1.

(a)

T_pM

$= \{ v : C^\infty(M) \to \mathbb{R} \mid v :$ linear and satisfies

$v(fg) = v(f)\cdot g(p) + f(p)\cdot v(g), \forall f,g \in C^\infty(M) \}$

(b)

a curve $\gamma : I \to M$ s.t.

$$X_{\gamma(t)} = \dot\gamma(t), \forall t \in I.$$

(c)

$\exists$ $(k-1)$-form η s.t. $d\eta = \omega$.

2.

(a)

Using an adapted chart of S (whose existence is guaranteed by S embedded in M), $\gamma : J \to S$ is cont.

$\Rightarrow \gamma : J \to S$ is smooth.

Thus, by considering $J \xrightarrow{\gamma} S \xhookrightarrow{i} M$,

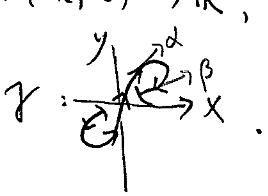
$\gamma'(t) = d\gamma_t(\frac{d}{dt}|_t) \cong di_{\gamma(t)}(d\gamma_t(\frac{d}{dt}|_t)) \in T_{\gamma(t)}S \subseteq T_{\gamma(t)}M$. #

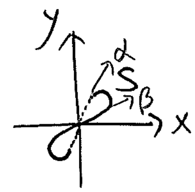
(b)

Consider S = the figure 8 immersed in $\mathbb{R}^2$.

Let $\gamma : (-\pi,\pi) \to \mathbb{R}^2$,

where γ :





Then, at the origin $(0,0)$, $\gamma'(t) = \alpha \notin T_{(0,0)}S$,

since $T_{(0,0)}S = sp_{\mathbb{R}}(\beta)$. #

3.

Since $\dim(\mathfrak{g}) = \dim(G) \overset{\text{since } F:\text{ local diffeo.}}{=} \dim(H) = \dim(\mathfrak{h})$,

it suffices to prove $\ker(F_*) = 0$.

Given $X \in \ker(F_*)$.

i.e. $F_*(X) = 0$.

i.e. $dF_g(X_g) = 0, \forall g \in G$.

$\because dF_g$: iso. $\therefore X_g = 0, \forall g \in G$.

i.e. $X = 0$ in $\mathfrak{g}$.

$\Rightarrow \ker(F_*) = 0$, as desired. #

5.

(a)

Let $E_1, \cdots, E_n$ be a basis of $\mathrm{Lie}(G)$.

Let $e^1, \cdots, e^n$ be its duals.

Define $\omega = e^1 \wedge \cdots \wedge e^n$.

Then $\omega(E_1, \cdots, E_n) = \det(e^i(E_j)) = 1 > 0$. $(*_0)$

Thus, ω is a nowhere vanishing n-form on G. $(*_1)$

Note that, for any $g \in G$,

$L_g^*(e^j)(E_i) = e^j((L_g)_*(E_i)) \overset{\text{since } E_i \in \mathrm{Lie}(G)}{=} e^j(E_i), \forall i,j$.

Thus $L_g^* e^j = e^j, \forall j$.

$\Rightarrow L_g^*\omega = L_g^*(e^1 \wedge \cdots \wedge e^n) = (L_g^* e^1) \wedge \cdots \wedge (L_g^* e^n)$

$= e^1 \wedge \cdots \wedge e^n = \omega$.

i.e. ω : left-invariant. $(*_2)$

Therefore, by $(*_1)$, $(*_2)$, ω is nowhere vanishing left-invariant. #

(b)

For existence, let $\tilde\omega = (1/\int_G \omega)\cdot\omega$. (so $\int_G \tilde\omega = 1$).

That $\tilde\omega$ is positively oriented comes from ω being positively oriented. (See $(*_0)$ above).

For uniqueness, suppose ω' is another such n-form.

Assume $\omega'_e = c\cdot\tilde\omega_e$.

By left-invariance,

$\omega'_g = L_{g^{-1}}^*\omega'_e = L_{g^{-1}}^* c\cdot\tilde\omega_e = c L_{g^{-1}}^*\tilde\omega_e = c\cdot\tilde\omega_g$.

Thus, $\omega'_g = c\cdot\tilde\omega_g, \forall g \in G$.

$\because \int_G \tilde\omega = \int_G \omega' \therefore c = 1 \Rightarrow \omega' = \tilde\omega$, proving uniqueness. #