MATH523 , Homework 4
Due October 31, 2017 in class

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1. [7 pts] The Bernstein polynomials of degree n is given by

$$B_n f(t) = \sum_{j=0}^{n} f(j/n) B_j^n(t), \quad t \in [0,1],$$

where

$$B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j}, \quad j = 0, \ldots, n.$$

(a) Show that $B_0^n(0) = 1$ and for $j = 1, 2, \ldots, n$,

$$\begin{aligned} \frac{d^r}{dt^r} B_j^n(t)|_{t=0} &= 0, \quad r = 0, 1, \ldots, j-1, \\ \frac{d^j}{dt^j} B_j^n(t)|_{t=0} &\neq 0. \end{aligned}$$

use symmetry

(b) What are the analogous properties at $t = 1$ and how are they most easily derived?

~~(c)~~ Plot the fourth-degree polynomials $B_j^4(t), j = 0, 1, \ldots, 4$ on interval $0 \leq t \leq 1$.

(d) Use (a) to show that the $\{B_j^n(t)\}$ is linearly independent on $[0,1]$ and spans the space Π_n. proving independence suffices; as for independence, prove by definition.

2. [7 pts] Define $\sigma : [0,1] \to [a,b]$ by $\sigma(t) = a + t(b-a)$ for $0 \leq t \leq 1$, and define a transformation $T_\sigma : \mathcal{C}[a,b] \to \mathcal{C}[0,1]$ by $(T_\sigma f)(t) = f(\sigma(t))$. Prove that T_σ satisfies:

definition check~~

(a) $T_\sigma(f+g) = T_\sigma(f) + T_\sigma(g)$ and $T_\sigma(cf) = cT_\sigma(f)$ for $c \in \mathbb{R}$.

(b) $T_\sigma(fg) = T_\sigma(f)T_\sigma(g)$. In particular, T_σ maps polynomials to polynomials.

(c) $T_\sigma(f) \leq T_\sigma(g)$ if and only if $f \leq g$.

(d) $\|T_\sigma(f)\|_\infty = \|f\|_\infty$.

(e) T_σ is both one-to-one and onto. Moreover $(T_\sigma)^{-1} = T_{\sigma^{-1}}$.

(f) Now use the above properties appropriately to show that if the Weierstrass theorem holds for $\mathcal{C}[0,1]$, then it is also holds for $C[a,b]$. For your information, the converse is also true and the proof is identical so you don't need to show the converse.

3. [7 pts] *Modulus of Continuity:*

(a) Let $f : [-1,1] \to \mathbb{R}$. If $x = cos\theta$, where $-1 \leq x \leq 1$ and if $g(\theta) = f(\cos\theta)$, show that $\omega_g([-\pi,\pi];\delta) = \omega_g([0,\pi];\delta) \leq \omega_f([-1,1];\delta)$, where

$$\omega_f([a,b];\delta) = \max_{x,y \in [a,b], |x-y| \leq \delta} |f(x) - f(y)|, \quad \delta > 0.$$

(b) If $h(x) = f(ax+b)$ for $c \le x \le d$, show that $\omega_h([c,d];\delta) = \omega_f([ac+b, ad+b]; a\delta)$.

4. [7 pts] Let $f \in \mathcal{C}[a,b]$ and $p \in \Pi_n$. If there exist at least $n+2$ points for which $f - p$ alternates between $\pm\|f-p\|_\infty$, prove that p is indeed the best approximator of f $\cancel{\in}$ in Π_n

5. [7 pts] Let $\{x_i\}_{i=0,\dots,n}$ be the Chebyshev nodes on $[-1,1]$. Let f satisfies the Dini condition,

$$|log\delta|\omega(\delta) \to 0, \quad \text{as } \delta \to 0,$$

where ω denotes the modulus of continuity of f on $[-1,1]$. Prove that the interpolating polynomial p_n of degree $\le n$ converges to f uniformly in $[-1,1]$ as $n \to \infty$. This shows that the Chebyshev nodes work for functions slightly better than continuous and are are almost the perfect choice for "most" functions.

6. [7 pts] Let $f \in \mathcal{C}^1[-1,1]$ and $E_n(f) = \min_{p\in\Pi_n} \|f-p\|_\infty$ be the error of best approximation in $L^\infty[-1,1]$. Proceed as follows to prove the generalization of Jackson Theorem,

$$E_n(f) \le \frac{6}{n} E_{n-1}(f').$$

(a) Show that $E_n(f) = E_n(f-p)$ for all $p \in \Pi_n$.

(b) If $p^*_{n-1} \in \Pi_{n-1}$ is the best approximation of f' within Π_{n-1}, set $p_n(x) = \int_0^x p^*_{n-1}(t)dt \in \Pi_n$ and use (a).

7. [8 pts] Given $f \in [a,b]$ and a partition $a = x_0 < \cdots < x_n = b$, consider the continuous piecewise linear approximation $P_n f(x)$ of f in the sense of least squares with weight $w(x) = 1$. A basis for the space of continuous piecewise linear functions $\mathcal{L}^n$ is the set of *hat* functions $\phi_i \in \mathcal{L}^n$ defined by $\phi_i(x_j) = 1$ if $i = j$ and 0 otherwise.

(a) Determine the normal equations. Prove that the system is tridiagonal and diagonally dominant.

(b) Prove that $\|P_n f\|_\infty \le 3\|f\|_\infty$. Hint: argue as with the matrix for splines.

(c) Let h be the meshsize. Conclude from (b) that

$$\|f - P_n f\|_\infty \le 4 \inf_{q\in\mathcal{L}^n} \|f-q\|_\infty \le \frac{h^2}{2}\|f''\|_\infty.$$