Product Structures

<u>Goal</u>: ① Define product structures on (1) the Čech-de Rham complex $C^*(\mathcal{U},\Omega^*)$

(2) the de Rham cohomology and (3) the Čech cohomology.

② Also discuss the product structures on a spectral sequence.

(a) (de Rham cohomology).

$Z := \{\text{closed forms}\}$, $B := \{\text{exact forms}\}$, on a mfd M.

Recall that $\boxed{d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^{\deg\omega}\omega\wedge d\eta}$ ← antiderivation property, for any diff. forms ω and η.

Thus, for $\omega,\eta\in Z$, $d(\omega\wedge\eta) = 0\wedge\eta + (-1)^{\deg\omega}\omega\wedge 0 = 0$.

$\Rightarrow \omega\wedge\eta\in Z$. i.e. Z is a subring of $\Omega^*(M)$, under multiplication $\wedge$.

Moreover, for $\omega\in Z, \eta\in B$, since $\eta = d\beta$, some β,

$$\omega\wedge\eta = \omega\wedge d\beta = [d(\omega\wedge\beta) - \underbrace{d\omega}_{=0}\wedge\beta](-1)^{\deg\omega} = (-1)^{\deg\omega} d(\omega\wedge\beta). \in B.$$

Similarly, $\eta\wedge\omega\in B$.

Thus, B is an ideal of Z and $H^*_{DR}(M) = Z/B$ inherits the product structure from $\Omega^*(M)$ by $[\omega]\wedge[\eta] := [\omega\wedge\eta]$, making $H^*_{DR}(M)$ into a graded algebra. We normally denote $\omega\wedge\eta$ by $\omega\eta$ and $[\omega]\wedge[\eta]$ by $[\omega][\eta]$.

(b) (the Čech-de Rham double complex $C^*(\mathcal{U},\Omega^*)$.)

Define, on $C^*(\mathcal{U},\Omega^*)$, where $\mathcal{U}$: open cover of M,

$\cup: C^p(\mathcal{U},\Omega^q)\otimes C^r(\mathcal{U},\Omega^s) \longrightarrow C^{p+r}(\mathcal{U},\Omega^{q+s})$ as follows:

for $\omega\in C^p(\mathcal{U},\Omega^q)$, $\eta\in C^r(\mathcal{U},\Omega^s)$,

$$\boxed{(\omega\cup\eta)(U_{\alpha_0\cdots\alpha_{p+r}}) = (-1)^{qr}\omega(U_{\alpha_0\cdots\alpha_p})\cdot\eta(U_{\alpha_p\cdots\alpha_{p+r}})},$$ where RHS (ω and η terms on) have been restricted to $\Omega^q(U_{\alpha_0\cdots\alpha_{p+r}})$ and $\Omega^s(U_{\alpha_0\cdots\alpha_{p+r}})$.

<u>Rmk</u>:

In the definition of $\cup$ above, we have regarded $C^p(\mathcal{U},\Omega^q) \overset{\text{def}}{=} \prod_{\alpha_0<\cdots<\alpha_p}\Omega^q(U_{\alpha_0\cdots\alpha_p})$ as the collection of functions $\{U_{\alpha_0\cdots\alpha_p} \mid \alpha_0<\cdots<\alpha_p\} \xrightarrow{\omega} \coprod_{\alpha_0<\cdots<\alpha_p}\Omega^q(U_{\alpha_0\cdots\alpha_p})$ w/ $\omega(U_{\alpha_0\cdots\alpha_p})\in\Omega^q(U_{\alpha_0\cdots\alpha_p})$.

<u>Rmk</u>: $D'+D''=\delta+D''$, $D''=(-1)^p d$.

① The sign $(-1)^{qr}$ is there to make D'' into an antiderivation relative to this product structure.

② (Exercise 14.26)

$*$ For $\omega\in K^{p,q}$, $\eta\in K^{r,s}$, we have

(1) $\delta(\omega\cup\eta)=(\delta\omega)\cup\eta+(-1)^{\deg\omega}\omega\cup(\delta\eta)$.

(2) $D''(\omega\cup\eta)=(D''\omega)\cup\eta+(-1)^{\deg\omega}\omega\cup(D''\eta)$

(3) $D(\omega\cup\eta)=(D\omega)\cup\eta+(-1)^{\deg\omega}\omega\cup(D\eta)$.

where $\deg\omega=p+q$.

i.e. δ, D'' (and hence D) are antiderivation relative to the product structure, where the degree is the "antidiagonal" degree $p+q$.

We usually write $\omega\cdot\eta$ or $\omega\eta$ for $\omega\cup\eta$.

(C) (Čech complex $C^*(\mathcal{U},\mathbb{R})$)

Recall that $C^*(\mathcal{U},\mathbb{R})$ is the kernel of $\begin{pmatrix} C^*(\mathcal{U},\Omega^1) \\ \uparrow d \\ C^*(\mathcal{U},\Omega^0) \end{pmatrix}$. in the double complex in (b).

Hence, it inherits the product structure from $C^*(\mathcal{U},\Omega^*)$.

(It is closed under multiplication since elements in $C^*(\mathcal{U},\mathbb{R})$ are exactly the locally const. functions).

Explicitly, for ω: p-cochain, η: r-cochain,

$$\boxed{(\omega\cdot\eta)_{\alpha_0\cdots\alpha_{p+r}}=\omega_{\alpha_0\cdots\alpha_p}\eta_{\alpha_p\cdots\alpha_{p+r}}}\quad (q\equiv 0\text{, thus, }(-1)^{qr}\equiv 1).$$

By (14.26), δ: antiderivation relative to this product.

$\Rightarrow$ The product structure reduces to homology level, making the Čech cohomology $H^*(\mathcal{U},\mathbb{R})$ into a graded algebra.

For a refinement $\mathcal{V}>\mathcal{U}$, the refinement map induces an algebra hom.

$H^*(\mathcal{U},\mathbb{R})\to H^*(\mathcal{V},\mathbb{R})$, giving the direct limit $H^*(M,\mathbb{R})$ an algebra structure.

<u>Rmk</u>: The construction also holds for topo. sp. X, giving product structure on the Čech cohomology $H^*(X,\mathbb{R})$.

Recall the inclusion maps

$r: \Omega^*(M) \to C^*(\mathcal{U}, \Omega^*)$

$i: C^*(\mathcal{U}, \mathbb{R}) \to C^*(\mathcal{U}, \Omega^*)$.

(*) The product structure just defined in (a), (b), (c) will make these two maps into algebra hom.

These two maps will induce bijections, when $\mathcal{U}$ is a good cover,

$H^*_{DR}(M) \simeq H_D\{C^*(\mathcal{U}, \Omega^*)\}$ and

$H^*(\mathcal{U}, \mathbb{R}) \cong H_D\{C^*(\mathcal{U}, \Omega^*)\}$.

Thus, $H^*_{DR}(M)$ and $H^*(\mathcal{U}, \mathbb{R})$ are algebra iso.

$\because$ $\mathcal{U}$ is a good cover $\therefore$ $H^*(\mathcal{U}, \mathbb{R}) \cong H^*(M, \mathbb{R})$. Hence, we have:

Thm (14.28)

The iso. between de Rham and Čech $H^*_{DR}(M) \cong H^*(M, \mathbb{R})$ is actually an algebra iso.

(d) (Product structure on spectral seq.)

Assume a double complex K has a product structure relative to which D is an antiderivation.

$\because$ d_r is induced from D and $E_r = H_{d_{r-1}}(E_{r-1})$

$\therefore$ d_r is an antiderivat'n on E_r, $\forall r$.

Thm (14.29)

As assumptions above.

Then $\exists$ spectral seq. $\{E_r, d_r : E_r^{p,q} \to E_r^{p+r, q-r+1}\}$ converging to $H_D(K)$ s.t.

① $E_2^{p,q} = H_\delta^{p,q} H_d(K)$

② E_r inherits the product structure from E_{r-1}, relative to which d_r is an antiderivation.

WARNING

Both E_∞ and $H_D(K)$ inherit product structures from K, but generally not ring iso.

Exercise (14.30)

A, B: two graded rings.

Define a product structure on $A \otimes B$ by

$$(a\otimes b)(c\otimes d) = (-1)^{\deg(b)\deg(c)}(ac\otimes bd), \quad a, c \in A,\ b, d \in B.$$

Rmk: (14.31).

The starting point of Leray's theorem is $K^{p,q} = C^p(\pi^{-1}(\mathcal{U}), \Omega^q)$, which is the Čech-de Rham complex, having $\cup$ as its product structure.

Hence, we have (as the theorem requires, $\mathcal{U}$ is now a good cover of M)

(1) each E_r has an algebra structure w/ d_r as an antiderivation.

Moreover, if M: simply connected and $H^*(F)$: finite dimensional, it can be checked that

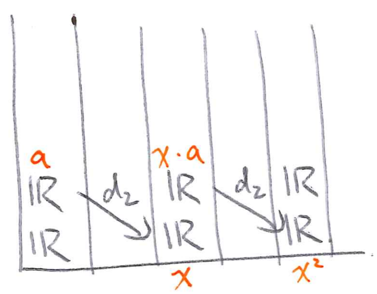
(2) $\underbrace{E_2}_{\text{product structure induced from Čech-de Rham.}} \cong \underbrace{H^*(M)\otimes H^*(F)}_{\text{product structure defined in (14.30)}}$, as graded algebras.

Example (14.32) (Computing ring structure of $H^*(\mathbb{C}P^n)$)

We restrict attention to $\mathbb{C}P^2$; arguments for $\mathbb{C}P^n$ are the same.

Recall

$$\begin{matrix} E_2 \\ \parallel \\ H^*(\mathbb{C}P^2)\otimes H^*(S^1) \end{matrix} =$$



, where d_2: iso. as vec. sp.

$a :=$ a generator of $E_2^{0,1} = H^0(\mathbb{C}P^2)\otimes H^1(S^1) \simeq H^1(S^1)$ $(*_1)$

$x := d_2 a \in E_2^{2,0} = H^2(\mathbb{C}P^2)\otimes H^0(S^1) \simeq H^2(\mathbb{C}P^2)$.

$\because d_2$: iso. of vec. sp. $\therefore x$: a generator of $H^2(\mathbb{C}P^2)$ $(*_2)$

By $(*_1)$, $(*_2)$, $x\cdot a$ is a generator of $H^2(\mathbb{C}P^2)\otimes H^1(S^1) = E_2^{2,1}$.

$\because d_2$: iso. of vec. sp. $\therefore d_2(x\cdot a) \underset{\uparrow}{=} (d_2 x)\cdot a + (-1)^2 x\cdot(d_2 a) = 0\cdot a + x\cdot x = x^2$.

d_2: antiderivation

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Thus, the ring structure of $H^*(\mathbb{C}P^2)$ is $\mathbb{R}[x]/(x^3)$. Similarly, $H^*(\mathbb{C}P^n) = \mathbb{R}[x]/(x^{n+1})$.