§10. Presheaves and Čech Cohomology.

Presheaves

Def A presheaf $\mathcal{F}$ on a topo. sp. X is a function assigning to each open set U in X an abelian gp $\mathcal{F}(U)$ and to each inclusion of open sets $i_U^V : V \hookrightarrow U$ a gp hom. $\mathcal{F}(i_U^V) : \mathcal{F}(U) \to \mathcal{F}(V)$, called restriction, s.t.

(1) $\mathcal{F}(i_V^V) = $ identity map

(2) $\mathcal{F}(i_V^W)\mathcal{F}(i_U^V) = \mathcal{F}(i_U^W)$. (transitivity)

We often denote $\mathcal{F}(i_U^V)$ by ρ_V^U.

A hom. of two presheaves, $f : \mathcal{F} \to \mathcal{G}$, is a collection of maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ s.t.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \rho_V^U \downarrow & & \rho_V^U \downarrow \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

commute, $\forall\, i_U^V : V \hookrightarrow U$.

Rmk:

Open(X) := the category w/ objects : open sets in X and morphisms : inclusions.

A presheaf on X is actually a contravariant functor from Open(X) to Ab.

A hom. of two presheaves is simply a natural transformation from a functor to another.

Def G : abelian gp. X : topo. sp.

The constant presheaf with group G is the presheaf $\mathcal{F}$ s.t.

$\mathcal{F}(U) = \{$locally const. functions $U \to G\}$. ρ_V^U are simply defined as the restriction of functions.

Rmk:

(1) We may regard $\mathcal{F}(U)$ as $\prod_{n(U)} G$, where $n(U) :=$ number of connected components of U.

(2) By abuse of notation, the const. presheaf w/ group $\mathbb{R}$ will be denoted by $\mathbb{R}$.

Example: ✱

$\pi : E \to M$ fiber bundle w/ fiber F

Define presheaf $\mathcal{H}^q$ on M by $\mathcal{H}^q(U) = H^q(\pi^{-1}(U))$, and for $V \subseteq U$, let

$\rho^{U}_{V} : H^q(\pi^{-1}U) \to H^q(\pi^{-1}V)$ be the natural restriction map.

$\boxed{\text{Fact}}$: If the base sp. of a fiber bundle is contractible, then the bundle is trivial.

Rmk:

(1) For contractible U, $\pi^{-1}(U) \cong U \times F$.

$\Rightarrow \mathcal{H}^q(U) \cong H^q(U\times F) \cong H^q(F)$. (The last iso. comes from a homotopy $U \sim \{pt\}$).

(2) For $V \subseteq U$, where U, V: contractible,

$\rho^{U}_{V} : \mathcal{H}^q(U) \to \mathcal{H}^q(V)$ is an iso,

(3) By (1), (2), $\mathcal{H}^q$ is a locally const. presheaf on any good cover $\mathcal{U}$. (as defined below).

Def ← $\boxed{P142}$–$\boxed{P143}$ $\boxed{\text{Detour in §13.}}$

X: topo. sp.

$\mathcal{U} = \{U_\alpha\}$: a good cover of X (meaning any finite nonempty intersection is contractible). ← $\boxed{P147}$

A presheaf $\mathcal{F}$ on $\mathcal{U}$ is a contravariant functor $\mathcal{F}$ on the subcategory of Open(X) consisting of all finite intersections $U_{\alpha_0\cdots\alpha_p}$ of open sets in $\mathcal{U}$. (to the category of abelian gps).

Denote it by Open$(X, \mathcal{U})$.

A constant presheaf with group G on $\mathcal{U}$ is defined as before, but replacing Open(X) by Open$(X,\mathcal{U})$.

A locally constant presheaf on $\mathcal{U}$ is a presheaf on $\mathcal{U}$ s.t. all gps are iso, and all arrows are iso.

Two presheeves $\mathcal{F}$ and $\mathcal{G}$ on X are isomorphic relative to $\mathcal{U}$ if for each $W = U_{\alpha_0\cdots\alpha_p} \in$ Open$(X,\mathcal{U})$, $\exists$ iso. $h_W : \mathcal{F}(W) \to \mathcal{G}(W)$ compatible w/ all arrows.

$\boxed{\text{Thm. (13.2)}}$ ← $\boxed{P146}$

$\mathcal{U}$: good cover on X, X: connected topo. sp.

$N(\mathcal{U})$: nerve of $\mathcal{U}$.

If $\pi_1(N(\mathcal{U})) = 0$, then every locally constant presheaf on $\mathcal{U}$ is const.

$\boxed{\text{Thm (13.4)}}$ ← $\boxed{P148}$ ← $\boxed{\text{This is a special case of "nerve lemma".}}$

X: topo. sp. having a good cover, say $\mathcal{U}$.

Then $\pi_1(X) \cong \pi_1(N(\mathcal{U}))$

Čech Cohomology

X : topo. sp. $\mathcal{F}$: presheaf on X.

$\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$: open cover of X, where J is an ordered set.

Define

$$C^0(\mathcal{U}, \mathcal{F}) := \{\text{functions } \mathcal{U} \xrightarrow{\omega} \coprod_{\alpha \in J} \mathcal{F}(U_\alpha) \text{ s.t. } \omega(U_\alpha) \in \mathcal{F}(U_\alpha),\ \forall \alpha \in J\}.$$

$$\cong \prod_{\alpha \in J} \mathcal{F}(U_\alpha).$$

$$\begin{array}{c} \mathcal{U} = \{\cdots U_\alpha \cdots\} \leftrightarrow J = \{\cdots \alpha \cdots\} \\ \downarrow \\ \{\cdots \mathcal{F}(U_\alpha) \cdots\} \end{array}$$

Similarly,

$$C^1(\mathcal{U}, \mathcal{F}) := \prod_{\alpha < \beta} \mathcal{F}(U_\alpha \cap U_\beta), \text{ etc.}$$

The inclusions

$$U_\alpha \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U_{\alpha\beta} \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} \cdots \quad \text{gives rise to}$$

$$\prod_{\alpha \in J} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha < \beta} \mathcal{F}(U_{\alpha\beta}) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots \quad \text{i.e. } C^0(\mathcal{U}, \mathcal{F}) \rightrightarrows C^1(\mathcal{U}, \mathcal{F}) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

Define $\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ as the alternating sum of the $\mathcal{F}(\partial_i)$'s :

$$\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}),\ \delta = \mathcal{F}(\partial_0) - \mathcal{F}(\partial_1).$$

$$\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}),\ \delta = \mathcal{F}(\partial_0) - \mathcal{F}(\partial_1) + \cdots + (-1)^{p+1} \mathcal{F}(\partial_{p+1}).$$

To be more explicit, for $\omega \in C^p(\mathcal{U}, \mathcal{F})$, we have

$$(\delta\omega) = \sum_{\alpha_0 < \cdots < \alpha_{p+1}} (\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}}, \text{ where } (\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}}.$$

(In fact, to be correct, $\omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}}$ on RHS should be $\mathcal{F}(\partial_i)(\omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}})$).

Rmk :

① This is simply a generalization of the construction of "generalized Mayer-Vietoris", replacing Ω^q by $\mathcal{F}$.

② $\delta^2 = 0$, as before. Thus, $C^*(\mathcal{U}, \mathcal{F})$ is a differential complex w/ differential δ.

③ The homology of this complex will be denoted by $H_\delta(C^*(\mathcal{U}, \mathcal{F}))$ or simply $H^*(\mathcal{U}, \mathcal{F})$, and, called the Čech cohomology of the cover $\mathcal{U}$ with values in $\mathcal{F}$.

④ If we start with a "covariant" functor, then all arrows will be reversed. (see P111).

Def

Let $\mathcal{U}=\{U_\alpha\}_{\alpha\in I}$, $\mathcal{V}=\{V_\beta\}_{\beta\in J}$ be open covers of X s.t. $\mathcal{U}<\mathcal{V}$.
Let $\phi: J\to I$ be a map w/ $V_\beta\subseteq U_{\phi(\beta)}$, $\forall \beta\in J$. (indicating the refinement).
This induces a map

$$\phi^{\#}: C^q(\mathcal{U},\mathcal{F})\to C^q(\mathcal{V},\mathcal{F}) \text{ by}$$

$$(\phi^{\#}\omega)(V_{\beta_0\cdots\beta_q}) = \omega(U_{\phi(\beta_0)\cdots\phi(\beta_q)}).$$

Note that $V_{\beta_0\cdots\beta_q}\neq\phi$ will imply $U_{\phi(\beta_0)\cdots\phi(\beta_q)}\neq\phi$ by $\mathcal{U}<\mathcal{V}$.

$\boxed{\text{Lemma (10.4.1)}}$

$\phi^{\#}$ commutes with δ. Hence, it is a chain map.

$\boxed{\text{Lemma (10.4.2)}}$

$\mathcal{U}=\{U_\alpha\}_{\alpha\in I}$: open cover. $\mathcal{V}=\{V_\beta\}_{\beta\in J}$: a refinement of $\mathcal{U}$.
$\phi,\psi: J\to I$: two refinement maps.
Then $\exists$ homotopy operator between $\phi^{\#}$ and $\psi^{\#}$.

Def A direct system of groups is a collection of groups $\{G_i\}_{i\in I}$, where I: directed set, s.t. $\forall\, a<b$ in I, $\exists$ gp hom. $f^a_b: G_a\to G_b$ s.t.

(1) f^a_a = identity on G_a.

(2) $f^a_c = f^b_c\circ f^a_b$, $\forall\, a<b<c$.

On $\coprod_i G_i$, define an equi. relation $\sim$ by $g_a(\in G_a)\sim g_b(\in G_b)$ if $\exists\, c$, $c>a,b$ s.t. $f^a_c(g_a)=f^b_c(g_b)$ in G_c.

The direct limit of the system, denoted by $\varinjlim_{i\in I} G_i$, is the quotient $\coprod_i G_i/\sim$, and the gp addition is defined by $[g_a]+[g_b]=[f^a_c(g_a)+f^b_c(g_b)]$, $\forall\, a,b\in I$ and any choice c with $c>a,b$.

Rmk: By (10.4.1), (10.4.2), we have a well-defined (indep. of choice of refinement map.) map, for each $\mathcal{U}<\mathcal{V}$, in cohomology $H^*(\mathcal{U},\mathcal{F})\to H^*(\mathcal{V},\mathcal{F})$, making $\boxed{\{H^*(\mathcal{U},\mathcal{F})\}_{\mathcal{U}} \text{ a direct system of groups.}}$

Def The direct limit of the system $\{H^*(\mathcal{U},\mathcal{F})\}_{\mathcal{U}}$ is called the Čech cohomology of X with values in $\mathcal{F}$, denoted by $H^*(X,\mathcal{F})$. (i.e. $H^*(X,\mathcal{F})=\varinjlim_{\mathcal{U}} H^*(\mathcal{U},\mathcal{F})$).

Prop (10.6)

- $\mathbb{R}$: const. presheaf on a mfd M.

Then the Čech cohomology of M with values in $\mathbb{R}$ is iso. to $H^*_{DR}(M)$.

[Pf]

The proof is easy!! ▨