Probability Outline:

Sec 2. 17 ~ 26.

field. $\mathcal{F}$-set, measurable $\mathcal{F}$. $\mathcal{B}_0$, $\mathcal{B}$.
σ-field.

$\sigma(\mathcal{A})$, σ-field generated by $\mathcal{A}$.

set function. / probability measure. on a field.

probability measure space / probability space.

support.
discrete probability measure.
inclusion-exclusion formula.
finite subadditivity (or Boole's inequality).

Thm 2.1 $A_n \uparrow A$, P: prob. measure on a field.
Then (1) Cont. from below (2) from above (3) Countably subadditive.
Thm 2.2 Leb. measure λ on $\mathcal{B}_0$ is a (countably additive) probability measure.

Sec 3. 36(b) ~ 44. (Existence and Extension).

Thm 3.1 A prob mea on a field has a unique extension to the generated σ-field.
$\boxed{\text{Existence}}$
outer measure P^* / inner measure P_*. / $\mathcal{M} := \{A \subseteq \Omega \mid P^*(E) = P^*(EA) + P^*(EA^c), \forall E\}$.

4 properties of P^*.

Lemma 1 $\mathcal{M}$ is a field.
Lemma 2 $A_1, A_2, \cdots \in \mathcal{M}$ disjoint. $E \subseteq \Omega$. $\Rightarrow P^*(E(\bigcup_n A_n)) = \sum_n P^*(EA_n)$.
Lemma 3 $\mathcal{M}$: σ-field. and $P^*|_{\mathcal{M}}$: countably additive.
} only use the 4 properties.
Lemma 4 $P^* := \inf\{\cdots\}$. Then $\mathcal{F}_0 \subseteq \mathcal{M}$. ← $\boxed{\text{use the explicit form of } P^*}$
Lemma 5. $P^* := \inf\{\cdots\}$. Then $P^*(A) = P(A)$, $\forall A \in \mathcal{F}_0$.

$\boxed{\text{Uniqueness}}$

π-system / λ-system. Rmk: σ-field $\Rightarrow$ λ-system. But not vice versa.

Lemma 6 $\mathcal{A}$: both π-system and λ-system. $\Rightarrow$ $\mathcal{A}$: σ-field.

Thm 3.2 (Dynkin's π-λ theorem).
$\mathcal{P}$: π-system. $\mathcal{L}$: λ-system. $\mathcal{P} \subseteq \mathcal{L}$.
$\Rightarrow \sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Thm 3.3 (Uniqueness of extension)
P_1, P_2 : prob. mea. on $\sigma(\mathcal{P})$, where $\mathcal{P}$: π-system.
$P_1 = P_2$ on $\mathcal{P}$.
$\Rightarrow P_1 = P_2$ on $\sigma(\mathcal{P})$.

Monotone Classes

monotone class.

Thm 3.4 (Halmos's monotone class theorem).
$\mathcal{F}_0$: field. $\mathcal{M}$: monotone class.
$\mathcal{F}_0 \subseteq \mathcal{M}$.
$\Rightarrow \sigma(\mathcal{F}_0) \subseteq \mathcal{M}$.

Sec 4. Denumerable Probabilities. (51~64).

$\underline{\lim}\, A_n$, $\overline{\lim}\, A_n$, $\lim A_n = A$.

Thm 4.1
(1) $P(\underline{\lim}\, A_n) \le \underline{\lim}\, P(A_n) \le \overline{\lim}\, P(A_n) \le P(\overline{\lim}\, A_n)$.
(2) $A_n \to A \Rightarrow P(A_n) \to P(A)$.

independence of events / independence of classes.

Rmk : Pairwise indep. does not imply indep.

Thm 4.2
$\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$: indep. π-systems.
$\Rightarrow \sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \cdots, \sigma(\mathcal{A}_n)$: indep.

Cor 1
$\mathcal{A}_\theta$, $\theta \in \Theta$: indep. π-systems.
$\Rightarrow \sigma(\mathcal{A}_\theta)$, $\theta \in \Theta$: indep.

Cor 2 Suppose
$$\begin{matrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots & \vdots & \end{matrix}$$
are indep.
$\mathcal{F}_i :=$ σ-field generated by the ith row.
Then $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \cdots$: indep.

Rmk : (by Thm 4.2)
$A_1, A_2, \cdots, A_n$: indep. $\Rightarrow P(A_1^c A_2^c \cdots A_k^c A_{k+1} \cdots A_n) = P(A_1^c)\, P(A_2^c) \cdots P(A_k^c)\, P(A_{k+1}) \cdots P(A_n)$, $k = 1, \cdots, n$.

Thm 4.3 (first Borel-Cantelli lemma)

$\sum_n P(A_n) < \infty \Rightarrow P(\limsup A_n) = 0$.

Thm 4.4 (second Borel-Cantelli lemma).

$\{A_n\}$: indep.

$\sum P(A_n) = \infty$

$\Rightarrow P(\limsup A_n) = 1$.

Def $A_1, A_2, A_3, \cdots \in \mathcal{F}$, where $(\Omega, \mathcal{F}, P)$: probability space.

$\mathcal{T} := \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \cdots)$, called the tail σ-field of the seq.

Elements in $\mathcal{T}$ are called tail events.

Rmk: $\liminf A_n, \limsup A_n \in \mathcal{T}$.

Thm 4.5 (Kolmogorov's zero-one law).

$A_1, A_2, A_3, \cdots$: indep.

$A \in \mathcal{T}$, where $\mathcal{T}$ is the tail σ-field of $A_1, A_2, \cdots$.

$\Rightarrow P(A) = 0$ or 1.

Use Cor 2 to prove it.

Goal : A is indep. of itself.

Section 10 General Measures. (158~164).

$\mathbb{R}^k$. $\mathcal{A} \cap \Omega_0 := \{A \cap \Omega_0 \mid A \in \mathcal{A}\}$. / σ-finite on $\mathcal{A} \subseteq \mathcal{F}$.

Thm 10.1 $\Omega_0 \subseteq \Omega$.

(1) $\mathcal{F}$: σ-field in Ω. $\Rightarrow \mathcal{F} \cap \Omega_0$: σ-field in Ω_0.

(2) $\mathcal{F} = \sigma(\mathcal{A}) \Rightarrow \mathcal{F} \cap \Omega_0 = \sigma(\mathcal{A} \cap \Omega_0)$. i.e. $(\sigma(\mathcal{A}) \cap \Omega_0 = \sigma(\mathcal{A} \cap \Omega_0))$.

Rmk:

If $\Omega_0 \in \mathcal{F}$, then $\mathcal{F} \cap \Omega_0 = \{A \in \mathcal{F} \mid A \subseteq \Omega_0\}$.

Thm 10.2

μ : measure on a field $\mathcal{F}$.

(i) Continuity from below.

(ii) Continuity from above.

(iii) Countable subadditivity.

(iv) If μ is σ-finite on $\mathcal{F}$, then $\mathcal{F}$ cannot contain an uncountable, disjoint collection of sets of positive μ-measure.

Thm 10.3

μ_1, μ_2 : measures on $\sigma(\mathcal{P})$, where $\mathcal{P}$: π-system.

μ_1, μ_2 : σ-finite on $\mathcal{P}$.

$\mu_1 = \mu_2$ on $\mathcal{P}$.

Then $\mu_1 = \mu_2$ on $\sigma(\mathcal{P})$.

Sec 11. Outer Measures.

Def outer measure μ^* / μ^*-measurable / $\mathcal{M}(\mu^*)$

Thm 11.1 μ^*: outer measure $\Rightarrow$ $\mathcal{M}(\mu^*)$: σ-field and $\mu^*|_{\mathcal{M}(\mu^*)}$: measure.

Thm 11.2 A measure on a field has an extension to the generated σ-field.

Def semiring.

Thm 11.3

μ: set function on $\mathcal{A}$, $\mathcal{A}$: semiring.

$\mu(\emptyset)=0$, μ: finitely additive and countably subadditive.

μ: has values in $[0,\infty]$.

$\Rightarrow$ μ extends to a measure on $\sigma(\mathcal{A})$.

Rmk: ① Thm 11.3 $\Rightarrow$ Thm 11.2.

② If μ: σ-finite on $\mathcal{A}$, then by Thm 10.3, the extension is unique.

③ The benefit of introducing semiring is that we don't have to bother ourselves by extending first to fields.

Lemma 1

$A, A_1, \cdots, A_n \in \mathcal{A}$, $\mathcal{A}$: semiring.

$\Rightarrow \exists$ disjoint $C_1, \cdots, C_m \in \mathcal{A}$ s.t. $A \cap A_1^c \cap \cdots \cap A_n^c = C_1 \cup \cdots \cup C_m$.

Thm 11.4 (Approximation Theorem)

$\mathcal{A}$: semiring

μ: measure on $\mathcal{F}=\sigma(\mathcal{A})$, μ: σ-finite on $\mathcal{A}$.

Then, for $B \in \mathcal{F}$ and $\varepsilon > 0$,

(i) $\exists$ countably many disjoint $A_1, A_2, \cdots \in \mathcal{A}$ s.t. $B \subseteq \bigcup_{n=1}^{\infty} A_n$ and $\mu(\bigcup_n A_n \setminus B) < \varepsilon$.

(ii) if $\mu(B) < \infty$, then $\exists$ finitely many disjoint $A_1, \cdots, A_n \in \mathcal{A}$ s.t. $\mu(B \triangle (\bigcup_{k=1}^{n} A_k)) < \varepsilon$.

Cor 1.

μ: finite measure on $\mathcal{F}$, where $\mathcal{F}$: σ-field generated by a field $\mathcal{F}_0$.

$\Rightarrow$ $\forall A \in \mathcal{F}$ and $\varepsilon > 0$, $\exists B \in \mathcal{F}_0$ s.t. $\mu(A \triangle B) < \varepsilon$.

Cor 2.

$\mathcal{A}$: semiring, $\mu_1(A) \le \mu_2(A) < \infty$, $\forall A \in \mathcal{A}$.

μ_1, μ_2: measures on $\mathcal{F}=\sigma(\mathcal{A})$ Then $\mu_1(B) \le \mu_2(B)$, $\forall B \in \mathcal{F}$.

μ_1, μ_2: σ-finite on $\mathcal{A}$

Lemma 2 (used in Sec 12).

μ: nonnegative, finitely additive, set function on $\mathcal{A}$, $\mathcal{A}$: semiring.

$A, A_1, \cdots, A_n \in \mathcal{A}$.

(i) If $\bigcup_{k=1}^{n} A_k \in A$, and $A_1, \cdots, A_n$: disjoint, then $\sum_{k=1}^{n} \mu(A_k) \leq \mu(A)$.

(ii) If $A \subseteq \bigcup_{k=1}^{n} A_k$, then $\mu(A) \leq \sum_{k=1}^{n} \mu(A_k)$.