Ref: Extending Persistence Using Poincaré and Lefschetz Duality. [2009] [D. C-S, et al].

For 2/14

2. Intuition.

$f: M \to \mathbb{R}$ Morse function. M: d-mfd.

$s_1, \cdots, s_m$: critical values of f.

$t_0 < t_1 < \cdots < t_m$ w/ $t_0 < s_1 < t_1 < \cdots < s_m < t_m$.

$\boxed{M_k} := f^{-1}((-\infty, t_k])$.

$\boxed{\text{Prop}}$ (from Morse theory)

see Def below.

$M_k \cong M_{k-1}$ $\boxed{\text{w/ an } r\text{-handle attached}}$, where

$r = \boxed{\text{index}}$ of the critical value s_k.

$:= \#\{$ negative eigenvalues of Hessian of f at "the" critical pt corresponding to $s_k\}$.

Def

M: d-mfd w/ bd. $D^k :=$ closed unit k-disk.

referring to "attaching".

$\alpha: S^{r-1} \times D^{d-r} \to \partial M$, an embedding.

$\boxed{H^r} = D^r \times D^{d-r}$, called an $\boxed{r\text{-handle}}$.

$\boxed{M \text{ w/ an } r\text{-handle attached via } \alpha}$ is defined as

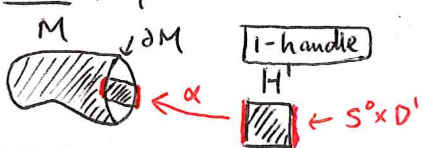
$$(M \sqcup H^r)/\sim \; = (M \sqcup D^r \times D^{d-r})/\sim, \text{ where}$$

$$(p, x) \in S^{r-1} \times D^{d-r} \leftrightarrow \alpha(p, x).$$

Example:

① $d = 2$

case 1 $r = 1$



case 2 $r = 2$

Def

$f: M \to \mathbb{R}$ smooth.

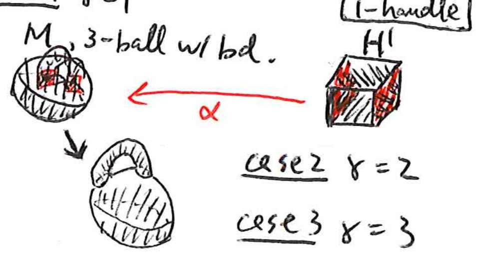
f is a $\boxed{\text{Morse function}}$ if

(i) all critical pts of f are non-degenerate,

(ii) no two critical pts share the same function value.

② $d = 3$

case 1 $r = 1$



case 2 $r = 2$

case 3 $r = 3$

all side faces.

all faces.

Prop (from Morse theory)

$M_k \cong M_{k-1}$ w/ an r-handle attached.

Then either $\beta_r(M_k) = \beta_r(M_{k-1}) + 1$ $(*_1)$

or $\beta_{r-1}(M_k) = \beta_{r-1}(M_{k-1}) - 1$. $(*_2)$.

Def

In case $(*_1)$, the critical pt is called $\boxed{\text{positive}}$ and

" " $(*_2)$, " " " " " $\boxed{\text{negative}}$.

Def

$f: M \to \mathbb{R}$ Morse, M: d-mfd.

s_i, t_i: as before.

Via persistence intervals of the filtration

$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m$, we have $\boxed{\text{persistence pairings}}$ as defined below:

If $[i, j)$ is a finite pers. interval, pair i w/ j.

A homology class that never dies is called an $\boxed{\text{essential homology class}}$.

Note: essential classes are not paired.

Goal: Extend the "pairing" concept above to essential classes.

Cohomology and relative homology.

$\boxed{\text{Thm (Poincaré duality)}}$

M: oriented closed d-mfd. (closed: cpt w/o bd.)

Then, for $0 \le r \le d$, $\exists$ canonical iso.

$$H_r(M) \cong H^{d-r}(M).$$

$\boxed{\text{Thm (Lefschetz duality)}}$

M: oriented cpt d-mfd w/ bd.

Then, for $0 \le r \le d$, $\exists$ canonical iso.

$$H^{d-r}(M) \cong H_r(M, \partial M).$$

By Poincaré duality, we have (M: oriented closed d-mfd)

$$0 = H_r(M_0) \to \cdots \to H_r(M_m)$$
$$\xrightarrow{\cong} H^{d-r}(M_m) \to \cdots \to H^{d-r}(M_0) = 0 \quad (*), \text{ where}$$

$M_0 = f^{-1}((-\infty, t_0]) = \emptyset$, $M_m = M$.

For $0 \le k \le m$, $M^{m-k} := f^{-1}([t_k, \infty))$.

By Lefschetz duality,

$$H^{d-r}(M_k) \cong H_r(M_k, \partial M_k).$$

By Excision,

$$H_r(M_k, \partial M_k) \cong H_r(M, M^{m-k}).$$

Hence, ⊛ can be rewritten as :

$$\boxed{\begin{aligned} 0 = H_r(M_0) \to \cdots \to H_r(M_m) \\ \xrightarrow{\cong} H_r(M, M^0) \to \cdots \to H_r(M, M^m) = 0. \end{aligned}} \quad (\hat{*})$$

Thm (EP duality).

$f: X \to \mathbb{R}$ Morse function, X : cpt d-mfd.

Then

$$\mathrm{Ord}_p(f) = \mathrm{Rel}^T_{d-p}(f), \text{ and}$$

$$\mathrm{Ext}_p(f) = \mathrm{Ext}^T_{d-p}(f), \ \forall\ 0 \le p \le d,$$

where $\boxed{T}$ = reflect'n w.r.t. main diagonal $\Delta = \{(x,x) \mid x \in \mathbb{R}\}$.

(i.e. $(x,y) \mapsto (y,x)$).

Thm (EP Symmetry).

$f: X \to \mathbb{R}$ Morse funct'n, X : cpt d-mfd.

Then

$$\mathrm{Ord}_r(f) = \mathrm{Ord}^R_{d-r-1}(-f)$$

$$\mathrm{Rel}_r(f) = \mathrm{Rel}^R_{d-r-1}(-f)$$

$$\mathrm{Ext}_r(f) = \mathrm{Ext}^O_{d-r}(-f), \text{ where}$$

$\boxed{R}$ = reflect'n w.r.t. minor diagonal

(i.e. $(x,y) \mapsto (-y,-x)$).

$\boxed{O}$ = reflect'n w.r.t. origin.

(i.e. $(x,y) \mapsto (-x,-y)$.)

Ref : Parametrized Homology via Zigzag Persistence. [2016] [G. Carlsson, etal]. $\boxed{\text{For } 2/17}$

1. Introduction.

Def $f: X \to \mathbb{R}$, where X : topo. sp., f : cont.

The pair (X, f) : $\boxed{\mathbb{R}\text{-space}}$, denoted $\boxed{\mathbb{X}}$

$\boxed{X^a} := f^{-1}(-\infty, a]$, $\boxed{X_a} := f^{-1}[a, \infty)$.

2. Algebraic Tools.

Def $\mathbb{V}$: zigzag module.

$\boxed{\text{shape}}$ of $\mathbb{V}$:= type of $\mathbb{V}$. (i.e. use the term "shape" instead of "type").

multiplicity of $[p,q]$ in $\mathbb{V}$ is denoted by $\boxed{\langle [p,q] \mid \mathbb{V} \rangle}$.

Thm 2.4 (Restriction Principle)

$\mathbb{V}$: zigzag module w/ two consecutive maps in the same direction.

$$V_1 \leftrightarrow V_2 \leftrightarrow \cdots \leftrightarrow V_{k-1} \xrightarrow{g} V_k \xrightarrow{h} V_{k+1} \leftrightarrow \cdots \leftrightarrow V_n.$$

$\mathbb{W}$:= the following zigzag module :

$$V_1 \leftrightarrow V_2 \leftrightarrow \cdots \leftrightarrow V_{k-1} \xrightarrow{hg} V_{k+1} \leftrightarrow \cdots \leftrightarrow V_n.$$

$[p,q]$: interval over index set of $\mathbb{W}$.

(i.e. $p \ne k \ne q$).

Then

$$\langle [p,q] \mid \mathbb{W} \rangle = \sum_{[\hat{p},\hat{q}]} \langle [\hat{p},\hat{q}] \mid \mathbb{V} \rangle, \text{ where}$$

the $\sum$ is over $[\hat{p},\hat{q}]$ s.t. $[\hat{p},\hat{q}]|_{\mathbb{W}} = [p,q]$.

$[\hat{p},\hat{q}]|_{\mathbb{W}} := \{ i \in \{1, \dots, k-1, k+1, \dots, n\} \mid \hat{p} \le i \le \hat{q} \}$.

Ex :

$$\mathbb{V} = V_1 \to V_2 \to V_3 \leftarrow V_4 \leftarrow V_5.$$

$$\mathbb{V}_{1,2,3,5} := V_1 \to V_2 \to V_3 \longleftarrow V_5.$$

$$\mathbb{V}_{1,3,4,5} := V_1 \longrightarrow V_3 \leftarrow V_4 \leftarrow V_5.$$

Then

$$\langle \circ\!-\!\circ\!-\!\bullet\!\!-\!\!-\!\!\bullet \mid \mathbb{V}_{1,2,3,5} \rangle = \langle \circ\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet \mid \mathbb{V} \rangle.$$

$$\langle \circ\!\!-\!\!-\!\!\circ\!-\!\bullet\!-\!\bullet \mid \mathbb{V}_{1,3,4,5} \rangle = \langle \circ\!-\!\circ\!-\!\bullet\!-\!\bullet\!-\!\bullet \mid \mathbb{V} \rangle + \langle \circ\!-\!\circ\!-\!\circ\!-\!\bullet\!-\!\bullet \mid \mathbb{V} \rangle.$$

Thm 2.6 (Diamond Principle)

$\mathbb{V}^+$, $\mathbb{V}^-$, ...

Rmk :

① No informat'n about $\langle [k,k] \mid \mathbb{V}^+ \rangle$ or $\langle [k,k] \mid \mathbb{V}^- \rangle$.

② In case M-V. thm holds, $\langle [k,k] \mid \mathbb{V}^{\cup} \rangle = \langle [k,k] \mid \mathbb{V}^{\cap} \rangle^+$.

However, the M-V. thm. is NOT always applicable.

Since X^a, X_a are indexed over $\mathbb{R}$, the interval decomposition may contain open, closed, half-open intervals. The following notations may make life easier:

Def

$\boxed{\text{Decorate real numbers}}$ w/ a $\boxed{+}$ or $\boxed{-}$ supscript.

Superscript $\boxed{*}$ may be used for an unspecified decorat'n.

$\boxed{\text{Points w/ ticks}}$ are used to denote decorated pairs in persistence diagram:

interval	decorated pair	point w/ tick
(p,q)	(p^+, q^-)	
$(p,q]$	(p^+, q^+)	
$[p,q)$	(p^-, q^-)	
$[p,q]$	(p^-, q^+).	

Def

$\boxed{\mathcal{H}} := \{(p,q) \mid -\infty \le p < q \le \infty\}$, called the $\boxed{\text{extended half-space}}$.

$\boxed{\text{(decorated) persistence diagram}}$

:= pers. diagram w/ ticks.

$\boxed{\text{undecorated persistence diagram}}$

:= " " w/o ticks.

$\boxed{\mathrm{Rect}(\mathcal{H})} := \{[a,b]\times[c,d] \subseteq \mathcal{H} \mid -\infty \le a<b<c<d \le +\infty\}$.

A $\boxed{\text{rectangle measure}}$ or $\boxed{\text{r-measure}}$ on $\mathcal{H}$ is a func'n

$$\mu : \mathrm{Rect}(\mathcal{H}) \to \{0,1,2,\dots\} \cup \{\infty\}. \text{ s.t.}$$

μ : additive w.r.t. splitting a rec. vertically or horizontally into two rec.'s.

Rmk:

μ: monotone w.r.t. inclusion of rec's.

$\boxed{\text{Thm 2.7 (Equivalence Theorem)}}$

There is a biject'n b/w

(1) Finite r-measures μ on $\mathcal{H}$, and

(2) Locally finite multisets A of decorated pts in $\mathcal{H}$,

where $\boxed{\text{"finite"}}$ in (1) means $\mu(R) < \infty$, $\forall R \in \mathrm{Rect}(\mathcal{H})$,

$\boxed{\text{"locally finite"}}$ in (2) means $\mathrm{card}(A|_R) < \infty$, $\forall R \in \mathrm{Rect}(\mathcal{H})$.

Explicitly,

- for a multiset A, the corresponding μ is given by $\mu(R) = \mathrm{card}(A|_R)$.
- for an r-measure μ, define the multiset A w/ multiplicity func'n

$$m_A(p^*, q^*) = \min\{\mu(R) \mid R \in \mathrm{Rect}(\mathcal{H}) \text{ w/ } (p^*, q^*) \in R\}.$$

Rmk:

$\because$ r-mea.: monotone

$\therefore$ $m_A(p^*, q^*)$ can be calculated as a limit.

E.g. $m_A(p^-, q^+) = \lim_{\varepsilon \to 0^+} \mu([p-\varepsilon, p] \times [q, q+\varepsilon])$.

Moreover, since μ takes values in natural numbers, the expression inside the limit stabilizes for sufficiently small $\varepsilon > 0$.

Def

μ: finite r-measure.

The $\boxed{\text{decorated diagram}}\boxed{\text{of } \mu}$, denoted $\mathrm{Dgm}(\mu)$, is the multiset of decorated pts corresponding to μ.

The $\boxed{\text{undecorated diagram}}\boxed{\text{of } \mu}$, denoted $\mathrm{Dgm}_u(\mu)$, is also thus defined.

Def (slightly different from that in the paper).

μ: r-measure. (no necessarily finite).

The $\boxed{\text{finite support}}$ of μ is defined to be the set $\{x \in \mathcal{H} \mid \exists \text{ rec. } R \in \mathrm{Rect}(\mathcal{H}) \text{ s.t. } x \in R \text{ and } \mu(R) < \infty\}$.

3. Parametrized Homology.

3.1. Four Measures.

Note:

Given $R = [a,b]\times[c,d] \subseteq \mathcal{H}$, where $-\infty \le a < b < c < d \le \infty$ (i.e. an arbitrary "non-degenerate" rectangle.)

Decorated pts in R correspond exactly to intervals that

(1) are supported over $[b,c]$ (i.e. contains $[b,c]$) and

(2) do not reach either end of (a,b) (i.e. contained by (a,b)).

Thus, for an $\mathbb{R}$-space $f: X \to \mathbb{R}$, we define:

Def (X,f): $\mathbb{R}$-space. $-\infty \le a < b < c < d \le \infty$. $\boxed{\text{For } 2/21}$

H: a homology functor w/ field coeff.

$$\boxed{\mathbb{X}_{\{a,b,c,d\}}} := X_a^a \rightarrow X_a^b \leftarrow X_b^b \rightarrow X_b^c \leftarrow X_c^c \rightarrow X_c^d \leftarrow X_d^d.$$

$$\boxed{H\mathbb{X}_{\{a,b,c,d\}}} := H(X_a^a) \rightarrow H(X_a^b) \leftarrow H(X_b^b) \rightarrow H(X_b^c) \leftarrow H(X_c^c) \rightarrow H(X_c^d) \leftarrow H(X_d^d)$$

To analyze features inside $R = [a,b]\times[c,d]$, define the following four quantities:

$\boxed{\mu^{\wedge}_{HX}(R)} = \langle$  $\mid H\mathbb{X}_{\{a,b,c,d\}}\rangle.$

$\boxed{\mu^{\shortparallel}_{HX}(R)} = \langle$  $\mid H\mathbb{X}_{\{a,b,c,d\}}\rangle.$

$\boxed{\mu^{/\!/}_{HX}(R)} = \langle$  $\mid H\mathbb{X}_{\{a,b,c,d\}}\rangle.$

$\boxed{\mu^{\vee}_{HX}(R)} = \langle$  $\mid H\mathbb{X}_{\{a,b,c,d\}}\rangle.$

Rmk:

① $\boxed{X_{\infty}^{\infty}} := \phi =: \boxed{X_{-\infty}^{-\infty}}$

② $\mu^{\boxed{\mathbb{X}}}_{H\mathbb{X}}$ would be used to represent an unspecified one of the 4 quantities above.

Prop 3.1 (Coordinate-Reversal Symmetry)

$\mathbb{X} = (X, f)$, $R = [a,b]\times[c,d] \in \mathrm{Rect}(\mathcal{H})$.

$\overline{\mathbb{X}} := (X, -f)$, $\overline{R} := [-d,-c]\times[-b,-a]$.

Then

$$\mu^{\wedge}_{H\overline{\mathbb{X}}}(\overline{R}) = \mu^{\wedge}_{H\mathbb{X}}(R).$$

$$\mu^{\shortparallel}_{H\overline{\mathbb{X}}}(\overline{R}) = \mu^{\shortparallel}_{H\mathbb{X}}(R)$$

$$\mu^{\shortparallel}_{H\overline{\mathbb{X}}}(\overline{R}) = \mu^{/\!/}_{H\mathbb{X}}(R)$$

$$\mu^{\vee}_{H\overline{\mathbb{X}}}(\overline{R}) = \mu^{\vee}_{H\mathbb{X}}(R)$$

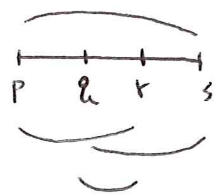
Goal: Identify conditions under which the four functions $\mu^{\mathbb{X}}_{H\mathbb{X}}$ are finite r-measures.

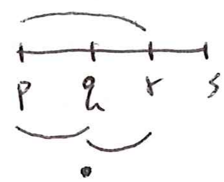
Outline: Additivity (S 3.2, 3.3), Finiteness (S 3.4).

3.2 Tautness.

For $p<q<r<s$, consider the following diamonds:

$$\boxed{\begin{tikzcd} & H_k(X_p^s) & \\ H_k(X_p^r) \arrow[ur] & & H_k(X_q^s) \arrow[ul] \\ & H_k(X_q^r) \arrow[ul] \arrow[ur] & \end{tikzcd}} \qquad \boxed{\begin{tikzcd} & H_k(X_p^r) & \\ H_k(X_p^q) \arrow[ur] & & H_k(X_q^r) \arrow[ul] \\ & H_k(X_q^q) \arrow[ul] \arrow[ur] & \end{tikzcd}}$$





$\because \mathring{X}_p^r \cup \mathring{X}_q^s = X_p^s$ (interior w.r.t. X_p^s)

$\therefore$ M-V. is applicable.

$\because \mathring{X}_p^q \cup \mathring{X}_q^r \neq X_p^r$ (interior w.r.t. X_p^r)

$\therefore$ M-V is NOT applicable.

Thus, the left diamond is exact while the right 〃 〃 NOT necessarily exact.

Q: Under what condition is the right diamond exact?

Def

U: a nbd of X_q^q.

$A := U\cap X^q = U \cap f^{-1}((-\infty, q])$, called the $\boxed{\text{lower-nbd}}$.

$B := U \cap X_q = U\cap f^{-1}([q,\infty))$, called the $\boxed{\text{upper-nbd}}$.

Then $U = A\cup B$ and $X_q^q = A\cap B$.

Let's look for conditions under which the following diamond is exact:

$$\begin{tikzcd} & H_k(U) & \\ H_k(A) \arrow[ur] & & H_k(B) \arrow[ul] \\ & H_k(A\cap B). \arrow[ul] \arrow[ur] & \end{tikzcd}$$

There are two natural pair inclusions:

$(A, X_q^q) \hookrightarrow (U, B)$.

$(B, X_q^q) \hookrightarrow (U, A)$.

Prop 3.2

The following two conditions are equivalent:

(1) $\alpha_{k+1}: H_{k+1}(A, X_q^q) \to H_{k+1}(U,B)$ epi, and
$\alpha_k: H_k(A, X_q^q) \to H_k(U,B)$ mono.

(2) $\beta_{k+1}: H_{k+1}(B, X_q^q) \to H_{k+1}(U,A)$ epi, and
$\beta_k: H_k(B, X_q^q) \to H_k(U,A)$ mono.

<Pf>

Criss-cross two long exact seq. as follows:

$$\begin{tikzcd}[column sep=small]
H_{k+1}(A,X^{h}_{q_h}) \arrow[rr, "\alpha_{k+1}", "\text{epi.}"'] \arrow[dr, "⑨"'] & & H_{k+1}(U,B) \arrow[rr, "② \, \partial"] & & H_k(B,X^{h}_{q_h}) \arrow[rr, "\beta_k"] \arrow[dr, "⑤"'] & & H_k(U,A) \\
& H_{k+1}(U,X^{h}_{q_h}) \arrow[ur, "①"] \arrow[dr, "⑧"] & & & & H_k(U,X^{h}_{q_h}) \arrow[ur, "⑥"'] \arrow[dr, "⑦"] & \\
H_{k+1}(B,X^{h}_{q_h}) \arrow[rr, "\beta_{k+1}"'] \arrow[ur, "⑩"] & & H_{k+1}(U,A) \arrow[rr, "\partial \, ④"'] & & H_k(A,X^{h}_{q_h}) \arrow[rr, "\alpha_k"', "\text{mono.}"] \arrow[ur, "③"] & & H_k(U,B)
\end{tikzcd}$$

By symmetry, proving (1) $\Rightarrow$ (2) is sufficient.

The diagram is clearly commutative. $\boxed{\text{The following is diagram chasing.}}$

$\because \alpha_{k+1}$: epi. $\therefore$ ① : epi. $\Rightarrow$ ② $= 0$.

$\because \alpha_k$: mono. $\therefore$ ③ : mono. $\Rightarrow$ ④ $= 0$.

<u>claim</u> : β_k : mono.

Given $v \in H_k(B,X^{h}_{q_h})$ s.t. $\beta_k(v) = 0$.

Then ⑥(⑤(v)) $= 0$. $\Rightarrow$ ⑤$(v) \in \ker$(⑥) $= \mathrm{im}$(③).

Let $v' \in H_k(A,X^{h}_{q_h})$ w/ ③(v') = ⑤(v).

By long exactness, ⑦(⑤(v)) $= 0$.

$\Rightarrow$ ⑦(③(v')) $= 0$. i.e. $\alpha_k(v') = 0$.

$\because \alpha_k$: mono. $\therefore v' = 0$. $\Rightarrow$ ⑤(v) = ③(v') $= 0$.

$\Rightarrow v \in \ker$(⑤) $= \mathrm{im}$(②) $= 0$. $\Rightarrow v = 0$.

Thus, β_k : mono. #

<u>claim</u> : β_{k+1} : epi.

Given $w \in H_{k+1}(U,A)$.

$\because$ ④$(w) = 0$ $\therefore w \in \ker$(④) $= \mathrm{im}$(⑧).

Let $w' \in H_{k+1}(U,X^{h}_{q_h})$ w/ ⑧$(w') = w$.

$\because \alpha_{k+1}$: epi. $\therefore \exists\ w'' \in H_{k+1}(A,X^{h}_{q_h})$ s.t. $\alpha_{k+1}(w'')$ = ①(w'). ⊛

Let $w_2' =$ ⑨(w'').

Then $\alpha_{k+1}(w'')$ = ①(⑨(w'')) = ①(w_2').

By ⊛, ①(w_2') = ①(w'). $\Rightarrow$ ①$(w' - w_2') = 0$.

$\Rightarrow w' - w_2' \in \ker$(①) $= \mathrm{im}$(⑩), say ⑩$(w''') = w' - w_2'$

Then $\beta_{k+1}(w''')$ = ⑧(⑩$(w'''))$ = ⑧$(w' - w_2')$

$= w -$ ⑧(⑨$(w''))$ $= w$. ⑧∘⑨ $= 0$

Thus, β_{k+1} : epi. #

<u>Def</u> In either of the above case, $X^{h}_{q_h}$ is said to be $\boxed{H_k\text{-taut in } U}$.