weird

Midterm of Stat 517, 10/17/2016

Problem 1. Suppose $\mathcal{A}$ is semiring on a nonempty set Ω. Let $\mathcal{B}$ be the collection of all finite disjoint unions of $\mathcal{A}$-sets. Prove that $\mathcal{B}$ is a field. Furthermore, show that $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.

Problem 2. Let Ω be a set with infinitely many elements. A co-finite subset of Ω is a set whose complement is a finite set. Let $\mathcal{A}$ be the collection of all finite and co-finite subsets of Ω. Show that $\mathcal{A}$ is a field. Use the set $\Omega = \{1, 2, \ldots\}$ to give a counter example that $\mathcal{A}$ is not a σ-field.

Problem 3. Suppose Ω is a nonempty set and $\mu^* : 2^\Omega \to [0, \infty]$ is set function that such that (i) $\mu^*(\varnothing) = 0$, and (ii) μ^* is finitely subadditive; that is, if $A_1, \ldots, A_n$ are subsets of Ω, then $\mu^*(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n \mu^*(A_i)$. Let

$$\mathcal{M}(\mu^*) = \{A \subseteq \Omega : \mu^*(E) = \mu^*(EA) + \mu^*(EA^c) \text{ for all } E \subseteq \Omega\}.$$

Show that $\mathcal{M}(\mu^*)$ is a field. (Hint, for the third condition for a field, use the equivalent condition $A \in \mathcal{F}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$).

Problem 4. In a sequence independent coin tossing, where the probability of getting a head at each tossing is 1/2. Show that the probability of getting infinitely many heads is 1, and the probability of getting finitely many heads is 0.

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Prob 1.

1° We first prove $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.

$\because$ $\mathcal{B}$ is the collection of "finite" disjoint union of $\mathcal{A}$ sets and $\sigma(\mathcal{A})$ is closed under countable union

$\therefore$ $\mathcal{B} \subseteq \sigma(\mathcal{A})$.

$\Rightarrow \sigma(\mathcal{B}) \subseteq \sigma(\sigma(\mathcal{A})) = \sigma(\mathcal{A})$.

On the other hand, since $\mathcal{A} \subseteq \mathcal{B}$, $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{B})$.

Thus, $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$. #

2° I think $\mathcal{B}$ is NOT necessarily a field on Ω.

But, it might be a field on $\bigcup_{B \in \mathcal{B}} B =: \Omega'$.

(i) $\emptyset \in \mathcal{B}$ is obvious since $\emptyset \in \mathcal{A}$.

(ii) Given $B \in \mathcal{B}$.

Then $B = \bigcup_{k=1}^{m} A_k$, where $A_1, \cdots, A_m$ are disjoint in $\mathcal{A}$.

$\Rightarrow \Omega' \setminus B = \bigcap_{k=1}^{m} (\Omega' \setminus A_k) = \bigcap_{k=1}^{m} \left(\bigcup_{B \in \mathcal{B}} B \setminus A_k \right) = \bigcap_{k=1}^{m} \left(\bigcup_{B \in \mathcal{B}} \left(\bigcup_{j=1}^{m_k} C_j \right) \right) =$

Prob 2

1° claim: $\mathcal{A}$ is a field.

(1) $\Omega^c = \emptyset$. $\Rightarrow \Omega$ is cofinite. $\Rightarrow \Omega \in \mathcal{A}$.

(2) Given $A \in \mathcal{A}$.

case 1 A is finite

Then $(A^c)^c = A$ is finite. $\Rightarrow A^c$ is cofinite. $\Rightarrow A^c \in \mathcal{A}$.

case 2 A is cofinite.

Then A^c is finite. $\Rightarrow A^c \in \mathcal{A}$.

(3) Given $A, B \in \mathcal{A}$. We shall prove $A \cup B$ is either finite or cofinite.

Suppose $A \cup B$ is NOT finite.

Then either $|A| = \infty$ or $|B| = \infty$, say $|A| = \infty$.

Then A is cofinite. i.e. A^c is finite.

$\Rightarrow (A \cup B)^c = A^c \cap B^c$ is finite.

$\Rightarrow A \cup B$ is cofinite. $\Rightarrow A \cup B \in \mathcal{A}$.

-1 expand on other cases.

By (1), (2), (3), $\mathcal{A}$ is a field. #

2° claim: $\mathcal{A}$ is NOT a σ-field when $\Omega = \{1, 2, 3, \cdots\}$.

Consider $A_k = \{2k+1\}$, $k = 0, 1, 2, \cdots$.

Then $|A_k| = 1$. $\Rightarrow A_k \in \mathcal{A}$, $\forall k$.

However, $\bigcup_{k=1}^{\infty} A_k = \{1, 3, 5, 7, 9, \cdots\}$ is neither finite nor cofinite.

$\Rightarrow \bigcup_{k=1}^{\infty} A_k \notin \mathcal{A}$. $\Rightarrow \mathcal{A}$ is NOT a σ-field. #

Prob 3

(i) $\forall E \subseteq \Omega$, $E\Omega = E$, $E\Omega^c = \phi$.

$\Rightarrow \mu^*(E) = \mu^*(E\Omega) + 0 = \mu^*(E\Omega) + \mu^*(\phi) = \mu^*(E\Omega) + \mu^*(E\Omega^c)$.

$\Rightarrow \Omega \in \mathcal{M}(\mu^*)$.

(ii) Given $A \in \mathcal{M}(\mu^*)$.

Then $\mu^*(E) = \mu^*(EA) + \mu^*(EA^c) = \mu^*(EA^c) + \mu^*(EA) = \mu^*(E(A^c)) + \mu^*(E(A^c)^c)$, $\forall E \subseteq \Omega$.

$\Rightarrow A^c \in \mathcal{M}(\mu^*)$.

(iii) For notational convenience, denote $\mu^*(E)$ by $|E|$ in the following, $\forall E \subseteq \Omega$.

Given $A, B \in \mathcal{M}(\mu^*)$.

Then $|E| = |EA| + |EA^c| = |EB| + |EB^c|$, $\forall E \subseteq \Omega$.

Moreover, $|EA| = |EAB| + |EAB^c| = (\text{I}) + (\text{II})$

$|EA^c| = |EA^cB| + |EA^cB^c| = (\text{III}) + (\text{IV})$

$(\text{I}) + (\text{III}) = |EAB| + |EA^cB| = |(EB)A| + |(EB)A^c| = |EB|$

Similarly, $(\text{II}) + (\text{IV}) = |EB^c|$.

Thus,

-3

~~You don't have all~~

-3 OK.

Prob 4

Let A_n be the event of getting a head at the nth toss, and
B " " " " " infinitely many heads.
C " " " " " finitely " heads.
Then $B = \limsup A_n$ & $C = \liminf A_n^c$.

Note that $P(A_n) = 1/2$ and $\{A_n\}$ is indep. by condition.

Thus, by 2nd Borel-Cantelli lemma,

since $\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} 1/2 = \infty$, $P(B) = 1$. #

Also, note that $C = B^c$.

Thus, $P(C) = 1 - P(B) = 1 - 1 = 0$. #

Final of Stat 517, 12/12/2016

Problem 1. Let $(\Omega, \mathcal{F})$, $(\Omega_1, \mathcal{F}_1), \dots, (\Omega_k, \mathcal{F}_k)$ be measurable spaces. Let $f_i : \Omega \to \Omega_i$, $i = 1, \dots, k$ be measurable functions with respect to $\mathcal{F}/\mathcal{F}_i$. Let $\mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ be the σ-field on $\Omega_1 \times \cdots \times \Omega_k$ generated by the class of sets $\{A_1 \times \cdots \times A_k : A_1 \in \mathcal{F}_1, \dots, A_k \in \mathcal{F}_k\}$. Let $f : \Omega \to \Omega_1 \times \cdots \times \Omega_k$ be defined by $f(\omega) = (f_1(\omega), \dots, f_k(\omega))$. Show that f is measurable with respect to $\mathcal{F}/(\mathcal{F}_1 \times \cdots \times \mathcal{F}_k)$.

Problem 2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : \Omega \to \overline{\mathbb{R}}$ be a nonnegative and measurable function with respect to $\mathcal{F}/\overline{\mathcal{R}}$.

1. Use the definition of $\int f d\mu$ to show that $f = 0$ a.e. μ if and only if $\int f d\mu = 0$.
2. Suppose

$$\int^* f d\mu = \inf_{\{A_i\} \in \mathcal{P}} \sum_{i=1}^{k} \left[\sup_{\omega \in A_i} f(\omega) \right] \mu(A_i)$$

   where $\mathcal{P}$ is the collection of all finite $\mathcal{F}$ partitions of Ω. Show that $\int^* f d\mu = 0$ if and only if $f = 0$ a.e. μ.

Problem 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}_{n=1}^{\infty}$ and f are functions from Ω to $\overline{\mathbb{R}}$ measurable with respect to $\mathcal{F}/\overline{\mathcal{R}}$.

1. Show that, if $f_n \geq g$ and g is integrable with respect to μ, then $\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu$.
2. Show that, if $f_n \leq g$ and g is integrable with respect to μ, then $\int \limsup_n f_n d\mu \geq \limsup_n \int f_n d\mu$.

Problem 4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\delta \geq 0$ be a function from Ω to $\overline{\mathbb{R}}$ measurable with respect to $\mathcal{F}/\overline{\mathcal{R}}$.

1. Let ν be the set function $\nu(A) = \int_A \delta d\mu$. Use MCT to show that ν is a measure on (Ω, F).
2. Use the three step argument to prove that $\int f d\nu = \int f \delta d\mu$ for any nonnegative function $f : \Omega \to \overline{R}$ that is measurable with respect to $\mathcal{F}/\overline{\mathcal{R}}$.

Problem 5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where μ is a finite measure. Suppose f is a function from Ω to $\overline{\mathbb{R}}$ measurable with respect to $\mathcal{F}/\overline{\mathcal{R}}$. Show that f is integral with respect to μ if and only if

$$\lim_{\alpha \to \infty} \int_{|f| \geq \alpha} |f| d\mu = 0.$$

$\mu(\Omega) < \infty.$