(c) Show that $A_n \to A$ and B_n together imply that $A_n \cup B_n \to A \cup B$ and $A_n \cap B_n \to A \cap B$.

Problem 2.6. For events $A_1, \ldots, A_n$, consider the 2^n equations

$$P(B_1 \cdots B_n) = P(B_1) \cdots P(B_n),$$

where $B_i = A_i$ or $B_i = A_i^c$ for each i. Show that $A_1, \ldots, A_n$ are independent if all these equations hold.

Problem 2.7. Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are π-systems and $\mathcal{A}_1 \perp\!\!\!\perp \cdots \perp\!\!\!\perp \mathcal{A}_n$. Let $\mathcal{B}_i = \mathcal{A}_i \cup \{\Omega\}$. Show that $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are π-systems and $\mathcal{B}_1 \perp\!\!\!\perp \cdots \perp\!\!\!\perp \mathcal{B}_n$.

Problem 2.8. Show that $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$.

Problem 2.9. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

1. Show that, for any sequence of independent $\mathcal{F}$-sets, say $\{B_n : n = 1, 2, \ldots\}$, we have

$$P(\cap_{n=1}^{\infty} B_n) = \prod_{n=1}^{\infty} P(B_n).$$

2. Use the above relation and the inequality in Problem 2.8 to prove the second Borel-Cantelli Lemma.

Problem 2.10. Show that a λ-system can be equivalently defined by these three conditions:

1. $\Omega \in \mathcal{L}$;
2. If $A \in \mathcal{L}$, $B \in \mathcal{L}$, and $A \subseteq B$, then $BA^c \in \mathcal{L}$;
3. If $A_1, A_2, \ldots$ are a disjoint sequence of members of $\mathcal{L}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{L}$.

3 HW3: due October 7, 2016

✓ **Problem 3.1.** Show that, in the definition of measure on a field, if condition (i) and (iii) hold, and if $\mu(A) < \infty$ for some $A \in \mathcal{F}$, then condition (ii) holds.

✓ **Problem 3.2.** On a σ-field of all subsets of $\Omega = \{1, 2, \ldots\}$, define the set function

$$\mu(A) = \begin{cases} \sum_{k \in A} 2^{-k} & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

Is μ finitely additive? Is μ countably additive?

✓ **Problem 3.3.**

1. In connection with Theorem 10.2 (ii), show that if $A_n \downarrow A$ and $\mu(A_k) < \infty$ for some k, then $\mu(A_n) \downarrow \mu(A)$.
2. Find an example in which $A_n \downarrow A$, $\mu(A_n) = \infty$ for all n, and $A = \varnothing$.

Problem 3.4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. The following is a generalization of Theorem 4.1, part (i).

√1. Show that

$$\mu\left(\liminf_n A_n\right) \leq \liminf_n \mu(A_n)$$

√2. If $\mu(\cup_{k \geq n} A_k) < \infty$ for some n then

$$\limsup_n \mu(A_n) \underset{\leq}{\cancel{\leq}} \mu\left(\limsup_n A_n\right).$$

Show that this inequality ~~equality~~ can fail if $\mu(\cup_{k \geq n} A_k) = \infty$ for all n.

The next three problems give an alternative approach to extend a measure from a field to the σ-field generated by it.

√ **Problem 3.5.** Extend Theorem 3.1 to finite measure. That is, a finite measure on a field has a unique extension to the generated σ-field. Hint: a finite measure can always be re-scaled to a probability measure.

√ **Problem 3.6.** Suppose Ω is a nonempty set, $\mathcal{F}_0$ is a field on Ω, and μ is a measure on $\mathcal{F}_0$. Let A be a nonempty set in $\mathcal{F}_0$ and $\mu(A) < \infty$. Let μ_A be μ restricted on $\mathcal{F}_0 \cap A$; that is, μ_A is the set function

$$\mathcal{F}_0 \cap A \to [0, \infty], \quad BA \mapsto \mu(BA).$$

1. Show that $\mathcal{F}_0 \cap A$ is a field;

2. μ_A is a measure on $\mathcal{F}_0 \cap A$;

3. μ_A has an extension $\hat{\mu}_A$ on $\mathcal{F} \cap A$, where $\mathcal{F} = \sigma(\mathcal{F}_0)$, and $\hat{\mu}_A$ is also a finite measure.

√ **Problem 3.7.** Define a set function $\hat{\mu}$ on $\mathcal{F}$ as follows. For any $E \in \mathcal{F}$, if there exists a sequence of disjoint $\mathcal{F}_0$-sets A_n such that $E \subseteq \cup_n A_n$ and $\mu(A_n) < \infty$, then let

~~I think this should be dropped.~~

$$\hat{\mu}(E) = \sum_n \hat{\mu}(E \cap A_n);$$

$\hat{\mu}_{A_n}(E \cap A_n)$.

if there exists no such sequence then let $\hat{\mu}(E) = \infty$.

√1. Show that this definition doesn't depend on the choice of sequence $\{A_n\}$.

2. Show that $\hat{\mu}$ is a measure on $\mathcal{F}$, and agrees with μ on $\mathcal{F}_0$.