1. (a) Apply the Newton - Cotes formula with $h = 1/4$, $n=4$.

$$\alpha_i = \int_0^4 \varphi_i(t)\, h\, dt \qquad \varphi_i(t) = \prod_{\substack{k=0 \\ k\neq i}}^{4} \frac{t-k}{i-k}, \quad i=0,1,\ldots,4.$$

$$\alpha_0 = \frac{h}{24}\int_0^4 (t-1)(t-2)(t-3)(t-4)\,dt = 7/90$$

$$\alpha_1 = -\frac{h}{6}\int_0^4 t(t-2)(t-3)(t-4)\,dt = 32/90$$

$$\alpha_2 = \frac{h}{4}\int_0^4 t(t-1)(t-3)(t-4)\,dt = 12/90$$

$\alpha_3 = \alpha_1$, $\alpha_4 = \alpha_0$ by symmetry.

Hence, the 4th order Newton's cote is the Milne's rule.

(b) $$E_4(f) = \frac{M_4}{6!} h^7 f^{(6)}(\xi), \qquad \Pi_5(t) = \prod_{i=0}^{4}(t-i)$$

$$M_4 = \int_0^4 t\,\Pi_5(t)\,dt = \int_0^4 t^2(t-1)(t-2)(t-3)(t-4)\,dt = -\frac{128}{21}$$

so $$c = M_4 h^7 = -\frac{128}{21}\frac{1}{4^7} = \frac{-1}{2688} = -3.7202\times10^{-4}$$

1, (c) Composite Milne's

$$M(f;a,b) = \sum_{i=0}^{n-1} M(f, x_i, x_{i+1}) \quad , \quad h_i = x_{i+1} - x_i$$

$$= \frac{1}{90}\sum_{i=0}^{n-1} h_i \left(7f(x_i) + 32 f\left(x_i + \frac{h_i}{4}\right) + 12 f\left(x_i+\frac{h_i}{2}\right) + 32 f\left(x_i + \frac{3h_i}{4}\right) + 7 f(x_{i+1})\right)$$

Error

$$E_4(f) = -\frac{8}{945} f^{(6)}(\eta_i)\, h_i^7$$ on each $[x_i, x_{i+1}]$, so

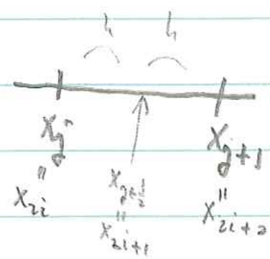
$\boxed{\text{assume } h = h_i \text{ so } \frac{b-a}{n} = h}$

$$|E(f)| \le \frac{8}{945}\sum_{i=0}^{n-1} h^7 |f^{(6)}(\eta_i)|$$

$$= \frac{8}{945} h^7 n |f^{(6)}(\xi)| = \frac{8(b-a)h^6 |f^{(6)}(\xi)|}{945}.$$

(d) let $$T_n^{(0)} = \sum_{i=0}^{n-1}\frac{h}{2}\left(f(x_i)+f(x_{i+1})\right)$$

$$T_{\frac{n}{2}}^{(0)} = \sum_{j=1}^{\frac{n}{2}-1} h \left(f(x_j) + f(x_{j+1})\right)$$

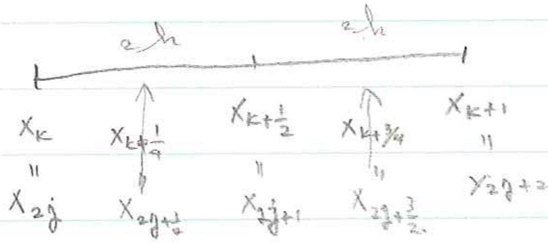


$$T_n^{(1)} = \frac{4T_n^{(0)} - T_{\frac{n}{2}}^{(0)}}{3} = \frac{1}{3}\sum_{j=0}^{\frac{n}{2}-1} \Big[\frac{4h}{2}\left(f(x_j)+f(x_{j+\frac{1}{2}})\right) + \frac{4h}{2}\left(f(x_{j+\frac{1}{2}}) + f(x_{j+1})\right)$$
$$- h\left(f(x_j)+f(x_{j+1})\right)$$

$$= \frac{h}{6}\sum_{j=0}^{\frac{n}{2}-1}\left[f(x_j) + 4f(x_{j+\frac{1}{2}}) + f(x_{j+1})\right] \longleftarrow \text{Composite Simpson}$$

Repeat this with

$$T_{\frac{n}{2}}^{(1)}(f) = \frac{1}{6} \sum_{k=0}^{n/4-1} 2h \left[f(x_k) + 4f(x_{k+\frac{1}{2}}) + f(x_{k+1}) \right]$$



$$T_n^{(2)} f = \frac{4^2 T_n^{(1)} f - T_{\frac{n}{2}}^{(1)} f}{15}.$$

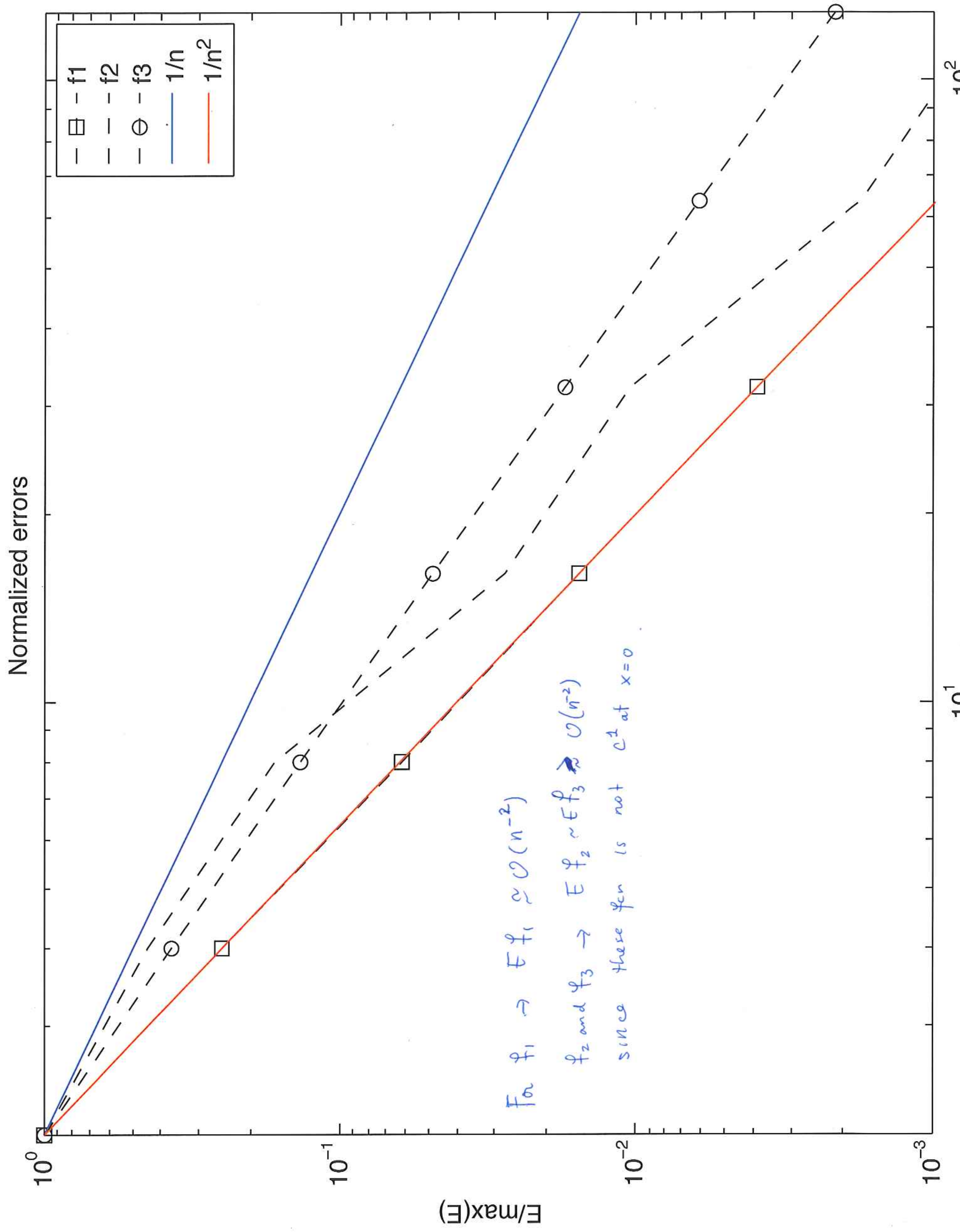
$$= \frac{1}{90} \sum_{k=0}^{\frac{n}{4}-1} 4^2 (2h) \Big(\left[f(x_k) + 4f(x_{k+\frac{1}{4}}) + f(x_{k+\frac{1}{2}}) \right] + \left[f(x_{k+\frac{1}{2}}) + 4f(x_{k+\frac{3}{4}}) + f(x_{k+1}) \right] \Big)$$

$$- 4h \left[f(x_k) + 4f(x_{k+\frac{1}{2}}) + f(x_{k+1}) \right]$$

$$= \frac{h}{90} \sum_{k=0}^{\frac{n}{4}-1} \left[28 f(x_k) + 128 f(x_{k+\frac{1}{4}}) + 48 f(x_{k+\frac{1}{2}}) + 128 f(x_{k+\frac{3}{4}}) + 28 f(x_{k+1}) \right]$$

for $h = 1/4$, we obtain the Milne's rule //.

$$E_n^{(2)}(f) = \frac{4^2 E_n^{(1)}(f) - E_{n/2}^{(1)}(f)}{15} = O(h^6). //$$



For $f_1 \to E f_1 \simeq O(n^{-2})$

f_2 and $f_3 \to E f_2 \sim E f_3 \gtrsim O(n^{-2})$

since these fcn is not C^1 at $x=0$.

Problem 3 solution. (5)


```
>> I

I =

    0.8427    1.0100    0.6667

>> driver_adaptquad

E =

   2.4452e-05


q1 =

    0.8427


E =

    0.0156


E =

   7.4015e-18


E =

    0.0011


q2 =

    1.0167


E =

    0.0012


q3 =

    0.6565

>>
```

1-step of Simpson is sufficent for obtaing $O(10^{-2})$ accuracy.

one step error > 0.01

2 steps of Simpson rule is used.

1 step of Simpson

(b) by means of the composite Simpson's rule,

For the composite Simpson's rule, we must be clear about the definition of N. Again, allow $N + \frac{b-a}{h}$ and take the uniform grid $x_j = a + jh$. The Simpson's rule requires 3 points, and thus the first interval for the composite Simpson's rule is $[x_0, x_2]$. The second is $[x_2, x_4]$, etc. Obviously, N must be even. Using this definition, the error is given by

$$E(f) = -\frac{f^{(4)}(\xi)}{180}(b-a)h^4.$$

Using Maple, the fourth derivative is given by

$$f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$$

and the maximum absolute value of this derivative is approximately 28.429 (occuring at $x \approx 0.852$). Thus

$$|E(f)| \leq \frac{28.249}{180}h^4 < 5 \times 10^{-6}$$

implies that $h < 0.07501$. Thus, the smallest N that will guarantee the desired accuracy is $N = 14$. While the formula for the composite Simpson's rule is slightly more involved than that of the composite trapezoid rule, we see here it requires much smaller N for the same accuracy, and is a better choice.

(c) by means of the composite corrected trapezoid rule.

As obtained in problem 1(c), the error for this rule is

$$E(f) = \frac{f^{(4)}(\xi)}{720}(b-a)h^4.$$

Using the same bound on the fourth derivative as obtained in part (b), this gives

$$E(f) \leq \frac{28.429}{720}h^4 < 5 \times 10^{-6}$$

which is true whenever $h < 0.1060$. Thus, this rule requires $N = 10$ subintervals to guarantee the desired accuracy. This is fewer intervals than the composite Simpson's rule, but comes at the price of knowing the function derivate at the endpoints.

~~3~~ 4. Define $S_n(x) = \frac{1}{n+1}T'_{n+1}(x), n \geq 0$, with $T_{n+1}(x)$ the Chebyshev polynomial of degree $n+1$. The polynomials $S_n(x)$ are called *Chebyshev polynomials of the second kind.*

(a) Show that $\{S_n(x) \mid n \geq 0\}$ is an orthogonal family on $[-1, 1]$ with respect to the weight function $w(x) = \sqrt{1-x^2}$.

First note that an explicit formula for the Cheybshev polynomials is

$$T_n(x) = \cos(n \arccos(x)).$$

Thus

$$T'_n(x) = \sin(n \arccos(x))n\frac{1}{\sqrt{1-x^2}}$$

and $S_n(x)$ can be written explicitely as

$$S_n(x) = \sin((n+1)\arccos(x))\frac{1}{\sqrt{1-x^2}}.$$

Using this definition, the inner product becomes

$$\begin{aligned}
\langle S_m, S_n \rangle &= \int_{-1}^{1} S_m(x) S_n(x) \sqrt{1-x^2} dx \\
&= \int_{-1}^{1} \sin((m+1)\arccos(x)) \sin((n+1)\arccos(x)) \frac{1}{1-x^2} dx.
\end{aligned}$$

Introduce the change of variables $\theta = \arccos x$. Thus $d\theta = -1/\sqrt{1-x^2}dx$ and the limits of integration give $x = -1 \Longrightarrow \theta = \pi, \quad x = 1 \Longrightarrow \theta = 0$. Applying this gives

$$\begin{aligned}
\langle S_m, S_n \rangle &= -\int_{\pi}^{0} \sin((m+1)\theta) \sin((n+1)\theta) d\theta \\
&= \int_{0}^{\pi} \sin((m+1)\theta) \sin((n+1)\theta) d\theta.
\end{aligned}$$

If we assume $m \neq n$ and then apply the trigonometric identity

$$\sin(s)\sin(t) = \frac{1}{2}(\cos(s-t) - \cos(s+t))$$

we obtain

$$\begin{aligned}
\langle S_m, S_n \rangle &= \frac{1}{2}\int_{0}^{\pi} \left(\cos((m-n)\theta) - \cos((m+n+2)\theta)\right) d\theta \\
&= \frac{1}{2}\left(\frac{1}{m-n} \sin((m-n)\theta)\Big|_{\theta=0}^{\theta=\pi} - \frac{1}{m+n+2} \sin((m+n+2)\theta)\Big|_{\theta=0}^{\theta=\pi} \right) \\
&= 0
\end{aligned}$$

since $m-n$ and $m+n+2$ are integers. If however, $m = n$, then

$$\begin{aligned}
\langle S_m, S_n \rangle &= \int_{0}^{\pi} \sin^2((n+1)\theta) d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

since n is an integer. Thus,

$$\langle S_m, S_n \rangle = \begin{cases} \frac{\pi}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

and $\{S_n \mid n \geq 0\}$ is an orthogonal family with respect to the weight function $w(x) = \sqrt{1-x^2}$.

(b) Show that the family $\{S_n(x)\}$ satisfies the same triple recursion relation as the family $\{T_n(x)\}$.

The recursion relation for $\{T_n(x)\}$ is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 1, \qquad T_1(x) = x.$$

Recall, as shown in part (a), that

$$S_n(x) = \sin((n+1)\arccos(x))\frac{1}{\sqrt{1-x^2}}.$$

Let $\theta = \arccos(x)$. Then

$$S_n(\cos\theta) = \sin((n+1)\theta)\frac{1}{\sqrt{1-\cos^2\theta}} = \sin((n+1)\theta)\frac{1}{\sin\theta}$$

since $\sin^2\theta + \cos^2\theta = 1$ for all θ. Applying the trigonometric identity

$$\sin(s-t) = \sin(s)\cos(t) - \cos(s)\sin(t)$$

gives

$$S_n(\cos\theta) = (\sin(n\theta)\cos(-\theta) - \cos(n\theta)\sin(-\theta))\,\frac{1}{\sin(\theta)}$$

$$= \sin(n\theta)\frac{\cos(\theta)}{\sin(\theta)} + \cos(n\theta).$$

This same process can be performed for $S_{n-2}(x)$, yielding

$$S_{n-2}(\cos\theta) = \sin((n-1)\theta)\frac{1}{\sin(\theta)}$$

$$= (\sin(n\theta)\cos(\theta) - \cos(n\theta)\sin(\theta))\,\frac{1}{\sin(\theta)}$$

$$= \sin(n\theta)\frac{\cos(\theta)}{\sin(\theta)} - \cos(n\theta).$$

Adding these togther we get

$$S_n(\cos\theta) + S_{n-2}(\cos\theta) = 2\sin(n\theta)\frac{\cos(\theta)}{\sin(\theta)}$$

and thus

$$S_n(\cos\theta) = 2\sin(n\theta)\frac{\cos(\theta)}{\sin(\theta)} - S_{n-2}(\cos\theta).$$

Returning to the original variable x gives

$$S_n(x) = 2\sin(n\arccos x)\frac{\cos(\arccos x)}{\sin(\arccos x)} - S_{n-2}(x)$$

$$= 2\sin(n\arccos x)\frac{x}{\sqrt{1-x^2}} - S_{n-2}(x).$$

However, by definition $S_{n-1} = \sin(n \arccos x)/\sqrt{1-x^2}$ and so

$$S_n(x) = 2xS_{n-1}(x) - S_{n-2}(x).$$

This is still true if we shift the index up by one, and thus

$$S_{n+1}(x) = 2xS_n(x) - S_{n-1}(x)$$

which is identical to the recurrence relation for $T_{n+1}(x)$. Note, however, that the values of $S_0(x)$ and $S_1(x)$ are not equivalent, as $S_0(x) = T_1'(x) = 1$ but $S_1(x) = T_2'(x)/2 = (2x^2-1)'/2 = 2x$. Thus, the polynomials are not identical, since the starting values for the recurrence are different.

(c) Given $f \in C[-1,1]$, solve the problem

$$\min \int_{-1}^{1} \sqrt{1-x^2}[f(x) - p_n(x)]^2 dx$$

where $p_n(x)$ is allowed to range over all polynomials of degree $\leq n$.

The integral $\int_{-1}^{1} \sqrt{1-x^2}[f(x)-p_n(x)]^2$ is minimized precisely when $(\int_{-1}^{1} \sqrt{1-x^2}[f(x)-p_n(x)]^2)^{1/2}$ is minimized, and thus this is a weighted least squares problem with weight $w(x) = \sqrt{1-x^2}$. For any basis $\{\phi_0, \ldots, \phi_n\}$ of $\mathbb{P}_n$ (the space of all polynomials of degree $\leq n$), the solution

$$p_n(x) = \sum_{j=0}^{n} c_j \phi_j(x)$$

is given by solving the normal equations

$$\sum_{j=0}^{n} \langle \phi_j, \phi_k \rangle = \langle f, \phi_k \rangle \qquad \text{where } \langle u, v \rangle = \int_{-1}^{1} u(x)v(x)\sqrt{1-x^2}dx.$$

Since $\{S_n(x)\}$ is an orthogonal family of polynomials and $S_n(x)$ is a polynomial of degree n, the set $\{S_0, \ldots, S_n\}$ is a basis for $\mathbb{P}_n$ (this was proved in homework 3 problem 1). However, as shown in part (a),

$$\langle S_m, S_k \rangle = \begin{cases} \frac{\pi}{2} & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}$$

and so using this basis the normal equations simplify to

$$c_j = \frac{2}{\pi}\langle f, S_j \rangle = \frac{2}{\pi}\int_{-1}^{1} f(x)S_j(x)\sqrt{1-x^2}dx = \frac{2}{\pi}\int_{-1}^{1} f(x)\sin((j+1)\arccos(x))dx.$$

Thus, the minimum polynomial is given by

$$\begin{aligned} p_n(x) &= \frac{2}{\pi}\sum_{j=0}^{n}\left(\int_{-1}^{1} f(x)S_n(x)\sqrt{1-x^2}dx\right) S_j(x) \\ &= \frac{2}{\pi}\sum_{j=0}^{n}\left(\int_{-1}^{1} f(x)\sin((j+1)\arccos(x))dx\right)\sin((j+1)\arccos(x))\frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

(d) For the integral

$$I = \int_{-1}^{1} \sqrt{1-x^2} f(x)dx$$

with weight $w(x) = \sqrt{1-x^2}$, find explicit formulae for the nodes and weights of Gaussian quadrature formula. Also give the error formula.

The nodes for the n point Gaussian quadrature formula are precisely the zeros of $S_n(x)$. Recall

$$S_n(x) = \sin((n+1)\arccos(x))\frac{1}{\sqrt{1-x^2}}, \; x \in (-1,1) \quad \Longrightarrow \quad S_n(\cos\theta) = \sin((n+1)\theta)\frac{1}{\sin\theta}, \; \theta \in (0,\pi).$$

Consider $\theta_k = \pi k/(n+1)$ for $k = 1, 2, \ldots, n$. Then

$$S_n(\cos\theta_k) = \sin\left((n+1)\frac{\pi k}{n+1}\right)\frac{1}{\sin(\pi k/(n+1))} = \frac{\sin(\pi k)}{\sin(\pi k/(n+1))}.$$

$0 < \theta_k < \pi$ for all $k = 1, 2, \ldots, n$ and thus $\sin(\pi k/(n+1)) \neq 0$. However, since k is an integer, $\sin(\pi k) = 0$ for all k. This gives $S_n(\cos\theta_k) = 0$ for all k. Transforming back to the original variable gives

$$x_k = \cos\left(\frac{\pi k}{n+1}\right), \qquad k = 1, 2, \ldots, n$$

and $S_n(x_k) = 0$. Each x_k is unique and thus this gives n distinct zeros. But $S_n(x)$ is a degree n polynomial, and so $S_n(x)$ has exactly n zeros. Thus, the x_k values are the zeros of $S_n(x)$, i.e. The set $\{x_1, x_2, \ldots, x_n\}$ is the set of quadrature points.

Using these points, the weights are readily determined. Define

$$\ell_k(x) = \prod_{j=1, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}.$$

The quadrature weights A_k are given by

$$A_k = \int_{-1}^{1} \ell_k(x)\sqrt{1-x^2}dx.$$

This gives the approxiimation

$$I = \int_{-1}^{1} \sqrt{1-x^2} f(x)dx \approx \sum_{j=1}^{n} A_k f(x_k).$$

As derived in class, the error for general Gaussian quadrature on $[a,b]$, assuming $f \in C^{2n}[a,b]$, is given by

$$E(f) = \frac{f^{(2n)}(\xi)}{(2n)!}\|\widetilde{\Psi}_n\|^2_{L^2,w}$$

where $\widetilde{\Psi}_n$ is the monic orthogonal polynomial of degree n with respect to $w(x)$ and $\xi \in (a,b)$. So in this case $\widetilde{\Psi}_n = \widetilde{S}_n$. Since

$$S_{n+1}(x) = 2xS_n(x) - S_{n-1}(x), \qquad S_0(x) = 1, \qquad S_1(x) = 2x$$

the leading coefficient of S_n is twice that of S_{n-1} and is given by 2^n. Thus $\widetilde{S}_n = S_n/2^n$ and the error is

$$
\begin{aligned}
E(f) &= \frac{f^{(2n)}(\xi)}{(2n)!}\|\widetilde{S}_n\|^2_{L^2,w} \\
&= \frac{f^{(2n)}(\xi)}{(2n)!}\left\langle \frac{1}{2^n}S_n, \frac{1}{2^n}S_n \right\rangle \\
&= \frac{f^{(2n)}(\xi)}{(2n)!}\frac{1}{2^{2n}}\langle S_n, S_n\rangle \\
&= \frac{f^{(2n)}(\xi)}{(2n)!}\frac{1}{2^{2n}}\frac{\pi}{2} \\
&= \frac{\pi}{(2n)!\,2^{2n+1}} f^{(2n)}(\xi)
\end{aligned}
$$

for some $\xi \in (-1, 1)$.

4. The *Gauss-Lobatto* quadrature rule is a Gaussian formula for integrating $I(f) = \int_{-1}^{1} f(x)dx$ except that it includes ± 1 as two fixed abscissas, that is, it has the form

$$\int_{-1}^{1} f(x)dx \approx A_1 f(-1) + A_n f(1) + \sum_{j=2}^{n-1} A_j f(x_j) = Lo_n(f),$$

where the abscissas x_j, $j = 2, \ldots, n-1$ and weights $A_j, j = 1, \ldots n$ are chosen so that the formula $Lo_n(f)$ has the maximum possible degree of exactness.

(a) What is the precise degree of exactness of $Lo_n(f)$?

There are n weights and $n-2$ nodes to be determined, so we expect the degree of exactness to be $n + (n-2) - 1 = 2n - 3$. To verify this, we must derive the quadrature rule. To this end, let $p(x) \in \mathbb{P}_{n-3}$ be any polynomial of degree $\leq 2n - 3$. We may write

$$p(x) = g(x)(1 - x^2) + r(x)$$

for some $g(x) \in \mathbb{P}_{n-5}$ and $r(x) \in \mathbb{P}_1$. If the problem was only to solve

$$\int_{-1}^{1} g(x)(1 - x^2)dx$$

and $g(x)$ was known we could use orthogonal polynomials with respect to the weighting function $w(x) = 1 - x^2$ to perform the integration. In fact, we would set the quadrature nodes equal to the zeros of the degree $n-2$ orthogonal polynomial. We begin by selecting these as the free nodes, and then show that this is indeed the correct choice. The Jacobi polynomials $\{P_n^{(1,1)}\}$ (with $\alpha = \beta = 1$) are orthogonal with respect to $w(x) = 1 - x^2$ on $[-1, 1]$. Thus, let $x_2, \ldots, x_{n-1}$

5(a) Apply Hermite interpolation to

$$(x_0, y_0),\ \underbrace{(x_1, y_1), (x_1, y_1')}_{n_1 = 2},\ \underbrace{(x_2, y_2), (x_2, y_2')}_{n_2 = 2},\ (x_3, y_3) \qquad (*)$$

$n_0 = 1$, $n_3 = 1$

$\exists\ p \in \Pi_n$ where $n = \sum_{j=0}^{3} n_j - 1 = 5$

by uniqueness, we can write this polynomial as follows:

$$p(x) = \sum_{i=0}^{3} \sum_{k=0}^{n_i - 1} y_i^{(k)} L_{ik}(x) \quad \text{satisfies constraints } (*),$$

where

$$L_{00}(x) = \frac{(x-x_1)^2 (x-x_2)(x-x_3)}{(x_0-x_1)^2 (x_0-x_2)^2 (x_0-x_3)} \equiv h_0(x)$$

$$L_{11}(x) = \frac{(x-x_1)(x-x_0)(x-x_2)^2 (x-x_3)}{(x_1-x_0)(x_1-x_2)^2 (x_1-x_3)} \equiv g_1(x)$$

$$L_{10}(x) = \left(1 - l_{10}'(x_1)(x-x_1)\right) l_{10}(x) \equiv h_1(x)$$

$$L_{21}(x) = \frac{(x-x_2)(x-x_0)(x-x_1)^2 (x-x_3)}{(x_2-x_0)(x_2-x_1)^2 (x_2-x_3)} \equiv g_2(x)$$

$$L_{20}(x) = \left(1 - (x-x_2)\, l_{20}'(x_2)\right) l_{20}(x) \equiv h_2(x)$$

$$L_{30}(x) = \frac{(x-x_1)^2 (x-x_2)^2 (x-x_0)}{(x_3-x_1)^2 (x_3-x_2)^2 (x_3-x_0)} \equiv h_3(x)$$

It is easy to check that $g_i(x_j) = 0$, $g_i'(x_j) = \delta_{ij}$

$h_i(x_j) = \delta_{ij}$, $h_i'(x_j) = 0$ //

5(b) $x_0 = -1$, $x_3 = 1$ so

$$g_1(x) = \frac{(x-x_1)(x-x_2)^2(x^2-1)}{(x_1+1)(x_1-x_2)^2(x_1-1)} = \alpha (x-x_1)(x-x_2)^2(1-x^2)$$

where $\alpha = \dfrac{1}{(x_1-x_2)^2(1-x_1^2)}$.

Similarly, we can write.

$$g_2(x) = \beta(1-x^2)(x-x_1)^2(x-x_2) \quad \text{where} \quad \beta = \frac{1}{(1-x_2^2)(x_2-x_1)^2}$$

5(c)

$$I(p) = \int_{-1}^{1} p(x)\,dx.$$

$$= \sum_{i=0}^{3} p(x_i) \underbrace{\int_{-1}^{1} h_i(x)\,dx}_{=w_i} + \sum_{j=1}^{2} p'(x_j) \int_{-1}^{1} g_j(x)\,dx$$

$$I(p) = L_2(f) + \sum_{j=1}^{2} p'(x_j) \int_{-1}^{1} g_j(x)\,dx$$

so for $I(p) = L_2(f) \Rightarrow \int_{-1}^{1} g_j(x)\,dx = 0$, $j = 1,2$ //.

5(d) Let $q = (x-x_1)(x-x_2) \perp \Pi_1$ with weight $w(x) = 1-x^2$.

Hence $q \perp \alpha(x-x_2)$, i.e. $\langle q, \alpha(x-x_2)\rangle = 0 = \int_{-1}^{1} \alpha(x-x_1)(x-x_2)^2(1-x^2)\,dx$

$$= \int_{-1}^{1} g_1(x)\,dx \quad //.$$

Similarly $q \perp \beta(x-x_1)$, i.e, $\langle q, \beta(x-x_1)\rangle = 0 = \int_{-1}^{1} \beta(x-x_1)^2(x-x_2)(1-x^2)\,dx$

$$= \int_{-1}^{1} g_2(x)\,dx \quad //.$$

5(e) Let $\varphi_0(x) = 1$, $\varphi_1(x) = x$, $\varphi_2(x) = x^2$.

using the Gram-Schmidt or the triple recursion formula of previous HW, we obtain

$q_0(x) = 1$

$q_1(x) = x$

$q_2(x) = x^2 - 1/5$.

5(f) $q(x) = (x - x_1)(x - x_2)$ from (d)

$q_2(x) = q(x) = x^2 - 1/5$ so, $x_1 = -\frac{1}{\sqrt{5}}$, $x_2 = \frac{1}{\sqrt{5}}$ //.

5(g) Compute the weights $w_i = \int_{-1}^{1} h_i(x)\,dx$. where $h_i(x)$ from 5(a) //.

where $x_0 = -1$, $x_1 = -\frac{1}{\sqrt{5}}$, $x_2 = \frac{1}{\sqrt{5}}$, $x_3 = 1$

we find that $w_0 = w_3 = 1/6$, $w_1 = w_2 = 5/6$.

5(h) Recall the error formula for Hermite interpolation $n=5$, $f \in C^{(6)}$, we have

$$f(x) - p(x) = \omega(x) \frac{f^{(6)}(\xi(x))}{6!}, \quad \omega(x) = (x^2-1)(x^2-\tfrac{1}{5})^2.$$

So, the integration error is

$$E(f) = I(f) - I(p) = I(f-p) = \int_{-1}^{1} \frac{f^{(6)}(\xi(x))}{6!}\, w(x)\,dx$$

$$= -\frac{f^{(6)}(\eta)}{6!} \int_{-1}^{1} \underbrace{(1-x^2)(x^2-1/5)^2}_{\text{same sign in } (-1,1)}\,dx \quad //.$$

By MVT //.

Problem (6)

6(a) >> boxmuller

N =

100000 ← # of pts in the box

Q =

0.1000 } integrate with $h = 0.8147$.

6(b) ans =

20000 ← min sample to achieve 1% for $p = 1$

ans =

~~960000~~ 1.0667×10^5 ← min sample for $p = 2$

ans =

~~6.5384e+12~~ 7.3216×10^6. ← min sample for $p = 5$

Theory

$$E\left[\left| x^{2p} - E[x^{2p}] \right| \geq \frac{k\sqrt{Var(x^{2p})}}{\sqrt{n}} \right] < \frac{1}{k^2}$$

Note that

$$E[x^{2p}] = 1 \cdot 3 \cdot \ldots \cdot (2p-1).$$

$$Var[x^{2p}] = E[x^{4p}] - E[x^{2p}]^2$$

$$= 1 \cdot 3 \cdot \ldots \cdot (4p-1) - [1 \cdot 3 \cdot \ldots \cdot (2p-1)]^2$$

and we choose n s.t.

$$\frac{k\sqrt{Var(x^{2p})}}{\sqrt{n}} = 0.01\, E(x^{2p})$$

for $k = 1$ std deviation, we have

$$n = \frac{Var(x^{2p})}{10^{-4} E(x^{2p})}.$$

6(c) Err2 =

~~0.0047~~ ← error is within 1% using 20,000 samples

0.0159

Err4 =

~~0.0045~~ ← error is also within 1% using 960,000 samples

0.0350

Err10 =

~~1.7077~~ ← error is NOT within 1%

>> 3.1579 b/c we only use 10^7 samples which is way below the theoretical estimate from 6(b).

For larger p, we need more sample because the variance increases.

7 $\int_0^1 x^\alpha f(x)\,dx \simeq A f(0) + B\int_0^1 f(x)\,dx$, $\alpha > -1$, $\alpha \neq 0$

(a) let $f(x) = 1$ and $f(x) = x$

$$\int_0^1 x^\alpha\,dx = A + B = \frac{1}{\alpha+1}$$

$$\int_0^1 x^{\alpha-1}\,dx = \frac{1}{2}B = \frac{1}{\alpha}$$

$B = \frac{2}{\alpha}$, $A = \frac{1}{\alpha+1} - \frac{2}{\alpha} = -\frac{(\alpha+2)}{\alpha(\alpha+1)}$

(b) $Ef = \int_0^1 x^\alpha f(x)\,dx - Af(0) - B\int_0^1 f(x)\,dx$

$$K_1(t) = E_{(x)}\,(x-t)_+$$

$$= \int_t^1 x^\alpha (x-t)\,dx - A\underbrace{(0-t)_+}_{=0} - \frac{2}{\alpha}\int_t^1 (x-t)\,dx$$

$$= \frac{1}{\alpha+2}x^{\alpha+2} - \frac{t}{\alpha+1}x^{\alpha+1}\Big]_t^1 - \frac{1}{\alpha}(x-t)^2\Big|_t^1$$

$$= \frac{1}{\alpha+2}\left(1-t^{\alpha+2}\right) - \frac{t}{\alpha+1}\left(1-t^{\alpha+1}\right) - \frac{1}{\alpha}(1-t)^2$$

$K_1(0) = \frac{1}{\alpha+2} - \frac{1}{\alpha} = -\frac{2}{\alpha(\alpha+2)}$, $K_1(1) = 0$

If $\alpha > 0$, then $K_1(0) < 0$, $K_1(1) = 0$

$$\frac{\partial K_1}{\partial t} = \frac{2}{\alpha}(1-t) - \frac{1}{\alpha+1}\left(1-t^{\alpha+1}\right) \geq \frac{2}{\alpha}(1-t) - \frac{2}{\alpha}\left(1-t^{\alpha+1}\right)$$

$$= \frac{2}{\alpha}\left(t - t^{\alpha+1}\right) \geq 0 \qquad \forall\ t \in [0,1].$$

So $K_1(t) \leq 0$, $\forall$ $t \in [0,1]$.

If $\alpha < 0$, $K_1(0) > 0$, $K_1(1) = 0$ $\Big\}$ so $K_1(t) \geq 0$

$$\frac{\partial K_1}{\partial t} = \underbrace{\frac{2}{\alpha}}_{\leq 0}\underbrace{(1-t)}_{\geq 0} - \underbrace{\frac{1}{\alpha+1}}_{\geq 0}\underbrace{\left(1-t^{\alpha+1}\right)}_{\geq 0} \leq 0.$$

(c) $e_2 = \int_0^1 K_1(t)\,dt$

$$= \frac{1}{\alpha+2}\int_0^1 (1-t^{\alpha+2})\,dt - \frac{1}{\alpha+1}\int_0^1 (t-t^{\alpha+2})\,dt - \frac{1}{\alpha}\int_0^1 (1-t)^2\,dt$$

$$= \frac{1}{\alpha+2}\left[t - \frac{t^{\alpha+3}}{\alpha+3}\right]_0^1 - \frac{1}{\alpha+1}\left[\frac{t^2}{2} - \frac{t^{\alpha+3}}{\alpha+3}\right]_0^1 + \frac{1}{3\alpha}(1-t)^3\Big|_0^1$$

$$= \frac{1}{\alpha+2}\left(1 - \frac{1}{\alpha+3}\right) - \frac{1}{\alpha+1}\left(\frac{1}{2} - \frac{1}{\alpha+3}\right) - \frac{1}{3\alpha}$$

$$= \frac{1}{\alpha+3} - \frac{1}{2(\alpha+3)} - \frac{1}{3\alpha} = \frac{1}{2(\alpha+3)} - \frac{1}{3\alpha} = \frac{\alpha-6}{6\alpha(\alpha+3)} \quad //$$