1 Gradient

Given $f \in C^\infty(M)$, define $\nabla: C^\infty(M) \longrightarrow \mathfrak{X}(M)$ by

$\nabla f = (df)^\#$. We call ∇f the gradient of f.

2 Lie Derivative (brief review)

For $V, W \in \mathfrak{X}(M)$, define $\mathcal{L}_V W \in \mathfrak{X}(M)$ (Lie derivative of W with respect to V) as

$$(\mathcal{L}_V W)_p = \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t}$$

Recall $\mathcal{L}_V W = [V, W]$

(this is on page 229, Lee)

Similarly, given a smooth k-covariant tensor field $\alpha \in \mathcal{T}^k(M)$ and $V \in \mathfrak{X}(M)$.

we define $\mathcal{L}_V\alpha \in \mathcal{T}^k(M)$, Lie derivative of α w.r.t V,

by $$\left(\mathcal{L}_V\alpha\right)_p = \lim_{t\to 0} \frac{d(\theta_t)_p^*(\alpha_{\theta_t(p)}) - \alpha_p}{t}$$

Recall, if $W_1, \cdots, W_k \in \mathfrak{X}(M)$, we have

$$\mathcal{L}_V(\alpha(W_1,\cdots,W_k)) = (\mathcal{L}_V\alpha)(W_1,\cdots,W_k) + \alpha(\mathcal{L}_V W_1,\cdots,W_k) + \cdots + \alpha(W_1,\cdots,\mathcal{L}_V W_k) \quad (*)$$

(this is on page 322, Lee)

Moreover, if $\alpha \in \Omega^k(M)$, then $\mathcal{L}_V\alpha \in \Omega^k(M)$, and

$$\mathcal{L}_V\alpha = d(i_V\alpha) + i_V(d\alpha) \qquad (**)$$

Here $i_V : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$i_V\alpha(V_2,\cdots,V_k) := \alpha(V, V_2, \cdots, V_k)$$

for $V_2, \cdots, V_k \in \mathfrak{X}(M)$

The formula (**) is called <u>Cartan's magic formula</u>, (on page 372, Lee)

3 <u>Divergence</u>

Say (M, g) is an oriented Riemannian manifold ω_g is the Riemannian volume form. Recall, in any oriented coordinates (x^i), we have

$$\omega_g = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n \quad \text{with } g := \det[g_{ij}]$$

Now let's define $\nabla\cdot : \mathfrak{X}(M) \to C^\infty(M)$ by

$$\underbrace{(\nabla \cdot V)}_{\text{divergence of } V} \omega_g = \mathcal{L}_V(\omega_g)$$

for all $V \in \mathfrak{X}(M)$.

If $\nabla \cdot W = 0$, we call W "divergence-free".

Now, our goal is to find local expression for $\nabla \cdot V$. Due to $(*)$, we have

$$\mathcal{L}_V(\omega_g)(\partial_1, \cdots, \partial_n)$$

$$= V(\underbrace{\cancel{\omega_g(\partial_1, \cdots, \partial_n)}}_{\sqrt{g}}) - \omega_g(\mathcal{L}_V \partial_1, \cdots, \partial_n)$$

$$- \cdots - \omega_g(\partial_1, \cdots, \mathcal{L}_V \partial_n)$$

Recall,

$$\mathcal{L}_V \partial_k = [V^i \partial_i, \partial_k] = V^i \overset{0}{\cancel{[\partial_i, \partial_k]}} - \partial_k(V^i)\partial_i$$

$$= -\partial_k(V^i)\partial_i$$

Here we use $[\partial_i, \partial_j] \equiv 0$ for all i, j (on page 187, Lee)

Then,

$$\omega_g(\partial_1, \cdots, \mathcal{L}_V \partial_i, \cdots, \partial_n) = -\partial_i(V^i)\, \overset{\sqrt{g}}{\cancel{\omega_g(\partial_1, \cdots, \partial_n)}}$$

no summation

So combining all the stuffs, we obtain.

$$\mathcal{L}_V(\omega_g)(\partial_1, \cdots, \partial_n) = V(\sqrt{g}) + \partial_i(V^i)\sqrt{g}$$
$$= V^i \partial_i(\sqrt{g}) + \partial_i(V^i)\sqrt{g}$$
$$= \partial_i(\sqrt{g}\, V^i)$$

So we conclude,

$$\sqrt{g}\,(\nabla \cdot V) = \partial_i(\sqrt{g}\, V^i)$$

or

$$\nabla \cdot V = \frac{1}{\sqrt{g}}\, \partial_i(\sqrt{g}\, V^i) \qquad (***)$$

$\boxed{4}$ Alternative definition of divergence:

Let $\nabla\cdot : \mathcal{X}(M) \to C^\infty(M)$ by $\nabla\cdot V = tr(\nabla V)$

covariant derivative

Recall $V^i{}_{;j} = \partial_j V^i + \Gamma^i{}_{jk} V^k$

So

$$\nabla\cdot V = V^i{}_{;i} = \partial_i V^i + \Gamma^i{}_{ij} V^j$$

when the underlying connection ∇ is of Levi-Civita,

we have

$$\frac{\partial g_{kj}}{\partial x^i} = \Gamma^l{}_{ik}\, g_{lj} + \Gamma^l{}_{ij}\, g_{kl}$$

Then,

$$\nabla\cdot V = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, V^i\right)$$

$$= \frac{\partial V^i}{\partial x^i} + \frac{1}{\sqrt{g}} \frac{1}{2\sqrt{g}} \frac{\partial g}{\partial x^i} V^i$$

$$= \frac{\partial V^i}{\partial x^i} + \frac{g}{2g} \operatorname{tr}\left([g^{ab}] \frac{\partial [g_{ab}]}{\partial x^i} \right) V^i$$

$$= \frac{\partial V^i}{\partial x^i} + \frac{1}{2} g^{jk} \frac{\partial (g_{kj})}{\partial x^i} V^i$$

$$= \frac{\partial V^i}{\partial x^i} + \frac{1}{2} g^{jk} \left(\Gamma^{l}{}_{ik} g_{lj} + \Gamma^{l}{}_{ij} g_{kl} \right) V^i$$

$$= \frac{\partial V^i}{\partial x^i} + \Gamma^{k}{}_{ik} V^i$$

Here we use the Jacobi formula,

$$\frac{\partial}{\partial x^i}\left(\det [A(x_i)]\right) = \operatorname{tr}\left(\operatorname{adj}[A] \frac{\partial [A]}{\partial x^i} \right)$$

$$= \operatorname{tr}\left(\det([A]) [A]^{-1} \frac{\partial [A]}{\partial x^i} \right)$$

5 Laplacian - Beltrami operator

On an oriented Riemannian manifold (M, g),

$\Delta_g : C^\infty(M) \longrightarrow C^\infty(M)$ is defined by

$$\underbrace{\Delta f}_{\text{Laplacian of } f} := \nabla \cdot (\overset{\text{gradient}}{\nabla} f) \qquad \forall f \in C^\infty(M)$$

Recall that $(\nabla f)^i = g^{ij} \dfrac{\partial f}{\partial x^j}$, so by (***),

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j}\right)$$

6 Alternative definition of Laplacian - Beltrami

Let $\nabla^2 = \nabla \circ \nabla$ (covariant derivative) be the covariant Hessian. Then,

$\Delta f := tr(\nabla^2 f)$, for all $f \in C^\infty(M)$.

$\boxed{7}$ Remarks

1) other versions of Laplacian Δ:

- Laplace-de Rham operator on $\Omega^k(M)$ $(k \geq 0)$, namely,

$$\underbrace{\Delta\alpha}_{\in\Omega^k(M)} = \delta d\alpha + d\delta\alpha \quad \text{for all } \alpha \in \Omega^k(M).$$

- fractional Laplacian Δ^s $(0 < s < 1)$

2) On page 43 (textbook), Lee introduced $\nabla \cdot V$ by $(\nabla \cdot V)\omega_g = d(i_V \omega_g)$. This definition is equivalent to ours due to (**).

3) Divergence theorem and "integration by parts" formula can be generalized to oriented Riemannian manifold with boundary (see page 43, textbook)

4) Let (M, g) be compact, connected, oriented Riemmanian manifold with ∂M. A function $u \in C^\infty(M)$ is said to be <u>harmonic</u> if $\Delta_g u = 0$. If two harmonic functions u, v agree on ∂M, then $u \equiv v$.

5) Let (M, g) be compact, oriented with $\partial M = \phi$. A number $\lambda \in \mathbb{R}$ is called an <u>eigenvalue</u> of Δ if $\exists \underset{\neq 0}{u} \in C^\infty(M)$ s.t. $\Delta u = \lambda u$. In this case, u is an <u>eigenfunction</u> w.r.t λ.

Exercise: if λ is an eigenvalue of Δ, then $\lambda \leq 0$.