Leray's Construction

Consider generally a map $\pi : X \to Y$ (not necessarily a fiber bundle). (mfds.)

$\mathfrak{U}$: open cover for Y. $\Rightarrow \pi^{-1}\mathfrak{U} = \{\pi^{-1}(U) \mid U \in \mathfrak{U}\}$: open cover for X.

By generalized Mayer-Vietoris,

$$\boxed{H^*(X) = H_D\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\}}.$$

Denote $K = C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$, the double-complex, on X.

By (14.14), the spectral seq. of K has

$$\boxed{E_\infty = H_D\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\}} \text{ and } \boxed{E_2^{p,q} = H_\delta^{p,q} H_d\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\}}.$$

Writing explicitly:

$$K = \begin{array}{|c|c|} & \prod \Omega^{q+1}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) \\ & \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^{q}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) \\ \hline \end{array}$$

$$H_d(K) = \begin{array}{|c|c|c} & \prod H^{q+1}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) & \prod H^{q+1}(\pi^{-1}U_{\alpha_0 \cdots \alpha_{p+1}}) \\ & \prod H^{q}(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) & \prod H^{q}(\pi^{-1}U_{\alpha_0 \cdots \alpha_{p+1}}) \\ \hline \end{array}$$

Define a presheaf $\underbrace{\mathcal{H}^q}_{\text{on } Y}$, for each q, by $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$.

With this def,
$$H_d^{p,q}(K) = \prod_{\alpha_0 < \cdots < \alpha_p} H^q(\pi^{-1}U_{\alpha_0 \cdots \alpha_p}) = C^p(\mathfrak{U}, \mathcal{H}^q).$$

Thus, in summary, there is a spectral seq. $\boxed{\text{converging to } H^*(X)}$ w/ E_2 term

$$\boxed{E^2_{p,q} = H^p(\mathfrak{U}, \mathcal{H}^q)}.$$

Rmk:
$\mathcal{H}^q$ is not necessarily locally const. (which is the case for fiber bundles.)

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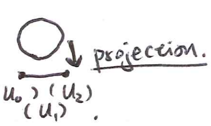
Ex. (14.34).

Consider the map $S^1 \to$ a segment as



Consider the open cover $\{U_0, U_1, U_2\}$ as shown.

With these, we can write $E_1 = H_d\{C^*(\pi^{-1}\mathfrak{U}, \Omega^*)\}$ as

$$\begin{array}{c|ccc} & (a,(b,c),d) & \longmapsto & (-a+b, -a+c, -b+d, -c+d). \\ 0 \to & \underset{U_0\ U_1\ U_2}{\mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}} & \to & \underset{U_{01}\ \ U_{12}}{\mathbb{R}^2 \oplus \mathbb{R}^2} \to 0 \\ \hline \end{array}$$

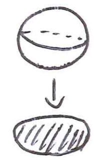
Thus,

$$E_2 = \begin{array}{|c:c:c} \mathbb{R} & \mathbb{R} & \\ \hline \end{array}$$

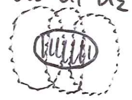
d_2 is then zero.

Thus, $E_\infty = E_2$. i.e. $H^*(S^1) = \mathbb{R} \oplus \mathbb{R} \oplus 0 \oplus \cdots$. #

Exercise (14.35)

Project S^2 to D^2 as . Use this map and Leray's construct'n to compute $H^*(S^2)$.

[Sol].

Consider the open cover $U_0\ U_1\ U_2$. 

Then

$$E_1 = \begin{array}{|c:c} 0\oplus\mathbb{R}\oplus 0 & \mathbb{R}\oplus\mathbb{R} \\ \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R} & \mathbb{R}\oplus\mathbb{R} \\ \hline U_0\ U_1\ U_2 & U_{01}\ \ U_{12} \end{array}$$

$d \overset{\delta}{\longmapsto} (d, -d)$. $\rightsquigarrow$ $\mathrm{Im}(\delta)$: 1-dim. $\ker(\delta)$: 0-dim.

$(a,b,c) \overset{\delta}{\longmapsto} (-a+b, -b+c)$ $\rightsquigarrow$ $\mathrm{Im}(\delta)$: 2-dim. $\ker(\delta)$: 1-dim.

$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$.

Thus, $E_2 = H_\delta(E_1) = \begin{array}{|c:c} 0 & \mathbb{R} \\ \mathbb{R} & 0 \\ \hline \end{array}$ and $d_2 = 0$. Thus, $H^*(S^2) = \mathbb{R} \oplus 0 \oplus \mathbb{R} \oplus 0 \oplus 0 \oplus \cdots$. #

Exercise (14.36)

Y : mfd. $\mathcal{U}$: finite good cover of Y.

$\beta_p :=$ # of nonempty $(p+1)$-fold intersection $U_{\alpha_0 \cdots \alpha_p}$.

Prove that $\chi(Y) = \sum (-1)^p \beta_p$.

[Sol].

By Leray's construction, w/ this open cover, and the identity map $Y \to Y$.

$$E_1 = \begin{array}{|c:c:c:c} \mathbb{R}^{\beta_0} & \mathbb{R}^{\beta_1} & \mathbb{R}^{\beta_2} & \cdots \\ \hline \end{array}$$

"Recall that the alternating sum of dimensions of a diff. complex will not vary after taking homology." ⊛

Thus, $E_2 = \begin{array}{|c:c:c:c} H^0_{dR}(Y) & H^1_{dR}(Y) & H^2_{dR}(Y) & \cdots \\ \hline \end{array}$ will have $\chi(Y) = \sum (-1)^p \dim H^p_{dR}(Y) = \sum (-1)^p \beta_p$. #

Exercise (14.37)

$\pi : X \to Y$ any map. $\mathcal{U}$: finite $\boxed{\text{good cover}}$ of Y.

Prove that $\chi(X) = \sum_{p,q} \sum_{\alpha_0 < \cdots < \alpha_p} (-1)^{p+q} \dim H^q(\pi^{-1} U_{\alpha_0 \cdots \alpha_p})$.

[Sol].

Use ⊛ in (14.36). I think good cover is NOT really necessary. #

好像不太對。因為 E_2 可能和 $H^*(X)$ 不一樣!!