Chap 3 Martingales in Discrete Time. Week 1

3.1 Seq. of R.V.

Def $\xi_1, \xi_2, \cdots$: seq. of r.v. $\omega \in \Omega$.

The seq. of numbers $\xi_1(\omega), \xi_2(\omega), \cdots$ is called a $\boxed{\text{sample path}}$

3.2 Filtrations.

Def A seq. of σ-fields $\mathcal{F}_1, \mathcal{F}_2, \cdots$ on Ω is called a $\boxed{\text{filtration}}$ if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$.

Rmk: $\mathcal{F}_n$ represents our knowledge at time n.

Def A seq. $\xi_1, \xi_2, \cdots$ of r.v. is $\boxed{\text{adapted}}$ to a filtration $\mathcal{F}_1, \mathcal{F}_2, \cdots$ if ξ_n is $\mathcal{F}_n$-mea., $\forall n$.

Ex:

1. $\mathcal{F}_n := \sigma(\xi_1, \cdots, \xi_n)$. Then $\xi_1, \cdots$ is adapted to $\mathcal{F}_1, \cdots$.
2. Indeed, $\mathcal{F}_n$ in 1. is the smallest filtration which $\xi_1, \cdots$ are adapted to.

3.3 Martingales.

Rmk: ① The concept of martingale has its origin in gambling, describing a fair game of chance.

② Martingale / Submartingale / Supermartingale $\longleftrightarrow$ Fair / Favorable / Unfavorable game of chance

Def A seq. $\xi_1, \xi_2, \cdots$ of r.v. is a $\boxed{\text{martingale}}$ w.r.t. a filtration $\mathcal{F}_1, \mathcal{F}_2, \cdots$ if

(i) ξ_n : integrable, $\forall n$.

(ii) $\xi_1, \xi_2, \cdots$: adapted to $\mathcal{F}_1, \mathcal{F}_2, \cdots$.

(iii) $E(\xi_{n+1} | \mathcal{F}_n) = \xi_n$ a.s., $\forall n$.

Ex:

$\eta_1, \eta_2, \cdots$: indep. integrable r.v. w/ $E(\eta_n) = 0, \forall n$.

$\xi_n := \eta_1 + \cdots + \eta_n$. $\mathcal{F}_n := \sigma(\eta_1, \cdots, \eta_n)$.

Then ξ_n is a martingale w.r.t. $\mathcal{F}_n$. (Use properties of cond. exp.)

Ex:

ξ : integ. r.v. $\mathcal{F}_1, \mathcal{F}_2, \cdots$: filtration.

$\xi_n := E(\xi | \mathcal{F}_n), \forall n$.

Then ξ_n : martingale w.r.t. $\mathcal{F}_n$.

Ex: ξ_n : martingale w.r.t. $\mathcal{F}_n$.

Then $E(\xi_1) = E(\xi_2) = \cdots$.

Ex: ξ_n : martingale w.r.t. $\mathcal{F}_n$. $\mathcal{G}_n := \sigma(\xi_1, \cdots, \xi_n)$.

Then ξ_n : martingale w.r.t. $\mathcal{G}_n$.

Ex: ξ_n : symmetric random walk.

i.e. $\xi_n = \eta_1 + \cdots + \eta_n$, where $\eta_1, \eta_2, \cdots$ iid Bernoulli w/ $p = 1/2$.

$\mathcal{F}_n := \sigma(\eta_1, \cdots, \eta_n)$.

Then $\xi_n^2 - n$: martingale w.r.t. $\mathcal{F}_n$.

Def $\xi_1, \xi_2, \cdots$ is a $\boxed{\text{supermartingale}}$ ($\boxed{\text{submartingale}}$) w.r.t. a filtration $\mathcal{F}_1, \mathcal{F}_2, \cdots$ if

(i) ξ_n : integrable, $\forall n$.

(ii) $\xi_1, \xi_2, \cdots$: adapted to $\mathcal{F}_1, \mathcal{F}_2, \cdots$.

(iii) $E(\xi_{n+1} | \mathcal{F}_n) \leq \xi_n$ ($E(\xi_{n+1} | \mathcal{F}_n) \geq \xi_n$, resp.), a.s. $\forall n$.

Ex:

ξ_n : square integrable, $\forall n$. and martingale w.r.t. $\mathcal{F}_n$

Then ξ_n^2 : submartingale w.r.t. $\mathcal{F}_n$. (Use Jensen's)

3.4 Games of Chance.

Consider the following scenario:

$\eta_1, \eta_2, \cdots$ a seq. of integrable r.v., where

η_n : $\boxed{\text{winnings}}$ (or $\boxed{\text{losses}}$) $\boxed{\text{per unit stake}}$ in game n.

If the stake in each game is 1 unit, then the total winnings after n games are

$$\xi_n = \eta_1 + \cdots + \eta_n.$$

$\underbrace{\mathcal{F}_n}_{\text{knowledge after } n \text{ games}} := \sigma(\eta_1, \cdots, \eta_n)$, $\xi_0 := 0$, $\mathcal{F}_0 := \{\emptyset, \Omega\}$. (For notational convenience.)

The game is fair if $E(\xi_n | \mathcal{F}_{n-1}) = \xi_{n-1}, \forall n$.

" " " favorable " " $\geq$ " ....

" " " un " " " $\leq$ " ....

These correspond to martingale, sub..., super..., resp.

If, instead of 1 unit, the stake is α_n in game n, then the $\boxed{\text{total winnings}}$ are

$$\xi_n = \alpha_1 \eta_1 + \cdots + \alpha_n \eta_n \qquad \xi_0 := 0 \text{ for convenience}$$

$$= \alpha_1(\xi_1 - \xi_0) + \cdots + \alpha_n(\xi_n - \xi_{n-1}).$$

Since $\mathcal{F}_n$: knowledge after n games, reasonable to assume α_n : $\mathcal{F}_{n-1}$-mea.

Def A $\boxed{\text{gambling strategy}}$ $\alpha_1, \alpha_2, \cdots$ w.r.t. a filtration $\mathcal{F}_1, \mathcal{F}_2, \cdots$ is a seq. of r.v. s.t. $\alpha_n : \mathcal{F}_{n-1}$-mea., $\forall n$, where $\mathcal{F}_0 := \{\phi, \Omega\}$.

Outside of gambling, such seq. are called $\boxed{\text{previsible}}$.

Prop 3.1

$\alpha_1, \alpha_2, \cdots$: a gambling strategy.

(i) If $\alpha_1, \alpha_2, \cdots$ is bdd and $\xi_1, \xi_2, \cdots$: martingale, then $\zeta_1, \zeta_2, \cdots$: martingale.

(ii) " $\alpha_1, \alpha_2, \cdots$ is nonnegative bdd and $\xi_1, \xi_2, \cdots$: super-..., then $\zeta_1, \zeta_2, \cdots$: super-martingale.

(iii) " $\alpha_1, \alpha_2, \cdots$: nonnegative bdd and $\xi_1, \xi_2, \cdots$: sub-..., then $\zeta_1, \zeta_2, \cdots$: submartingale.

Rmk:

(i) means one cannot have advantage by varying strategies in a fair game.

(ii) and (iii) means, if one is not in a position to wager negative sums of money (e.g. run a casino), then it's impossible to turn unfavorable games into favorable ones, and vice versa.

3.5 Stopping Times.

$\tau :=$ the number of rounds played before quitting the game.

Rmk: τ is a r.v. w/ values in $\{1, 2, \cdots\} \cup \{\infty\}$.

Motivation:

At step n (after the nth game), one should be able to decide whether to stop playing. Thus, reasonable to assume $\{\tau = n\} \in \mathcal{F}_n$, $\forall n$.

Def A r.v. τ w/ values in $\{1, 2, \cdots\} \cup \{\infty\}$ is called a $\boxed{\text{stopping time}}$ w.r.t. a filtration $\mathcal{F}_n$ if $\{\tau = n\} \in \mathcal{F}_n$, $\forall n$.

Ex: τ : r.v. w/ values in $\{1, 2, \cdots\} \cup \{\infty\}$, $\mathcal{F}_n$: filtration.
Then $\{\tau \le n\} \in \mathcal{F}_n$, $\forall n \iff \{\tau = n\} \in \mathcal{F}_n$, $\forall n$.

Ex (First Hitting Time).

Suppose one starts the game w/ \$5 and decide to play until one has \$10, or loses everything.

$\xi_n :=$ amount at step n.

Then $\tau = \min\{n \mid \xi_n = 0 \text{ or } 10\}$, called the $\boxed{\text{first hitting time}}$ (of 10 or 0 by the seq. ξ_n), is the first time of n s.t. $\xi_n = 0$ or 10.

$\because \{\tau = n\} = \{\xi_1 \in (0,10)\} \cap \{\xi_2 \in (0,10)\} \cap \cdots \cap \{\xi_{n-1} \in (0,10)\} \cap \{\xi_n = 0 \text{ or } 10\}$.

$\therefore \{\tau = n\} \in \mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$, $\forall n$.

Thus, τ is a stopping time w.r.t. $\mathcal{F}_n$.

Ex:

ξ_n : a seq. of r.v. adapted to $\mathcal{F}_n$.

$B \in \mathbb{R}$: Borel.

Then $\tau := \min\{n \mid \xi_n \in B\}$, called $\boxed{\text{time of first entry}}$ of ξ_n into B, is a stopping time.

Notation:

For $a, b \in \mathbb{R}$, $\boxed{a \wedge b} := \min\{a, b\}$.

Def

ξ_n : a seq. of r.v.

τ : r.v. w/ values in $\{1, 2, \cdots\} \cup \{\infty\}$.

The seq. $\xi_{n \wedge \tau}$ is called the seq. $\boxed{\text{stopped at } \tau}$, often denoted by ξ_n^τ. Explicitly, $\xi_n^\tau(\omega) := \xi_{\tau(\omega) \wedge n}(\omega)$, $\forall \omega \in \Omega$.

Ex:

ξ_n : adapted to $\mathcal{F}_n$. τ : stopping time w.r.t. $\mathcal{F}_n$.

Then $\xi_{n \wedge \tau}$: adapted to $\mathcal{F}_n$.

<Pf>

$$\{\xi_{n \wedge \tau} \in B\} = \bigcup_{k=1}^{n-1} (\{\tau = k\} \cap \{\xi_k \in B\}) \cup \{\xi_n \in B\}, \quad \forall \text{ Borel } B. \quad \#$$

Prop 3.2.

τ : stopping time.

(i) If ξ_n : martingale, then $\xi_{n \wedge \tau}$: martingale.

(ii) " " sub... , " " : sub... .

(iii) " " super... , " " : super... .

<Pf>

Regard stopping time as a certain kind of gambling strategy. #

Rmk: Prop 3.2 means we cannot turn fair/unfavorable/favorable game into other types w/ stopping time.

Rmk:
Possible to beat the system if one had unlimited capital and unlimited playing time. (See Ex 3.6).

3.6 Optional Stopping Time.

Rmk: In general, for a stopping time τ and martingale ξ_n, $E(\xi_1)$ is not necessarily equal to $E(\xi_\tau)$ even $E(\xi_1)=E(\xi_n), \forall n$.
However, it is often useful to have
$$E(\xi_1)=E(\xi_\tau).$$
The following is a sufficient condition for it.

Thm 3.1 (Optional Stopping Theorem).
ξ_n: martingale, τ: stopping time w.r.t. a filtrat'n $\mathcal{F}_n$.
Suppose
(i) $\tau<\infty$ a.s.
(ii) ξ_τ: integrable.
(iii) $E(\xi_n \cdot 1_{\{\tau>n\}}) \to 0$ as $n\to\infty$.
Then
$$E(\xi_\tau)=E(\xi_1).$$

Ex: (Expectat'n of first hitting time of a random walk)
ξ_n: symmetric random walk.
$K\in\{1,2,3,\cdots\}$.
$\tau:=\min\{n \mid |\xi_n|=K\}$, first hitting time of $\pm K$ by ξ_n.
Recall that ξ_n^2-n is a martingale.
It can be checked that Optional Stopping Theorem can be applied.
Thus, $E(\xi_\tau^2-\tau)=E(\xi_1^2-1)=0$.
$$\Rightarrow E(\tau)=E(\xi_\tau^2)\underset{|\xi_\tau|=K}{=}K^2.$$
So the expected first hitting time of $\pm K$ is K^2. #

Chap 4 Martingale Inequalities and Convergence.

4.1 Doob's Martingale Inequalities.

Week 2

Prop 4.1 (Doob's maximal inequality).
ξ_n: submartingale w.r.t. $\mathcal{F}_n$, $\xi_n\geq 0$.
$\xi_n^*:=\max_{k\leq n}\xi_k$, $n=1,2,\cdots$.
Then, for $\lambda>0$, $\lambda P(\xi_n^*\geq\lambda)\leq E(\xi_n\cdot 1_{\{\xi_n^*\geq\lambda\}})$.

Thm 4.1 (Doob's maximal L^2 inequality)
ξ_n: nonnegative square integrable submartingale w.r.t. $\mathcal{F}_n$.
Then
$$E(|\xi_n^*|^2)\leq 4E(|\xi_n|^2), \text{ where } \xi_n^*=\max_{k\leq n}\xi_k.$$

Def
ξ_n: adapted to $\mathcal{F}_n$.
$a<b$ in $\mathbb{R}$.
Define the following gambling strategy:
$\alpha_1:=0$.
For $n=1,2,\cdots$,
$$\alpha_{n+1}:=\begin{cases}1 & \text{if } \alpha_n=0 \text{ and } \xi_n<a.\\ 1 & \text{if } \alpha_n=1 \text{ and } \xi_n\leq b.\\ 0 & \text{otherwise}.\end{cases}$$
This is called the **upcrossing strategy**.
Any k s.t. $\alpha_k=1$ but $\alpha_{k+1}=0$ is called an **upcrossing**.
The upcrossings form a (finite or infinite) increasing seq.: $u_1<u_2<\cdots$.
$\boxed{U_n[a,b]}:=\max\{k \mid u_k\leq n\}$ (=0 if no such k).
= number of upcrossings made up to time n.

Rmk:
① The meaning of the above process:
Play until ξ_n upcrosses b and stop playing until ξ_n drops below a and then resume playing.
Upcrossings are the times when ξ_n upcrosses b.
② The upcrossing strategy is a gambling strategy.

Lemma 4.1 (Upcrossing inequality).
ξ_n: supermartingale.
$a<b$.
Then
$$(b-a)E(U_n[a,b])\leq E((\xi_n-a)^-),$$
where $f^-:=$ negative part of f
$=\max\{0,-f\}$.

4.2 Doob's Martingale Convergence Theorem.

Thm 4.2 (Doob's Mart. Con. Thm.)

ξ_n : supermartingale w.r.t. $\mathcal{F}_n$.

$\sup_n E(|\xi_n|) < \infty$.

Then $\exists$ integrable ξ s.t. $\xi_n \to \xi$ a.s.

Rmk:

① Thm 4.2 works for martingales (since a mart. is a supermart.)
Also, it works for submart. since its minus is a supermart.

② No con. in L^1 asserted. Only con. a.s.

Dealt w/ later

Ex:

ξ_n : nonnegative supermart.

Then it con. a.s. to an integrable r.v.

<Pf> claim: $E(|\xi_n|)$ unif. bdd.

4.3 Unif. Integ. and L^1 Con. of Mart.

Ex:

An r.v. ξ is integrable iff

$\forall \varepsilon > 0, \exists M > 0$ s.t. $\int_{\{|\xi| > M\}} |\xi| \, dP < \varepsilon$.

Def

$\{\xi_n\}_{n=1}^{\infty}$: r.v.'s, is called $\boxed{\text{unif. integ.}}$ if

$\forall \varepsilon > 0, \exists M > 0$ s.t. $\int_{\{|\xi_n| > M\}} |\xi_n| \, dP < \varepsilon, \forall n$.

Ex:

$\xi_n := n \cdot 1_{(0, 1/n)}$, $n = 1, 2, \cdots$, is not unif. integ.

Prop 4.2

A seq. $\xi_1, \xi_2, \cdots$ of integ. r.v.'s con. in L^1.

$\Rightarrow$ It is unif. integ.

i.e. Unif. integ. is necessary for con. in L^1.

Lemma 4.2

ξ : integ. Then $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$\int_A |\xi| \, dP < \varepsilon, \forall P(A) < \delta$.

Ex:

ξ : integ. $\mathcal{F}_1, \mathcal{F}_2, \cdots$: filtration.

Then $\xi_n := E(\xi | \mathcal{F}_n)$: unif. integ. mart.

Ex:

Unif. integ. seq. of r.v. is bdd in L^1.

(Thus, satisfies the conditions of Doob's thm.

Thm 4.3

Every unif. integ. supermart. (or submart.) con. in L^1.

Rmk: (Combining Prop 4.2 and Thm 4.3).

A supermart. (or submart.) con. in L^1

$\Leftrightarrow$ it is unif. integ.

Thm 4.4

ξ_n : unif. integ. mart. (so con. in L^1).

$\xi := \lim_{n\to\infty} \xi_n$. $\mathcal{F}_n := \sigma(\xi_1, \cdots, \xi_n)$.

Then $\xi_n = E(\xi | \mathcal{F}_n), \forall n$.

i.e. Unif. integ. mart. must be of the form

$E(\xi | \mathcal{F}_n)$ w/ ξ being their limit.

Ex: (Apply Thm 4.4)

ξ_n : mart. w/ $\xi_n \to a$ in L^1, $a \in \mathbb{R}$.

Then $\xi_n = a$ a.s. $\forall n$.

Thm 4.5 (Kolmogorov's 0-1 law)

$\eta_1, \eta_2, \cdots$: seq. of indep. r.v.

$\mathcal{T} := \mathcal{T}_1 \cap \mathcal{T}_2 \cap \cdots$, where $\mathcal{T}_n := \sigma(\eta_n, \eta_{n+1}, \cdots)$,

called the tail σ-field.

Then $P(A) = 0$ or 1, $\forall A \in \mathcal{T}$.

Ex:

① $A_n \in \sigma(\xi_n), \forall n. \Rightarrow \underline{\lim} A_n, \overline{\lim} A_n \in \mathcal{T}$.

② (Apply Kolmogorov's 0-1 law).
In a seq. of coin tosses, $\exists$ a.s. infinitely many heads.

Chap 5 Markov Chains. Week 3

5.1 First Examples and Definitions.

Ex 5.1

Scenario: The use of telephone in some home.

Assumptions:

(i) Free during the nth minute

$\Rightarrow$ W/ prob. p, $0<p<1$, busy during the next minute.

(ii) Busy during the nth minute

$\Rightarrow$ W/ prob. q, $0<q<1$, free during the next minute.

$\boxed{A_n}$:= the event phone is free during the nth minute.

$\boxed{B_n} := \Omega \setminus A_n$.

Then $P(B_{n+1}|A_n) = p$ Also assume $\boxed{P(A_0)=1}$.

$P(A_{n+1}|B_n) = q$.

$\boxed{x_n} := P(A_n)$ = the prob. that phone is free during the nth time

By total prob. formula,

$$\begin{aligned} x_{n+1} = P(A_{n+1}) &= P(A_{n+1}|A_n)P(A_n) + P(A_{n+1}|B_n)\cdot P(B_n) \\ &= (1-p)x_n + q\cdot(1-x_n) \\ &= q + (1-p-q)x_n. \end{aligned}$$

By solving $\square = q + (1-p-q)\square$, we have

$$\square = \frac{q}{p+q}.$$

Thus,

$$x_{n+1} = q + (1-p-q)x_n$$

$$-) \quad \frac{q}{p+q} = q + (1-p-q)\cdot\frac{q}{p+q}.$$

$$\left(x_{n+1} - \frac{q}{p+q}\right) = (1-p-q)\left(x_n - \frac{q}{p+q}\right).$$

$$\Rightarrow x_n - \frac{q}{p+q} = \left(1 - \frac{q}{p+q}\right)(1-p-q)^n, \quad n = 1, 2, \cdots$$

$$\Rightarrow x_n = \frac{q}{p+q} + \frac{p}{p+q}(1-p-q)^n, \quad n = 1, 2, \cdots.$$

Also, $\lim_{n\to\infty} x_n = \frac{q}{p+q}$.

$\boxed{y_n} := P(B_n)$ = the prob. that phone is busy during the nth minute.

Then

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \ \forall n.$$

Def S := a finite or countable set. $(\Omega, \mathcal{F}, P)$.

A seq. of S-valued rv $(\xi_n)_{n=0}^{\infty}$ is called a $\boxed{\text{Markov chain on } S}$ or an $\boxed{S\text{-valued Markov chain}}$ if

$$P(\xi_{n+1} = s | \xi_0, \cdots, \xi_n) = P(\xi_{n+1} = s | \xi_n), \quad (*)$$

$\forall s \in S$, $n \in \mathbb{N}\cup\{0\}$.

The (n+1)th result only depends on the nth result.

(*) is called the $\boxed{\text{Markov property}}$.

S is " " $\boxed{\text{state space}}$.

Elements in S are called $\boxed{\text{states}}$.

Prop 5.1

Ex 5.1 can be described as a Markov process.

Def

An S-valued Markov chain ξ_n, $n \in \mathbb{N}\cup\{0\}$, is called $\boxed{\text{time-homogeneous}}$ or $\boxed{\text{homogeneous}}$ if

$$P(\xi_{n+1} = j | \xi_n = i) = P(\xi_1 = j | \xi_0 = i), \ \forall \begin{matrix} n \in \mathbb{N}\cup\{0\} \\ i, j \in S. \end{matrix}$$

In this case,

$$\boxed{p(j|i)} := P(\xi_{n+1} = j | \xi_n = i) = P(\xi_1 = j | \xi_0 = i),$$

called the $\boxed{\text{transition probability}}$ from state i to state j.

The matrix

$$\boxed{P} = [p(j|i)]_{j,i\in S}$$

is called the $\boxed{\text{transition matrix}}$ of the chain ξ_n.

Def

$A = [a_{ji}]_{j,i\in S}$ is called a $\boxed{\text{Stochastic matrix}}$

if (1) $a_{ji} \geq 0$, $\forall j, i \in S$, and

(2) the sum of entries in each column is 1.

A is called a $\boxed{\text{double stochastic matrix}}$ if both A and A^t are stochastic.

Prop 5.2

A stochastic matrix is double stochastic iff the sum of entries in each row is 1.

Rmk:

Product of stochastic matrices is again stochastic.

In particular, powers of sto. mtx is again sto.

Both are true for double sto. mtx.

Def ξ_n: homogeneous Markov chain.

$\boxed{P_n} = [p_n(j|i)]_{j,i\in S}$, where $\boxed{p_n(j|i)} = P(\xi_n = j | \xi_0 = i)$, is called the $\boxed{\text{n-step transition matrix}}$.

Rmk:

$P_n = P_1^n$, $n=1,2,\cdots$.

Prop 5.3 (Chapman–Kolmogorov equation)

ξ_n: homogeneous S-valued Markov chain.

$p_n(j|i)$: n-step transition prob.

Then, $\forall\, n,k \in \mathbb{N}\cup\{0\}$, and $i,j\in S$,

$$p_{n+k}(j|i) = \sum_{s\in S} p_n(j|s)\, p_k(s|i).$$

Exercise 5.8 (random walk).

$S := \mathbb{Z}$.

η_n, $n\geq 1$, iid w/ $P(\eta_n=1)=p$, $P(\eta_n=-1)=1-p$.

$\xi_n := \eta_1+\cdots+\eta_n$, $\xi_0 := 0$.

Then ξ_n: homogeneous Markov chain w/ transit'n prob.

$$p(j|i) = \begin{cases} p & \text{if } j=i+1 \\ 1-p & \text{if } j=i-1 \\ 0 & \text{o.w.} \end{cases}$$

This is called a $\boxed{\text{random walk}}$ starting at 0.

If $\xi_0 := i_0$, it is called a $\boxed{\text{random walk}}$ starting at i_0.

Exercise 5.9

For ξ_n as above, we have

$$P(\xi_n=j\,|\,\xi_0=i) = \begin{cases} \binom{n}{\frac{n+j-i}{2}} p^{\frac{n+j-i}{2}}\cdot(1-p)^{\frac{n-j+i}{2}}, & \text{if } n+j-i\equiv 0 \bmod 2 \text{ and } n+j-i\geq 0. \\ 0 & \text{, o.w.} \end{cases}$$

Prop 5.4

For $p\in(0,1)$,

$$P(\xi_n=i\,|\,\xi_0=i)\to 0 \text{ as } n\to\infty.$$

Prop 5.5

The prob. that the random walk ever returns to the starting pt is $1-|2p-1| = 1-|p-q|$, where $q=1-p$.

Rmk: (SKIPPED PARTS).

Exercise 5.11, 5.12, Prop 5.6, Exercise 5.13 are related examples.

5.2 Classification of States.

In this sect'n, we fix a S-valued Markov chain w/ transit'n matrix $P=[p(j|i)]_{j,i\in S}$ (so that it is homogeneous is implicitly assumed), where S is countable.

Def

$i\in S$ is $\boxed{\text{recurrent}}$ if $P(\xi_n=i, \text{ for some } \overset{1}{n}\,|\,\xi_0=i)=1$.

$i\in S$ is $\boxed{\text{transient}}$ if it's not recurrent.

Thm 5.1

For a random walk on $\mathbb{Z}$ w/ para. $p\in(0,1)$, the state 0 is recurrent $\iff p=\frac{1}{2}$.

Indeed, it holds for every state $i\in\mathbb{Z}$.

Def $i,j\in S$.

State i $\boxed{\text{communicates}}$ w/ state j if

$$P(\xi_n=j \text{ for some } n\geq 0\,|\,\xi_0=i)>0, \text{ written } \boxed{i\to j}.$$

State i $\boxed{\text{intercommunicates}}$ w/ state j if

$i\to j$ and $j\to i$, written $\boxed{i\leftrightarrow j}$.

Prop

(1) $i\leftrightarrow i$. (2) if $i\leftrightarrow j$, then $j\leftrightarrow i$.

(3) if $i\leftrightarrow j$, $j\leftrightarrow k$, then $i\leftrightarrow k$.

i.e. $\leftrightarrow$ is an equivalence relat'n on S.

Exe 5.17 / 5.18

For $|x|<1$, $i,j\in S$, define

$$\boxed{P_{ji}(x)} = \sum_{n=0}^{\infty} p_n(j|i)x^n,$$

$$\boxed{F_{ji}(x)} = \sum_{n=1}^{\infty} f_n(j|i)x^n, \text{ where}$$

$$\boxed{f_n(j|i)} = P(\xi_n=j, \xi_k\neq j, k=1,\cdots,n-1\,|\,\xi_0=i),\ n\geq 1.$$

Then

(1) $P_{ji}(x)$, $F_{ji}(x)$ (the power series) are abs. con. for $|x|<1$.

(2) $P_{ji}(x) = F_{ji}(x)P_{jj}(x)$, $\forall\, j\neq i$

$P_{ii}(x) = F_{ii}(x)P_{ii}(x)+1$.

(3) $\lim_{x\to 1^-} P_{jj}(x) = \sum_{n=0}^{\infty} p_n(j|j)$, $\lim_{x\to 1^-} F_{jj}(x) = \sum_{n=1}^{\infty} f_n(j|j)$.

Exe 5.19

(1) A state j is recurrent iff $\sum_n p_n(j|j) = \infty$.

Hence, a state j is transient iff $\sum_n p_n(j|j) < \infty$.

(2) If j: transient, then $\forall i \in S$, $\sum_n p_n(j|i) < \infty$.

Exe 5.21

Suppose $|S| < \infty$, then $\exists j \in S$ s.t. j: recurrent.

Thm 5.2

(1) $j \in S$: recurrent

$\iff P(\xi_n = j$ for infinitely many $n \mid \xi_0 = j) = 1$.

(2) $j \in S$: transient

$\iff P(\xi_n = j$ for infinitely many $n \mid \xi_0 = j) = 0$.

Def

$i \in S$: recurrent.

$\boxed{m_i} := \sum_{n=1}^{\infty} n f_n(i|i)$, called the mean recurrence time of i.

i is called $\boxed{\text{null-recurrent}}$ if $m_i = \infty$.

i is called $\boxed{\text{positive-recurrent}}$ if $m_i < \infty$.

Rmk

A recurrent state i is null-recurrent iff $p_n(i|i) \to 0$.

Exe 5.22

Given a symmetric random walk on $\mathbb{Z}$.

Then 0 is a null-recurrent state.

Def $i \in S$.

$\boxed{d(i)} := \gcd\{n \in \mathbb{N} \mid p_n(i|i) > 0\}$, called the $\boxed{\text{period of the state } i}$.

i is called a $\boxed{\text{periodic}}$ state if $d(i) \geq 2$.

" " " $\boxed{\text{aperiodic}}$ if $d(i) = 1$.

" " " $\boxed{\text{ergodic}}$ if $d(i) = 1$ and i: positive-recurrent.

Prop 5.7

$i, j \in S$, $i \leftrightarrow j$. Then

(1) i: transient $\iff$ j is.

(2) i: recurrent $\iff$ j is.

(3) i: null-recurrent $\iff$ j is.

(4) i: positive-recurrent $\iff$ j is.

(5) i: periodic $\iff$ j is. In this case, $d(i) = d(j)$.

(6) i: ergodic $\iff$ j is.

Exe 5.26

i: recurrent, $i \to j$.

$\Rightarrow j \to i$, and hence $i \leftrightarrow j$, and j: recurrent.

Hence, if i: recurrent and a chain starts from i and visit j, then it is ~~impossible~~ for the chain not to visit i again.

Def

(1) $C \subseteq S$ is $\boxed{\text{closed}}$ if

$P(\xi_k \in S \setminus C, \text{ some } k \geq 1 \mid \xi_0 \in C) = 0$.

i.e. once the chain enters C, it will never leave it.

(2) $C \subseteq S$ is $\boxed{\text{irreducible}}$ if $\forall i, j \in C$, $i \leftrightarrow j$.

i.e. $\forall i, j \in C$, $\exists n \in \mathbb{N} \cup \{0\}$ s.t. $p_n(j|i) > 0$.

Thm 5.3

I think this might be infinite.

$S = T \cup \left(\bigcup_{j=1}^{N} C_j\right)$, (disjoint union), where

$T = \{$all transient states in $S\}$, and each

C_j: closed irr. set of recurrent states.

Exe 5.27

$C \subseteq S$ is closed iff $p(j|i) = 0$, $\forall i \in C, j \in S \setminus C$.

5.3 Long-Time Behavior of Markov Chains: General Case.

Denote S either by $\{1, 2, 3, \cdots\}$ or $\{1, 2, 3, \cdots, n\}$.

Prop 5.8

$P = [p(j|i)]_{j, i \in S}$: transition matrix of a (homogeneous) M.C.

For all $i, j \in S$, $\lim_{n \to \infty} p_n(j|i) =: \boxed{\pi_j}$ (indep. of i).

assume existence.

Then

(1) $\sum_j \pi_j \leq 1$.

(2) $\sum_i p(j|i) \pi_i = \pi_j$.

(3) either $\sum_j \pi_j = 1$ or $\pi_j = 0$, $\forall j \in S$.

Def

A prob. mea. $\mu = \sum_{j \in S} \mu_j \delta_j$ is an $\boxed{\text{invariant measure}}$ of a M.C. (homo.) if $\sum_{i \in S} p_n(j|i) \mu_i = \mu_j$, $\forall n \in \mathbb{N} \cup \{0\}$, $j \in S$.

i.e. $P\begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}$, where $P = [p(j|i)]$ is the transition matrix.

Exe 5.28 (*)

Under assumption of Prop 5.8, if $\sum_{j \in S} \pi_j = 1$, then $\mu := \sum_{j\in S} \pi_j \delta_j$ is the unique invariant measure of the M.C.

Rmk:
The proof indicates that if such π_j (the limit) does exist, then $\sum_{j\in S} \pi_j \delta_j$ is the only possibility for the invariant measure.

Thus, if $\pi_j = 0, \forall j$, then $\nexists$ any invariant measure.

Exe 5.31

μ, ν : invariant measures.

$\theta \in [0,1]$.

Then $(1-\theta)\mu + \theta \nu$: invariant measure.

Exe 5.32

μ : invariant measure.

$T := \{$all transient states$\}$.

Then $\operatorname{supp}(\mu) \subseteq S \setminus T$.

Hint: If j : transient, then $P_n(j|i) \to 0, \forall i \in S$

Thm 5.4

Suppose $S = T \cup C$, where $T = \{$all transient states$\}$, C = an irr. closed set of recurrent states.

i.e. we assume there is only 1 irr. closed set.

Then $\exists$ invariant measure

$\iff$ every element in C is positive-recurrent.

Moreover, in this case, the invariant mea. is unique and given by $\mu = \sum_{i \in C} \mu_i \delta_i$, where $\mu_i = \frac{1}{m_i}$.

mean recurrence time of state i

Rmk

If, instead of being just one piece, $C = \bigcup_{j=1}^{N} C_j$, where each C_j : irr. closed set of recurrent states, then the above result holds except for uniqueness.

(In fact, apply Thm 5.4 on each C_j, normalize and use convex combination).

Thm 5.5

$j \in S$: recurrent.

(1) j : aperiodic. $\Rightarrow P_n(j|j) = \frac{1}{m_j}$, $\forall i \in S$, $P_n(j|i) \to \frac{F_{ji}(1)}{m_j}$.

(2) j : periodic, $d := d(j)$. (hence ≥ 2).

Then $P_{nd}(j|j) \to \frac{d}{m_j}$, as $n \to \infty$.

Def

A M.C. is $\boxed{\text{ergodic}}$ if every $i \in S$ is ergodic.

i.e. positive-recurrent, aperiodic.

Exe 5.34

Fact: If j : recurrent and $i \leftrightarrow j$, then $F_{ji}(1) = 1$. ∎

Given ergodic irr. M.C.

Then $P_n(j|i) \to \pi_j$ as $n \to \infty$, $\forall i, j \in S$, where $\sum_j \pi_j \delta_j$ is the unique invariant measure.

Week 5

Thm 5.6

$(\xi_n)_{n \in \mathbb{N} \cup \{0\}}$: irr. aperiodic M.C. w/ statesp. S.

Then it is ergodic iff

it has a unique invariant measure.

5.4 Long-Time Behavior of Markov Chains w/ Finite State Spaces.

Rmk: Existence of π_j plays an important role in the existence of invariant measures.

Thm 5.7 (Existence of π_j under $|S| < \infty$ and ...)

$|S| < \infty$.

$P = [p(j|i)]$: a transition matrix of a M.C. on S.

$\exists n_0 \in \mathbb{N}$, $\varepsilon > 0$ s.t.

$P_{n_0}(j|i) \geq \varepsilon, \forall i, j \in S$. (*₀)

Then, $\forall i, j \in S$, the limit

$\lim_{n\to\infty} P_n(j|i)$ exists and is indep. of i, denoted π_j. (*₁)

Moreover, $\pi_j > 0, \forall j \in S$ and $\sum_{j\in S} \pi_j = 1$. (*₂)

Conversely, if (*₁) and (*₂) are satisfied, then (*₀) holds.

Thm 5.8

Assume $(*_0)$.

Then $\exists!$ invariant measure (i.e. $\sum_{j\in S}\pi_j\delta_j$).

Moreover, $\exists\ A>0$ and $\alpha<1$ s.t.

$$|p_n(j|i)-\pi_j|\le A\alpha^n,\ \forall\ i,j\in S,\ n\in\mathbb{N}.$$

Rmk:

Skip Exercise 5.38 - 5.42.

Chap 6 Stochastic Processes in Continuous Time.

6.1 General Notions.

Def

A stochastic process: $\{\xi(t)\}_{t\in T}$, where $T\subseteq\mathbb{R}$ (each $\xi(t)$: r.v.)

When $T=\{1,2,\cdots\}$: discrete.

" $T=[0,\infty)$: continuous.

For $\omega\in\Omega$ (sample space of each $\xi(t)$), the function

$T\ni t\mapsto\xi(t)(\omega)$ is called a path (or sample path) of $\xi(t)$.

Def $T\subseteq\mathbb{R}$.

$\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, where $\mathcal{F}_t$: σ-field on Ω, is called a filtration if $\mathcal{F}_t\subseteq\mathcal{F}_s$, $\forall\ t\le s$.

Def $T\subseteq\mathbb{R}$. $\{\mathcal{F}_t\}_{t\in T}$: filtration.

$\{\xi(t)\}_{t\in T}$: stochastic process, is called a martingale (sub-/sup-) w.r.t. $\{\mathcal{F}_t\}_{t\in T}$, if

(i) $\xi(t)$: integrable, $\forall\ t\in T$.

(ii) $\xi(t)$: $\mathcal{F}_t$-mea., $\forall\ t\in T$. (i.e. $\{\xi(t)\}$ adapted to $\{\mathcal{F}_t\}$).

(iii) $\xi(s)=E(\xi(t)|\mathcal{F}_s)$, $\forall\ s\le t$ in T.

($\le$ or $\ge$ for sub-/sup-, resp.)

6.2 Poisson Process

6.2.1 Exp. Dist. and Lack of Memory.

Def η: r.v.

η has the exponential distribution of rate $\lambda>0$

if $P\{\eta>t\}=e^{-\lambda t}$, $\forall\ t\ge 0$.

Rmk:

In reality,

P(no particle emitted (or no call is made) up to time t) decays exponentially as t increases.

i.e. if η = waiting time until next particle emitted
(or " " " " call made).

then $P(\eta>t)=e^{-\lambda t}$, some const. $\lambda>0$.

Exe 6.3

η: r.v. having exp. dist.

Then $P\{\eta>t+s\}=P\{\eta>t\}P\{\eta>s\}$, $\forall\ s,t\ge 0$.

Exe 6.4

The above equality can be restated as:

$P\{\eta>t+s|\eta>s\}=P\{\eta>t\}$, $\forall\ s,t\ge 0$.

This is called the lack of memory property.

Exe 6.5

The exp. dist. is the only prob. dist. w/ the lack of memory property.

6.2.2 Construction of the Poisson Process.

Def

$\eta_1,\eta_2,\cdots$: seq. of indep. r.v. w/ the same exp. dist. of rate λ.

$\xi_n:=\eta_1+\eta_2+\cdots+\eta_n$. $\xi_0:=0$.

Note that, for $t\ge 0$,

$N(t):=\max\{n\mid\xi_n\le t\}$, is a r.v. (when t fixed), counting the number of emissions (or calls) made up to time t (inclusive).

$\{N(t)\}_{t\in[0,\infty)}$ is called a Poisson process.

Q: What is the distribution of $N(t)$ (as a r.v. w/ fixed t)?

Def ν: r.v.

ν has Poisson distribution w/ parameter $\alpha>0$ if

$P\{\nu=n\}=e^{-\alpha}\dfrac{\alpha^n}{n!}$, for $n=0,1,2,\cdots$.

A: The following prop.

Prop 6.1

$N(t)$ has Poisson dist. w/ para. λt.

i.e. $P(N(t)=n) = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$, $n=0,1,2,\cdots$.

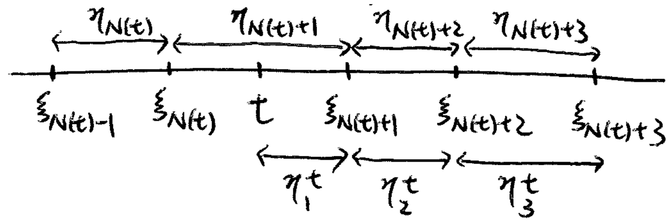
Exe 6.6

$E(N(t)) = \lambda t$.

6.2.3 Poisson Process Starts from Scratch at Time t.

Def Fix $t \geq 0$.

$\eta_1^t := \xi_{N(t)+1} - t$, $\eta_n^t := \eta_{N(t)+n}$, $n=2,3,\cdots$.



$\xi_n^t := \eta_1^t + \cdots + \eta_n^t$.

$N^t(s) := \max\{n \mid \xi_n^t \leq s\}$.

Exe 6.8: $N^t(s) = N(t+s) - N(t)$.

Thm 6.1

For fixed $t \geq 0$,

$N^t(s) (= N(t+s) - N(t))$, $s \geq 0$,

is a Poisson process.

Moreover, $N^t(s)$ is indep. of $N(t)$ and

$N^t(s)$ and $N(s)$ have the same dist.

Thm 6.2

For any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$,

$N(t_1), N(t_2)-N(t_1), \cdots, N(t_n)-N(t_{n-1})$ are indep. and have the same prob. dist. as

$N(t_1), N(t_2-t_1), \cdots, N(t_n-t_{n-1})$.