MATH523 , Homework 1

Due September 7, 2017 in class

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1. [5 pts] Let $v_1, \ldots, v_n$ be a basis of a vector space V.

   (a) Suppose that

   $$w = a_1 v_1 + \ldots + a_n v_n, \quad a_j \neq 0.$$

   Show that $v_1, \ldots, v_{j-1}, w, v_{j+1}, \ldots, v_n$ also forms a basis for V.

   (b) Use part (a) to show any n linearly independent vectors of V form a basis of V.

2. [5 pts] Let $x \in \mathbb{R}^n$.

   (a) Show that

   $$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty.$$

   (b) Use part (a) to show that

   $$\frac{1}{n}\|A\|_\infty \leq \|A\|_1 \leq m\|A\|_\infty,$$

   where A is an $m \times n$ matrix.

3. [5 pts] Show that for any $x \in \mathbb{R}^n$,

   $$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty.$$

4. [5 pts] Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times s}$, the sub-multiplicative property holds for p-norm,

   $$\|AB\|_p \leq \|A\|_p \|B\|_p.$$

   Does this inequality hold for norm different than the $p-$norm (prove or disprove)?

5. [5 pts] A matrix $A = \{a_{ij}\}$ that satisfies

$$\sum_{i=1, i \neq j}^{n} |a_{ij}| < |a_{jj}|, \quad j = 1, 2, \ldots, n.$$

is called a strictly column diagonal dominant matrix. Show that no permutation is required in the first step of Gaussian elimination process! After one-step of Gaussian elimination to A, we obtain

$$\begin{pmatrix} a_{11} & * \\ 0 & A_1 \end{pmatrix}.$$

Show that the $(n-1) \times (n-1)$ matrix A_1 is a strictly column diagonally dominant matrix. This result suggests that there is no permutation ~~is~~ required if A is strictly column diagonally dominant.

6. [5 pts] Given $A \in \mathbb{R}^{2n \times 2n}$, where $n > 0$ is an even positive integer, your computer can solve the n problems $Ax = b_j, j = 1, \ldots, n$ in one minute, using $A = LU$ factorization method. How many minutes (or seconds) was your computer working on the LU factorization? (2n+1)/(2n+4) minutes

7. [10 pts] Programming assignment. Given a sequence of data

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{m+1}, y_{m+1}),$$

write a program to interpolate the data using the following model

$$y(x) = a_0 + a_1 x + \ldots + a_m x^m,$$

where a_i are the unknown to be determined.

   (a) Derive the linear system of equations for the above interpolation problem.

   (b) Let $x_i = (i-1)/m, i = 1, \ldots, m+1$ and $y_i = \sin(\pi x_i)$. Solve the linear problem using your **own** Gaussian-elimination with pivoting (hand in your MATLAB code which includes the LU factorization, the forward, and backward iterations for solving triangular systems). Make sure you record the permutation matrix P, such that $PA = LU$, which will be needed for solving the triangular system!

(c) Plot the error $|y(x) - sin\pi(x)|$ and $\|y(x) - sin\pi(x)\|_2$ for $m = 4, 8, 16, 32, 64, 128, 256$ with denser sample points on interval $[0, 1]$ as a function of x (use semilogy to see the absolute error pattern on the first one) and as a function of m, respectively. You can use the MATLAB build-in function polyval.m to evaluate polynomial $y(x)$. Discuss the results, how does the higher order polynomial fit perform?