1/4 Chap III Spectral seq. and Applications.

§ 14. The spectral seq. of a filtered complex.

- The $\boxed{\text{double complex}}$ introduced previously is a very degenerate case of $\boxed{\text{spectral seq.}}$
- Chap III constructs the $\boxed{\text{spectral seq. of a filtered complex}}$ and applies it to some situations.

Exact Couples

Def An $\boxed{\text{exact couple}}$ is an exact seq. of Ab. gps of the form

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

Define $\boxed{d: B \to B}$ by $d = jk$.

Then $d^2 = jkjk = j(kj)k = 0$.

Hence, $\boxed{H(B)} = \ker(d)/\operatorname{im}(d)$ is well-defined.

An exact couple $\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$ gives rise to another exact couple, called the $\boxed{\text{derived couple}}$,

$$\boxed{\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}}$$

as follows: $\boxed{(14.1)}$

(1) $\boxed{A'} := i(A)$, $\boxed{B'} := H(B)$.

(2) $\boxed{i'}(ia) := i(ia)$, $\forall a \in A$

$\boxed{j'}(ia) := [ja]$, $\forall a \in A$

$\boxed{k'}([b]) := kb$, $\forall b \in \ker(d)$.

Well-definedness of i', j', and k':

$\boxed{i'}$ $i(ia) \in i(A)$ since $ia \in A$.

$\boxed{j'}$ $d(ja) = jkja = j(kj)a = 0$. $\Rightarrow ja \in \ker(d)$.

$ia = i\bar{a} \Rightarrow i(a-\bar{a}) = 0 \Rightarrow a - \bar{a} \in \ker(i) = \operatorname{im}(k)$. $\Rightarrow a - \bar{a} = kb$, some $b \in B$.

$\Rightarrow ja - j\bar{a} = j(a-\bar{a}) = jkb = db \in \operatorname{im}(d)$.

$\boxed{k'}$ $b \in \ker(d) \Rightarrow db = 0 \Rightarrow jkb = 0$. $\Rightarrow kb \in \ker(j) = \operatorname{im}(i) = i(A)$.

$[b] = [\bar{b}] \Rightarrow b - \bar{b} \in \operatorname{im}(d) \Rightarrow b - \bar{b} = d\beta$, some $\beta \in B$. $\Rightarrow b - \bar{b} = jk\beta$

$\Rightarrow kb - k\bar{b} = k(b-\bar{b}) = kjk\beta = 0$. $\Rightarrow kb = k\bar{b}$.

That the derived comple is exact is easy to check!!

The Spectral Sequence of a Filtered Complex.

1/6 **Def** A differential complex K is an Abelian gp w/ a differential operator $K \xrightarrow{D} K$ s.t. $D^2 = 0$.

(*) Usually, K comes w/ a grading: $K = \bigoplus_{k \in \mathbb{Z}} C^k$, $D : C^k \to C^{k+1}$. But this is not absolutely necessary.

A $\underbrace{\text{subcomplex}}_{K'}$ of K is a subgp $K' \subseteq K$ s.t. $DK' \subseteq K'$.

A seq. of subcomplexes $K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ is called a filtration of K.

K w/ a filtration is called a filtered complex.

$\boxed{GK} := \bigoplus_{p=0}^{\infty} K_p / K_{p+1}$ is called the associated graded complex.

GK is again a differential complex w/ diff. operator induced from D.

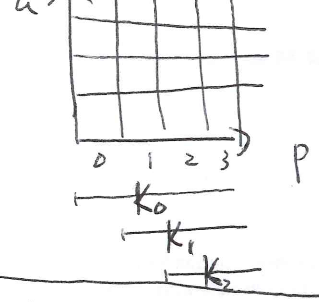
$\boxed{K_p := K, \ \forall p < 0}$ (For notational reasons).

<u>Example</u>:

$K = \bigoplus K^{p,q}$: double complex w/ horizontal operator δ and vertical operator d.

Then $K = \bigoplus C^k$, where $C^k = \bigoplus_{p+q=k} K^{p,q}$, $D : C^k \to C^{k+1}$ defined by $D = \delta + (-1)^p d$ makes K a differential complex.

$K_p := \bigoplus_{i \geq p} \bigoplus_{q \geq 0} K^{i,q}$, $p = 0, 1, 2, \cdots$ is then a filtration of K.



K : a filtered complex. ($K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$, $K_p = K$, $\forall p < 0$).

$\boxed{A} := \bigoplus_{p \in \mathbb{Z}} K_p$. A is again a d.c. w/ operator D.

$\boxed{i} : A \to A$, the inclusion : $K_{p+1} \hookrightarrow K_p$.

$\boxed{B}$:= the quotient $0 \to A \to A \to B \to 0$. i.e. $B = A / iA$.

(*) B is exactly the associated graded complex GK.

<u>case1</u> If K comes w/ a grading, then $0 \to A \to A \to B \to 0$ induces a long exact seq. of cohomology gps:

$$\cdots \to H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \to \cdots$$

Writing $H(A) = \oplus H^k(A)$, $H(B) = \oplus H^k(B)$, we then have an exact couple:

$$\begin{tikzcd} H(A) \arrow[rr, "i_1"] & & H(A) \arrow[dl, "j_1"] \\ & H(B) \arrow[ul, "k_1"] & \end{tikzcd} =: \boxed{\begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd}}$$

(The subscript of i is suppressed to avoid notational cumbersomeness later).

(*) This "i" needs no longer to be the inclusion map.

Case 2 K does not have a grading.

$$\boxed{\begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd}} := \begin{tikzcd} A \arrow[rr, hook, "K_{p+1} \hookrightarrow K_p"] & & A \arrow[dl, "\text{quotient map}"] \\ & B \arrow[ul, "0"] & \end{tikzcd}$$

, simply writing $0 \to A \to A \to B \to 0$ in triangular form.

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By (14.1), we obtain a seq. of exact couples:

$$\boxed{\begin{tikzcd} A_r \arrow[rr, "i"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd}}$$

Example: (An illustrative example).

K: filtered complex w/ grading and terminates after K_3:

$$\cdots = K_{-1} = K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq 0.$$

Then A_1 = direct sum of all terms of

$$\cdots \xleftarrow{\sim} H(K) \xleftarrow{\sim} H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow 0,$$

where the arrows is the map $A_1 \xrightarrow{i} A_1$.

A_2 = ...

$$\cdots \xleftarrow{\sim} H(K) \xleftarrow{\sim} H(K) \hookleftarrow iH(K_1) \leftarrow iH(K_2) \leftarrow iH(K_3) \leftarrow 0,$$

.... $A_2 \xrightarrow{i} A_2$.

A_3 = ...

$$\cdots \xleftarrow{\sim} H(K) \xleftarrow{\sim} H(K) \hookleftarrow iH(K_1) \hookleftarrow iiH(K_2) \leftarrow iiH(K_3) \leftarrow 0.$$

A_4 = ...

$$\cdots \xleftarrow{\sim} H(K) \xleftarrow{\sim} H(K) \hookleftarrow iH(K_1) \hookleftarrow iiH(K_2) \hookleftarrow iiiH(K_3) \hookleftarrow 0.$$

(*) The above seq. are not exact.

$\because$ All maps become inclusion in A_4

$\therefore A_4 = A_5 = A_6 = \cdots$ i.e. $\boxed{A \text{ is stationary after } A_4}$

Define $\boxed{A_\infty}$ to be the stationary value. (i.e. $A_\infty := A_4 = A_5 = A_6 = \cdots$)

Consider the exact couple:

$$\begin{tikzcd} A_4 \arrow[rr, "i"] & & A_4 \arrow[dl] \\ & B_4 \arrow[ul, "K_4"] & \end{tikzcd}$$

$\because$ i is inclusion and the couple is exact

$\therefore K_4 = 0 \Rightarrow K_5 = 0 \Rightarrow K_6 = 0 \Rightarrow K_p = 0,\ \forall p \geq 4.$

$\Rightarrow d_4 = j_4 K_4 = 0,\ d_5 = 0, \Rightarrow d_p = 0,\ \forall p \geq 4.$

Hence, $B_4 = B_5 = B_6 = \cdots$. i.e. $\boxed{B \text{ is stationary after } B_4.}$

Define $\boxed{B_\infty} = B_4\ (= B_5 = B_6 = \cdots)$.

After the 4th step, we have stationary exact couple:

$$\begin{tikzcd} A_\infty \arrow[rr, "i_\infty", "\text{inclusion}"'] & & A_\infty \arrow[dl] \\ & B_\infty \arrow[ul, "K_\infty = 0"] & \end{tikzcd}$$

A_∞ = direct sum of $\boxed{(14.4)}$

$$\cdots = H(K) = H(K) \supseteq iH(K_1) \supseteq iiH(K_2) \supseteq iiiH(K_3) \supseteq 0.$$

i_∞ = inclusion and identity map in (14.4).

Note the above exact couple is equivalent to

$$0 \to A_\infty \xrightarrow{i_\infty} A_\infty \to B_\infty \xrightarrow{K_\infty} 0.$$

Hence, $B_\infty = A_\infty / i_\infty A_\infty$ = associated graded complex of the filtration (14.4) on $H(K)$.

[redacted]

We now turn to general case:

K: filtered complex w/ grading: $\cdots = K = K \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$.

$\rightsquigarrow \cdots \leftleftarrows H(K) \leftleftarrows H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow \cdots$

$\boxed{F_p} :=$ image of $H(K_p)$ in $H(K)$ $(= \underbrace{ii\cdots i}_{p} H(K_p))$

Then we have a filtration

$$H(K) = F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots \quad \boxed{(14.5)}$$, called the $\boxed{\text{induced filtration}}$ on $H(K)$.

Def A filtration K_p on K has $\boxed{\text{length } \ell}$ if $K_\ell \neq 0$ but $K_p = 0,\ \forall p > \ell$.

Rmk: By the same arguments in "illustrative example", if a filtration on K has finite length, then

① A_r, B_r: eventually stationary, denoted by A_∞ and B_∞

② $B_\infty = \bigoplus F_p / F_{p+1}$, the associated graded complex of (14.5).

Conventionally, we write $\boxed{E_r}$ for B_r so that

$E_1 = H(B)$ w/ $d_1 = j_1 k_1$,

$E_2 = H(E_1)$ w/ $d_2 = j_2 k_2$,

$E_3 = H(E_2)$, etc.

"abelian" gps w/ differential.

Def A seq. of differential groups $\{E_r, d_r\}$ is called a $\boxed{\text{speceral sequence}}$ if $E_r = H(E_{r-1})$, $\forall r$.

If E_r : evenenually stationary, denote the stationary value by $\boxed{E_\infty}$.

If E_∞ = associated graded gp of some filtered gp H, we then say that the speceral seq. converges to H

i.e. $E_\infty = GH$, where H is some filtered gp.

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Def K : filtered complex w/ a grading.

$K = \bigoplus_{n \in \mathbb{Z}} K^n$. We shall call n the $\boxed{\text{dimension}}$ to distinguish it from the fileration deg p.

$\{K_p\}$: fileration on K.

$\boxed{K_p^n} := K^n \cap K_p$. Then $\{K_p^n\}_p$: fileration on K^n.

Thm (14.6)

$K = \bigoplus_{n \in \mathbb{Z}} K^n$: graded filtered complex w/ filtration $\{K_p\}$.

$H_D^*(K)$: cohomology of K w/ fileration given by (14.5)

i.e. $H_D^*(K) = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$

For each dim. n, the fileration $\{K_p^n\}$ has finite length.

$\Rightarrow$ the short exact seq.

$$0 \to \bigoplus K_{p+1} \to \bigoplus K_p \to \bigoplus K_p/K_{p+1} \to 0$$

induces a speceral seq. which converges to $H_D^*(K)$.