List of materials to be tested in final Exam. Fall. 2016. Stat517.

Midterm covered. First 11 sections.

- Lebesgue measure in $\mathbb{R}^K$

Theorem 12.5

$$F : \mathbb{R}^K \to \mathbb{R}$$

$$\Delta_A F \geq 0.$$

$\exists$ unique measure μ on $\mathbb{R}^K$ s.t.

$\mu(A) = \Delta_A F \quad \forall A$ bounded rectangles.

When $F = x_1 \cdots x_K$.

μ is the Lebesgue measure $\mathbb{R}^K$

(iii) Measurable functions.

* $(\Omega, \mathcal{F}), \ (\Omega', \mathcal{F}')$

  $T : \Omega \to \Omega'$ (m) $\mathcal{F}/\mathcal{F}'$

  $\Leftrightarrow \ T^{-1}(\mathcal{F}') \subseteq \mathcal{F}$

  $T^{-1}(\mathcal{F}') = \{ T^{-1}(A') : A' \in \mathcal{F}' \}$

* induces a measure $\mu \circ T^{-1}$ on $\mathcal{F}'$

* Set inverse and its properties.

  Remember: this function is particularly <u>nice</u>.

* Theorem 13.1

  T (m) $\mathcal{F}/\mathcal{F}'$, T_1 (m) $\mathcal{F}'/\mathcal{F}''$

  $\Rightarrow T_1 \circ T$ (m) $\mathcal{F}/\mathcal{F}''$

Relies heavily on set inverse function

* Theorem 13.2

Continuous functions are measurable

* Theorem 13.3.

marginally measurable ⇒ Jointly measurable.

Problem idea. Show this beyond Euclidean space.

(ii) Limit and measurability

Theorem 13.4. f_n @ $\mathcal{F}/\mathcal{R}$

$\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$

$\lim_n f_n$, @ $\mathcal{F}/\mathcal{R}$.

and other results.

Approximating (m) function by simple functions.

Theorem 13.5

$f: \Omega \to \bar{\mathbb{R}}$ (m) $\mathcal{F}/\mathcal{R}$. $\exists \{f_n\}$ simple

s.t. $0 \le f_n(\omega) \uparrow f(\omega)$ $f(\omega) \ge 0$

$0 \ge f_n(\omega) \downarrow f(\omega)$ $f(\omega) \le 0$

Measure induced by transformation

$\mu \circ T^{-1}(A') : A' \in \mathcal{R}'$

This is a measure.

(rely on inverse set function properties).

Section 14. Distribution

Random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{R})$

Random vector. $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^k, \mathcal{R}^k)$.

Random element. $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$.

Distribution:

$X : \Omega \to \Omega'$. $\mathcal{F}/\mathcal{F}'$,

$P \circ X^{-1}$ is the distribution of X.

Distribution function. for random variable X.

$$F(x) = PX^{-1}(-\infty, x] = P(X \in (-\infty, x])$$

Right continuous, left limit is $P(X < x)$. Jump: $P(X = x)$.

At most countably many jumps.

Theorem 14.1

distribution function F determines a probability measure.

Section 15 Integration

$(\Omega, \mathcal{F}, \mu)$. $f : \Omega \to \overline{\mathbb{R}}$ (m) $\mathcal{F}/\overline{\mathcal{R}}$

(///) $\mathcal{P}$: the collection of all finite $\mathcal{F}$-partitions.

(///) $f \geq 0$.

$$\int f \, d\mu$$

$$= \sup_{\{A_i\} \in \mathcal{P}} \sum_{i=1}^{k} \underbrace{\left[\inf_{A_i} f(\omega)\right] \mu(A_i)}.$$

$0 \cdot \infty = 0$

$\infty \cdot 0 = 0$

$\infty \cdot x = \infty$

$x \cdot \infty = \infty$.

$\infty \cdot \infty = \infty$

f @ μ $\qquad \int f \, d\mu < \infty$.

f^+, f^-.

Suppose one of.

$\int f^+ d\mu \qquad \int f^- d\mu$

is finite, then

$$\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu .$$

$\int f^+ d\mu$	$\int f^- d\mu$	
$< \infty$	$< \infty$	f @ μ.
$< \infty$	$= \infty$	f $\cancel{@}$ μ. but has definite integral.
$= \infty$	$< \infty$	f $\cancel{@}$ μ. but has definite integral.
$= \infty$	$= \infty$	f doesn't have definite integral.

⊘ Properties integral non neg case.

$(\Omega, \mathcal{F})$. $f, g : \Omega \to \bar{\mathbb{R}}$ ⓜ $\mathcal{F}/\bar{\mathcal{R}}$.

Theorem 15.1

- $f = \sum x_i I_{A_i} \geq 0$. simple.

$$\int f \, d\mu = \sum x_i \mu(A_i)$$

- $0 \leq f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$.
- $0 \leq f_n \uparrow f \Rightarrow \int f_n \, d\mu \to \int f \, d\mu$.
- $f \geq 0$, $g \geq 0$, $\alpha \geq 0$, $\beta \geq 0$.

$$\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu.$$

⊘ Almost everywhere μ.

$(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$.

$f : \Omega \to \Omega'$, ⓜ $\mathcal{F}/\mathcal{F}'$

$B' \in \mathcal{F}'$.

$f \in B'$ a.e. μ if $\mu(f \notin B')$
$= \mu f^{-1}((B')^c) = 0$.

Theorem 15.2. $f \geq 0$, $g \geq 0$

- $f = 0$ a.e. $\mu \Rightarrow \int f \, d\mu = 0$.
- $\int f \, d\mu \neq 0 \Rightarrow f \neq 0$ a.e. μ.
- $\int f \, d\mu < \infty \Rightarrow f < \infty$ a.e. μ.
- $f \leq g$ a.e. $\Rightarrow \int f \, d\mu \leq \int g \, d\mu$.
- $f = g$ a.e. $\mu \Rightarrow \int f \, d\mu = \int g \, d\mu$.

Section 16. Properties of integral (integrable case).

Theorem 16.1

- f, g @ μ. $f \leq g$ a.e. $\Rightarrow \int f d\mu \leq \int g d\mu$
- f, g @ μ, $\alpha, \beta \in \mathbb{R}$

$$\int \alpha f + \beta g \, d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

Theorem 16.2. (MCT) $0 \leq f_n \uparrow f$ a.e. μ.

$$\Rightarrow \int f_n d\mu \to \int f d\mu$$

Theorem 16.3 (Fatou)

$f_n \geq 0$ @ $\mathcal{F} / \bar{\mathbb{R}}$

$$\Rightarrow \int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n d\mu.$$

Theorem 16.4 DCT.

$|f_n| \le g$. g ⓘ μ. $f_n \to f$ a.e. μ

$\Rightarrow \int f_n \, d\mu \to \int f \, d\mu$

Theorem 16.5 BCT.

$\mu(\Omega) < \infty$, $|f_n| \le M$.

$f_n \to f$ a.e. μ $\Rightarrow \int f_n \, d\mu \to \int f \, d\mu$

Theorem 16.6 (Series MCT).

$f_n \geq 0$ $\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$.

Theorem 16.7 Series DCT

$\left|\sum_{k=1}^{n} f_k\right| \le g$. $\sum_n f_n$ converges a.e. μ.

g ⓘ μ. $\Rightarrow \int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$

Theorem 16.8. Apply DCT to continuity & differentiability of

$$t \mapsto \int f(\omega, t)\, d\mu(\omega).$$

- Integral over set

- $\int_A f\, d\mu = \int f I_A\, d\mu$.

- Theorem 16.9 A_n disjoint.

$$\int_{\cup_n A_n} f\, d\mu = \sum_n \int_{A_n} f\, d\mu$$

(nonneg. integrable case).

- Theorem 16.9

$$\int_A f\, d\mu = \int_A g\, d\mu \quad \forall A \in \mathcal{F}$$

$$\Rightarrow f = g \quad a.e.\ \mu.$$

(nonneg. integrable)

$f \cdot g$ @ μ.

$\int_A f \, d\mu = \int_A g \, d\mu$ $\quad A \in \pi$ system generating $\mathcal{F}$

Ω is countable union of P sets

$f = g$ a.e. μ.

③ Density

Theorem 16.11 (density theorem).

$d\nu = \delta \, d\mu \Rightarrow \int f \, d\nu = \int f \delta \, d\mu$

(nonneg. integrable.

Theorem 16.12 (Scheffe).

$d\nu_n = \delta_n \, d\mu$

$d\nu = \delta \, d\mu$.

$\nu_n(\Omega) = \nu(\Omega) < \infty$.

$\delta_n \to \delta$.

$$\Rightarrow \sup_{A \in \mathcal{F}} | \nu_n(A) - \nu(A) | \le \int |\delta_n - \delta| \, d\mu \to 0.$$

(III) Change of variable:

Theorem 16.13.

$$(\Omega, \mathcal{F}) \xrightarrow{T} \overset{A'}{(\Omega', \mathcal{F}')} \xrightarrow{f} (\bar{\mathbb{R}}, \bar{\mathcal{R}})$$

$$\int_{A'} f \, d\mu T^{-1} = \int_{T^{-1}(A')} (f \circ T) \, d\mu$$

(nonneg. ② cases)

(11) Uniform Integrability

$$\lim_{\alpha} \sup_{n} \int_{|f_n| \geq \alpha} |f_n| \, d\mu < \infty \qquad f_n \text{ (u) } \mu$$

Theorem 16.14. $\mu(\Omega) < \infty$.

(i) $f_n \to f$ a.e. μ f_n (u) μ

$$\Rightarrow \int f_n \, d\mu \to \int f \, d\mu.$$

(ii) $f \geq 0$, $f_n \geq 0$ f (i) μ. f_n (i) μ

$$\int f_n \, d\mu \to \int f \, d\mu \Rightarrow f_n \text{ (u) } \mu.$$