Math 528 H.W. 1. Min-Chun Wu.

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Exercise 1-7. :

(a)

claim : For $x \in S^n \setminus \{N\}$, $\sigma(x) = u$, where $(u,0)$ is the pt where the line through N and x intersects the lineer space $x^{n+1}=0$.

$N = (0,\cdots,0,1)$

$x = (x^1,\cdots,x^{n+1})$. The line through N and x can be parametrized as

$$L(t) = N + t(x-N),\ t \in \mathbb{R}.$$
$$= (tx^1,\cdots,tx^n, 1+t(x^{n+1}-1)).$$

Thus, the intersecth of L and the lineer space $x^{n+1}=0$ happens when $1+t(x^{n+1}-1)=0$.

i.e. $t = \frac{1}{1-x^{n+1}}$. i.e. at $\frac{(x^1,\cdots,x^n,0)}{1-x^{n+1}}$, as expected. #

claim : Similar for $x \in S^n \setminus \{S\}$.

$S = (0,\cdots,0,-1)$

$x = (x^1,\cdots,x^{n+1})$ Line through x and S : $\tilde{L}(t) = S + t(x-S)$

$$= (tx^1,\cdots,tx^n, -1+t(x^{n+1}+1)).$$

Intersecth : $-1+t(x^{n+1}+1) = 0 \Leftrightarrow t = \frac{+1}{x^{n+1}+1}$.

$\Rightarrow Pt = \frac{(x^1,\cdots,x^n,0)}{+(x^{n+1}+1)} = (-\sigma(-x),0)$, as expeced. #

✓

(b)

Denote $\tau : \mathbb{R}^n \to \mathbb{R}^{n+1}$, where $\tau(u^1,\cdots,u^n) = \frac{(2u^1,\cdots,2u^n,|u|^2-1)}{|u|^2+1}$.

1° $\left|\frac{(2u^1,\cdots,2u^n,|u|^2-1)}{|u|^2+1}\right|^2 = \frac{1}{(|u|^2+1)^2}\left|(2u^1)^2+\cdots+(2u^n)^2+(|u|^2-1)^2\right| = \frac{1}{(|u|^2+1)^2}\cdot\left|4|u|^2+(|u|^2-1)^2\right|$

$= \frac{(|u|^2+1)^2}{(|u|^2+1)^2} \equiv 1$. Thus, $\tau : \mathbb{R}^n \to S^n$.

Also note that $\frac{|u|^2-1}{|u|^2+1} = 1 \Leftrightarrow |u|^2-1 = |u|^2+1 \Leftrightarrow 1 = -1$, which is impossible.

Thus, $\tau : \mathbb{R}^n \to S^n \setminus \{N\}$.

2° claim : $\sigma\tau(u^1,\cdots,u^n) = (u^1,\cdots,u^n)$ and $\tau\sigma(x^1,\cdots,x^{n+1}) = (x^1,\cdots,x^{n+1})$, $\forall\ (u^1,\cdots,u^n) \in \mathbb{R}^n$, $(x^1,\cdots,x^{n+1}) \in S^n\setminus\{N\}$

$$\sigma\tau(u^1,\cdots,u^n) = \sigma\left(\frac{(2u^1,\cdots,2u^n,|u|^2-1)}{|u|^2+1}\right) = \frac{(2u^1,\cdots,2u^n)/(|u|^2+1)}{1-\left((|u|^2-1)/(|u|^2+1)\right)}$$

$$= \frac{(2u^1,\cdots,2u^n)/(|u|^2+1)}{2/(|u|^2+1)} = (u^1,\cdots,u^n),\ \forall\,(u^1,\cdots,u^n)\in\mathbb{R}^n.$$

$$\tau\sigma(x^1,\cdots,x^{n+1}) = \tau\left(\frac{(x^1,\cdots,x^n)}{1-x^{n+1}}\right) = \frac{\left(2x^1/(1-x^{n+1}),\cdots,2x^n/(1-x^{n+1}),\ \sum_{i=1}^{n}\left(x^i/(1-x^{n+1})\right)^2-1\right)}{\sum_{i=1}^{n}\left(x^i/(1-x^{n+1})\right)^2+1}.$$

$$= \frac{\left(2x^1(1-x^{n+1}),\cdots,2x^n(1-x^{n+1}),\ \sum_{i=1}^{n}(x^i)^2-(1-x^{n+1})^2\right)}{\sum_{i=1}^{n}(x^i)^2+(1-x^{n+1})^2}$$

(*) $\because (x^1,\cdots,x^{n+1})\in S^n$ $\therefore (x^1)^2+\cdots+(x^{n+1})^2=1.$

$$= \frac{\left(2x^1(1-x^{n+1}),\cdots,2x^n(1-x^{n+1}),\ 1-(x^{n+1})^2-(1-x^{n+1})^2\right)}{1-(x^{n+1})^2+(1-x^{n+1})^2}.$$

$$= \frac{\left(2x^1(1-x^{n+1}),\cdots,2x^n(1-x^{n+1}),\ (1-x^{n+1})(2x^{n+1})\right)}{(1-x^{n+1})(2)} \underset{\boxed{\because\, x^{n+1}\neq 1}}{=} (x^1,\cdots,x^{n+1}),\ \forall\,(x^1,\cdots,x^{n+1})\in S^n\setminus\{N\}.$$

Hence, σ is bijective and $\tau=\sigma^{-1}$. (#) ✓

(c)

$$\tilde{\sigma}(x) = -\sigma(-x) = -\frac{(-x^1,\cdots,-x^n)}{1+x^{n+1}} = \frac{(x^1,\cdots,x^n)}{1+x^{n+1}}.$$

$$\sigma^{-1}(u^1,\cdots,u^n) = \frac{(2u^1,\cdots,2u^n,|u|^2-1)}{|u|^2+1}.$$

$$\Rightarrow \tilde{\sigma}\circ\sigma^{-1}(u^1,\cdots,u^n) = \frac{(2u^1,\cdots,2u^n)/(|u|^2+1)}{1+(|u|^2-1)} = \frac{2(u^1,\cdots,u^n)}{|u|^2(|u|^2+1)}.$$ (*) (#)

careful

$\because (S^n\setminus\{N\})\cap(S^n\setminus\{S\}) = S^n\setminus\{N,S\}$ $\therefore$ $0\in\mathbb{R}^n$ is excluded from the domain of $\tilde{\sigma}\circ\sigma^{-1}$.

Hence, by (*), $\tilde{\sigma}\circ\sigma^{-1}$ is smooth.

$\because \tilde{\sigma}(x) = -\sigma(-x)$ $\therefore x = -\sigma^{-1}(-\tilde{\sigma}(x))$. $\Rightarrow \tilde{\sigma}^{-1}(u) = -\sigma^{-1}(-u) = \dfrac{(-2u^1,\cdots,-2u^n,|u|^2-1)}{-(|u|^2+1)}.$

$$\Rightarrow \sigma\circ\tilde{\sigma}^{-1}(u^1,\cdots,u^n) = \sigma\left(\frac{(2u^1,\cdots,2u^n,1-|u|^2)}{|u|^2+1}\right) = \frac{(2u^1,\cdots,2u^n)/(|u|^2+1)}{|u|^2} = \frac{2(u^1,\cdots,u^n)}{|u|^2(|u|^2+1)}.$$

Similar to $\tilde{\sigma}\circ\sigma^{-1}$, $0\in\mathbb{R}^n$ is excluded from domain of it, and hence it is smooth. Thus, $\{(S^n\setminus\{N\},\sigma),(S^n\setminus\{S\},\tilde{\sigma})\}$ defines a C^∞ structure on S^n. (#)

(d)

Suffice to prove these two atlases are compatible.

Note that in either atlas, the charts can be written down in terms of coordinates as smooth functions.

Thus, ~~the~~ so are the transition functions. i.e. they two are compatible.

Hence, the defined smooth structures are the same. #

Exercise 1-12.

Recall that a product of smooth mfds w/o bd is again a smooth mfd w/o bd.

Thus, it suffices to prove $M \times N$ is a mfd w/ bd if M : mfd w/o bd and N : mfd w/ bd.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be an atlas of M and

$\{(V_\beta, \varphi_\beta)\}_{\beta \in B}$ 〃 〃 〃 〃 N. Assume $\dim(M) = m$ and $\dim(N) = n$.

Then each $\psi_\alpha(U_\alpha)$: open in $\mathbb{R}^m$ and $\varphi_\beta(V_\beta)$: open in $\mathbb{H}^n = \{(y_1, \cdots, y_n) \mid y_n \geq 0\}$.

$\Rightarrow \psi_\alpha(U_\alpha) \times \varphi_\beta(V_\beta)$: open in $\mathbb{R}^m \times \mathbb{H}^n = \mathbb{H}^{m+n}$. This is the key pt.

$U_\alpha \times V_\beta \to \mathbb{H}^{m+n}$.

Note that $\psi_\alpha(U_\alpha) \times \varphi_\beta(V_\beta) = (\psi_\alpha \times \varphi_\beta)(U_\alpha \times V_\beta)$, where $\psi_\alpha \times \varphi_\beta : (p, q) \mapsto (\psi_\alpha(p), \varphi_\beta(q))$

Hence, $\{(U_\alpha \times V_\beta, \psi_\alpha \times \varphi_\beta)\}_{(\alpha,\beta) \in A \times B}$ is an atlas of $M \times N$, making it a mfd w/ bd. #

Exercise 2-10.

(a)

$$F^*(f+g) = (f+g) \circ F = (f \circ F) + (g \circ F) = F^*(f) + F^*(g), \ \forall f, g \in C(N)$$

$$F^*(\alpha f) = (\alpha f) \circ F = \alpha (f \circ F) = \alpha F^*(f), \ \forall \alpha \in \mathbb{R}, f \in C(N).$$

Hence, F^* : $\mathbb{R}$-linear. #

(b)

($\Rightarrow$)

Given $f \in C^\infty(N)$.

$\because F : M \to N : C^\infty \ \therefore f \circ F : C^\infty$. i.e. $F^*(f) \in C^\infty(M)$. #

($\Leftarrow$)

Given $p \in M$, (U, φ) : chart on M around p

(V, ψ) : 〃 〃 N 〃 $F(p)$.

Denote $\psi = (\psi_1, \cdots, \psi_n)$.

Then each $\psi_k \in C^\infty(V)$. Shrink V to a smaller closed nbd of $F(p)$, and extend each ψ_k to $\widetilde{\psi}_k \in C^\infty(N)$. ($\psi_k = \widetilde{\psi}_k$ on some smaller nbd of $F(p)$ contained in V).

By condition, $F^*(\widetilde{\psi}_k) = \widetilde{\psi}_k \circ F \in C^\infty(M)$, $\forall k$.

In particular, $\psi_k \circ F$ is C^∞ in some nbd of $F(p)$ contained in V.

Thus, $\psi \circ F \circ \varphi^{-1}$: C^∞ in some nbd of p.

$\because p \in M$ is arbitrary $\therefore F$ is C^∞. #

(c)

($\Rightarrow$)

By ($\Rightarrow$) of (b), $F^* : C^\infty(N) \to C^\infty(M)$, and is $\mathbb{R}$-linear by (a).

It remains to prove F^* is bijective.

Consider $F^{-1} : N \to M$, which is also C^∞. (since F : diffeo.)

It induces $(F^{-1})^* : C^\infty(M) \to C^\infty(N)$.

For $f \in C^\infty(M)$, $F^*((F^{-1})^*(f)) = f \circ F^{-1} \circ F = f$ and for $g \in C^\infty(N)$, $(F^{-1})^*(F^*(g)) = g \circ F \circ F^{-1} = g$.

Thus, F^* : bijective w/ inverse $(F^{-1})^*$. #

($\Leftarrow$)

$\because F$: homeo $\therefore F$: invertible and F^{-1} : cont.

We need to prove F^{-1} : C^∞.

Given $F(p) \in N$, $p \in M$, and charts (V, ψ), (U, φ) for $F(p)$ and p, resp.

Extend φ to $\widetilde{\varphi} : M \to \mathbb{R}^m$, so that $\widetilde{\varphi} = \varphi$ on some smaller nbd of p. $\widetilde{\varphi} = (\widetilde{\varphi}^1, \cdots, \widetilde{\varphi}^m)$.

We prove F^{-1} to be C^∞ below; that F is C^∞ can be similarly proved.

Consider $\widetilde{\varphi}^k \circ F^{-1} \in C(N)$. $\because F^*(\widetilde{\varphi}^k \circ F^{-1}) = \widetilde{\varphi}^k \circ F^{-1} \circ F = \widetilde{\varphi}^k \in C^\infty(M)$

$\therefore \widetilde{\varphi}^k \circ F^{-1} \in C^\infty(N)$.

Hence, $\widetilde{\varphi} \circ F^{-1} = (\widetilde{\varphi}^1, \cdots, \widetilde{\varphi}^m) \circ F^{-1}$ is C^∞. Note that it is identical to $\varphi \circ F^{-1}$ around $F(p)$.

Thus, $\varphi \circ F^{-1} \circ \psi^{-1}$ is C^∞ around $\psi(F(p))$. ✓

$\because F(p) \in N$ is arbitrary $\therefore F^{-1}$ is C^∞. #

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Exercise 3-2:

$d_j := \dim(M_j)$.

Locally, π_j = proj. of $\mathbb{R}^{d_1+\cdots+d_k} \to \mathbb{R}^{d_j}$.

$\Rightarrow d(\pi_j)_p$ = proj. of $\mathbb{R}^{d_1+\cdots+d_k} \to \mathbb{R}^{d_j}$.

$\Rightarrow d(\pi_j)_p$: has rank d_j.

$\Rightarrow \alpha$ has rank $d_1+\cdots+d_k$. (so α: surj.)

Clearly α is linear.

Moreover, $\dim(T_p(M_1\times\cdots\times M_k)) = d_1+\cdots+d_k$

$= \dim(T_pM_1 \oplus \cdots \oplus T_pM_k)$.

Thus, α: inj.

In conclusion, α: iso. #

Similar arguments work for mfds w/ bd. #

Exercise 3-4.:

Parametrize S^1 by (N: north pole, S: south pole)

$S^1\setminus\{N\} = \{e^{it} \mid t \in (\frac{\pi}{2}, \frac{5}{2}\pi)\} \cong \mathbb{R}$

$S^1\setminus\{S\} = \{e^{i\hat{t}} \mid \hat{t} \in (-\frac{\pi}{2}, \frac{3}{2}\pi)\} \cong \mathbb{R}$.

Define $\alpha: TS^1 \to S^1\times\mathbb{R}$ by

$(p, \gamma\frac{d}{dt}) \mapsto (p,\gamma)$, $p \in S^1\setminus\{N\}$

$(p, \hat{\gamma}\frac{d}{d\hat{t}}) \mapsto (p,\hat{\gamma})$, $p \in S^1\setminus\{S\}$.

It's obviously well-defined and on each chart, α is simply $\mathbb{R}\times\mathbb{R} \hookrightarrow S^1\times\mathbb{R}$, which is smooth.

Thus, α: C^∞. This requires compatibility on the charts.

Moreover, since it is an inclusion, it is an immersion.

$\because \dim(TS^1) = \dim(S^1\times\mathbb{R}) = 2$

$\therefore \alpha$ is also a submersion. $\Rightarrow \alpha$: local diffeo.

A bijective local diffeo. is absolutely a diffeo.

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Thus, $TS^1 \cong S^1\times\mathbb{R}$. #

Exercise 4-5:

(a)

By definition, π is surj. $(*_1)$

Given $p \in \mathbb{C}^{n+1}\setminus\{0\}$, $p = (z_1, \cdots, z_{n+1})$

WLOG, assume $z_1 \neq 0$.

Then $\pi(p) = [z_1^0, \cdots, z_{n+1}^0] = [1, \frac{z_2^0}{z_1^0}, \cdots, \frac{z_{n+1}^0}{z_1^0}]$.

Let $U_1 = \{[z_1,\cdots,z_{n+1}] \mid z_1 \neq 0\}$, an open set in $\mathbb{CP}^n$, w/ $\varphi: U_1 \xrightarrow{\cong} \mathbb{C}^n$ defined by

$\varphi([z_1,\cdots,z_{n+1}]) = \varphi([1, \frac{z_2}{z_1}, \cdots, \frac{z_{n+1}}{z_1}]) = (\frac{z_2}{z_1}, \cdots, \frac{z_{n+1}}{z_1})$.

Under this chart, $\pi: (z_1,\cdots,z_{n+1}) \mapsto (\frac{z_2}{z_1}, \cdots, \frac{z_{n+1}}{z_1})$, which is smooth. $(*_2)$

Moreover, $U_1 \to \mathbb{C}^{n+1}\setminus\{0\}$, $[z_1,\cdots,z_{n+1}] \mapsto (1, \frac{z_2}{z_1}, \cdots, \frac{z_{n+1}}{z_1})$ (nonzero) is a local section.

Thus, π is a submersion. $(*_3)$

By $(*_1)$, $(*_2)$, $(*_3)$, done. #

(b)

$\mathbb{CP}^1 = U_1 \cup \{[0,1]\} = U_2 \cup \{[1,0]\}$, where $U_i = \{[z_1,z_2] \mid z_i \neq 0\}$, $i=1,2$, where $\{(U_1,\varphi_1),(U_2,\varphi_2)\}$ is an atlas,

$\varphi_1: [z_1,z_2] \mapsto z_2/z_1$, $\varphi_2: [z_1,z_2] \mapsto z_1/z_2$.

Thus, the transition function is $z_2/z_1 \mapsto z_1/z_2$,

i.e. $z \mapsto 1/z$, or $a+bi \mapsto \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$.

Recall that the transition for stereographic proj. is $(a,b) \mapsto (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$.

Also note that $U_1, U_2 \cong \mathbb{R}^2$ naturally.

Thus, $\mathbb{CP}^1 \cong S^2$, where $[0,1]$ as north pole

$[1,0]$ " south ". #

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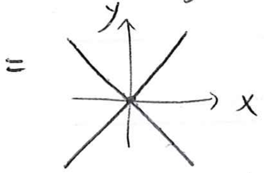
Exercise 5-11:

(a)

$\Phi^{-1}(0) = \{(x,y) \mid x^2 - y^2 = 0\}$

$= \{(x,y) \mid x = y\}$ or $\{(x,y) \mid x = -y\}$

= 

Any nbd of 0 in $\Phi^{-1}(0)$ is of the form 

Removing the center 0 will result in 4 disconnected parts.

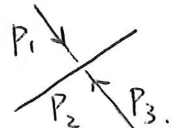
However, for any open ball in $\mathbb{R}^d$, removing a pt will either result in 2 disconnected parts or not disconnect anything.

Thus, around 0, $\Phi^{-1}(0)$ cannot be embedded. (#)

(b)

Yes.

Split $\Phi^{-1}(0)$ into 3 parts:



Each part of the 3 is diffeo. to $\mathbb{R}$.

Hence, the immersion can be achieved by $\Phi^{-1}(0) \to P_1 \sqcup P_2 \sqcup P_3$. (#)

Exercise 5-18: $m := \dim(M)$, $k := \dim(S)$.

(a)

($\Rightarrow$)

At each $p \in S$, since S is embedded, $\exists$ slice chart (U, ϕ) around p. (U,ϕ) $\phi = (x^1, \dots, x^m)$.

W.l.o.g., choose slice charts s.t. $U \cap S = \{x^{k+1} = \dots = x^m = 0\}$

Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be the collect'n of all such slice charts.

For each $\alpha \in I$, $f|_{U_\alpha \cap S} = f(x^1_\alpha, \dots, x^k_\alpha)$.

Extend $f|_{U_\alpha \cap S}$ to $\tilde{f}_\alpha : U_\alpha \to \mathbb{R}$ naturally by $\tilde{f}_\alpha(x^1_\alpha, \dots, x^m_\alpha) = f(x^1_\alpha, \dots, x^k_\alpha)$.

$\tilde{f}_\alpha|_{U_\alpha \cap S} = f|_{U_\alpha \cap S}$.

$\boxed{\text{P.1}}$

Let $\{p_\alpha\}_{\alpha \in I}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in I}$ (of $\bigcup_{\alpha \in I} U_\alpha$).

Define $\tilde{f} = \sum_{\alpha \in I} p_\alpha \cdot \tilde{f}_\alpha : \bigcup_{\alpha \in I} U_\alpha \to \mathbb{R}$.

Clearly well-defined (by local finiteness of $\{p_\alpha\}$) and smooth (by the fact $\operatorname{supp}(p_\alpha) \subseteq U_\alpha$).

Moreover, $\tilde{f}|_S = f$ (by the fact $\sum_\alpha p_\alpha = 1$).

i.e. $\tilde{f}$ is an extension of f to $\bigcup_{\alpha \in I} U_\alpha$, a nbd of S. (#)

($\Leftarrow$)

Suppose S is NOT embedded.

Then $\exists\, p \in S$ s.t. any chart on M around p is NOT slice.

Let U be open in S s.t. $p \in U$ and $i|_U : U \to M$ is an embedding, where $i : S \hookrightarrow M$.

(This U exists since immersion is a local embedding).

Consider $f : S \to \mathbb{R}$, a bump function supported in U, w/ $f(p) = 1$.

By condition, f can be extended to $\tilde{f} : M \to \mathbb{R}$. nbd of S

For a nbd V of p in M, since no chart around p can be assigned making V slice, $V \cap S$ contains $q_V \in S$ outside of U (o.w. it becomes a slice chart). $f(q_V) = 0$.

$\Rightarrow \tilde{f}(q_V) = 0$.

Shrink V closer and closer to p, we have a seq. $q_1, q_2, \dots \to p$, $\tilde{f}(q_k) = 0$, $\forall k$.

$\Rightarrow \tilde{f}(p) = \lim_{k \to \infty} \tilde{f}(q_k) = 0$. However, $\tilde{f}(p) = 1$. ※

Thus, S must be embedded. (#)

(b)

($\Rightarrow$)

claim: properly embedded submfds are closed.

<pf of claim>. $i : S \hookrightarrow M$.

Suppose NOT. Then $\exists$ seq. $p_1, p_2, \dots \to p$ w/ $p_k \in S$, $p \notin S$.

$K := \{p_1, p_2, \dots\} \cup \{p\}$.

Then K: cpt in M but $i^{-1}(K) = \{p_1, p_2, \dots\}$ is NOT cpt anymore. ※

Thus, S is closed. ⊕ of claim.

Adopting notations in (a)($\Rightarrow$).

Now, consider the open cover of M:

$\{U_\alpha\}_{\alpha\in I} \cup \{M\setminus S\}$.

As in (a)($\Rightarrow$), using p.o.u., we can construct an extension of f to all M. #

($\Leftarrow$)

claim: If S is embedded, then S: proper $\Leftrightarrow$ S: closed.

⟨Pf of claim⟩.

($\Rightarrow$) is proved in ($\Rightarrow$) above.

($\Leftarrow$) is obvious since cpt $\cap$ closed $=$ cpt. $\boxed{\text{⊕ of claim.}}$

By (a)($\Leftarrow$), S is embedded $(*_1)$

Suppose S is not proper. i.e. not closed.

Choose $p_1, p_2, \cdots \to p$ s.t. $p_k \in S$, $p \notin S$.

Let U_k be open sets in S s.t. $p_k \in U_k, \forall k$, and $U_k \cap U_j = \emptyset$, $\forall\, k \neq j$. (using Hausdorff).

Let $f_k : S \to \mathbb{R}$ be bump functions supp. in U_k.

Let $f = \sum_{k=1}^{\infty} k \cdot f_k : S \to \mathbb{R}$.

By condition, f can be extended to $\tilde{f}: M \to \mathbb{R}$.

Then $\tilde{f}(p) = \lim_{k\to\infty} \tilde{f}(p_k) = \lim_{k\to\infty} f(p_k) = \lim_{k\to\infty} k = \infty$. $\divideontimes$

Thus, S must be proper. $(*_2)$

By $(*_1)$ and $(*_2)$, done. #

Exercise 6-2:

By Whitney embedding, embed M into $\mathbb{R}^{2n+1}$.

$M \xhookrightarrow{\text{embed}} \mathbb{R}^{2n+1}$.

Define $UM = \{(p,v) \mid \|v\|_p = 1, v \in T_pM\}$
$\subseteq TM \subseteq T\mathbb{R}^{2n+1}$, where $\|\cdot\|_p$ is from $T_pM \subseteq T_p\mathbb{R}^{2n+1} \cong \mathbb{R}^{2n+1}$.

Note $\dim(UM) = 2n-1$.

Define $G: UM \to \mathbb{RP}^{2n}$ by $(p,v) \mapsto [v]$.

$\because \dim(UM) = 2n-1 < 2n = \dim(\mathbb{RP}^{2n})$

$\therefore$ All pts in UM are critical pts of G.

$\Rightarrow$ {critical values of G} $= \operatorname{im}(G)$.

By Sard's thm, {critical val. of G} has measure 0 in $\mathbb{RP}^{2n}$.

$\Rightarrow \exists\, v \in \mathbb{R}^{2n+1}\setminus\{0\}$ s.t. $[v] \notin \operatorname{im}(G)$.

Define $\pi_v : \mathbb{R}^{2n+1} \to \mathbb{R}v^{\perp} \cong \mathbb{R}^{2n}$ to be the orthogonal projection along $\mathbb{R}v$.

claim: $\pi_v|_M : M \to \mathbb{R}^{2n}$ is an immersion.

For $p \in M$,

$$d(\pi_v|_M)_p = d(\pi_v)_p|_{T_pM} = \pi_v|_{T_pM}$$

$\because [v] \notin \operatorname{im}(G)$

$\therefore v \notin T_pM$. $\Rightarrow \pi_v|_{T_pM} = \cancel{i}_{T_pM} : T_pM \to \mathbb{R}^{2n}$. $\boxed{\text{inclusion}}$

Thus, $d(\pi_v|_M)_p$ is mono. $\pi_v(w) = w - \frac{\langle w,v\rangle}{\langle v,v\rangle} v$

i.e. $\pi_v|_M$ is an immersion. #

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Exercise 6-11. 10/10

$$\begin{array}{ccccc} M & \xrightarrow{F} & N & \xrightarrow{G} & P \\ \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid \\ F^{-1}(G^{-1}(X)) & & G^{-1}(X) & & X \end{array}$$

That F: transverse to $G^{-1}(X)$ means (*1)

$$dF_z(T_zM)+T_{F(z)}(G^{-1}(X)) = T_{F(z)}N, \ \forall z\in F^{-1}(G^{-1}(X)).$$

That GF: transverse to X means (*2)

$$d(GF)_z(T_zM)+T_{GF(z)}X = T_{GF(z)}P, \ \forall z \in F^{-1}(G^{-1}(X)).$$

For $z\in F^{-1}(G^{-1}(X))$, simplify notations as follows:

$U := T_zM$, $V := T_{F(z)}N$, $W := T_{GF(z)}P$.

$S := dF_z$, $T := dG_{F(z)}$.

$W' := T_{GF(z)}X$, $V' := T_{F(z)}(G^{-1}(X))$.

Then $U \xrightarrow{S} V \xrightarrow{T} W$, and $V' \subseteq V$, $W' \subseteq W$.

(*1) means $S(U)+V'=V$.

(*2) means $TS(U)+W'=W$.

Notice that G restricts to $G: G^{-1}(X) \to X$.

Thus, $(dG_{F(z)})^{-1}(T_{GF(z)}X) = T_{F(z)}(G^{-1}(X))$

i.e. $T^{-1}(W') = V'$. This uses G transverse to X; e.g. $G(x)=(x,x^2)$; $X=\{(x,0)\}$.

Also, the condition G transverse to X can be rephrased as $T(V)+W'=W$

((*1) ⇒ (*2))

$$W \overset{\text{condition}}{=} T(V)+W' \overset{(*1)}{=} T(S(U)+V')+W'$$

$$= TS(U)+T(V')+W' = TS(U)+W' \ (\#)$$

$T(V')\subseteq W'$

((*2) ⇒ (*1))

Clearly, $S(U)+V' \subseteq V$.

Given $v\in V$. By (*2), $T(v) = TS(u)+w'$, some $u\in U$, $w'\in W'$.

Thus $T(v-S(u)) = w' \in W'$. [P1]

$\Rightarrow v-S(u) \in T^{-1}(W') = V'$, say $v-S(u)=v'$, some $v'\in V'$.

$\Rightarrow v = S(u)+v' \Rightarrow v \in S(U)+V'$. (#)

Exercise 7-4.

(a)
$$A = \begin{bmatrix} A^1_1 & \cdots & A^n_1 \\ A^1_2 & \ddots & A^n_2 \\ \vdots & & \vdots \\ A^1_n & \cdots & A^n_n \end{bmatrix}.$$

$$\Rightarrow I_n + tA = \begin{bmatrix} 1+tA^1_1 & \cdots & \cdots & tA^n_1 \\ tA^1_2 & 1+tA^2_2 & & \vdots \\ \vdots & & \ddots & \\ tA^1_n & \cdots & \cdots & 1+tA^n_n \end{bmatrix}.$$

For $\sigma\in S_n$, say $\{i_1,\dots,i_k\} \subseteq \{1,\dots,n\}$ are those who are fixed by σ, and $\{1,\dots,n\}\setminus\{i_1,\dots,i_k\} =: \{j_1,\dots,j_{n-k}\}$.

Then

$$f_\sigma(t) := \operatorname{sgn}(\sigma)(I_n+tA)^{\sigma(1)}_1 \cdots (I_n+tA)^{\sigma(n)}_n$$

$$= \operatorname{sgn}(\sigma)(1+tA^{i_1}_{i_1})\cdots(1+tA^{i_k}_{i_k})\, t^{n-k} A^{j_1}_{j_1}\cdots A^{n-k}_{j_{n-k}}.$$

case1 $k=n$

Then $\sigma=(1)$ and $\frac{d}{dt}\Big|_{t=0} f_\sigma(t) = \frac{d}{dt}\Big|_{t=0}(1+tA^1_1)\cdots(1+tA^n_n)$

$= A^1_1+\cdots+A^n_n$. (generalized produce rule from calculus)

case2 $k=n-1$.

This is impossible since a permutation in S_n fixing $n-1$ elements must fix n elements.

case3 $k\le n-2$

Then $f_\sigma(t) = \operatorname{sgn}(\sigma)\, t^{n-k}(\cdots)$. ($n-k\ge 2$)

By produce rule, $\frac{d}{dt}\Big|_{t=0} f_\sigma(t) = 0$.

Thus, $\frac{d}{dt}\Big|_{t=0}\det(I_n+tA) = A^1_1+\cdots+A^n_n = \operatorname{tr}(A)$. (#)

(b)
$d(\det)_X(B)$

$$= \frac{d}{dt}\Big|_{t=0}\det(X+tB) = \frac{d}{dt}\Big|_{t=0}\det(X)\det(I+tX^{-1}B)$$

$$= \det(X)\frac{d}{dt}\Big|_{t=0}\det(I+tX^{-1}B)$$

$$\underset{\text{by (a)}}{=} \det(X)\operatorname{tr}(X^{-1}B) \ (\#)$$

Exercise 8-13.

Define $X = \frac{x}{1+x^2+y^2}\frac{\partial}{\partial x} + \frac{y}{1+x^2+y^2}\frac{\partial}{\partial y}$.

As $\|(x,y)\| \to \infty$, both $\frac{x}{1+x^2+y^2}$ and $\frac{y}{1+x^2+y^2} \to 0$.

Thus, $X \to 0$ as $\|(x,y)\| \to \infty$.

Use stereographic projection, $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

Define $\tilde{X}$ on S^2 by $\tilde{X} = X$ on $\mathbb{R}^2$ and $\tilde{X} = 0$ on $\{\infty\}$.

Then

$\tilde{X}$ is a vector field on S^2 vanishing at exactly one point.

To prove smoothness, use ~~another~~ the other chart in stereographic projection containing ∞. The formula can be written down explicitly. (#)

This is worth doing at least once

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Exercise 9-19.

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claim: V, W commute.

Given $f \in C^\infty(M)$.

Then

VWf

$= V\left(f_y + \frac{x}{x^2+y^2} f_z\right)$

$= \left(\frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z}\right)\left(f_y + \frac{x}{x^2+y^2} f_z\right)$

$= f_{xy} + \frac{x}{x^2+y^2} f_{xz} + \frac{y^2-x^2}{(x^2+y^2)^2} f_z - \frac{x}{x^2+y^2} f_{yz} - \frac{y}{x^2+y^2}\cdot\frac{x}{x^2+y^2} f_{zz}$.

WVf

$= W\left(f_x - \frac{y}{x^2+y^2} f_z\right)$

$= \left(\frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\right)\left(f_x - \frac{y}{x^2+y^2} f_z\right)$

$= f_{xy} - \frac{y}{x^2+y^2} f_{yz} - \frac{x^2-y^2}{(x^2+y^2)^2} f_z + \frac{x}{x^2+y^2} f_{xz} - \frac{x}{x^2+y^2}\cdot\frac{y}{x^2+y^2} f_{zz}$.

Comparing terms, $VWf = WVf$.

Hence, V and W commute #

Write V and W in standard coordinates:

$V|_{(x,y,z)} = \left(1, 0, \frac{-y}{x^2+y^2}\right)$.

$W|_{(x,y,z)} = \left(0, 1, \frac{x}{x^2+y^2}\right)$.

Denote an integral curve of V and W by $\theta(t)$ and $\psi(s)$.

1°

$\theta(t) = (x(t), y(t), z(t))$

$\theta'(t) = \left(1, 0, \frac{-y(t)}{x^2(t)+y^2(t)}\right)$.

$\Rightarrow$

$$\theta(t) = \begin{cases} x(t) = t+a \\ y(t) = b \\ z(t) = \int_0^t \frac{-b}{(u+a)^2+b^2}\,du + c \end{cases},$$

where a, b, c are const.

2°

$\psi(s) = (x(s), y(s), z(s))$

$\psi'(s) = \left(0, 1, \frac{x(s)}{x^2(s)+y^2(s)}\right)$.

$\Rightarrow$

$$\psi(s) = \begin{cases} x(s) = a \\ y(s) = s+b \\ z(s) = \int_0^s \frac{a}{a^2+(u+b)^2}\,du + c \end{cases},$$

where a, b, c are const.

Let's start from the point $(0,1,0)$.

Then

$(0,1,0) \overset{\theta_t}{\rightsquigarrow} \left(t, 1, \int_0^t \frac{-1}{u^2+1}\,du\right)$

$\overset{\psi_s}{\rightsquigarrow} \left(t, s+1, \int_0^s \frac{t}{t^2+(u+1)^2}\,du + \int_0^t \frac{-1}{u^2+1}\,du\right)$ $(*_1)$

$(0,1,0) \overset{\psi_s}{\rightsquigarrow} (0, s+1, 0)$

$\overset{\theta_t}{\rightsquigarrow} \left(t, s+1, \int_0^t \frac{-(s+1)}{u^2+(s+1)^2}\,du\right)$ $(*_2)$

Setting $t = \pi/4$, we can use substitution trick to obtain

$(*_1) = \left(\frac{\pi}{4}, s+1, \tan^{-1}\frac{4(s+1)}{\pi} - \tan^{-1}\frac{4}{\pi} - 1\right)$, and

$(*_2) = \left(\frac{\pi}{4}, s+1, -\tan^{-1}\frac{\pi}{4(s+1)}\right)$.

Take $s = 1/2$;

$(*_1) = \left(\frac{\pi}{4}, \frac{3}{2}, -0.81657\ldots\right)$

$(*_2) = \left(\frac{\pi}{4}, \frac{3}{2}, -0.59335\ldots\right)$

i.e. $(*_1) \neq (*_2)$, as desired.

#

Exercise 10-7.

Recall that the transition map of stereographic proj'ns of S^2 is

$(u_1, u_2) \mapsto \frac{(u_1, u_2)}{u_1^2+u_2^2}$.

For a 2-dim mfd M and two overlapping charts $\psi = (x_1, x_2)$, $\varphi = (y_1, y_2)$ of M, the transition map of the corresponding charts on TM is

$(x_1, x_2, a_1, a_2) \mapsto (y_1, y_2, b_1, b_2)$, where both represent pts $(p, v) \in TM$ w/

$v = a_i \frac{\partial}{\partial x_i}\Big|_p = b_j \frac{\partial}{\partial y_j}\Big|_p \Rightarrow b_j = a_i \frac{\partial y_j}{\partial x_i}\Big|_p$.

Thus, $(x_1, x_2, a_1, a_2) \mapsto (y_1, y_2, a_i \frac{\partial y_1}{\partial x_i}, a_i \frac{\partial y_2}{\partial x_i})$.

Applying this onto the TS^2 case, we have

$(u_1, u_2, a_1, a_2) \mapsto$

$$\left(\frac{u_1}{u_1^2+u_2^2}, \frac{u_2}{u_1^2+u_2^2}, a_1 \frac{\partial}{\partial u_1}\frac{u_1}{u_1^2+u_2^2} + a_2 \frac{\partial}{\partial u_2}\frac{u_1}{u_1^2+u_2^2},\right.$$
$$\left. a_1 \frac{\partial}{\partial u_1}\frac{u_2}{u_1^2+u_2^2} + a_2 \frac{\partial}{\partial u_2}\frac{u_2}{u_1^2+u_2^2}\right).$$

$$= \left('', '', a_1 \cdot \frac{u_2^2 - u_1^2}{(u_1^2+u_2^2)^2} + a_2 \cdot \frac{-2u_1u_2}{(u_1^2+u_2^2)^2}, a_1 \cdot \frac{-2u_1u_2}{(u_1^2+u_2^2)^2} + a_2 \cdot \frac{u_1^2-u_2^2}{(u_1^2+u_2^2)^2}\right).$$

\#

Exercise 11-11.

WLOG, assume 0 is the reguler value of Φ making $U \cap C = \Phi^{-1}(0)$.

Thus, $d\Phi_q$ is surj. $\forall q \in U \cap C$.

$\Rightarrow$ We may resetrict Φ to a smeller nbd of C making it a submersion.

By abuse of notetion, still denote this nbd U.

By Const. Rank Thm, $\exists$ chert around p meking Φ a projectin onto first k coordinetes.

By abuse of notetn, also denote this chert U.

So, we have a chert $\tilde{\Phi} : U \to \mathbb{R}^n$ s.t.

$\tilde{\Phi} = (\Phi^1, \cdots, \Phi^k, \Phi^{k+1}, \cdots, \Phi^n)$, where $\Phi = (\Phi^1, \cdots, \Phi^k)$.

Moreover, $U \cap C = \{\Phi^1 = \cdots = \Phi^k = 0\}$ and $(\Phi^{k+1}, \cdots, \Phi^n)$ is a chert for $U \cap C$.

$\Rightarrow \left\{\frac{\partial}{\partial \Phi^{k+1}}\Big|_p, \cdots, \frac{\partial}{\partial \Phi^n}\Big|_p\right\}$: a basis of T_pC.

<u>claim</u>: $df_p \equiv 0$ on T_pC.

Given $v \in T_pC$, w/ curve $\gamma : (-\varepsilon, \varepsilon) \to C$ s.t. $\gamma'(0) = v$.

Then $df_p(v) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma(t) = 0$, since $f \circ \gamma$ has locel extrema at $t = 0$. \# of claim.

Note thet, since $\{d\Phi^1_p, \cdots, d\Phi^n_p\}$ is a basis of T_p^*M,

$df_p = \lambda_i \, d\Phi^i_p$, where $\lambda_i = df_p\left(\frac{\partial}{\partial \Phi^i}\Big|_p\right)$.

For $i \geq k+1$, $\frac{\partial}{\partial \Phi^i}\Big|_p \in T_pC$.

By <u>claim</u>, $df_p\left(\frac{\partial}{\partial \Phi^i}\Big|_p\right) = 0$, $\forall i \geq k+1$.

Hence,

$$df_p = \lambda_1 \, d\Phi^1|_p + \cdots + \lambda_k \, d\Phi^1|_p. \quad \#$$

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Exercise 12-12.

The 4 conditions are clearly satisfied by $\mathcal{L}_V$.

Let's prove by induction on k.

For $k=0$, (b) holds, nothing to prove.

For clarity, denote a map satisfying (a), ..., (d) by F_V.

For $k=1$, $\omega \in \mathcal{T}^1(M)$, by (d),

$$F_V(\omega(X)) = (F_V\omega)(X) + \omega([V,X]),\ \forall X \in \mathfrak{X}(M)$$

By the case of $k=0$, since $\omega(X) \in \mathcal{T}^0(M)$,

$$F_V(\omega(X)) = \mathcal{L}_V(\omega(X)).$$

Thus, $\mathcal{L}_V(\omega(X)) = (F_V\omega)(X) + \omega([V,X])$.

$$\Rightarrow F_V(\omega)(X) = \mathcal{L}_V(\omega(X)) - \omega([V,X]) = (\mathcal{L}_V\omega)(X).$$

Thus, $F_V = \mathcal{L}_V$ on $\mathcal{T}^1(M)$, proving $k=1$.

Assume results for k.

For $A \in \mathcal{T}^{k+1}(M)$, locally

$$A = \sum_{i_1,\cdots,i_{k+1}} C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_{k+1}}, \text{ where}$$

$C_{i_1\cdots i_{k+1}}$ are C^∞ and $i_1,\cdots,i_{k+1} \in \{1,\cdots,n\}$.

For each individual $C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_{k+1}}$,

$$F_V(C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_{k+1}})$$

$$= F_V(C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_k})\otimes dx^{i_{k+1}} + C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_k}\otimes F_V(dx^{i_{k+1}}).$$

$$\overset{\boxed{\text{induction.}}}{=} \mathcal{L}_V(C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_k})\otimes dx^{i_{k+1}} + C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_k}\otimes \mathcal{L}_V(dx^{i_{k+1}}).$$

$$= \mathcal{L}_V(C_{i_1\cdots i_{k+1}}\, dx^{i_1}\otimes\cdots\otimes dx^{i_k}).$$

As for A,

$$F_V(\Sigma\cdots) \overset{\boxed{\text{by (a)}}}{=} \Sigma(F_V(\cdots)) \overset{\boxed{\text{by individual case}}}{=} \Sigma\,\mathcal{L}_V(\cdots)$$

$= \mathcal{L}_V(\Sigma\cdots)$. i.e. $F_V(A) = \mathcal{L}_V(A)$.

Therefore, $F_V = \mathcal{L}_V$. (#)

Exercise 14-5.

Extend $(\omega^1,\cdots,\omega^k)$ to $(\omega^1,\cdots,\omega^n)$ s.t. at every $p\in U$, $\omega^1|_p,\cdots,\omega^n|_p$ is linearly indep., thus a basis of T_p^*U.

$$\Rightarrow \alpha^i|_p = \sum_{j=1}^n f_{ij}(p)\,\omega^j|_p.$$

By condition,

$$f_{11}\cancel{\omega^1\wedge\omega^1} + f_{12}\,\omega^2\wedge\omega^1 + \cdots + f_{1n}\,\omega^n\wedge\omega^1$$
$$+ f_{21}\,\omega^1\wedge\omega^2 + f_{22}\cancel{\omega^2\wedge\omega^2} + \cdots + f_{2n}\,\omega^n\wedge\omega^2$$
$$+ \cdots$$
$$+ f_{k1}\,\omega^1\wedge\omega^k + f_{k2}\,\omega^2\wedge\omega^k + \cdots + f_{kn}\,\omega^n\wedge\omega^k = 0.$$

Notice that for f_{ij} w/ $j>k$, since we only have $\omega^j\wedge\omega^i$ but no $\omega^i\wedge\omega^j$, it $(\omega^i\wedge\omega^j)$ only appears once. $\Rightarrow f_{ij}=0,\ \forall j>k$.

i.e. $\alpha^i|_p = \sum_{j=1}^k f_{ij}(p)\,\omega^j|_p$.

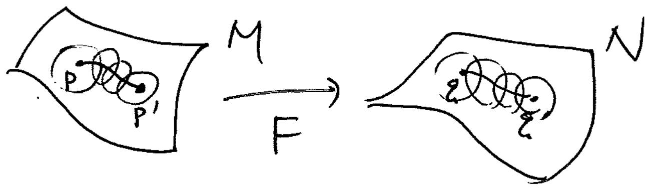
i.e. each α^i is a C^∞ linear combination of $\omega^1,\cdots,\omega^k$. (#)

Exercise 15-2.

Let $\{U_\alpha\}_{\alpha\in A}$, $\{V_\beta\}_{\beta\in B}$ be atlases of M and N compatible w/ their resp. orientations.

It suffices to prove: $\forall p, p' \in M$ w/ $p\in U_\alpha$, $p'\in U_{\alpha'}$, $F(p)\in V_\beta$, $F(p')\in V_{\beta'}$, the det of dF w.r.t. (U_α, V_β) and $(U_{\alpha'}, V_{\beta'})$ have the same sign.

$\because M$ is connected $\therefore \exists$ path γ on M from p to p'.



$\because$ im(γ) is cpt $\therefore \exists$ finitely many U_α covering im(γ).

WLOG, we may assume
(in practice, by shrinking U_α) $F|_{U_\alpha}$: diffeo.
So, $F(U_\alpha)$ thus serves as a coordinate chart.
On each U_α, by local diffeo, det(dF) has the same sign.
Using overlapping, we can pass this sign down.

Eventually, since there are only finitely many U_α along im(γ), we can pass the sign from U_α to $U_{\alpha'}$ and V_β to $V_{\beta'}$ and the result follows. #