Theorem 1.6 is just a foreshadowing of the true state of affairs, however. As we shall show next, the curve $y = e(x)$ must touch the lines $y = \pm E_n(f)$ alternately at least $n + 2$ times, and this property characterizes the best uniform approximation of a continuous function by a polynomial of degree at most n. A set of $k + 1$ distinct points $x_0, \ldots, x_k$, satisfying $a \leq x_0 < x_1 < \cdots < x_{k-1} < x_k \leq b$ is called an *alternating set* for the error function $f - p_n$ if

$$|f(x_j) - p_n(x_j)| = \|f - p_n\|, \qquad j = 0, \ldots, k \tag{1.2.4}$$

and

$$[f(x_j) - p_n(x_j)] = -[f(x_{j+1}) - p_n(x_{j+1})], \qquad j = 0, \ldots, k - 1. \tag{1.2.5}$$

THEOREM 1.7. *Suppose $f \in C[a, b]$; p_n^* is a best uniform approximation on $[a, b]$ to f out of P_n if and only if there exists an alternating set for $f - p_n^*$ consisting of $n + 2$ points.*

Proof. (i) Suppose $x_0, \ldots, x_{n+1}$ form an alternating set for $f - p_n^*$. We show that p_n^* is a best approximation. If it is not, then there exists $q_n \in P_n$ such that

$$\|f - q_n\| < \|f - p_n^*\|. \tag{1.2.6}$$

Hence, in particular, since $x_0, \ldots, x_{n+1}$ form an alternating set,

$$|f(x_j) - q_n(x_j)| < \|f - p_n^*\| = |f(x_j) - p_n^*(x_j)|, \qquad j = 0, \ldots, n + 1. \tag{1.2.7}$$

(1.2.7) and (1.2.5) imply that the difference

$$[f(x_j) - p_n^*(x_j)] - [f(x_j) - q_n(x_j)] = q_n(x_j) - p_n^*(x_j)$$

alternates in sign as j runs from 0 to $n + 1$. Thus the polynomial $q_n(x) - p_n^*(x) \in P_n$ has a zero in each interval (x_j, x_{j+1}), $j = 0, \ldots, n$, for a total of $n + 1$ zeros, which implies $q_n = p_n^*$. This contradicts (1.2.6), hence implies that p_n^* is a best approximation and concludes the easier half of our proof.

(ii) Suppose that p_n^* is a best approximation to f and $f \notin P_n$. (If $f \in P_n$, the whole question is trivial.) Let a largest alternating set for $f - p_n^*$ consist of the $k + 1$ points $x_0, \ldots, x_k$ satisfying $a \leq x_0 < x_1 < \cdots < x_{k-1} < x_k \leq b$. In view of Theorem 1.6, $k \geq 1$. We wish to prove that $k \geq n + 1$. Suppose, then, that $k \leq n$, and let us put

$$\|f - p_n^*\| = \rho \qquad (>0).$$

Let $t_0, \ldots, t_s$ be points of $[a, b]$ chosen so that $a = t_0 < t_1 < \cdots < t_s = b$ and so that $e(x) = f(x) - p_n^*(x)$ satisfies

$$|e(\xi) - e(\eta)| < \tfrac{1}{2}\rho \tag{1.2.8}$$