§15. Cohomology with Integer Coefficients.

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The Cone Construction

The Cone Construction $K : S_*(\mathbb{R}^n) \to S_{*+1}(\mathbb{R}^n)$ can be proved to satisfy

$$\partial K - K\partial = (-1)^{q+1} \quad \text{on } S_q(\mathbb{R}^n),\ \forall\, q \geq 1.$$

$\Rightarrow$ K : chain homotopy between $1_{S_*(\mathbb{R}^n)}$ and 0 map.

$$\Rightarrow H_q(\mathbb{R}^n) = \begin{cases} 0 & , q \geq 1 \\ \mathbb{Z} & , q = 0. \end{cases}$$

The Mayer-Vietoris Sequence for Singular Chains.

$\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$: open cover of X. (X : topo. sp.), where I is countable and total ordered.

Def

The gp $\boxed{S_*^{\mathcal{U}}(X)}$ of $\boxed{\mathcal{U}\text{-small cochains of } X}$, is defined to be the free abelian gp generated by simplices whose images lie in U_α for some $\alpha \in I$.

Fact :

① The inclusion $i : S_*^{\mathcal{U}}(X) \to S_*(X)$ is clearly a chain map.

② The inclusion in ① is actually a chain equivalence. (Proof can be found in Vick).

Thus, $H(S_*(X)) = H(S_*^{\mathcal{U}}(X))$.

As before, we have the inclusions:

$$X \leftarrow \coprod_{\alpha_0} U_{\alpha_0} \leftleftarrows \coprod_{\alpha_0<\alpha_1} U_{\alpha_0\alpha_1} \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \coprod_{\alpha_0<\alpha_1<\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots,$$ inducing maps in chain complexes.

Define

$$\delta : \bigoplus_{\alpha_0<\cdots<\alpha_p} S_q(U_{\alpha_0\cdots\alpha_p}) \to \bigoplus_{\alpha_0<\cdots<\alpha_{p-1}} S_q(U_{\alpha_0\cdots\alpha_{p-1}}) \quad \text{by}$$

$$(\delta c)_{\alpha_0\cdots\alpha_{p-1}} = \sum_\alpha c_{\alpha\alpha_0\cdots\alpha_{p-1}},$$ where we adopt the convention that interchanging two indices in $c_{\alpha_0\cdots\alpha_p}$ introduces a minus sign.

$\delta^2 = 0$ can be proved as before.

Denote $\bigoplus S_q(U_{\alpha_0}) \to S_q^{\mathcal{U}}(X)$, simply the sum, by ε.

Then, we have :

Prop (15.2) (The Mayer-Vietoris Seq. for Singular Chains).

The seq.

$$0 \leftarrow S_q^{\mathcal{U}}(X) \xleftarrow{\varepsilon} \bigoplus_{\alpha_0} S_q(U_{\alpha_0}) \xleftarrow{\delta} \bigoplus_{\alpha_0<\alpha_1} S_q(U_{\alpha_0\alpha_1}) \xleftarrow{\delta} \cdots$$

is exact.

To prove it, we recall a simple lemma:

Lemma (15.3)

$0 \to A \to B \to C \to 0$: short exact of diff. complexes.

If two out of A, B, C have zero homology, then so does the third.

<Proof>

Consider the long exact seq. induced by this short exact seq. #

<Proof of (15.2)>

We shall prove the finite case ($|I|<\infty$) first by induction and then extend the result to countably infinite case ($|I| = \aleph_0$).

For two open sets U_0, U_1,

$$0 \leftarrow S_q^{\mathcal{U}}(U_0 \cup U_1) \xleftarrow{\text{sum}} S_q(U_0) \oplus S_q(U_1) \leftarrow S_q(U_{01}) \leftarrow 0 \quad \text{is clearly exact by def.}$$

$$(C_{10}, C_{01}) \longleftarrow\!\!| \; C_{01}$$

For three open sets U_0, U_1, U_2, the seq. is

$$S_q(U_{012}) \leftarrow 0.$$

$$0 \leftarrow S_q^{\mathcal{U}}(U_0 \cup U_1 \cup U_2) \xleftarrow{\text{sum}} S_q(U_0) \oplus S_q(U_1) \oplus S_q(U_2) \leftarrow S_q(U_{01}) \oplus S_q(U_{02}) \oplus S_q(U_{12})$$

$$(C_{10}+C_{20}, C_{01}+C_{21}, C_{02}+C_{12}) \longleftarrow\!\!| \; (C_{01}, C_{02}, C_{12})$$

$$C_{012} \longmapsto (C_{201}, C_{102}, C_{012})$$

Now, consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & S^{\mathcal{U}}(U_0 \cup U_1) \arrow[l] \arrow[d] & S(U_0) \oplus S(U_1) \arrow[l] \arrow[d] & S(U_{01}) \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 & S^{\mathcal{U}}(U_0 \cup U_1 \cup U_2) \arrow[l] \arrow[d] & S(U_0) \oplus S(U_1) \oplus S(U_2) \arrow[l] \arrow[d] & S(U_{01}) \oplus S(U_{02}) \oplus S(U_{12}) \arrow[l] \arrow[d] & S(U_{012}) \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & \dfrac{S^{\mathcal{U}}(U_0 \cup U_1 \cup U_2)}{S^{\mathcal{U}}(U_0 \cup U_1)} \arrow[l] \arrow[d] & S(U_2) \arrow[l, "\beta"'] \arrow[d] & S(U_{02}) \oplus S(U_{12}) \arrow[l, "\delta"'] \arrow[d] & S(U_{012}) \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

The 1st row is Mayer-Vietoris for U_0, U_1, the 2nd row is Mayer-Vietoris for U_0, U_1, U_2, and the 3rd row is the quotient of the 2nd by the 1st.

Thus, the columns are short exact, and we may view the diagram as a short exact seq. of diff. complexes (the rows).

The 1st row is exact by induction hypothesis and hence has 0 homology.

The 3rd row is almost exact (in fact, it is the Mayer-Vietoris for U_{02}, U_{12} from the right for 3 terms), except possibly for $S(U_2)$-term.

(That β is surj. comes from the fact that $S^{\mathcal{U}}(U_0 \cup U_1 \cup U_2)/S^{\mathcal{U}}(U_0 \cup U_1)$ is generated by simplices in U_2 which do not lie entirely in U_0 or U_1).

$\beta\delta = 0. \Rightarrow \operatorname{im}(\delta) \subseteq \ker(\beta)$.

If $\beta(c) = 0$. i.e. $\operatorname{im}(c)$ is entirely in U_0 or U_1. $\Rightarrow c \in S(U_{02})$ or $S(U_{12})$.

$\Rightarrow c \in \operatorname{im}(\delta)$. Thus, $\ker(\beta) \subseteq \operatorname{im}(\delta)$.

Thus $\ker(\beta) = \operatorname{im}(\delta)$ and row 3 is exact and has zero homology.

By lemma, the 2nd row has zero homology and is exact.

The same process can be used from the rth step to the $(r+1)$st step in the induction and proves the $|I| < \infty$ case.

For the case $|I| = \aleph_0$, since each chain only involves finitely many open sets, it follows from the finite case that the seq. is still exact for countably infinite case. #

Rmk:

Replacing the $\mathbb{Z}$-coefficient by an arbitrary G-coefficient (G: abelian), the above proof still works and we have exact Mayer-Vietoris for an arbitrary G-coefficient as well.

Singular Cohomology

Def X: topo. sp.

$\boxed{S^q(X)} := \operatorname{Hom}(S_q(X), \mathbb{Z})$. Elements in $S^q(X)$ are called singular q-chains on X.

$\boxed{\text{Coboundary operator}}$ $\boxed{d}$ is defined by $(d\omega)(c) = \omega(\partial c)$. Then $d^2 = 0$.

$\boxed{H^*(X)} := H_d(S^*(X))$. We may replace the coefficient $\mathbb{Z}$ by an arbitrary abelian gp G and get $\boxed{H^*(X;G)}$.

Rmk:

① $H^0_{sing}(X) = \mathbb{Z}^r$, where $r = \#$ path components of X.

② The singular cohomology of $\mathbb{R}^n$ can be computed by considering the dual map $L: S^q(\mathbb{R}^n) \to S^{q-1}(\mathbb{R}^n)$ of the cone construction K. The result is $H^q(\mathbb{R}^n) = \begin{cases} \mathbb{Z}, & q=0 \\ 0, & q>0. \end{cases}$

Recall:

$\cdots \to A_1 \to A_2 \to A_3 \to \cdots$ exact seq. of free abelian gps.

G : abelian gp.

$\Rightarrow \cdots \leftarrow \mathrm{Hom}(A_1, G) \leftarrow \mathrm{Hom}(A_2, G) \leftarrow \cdots$ exact. ▨

Applying $\mathrm{Hom}(\ ,\mathbb{Z})$ to the Mayer-Vietoris seq. for singular chains, we have an exact seq.

$$0 \to S^*_{\mathcal{U}}(X) \xrightarrow{\varepsilon^*} \bigoplus_{\alpha_0} S^*(U_{\alpha_0}) \xrightarrow{\delta^*} \bigoplus_{\alpha_0<\alpha_1} S^*(U_{\alpha_0\alpha_1}) \to \cdots$$

, called the Mayer-Vietoris seq. for singular cochains.

Rmk (Exercise 15.7.1)

ε^* : restriction map.

δ^* : the alternating sum $(\delta^*\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1}(-1)^i\omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}$. ▨

Define $C^*(\mathcal{U}, S^*)$ by $C^p(\mathcal{U}, S^q) = \bigoplus_{\alpha_0<\cdots<\alpha_p} S^q(U_{\alpha_0\cdots\alpha_p})$.

Then we have a double complex, $\xrightarrow{\delta^*}$ is the Mayer-Vietoris and $\uparrow d$ is the usual S^*.

$\because \xrightarrow{\delta^*}$ is exact

$\therefore$ By taking the vertical filtration direction, we have $E_1 = H_{\delta^*} =$

$$\begin{array}{c|c|c}
\uparrow & & \\
S^2_{\mathcal{U}}(X) & 0 & \\
\uparrow & & \\
S^1_{\mathcal{U}}(X) & 0 & \\
\uparrow & & \\
S^0_{\mathcal{U}}(X) & 0 & \\
\hline
0 & 1 &
\end{array}$$

$\Rightarrow E_2 = H_d H_{\delta^*} =$

$$\begin{array}{|c|c|c}
H^2(X) & 0 & \\
\uparrow & & \\
H^1(X) & 0 & \\
\uparrow & & \\
H^0(X) & 0 & \\
\hline
\end{array}$$

, which stabilizes.

Thus, $GH_D\{C^*(\mathcal{U}, S^*)\} = E_\infty = E_2 = H^*(X)$.

Fact:

X : triangulable.

$\Rightarrow X$ admits a good cover and the good covers on X are cofinal in the set of all open covers of X.

Discussion:

$\mathcal{U}$: good cover on X.

Then $H_d(C^*(\mathcal{U}, S^*)) =$

$$\begin{array}{|c|c|c|c}
0 & 0 & 0 & \\
C^0(\mathcal{U},\mathbb{Z}) & C^1(\mathcal{U},\mathbb{Z}) & C^2(\mathcal{U},\mathbb{Z}) & \cdots \\
\hline
\end{array}$$

for $U_{\alpha_0\cdots\alpha_p} \neq \phi$

$\begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * > 0. \end{cases}$

This is because $H_d(C^p(\mathcal{U}, S^*)) = H_d(\bigoplus_{\alpha_0<\cdots<\alpha_p} S^*(U_{\alpha_0\cdots\alpha_p})) = \bigoplus_{\alpha_0<\cdots<\alpha_p} H_d(S^*(U_{\alpha_0\cdots\alpha_p})) = \bigoplus_{\substack{\alpha_0<\cdots<\alpha_p \\ U_{\alpha_0\cdots\alpha_p}\neq\phi}} \mathbb{Z} = C^p(\mathcal{U}, \mathbb{Z})$.

$\Rightarrow H_\delta H_d(C^*(\mathcal{U}, S^*)) = H^*(\mathcal{U}, \mathbb{Z})$.

But, by spectral seq. theory, it is also $GH_D(C^*(\mathcal{U}, S^*))$.

For X triangulable, good covers are cofinal in all covers of X.
$\Rightarrow H^*(X,\mathbb{Z}) = H^*(\mathcal{U},\mathbb{Z})$ and we have the following theorem:

Thm (15.8)

X: triangulable.

Then $$H^*_{sing}(X,\mathbb{Z}) \simeq H^*_{\check{C}ech}(X,\mathbb{Z}).$$

Moreover, for any good cover $\mathcal{U}$, we have $H^*(\mathcal{U},\mathbb{Z})$ iso. to any of them.

<Proof>

$$H^*_{sing}(X,\mathbb{Z}) \simeq GH_D(C^*(\mathcal{U},\mathbb{Z})) = H^*(\mathcal{U},\mathbb{Z}) \simeq H^*_{\check{C}ech}(X,\mathbb{Z}) \quad \#$$

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$\pi: E \to X$ fiber bundle, w/ fiber F, where X: triangulable.
$\mathcal{U}$: good cover of X. (good covers exist, for X: triangulable).
From the double complex $C^*(\pi^{-1}\mathcal{U}, S^*)$, we have a spectral seq. converging to $H^*(E)$
w/ $E_2^{p,q} = H^p(\mathcal{U}, \mathcal{H}^q(F))$, where $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$, $\mathcal{H}^q = \mathcal{H}^q(F)$: locally const. presheaf.
If $\mathcal{H}^q(F)$ happens to be the const. presheaf $\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}$ on $\mathcal{U}$, then

$$E_2^{p,q} = \underbrace{H^p(\mathcal{U},\mathbb{Z})\oplus\cdots\oplus H^p(\mathcal{U},\mathbb{Z})}_{\dim H^q(F)\text{ terms}} = H^p(X)\oplus\cdots\oplus H^p(X) = H^p(X)\otimes H^q(F).$$

Def

$(A_0,\cdots,A_q)$: q-simplex in X.
$(A_0\cdots A_r)$: the $\boxed{\text{front } r\text{-face}}$ of X and $(A_{q-r}\cdots A_q)$: the $\boxed{\text{back } r\text{-face}}$ of X.

ω: p-cochain, η: q-cochain, both on X.
Their $\boxed{\text{cup product}}$, denoted $\boxed{\omega\cup\eta}$, is defined by

$$(\omega\cup\eta)(A_0\cdots A_{p+q}) = \omega(A_0\cdots A_p)\,\eta(A_p\cdots A_{p+q}).$$

Rmk: (15.10)

d is an antiderivation relative to the cup product:

$$d(\omega\cup\eta) = (d\omega)\cup\eta + (-1)^{\deg\omega}\,\omega\cup(d\eta).$$

Thus, $\cup$ induces a product structure on $H^*(X)$ by $[\omega][\eta] := [\omega\cup\eta]$.

W/ arguments as before, we have

Thm (15.11) (Leray's Theorem for Singular Cohomology w/ A-coeff. A: com. ring.)

$\pi: E \to X$ fiber bundle w/ fiber F.

$\mathcal{U}$: good cover of X.

Then

(1) $\exists$ spec. seq. converging to $H^*(E;A)$ w/ $E_2^{p,q} = H^p(\mathcal{U}, \mathcal{H}^q(F;A))$.

Each E_r can be given a product structure making d_r: anti-derivation.

(2) if X: simply connected, then $E_2^{p,q} = H^p(X, H^q(F;A))$.

if, in addition, $H^*(F;A)$: f.g. free A-module,

then $E_2 = H^*(X;A) \otimes H^*(F;A)$ as algebras over A.

Rmk: (15.12) (Künneth formula for singular cohomology).

X: topo. sp. w/ a good cover.

Y: a topo. sp.

Then $H^n(X \times Y) = \bigoplus_{p+q=n} H^p(X, H^q(Y))$.

(Proof: Use the trivial bundle $\pi: X \times Y \to X$ and spec. seq.).

Rmk:

Künneth formula in the form $H^*(M \times F) = H^*(M) \otimes H^*(F)$ fails since there might be torsion in $H^*(F)$.

Def

A: abelian gp.

A short exact seq. $0 \to R \xrightarrow{i} F \xrightarrow{p} A \to 0$ is called a $\boxed{\text{free resolution}}$ if R and F are free abelian gps.

(This comes from the following construction:

Take a generating set $\{\alpha_i\}_{i \in I}$ of A. $F := \mathcal{F}(I)$, free abelian, generated by I.

$p: F \to A$, $i \mapsto \alpha_i$. $R := \ker(p)$.

$\because$ F: free abelian and $R \leq F$ $\therefore$ R: free abelian.

Thus, we have $0 \to R \to F \to A \to 0$ short exact and R, F: free abelian.)

Rmk:

Given a free resolution $0 \to R \xrightarrow{i} F \xrightarrow{p} A \to 0$ of A. G: abelian gp.

Then $(*_1)$ $0 \to \mathrm{Hom}(A,G) \to \mathrm{Hom}(F,G) \xrightarrow{i^*} \mathrm{Hom}(R,G)$ and

$(*_2)$ $R \otimes G \xrightarrow{i\otimes 1} F \otimes G \to A \otimes G \to 0$ are both exact.

The following two quantities measure the "failure of these two exact seq. to be short exact":

Def

$\boxed{\mathrm{Ext}(A,G)} := \mathrm{coker}(i^*) = \mathrm{Hom}(R,G)/\mathrm{im}(i^*)$.

$\boxed{\mathrm{Tor}(A,G)} := \ker(i\otimes 1)$.

Rmk:

(1) $\mathrm{Ext}(A,G) = 0 \Leftrightarrow (*_1)$ short exact.

$\mathrm{Tor}(A,G) = 0 \Leftrightarrow (*_2)$ short exact.

(2) $\mathrm{Ext}(A,G)$ and $\mathrm{Tor}(A,G)$ do not depend on choices of free resolution of A.

Prop (15.13.4)

m, n: positive integers. Then

Ext	$\mathbb{Z}$	$\mathbb{Z}_n$
$\mathbb{Z}$	0	0
$\mathbb{Z}_m$	$\mathbb{Z}_m$	$\mathbb{Z}_{(m,n)}$

and

Tor	$\mathbb{Z}$	$\mathbb{Z}_n$
$\mathbb{Z}$	0	0
$\mathbb{Z}_m$	0	$\mathbb{Z}_{(m,n)}$

.

With Ext and Tor, we can now state "Universal Coefficient Theorem".

Thm. (15.14) (Universal Coefficient Theorem).

X: topo. sp. G: abelian gp.

Then

(a) $H_q(X;G)$ has a splitting: $H_q(X;G) \cong H_q(X)\otimes G \oplus \mathrm{Tor}(H_{q-1}(X), G)$.

(b) $H^q(X;G)$ " " " : $H^q(X;G) \cong \mathrm{Hom}(H_q(X), G) \oplus \mathrm{Ext}(H_{q-1}(X), G)$. ∎

Taking $G = \mathbb{Z}$ in (b), we have

Cor (15.14.1)

X: topo. sp. $H_q(X), H_{q-1}(X)$: f.g. $\mathbb{Z}$-module.

Then $H^q(X) \cong F_q \oplus T_{q-1}$, where F_q: free part of $H_q(X)$ and T_{q-1}: torsion part of $H_{q-1}(X)$. ∎

Rmk:

Splittings in universal coefficient thm are not compatible w/ the induced hom.
Thus we often call them $\boxed{\text{unnatural splittings}}$.