MATH523

Take-home Midterm (I expect no collaborative work)

Due October 27 4, 2017 11.30AM (turn it into my office at McAllister 214)

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1. [5 pts] Consider approximating the function $f(x) = \sqrt{x}$ with a linear polynomial $p(x)$ on $[0, 1]$.

   (a) Find the best approximation in $L^2[0, 1]$, namely the least squares approximation with the standard scalar product.

   (b) Find the best approximation in $L^\infty[0, 1]$.

2. [5 pts] Given a function $f \in \mathcal{C}^3[x_0, x_2]$, let q be a quadratic polynomial that interpolates f at the nodes $x_0 < x_1 < x_2$, with variable spacing $h_1 = x_1 - x_0$ and $h_2 = x_2 - x_1$. Let $h = \max(h_1, h_2)$.

   (a) Determine the Newton form of q and prove that

   $$\begin{aligned} q'(x_1) &= f[x_1, x_2]\frac{x_1 - x_0}{x_2 - x_0} + f[x_1, x_0]\frac{x_2 - x_1}{x_2 - x_0}, \\ q''(x_1) &= 2f[x_0, x_1, x_2]. \end{aligned}$$

   These values are finite difference approximations to $f'(x_1)$ and $f''(x_1)$.

   (b) Use Taylor expansion to show

   $$\begin{aligned} |f'(x_1) - q'(x_1)| &\leq Ch^2 \max_{x_0 \leq x \leq x_2} |f^{(3)}(x)|, \\ |f''(x_1) - q''(x_1)| &\leq Ch \max_{x_0 \leq x \leq x_2} |f^{(3)}(x)|. \end{aligned}$$

   You cannot use the formula derived in class for the error $f(x) - q(x)$. Explain why not!

   (c) Consider the case of equally spaced nodes, i.e., $h_1 = h_2 = h$. Derive the *centered* difference formulas,

   $$q'(x_1) = \frac{1}{2h}(f(x_2) - f(x_0)), \quad q''(x_1) = \frac{1}{h^2}(f(x_0) - 2f(x_1) + f(x_2)).$$

   (d) Use Taylor expansion again to derive the improved error bound

   $$|f''(x_1) - q''(x_1)| \leq Ch^2 \max_{x_1 \leq x \leq x_2} |f^{(4)}(x)|,$$

   provided that $f \in \mathcal{C}^4[x_0, x_2]$ and $h = h_1 = h_2$. Explain what would happen if f has only three continuous derivatives. Could you get an improved error of order $\mathcal{O}(h^3)$ for the first derivative? Explain.

3. [5 pts] Let $b(x)$ be the piecewise cubic interpolating the quantities (x_i, f_i, s_i) where the slopes s_i are the first derivatives $q_i'(x_i)$ of a quadratic polynomial q_i that fits the data (x_j, f_j) for $j = i-1, i, i+1$ if $1 \le i < n$ and so $s_0 = f'(x_0)$ and $s_n = f'(x_n)$. Prove the error estimate

$$|f(x) - b(x)| \le Ch^3, \quad \forall x_0 < x < x_n,$$

provided $f \in \mathcal{C}^3[x_0, x_n]$. Explain whether or not this estimate improves for equally spaced nodes and compare with spline.

Hint: Decompose the error as $e(x) = (f(x) - H(x)) + (H(x) - b(x))$, where $H(x)$ stands for the piecewise-cubic Hermite interpolant. You can't use the standard error formula as well since f is only $\mathcal{C}^3$ so you have to deduce the errors in $f - H$ by exploiting the Taylor expansion. Secondly, notice that $g \equiv H - b$ is a Hermite interpolation functions of $g(x_i) = 0$ and $g'(x_i) = H'(x_i) - s_i$. You may need to use results from Problem 2 above appropriately.

4. [5 pts] Let $s(x)$ denote the complete cubic spline of $f(x)$ on the interval $[a, b]$ with knots $a = x_0 < \ldots < x_n = b$; thus $s'(a) = f'(a)$ and $s'(b) = f'(b)$. Set $e(x) = f(x) - s(x)$.

   (a) Integrate twice by parts on each subinterval (x_{i-1}, x_i) to derive the (orthogonality) relation

   $$\int_a^b e''(x)\psi(x)dx = 0,$$

   for all piecewise linear functions ψ.

   (b) Use (a) to prove the identity (Phytagoras ~~in~~equality)

   $$\int_a^b |f''(x)|^2 dx = \int_a^b |s''(x)|^2 dx + \int_a^b |f''(x) - s''(x)|^2 dx.$$

   (c) Use (b) to conclude that $s(x)$ minimizes the integral $\int_a^b |\phi''(x)|^2 dx$ among all functions ϕ that have two continuous derivatives, interpolate f at the knots (i.e., $\phi(x_i) = f(x_i)$) and satisfy $\phi'(a) = f'(a), \phi'(b) = f'(b)$. Take f = \phi

5. [5 pts] Let $\alpha = \min \left[\max_{|x| \le 1} |x^6 - x^3 - p_5(x)| \right]$, where the minimum is taken over all polynomials of degree ≤ 5.

   (a) Find the polynomial $p_5(x)$ for which the minimum α is attained.

   (b) Find α.