n (A_n \cap B_n),\\
(\limsup_n A_n) \cup (\limsup_n B_n) &= \limsup_n (A_n \cup B_n),\\
(\lim_n \overset{\text{inf}}{\sup} A_n) \cap (\liminf_n B_n) &= \liminf_n (A_n \cap B_n),\\
(\liminf_n A_n) \cup (\liminf_n B_n) &\subseteq \liminf_n (A_n \cup \underset{B_n.}{\cancel{A_n}}).
\end{aligned}
$$

(b) Show that

$$
\begin{aligned}
\limsup_n A_n^c &= (\liminf_n A_n)^c,\\
\liminf_n A_n^c &= (\limsup_n A_n)^c,\\
\limsup_n A_n \setminus \liminf_n A_n &= \limsup_n (A_n \cap A_{n+1}^c) = \limsup_n (A_n^c \cap A_{n+1}).
\end{aligned}
$$

(c) Show that $A_n \to A$ and $\overset{\to B}{B_n}$ together imply that $A_n \cup B_n \to \overset{A\cup B?}{A_B}$ and $A_n \cap B_n \to A \cap B$.

**Problem 2.6.** For events $A_1, \ldots, A_n$, consider the $2^n$ equations

$$P(B_1 \cdots B_n) = P(B_1) \cdots P(B_n),$$

where $B_i = A_i$ or $B_i = A_i^c$ for each $i$. Show that $A_1, \ldots, A_n$ are independent if all these equations hold.

**Problem 2.7.** Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are $\pi$-systems and $\mathcal{A}_1 \perp\!\!\!\perp \cdots \perp\!\!\!\perp \mathcal{A}_n$. Let $\mathcal{B}_i = \mathcal{A}_i \cup \{\Omega\}$. Show that $\underset{B_1}{\mathcal{B}_1}, \ldots, \underset{B_n}{\mathcal{B}_n}$ are $\pi$-systems and $\underset{B_1}{\mathcal{B}_1} \perp\!\!\!\perp \cdots \perp\!\!\!\perp \underset{B_n}{\mathcal{B}_n}$.

**Problem 2.8.** Show that $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$.

**Problem 2.9.** Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

1. Show that, for any sequence of independent $\mathcal{F}$-sets, say $\{B_n : n = 1, 2, \ldots\}$, we have

$$P(\cap_{n=1}^{\infty} B_n) = \prod_{n=1}^{\infty} P(B_n).$$

2. Use the above relation and the inequality in Problem 2.8 to prove the second Borel-Cantelli Lemma.