[2003][J. Moller, etal] Pt Process.

Chap 2 Intro. to Pt Processes.

See App. B for basic formal treatments.

Def

$u : S \to [0,1]$ w/ $\{\xi \in S \mid u(\xi) < 1\}$: bdd in S.

The generating functional for a pt process X on S is $G_X(u) := E\left[\prod_{\xi \in X} u(\xi)\right]$, for u as described above.

Prop

The dist. of X is uniquely determined by G_X.

(Pf) Fix $B \in \mathcal{B}_0$.

For $0 \le t \le 1$, define $u_t(\xi) = t^{1[\xi \in B]}$.

Then $G_X(u_t) = E[t^{N(B)}]$, which is the prob. generating fn. of $N(B)$.

Thus, dist. of $N(B)$ can be recovered by G_X. By Thm B.1, the result follows. #

Chap 3 Poisson point processes.

Purpose: ① As a tractable model class for "no interaction" or "complete spatial randomness".

② As a reference process.

3.1 Basic Properties.

Def

An intensity function is a fn. $\rho : S \to [0, \infty)$ which is locally integrable. (i.e. $\int_B \rho(\xi) d\xi < \infty$, $\forall B \in \mathcal{B}_0$).

Moller's Pt Process Book (App).:

App B : Measure theoretical details.

B.1 Preliminaries.

S : metric sp. w/ metric $d(\cdot,\cdot)$.

Def $(\Omega, \mathcal{F}, \mu)$: measure sp.
Any $\mathcal{A} \subseteq \mathcal{F}$ is called a paving.

Lemma B.1.

μ_1, μ_2: mea. defined on $(\Omega, \mathcal{F})$.

$\mathcal{A}$: a paving s.t. $\mathcal{A}$: closed under finite intersectn

and $\sigma(\mathcal{A}) = \mathcal{F}$.

$\mu_1(\Omega) = \mu_2(\Omega) < \infty$, and

$\mu_1 = \mu_2$ on $\mathcal{A}$.

Then $\mu_1 = \mu_2$ on $\mathcal{F}$.

B.2 Formal def. of pt processes.

Def

$\mathcal{B}$:= the Borel σ-algebra on S.

$\mathcal{B}_0$:= { bdd Borel sets }.

N_{lf} := { locally finite subsets of S }.

$$\mathcal{N}_{lf} := \sigma\Big(\{\{x \in N_{lf} \mid n(\underbrace{x_B}_{x\cap B}) = m\} : B \in \mathcal{B}_0, m \in \underbrace{\mathbb{N}_0}_{\mathbb{N}\cup\{0\}}\}\Big)$$

$n(x_B)$: $\boxed{\#(x_B)}$

Def

A point process defined on S is a measurable map $X : (\Omega, \mathcal{F}, P) \to (N_{lf}, \mathcal{N}_{lf})$.

The distribution of X, denoted P_X, is a prob. mea. on $(N_{lf}, \mathcal{N}_{lf})$ defined by

$$P_X(F) = P(X^{-1}(F)) = P(\{\omega \mid X(\omega) \in F\}),$$

$\forall F \in \mathcal{N}_{lf}$.

Def

For $B \in \mathcal{B}_0$, $N(B) : (\Omega, \mathcal{F}) \to (\mathbb{N}_0, 2^{\mathbb{N}_0})$ is defined by $(N(B))(\omega) = n(B \cap X(\omega))$.

Prop

X is measurable $\Leftrightarrow$ $N(B)$ is measurable $\forall B \in \mathcal{B}_0$.

<Pf>

$(\Rightarrow)$ is obvious.

$(\Leftarrow)$

It suffices to prove, $\forall m \in \mathbb{N}_0, B \in \mathcal{B}_0$, we have

$$X^{-1}(\{x \in N_{lf} \mid n(x_B) = m\}) \in \mathcal{F}.$$

This is because $\mathcal{N}_{lf}$ is generated by such sets.

Notice this preimage is exactly $N(B)^{-1}(m)$ and hence the result follows. #

Rmk:

Due to this prop, we can define, for every $B \in \mathcal{B}$, $N(B)$ as $\sum_{i=1}^{\infty} N(B_i \cap B)$, where $S = \coprod_{i=1}^{\infty} B_i$ w/ $B_i \in \mathcal{B}_0, \forall i$.

Lemma B.2

The distribution P_X of a point process X is determined by the finite dimensional distributions of its count functions.

i.e. the joint dist. of $N(B_1), \cdots, N(B_m)$ for any $B_1, \cdots, B_m \in \mathcal{B}_0$ and $m \in \mathbb{N}$.

<Pf>.

$$\mathcal{A} := \{\{x \in N_{lf} \mid n(x_{B_i}) = n_i, i = 1, \cdots, m\} \mid n_i \in \mathbb{N}_0, B_i \in \mathcal{B}_0, i = 1, \cdots, m, m \in \mathbb{N}\}.$$

Then $\mathcal{A}$ is closed under finite intersection and $\sigma(\mathcal{A}) = \mathcal{N}_{lf}$.

$\because$ P_X is a prob. mea. on $\mathcal{N}_{lf}$ and its values are the finite dimensional dist. of the count functions

$\therefore$ By Lemma B.1, the result follows. #

Rmk:

① We may identify a point process X w/ a locally finite random counting measure.

② Here, we will only consider simple point processes.
i.e. $P(N(\{\xi\}) \le 1) = 1, \forall \xi \in S$.

③ A point process w/ multiple pts may be viewed as a marked pt process. (which may be viewed as simple)

B.3 Some useful conditions and results.

Henceforth, we assume S is separable.
i.e. S contains a countable dense set.

Lemma B.3

$$\mathcal{N}_{lf}^{0} := \{ \{x \in N_{lf} \mid x(B_x)=0\} : B \in \mathcal{B}_0 \},$$

called the class of void events.

Then
$$\mathcal{N}_{lf} = \sigma(\mathcal{N}_{lf}^{0}).$$

Thm B.1 (extreme useful (as the authors said)).

A pt process is uniquely determined by its void probabilities.

i.e. $P_X(\{x \in N_{lf} \mid x(B_x)=0\})$, $\forall B \in \mathcal{B}_0$.

<Pf>.

Immediate from Lemma B.1 and B.3 #

Def

A metric sp. S is called a Polish space if it is complete and separable.

Prop B.1

If S is Polish, then $\mathcal{N}_{lf}$ is $\boxed{\text{separable}}$ (i.e. countably generated).

S: complete separable metric sp.

$x \subseteq S$, $B \subseteq S$, $x_B := x \cap B$.

Object	Element
$\mathcal{N}_{lf}$	pt cloud set
$\mathcal{B}, \mathcal{B}_0, N_{lf}$	pt cloud
S	pt

① x: locally finite if $n(x_B) < \infty$, $\forall$ bdd B.

② $N_{lf} := \{x \subseteq S \mid x: \text{locally finite}\}$.

An element in N_{lf} is called a locally finite $\overset{\text{pt}}{\vee}$ configuration.

③ $\mathcal{B} :=$ Borel σ-algebra of $S = \sigma(\text{open sets in } S)$.

$\mathcal{B}_0 := \{\text{bdd Borel sets}\}$.

④ Given $B \in \mathcal{B}_0$, $m \in \mathbb{N}$. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

$\mathcal{N}_{lf} := \sigma(\{x \in N_{lf} \mid n(x_B) = m\} : B \in \mathcal{B}_0, m \in \mathbb{N}_0)$ ← a σ-algebra on N_{lf}.

⑤ A pt process is a random element

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (N_{lf}, \mathcal{N}_{lf}).$$

induces $P_X \leftarrow$ $\boxed{\text{a distribution}}$ on $(N_{lf}, \mathcal{N}_{lf})$ by $P_X(F) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in F\})$

$= \mathbb{P}(X^{-1}(F))$, $\forall F \in \mathcal{N}_{lf}$.

For $B \in \mathcal{B}$, define

$$\underset{\boxed{N_X(B)}}{\overset{\parallel}{N(B)}} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{N}_0, 2^{\mathbb{N}_0}) \text{ by } (N_X(B))(\omega) = n(X(\omega) \cap B).$$

(emphasizing dependence on X)

Prop

X: measurable $\iff$ $N_X(B)$: measurable, $\forall B \in \mathcal{B}$.

Def

By abuse of notation, $X \subseteq S$ means $X \in \mathcal{B}$, and

$F \subseteq N_{lf}$ 〃 $F \in \mathcal{N}_{lf}$.

1.1.1 Characterization of Pt Processes.

Spoiler: A pt process is characterized (i.e. uniquely determined) by either of the following: its finite dimensional distributions, its void distributions, or its generating functional.

Def X: a pt process.

The collection of joint dist. of $(N(B_1), \cdots, N(B_m))$, running over all $m \in \mathbb{N}$ and $B_1, \cdots, B_m \in \mathcal{B}_0$, is called the family of finite dimensional distributions of X.