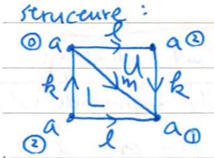
Math 597C . H.W.4. Min-Chun Wu . 9/9

Prob 1 2/2

Use the def. of simplicial cohomology to compute $H^*(KL;\mathbb{Z})$ and $H^*(KL;\mathbb{Z}_2)$, where KL = Klein Bottle.

[Sol].

Klein bottle can be given the following Δ-complex structure :



To compute how δ works, we should know how ∂ works first. We use general coeff. G first.

$$0 \leftarrow C_0(KL) \leftarrow C_1(KL) \leftarrow C_2(KL) \leftarrow 0.$$

$C_0 = \langle a \rangle$ $\quad \partial a = 0$

$C_1 = \langle k, l, m \rangle$ $\quad \partial k = \partial l = \partial m = a - a = 0.$

$C_2 = \langle U, L \rangle.$ $\quad \partial U = -k - l + m.$

$\partial L = -l + k + m = k - l + m.$

Now, we turn to cohomology.

$C^j(KL) = \mathrm{Hom}(C_j(KL), G)$ and $\varphi \in C^j(KL)$ are uniquely determined by how they map the j-simplices (generators) of KL.

Thus, (for the case $G = \langle 1 \rangle$),

$$C^0 = \langle \varphi_a : a \mapsto 1 \rangle$$

$$\delta \downarrow$$

$$C^1 = \left\langle \varphi_k : \begin{matrix} k \mapsto 1 \\ l \mapsto 0 \\ m \mapsto 0 \end{matrix},\ \varphi_l : \begin{matrix} k \mapsto 0 \\ l \mapsto 1 \\ m \mapsto 0 \end{matrix},\ \varphi_m : \begin{matrix} k \mapsto 0 \\ l \mapsto 0 \\ m \mapsto 1 \end{matrix} \right\rangle$$

$$\delta \downarrow$$

$$C^2 = \left\langle \varphi_U : \begin{matrix} U \mapsto 1 \\ L \mapsto 0 \end{matrix},\ \varphi_L : \begin{matrix} U \mapsto 0 \\ L \mapsto 1 \end{matrix} \right\rangle.$$

$$\delta\varphi_a : \begin{matrix} k \mapsto \varphi_a(\partial k) = 0 \\ l \mapsto \varphi_a(\partial l) = 0 \\ m \mapsto \varphi_a(\partial m) = 0 \end{matrix} \Rightarrow \delta\varphi_a = 0.$$

$$\delta\varphi_k : \begin{matrix} U \mapsto \varphi_k(\partial U) = -1 \\ L \mapsto \varphi_k(\partial L) = 1. \end{matrix}$$

$$\delta\varphi_l : \begin{matrix} U \mapsto -1 \\ L \mapsto -1 \end{matrix},\quad \delta\varphi_m : \begin{matrix} U \mapsto 1 \\ L \mapsto 1. \end{matrix}$$

$\Rightarrow \delta\varphi_k = -\varphi_U + \varphi_L,\ \delta\varphi_l = -\varphi_U - \varphi_L,\ \delta\varphi_m = \varphi_U + \varphi_L.$

$\delta\varphi_U = \delta\varphi_L = 0.$

$B^0 = 0,\ Z^0 = C^0 = \langle \varphi_a \rangle$

$B^1 = 0,\ Z^1 = \left\{ n_1\varphi_k + n_2\varphi_l + n_3\varphi_m \,\middle|\, \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$B^2 = \langle -\varphi_U + \varphi_L, -\varphi_U - \varphi_L, \varphi_U + \varphi_L \rangle,\ Z^2 = C^2 = \langle \varphi_U, \varphi_L \rangle$

$= \langle \varphi_U + \varphi_L, 2\varphi_L \rangle$

$H^0 = Z^0/B^0 = \langle \varphi_a \rangle / 0 = G.$

$H^1 = Z^1/B^1 = Z^1 = \left\{ n_1\varphi_k + n_2\varphi_l + n_3\varphi_m \,\middle|\, \begin{matrix} n_1 - n_2 + n_3 = 0 \\ 2n_2 - 2n_3 = 0 \end{matrix} \right\}$

$H^2 = Z^2/B^2 = C^2/B^2 = \langle \varphi_U + \varphi_L, \varphi_L \rangle / \langle \varphi_U + \varphi_L, 2\varphi_L \rangle$

$= \langle \varphi_L \rangle / \langle 2\varphi_L \rangle = G/2G.$

case1 $G = \mathbb{Z}$.

$H^0 = \mathbb{Z}.$

$H^2 = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$

$H^1 = Z^1 = \{ n_1\varphi_k + n_2\varphi_l + n_3\varphi_m \mid n_1 = 0, n_2 = n_3 \}$

$= \langle \varphi_l + \varphi_m \rangle = \mathbb{Z}.$

Thus, $H^k(KL;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}, & k = 1 \\ \mathbb{Z}_2, & k = 2 \\ 0, & \text{o.w.} \end{cases}$ #

case2 $G = \mathbb{Z}_2$.

$H^0 = \mathbb{Z}_2.$

$H^2 = \mathbb{Z}_2 / 2\mathbb{Z}_2 = \mathbb{Z}_2,$

$H^1 = \{ n_1\varphi_k + n_2\varphi_l + n_3\varphi_m \mid n_1 + n_3 = n_2 \}$

$= \{ n_1(\varphi_k + \varphi_l) + n_3(\varphi_l + \varphi_m) \} = \langle \varphi_k + \varphi_l, \varphi_l + \varphi_m \rangle.$

$= \mathbb{Z}_2 \oplus \mathbb{Z}_2.$

Thus, $H^k(KL;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & k = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & k = 1 \\ \mathbb{Z}_2, & k = 2 \\ 0, & \text{o.w.} \end{cases}$ #

Prob 2 2/2

Formulate and prove Mayer-Vietoris long exact seq. thm for singular cohomology.

[Sol]

Formulation :

Let $U, V \subseteq X$ s.t. $X = U \cup V$.

Then $\exists$ long exact seq. in cohomology :

$$\cdots \leftarrow H^n(U \cap V) \leftarrow H^n(U) \oplus H^n(V) \leftarrow H^n(X)$$
$$\leftarrow H^{n-1}(U \cap V) \leftarrow H^{n-1}(U) \oplus H^{n-1}(V) \leftarrow \cdots$$

⟨Proof⟩

Consider the following short exact seq. :

$$0 \to S_n(U \cap V) \to S_n(U) \oplus S_n(V) \to S_n(X) \to 0.$$

Take $\mathrm{Hom}(-, G)$. Then we have

$$0 \leftarrow S^n(U \cap V) \leftarrow S^n(U) \oplus S^n(V) \leftarrow S^n(X) \leftarrow 0.$$

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Note this is also short exact, and thus induces the desired long exact seq. in cohomology. #

Prob3 2/2

Prove that if $f: S^n \to S^n$ has degree d, then $f^*: H^n(S^n) \to H^n(S^n)$ is multiplication by d.

[Sol] Denote the domain S^n by X and Y for the other.

We first examine how f^* works.

$f_\#: C_n(S^n_X) \to C_n(S^n_Y)$
$\sigma \longmapsto f\circ\sigma$.

$f_*: H_n(S^n_X) \to H_n(S^n_Y)$
$[\alpha] \longmapsto [f_\#(\alpha)]$.

$f^\#: C^n(S^n_Y) \to C^n(S^n_X)$
$\varphi \longmapsto \varphi\circ f_\#$.

$f^*: H^n(S^n_Y) \to H^n(S^n_X)$
$[\varphi] \longmapsto [\varphi\circ f_\#]$.

We want to prove $[\varphi\circ f_\#] = d[\varphi]$.

Our condition is $[f_\#(\alpha)] = d[\alpha]$, $\forall \alpha \in Z_n(S^n)$.

Let $[\varphi]$ be a generator of $H^n(S^n)$.

Then, since $f^*([\varphi]) \in H^n(S^n)$, $f^*([\varphi]) = r[\varphi]$, for some $r \in \mathbb{Z}$.

Suffice to prove $r = d$.

Let $[\alpha]$ be a generator of $H_n(S^n)$ (so that $\varphi(\alpha) \neq 0$.)

$f^*[\varphi] = [\varphi\circ f_\#]$. $\Rightarrow \varphi\circ f_\# - r\varphi = \delta\psi$, some ψ.

$[f_\#(\alpha)] = d[\alpha]$. $\Rightarrow f_\#(\alpha) - d\alpha = \partial\beta$, some β.

Thus,

$0 = \varphi\circ f_\#(\alpha) - r\varphi(\alpha) - \delta\psi(\alpha)$

$= \varphi(d\alpha + \partial\beta) - r\varphi(\alpha) - \delta\psi(\alpha)$

$= (d-r)\varphi(\alpha) + \underbrace{\delta\varphi(\beta)}_{0} - \psi(\underbrace{\partial\alpha}_{0}) = (d-r)\varphi(\alpha)$.

Thus, $d - r = 0/\varphi(\alpha) = 0$. $\Rightarrow d = r$.

$\because [\varphi]$: generator of $H^n(S^n)$

$\therefore f^*$: multiplication by d. #

Prob4 2/2 there is a much simpler solution.

(a) $\{U_i\}_{i=1}^\infty$: cpt convex in $\mathbb{R}^n$ s.t. every $(n+1)$-tuple has a common pt.

Prove that $\bigcap_{i=1}^\infty U_i \neq \emptyset$.

(b) Find a counterexample if cpt in (a) is dropped.

[Sol]

(a)

Recall the following $\boxed{\text{Cor of Alexander duality}}$:

$\boxed{\text{If } K \text{: cpt orientable and locally contractible w/ } K \neq S^n \text{ and } H^\ell(K, G) \neq 0, \text{ for some } \ell \geq n, \text{ then } K \text{ can not be embedded in } S^n.}$

We first prove the finite case. i.e. $\bigcap_{i=1}^m U_i \neq \emptyset$, $\forall m$.

Instead of proving $\bigcap_{i=1}^m U_i \neq \emptyset$, $\forall m$, we prove a stronger version: every m-tuple of $\{U_i\}_{i=1}^\infty$ has nonempty intersection, $\forall m$.

Induction on m.

For $m \leq n+1$, by condition, nothing to do.

Assume $m \geq n+2$ and every k-tuple, $k \leq n+1$, has nonempty intersection.

Suppose $\exists$ some m-tuple with empty intersection, say $U_{i_1}, \dots, U_{i_m}$, $\bigcap_{j=1}^m U_{i_j} = \emptyset$.

$\because$ every $(m-1)$-tuple of $U_{i_1}, \dots, U_{i_m}$ has nonempty intersection

$\therefore$ Nerve $(\{U_{i_j}\}_{j=1}^m) = S^{m-1}$.

By nerve lemma, $\bigcup_{j=1}^m U_{i_j} \sim S^{m-1}$!

$\Rightarrow H^{m-1}(\bigcup_{j=1}^m U_j; \mathbb{R}) = H^{m-1}(S^{m-1}; \mathbb{R}) = \mathbb{R} \neq 0$.

$\because m-1 \geq n+1 \geq n$ $\therefore$ By Cor, $\bigcup_{j=1}^m U_{i_j}$ can not be embedded in $S^n = \mathbb{R}^n \cup \{\infty\}$.

However, $U_{i_j} \subseteq \mathbb{R}^n$, $\forall j \Rightarrow \bigcup_{j=1}^m U_{i_j} \subseteq \mathbb{R}^n \subseteq S^n$.

$\Rightarrow \bigcup_{j=1}^m U_{i_j}$: embedded in S^n. $\ast$

Thus, every m-tuple of $\{U_i\}_{i=1}^\infty$ has nonempty intersection, $\forall m$.

$\because$ They are all cpt $\therefore$ Infinite case follows from the finite intersection property. # i U_i

(b). Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. $U_i := \{re_1 \mid i \leq r < \infty\}$.

1/1 Then U_i: convex, $\forall i$, $\bigcap_{i=1}^m U_i \neq \emptyset$, $\forall m$, and $\bigcap_{i=1}^\infty U_i = \emptyset$. #

Math 597C H.W. #5. Min-Chun Wu. 12/13

Problem 1 3/3

Use existence of partition of unity to prove an n-dim. mfd M is orienteable $\iff$ $\exists$ nowhere vanishing $\omega \in \Omega^n(M)$.

[Sol]

($\Rightarrow$)

Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ be a chart of M s.t.

$$\det\left(d(\varphi_\alpha \circ \varphi_\beta^{-1})_p\right) > 0, \quad \forall\, p \in \varphi_\beta(U_\alpha \cap U_\beta),\ \alpha \neq \beta \text{ in } I. \quad (*)$$

(This chart exists by orienteability of M).

Denote $\varphi_\alpha = (x_\alpha^1, \cdots, x_\alpha^n)$.

Let $\{\rho_\alpha\}_{\alpha\in I}$ be a partition of unity subordinate to $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$.

Note $\rho_\alpha \cdot dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ is globally defined w/ support contained in U_α.

Define $\omega = \sum_\alpha \rho_\alpha\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$.

ω is well-defined since $\{\operatorname{supp} \rho_\alpha\}_{\alpha\in I}$ is locally finite.

We claim that ω is nowhere vanishing.

For $q \in M$, assume $\rho_{\alpha_1}, \cdots, \rho_{\alpha_m}$ are those w/ positive values at q.

Choose $(U_{\alpha_1}, \varphi_{\alpha_1})$ to represent ω_q.

Then
$$\omega_q = \rho_{\alpha_1}(q)\, dx_{\alpha_1}^1 \wedge \cdots \wedge dx_{\alpha_1}^n + \sum_{k=2}^m \rho_{\alpha_k}(q) \left.\frac{\partial(x_{\alpha_k}^1, \cdots, x_{\alpha_k}^n)}{\partial(x_{\alpha_1}^1, \cdots, x_{\alpha_1}^n)}\right|_q dx_{\alpha_1}^1 \wedge \cdots \wedge dx_{\alpha_1}^n$$

By $(*)$, $\dfrac{\partial(x_{\alpha_k}^1, \cdots, x_{\alpha_k}^n)}{\partial(x_{\alpha_1}^1, \cdots, x_{\alpha_1}^n)} > 0$, $\forall k$.

$$\Rightarrow \rho_{\alpha_1}(q) + \sum_{k=2}^m \rho_{\alpha_k}(q) \left.\frac{\partial(x_{\alpha_k}^1, \cdots, x_{\alpha_k}^n)}{\partial(x_{\alpha_1}^1, \cdots, x_{\alpha_1}^n)}\right|_q > 0.$$

$\Rightarrow \omega_q \neq 0$.

Thus, ω is nowhere vanishing. #

($\Leftarrow$)

Given a chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ of M. (we assume U_α: connected)

We shall modify this chart so that it becomes an oriented chart.

Let $\omega \in \Omega^n(M)$ be a nowhere vanishing n-form.

Denote $\varphi_\alpha = (x_\alpha^1, \cdots, x_\alpha^n)$.

Then, on U_α, $\omega = f_\alpha\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$.

$\because$ ω is nowhere vanishing

$\therefore$ f_α " " " (since U_α: connected)

$\Rightarrow$ f_α is always positive or always negative.

If f_α is always positive, leave $(U_\alpha, \varphi_\alpha)$ unchanged. i.e. Define $(U_\alpha', \varphi_\alpha') = (U_\alpha, \varphi_\alpha)$.

If f_α is always negative, replace $\varphi_\alpha = (x_\alpha^1, \cdots, x_\alpha^n)$ by $\varphi_\alpha' = (-x_\alpha^1, x_\alpha^2, \cdots, x_\alpha^n)$.

Then $\omega = (-f_\alpha)(d(-x_\alpha^1)) \wedge dx_\alpha^2 \wedge \cdots \wedge dx_\alpha^n$ w/ $(-f_\alpha)$ always positive.

Now, we claim $\{(U_\alpha', \varphi_\alpha')\}_{\alpha\in I}$ is an oriented chart. Denote $(U_\alpha', \varphi_\alpha') = (U_\alpha', (y_\alpha^1, \cdots, y_\alpha^n))$.

Given $\alpha \neq \beta$ w/ $U_\alpha \cap U_\beta \neq \emptyset$.

Then $\omega = f_\alpha\, dy_\alpha^1 \wedge \cdots \wedge dy_\alpha^n = f_\beta\, dy_\beta^1 \wedge \cdots \wedge dy_\beta^n$, where f_α and f_β are always positive.

$\because$ $f_\beta = f_\alpha \cdot \partial(y_\beta^1, \cdots, y_\beta^n)/\partial(y_\alpha^1, \cdots, y_\alpha^n)$

$\therefore$ $\partial(y_\beta^1, \cdots, y_\beta^n)/\partial(y_\alpha^1, \cdots, y_\alpha^n)$ is always positive.

$\Rightarrow$ $\{(U_\alpha', \varphi_\alpha')\}_{\alpha\in I}$: oriented chart for M.

Thus, M: orienteable. #

Problem 3

(a) Compute $H_c^*(\{pt\})$ and $H_c^*(\mathbb{R})$. 2/2

[Sol].

$\{pt\}$ is a 0-dim. mfd. Thus,

$0 \to \Omega_c^0(\{pt\}) \to 0$.

Elements in $\Omega_c^0(\{pt\})$ are the same as $\Omega^0(\{pt\})$ since $\{pt\}$ itself is cpt.

$\Rightarrow \Omega_c^0(\{pt\}) = \{f : \{pt\} \to \mathbb{R}\} \cong \mathbb{R}$ via $f \mapsto f(pt)$.

Thus, $H_c^k(\{pt\}) = \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ 0, & \text{o.w.} \end{cases}$ #

$\mathbb{R}$ is a 1-dim. mfd.

$0 \to \Omega_c^0(\mathbb{R}) \xrightarrow{d} \Omega_c^1(\mathbb{R}) \to 0$.

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$\Omega_c^0(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R}, C^\infty \mid \exists R_f$ s.t. $f(x)=0$, $\forall |x| > R_f \}$.

$\ker(d) = \{ f \in \Omega_c^0(\mathbb{R}) \mid f' \equiv 0 \}$.

$\because f' \equiv 0 \therefore f = \text{const}$ on $\mathbb{R}$.

However, $f(x)=0, \forall |x| \geq R_f. \Rightarrow f \equiv 0$.

Thus, $\ker(d) = \{0\}. \Rightarrow H_c^0(\mathbb{R}) = \ker(d) = \{0\}$. #

$\Omega_c^1(\mathbb{R}) = \{ f\,dt \mid f \in \Omega_c^0(\mathbb{R}) \}$.

Define $\Phi: \Omega_c^1(\mathbb{R}) \to \mathbb{R}$ by

$(f\,dt) \mapsto \int_{-\infty}^{\infty} f(t)\,dt$.

claim: $\ker(\Phi) = \operatorname{im}(d)$ and Φ: surj.

By considering bump functions, Φ is clearly surj.

As for $\ker(\Phi) = \operatorname{im}(d)$,

$\subseteq$

$(f\,dt) \in \ker(\Phi) \Rightarrow \int_{-\infty}^{\infty} f(t)\,dt = 0$.

Let $R>0$ s.t. $f(x)=0, \forall |x| \geq R$.

Define $g(t) = \int_{-R}^{t} f(s)\,ds$.

$\because \int_{-\infty}^{-R} f(s)\,ds = 0$ and $\int_{-\infty}^{\infty} f(s)\,ds = 0$

$\therefore g(t) = 0, \forall |t| \geq R. \Rightarrow g(t) \in \Omega_c^0(\mathbb{R})$.

Note $d(g(t)) = g'(t)\,dt = f(t)\,dt$.

$\Rightarrow (f(t)\,dt) \in \operatorname{im}(d). \Rightarrow \ker(\Phi) \subseteq \operatorname{im}(d)$.

$\supseteq$

For $d\,g(t) = g'(t)\,dt \in \operatorname{im}(d)$, let $R>0$ s.t. $g(t)=0, \forall |t| \geq R$.

Then $\int_{-\infty}^{\infty} g'(t)\,dt = \int_{-2R}^{2R} g'(t)\,dt = g(t)\Big|_{-2R}^{2R} = 0$.

i.e. $\Phi(dg(t)) = 0. \Rightarrow dg(t) \in \ker(\Phi)$.

$\Rightarrow \operatorname{im}(d) \subseteq \ker(\Phi)$. # of claim.

By claim and 1st iso. thm,

$\Omega_c^1(\mathbb{R}) / \operatorname{im}(d) \cong \mathbb{R}$.

i.e. $H_c^1(\mathbb{R}) \cong \mathbb{R}$. #

In summary,

$$H_c^k(\mathbb{R}) = \begin{cases} 0, & \text{if } k=0 \\ \mathbb{R}, & \text{if } k=1 \\ 0, & \text{o.w.} \end{cases}$$ #

1/1

(b) Does $H_c^*(-)$ respect the homotopy invariance?

[Sol].

Ans: No.

For example, in (a), $\{pt\}$ and $\mathbb{R}$ have the same homotopy type while their respective $H_c^*(-)$ are not isomorphic. #

(c) Explain why pullback is not well-defined on $\Omega_c^*(-)$.

[Sol] 1/1

Consider $M = \mathbb{R}$, $N = \{0\}$.

$M \xrightarrow{\phi} N$, $x \mapsto 0, \forall x$.

$N \xrightarrow{f} \mathbb{R}$, $f(0) = 0$.

Then $f \in \Omega_c^0(N)$ but $\operatorname{supp}(\phi^* f) = \mathbb{R}$ is not cpt. i.e. $\phi^* f \notin \Omega_c^0(M)$.

Thus, ϕ^* is not well-defined. #

(d) $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$. Use integration to define the pairing

$$H^k(U) \otimes H_c^{n-k}(U) \to \mathbb{R}.$$

Here, you can use Stokes theorem w/o proof.

[Sol] 2/2

claim: For $[\omega] \in H_c^n(U)$, $\int_U [\omega] := \int_U \omega$ is well-defined.

<Pf of claim>.

For $\omega' = \omega + d\tau$, some $\tau \in \Omega_c^{n-1}(U)$,

$\int_U \omega' = \int_U \omega + \int_U d\tau \overset{\text{Stokes}}{=} \int_U \omega + \int_{\partial U} \tau$

$\because \operatorname{supp} \tau \subseteq U \therefore \tau \equiv 0$ on ∂U.

$\Rightarrow \int_{\partial U} \tau = 0. \Rightarrow \int_U \omega' = \int_U \omega$. # of claim.

Define $H^k(U) \otimes H_c^{n-k}(U) \to \mathbb{R}$ by

$[\alpha] \otimes [\beta] \mapsto \int_U [\alpha \wedge \beta]$.

$\because [\alpha \wedge \beta] = [\alpha] \wedge [\beta] \therefore$ This map is well-defined.

(We implicitly use the fact $\int_U [\alpha \wedge \beta]$ is indep. of the choice of rep. of $\alpha \wedge \beta$).

It's clear that this map is linear in each component and hence a pairing. #

Problem 2 3/4

Compute $H^*_{dR}(S^1)$ and $H^*_{dR}(T^2)$ by def.

[Sol]

(1) $H^*_{dR}(S^1)$.

$0 \to \Omega^0(S^1) \xrightarrow{d} \Omega^1(S^1) \to 0$.

$f \in \Omega^0(S^1)$ may be regarded as periodic function on $\mathbb{R}$ of period 1.

$df = f'(t)\,dt$.

$\ker(d) = \{ f \mid f'(t) \equiv 0 \text{ and } f: \text{periodic} \}$
$= \{ \text{const. functions on } \mathbb{R} \} \cong \mathbb{R}$.

Thus, $H^0(S^1) = \ker(d)/0 \cong \mathbb{R}$.

$\Omega^1(S^1) = \{ f\,dt \mid f: \text{periodic on } \mathbb{R} \text{ of period } 1 \}$.

claim: $\Phi : \Omega^1(S^1) \to \mathbb{R}$, $f\,dt \mapsto \int_0^1 f(t)\,dt$ is surj. w/ $\ker(\Phi) = B^1(S^1)$.

(Pf of claim)

$\because$ $dt \mapsto \int_0^1 dt = 1$ $\therefore$ Φ is surj.

($\ker(\Phi) \subseteq B^1$).

Give $f\,dt \in \ker(\Phi)$. Then $\int_0^1 f(t)\,dt = 0$.

Define $g(t) = \int_0^t f(s)\,ds$.

Then, since $\int_0^1 f(t)\,dt = 0$ and f: periodic of period 1, $g(t)$ is periodic of period 1.

Moreover, by Fundamental theorem of calculus, $g'(t) = f(t)$. $\Rightarrow dg = g'\,dt = f\,dt$.

Thus, $f\,dt \in B^1$ and $\ker(\Phi) \subseteq B^1$.

($\ker(\Phi) \supseteq B^1$)

$B^1 = \{ f'\,dt \mid f: \text{periodic of period } 1 \}$.

$\int_0^1 f'\,dt = f(t)\big|_0^1 = f(1) - f(0) \overset{f:\text{ period } 1}{=} 0$.

$\Rightarrow B^1 \subseteq \ker(\Phi)$. # of claim.

Hence, $H^1(S^1) \cong \Omega^1(S^1)/B^1 = \Omega^1(S^1)/\ker(\Phi) \simeq \mathbb{R}$.

In summary, $H^k(S^1) = \begin{cases} \mathbb{R}, & k = 0,1 \\ 0, & \text{o.w.} \end{cases}$ #

? what is the algebra structure?

(2) $H^*_{dR}(T^2)$.

$0 \to \Omega^0(T^2) \xrightarrow{d_0} \Omega^1(T^2) \xrightarrow{d_1} \Omega^2(T^2) \to 0$.

Similar to (1), functions on T^2 may be regarded as "doubly periodic" functions on $\mathbb{R}^2$, w/ periods 1 and 1. i.e. $f(x+1,y) = f(x,y) = f(x,y+1)$, $\forall (x,y) \in \mathbb{R}^2$.

$Z^0 = \ker(d_0) = \{ f \mid df = 0 \} = \{ f \mid f_x = 0 = f_y \}$,

$B^1 = \operatorname{im}(d_0) = \{ f_x\,dx + f_y\,dy \}$

$Z^1 = \ker(d_1) = \{ f\,dx + g\,dy \mid g_x - f_y = 0 \}$

$B^2 = \operatorname{im}(d_1) = \{ (g_x - f_y)\,dx\,dy \}$.

$Z^2 = \Omega^2(T^2) = \{ f\,dx\,dy \}$,

where f, g are always doubly periodic w/ period 1.

For Z^0, since $f_x \equiv 0 \equiv f_y$, f is const.

$\Rightarrow H^0 = Z^0/0 = \{ \text{const. on } \mathbb{R}^2 \} \cong \mathbb{R}$. #

For Z^1/B^1, consider

$\Phi_1 : Z^1 \to \mathbb{R} \oplus \mathbb{R}$,

$(f\,dx + g\,dy) \mapsto \left(\int_0^1 f(t,0)\,dt, \int_0^1 g(0,t)\,dt \right)$.

Φ_1 is clearly surj. since $dx \mapsto (1,0)$ and $dy \mapsto (0,1)$.

claim: $B^1 = \ker(\Phi_1)$ so that $H^1 \cong \mathbb{R} \oplus \mathbb{R}$.

($\subseteq$)

$\int_0^1 f_x(t,0)\,dt = \int_0^1 \frac{d}{dt} f(t,0)\,dt = f(1,0) - f(0,0) \overset{f:\text{doubly periodic}}{=} 0$

$\int_0^1 f_y(0,t)\,dt = \int_0^1 \frac{d}{dt}(f(0,t))\,dt = f(0,1) - f(0,0) = 0$.

($\supseteq$) ※

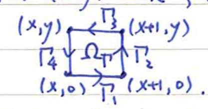
Given $f\,dx + g\,dy \in \ker(\Phi_1)$.

Define $F(x,y) = \int_0^1 x f(tx,ty) + y f(tx,ty)\,dt$.

$\because$ $g_x = f_y$ $\therefore$ $F_x = f$, $F_y = g$.

It suffices to prove F: doubly periodic w/ periods 1 and 1, so that $dF = F_x\,dx + F_y\,dy = f\,dx + g\,dy$.

For $(x,y) \in \mathbb{R}^2$, consider



$\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

$\because$ $g_x = f_y$ $\therefore$ By Green's thm, $\int_\Gamma f\,dx + g\,dy = \int_{\Omega_\Gamma} g_x - f_y\,dx\,dy = 0$.

i.e. $\left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} \right)(f\,dx + g\,dy) = 0$.

i.e. $\int_x^{x+1} f(t,0)\,dt + \int_0^y g(x+1,t)\,dt + \int_x^{x+1} f(t,y)(-dt) + \int_0^y g(x,t)(-dt) = 0$.

cancel out (since g: doubly periodic).

No.

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Hence, $\int_x^{x+1} f(t,0)\,dt = \int_x^{x+1} f(t,y)\,dt$. --- ⓪

$\because f\,dx + g\,dy \in \ker(\Phi_1)$ $\therefore \int_0^1 f(t,0)\,dt = 0$ --- ①

$\because f$: periodic $\therefore \int_x^{x+1} f(t,0)\,dt = \int_0^1 f(t,0)\,dt$ -- ②

$\int_x^{x+1} f(t,y)\,dt = \int_x^{x+1} \frac{d}{dt} F(t,y)\,dt = F(x+1,y) - F(x,y)$ --- ③

Combining them together,

$F(x+1,y) - F(x,y) \underset{③}{=} \int_x^{x+1} f(t,y)\,dt \underset{⓪}{=} \int_x^{x+1} f(t,0)\,dt$

$\underset{②}{=} \int_0^1 f(t,0)\,dt \underset{①}{=} 0$.

$\Rightarrow F(x+1,y) = F(x,y),\ \forall (x,y)$.

Similarly, $F(x,y+1) = F(x,y)$, $\forall (x,y)$

Thus, F : doubly periodic of period 1 and 1, and the result $H^1(T^2) \cong \mathbb{R} \oplus \mathbb{R}$ follows. #

For $Z^2/B^2 = \Omega^2/B^2$, consider

$\Phi_2 : Z^2 = \Omega^2 \longrightarrow \mathbb{R}$ defined by

$$f\,dx\wedge dy \longmapsto \int_0^1\int_0^1 f(x,y)\,dx\,dy.$$

Claim : $\ker \Phi_2 = B^2$ so that

$$H^2 = Z^2/B^2 = Z^2/\ker(\Phi_2) \cong \mathbb{R}.$$ #

what about the algebra structure?