MATH523 , Homework 3

Due October 12, 2017 in class

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1. [5 pts] Determine the coefficients of $a_3x^3 + x^2 + a_1x + a_0$ such that it has the smallest deviation from zero on the interval $[0, 2]$, i.e., find a_i ~~such that~~ minimizing

$$\cancel{\min} \|a_3x^3 + x^2 + a_1x + a_0\|_{L^\infty[0,2]},$$

where the minimum is taken over all real a_3, a_1, a_0 with $a_3 \neq 0$.

Hint: Find the best interpolation function for x^3 out of quadratic polynomials of the form $p_2(x) = -(x^2 + a_1x + a_0)/a_3$.

2. [5 pts] *Hermite Interpolation*

   (a) Use Hermite interpolation to find a polynomial of lowest degree satisfying

   $$p(-1) = p'(-1) = 0, \quad p(0) = 1, \quad p(1) = p'(1) = 0.$$

   Simplify your expression for p as much as possible.

   (b) Suppose the polynomial p from (a) is used to approximate the function $f(x) = [\cos(\pi x/2)]^2$ on $-1 \leq x \leq 1$.

   i. Express the error $e(x) = f(x) - p(x)$ (for some fixed x in $[-1, 1]$) in terms of an appropriate derivative of f.

   ii. Find an upper bound for $|e(x)|$ (still for a fixed $x \in [-1, 1]$).

   iii. Estimate $\max\limits_{x\in[-1,1]} |e(x)|$.

3. [5 pts] *Piecewise Linear Interpolation:* Let $0 \leq a < b$, and let $p(x)$ be the straight line which interpolates $f(x) = \sqrt{x}$ at a and b.

   (a) Show that

   $$\|f - p\|_{L^\infty[a,b]} = \frac{(\sqrt{b} - \sqrt{a})^2}{4(\sqrt{b} + \sqrt{a})}.$$

   (b) Prove that the distance of $\sqrt{x}$ on $[0, 1]$ to the space of piecewise linear polynomials over a *uniform* partition with $n + 1$ equally spaced nodes x_i goes to zero no faster than $n^{-1/2}/4$.

(c) Consider now a *graded* partition with nodes $x_i = (i/n)^4$. Show that the maximum error over $[x_{i-1}, x_i]$ is

$$\frac{1}{4n^2}\Big(2 - \frac{1}{i^2 + (i-1)^2}\Big).$$

Conclude that the global maximum error is $n^{-2}/2$. Compare with (b) and draw conclusions.

~~4.~~ [5 pts] Let $s_1(x) = 1 + c(x+1)^3$ on $-1 \le x \le 0$, where c is a real parameter. Determine $s_2(x)$ on $0 \le x \le 1$ so that

$$s(x) = \begin{cases} s_1(x), & \text{if} \quad -1 \le x \le 0 \\ s_2(x), & \text{if} \quad 0 \le x \le 1 \end{cases}$$

is a natural cubic spline with knots at $-1, 0, 1$. How must c be chosen if one wants $s(1) = -1$?

5. [10 pts] *Cubic spline with "not-a-knot" boundary condition*

(a) Derive the two extra equations for $\{p_i''(x_i) = M_i\}_{i=0,1,\dots,n}$ that result from the "not-a-knot" boundary condition imposed on the cubic spline interpolant $p \in \mathcal{C}^2[a,b]$ and $p|_{[x_i, x_i+1]} \equiv p_i \in \Pi_3$ with nodes

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

(b) Adjoin these two equations to the rest of the equations (obtained in class for $j = 1, \dots n-1$) and write the system in a matrix form. Is the tridiagonal system so obtained diagonally dominant?

(c) Write your own MATLAB function for cubic spline interpolation with "not-a-knot" boundary conditions (turn in the m-files by email) that takes on equi-spaced nodes, $x_i = a + (b-a)i/n$. Your program should take inputs of the domain boundaries $[a,b]$, the number of nodes $n+1$, and $f_i = f(x_i), i = 0, \dots, n$ and produce coefficients $c_{i,j}, j = 0, \dots, 3$, where

$$p_i(x) = c_{i,0} + c_{i,1}(x - x_i) + c_{i,2}(x - x_i)^2 + c_{i,3}(x - x_i)^3, \quad x_i \le x \le x_{i+1}.$$

Test your program on $f(x) = x^{5/2}$ on $[a, b] = [0, 1]$ and $n = 11$. Plot the interpolant polynomial on finer grid points `xx = linspace(0,1,100)` and the (x_i, f_i).

6. [10 pts] *Adaptive Piecewise Linear Interpolation:* This problem describes how to detect the *nonlinear* behavior of a given function $f(x)$ adaptively and locate the interpolation nodes accordingly. Suppose tolerances $\delta, h_0 > 0$ are given. We say that a subinterval $[xL, xR]$ is acceptable if

$$\left| f\Big(\frac{xL + xR}{2}\Big) - \frac{f(xL) + f(xR)}{2} \right| \leq \delta,$$

or if $xR - xL \leq h_0$. A partition $x_1 < x_2 < \cdots < x_n$ is acceptable if each subinterval is acceptable.

(a) Design and implement a *recursive* MATLAB function

```
function [x,y] = PWLAdapt (fname,xL,fL,xR,fR,delta,hzero)
```

which performs adaptive piecewise linear interpolation for a given function `fname` and inputs `fL = fname(xL), fR=fname(xR)`. The final partition `xL = x(1) < x(2) <` $\cdots$ `< x(n)=xR` with `y(i) = fname(x(i))` should be acceptable.

Hint: For a given interval `[xL,xR]` *define* `mid=(xL+xR)/2` *and evaluate whether* `[xL,xR]` *is acceptable or not. If the check is positive accept the interval; otherwise call again* `PWLAdapt` *twice, first for* `[xL,mid]` *and next for* `[mid,xR]`*, and update the vectors* `x` *and* `y`*. The update can be easily done by concatenating vectors. Recall that to evaluate fname within* `PWLAdapt` *you need* `feval`*.*

(b) Check the `PWLAdapt` on the built-in function `humps(x)` on `[0,3]`,

$$humps(x) = \frac{1}{(x - 0.3)^2 + 0.01} + \frac{1}{(x - 0.9)^2 + 0.04},$$

and $\sqrt{x}$ on $[0, 1]$. Plot the piecewise linear polynomial together with the interpolating points in the same graph.