The "great hyperbolas".
i.e. intersections of $\mathbb{H}^n_R$ w/ 2-planes through the origin.

Ball Model:

The line segments through the origin, and the circular arcs intersecting ∂B^n_R orthogonally.

Half-Space Model:

The vertical half-lines and the semicircles w/ centers on the $y=0$ hyperplane.

〈Pf〉
Also using the homogeneity and isotropicity.
See P84. #

Chap 6 Geodesics and Distance.

$\boxed{\text{To Study}}$: (on Riem mfd).
Relation b/w geodesics, lengths, and distances.

$\boxed{\text{Main Results}}$:

① All length-minimizing curves are geodesics.
② All geodesics are locally length minimizing.
③ A Riem mfd is geodesically complete iff it is complete as a metric sp.

$\boxed{\text{Caution}}$:
Most results in this Chap do not apply to pseudo-Riem. metrics.

Lengths of Curves

Def

$\gamma:[a,b]\to M$, a curve segment.

① The $\boxed{\text{length of } \gamma}$, denoted $\boxed{L(\gamma)}$ or $\boxed{L_g(\gamma)}$, is $L(\gamma)=\int_a^b|\dot\gamma(t)|\,dt$.

② A $\boxed{\text{reparameterization}}$ of γ is a curve segment $\tilde\gamma=\gamma\circ\varphi$, where $\varphi:[c,d]\to[a,b]$ is C^∞ w/ C^∞ inverse. It is called a $\boxed{\text{forward / backward}}$ $\boxed{\text{repara}}$ if φ is orientation preserving/reversing.

$\boxed{\text{Lem 6.1}}$

$\gamma:[a,b]\to M$, a curve segment.
$\tilde\gamma$: a repara. of γ.
Then $L(\gamma)=L(\tilde\gamma)$.

〈Pf〉
Change of variable. #

Def

① A $\boxed{\text{regular curve}}$ is a curve $\gamma:I\to M$ s.t. $|\dot\gamma(t)|\neq 0,\ \forall t$. (to guarantee $\mathrm{im}(\gamma)$ an immersed submfd of M)

② A cont. $\gamma:[a,b]\to M$ is called a $\boxed{\text{piecewise regular curve segment}}$ if $\exists$ $a=a_0<a_1<\cdots<a_k=b$ s.t. $\gamma|_{[a_{i-1},a_i]}$ is regular, $\forall i$.

For simplicity, we shall call a piecewise regular curve segment by an $\boxed{\text{admissible curve}}$.

③ By def, for each i, $\boxed{\dot\gamma(a_i^-)}:=\lim_{t\to a_i^-}\dot\gamma(t)$ and $\boxed{\dot\gamma(a_i^+)}:=\lim_{t\to a_i^+}\dot\gamma(t)$ exist.
For admissible curves, define $\boxed{L(\gamma)}=\sum_{i=1}^k L(\gamma|_{[a_{i-1},a_i]})$.

$\boxed{\text{Repara.}}$ can be similarly defined (allowing φ to be a homeo. w/ subdivision $c=c_0<c_1<\cdots<c_k=d$ s.t. $\varphi|_{[c_{i-1},c_i]}$ is C^∞, $\forall i$).

④ $\gamma:[a,b]\to M$, an admissible curve.

$\boxed{s(t)}:=L(\gamma|_{[a,t]})=\int_a^t|\dot\gamma(u)|\,du$

is called the $\boxed{\text{arc length function}}$ of γ.

Rmk:
$\dot s(t)=|\dot\gamma(t)|$, $\forall t$ s.t. γ is C^∞ at t.

Prop (Exe 6.2)

$\gamma:[a,b]\to M$, an admissible curve.
$\ell:=L(\gamma)$.
Then
① $\exists!$ forward para $\tilde\gamma:[0,\ell]\to M$ of γ s.t. $|\dot{\tilde\gamma}(t)|\equiv 1$.
② the arc length fun. $s(t)$ of $\tilde\gamma$ satisfies $s(t)=t$. (Curves w/ this property is called $\boxed{\text{parametrized by arc length}}$

⟨Pf⟩

Note $s:[a,b]\to[0,\ell]$ can be used for repara. ㊊

Def

$\gamma:[a,b]\to M$, an admissible curve.

$f\in C^\infty[a,b]$.

Define $\boxed{\int_\gamma f\,ds} := \int_a^b f(t)\,|\dot\gamma(t)|\,dt$, called

the $\boxed{\text{integral of } f \text{ w.r.t. the arc length of } \gamma}$.

Rmk:

$\int_\gamma f\,ds$ is indep. of para. of γ in the sense

that $\int_{\tilde\gamma}\tilde f\,ds = \int_\gamma f\,ds$, where $\tilde\gamma=\gamma\circ\varphi$, and $\tilde f = f\circ\varphi$.

Def γ : admissible curve.

$V:[a,b]\to TM$ w/ $V_t\in T_{\gamma(t)}M$, $\forall t$, is called

a $\boxed{\text{piecewise smooth vec. field along } \gamma}$ if it is

cont. and $\exists$ (possibly finer than that for γ)

$a=\tilde a_0<\tilde a_1<\cdots<\tilde a_m=b$ s.t. $V|_{[\tilde a_{i-1},\tilde a_i]}$ is smooth $\forall i$.

Rmk: ($\boxed{\text{parallel translate}}$)

For $V_a\in T_{\gamma(a)}M$, we can uniquely parallel translate it along all of γ. (simply translate piece by piece)

The Riem. Distance Fun.

Def

M: connected Riem. mfd.

For $p,q\in M$, define

$\boxed{d(p,q)} = \inf\{L(\gamma)\mid \gamma : \text{admissible from } p \text{ to } q\}$.

This is called the $\boxed{\text{Riemannian distance}}$.

$\boxed{\text{Lem 6.2}}$

W/ d defined above, any connected Riem. mfd is a metric sp. whose induced topology is the same as the given mfd topology.

⟨Pf⟩

See P94-95. The property $d(x,y)>0$, $\forall x\neq y$ is the only one needing to be dealt with. ㊊

Geodesics and Minimizing Curves.

Def

An admissible curve γ in a Riem mfd is called minimizing if $L(\gamma)\le L(\tilde\gamma)$ for any admissible $\tilde\gamma$ having the same endpts of γ.

Goal:

We shall use ideas of "calculus of variat'n" to prove minimizing curves are geodesics.

Def

① $\Gamma:(-\varepsilon,\varepsilon)\times[a,b]\to M$ is called an admissible family of curves if $\exists$ $a=a_0<a_1<\cdots<a_k=b$ s.t.

$\Gamma_s(t):=\Gamma(s,t)$ is an admissible curve $\forall s\in(-\varepsilon,\varepsilon)$

and Γ is smooth on each $(-\varepsilon,\varepsilon)\times[a_i,a_{i+1}]$.

② A vec. field along an admissible family Γ is a cont. $V:(-\varepsilon,\varepsilon)\times[a,b]\to TM$ s.t.

$V(s,t)\in T_{\Gamma(s,t)}M$, $\forall s,t$ and

$\exists$ subdivision $a=\tilde a_0<\cdots<\tilde a_k=b$ (possibly finer than Γ)

s.t. each $V|_{(-\varepsilon,\varepsilon)\times[\tilde a_i,\tilde a_{i+1}]}$ is smooth

Def

Γ : admissible family of curves.

① Curves of the form $\Gamma_s(t):=\Gamma(s,t)$, $t\in[a,b]$, are called main curves.

② $\Gamma^{(t)}(s):=\Gamma(s,t)$, $s\in(-\varepsilon,\varepsilon)$, are called transverse curves.

Rmk:

① Transverse curves are always smooth on $(-\varepsilon,\varepsilon)$, $\forall t$.

② Main curves are only piecewise regular in general.

Def

$\partial_t\Gamma(s,t):=\frac{d}{dt}\Gamma_s(t)$, $\partial_s\Gamma(s,t):=\frac{d}{ds}\Gamma^{(t)}(s)$.

Rmk:

① $\partial_t\Gamma$ is not usually cont. at $t=a_i$. Thus not necessarily a vec. field.

② $\partial_s \Gamma$ is always cont. on the whole $(-\varepsilon,\varepsilon)\times[a,b]$.
(indeed, since its values on $(-\varepsilon,\varepsilon)\times\{a_i\}$ only dep. on the values of Γ on $(-\varepsilon,\varepsilon)\times\{a_i\}$ and $\partial_s\Gamma$ is cont. on each $(-\varepsilon,\varepsilon)\times[a_i,a_{i+1}]$).

Def
V : vec. field along Γ.
$D_t V :=$ covariant derivatives along mein curves.
$D_s V :=$ 〃 〃 〃 transverse 〃.

Lem 6.3 (Symmetry Lemma).
$\Gamma : (-\varepsilon,\varepsilon)\times[a,b]\to M$: admissible family, in a Riem. mfd M.
Then, on any rectangle $(-\varepsilon,\varepsilon)\times[a_i,a_{i+1}]$ where Γ is smooth, we have

$$D_s\partial_t\Gamma = D_t\partial_s\Gamma.$$

<Pf> (Rmk: Works for any symmetric connection).
Since the equality is local, it suffices to prove in coordinates.
Given (s_0,t_0) and a chart $(x^1,\cdots,x^n)$ around $\Gamma(s_0,t_0)$.
Write $\Gamma(s,t)=(x^1(s,t),\cdots,x^n(s,t))$.
Then $\partial_t\Gamma = \frac{\partial x^k}{\partial t}\partial_k$ and $\partial_s\Gamma=\frac{\partial x^k}{\partial s}\partial_k$.
Thus, by (4.10),

$$D_s\partial_t\Gamma = \left[\frac{\partial}{\partial s}\frac{\partial x^k}{\partial t}+\frac{\partial x^j}{\partial t}\frac{\partial x^i}{\partial s}\Gamma_{ij}^k\right]\partial_k, \text{ and}$$

$$D_t\partial_s\Gamma = \left[\frac{\partial}{\partial t}\frac{\partial x^k}{\partial s}+\frac{\partial x^j}{\partial s}\frac{\partial x^i}{\partial t}\Gamma_{ij}^k\right]\partial_k$$

$$= \Big[\underbrace{\frac{\partial}{\partial s}\frac{\partial x^k}{\partial t}}_{\left(\frac{\partial^2}{\partial s\partial t}=\frac{\partial^2}{\partial t\partial s}\right)}+\underbrace{\frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}}_{\text{(just shift } i,j \text{ notation)}}\Gamma_{ji}^k\Big]\partial_k$$

$$= \left[\frac{\partial}{\partial s}\frac{\partial x^k}{\partial t}+\frac{\partial x^j}{\partial t}\frac{\partial x^i}{\partial s}\Gamma_{ij}^k\right]\partial_k$$

$$= D_s\partial_t\Gamma. \quad \#$$

(by symmetry of connection)

(Recall)
Rmk: (For convenient ref). (4.10)

$$D_t V(t_0) = [\dot V^k(t_0)+V^j(t_0)\dot\gamma^i(t_0)\Gamma_{ij}^k(\gamma(t_0))]\partial_k.$$

Def $\gamma:[a,b]\to M$, an admissible curve. RM P75
① A variation of γ is an admissible family Γ w/ $\Gamma_0(t)=\gamma(t)$, $\forall t\in[a,b]$.
② A proper variation (or fixed-endpt variat'n) of γ is a variation Γ of γ s.t.
$\Gamma_s(a)=\gamma(a)$ and $\Gamma_s(b)=\gamma(b)$, $\forall s$.
③ The variation field of Γ (a variation of γ) is the vector field $V_\Gamma(t)=\partial_s\Gamma(0,t)$ along γ.
④ A vec. field V along γ is called proper if $V(a)=V(b)=0$.

Rmk:
If Γ is proper, then V_Γ is proper.

Lem 6.4
γ: admissible curve. ($\gamma:[a,b]\to M$).
$V\in\mathcal{T}(\gamma)$ (i.e. V is a vec. field along γ).
Then $\exists$ variation Γ of γ s.t. $V_\Gamma = V$.
If V is proper, then we may choose Γ to be proper s.t. $V_\Gamma=V$.

<Pf>
$\Gamma(s,t):=\exp(sV(t))$.
$\because [a,b]$ is cpt $\therefore \exists\,\varepsilon>0$ s.t. $\Gamma(s,t)$ is defined on $(s,t)\in(-\varepsilon,\varepsilon)\times[a,b]$.
$V_\Gamma(t)=\partial_s\Gamma(0,t)=\frac{d}{ds}\Big|_{s=0}\Gamma(s,t)$
$= V(t)$ (by def. of exp and rescaling lemma).
i.e. $V_\Gamma = V$. #
If V is proper, then $V(a)=0=V(b)$.
$\Rightarrow \Gamma(s,a)=\exp(0\cdot V(a))=\gamma(a)$ and
$\Gamma(s,b)=\exp(0\cdot V(b))=\gamma(b)$. (i.e. Γ: proper). #

Minimizing Curves are Geodesics.

Prop 6.5 (1st Variation Formula).
$\gamma:[a,b]\to M$ admissible curve w/ unit speed.
Γ : a proper variation of γ.
w/ partition $a=a_0<a_1<\cdots<a_k=b$.
Then

$$\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s)$$

$$= -\int_a^b \langle V_\Gamma, D_t\dot\gamma\rangle\, dt - \sum_{i=1}^{k-1} \langle V_\Gamma(a_i), \Delta_i \dot\gamma\rangle,$$

where $\Delta_i \dot\gamma := \dot\gamma(a_i^+) - \dot\gamma(a_i^-)$.

i.e. the "jump" of $\dot\gamma$ at a_i.

⟨Pf⟩

Denote $T(s,t) = \partial_t \Gamma(s,t)$ and $S(s,t) = \partial_s \Gamma(s,t)$.

On each $[a_{i-1}, a_i]$,

$$\frac{d}{ds} L(\Gamma_s|_{[a_{i-1},a_i]}) = \frac{d}{ds}\int_{a_{i-1}}^{a_i} \langle T,T\rangle^{1/2}\, dt$$

chain rule and compatibility

$$= \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial s}\langle T,T\rangle^{1/2}\, dt = \int_{a_{i-1}}^{a_i} \frac{1}{2}\langle T,T\rangle^{-1/2}\, 2\langle D_sT, T\rangle\, dt$$

symmetry lem

$$= \int_{a_{i-1}}^{a_i} \frac{\langle D_sT, T\rangle}{|T|}\, dt = \int_{a_{i-1}}^{a_i} \frac{1}{|T|}\langle D_tS, T\rangle\, dt.$$

Note that $S(0,t) = V_\Gamma(t)$ and $T(0,t) = \dot\gamma(t)$.

Thus

$$\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s|_{[a_{i-1},a_i]})$$

$$= \int_{a_{i-1}}^{a_i} \frac{1}{|T|}\langle D_tV_\Gamma, \dot\gamma\rangle\, dt$$

1 (unit speed of γ)

$$= \int_{a_{i-1}}^{a_i} \left(\frac{d}{dt}\langle V_\Gamma, \dot\gamma\rangle - \langle V_\Gamma, D_t\dot\gamma\rangle\right) dt$$

$$= \left[\langle V_\Gamma(a_i), \dot\gamma(a_i^-)\rangle - \langle V_\Gamma(a_{i-1}), \dot\gamma(a_{i-1}^+)\rangle\right] - \int_{a_{i-1}}^{a_i}\langle V_\Gamma, D_t\dot\gamma\rangle\, dt.$$

Summing over $i = 1, \cdots, k$ (and noting $V_\Gamma = 0$ at a_0 and a_k) (since Γ is proper) the result follows. #

Rmk:

∵ Every admissible curve can be repara. as unit speed

∴ The assumption of unit speed in Prop 6.5 is just for computational convenience.

Thm 6.6 RM P26

Every minimizing curve is a geodesic when it is given a unit speed para.

⟨Pf⟩

Given minimizing $\gamma : [a,b] \to M$ w/ unit speed and a subdivision $a = a_0 < \cdots < a_k = b$ w/ $\gamma|_{[a_{i-1},a_i]}$ being smooth, $\forall i$.

i.e. γ is "broken geodesic".

claim: $D_t\dot\gamma = 0$ on each $[a_{i-1}, a_i]$.

Choose a bump fun. $\varphi \in C^\infty(\mathbb{R})$ w/ $\varphi > 0$ on (a_{i-1}, a_i), $\varphi = 0$ elsewhere.

Choose $V = \varphi D_t\dot\gamma$. (proper along γ).

By Lem 6.4, $\exists$ proper variation Γ s.t. $V = V_\Gamma$.

By Calculus, since γ is minimizing, by Prop 6.5,

$$\int_{a_{i-1}}^{a_i} \varphi\langle D_t\dot\gamma, D_t\dot\gamma\rangle\, dt = 0. \Rightarrow D_t\dot\gamma = 0 \text{ on } [a_{i-1}, a_i].$$

claim: $\Delta_i\dot\gamma = 0$, $\forall i$. i.e. γ has no corner. # of claim

Use a coordinate chart around $\gamma(a_i)$ to construct a vec. field V along γ w/ $V(a_i) = \Delta_i\dot\gamma$ and $V(a_j) = 0$, $\forall j \neq i$.

By the previous claim, the 1st part of RHS of Prop 6.5 vanishes.

By condition of V, we now have $0 = -|\Delta_i\dot\gamma|^2$.

$\Rightarrow \Delta_i\dot\gamma = 0$. # of claim

claim: γ is a geodesic.

∵ $D_t\dot\gamma \equiv 0$ ∴ γ is piecewise geodesic

∵ $\Delta_i\dot\gamma = 0$, $\forall i$ ∴ At overlapping pts, $\dot\gamma(a_i^+) = \dot\gamma(a_i^-)$.

By uniqueness of geodesic, γ is a geodesic itself.

#

Rmk:

We only use the fact $\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s) = 0$, for any proper variation Γ of γ.

Def

An admissible curve γ is called a critical point for L if $\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s) = 0$, $\forall$ proper variat'n Γ of γ.

Cor 6.7

A unit speed admissible curve γ is a geodesic iff it is a critical pt for L.

Rmk:
The geodesic equation $D_t\dot{\gamma}=0$ characterizes the critical pts of the "length functional".

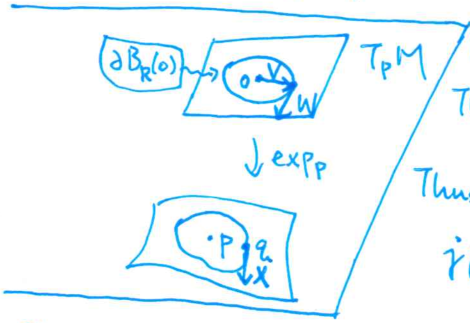
Def
The equation characterizing critical pts of a functional on a space of maps is called the variational equation or the Euler-Lagrange equation of the functional.

Rmk:
Three more such eq'n in this book:
Einstein equation (Chap 7), Yamabe eq'n (Chap 7) and minimal surface eq'n. (Chap 8).

Geodesics are Locally Minimizing.

Thm 6.8 (The Gauss Lemma).
U: a geodesic ball centered at $p\in M$.
Then $\frac{\partial}{\partial r}$, the unit radial vec. field is g-orthogonal to the geodesic spheres in U.

⟨Pf⟩
Given $q\in U$, say $q\in \exp_p(\partial B_R(0))$.
Given $X\in T_q(\exp_p(\partial B_R(0)))\leq T_qM$.
We need to prove $X\perp \frac{\partial}{\partial r}\big|_q$.
$\because$ $\exp_p$ is diffeo. onto U
$\therefore$ $\exists!\, V\in T_pM$ s.t. $\exp_p(V)=q$.
Also, $\exists!\, W\in T_V(T_pM)=T_pM$ s.t. $(\exp_p)_*(W)=X$.



Define $\gamma(t)=\exp_p(tV)$.
Then $\dot{\gamma}(t)=R\frac{\partial}{\partial r}$. ← Check in normal coordinates.
Thus, it suffices to prove $\dot{\gamma}(1)\perp X$.

Choose $\sigma:(-\varepsilon,\varepsilon)\to\partial B_R(0)\ (\subseteq T_pM)$ s.t. $\sigma(0)=V$ and $\dot{\sigma}(0)=W$.

Define $\Gamma(s,t)=\exp_p(t\,\sigma(s))$.
$S:=\partial_s\Gamma$ and $T:=\partial_t\Gamma$.

Then $S(0,0)=\frac{d}{ds}\big|_{s=0}\exp_p(0)=0$
$T(0,0)=\frac{d}{dt}\big|_{t=0}\exp_p(tV)=V$
$S(0,1)=\frac{d}{ds}\big|_{s=0}\exp_p(\sigma(s))=(\exp_p)_*(W)=X$
$T(0,1)=\frac{d}{dt}\big|_{t=1}\exp_p(tV)=\dot{\gamma}(1)$.

Claim: $\frac{d}{dt}\langle S,T\rangle\equiv 0$.

$\frac{d}{dt}\langle S,T\rangle=\langle D_tS,T\rangle+\langle S,D_tT\rangle$ ← $=0$ since $\Gamma_s(t)$ is a geod.
$=\langle D_sT,T\rangle+0=\frac{1}{2}\frac{d}{ds}\langle T,T\rangle$.
Note that $\langle T(s,t),T(s,t)\rangle=\left|\frac{d}{dt}\exp_p(t\sigma(s))\right|^2$
$=|\sigma(s)|^2\equiv R^2$.
Thus, $\frac{d}{dt}\langle S,T\rangle=\frac{1}{2}\frac{d}{ds}R^2\equiv 0$. # of claim

$\Rightarrow\langle S,T\rangle=$ const.
$\Rightarrow\langle S(0,1),T(0,1)\rangle=\langle S(0,0),T(0,0)\rangle=\langle 0,V\rangle=0$.
i.e. $\langle X,\dot{\gamma}(1)\rangle=0$. #

Cor 6.9
(x^i): normal coordinates on a geod. ball U, centered at p.
$r:=$ the radial dist. fun.
Then $\operatorname{grad} r=\partial/\partial r$ on $U\setminus\{p\}$.

⟨Pf⟩
Given $q\in U\setminus\{p\}$ and $Y\in T_qM$.
It suffices to prove $dr(Y)=\langle\partial/\partial r,Y\rangle$. ← def. for raising index.
By Gauss lemma, $Y=\alpha\frac{\partial}{\partial r}+X$ w/ X tangent to the "sphere" and $X\perp\partial/\partial r$.
By computation in coordinates, $dr(\partial/\partial r)=1$.
$\because$ X is tangent to a level set of r
$\therefore$ $dr(X)=0$.
Thus $dr(Y)=\alpha$. (LHS).
$\langle\partial/\partial r,Y\rangle=\langle\partial/\partial r,\alpha\,\partial/\partial r+X\rangle=\alpha\,|\partial/\partial r|^2+0=\alpha$. (RHS) ← $|\partial/\partial r|\equiv 1$
Hence, LHS=RHS, as desired. #

Prop 6.10

$p \in M$, $q \in U$, a geod. ball around p.

Then, (up to repara),

the radial geod. from p to q is the unique minimizing curve from p to q in M.

<Pf>

Choose $\varepsilon > 0$ s.t. $\exp_p(B_\varepsilon(0))$: geod. ball containing q.

$\gamma : [0, R] \to M$: the radial geod. from p to q w/ unit speed; denote $\gamma(t) = \exp_p(tV)$ w/ $|V| = 1$.

$S_R := \exp_p(\partial B_R(0))$.

"Minimizing of γ".

Given $\sigma : [0, b] \to M$, unit speed from p to q.

$a_0 := \sup\{t \in [0,b] \mid \sigma(t) = p\}$.

$b_0 := \inf\{t > a_0 \mid \sigma(t) \in S_R\}$.

By Gauss lemma, $\dot\sigma(t) = \alpha(t)\, \partial/\partial r + X(t)$, w/ X : tangent to spheres and $\partial/\partial r \perp X$.

Then $|\dot\sigma(t)|^2 = \alpha(t)^2 + |X(t)|^2 \geq \alpha(t)^2$.

Moreover, by Cor 6.9, $\alpha(t) = dr(\dot\sigma(t))$.

Hence, $L(\sigma) \overset{①}{\geq} L(\sigma|_{[a_0, b_0]}) = \int_{a_0}^{b_0} |\dot\sigma(t)|\, dt$

$\overset{②}{\geq} \int_{a_0}^{b_0} \alpha(t)\, dt = \int_{a_0}^{b_0} dr(\dot\sigma(t))\, dt = \int_{a_0}^{b_0} \frac{d}{dt} r(\sigma(t))\, dt$

$= r(\sigma(b_0)) - r(\sigma(a_0)) = R - 0 = R = L(\gamma)$. #

"Uniqueness of minimizing curve".

Assume $L(\sigma) = R$.

Then ① $a_0 = 0$ and $b_0 = b = R$

② $|X(t)| \equiv 0$, $\alpha(t) > 0$, $\dot\sigma(t) = \alpha(t)\, \partial/\partial r$.

$\because |\partial/\partial r| \equiv 1$ $\therefore \alpha(t) = |\dot\sigma(t)| / |\partial/\partial r| \equiv 1$.

Thus $\dot\sigma(t) = \partial/\partial r$. $\Rightarrow$ σ and γ are both integral curves of $\partial/\partial r$.

$\Rightarrow \sigma = \gamma$. #

Cor 6.11

Within any geod. ball around $p \in M$,

$r(x)$ = Riemannian distance from p to x.

<Pf>

Use the fact that $\partial/\partial r$ is exactly the velocity of radial geodesics from p to x w/ unit speed. #

Def (simplifying notation).

Within a geod. ball around $p \in M$,

$B_R(p) := \exp_p(B_R(0))$,

$\overline{B}_R(p) := \exp_p(\overline{B}_R(0))$, and

$S_R(p) := \exp_p(\partial B_R(0))$.

Rmk:

By Prop 6.10 and Cor 6.11, $B_R(p)$, etc, are exactly the open R-ball around p w.r.t. Riem. dist.

Def

A curve $\gamma : I \to M$ is called local minimizing if $\forall t_0 \in I$, $\exists$ nbd U of t_0 in I s.t.

$\gamma|_U$ is minimizing b/w each pair of pts in U.

Rmk:

Minimizing curves are automatically local minimizing.

Thm 6.12

Every Riem geod. is locally minimizing.

<Pf>

Given $\gamma : I \to M$, a geod. and $t_0 \in I$.

Choose a unif. normal nbd W of $\gamma(t_0)$.

$U :=$ the connected component of $\gamma^{-1}(W)$ containing t_0. (say $t_1 < t_2$).

For $t_1, t_2 \in U$, since $\gamma(t_1), \gamma(t_2) \in W$, the radial geod. from $\gamma(t_1)$ to $\gamma(t_2)$ is also in W.

$\because$ U is a connected component of $\gamma^{-1}(W)$

$\therefore \gamma|_{[t_1, t_2]} \in W$.

By Prop 6.10, $\gamma|_{[t_1,t_2]}$ is exactly the unique minimizing curve. #

<Another Pf of Thm 6.6>

Recall: (Thm 6.6)
Every minimizing curve is a geod.

Given minimizing $\gamma: I \to M$.
For $t_0 \in I$, as the above proof, $\exists$ nbd U of t_0 in I s.t. $\gamma(U) \subseteq W$ w/ W unif. normal.
For $\underbrace{t_1 < t_0 < t_2}_{\text{in } U}$, $\gamma|_{[t_1,t_2]}$ is still minimizing.
Moreover, since $\gamma([t_1,t_2]) \subseteq W$, by Prop 6.10, $\gamma|_{[t_1,t_2]}$ must be a geod. (the radial geod.)
Hence γ is smooth at t_0 and geod. around t_0.
$\because t_0$ is arbitrary $\therefore \gamma$ is a geod. #

Completeness

<Another Pf of Lem 6.2>

Recall: (Lem 6.2)
$d(p,q) := \inf\{L(\gamma) \mid \gamma: \text{admissible from } p \text{ to } q\}$, for $p,q \in M$, is a metric on M whose induced topology is the same as the topology of M.

It is clear that $d(p,q) \geq 0$, $\forall p,q$, $d(p,p)=0$, $\forall p$.
Triangle inequality follows from the fact that the adjoining of two consecutive admissible curves is admissible.
It remains to prove $d(p,q) > 0$, $\forall p \neq q$.
Let $\varepsilon > 0$ s.t. $\exp_p(B_\varepsilon(0))$ is a geod. ball around p.
If $q \in \exp_p(B_\varepsilon(0))$, then, by Prop 6.10, $d(p,q) > 0$. $\forall q \neq p$.
If $q \notin \exp_p(B_\varepsilon(0))$, since every curve from p to q passes through $\exp_p(\partial B_{\varepsilon/2}(0))$, $d(p,q) > \varepsilon/2 > 0$.
Thus, we do have $d(p,q) > 0$, and d is a metric.
As for the topology, since within any geod. ball the R-ball w.r.t. d is open in M, we must have $\mathcal{T}_M \supseteq \mathcal{T}_{(M,d)}$. Conversely, since $\mathcal{T}_M$ is generated by such R-balls in normal coordinate charts, we also have $\mathcal{T}_M \subseteq \mathcal{T}_{(M,d)}$. Hence, $\mathcal{T}_M = \mathcal{T}_{(M,d)}$. #

RM P29

Def
A Riem. mfd M is called $\boxed{\text{geodesically complete}}$ if every maximal geod. is defined $\forall t \in \mathbb{R}$.

$\boxed{\text{Thm 6.13}}$ (Hopf-Rinow).
A connected Riem mfd M is geod. complete iff it is complete as a metric sp.

<Pf>
($\Leftarrow$) (metric complete $\Rightarrow$ geod. complete).
Suppose not.
Then $\exists$ unit speed geod. $\gamma: [0,b) \to M$ not extendible to $[0, b+\varepsilon)$, $\forall \varepsilon > 0$.
Choose $t_i \in [0,b)$ s.t. $t_i \nearrow b$.
Denote $q_i = \gamma(t_i)$.
Then $d(q_i, q_j) \leq |t_i - t_j|$, $\forall i,j$.
Thus, $\{q_i\}_{i=1}^\infty$ is Cauchy, say $q_i \to q$ in M.
Let W be a unif. normal nbd of q, say $\delta > 0$ is such that $W \subseteq B_{2\delta}(p)$, the 2δ-geod. ball of p, $\forall p \in W$.
Choose j large enough s.t. $t_j > b - \delta$. We also need $q_j \in W$.
Let σ be the geod. w/ $\sigma(0) = q_j$ and $\dot\sigma(0) = \dot\gamma(t_j)$.
Then σ is defined at least on $[0,\delta]$ and, by uniqueness, $\sigma(t) = \gamma(t_j + t)$.
$\Rightarrow \gamma$ can be extended to be defined on $[0, t_j+\delta) \supsetneq [0,b)$. $\times$
Thus, M is geod. complete. #

($\Rightarrow$) (geod. complete $\Rightarrow$ metric complete).

Def Given $\gamma: [0,b] \to M$, a geod. segment, and $q \in M$.
We say that γ $\boxed{\text{aims at } q}$ if
γ is minimizing and
$d(\gamma(0), q) = d(\gamma(0), \gamma(b)) + d(\gamma(b), q)$.

claim : (Stronger Statement).

If $\exists\, p \in M$ s.t. $\exp_p$ is defined on all T_pM, then M is a complete metric space.

⟨Pf of claim⟩.

Claim 1 : Given such p. Then, $\forall q \in M$, $\exists$ minimizing geod. segment from p to q.

Choose $\varepsilon > 0$ s.t. $\overline{B}_\varepsilon(p)$ is a closed geod. ball.

$$x := \operatorname*{argmin}_{x \in S_\varepsilon(p)} d(x,q).$$

$\gamma :=$ unit speed radial geod. from p to x.

By geod. completeness of M, γ is defined on $\mathbb{R}$.

Subclaim : $\gamma|_{[0,\varepsilon]}$ aims at q.

i.e. $\gamma|_{[0,\varepsilon]}$ is minimizing and

$$d(p,q) = d(p,x) + d(x,q). \quad ⊛$$

That $\gamma|_{[0,\varepsilon]}$ is minimizing comes from Prop 6.10.

By triangle ineq, ⊛ fails only when

$d(p,q) < d(p,x) + d(x,q)$.

$\Rightarrow \exists\, \sigma$ from p to q s.t. $L(\sigma) < d(p,x) + d(x,q)$.

$\sigma_1 := \sigma$ inside $\overline{B}_\varepsilon(p)$

$\sigma_2 :=$ ″ outside ″ .

Then $d(p,x) + d(x,q) > L(\sigma) = L(\sigma_1) + L(\sigma_2)$

$\geq \varepsilon + L(\sigma_2) = d(p,x) + L(\sigma_2)$.

$\Rightarrow d(x,q) > L(\sigma_2)$. ✶ (since $d(x,q)$ is minimized).

Thus $\gamma|_{[0,\varepsilon]}$ aims at q.

$T := d(p,q)$

$S := \{ b \in [0,T] \mid \gamma|_{[0,b]} \text{ aims at } q \}$.

By conti. of dist. fun., S is closed.

$A := \sup(S)$. Then $A \in S$.

Suppose $A < T$. (and try to get a contradict'n).

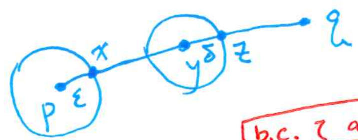
$y := \gamma(A)$.

Choose $\delta > 0$ s.t. $\overline{B}_\delta(y)$ is a closed geod. ball.

$$z := \operatorname*{argmin}_{z \in S_\delta(y)} L(z,q).$$

Let $\tau : [0,\delta] \to M$ be the unit speed radial geod. from y to z.

As before, τ aims at q.



Thus $d(z,q) \overset{\text{b.c. } \tau \text{ aims at } q}{=} d(y,q) - d(y,z)$

$= \underbrace{(T-A)}_{\text{b.c. } A \in S} - \underbrace{\delta}_{\text{Prop 6.10}}$

Δ ineq.

$\Rightarrow d(p,z) \geq d(p,q) - d(z,q)$

$= T - [(T-A) - \delta] = A + \delta$

$= d(p,y) + d(y,z) \geq d(p,z)$ (Δ - ineq.)

Thus, $d(p,z) = A + \delta$ and

$\gamma|_{[0,A]}$ followed by $\tau|_{[0,\delta]}$ is a minimizing curve(, and thus has no corner).

Moreover, $d(p,z) + d(z,q) = (A+\delta) + (T-A) - \delta = T = d(p,q)$.

$\Rightarrow A + \delta \in S$. ✶

Thus $A = T$ and $\gamma|_{[0,T]}$ is a minimizing curve from p to q. # of claim 1

Now we turn to metric completeness.

Given $\{q_i\}$ Cauchy in M.

By assumption of p (and claim 1), $q_i = \exp_p(d_i V_i)$, w/ $d_i = d(p,q_i)$ and $\|V_i\| = 1$ in T_pM.

$\because \{q_i\}$ is Cauchy $\therefore \{d_i\}$ is bdd.

$\Rightarrow \{d_i V_i\}$ is bdd in T_pM.

$\Rightarrow \exists$ subseq. $d_{i_k} V_{i_k} \to V$ in T_pM.

(conti. of $\exp_p$) $\Rightarrow q_{i_k} = \exp_p(d_{i_k} V_{i_k}) \to \exp_p(V) =: q$ in M.

$\because \{q_i\}$: Cauchy $\therefore q_i \to q$ in M.

Therefore, M is complete as a metric sp.

#

Cor 6.14

If $\exists\, p \in M$ s.t. $\exp_p$ is defined on all of T_pM, then M is complete.

⟨Pf⟩ This is the claim in Hopf-Rinow's proof. #

Cor 6.15

M is complete iff every two pts in M can be joined by a minimizing geod. segment.

⟨Pf⟩

($\Rightarrow$)

By claim I in the proof of Hopf-Rinow #

($\Leftarrow$)

Adopt the same argument of the last part of Hopf-Rinow's proof. #

Cor 6.16

If M is cpt, then every geod. can be defined for all time.