§5. The Mayer-Vietoris Argument.

1/24 Existence of a Good Cover.

Def M: mfd. $\mathcal{U} = \{U_\alpha\}$: open cover of M.

$\mathcal{U}$ is called a good cover if every nonempty intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeo. to $\mathbb{R}^n$, where $n = \dim(M)$.

A mfd having a good cover is said to be of finite type.

$\boxed{\text{Thm. (5.1)}}$

Every mfd has a good cover. In particular, every cpt mfd is of finite type.

Def $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$, $\mathcal{V} = \{V_\beta\}_{\beta \in J}$: two open covers of M.

$\mathcal{V}$ is a refinement of $\mathcal{U}$, denoted by $\mathcal{U} < \mathcal{V}$, if $\exists\, \phi: J \to I$ s.t. $V_\beta \subseteq U_{\phi(\beta)}$, $\forall\, \beta \in J$.

Rmk: Slight modification of Proof of Thm (5.1) gives:

$\boxed{\text{Every open cover on a mfd has a refinement that is a good cover.}}$

Def A directed set is a set I w/ a relation $<$ s.t.

(1) (reflexive) $a < a$, $\forall\, a \in I$.

(2) (transitive) $a < b$, $b < c \Rightarrow a < c$, $\forall\, a, b, c \in I$.

(3) (upper bound) $\forall\, a, b \in I$, $\exists\, c \in I$ s.t. $a < c$, $b < c$.

Def Given a directed set I. $J \subseteq I$.

J is called cofinal in I if, $\forall\, i \in I$, $\exists\, j \in J$ s.t. $i < j$.

Rmk:

(1) $I = \{\text{open covers on a mfd } M\}$ is a directed set.

(2) If J is cofinal in I, where I: directed set, then J itself is also a directed set.

$\boxed{\text{Cor (5.2)}}$

$J = \{\text{good covers on a mfd } M\}$ is cofinal in $I = \{\text{open covers on a mfd } M\}$.

$\boxed{\text{These concepts will be used when defining the Čech cohomology of a mfd.}}$

~TBC~ not end of §5.

§5. The Mayer-Vietoris Argument.

The Künneth Formula and the Leray-Hirsch Theorem

Thm. (Künneth Formula).

$H^*(M\times F) = H^*(M)\otimes H^*(F)$, meaning that

$$H^n(M\times F) = \bigoplus_{p+q=n} H^p(M)\otimes H^q(F), \text{ for any } n\geq 0.$$

Except for products, we consider a more general concept, the fiber bundle:

Def Let G be a gp acting on X. We say that G acts on X effectively if $g\cdot y = y,\ \forall y\in X \Rightarrow g = 1\in G$. i.e. the only element of G acting trivially on X is the identity.

Def G: Lie gp. E, B, F: mfds. G acts on F effectively.

A surj. $\pi: E\to B$ is called a fiber bundle with fiber F and structure group G if $\exists$ open cover $\{U_\alpha\}$ of B and fiber-preserving diffeo. $\phi_\alpha: E|_{U_\alpha}\to U_\alpha\times F$, where $E|_{U_\alpha} = \pi^{-1}(U_\alpha)$ s.t. the transition functions

$$g_{\alpha\beta}(x) = \phi_\alpha\phi_\beta^{-1}\Big|_{\{x\}\times F}: F\to F \text{ is in } G,\ \forall x\in U_\alpha\cap U_\beta,\ \forall \alpha,\beta. \quad (*)$$

E: total space. B: base space. $E_x := \pi^{-1}(x)$ is called the fiber at x, $\forall x\in B$.

Emphasizing the structure gp G, we sometimes use the name G-bundle.

Rmk:

(1) ($(*)$ above) Since G acts on F effectively, we may view G as a subgp of $\mathrm{Diff}(F)$ $= \{\text{diffeo. } F\to F\}$ via $G\to \mathrm{Diff}(F)$, $g\mapsto \tau_g$, where $\tau_g: F\to F$, $x\mapsto gx$. $\tau_g\in\mathrm{Diff}(F)$ by effectiveness of the action.

(2) The transition functions $g_{\alpha\beta}: U_\alpha\cap U_\beta\to G$ satisfy the cocycle condition:

$g_{\alpha\beta}\cdot g_{\beta\gamma} = g_{\alpha\gamma}$, $\forall \alpha,\beta,\gamma$.

(3) Given $\{U_\alpha\}$: open cover of B, $G\leq \mathrm{Diff}(F)$, $\{g_{\alpha\beta}: U_\alpha\cap U_\beta\to G\}$ a cocycle w/ values in G (i.e. for each $U_\alpha\cap U_\beta$, $g_{\alpha\beta}: U_\alpha\cap U_\beta\to G$ and, $\forall x\in U_\alpha\cap U_\beta\cap U_\gamma$, they satisfy the cocycle conditions $g_{\alpha\beta}\cdot g_{\beta\gamma} = g_{\alpha\gamma}$).

Define $E = (\coprod U_\alpha\times F)/(x,y)\sim(x, g_{\alpha\beta}(x)y)$, $\forall (x,y)\in U_\beta\times F$ and $(x, g_{\alpha\beta}(x)y)\in U_\alpha\times F$.

Then E is a fiber bundle having $\{g_{\alpha\beta}\}$ as its transition functions.

In other words, $\{g_{\alpha\beta}\}$ tells us how to patch these produces $U_\alpha \times F$ together.

(4) We can also define the concept of fiber bundle on topo. sp. by G: topo. gp., E, B, F: topo. sp. In this case, $G \leq \text{Homeo}(F)$.

$\boxed{\text{Thm. (5.11) (Leray-Hirsch theorem)}}$

E: fiber bundle over M w/ fiber F.

M: has a finite good cover.

$\exists$ global cohomology classes $e_1, \cdots, e_r$ on E which, when restricted to each fiber, freely generate the cohomology of the fiber.

Then $H^*(E) \cong H^*(M) \otimes \mathbb{R}\{e_1, \cdots, e_r\} \cong H^*(M) \otimes H^*(F)$.

Rmk (5.12) $\boxed{\text{(Kunneth formula for cpt cohomology)}}$

M, N: mfds having a finite good cover.

Then $H_c^*(M \times N) = H_c^*(M) \otimes H_c^*(N)$.

~TBC~ not end of §5.