Math 523 H.W. 6 Min-Chun Wu.

1.

(a)

$I_4(f) = h \cdot \sum_{i=0}^{4} w_i f(x_i)$, where $x_0, x_1, x_2, x_3, x_4 = 0, 1/4, 2/4, 3/4, 4/4$, and $h = 1/4$.

$w_i = \int_0^4 \varphi_i(t)\,dt$, where $\varphi_i(t) = \prod_{k=0,\, k\neq i}^{n} \frac{t-k}{i-k}$.

Thus $\varphi_0(t) = \frac{(t-1)(t-2)(t-3)(t-4)}{(0-1)(0-2)(0-3)(0-4)}$, ..., $\varphi_4(t) = \frac{(t-0)(t-1)(t-2)(t-3)}{(4-0)(4-1)(4-2)(4-3)}$.

By integral calculator, $w_0 = 14/45$, $w_1 = 64/45$, $w_2 = 24/45$, $w_3 = 64/45$, $w_4 = 14/45$.

Therefore, $I_4(f) = \frac{1}{4}\sum_{i=0}^{4} w_i f(x_i) = \frac{1}{90}\left(7f(0) + 32f(1/4) + 12f(1/2) + 32f(3/4) + 7f(1)\right)$ #.

(b)

Using Thm 9.2 of Quarteroni's book, we should take $n=4$ and

$$E(f;0,1) = \frac{M_4}{6!} h^{4+3} f^{(4+2)}(\xi) \quad (\text{i.e. } k=6).$$

$$\Rightarrow 6!\,C = M_4 \cdot h^7 = \left(\int_0^4 t\,\pi_5(t)\,dt\right)\cdot\left(\tfrac{1}{4}\right)^7 = \left(\int_0^4 t\cdot t(t-1)(t-2)\cdots(t-4)\,dt\right)\cdot\left(\tfrac{1}{4}\right)^7$$

$$= -\frac{128}{21}\cdot\left(\tfrac{1}{4}\right)^7 \qquad \Rightarrow C = \left(\frac{-128}{21}\right)\cdot\left(\tfrac{1}{4}\right)^7\cdot\frac{1}{6!} \text{ #}$$

(c)

Let's apply Thm 9.3 of Quarteroni's book. Notice the m in Thm 9.3 is n in the prob.

Take $n = 4$ (so that $M_n = M_4 = -128/21$, as computed in (b)).

$\boxed{\text{as in Thm 9.3}}$

$\because$ We are dealing with closed formula, $\therefore\ \gamma_n = \gamma_4 = 4$.

H here is h, as stated in the problem.

Thus, by Thm 9.3, the desired quantity is $\frac{b-a}{6!}\cdot\frac{(-128/21)}{4^7} h^6 \cdot f^{(6)}(\xi)$, some $\xi \in (a,b)$. #

The composite rule can be stated as follows:

$$\sum_{k=0}^{n-1} \frac{h}{90}\left[7\cdot f(a+kh) + 32f(a+kh+\tfrac{h}{4}) + 12f(a+kh+\tfrac{h}{2}) + 32f(a+kh+\tfrac{3h}{4}) + 7f(a+(k+1)h)\right],$$

where $h = b-a/n$. #

4.

(a) $T_n(\cos\theta) = \cos n\theta. \Rightarrow T_{n+1}(\cos\theta) = \cos(n+1)\theta \Rightarrow T_{n+1}'(\cos\theta)(-\sin\theta) = -\sin(n+1)\theta\cdot(n+1)$

$$\Rightarrow S_n(\cos\theta) = \frac{T_{n+1}'(\cos\theta)}{n+1} = \frac{\sin(n+1)\theta}{\sin\theta}.$$

$x = \cos\theta,\ dx = -\sin\theta\, d\theta.$

Thus, $\displaystyle\int_{-1}^{1} S_n(x)S_m(x)\sqrt{1-x^2}\,dx \overset{\swarrow}{=} \int_0^{\pi} S_n(\cos\theta)S_m(\cos\theta)\sin^2\theta\, d\theta$

$$= \int_0^{\pi}\sin(n+1)\theta\,\sin(m+1)\theta\, d\theta = \begin{cases} 0, & \text{if } n\neq m \\ \pi/2, & \text{if } n=m. \end{cases} \quad \#$$

(b)

Recall that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ \forall\, n\geq 1.$

Thus, our goal is to prove $S_{n+1}(x) = 2xS_n(x) - S_{n-1}(x),\ \forall\, n\geq 1.$

Taking $x = \cos\theta$,

$$\text{LHS} = S_{n+1}(\cos\theta) = \frac{\sin(n+2)\theta}{\sin\theta}$$

$$\text{RHS} = 2\cos\theta\cdot\frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin n\theta}{\sin\theta}.$$

Recall that $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$

Hence $\displaystyle \text{LHS} + \frac{\sin n\theta}{\sin\theta} = \frac{1}{\sin\theta}\cdot\left[2\sin(n+1)\theta\cos\theta\right] = \text{RHS} + \frac{\sin n\theta}{\sin\theta}.$

Thus LHS = RHS, as desired. #

(c)

For $f, g \in C[-1,1]$, define $\langle f,g\rangle := \frac{2}{\pi}\int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}\,dx.$

Then $\{S_m(x)\}_{m=0}^{n}$ is an orthonormal basis for $\Pi_n[-1,1]$.

Thus, the sol'n to the problem is $\displaystyle\sum_{m=0}^{n}\frac{\langle f, S_m\rangle}{\langle S_m, S_m\rangle}\cdot S_m(x).$ #

(d)

By Thm 3.2.1 of Gautschi's book ([2011] Numerical Analysis),

the nodes are exactly the zeros of $S_n(x)$.

i.e. $x = \cos\theta$ s.t. $\dfrac{\sin(n+1)\theta}{\sin\theta} = 0.$

Hence, $(n+1)\theta = k\pi$, w/ $k \in \mathbb{Z}$ but $\theta \in (0,\pi)$. denote $x_k = \cos\frac{k\pi}{n+1}$.

i.e. $\theta = \frac{\pi}{n+1}, \frac{2\pi}{n+1}, \cdots, \frac{n\pi}{n+1}$. $\Rightarrow x = \cos\frac{\pi}{n+1}, \cos\frac{2\pi}{n+1}, \cdots, \cos\frac{n\pi}{n+1}$. #.

For $k=1,2,\cdots,n$, the weights $w_k = \int_{-1}^{1} \frac{S_n(x)}{(x-x_k)\cdot S_n'(x_k)}\sqrt{1-x^2}\,dx$, as stated in (3.42) in Gautschi's book. #

By Thm 3.6.24 of Stoer & Bulirsch's book [2002],
the error estimate is given by $\frac{f^{(2n)}(\xi)}{(2n)!}(\tilde{S}_n, \tilde{S}_n)$, where

$$\tilde{S}_n = \frac{1}{2^n}\cdot S_n \text{ and hence } (\tilde{S}_n, \tilde{S}_n) = \frac{1}{2^n}\cdot\frac{1}{2^n}\cdot\int_{-1}^{1} S_n(x)\, S_n(x)\sqrt{1-x^2}\,dx = \frac{1}{4^n}\cdot\frac{\pi}{2}.$$

$$\Rightarrow \text{error} = \frac{f^{(2n)}(\xi)}{(2n)!}\cdot\frac{1}{4^n}\cdot\frac{\pi}{2}. \quad \#$$

5.

(a)

Note that we have $h_0, h_1, h_2, h_3, g_1, g_2$.

claim: $\{h_i\}_{i=0}^{3} \cup \{g_j\}_{j=1}^{2}$ is linearly indep.

Suppose $\alpha_i \in \mathbb{R}, \beta_j \in \mathbb{R}$ s.t. $\alpha_0 h_0 + \cdots + \alpha_3 h_3 + \beta_1 g_1 + \beta_2 g_2 = 0$.

Taking $x = x_i$, we arrive at $\alpha_i = 0$.

Differentiating at taking $x = x_j$, we have $\beta_j = 0$. Thus, claim is proved.

$\because \dim \Pi_5 = 6 \quad \therefore \{h_i\}_{i=0}^{3} \cup \{g_j\}_{j=1}^{2}$ is a basis for Π_5.

claim: $p(x) = \sum_{i=0}^{3} p(x_i) h_i(x) + \sum_{j=1}^{2} p'(x_j) g_j(x)$.

$\because \{h_i\}_{i=0}^{3} \cup \{g_j\}_{j=1}^{2}$ is a basis of Π_5.

$\therefore$
$$p(x) = \sum_{i=0}^{3} \alpha_i h_i(x) + \sum_{j=1}^{2} \beta_j g_j(x).$$

Taking $x = x_i$, we have $\alpha_i = p(x_i), \forall i$. #

Differentiating and taking $x = x_j$, we have $\beta_j = p'(x_j), \forall j$. #

(b)

By condition, $g_1(x_j) = 0$, $j = 0,1,2,3$ and $g_1'(x_2) = 0$.

Thus, g_1 has roots $x_0, x_1, \cdots, x_3$ and x_2 is a double root.

$\Rightarrow g_1(x) = \tilde{\alpha}(x-x_0)(x-x_1)(x-x_2)^2(x-x_3)$ for some const. $\tilde{\alpha}$.

$= \tilde{\alpha}(x^2-1)(x-x_1)(x-x_2)^2 = \alpha(1-x^2)(x-x_1)(x-x_2)^2$, where $\alpha = -\tilde{\alpha}$. #

Similar arguments work for g_2. #

(c)

By (a), $L_2(f) = \int_{-1}^{1} p(x)\,dx = \underbrace{\sum_{i=0}^{3} \left(\int_{-1}^{1} h_i(x)\,dx\right) \cdot p(x_i)}_{\text{of form (1)}} + \sum_{j=1}^{2} \left(\int_{-1}^{1} g_j(x)\,dx\right) \cdot p'(x_j)$.

Thus, making $\int_{-1}^{1} g_j(x)\,dx = 0$, for $j = 1, 2$, is sufficient to make $L_2(f)$ of form (1). #

(d)

Assume the condition (i.e. $q(x)$ orthogonal to Π_1 w.r.t. $w(x) = 1-x^2$) holds.

Then ① $q(x) \perp (x-x_1)$ w.r.t. $(1-x^2) \Rightarrow 0 = \beta \int_{-1}^{1} q(x)(x-x_1)(1-x^2)\,dx = \int_{-1}^{1} g_2(x)\,dx$. #

② $q(x) \perp (x-x_2)$ w.r.t. $(1-x^2)$. $\Rightarrow 0 = \alpha \int_{-1}^{1} q(x)(x-x_2)(1-x^2)\,dx = \int_{-1}^{1} g_1(x)\,dx$. #

(e)

Start w/ the basis $1, x, x^2$. Apply Gram-Schmidt, (w.r.t. $\langle \cdot, \cdot \rangle = \int_{-1}^{1} \cdot x \cdot x (1-x^2) dx$.)

$q_0(x) = 1.$

$q_1(x) = x - \frac{\langle x, q_0 \rangle}{\langle q_0, q_0 \rangle} \cdot q_0 = x - 0 = x.$

$q_2(x) = x^2 - \frac{\langle x^2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 - \frac{\langle x^2, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 = x^2 - 0 - \frac{4/15}{4/3} = x^2 - 1/5.$ #

(f)

$q(x) = q_2(x) = (x - 1/\sqrt{5})(x + 1/\sqrt{5})$. Thus, $x_1 = -1/\sqrt{5}$, $x_2 = 1/\sqrt{5}$. #

(g)

$w_i = \int_{-1}^{1} h_i(x) dx$, $i = 0, 1, 2, 3$. (as seen in (c)).

Denote $\omega_4(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)$, and $l_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{3} \frac{(x - x_k)}{(x_i - x_k)}$.

Then $h_i(x) = \left[1 - \frac{\omega_4''(x_i)}{\omega_4'(x_i)} \cdot (x - x_i) \right] \cdot l_i^2(x)$ (from the conditions of h_i).

Thus, by taking $x_1 = -1/\sqrt{5}$, $x_2 = 1/\sqrt{5}$, $x_0 = -1$, $x_3 = +1$, we have

$w_0 = 1/6 = w_3$, $w_1 = 5/6 = w_2$. #

(h)

Recall the Hermite interpolation error below: (see (8.37) of Quarteroni's book).

$$f(x) - p(x) = \frac{f^{(6)}(\xi)}{6!} \cdot (x+1)(x + 1/\sqrt{5})^2 (x - 1/\sqrt{5})^2 \cdot (x-1) = \frac{f^{(6)}(\xi)}{6!} (x^2 - 1)(x^2 - 1/5)^2.$$

Thus, $I(f) - L_2(f) = \int_{-1}^{1} f(x) dx - \int_{-1}^{1} p(x) dx = -\frac{f^{(6)}(\xi)}{6!} \int_{-1}^{1} (1 - x^2)(x^2 - 1/5)^2 dx.$ #

6.

(a) Run Prob6a.m

$\mathbb{E}[X_j^{2p}], \forall j.$

(b) $X_1, X_2, \cdots$ iid sampled from standard normal. $S_n := \frac{X_1^{2p} + \cdots + X_n^{2p}}{n}$. $\mathbb{E}[S_n] =: \mu$

By Chebyshev inequality, $\Pr[\,|S_n - \mu| \geq 0.01\mu\,] \leq \frac{\sigma^2/n}{(0.01\mu)^2} = \frac{10^4}{n} \cdot \frac{\sigma^2}{\mu^2}$.

Note

$\sigma^2 = Var[X^{2p}] = \mathbb{E}[X^{4p}] - \mathbb{E}[X^{2p}]^2 = [1 \cdot 3 \cdot \cdots \cdot (4p-1)] - [1 \cdot 3 \cdot \cdots \cdot (2p-1)]^2.$

For

$p=1,\ Var[X^{2p}] = 1 \cdot 3 - 1^2 = 2.$ #

$p=2,\ Var[X^{2p}] = 1 \cdot 3 \cdot 5 \cdot 7 - (1 \cdot 3)^2 = 105 - 9 = 96.$

$p=5,\ Var[X^{2p}] = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 - (1 \cdot 3 \cdot 5 \cdot 7 \cdot 9)^2 = 653836050.$

Set $\frac{10^4}{n} \cdot \frac{\sigma^2}{\mu^2} = 5\%. \Rightarrow n = \frac{10^4 \sigma^2}{\mu^2} \cdot \frac{100}{5} = \frac{2 \cdot 10^5 \sigma^2}{\mu^2}.$

For

$p=1,\ \mu = 1. \qquad \sigma/\mu = \sqrt{2} \qquad n = 4 \cdot 10^5.$

$p=2,\ \mu = 1 \cdot 3 = 3. \Rightarrow \sigma/\mu = \sqrt{96}/3 \Rightarrow n = 2.13 \cdot 10^6.$ #

$p=5,\ \mu = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945. \qquad \sigma/\mu = 27.05 \qquad n = 1.46 \times 10^8.$

(c)

Yes. They are all within 1% accuracy. #

By formula above ($\frac{2 \cdot 10^5 \sigma^2}{\mu^2} = n$), since larger p gives larger σ/μ, it also gives larger n. #

7.

(a) $f(x) = ax + b.$

$$\int_0^1 x^\alpha (ax+b)\,dx = \frac{a}{\alpha+2} + \frac{b}{\alpha+1}.$$

$$Af(0) + B\int_0^1 f(x)\,dx = Ab + B\cdot\left(\frac{a}{2}+b\right).$$

To make them equal $\forall$ a, b, we need $\begin{cases} B/2 = 1/\alpha+2 \\ A+B = 1/\alpha+1 \end{cases} \Leftrightarrow \begin{cases} A = \frac{-\alpha}{(\alpha+1)(\alpha+2)} \\ B = 2/\alpha+2 \end{cases}$ #.

(b)

$$K_1(t) = \int_0^1 x^\alpha (x-t)_+\,dx + \frac{\alpha}{(\alpha+1)(\alpha+2)}(0-t)_+ - \frac{2}{\alpha+2}\int_0^1 (x-t)_+\,dx$$

$$= \int_t^1 x^\alpha (x-t)\,dx - \frac{2}{\alpha+2}\int_t^1 x-t\,dx = \frac{x^{\alpha+2}}{\alpha+2}\Big|_t^1 - t\cdot\frac{x^{\alpha+1}}{\alpha+1}\Big|_t^1 - \frac{2}{\alpha+2}\left[\frac{x^2}{2}\Big|_t^1 - tx\Big|_t^1\right]$$

$$= \frac{1}{\alpha+2} - \frac{t^{\alpha+2}}{\alpha+2} - \frac{t}{\alpha+1} + \frac{t^{\alpha+2}}{\alpha+1} - \frac{2}{\alpha+2}\left[\frac{1}{2} - \frac{t^2}{2} - t + t^2\right]$$

$$= \frac{t^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{t^2}{\alpha+2} + \frac{\alpha t}{(\alpha+1)(\alpha+2)} = \frac{t}{(\alpha+1)(\alpha+2)}\cdot\underbrace{\left[t^{\alpha+1} - (\alpha+1)t + \alpha\right]}_{=: g(t)}.$$

<u>case1</u> $\alpha < 0$.

$g(0) = \alpha < 0$, $g(1) = 0$. $g'(t) = (\alpha+1)(t^\alpha - 1) \geq 0$, $\forall t \in [0,1]$.

Thus, $g(t) \leq 0$, $\forall t \in [0,1]$. $\Rightarrow K_1(t) \leq 0$, $\forall t \in [0,1]$. #

<u>case2</u> $\alpha > 0$.

$g(0) = \alpha > 0$, $g(1) = 0$. $g'(t) = (\alpha+1)(t^\alpha - 1) \leq 0$, $\forall t \in [0,1]$.

Thus, $g(t) \geq 0$, $\forall t \in [0,1]$. $\Rightarrow K_1(t) \geq 0$, $\forall t \in [0,1]$. #

(c)

$$e_2 = E(t^2/2!) = \int_0^1 x^\alpha\cdot\frac{x^2}{2}\,dx + \frac{\alpha}{(\alpha+1)(\alpha+2)}\cdot 0 - \frac{2}{\alpha+2}\int_0^1 \frac{x^2}{2}\,dx$$

$$= \frac{x^{\alpha+3}}{2(\alpha+3)}\Big|_0^1 + 0 - \frac{2}{\alpha+2}\cdot\frac{x^3}{6}\Big|_0^1 = \frac{1}{2(\alpha+3)} - \frac{2}{\alpha+2}\cdot\frac{1}{6} = \frac{\alpha}{6(\alpha+2)(\alpha+3)}.$$ #