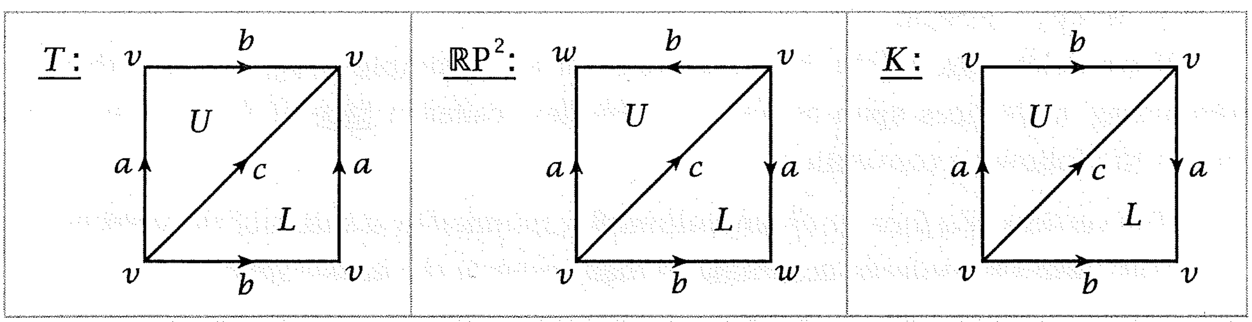
2.1 Simplicial and Singular Homology

The most important homology theory in algebraic topology, and the one we shall be studying almost exclusively, is called singular homology. Since the technical apparatus of singular homology is somewhat complicated, we will first introduce a more primitive version called simplicial homology in order to see how some of the apparatus works in a simpler setting before beginning the general theory.

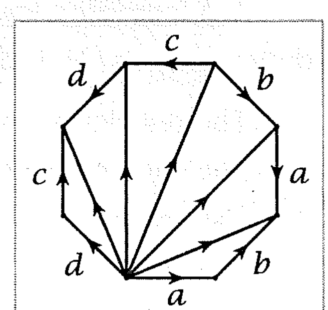
The natural domain of definition for simplicial homology is a class of spaces we call Δ-complexes, which are a mild generalization of the more classical notion of a simplicial complex. Historically, the modern definition of singular homology was first given in [Eilenberg 1944], and Δ-complexes were introduced soon thereafter in [Eilenberg-Zilber 1950] where they were called semisimplicial complexes. Within a few years this term came to be applied to what Eilenberg and Zilber called complete semisimplicial complexes, and later there was yet another shift in terminology as the latter objects came to be called simplicial sets. In theory this frees up the term semisimplicial complex to have its original meaning, but to avoid potential confusion it seems best to introduce a new name, and the term Δ-complex has at least the virtue of brevity.

Δ-Complexes

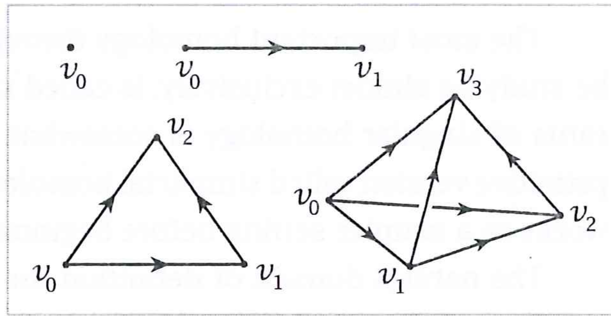
The torus, the projective plane, and the Klein bottle can each be obtained from a square by identifying opposite edges in the way indicated by the arrows in the following figures:



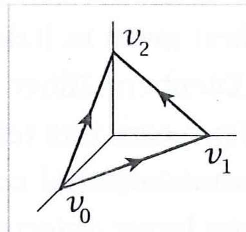
Cutting a square along a diagonal produces two triangles, so each of these surfaces can also be built from two triangles by identifying their edges in pairs. In similar fashion a polygon with any number of sides can be cut along diagonals into triangles, so in fact all closed surfaces can be constructed from triangles by identifying edges. Thus we have a single building block, the triangle, from which all surfaces can be constructed. Using only triangles we could also construct a large class of 2-dimensional spaces that are not surfaces in the strict sense, by allowing more than two edges to be identified together at a time.



The idea of a Δ-complex is to generalize constructions like these to any number of dimensions. The n-dimensional analog of the triangle is the **n-simplex**. This is the smallest convex set in a Euclidean space $\mathbb{R}^m$ containing $n+1$ points $v_0, \cdots, v_n$ that do not lie in a hyperplane of dimension less than n, where by a hyperplane we mean the set of solutions of a system of linear equations. An equivalent condition would be that the difference vectors $v_1 - v_0, \cdots, v_n - v_0$ are linearly independent. The points v_i are the **vertices** of the simplex, and the simplex itself is denoted $[v_0, \cdots, v_n]$. For example, there is the standard n-simplex



$$\Delta^n = \{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \textstyle\sum_i t_i = 1 \text{ and } t_i \ge 0 \text{ for all } i \}$$



whose vertices are the unit vectors along the coordinate axes.

For purposes of homology it will be important to keep track of the order of the vertices of a simplex, so 'n-simplex' will really mean 'n-simplex with an ordering of its vertices.' A by-product of ordering the vertices of a simplex $[v_0, \cdots, v_n]$ is that this determines orientations of the edges $[v_i, v_j]$ according to increasing subscripts, as shown in the two preceding figures. Specifying the ordering of the vertices also determines a canonical linear homeomorphism from the standard n-simplex Δ^n onto any other n-simplex $[v_0, \cdots, v_n]$, preserving the order of vertices, namely, $(t_0, \cdots, t_n) \mapsto \sum_i t_i v_i$. The coefficients t_i are the **barycentric coordinates** of the point $\sum_i t_i v_i$ in $[v_0, \cdots, v_n]$.

If we delete one of the $n+1$ vertices of an n-simplex $[v_0, \cdots, v_n]$, then the remaining n vertices span an $(n-1)$-simplex, called a **face** of $[v_0, \cdots, v_n]$. We adopt the following convention:

The vertices of a face, or of any subsimplex spanned by a subset of the vertices, will always be ordered according to their order in the larger simplex.

The union of all the faces of Δ^n is the **boundary** of Δ^n, written $\partial\Delta^n$. The **open simplex** $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$, the interior of Δ^n.

A **Δ-complex structure** on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

(i) The restriction $\sigma_\alpha | \mathring{\Delta}^n$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha | \mathring{\Delta}^n$.

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

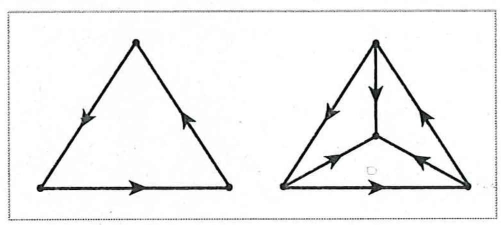
(iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

also, condit'n (iii) recovers the topology of X from a Δ-complex structure.

Among other things, this last condition rules out trivialities like regarding all the points of X as individual vertices. The earlier decompositions of the torus, projective plane, and Klein bottle into two triangles, three edges, and one or two vertices define Δ-complex structures with a total of six σ_α's for the torus and Klein bottle and seven for the projective plane. The orientations on the edges in the pictures are compatible with a unique ordering of the vertices of each simplex, and these orderings determine the maps σ_α.

A consequence of (iii) is that X can be built as a quotient space of a collection of disjoint simplices Δ_α^n, one for each $\sigma_\alpha : \Delta^n \to X$, the quotient space obtained by identifying each face of a Δ_α^n with the Δ_β^{n-1} corresponding to the restriction σ_β of σ_α to the face in question, as in condition (ii). One can think of building the quotient space inductively, starting with a discrete set of vertices, then attaching edges to these to produce a graph, then attaching 2-simplices to the graph, and so on. From this viewpoint we see that the data specifying a Δ-complex can be described purely combinatorially as collections of n-simplices Δ_α^n for each n together with functions associating to each face of each n-simplex Δ_α^n an $(n-1)$-simplex Δ_β^{n-1}.

More generally, Δ-complexes can be built from collections of disjoint simplices by identifying various subsimplices spanned by subsets of the vertices, where the identifications are performed using the canonical linear homeomorphisms that preserve the orderings of the vertices. The earlier Δ-complex structures on a torus, projective plane, or Klein bottle can be obtained in this way, by identifying pairs of edges of two 2-simplices. If one starts with a single 2-simplex and identifies all three edges to a single edge, preserving the orientations given by the ordering of the vertices, this produces a Δ-complex known as the 'dunce hat.' By contrast, if the three edges of a 2-simplex are identified preserving a cyclic orientation of the three edges, as in the first figure at the right, this does not produce a Δ-complex structure, although if the 2-simplex is subdivided into three smaller 2-simplices about a central vertex, then one does obtain a Δ-complex structure on the quotient space.



Thinking of a Δ-complex X as a quotient space of a collection of disjoint simplices, it is not hard to see that X must be a Hausdorff space. Condition (iii) then implies that each restriction $\sigma_\alpha | \mathring{\Delta}^n$ is a homeomorphism onto its image, which is thus an open simplex in X. It follows from Proposition A.2 in the Appendix that these open simplices $\sigma_\alpha(\mathring{\Delta}^n)$ are the cells e_α^n of a CW complex structure on X with the σ_α's as characteristic maps. We will not need this fact at present, however.

Simplicial Homology

Our goal now is to define the simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e_α^n of X. Elements