<u>Def</u> A <u>curve (segment)</u> in M is a smooth map $\gamma: I \to M$, where $I = [a,b] \subseteq \mathbb{R}$. Suppose $\varphi: [a,b] \to [c,d]$ is diffeomorphism, we call $\tilde{\gamma} = \gamma \circ \varphi$ a <u>reparametrization</u> of γ.

<u>Def</u> A <u>regular curve</u> $\gamma: I \to M$ is a curve w/ $\dot{\gamma}(t) \neq 0$ for all $t \in I$. A regular curve is an immersion of I into M.

<u>Ex1</u>. For any <u>regular curve</u> γ, there exists a reparametrization $\tilde{\gamma}$ s.t. $\tilde{\gamma}$ is a unit speed curve, i.e, $|\dot{\tilde{\gamma}}(t)| = 1$ for all t.

<u>Def</u>. The <u>length</u> of a curve $\gamma: [a,b] \to M$ is defined as $L(\gamma) := \int_a^b |\dot{\gamma}(t)|\, dt$. The <u>arc length function</u> of γ is a function $s: [a,b] \to \mathbb{R}$ given by

$$s(t) := \int_a^t |\dot{\gamma}(u)|\, du$$

<u>Def</u>. A <u>piecewise regular curve</u> is a continuous map $\gamma: [a,b] \to M$ with finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ s.t. $\gamma|_{[a_i, a_{i+1}]}$ is a regular curve. For such γ, it has non-zero, one-sided velocity when approaching each a_i, which we denote by

$$\dot{\gamma}(a_i^-) = \lim_{t \uparrow a_i} \dot{\gamma}(t) \quad \text{and} \quad \dot{\gamma}(a_i^+) = \lim_{t \downarrow a_i} \dot{\gamma}(t).$$

<u>Rmk</u>. Consider $\varphi: [c,d] \to [a,b]$ with (1) φ is a homeomorphism (2) there exists a finite subdivision $c = c_0 < c_1 < \cdots < c_k = d$ s.t. each $\varphi|_{[c_{i-1}, c_i]}$ is smooth with smooth inverse. Then we set $\tilde{\gamma} = \gamma \circ \varphi$ as a reparametrization of any piecewise regular curve $\gamma: [a,b] \to M$.

<u>Rmk</u>. We also call a piecewise regular curve as an <u>admissible curve</u>.

Ex2. For any curve γ and its reparametrization $\tilde{\gamma}$, we have $L(\gamma) = L(\tilde{\gamma})$.

Ex3. For any admissible curve γ, there is a reparametrization of γ with unit speed.

Def An admissible curve γ is minimizing if $L(\gamma) \leq L(\tilde{\gamma})$ for any admissible curve $\tilde{\gamma}$ with same endpoints.

Rmk. For p, q in the same connected component of M, there exists an admissable curve with p, q being endpoints.

Def. An admissible family of curves is a continuous map $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ s.t.
(1) for some finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$, Γ is smooth on each rectangle $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ (2) $\Gamma_s(t) := \Gamma(s, t)$ is an admissible curve for all $s \in (-\varepsilon, \varepsilon)$.

We call $\Gamma_s(t) := \Gamma(s, t)$ the main curves, and $\Gamma^t(s) := \Gamma(s, t)$ the transverse curves.

Also, we denote:

$$\partial_t \Gamma(s, t) := \frac{d}{dt} \Gamma_s(t)$$

$$\partial_s \Gamma(s, t) := \frac{d}{ds} \Gamma^t(s)$$

Lemma (symmetry lemma)

Suppose ∇ is a symmetric linear connection on M. Let $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be an admissible family. On any rectangle $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ where Γ is smooth, we have

$$D_s \partial_t \Gamma = D_t \partial_s \Gamma$$

Proof: suppose $\Gamma(s, t) = (x^1, \cdots, x^n)$, then

$$\partial_t \Gamma = \frac{\partial x^k}{\partial t} \partial_k \quad \text{and} \quad \partial_s \Gamma = \frac{\partial x^k}{\partial s} \partial_k$$

Recall that

$$D_t V = \dot V^j \partial_j + V^j \dot\gamma^i \Gamma^k_{ij} \partial_k$$
$$= (\dot V^k + V^j \dot\gamma^i \Gamma^k_{ij}) \partial_k$$

So

$$D_s \partial_t \Gamma = \left(\frac{\partial^2 x^k}{\partial s \partial t} + \frac{\partial x^j}{\partial t} \frac{\partial x^i}{\partial s} \Gamma^k_{ij} \right) \partial_k$$

$$D_t \partial_s \Gamma = \left(\frac{\partial^2 x^k}{\partial t \partial s} + \frac{\partial x^j}{\partial s} \frac{\partial x^i}{\partial t} \Gamma^k_{ij} \right) \partial_k$$

Due to symmetry of ∇ and smoothness of Γ (in rectangles), we have $D_s \partial_t \Gamma = D_t \partial_s \Gamma$ ▨

Def If $\gamma : [a,b] \to M$ is an admissible curve, a variation of γ is an admissible family Γ with $\Gamma_0 = \gamma$. If $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all s, we call it a proper variation of γ.

Def Given Γ as a variation of γ, define $V(t) = \partial_s \Gamma(0,t)$ as the variation field of Γ. V is proper if $V(a) = V(b) = 0$.

Rmk. If Γ is proper, then so is V.

Lemma. Given an admissible curve γ. Suppose V is a vector field along γ. Then V is the variation field for some variation of γ. Furthermore, if V is proper, then that variation is proper as well.

Proof: set $\Gamma(s,t) := \exp(sV(t))$. Then we have

(1) for some $\varepsilon > 0$, Γ is defined on $(-\varepsilon, \varepsilon) \times [a,b]$.

(2) there exists finite subdivision s.t. Γ is smooth on each $(-\varepsilon,\varepsilon) \times [a_{i-1}, a_i]$; V is smooth on $[a_{i-1}, a_i]$ as well.

(3) Γ is a variation of γ b/c $\Gamma_0(t) = \gamma(t)$

(4) if $V(a)=V(b)=0$, then $\Gamma_s(a)=\gamma(a)$ and $\Gamma_s(b)=\gamma(b)$ for all s. ☒

Prop (First variation formula)

Let $\gamma:[a,b]\to M$ be any admissible curve with unit speed. Let Γ be a proper variation of γ with variation field V. Then,

$$\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s) = -\int_a^b \langle V, D_t\dot\gamma\rangle dt - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\dot\gamma\rangle$$

where $\Delta_i\dot\gamma = \dot\gamma(a_i^+) - \dot\gamma(a_i^-)$ for each velocity "jump" at a_i.

Proof: for convenience, set $T(s,t)=\partial_t\Gamma(s,t)$ and $S(s,t)=\partial_s\Gamma(s,t)$. By symmetry lemma, $D_tS = D_sT$ whenever Γ is smooth. Now, let's compute,

$$\frac{d}{ds}L(\Gamma_s|_{[a_{i-1},a_i]}) = \frac{d}{ds}\int_{a_{i-1}}^{a_i}\left\langle \frac{d}{dt}\Gamma_s, \frac{d}{dt}\Gamma_s\right\rangle^{\frac{1}{2}} dt$$

$$= \int_{a_{i-1}}^{a_i}\frac{\partial}{\partial s}\langle T,T\rangle^{\frac{1}{2}}dt$$

$$= \int_{a_{i-1}}^{a_i}\frac{1}{2}\langle T,T\rangle^{-\frac{1}{2}}2\langle D_sT,T\rangle dt$$

$$= \int_{a_{i-1}}^{a_i}\frac{1}{|T|}\langle D_tS, T\rangle dt$$

Note that

(1) $T(0,t)=\partial_t\Gamma(0,t)=\dot\gamma(t)$ is unit vector

(2) $S(0,t)=V(t)$

Thus,

$$\left.\frac{d}{ds}\right|_{s=0}L(\Gamma_s|_{[a_{i-1},a_i]}) = \left.\int_{a_{i-1}}^{a_i}\frac{1}{|T|}\langle D_tS,T\rangle dt\right|_{s=0}$$

$$= \int_{a_{i-1}}^{a_i}\langle D_tV,\dot\gamma\rangle dt$$

$$= \int_{a_{i-1}}^{a_i} \left(\frac{d}{dt}\langle V, \dot{\gamma}\rangle - \langle V, D_t\dot{\gamma}\rangle \right) dt$$

$$= \langle V(a_i), \dot{\gamma}(a_i^-)\rangle - \langle V(a_{i-1}), \dot{\gamma}(a_{i-1}^+)\rangle$$

$$- \int_{a_{i-1}}^{a_i} \langle V, D_t\dot{\gamma}\rangle dt.$$

Summing over i and noting $V(a_0) = V(a_k) = 0$,

So we obtain

$$\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s) = -\int_a^b \langle V, D_t\dot{\gamma}\rangle dt - \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i\dot{\gamma}\rangle$$

Thm suppose $\gamma: [a,b] \to M$ is minimizing with unit speed. Then it is a geodesic.

Proof: Let $a = a_0 < \cdots < a_k = b$ be ~~the~~ a subdivision s.t. γ is smooth on each $[a_i, a_{i+1}]$. Because γ is minimizing, for any proper variation Γ and its variation field V, we must have

$$-\int_a^b \langle V, D_t\dot{\gamma}\rangle dt - \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i\dot{\gamma}\rangle = 0$$

Step 1: on each (a_{i-1}, a_i), γ is geodesic.

Choose $\varphi \in C^\infty(\mathbb{R})$ s.t. $\varphi > 0$ on (a_{i-1}, a_i) and $\varphi = 0$ elsewhere (bump function).

Let $V = \varphi D_t\dot{\gamma}$, we have

$$-\int_{a_{i-1}}^{a_i} \varphi |D_t\dot{\gamma}|^2 dt.$$

This gives $D_t\dot{\gamma} \equiv 0$ on (a_{i-1}, a_i)

Step 2: fix $i \in \{0, \cdots, k\}$. We already showed $D_t\dot{\gamma} \equiv 0$ on (a_{i-1}, a_i) and (a_i, a_{i+1}).

Set V along γ s.t. $V(a_i) = \Delta_i\dot{\gamma}$ and $\overline{\mathrm{supp}(V)} \subseteq (a_{i-1}, a_{i+1})$. then same argument shows $\Delta_i\dot{\gamma} = 0$.

Step 3: Now $\dot{\gamma}$ is continuous at a_i, with γ is geodesic on (a_{i-1}, a_i) $\forall i$

By uniqueness of geodesic, γ is smooth on $[a,b]$.

<u>Corollary</u>. A unit speed admissible curve γ is a critical point for L, i.e. for any proper variation Γ_s of γ, $dL(\Gamma_s)/ds = 0$ when $s=0$, if and only if it is a geodesic.

<u>proof</u>: ($\Rightarrow$) is already discussed.

($\Leftarrow$) if γ is a geodesic, then $\int_a^b \langle V, D_t\dot{\gamma}\rangle dt = 0$ and $\sum_{i=1}^{k-1} \langle V(a_i), \Delta_i \dot{\gamma}\rangle = 0$, so $dL(\Gamma_s)/ds = 0 \big|_{s=0}$ for all proper variation Γ_s. ▨

<u>Rmk</u>: $D_t\dot{\gamma} = 0$ characterizes the critical points of the length functional. Historically, it is called the variational equation.

<u>Thm</u> (The Gauss lemma)

Let $\mathcal{U}$ be a geodesic ball centered at $p \in M$. The unit radial vector field $\frac{\partial}{\partial r}$ is orthogonal to the geodesic sphere in $\mathcal{U}$.

<u>Proof</u>:

Let $q \in \mathcal{U}$, $X \in T_qM$ tangent to geodesic sphere through q. So there is $V \in T_pM$ with $q = \exp_p V$. Also there is $W \in T_V(T_pM)$ s.t. $X = (\exp_p)_* W$. The radial geodesic from p to q is $\gamma(t) = \exp_p(tV)$ with $\dot{\gamma}(t) = R\frac{\partial}{\partial r}$, where R is the radius of the geodesic ball.

choose a curve $\sigma: (-\varepsilon, \varepsilon) \to T_pM$ with $\sigma(s) \in \partial B_R(0)$, $\sigma(0) = V$, $\dot{\sigma}(0) = W$. We define a variation Γ of γ by

$$\Gamma(s,t) := \exp_p(t\sigma(s))$$

Fix s. each Γ_s is a geodesic with constant speed R.

Set (again) $S = \partial_s \Gamma$ and $T = \partial_t \Gamma$. We have,

$$S(0,0) = \frac{d}{ds}\Big|_{s=0} \exp_p(0) = 0$$

$$T(0,0) = \frac{d}{dt}\Big|_{t=0} \exp_p(tV) = V$$

$$S(0,1) = \frac{d}{ds}\Big|_{s=0} \exp_p(\sigma(s)) = X$$

$$T(0,1) = \frac{d}{dt}\Big|_{t=1} \exp_p(tV) = \dot{\gamma}(1)$$

So when $s=t=0$, $\langle S, T\rangle = 0$; when $s=0$, $t=1$, $\langle S,T\rangle = \langle X, \dot{\gamma}(1)\rangle$. But then,

$$\begin{aligned}\frac{\partial}{\partial t}\langle S,T\rangle &= \langle D_t S, T\rangle + \langle S, D_t T\rangle \\ &= \langle D_s T, T\rangle + 0 \\ &= \frac{1}{2}\frac{\partial}{\partial s}|T|^2 \\ &= 0\end{aligned}$$

So we conclude $\langle X, \dot{\gamma}(1)\rangle = 0$ ∎

<u>Corollary</u> Let (x^i) be a normal coordinates on a geodesic ball $\mathcal{U}$ centered at $p \in M$. let r be the radial distance function, i.e,

$$r(x) = \left(\sum_i (x^i)^2\right)^{\frac{1}{2}}$$

then $\operatorname{grad} r = \frac{\partial}{\partial r}$ on $\mathcal{U}\setminus\{p\}$.

<u>Proof</u>: Given $q \in \mathcal{U}\setminus\{p\}$ and $Y \in T_qM$. Because $\frac{\partial}{\partial r}$ is transverse to the geodesic sphere through q, then $Y = X + \frac{\partial}{\partial r}\alpha$ for some $\alpha > 0$ and some X tangent to the sphere. $dr(X) = 0$ due to the fact X is tangent to a level set of r. Also $dr(\frac{\partial}{\partial r}) = 1$. So,

$$dr(\alpha\frac{\partial}{\partial r} + X) = \alpha\, dr(\frac{\partial}{\partial r}) + dr(X) = \alpha$$

$$\langle \frac{\partial}{\partial r}, \alpha\frac{\partial}{\partial r} + X\rangle = \alpha\left|\frac{\partial}{\partial r}\right|^2 + \langle\frac{\partial}{\partial r}, X\rangle = \alpha$$

∎

Prop Suppose $p \in M$ and q is contained in a geodesic ball centered at p. Then the radial geodesic from p to q (with unit speed parametrization) is the unique minimizing curve in M.

Proof: ~~Suppose~~ $\gamma(t) = \exp_p(tV)$ is the radial geodesic from p to q. Due to $\dot\gamma(t) = R\frac{\partial}{\partial r}$, we have $L(\gamma) = R$.

Let $S_R = \exp_p(\partial B_R(0))$. Let $\sigma : [a,b] \to M$ be a unit speed curve from p to q.

let a_0 be the last time $\sigma(t) = p$, b_0 be the first time (after a_0) $\sigma(t) \in S_R$.

We want to show $L(\sigma) \geq R$.

Now, for any $t \in (a_0, b_0]$, write

$$\dot\sigma(t) = \alpha(t)\frac{\partial}{\partial r} + X(t)$$

where $X(t)$ is tangent to geodesic spheres, .

By Gauss lemma,

$$|\dot\sigma(t)|^2 = \alpha(t)^2 + |X(t)|^2 \geq \alpha(t)^2$$

Thus,

$$
\begin{aligned}
L(\sigma) &\geq L(\sigma|_{[a_0,b_0]}) \\
&= \lim_{\varepsilon\to 0} \int_{a_0+\varepsilon}^{b_0} |\dot\sigma(t)|\, dt \\
&\geq \lim_{\varepsilon\downarrow 0} \int_{a_0+\varepsilon}^{b_0} \alpha(t)\, dt \\
&= \lim_{\varepsilon\downarrow 0} \int_{a_0+\varepsilon}^{b_0} dr(\dot\sigma(t))\, dt \\
&= \lim_{\varepsilon\downarrow 0} \int_{a_0+\varepsilon}^{b_0} \frac{d}{dt} r(\sigma(t))\, dt \\
&= r(\sigma(b_0)) - r(\sigma(a_0)) = R.
\end{aligned}
$$

Now let's assume $L(\sigma)=R$. Then

$$\lim_{\varepsilon\downarrow 0}\int_{a_0+\varepsilon}^{b_0}|\dot\sigma(t)|\,dt=\lim_{\varepsilon\downarrow 0}\int_{a_0+\varepsilon}^{b_0}\alpha(t)\,dt=R$$

Because we require $|\dot\sigma(t)|=1$, so $a_0=0$, $b_0=R$. $X(t)\equiv 0$ and $\alpha(t)\equiv 1$.

So. $\dot\sigma(t)=\frac{\partial}{\partial r}$, meaning σ is the integral curve of $\frac{\partial}{\partial r}$, so $\sigma\equiv\gamma$.

∎

<u>Theorem</u>: every Riemannian geodesic is <u>locally minimizing</u>, i.e, for any $t_0\in I$, there is a neighbourhood U of t_0 s.t. $\gamma|_U$ is minimizing between each pair of its points in U.

<u>Proof</u>:

Let $\gamma: I\to M$ be a geodesic. let $t_0\in I$. Let W be a uniformly normal neighbourhood of $\gamma(t_0)$, set $U:=$ connected component of $\gamma^{-1}(W)$ containing t_0. Given $t_1,t_2\in U$, Set $q_1=\gamma(t_1)$, $q_2=\gamma(t_2)$. this implies q_2 is contained in a geodesic ball around q_1. So radial geodesic from q_1 to q_2 is the unique minimizing curve. Because q_1 and q_2 are in the same geodesic ball, so radial geodesic is just part of γ.

∎

<u>Def.</u> For $p,q\in M$ in the same connected components, we define the Riemmanian distance $d(p,q)$ to be the infimum of lengths of all admissible curves from p to q.

<u>Lemma</u> With this distance d, any connected Riemannian manifold is a metric space.

Moreover, the induced topology is the same as the given manifold topology.

Proof:

(1) $d(p,q) = d(q,p) \geq 0$, and $d(p,p)=0$ are immediate.

(2). triangle inequality ?

(3). $d(p,q)>0$ when $p\neq q$:

Let $p\in M$, (x^i) be normal coordinates around p. There exists a closed geodesic ball $\bar{y}$ of radius ε around p and positive constants c, C s.t. $c|V|_{\bar{g}} \leq |V|_g \leq C|V|_{\bar{g}}$ whenever $V\in T_xM$ and $x\in \bar{y}$.

So if an admissible curve γ is in $\bar{y}$, we have

$$cL_{\bar{g}}(\gamma) \leq L_g(\gamma) \leq CL_{\bar{g}}(\gamma).$$

if $p\neq q$, we can shrink ε s.t. $q\notin \bar{y}$. So

$$\begin{aligned} d(p,q) &\geq L_g(\gamma) \geq L_g(\gamma|_{[a_0,t_0]}) \\ &\geq cL_{\bar{g}}(\gamma|_{[a,t_0]}) \\ &\geq c d_{\bar{g}}(p, \gamma(t_0)) \\ &= c\varepsilon > 0 \end{aligned}$$

Combining (1)(2)(3) $\Rightarrow$ d is a metric.

Now, b/c Euclidean distance and Riemannian distance are equivalent, so two topologies are the same.