PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, May 9, 2016

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

0 1. Let λ be the Lebesgue measure on $[0,1]$ and $\mathcal{A}$ be the σ-algebra of Lebesgue measurable sets in $[0,1]$. Suppose that a set $N \subset [0,1]$ is not in $\mathcal{A}$. Prove that λ can be extended to a measure on a σ-algebra $\mathcal{B}$ that contains $\mathcal{A}$ and N.

✓ 2. Let μ be a finite Borel measure on a metric space (X,d). Prove that for each $x \in X$ and for each $\varepsilon > 0$ there exists a number $0 < r < \varepsilon$ such that

$$\lim_{\rho \to r} \mu(B_x(\rho)) = \mu(B_x(r)), \tag{1}$$

where $B_x(r)$ denotes the open ball in X of radius r centered at x.

✓ 3. Let f be a function which is in $L^p = L^p((0,1),\lambda)$ for each $p \geq 1$, where λ is the Lebesgue measure on the interval $(0,1)$. Give an example of such a function which is not in $L^\infty = L^\infty((0,1),\lambda)$. ✓ Prove that if there is a constant C so that $\|f\|_p \leq C$ for all $p \geq 1$, then $f \in L^\infty$.

✓ 4. Let f be a function in $L^1 = L^1(\mathbb{R},\lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$. Let E be a measurable set with $\lambda(E) < \infty$. Prove that the following function is continuous on $\mathbb{R}$

$$g(x) = \int_E f(x-t)\,d\lambda(t).$$

You may use without proof that for each $\varepsilon > 0$ there exists a continuous function ϕ on $\mathbb{R}$ with compact support such that $\|f - \phi\|_1 < \varepsilon$.

PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, December 12, 2015

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Note: In the following, λ denotes the Lebesgue measure on $\mathbb{R}^m$.

1. Denote by λ^* the Lebesgue outer measure on $\mathbb{R}^m$. Assume that $E_1, E_2 \subset \mathbb{R}^m$ satisfy $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2$ is Lebesgue measurable with $\lambda(E_1 \cup E_2) < \infty$, and
$$\lambda(E_1 \cup E_2) = \lambda^*(E_1) + \lambda^*(E_2).$$
Prove that E_1, E_2 are Lebesgue measurable.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(f_n) \subset L^1(X, \mathcal{A}, \mu)$, and $f \in L^1(X, \mathcal{A}, \mu)$. Assume that
$$\lim_{n\to\infty} \|f_n - f\|_1 = 0.$$
Prove that there exists a subsequence (f_{n_k}) such that $f_{n_k} \to f$ μ-a.e.

3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $1 < p < \infty$. Assume that the sequence $(f_k) \subset L^p(X, \mathcal{A}, \mu)$ satisfies
$$\|f_k\|_p \le C \quad \text{and} \quad \lim_{k\to\infty} f_k(x) = 0 \quad \text{for all } x \in X.$$
Prove that $f_k \to 0$ in $L^1(X, \mathcal{A}, \mu)$.

4. Let $f\colon \mathbb{R} \to (0, \infty)$ be a Borel measurable function and let E be a Borel measurable subset of $\mathbb{R}$ such that $\lambda(E) > 0$. Define
$$F(t) = \int_E f(t + x)\, d\lambda(x), \quad t \in \mathbb{R}.$$
Prove that F is Borel measurable. Prove further that if $F \in L^1(\mathbb{R})$, then $f \in L^1(\mathbb{R})$ and $\lambda(E) < \infty$.

PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, May 13, 2015

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Assume that $f\colon [0,1] \to [0,\infty)$ is a Lebesgue measurable function such that $f(x) > 0$ for a.e x. Show that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every Lebesgue measurable set E with Lebesgue measure $\lambda(E) \geq \varepsilon$ we have $\int_E f d\lambda \geq \delta$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space with a finite measure μ and let f be a measurable function such that $\int_X |f|^q \, d\mu < \infty$ for some $0 < q < \infty$. Show that
$$\lim_{p \to 0+} \int_X |f|^p \, d\mu = \mu(\{x \in X : f(x) \neq 0\}).$$

3. Let f and g be integrable functions on $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, respectively, and let $F(x,y) = f(x)g(y)$. Show that F is measurable and integrable on $X \times Y$ and that
$$\int F \, d(\mu \times \nu) = \int f \, d\mu \int g \, d\nu.$$

4. Let $f\colon [0,\infty) \to \mathbb{R}$ be Lebesgue integrable. Show that if f is uniformly continuous, then $\lim_{x\to\infty} f(x) = 0$.

PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, August 19, 2014

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Note: In the following, λ denotes the Lebesgue measure on $\mathbb{R}^m$.

1. Assume that E is Borel subset of $\mathbb{R}^2$. Show that for every $y \in \mathbb{R}$, the slice $E^y = \{x \in \mathbb{R} \mid (x, y) \in E\}$ is a Borel subset of $\mathbb{R}$.

2. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous and increasing function. Show that f is absolutely continuous if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\lambda^*(f(E)) < \varepsilon$$

for Lebesgue measurable subsets $E \subset [a, b]$ satisfying $\lambda(E) < \delta$. (Here λ^* is the outer Lebesgue measure.)

3. Assume that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of Lebesgue measurable functions such that $\int_{\mathbb{R}} |f_n| \leq 1/n^2$. Show that $f_n \to 0$ λ-a.e.

4. Let $0 < \alpha < d$ and $K(x) = \dfrac{1}{|x|^\alpha}$ for $x \in \mathbb{R}^d$. For nonnegative Lebesgue integrable function $f : \mathbb{R}^d \to \mathbb{R}$, define

$$g(x) = \int_{\mathbb{R}^d} f(x - y)K(y) \, d\lambda(y).$$

Show that $g(x)$ is finite for λ-a.e. $x \in \mathbb{R}^d$.

PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, May 14, 2014

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Note: In the following, λ denotes the Lebesgue measure on $\mathbb{R}^m$.

✓ 1. Let X be a compact metric space and let μ be a finite measure on the σ-algebra of Borel measurable subsets of X. Assume that $\mu(\{x\}) = 0$ for every $x \in X$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu(B) < \varepsilon$$

for every Borel subset B of X satisfying diam $(B) < \delta$.

✓ 2. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Show that the following conditions are equivalent:

(a) For every $N \subset [a,b]$ satisfying $\lambda(N) = 0$, $\lambda(f(N)) = 0$.

(b) For Lebesgue measurable subset $E \subset [a,b]$, $f(E)$ is Lebesgue measurable.

✓ 3. Let (f_n) be a sequence of Lebesgue integrable functions $f_n\colon \mathbb{R}^k \to [-\infty,\infty]$ such that $\|f_n\|_1 \leq 1$ for all $n \geq 1$ and $f_n \to f$ λ-a.e. in $\mathbb{R}^k$. Show that $f \in L^1(\mathbb{R}^k)$, $\|f\|_1 \leq 1$, and

$$\|f\|_1 = \lim_{n\to\infty} \left(\|f_n\|_1 - \|f_n - f\|_1\right).$$

✓ 4. Assume that f is a Lebesgue integrable function on $[0,a]$ and

$$g(x) = \int_x^a \frac{f(t)}{t}\, dt \quad \text{for } 0 < x \leq a.$$

Show that g is Lebesgue integrable on $[0,a]$ and

$$\int_0^a g(x)\, dx = \int_0^a f(t)\, dt.$$

PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, August 22, 2013

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

√ √ 1. Let X be a metric space and $\{f_n\}$ a sequence of continuous functions from X to a metric space Y. Assume that f_n converge to a function f uniformly on each compact subset $K \subset X$. Show that f is continuous.

√ √ 2. Let $g(x, y) = x^{-3/2} \cos\left(\frac{\pi y}{2x}\right)$.

(a) Prove that

$$\int_0^1 \int_0^x |g(x, y)| dy dx < \infty.$$

(b) Evaluate the integral

$$\int_0^1 \int_y^1 g(x, y)\, dx dy.$$

Justify your reasoning carefully.

√ √ 3. Let $\{f_n\}$ be a sequence of integrable functions on $[0, 1]$ (i.e., function in $L^1([0, 1], m)$, where m is the Lebesgue measure on $[0, 1]$). Suppose that f_n converges almost everywhere to a function $f \in L^1([0, 1], m)$. Prove that $\|f_n - f\|_1 \to 0$ if and only if $\|f_n\|_1 \to \|f\|_1$.

√ √ 4. Let μ and ν be finite positive measures on a measurable space $(X, \mathcal{A})$, which are equivalent (i.e., μ is absolutely continuous with respect to ν and ν is absolutely continuous with respect to μ). Let $\lambda = \mu + \nu$. Show that the Radon-Nikodym derivative $d\nu/d\lambda$ satisfies

$$0 < \frac{d\nu}{d\lambda} < 1$$

almost everywhere.

PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, May 8th, 2013

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let μ and ν be finite measures on a measure space $(X, \mathcal{A})$. Show that there is a nonnegative measurable function f on X such that for all $E \in \mathcal{A}$,

$$\int_E (1-f)\, d\mu = \int_E f\, d\nu.$$

2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\{f_n\}$ be a sequence of real-valued measurable functions on X. Suppose that there is a constant M such that $|f_n(x)| \leq M$ for all n and $x \in X$. Suppose also that the sequence $f_n(x)$ converges almost everywhere to a function f. Show that f is measurable and for each function $g \in L^2(X, \mu)$

$$\int f g d\mu = \lim_{n\to\infty} \int f_n g d\mu.$$

3. Construct an example of a continuous function $f\colon [0,1] \to [0,1]$, which has the following property: there exists a Lebesgue measurable set $E \subset [0,1]$ such that for every $n > 0$ the set

$$F_n = \left\{ y \in \left[\frac{1}{n+1}, \frac{1}{n}\right] : y = f(x) \text{ for some } x \in E \right\}$$

is not Lebesgue measurable.

4. Let f be a non-negative function defined on a Lebesgue measurable subset $E \subseteq [0,1]$. Show that f is Lebesgue measurable if the region

$$\{(x,y) : x \in E,\ f(x) \geq y\}$$

is a Lebesgue measurable subset of $\mathbb{R}^2$.

PhD QUALIFYING EXAMINATION IN ANALYSIS
Part A, August 21, 2012

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $\{f_n\}$ be a non-increasing sequence of non-negative integrable functions on a measure space $(X, \mathcal{A}, \mu)$. Assume that

$$\lim_{n\to\infty} \int f_n \, d\mu = 0.$$

Show that $\lim_{n\to\infty} f_n(x) = 0$ a.e. Does this statement remain true if the assumption that "the sequence $\{f_n\}$ is non-increasing" is dropped? No.

2. Let $C([0,1])$ be the metric space of all continuous functions on the unit interval $[0,1]$ equipped with the usual supremum metric. Fix $k > 0$ and let E be the set of all polynomial functions on $[0,1]$ of degree $\leq k$. Prove that E is a closed and nowhere dense subset of $C([0,1])$.

3. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $V(f) \leq 3$, where $V(f)$ is the total variation of the function f.

4. Prove that if f is a Lipschitz function on an interval $[a,b]$ then f is absolutely continuous and $f' \in L^\infty([a,b])$.

PhD QUALIFYING EXAMINATION IN ANALYSIS
Paper A, May 7th, 2012

Instructions: To pass the exam you must correctly solve two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity. Please write your solutions carefully and clearly.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

✓ 1. Let E be a noncompact subset of $\mathbb{R}^n$. Show that there is a bounded continuous real-valued function on E that does not assume its maximum on E.

✓ 2. Let (X, μ) be a measure space. Show that if $f \in L^p(X, \mu)$ with $1 \le p < \infty$, then

$$\lim_{t \to \infty} t^p \mu(\{x \in X : |f(x)| > t\}) = 0.$$

✓ 3. Let (X, μ) be a measure space of finite measure and let f be a nonnegative integrable function on X such that for every integer $n = 1, 2, \ldots,$

$$\int_X f(x)^n \, dx = \int_X f(x) \, dx.$$

Show that f must be almost everywhere equal to the characteristic function χ_E of some measurable set E.

✓ 4. Show that the graph of a continuous function $f\colon [0, 1] \to \mathbb{R}$ has measure zero with respect to the two-dimensional Lebesgue measure.