3. Stochastic Integrals.

3.1 Background.

We wanna define $\int_0^T f(s,\omega)\, dW_s(\omega)$, as the limit of certain partition process of $[0,T]$.

E.g.

$$I := \int_0^T W_s(\omega)\, dW_s(\omega).$$

Two possible approaches are

1° $$I_1 = \sum_{n=0}^{N-1} W_{t_n}(W_{t_{n+1}} - W_{t_n}), \text{ and}$$

2° $$I_2 = \sum_{n=0}^{N-1} W_{t_{n+1}}(W_{t_{n+1}} - W_{t_n}),$$

Note: These are r.v.

where $0 = t_0 < t_1 < \cdots < t_N = T$ is a partition of $[0,T]$.

Notice that

indep. of W_{t_n} and $W_{t_{n+1}} - W_{t_n}$

$$\mathbb{E}[I_1] = \sum_{n=0}^{N-1} \underbrace{\mathbb{E}[W_{t_n}]}_{=0}\, \underbrace{\mathbb{E}[W_{t_{n+1}} - W_{t_n}]}_{=0}$$

$$= 0$$

$$\mathbb{E}[I_2] = \sum_{n=0}^{N-1} \mathbb{E}[W_{t_{n+1}}(W_{t_{n+1}} - W_{t_n})].$$

$$= \sum_{n=0}^{N-1} \mathbb{E}[(W_{t_{n+1}} - W_{t_n} + W_{t_n})(W_{t_{n+1}} - W_{t_n})]$$

$$= \sum_{n=0}^{N-1} \underbrace{\mathbb{E}[(W_{t_{n+1}} - W_{t_n})^2]}_{= t_{n+1} - t_n} + \underbrace{\mathbb{E}[W_{t_n}(W_{t_{n+1}} - W_{t_n})]}_{=0}$$

$$= \sum_{n=0}^{N-1} t_{n+1} - t_n = t_N - t_0 = T.$$

Thus, using I_1 and I_2, resp., will give us different limits.

I_1 gives rise to the Itô integral while

I_2 " " " " Stratonovich integral.

3.2 The Itô integral.

Def 3.1

W : a Wiener process.

Define $\boxed{\mathcal{A}_t} = \sigma(\{W_s \mid s \le t\})$.

$\mathcal{A}_t$ can be thought of as the history of W up to time t.

Thus $\{\mathcal{A}_t\}$ is a $\boxed{\text{filtration}}$ of σ-algebras.

i.e. $\mathcal{A}_s \subseteq \mathcal{A}_t$, $\forall s \le t$.

Rmk: Given $s \le t$.

$\because$ $W_t - W_s$ is indep. of $\mathcal{A}_s$ (by def).

$\therefore$ $\mathbb{E}[W_t - W_s \mid \mathcal{A}_s] = \mathbb{E}[W_t - W_s] = 0$.

$\Rightarrow$ $\mathbb{E}[W_t \mid \mathcal{A}_s] = \mathbb{E}[W_s \mid \mathcal{A}_s] = W_s$.

Def 3.2 Given a filtration $\{\mathcal{M}_t\}_{t \ge 0}$. A stochastic pr. $g : [0,\infty) \times \Omega \to \mathbb{R}$ is called $\boxed{\mathcal{M}_t\text{-adapted}}$ if, $\forall t \ge 0$, the map $\omega \mapsto g(t,\omega)$ is $\mathcal{M}_t$-measurable.

Def 3.3

For $0 < T < \infty$, define $\boxed{\mathcal{L}^2_T}$ to be the class of functions $f : [0,T] \times \Omega \to \mathbb{R}$ s.t.

① f is $\mathcal{L} \times \mathcal{A}$-measurable. ($\mathcal{A}$: σ-alg. on Ω).

② f : $\mathcal{A}_t$-adapted. ($\mathcal{A}_t$: see Def 3.1).

③ $\int_0^T \mathbb{E}[(f(t,\cdot))^2]\, dt < \infty$.

Rmk:

① ③ above implies $\mathbb{E}[(f(t,\cdot))^2] < \infty$ for a.e. t.

almost every

② Identifying functions differing on measure zero sets, $\mathcal{L}^2_T$ is complete w.r.t. the norm

$$\boxed{\|f\|_{2,T}} := \left(\int_0^T \mathbb{E}[(f(t,\cdot))^2]\, dt\right)^{1/2}.$$

SDE C3 P2

Def

$f:[0,T]\times\Omega\to\mathbb{R}$ is called a $\boxed{\text{step function}}$ if $\exists$ partition $0=t_0<t_1<\cdots<t_n=T$ and $f_0,f_1,\cdots,f_{n-1}:\Omega\to\mathbb{R}$ s.t.

$$f(t,\omega)=f_j(\omega),\ \forall\, t\in[t_j,t_{j+1}).$$

The subset of all step functions in $\mathcal{L}_T^2$ is denoted by $\boxed{\mathcal{S}_T^2}$.

Def Given a step function $f\in\mathcal{S}_T^2$ as above. Define its $\boxed{\text{Itô integral}}$ by (of step functions)

$$\boxed{I[f]}(\omega)=\int_0^T f(s,\omega)\,dW_s(\omega)$$

$$:=\sum_{j=0}^{n-1} f_j(\omega)\left(W_{t_{j+1}}(\omega)-W_{t_j}(\omega)\right)$$

a r.v.

Rmk: $\because f\in\mathcal{S}_T^2\subseteq\mathcal{L}_T^2$ $\therefore f_j$ is $\mathcal{A}_{t_j}$-measurable.

$\because \mathcal{A}_{t_j}\subseteq\mathcal{A}_T,\ \forall j$ $\therefore I[f]$ is $\mathcal{A}_T$-measurable.

$\boxed{\text{Lem 3.4}}$ (3.2.2 in [KP92]).

Given $f,g\in\mathcal{S}_T^2$ and $\alpha,\beta\in\mathbb{R}$. Then

① $I[f]$ is $\mathcal{A}_T$-measurable.

② $\mathbb{E}[I[f]]=0$.

③ $\mathbb{E}[(I[f])^2]=\int_0^T\mathbb{E}[(f(t,\cdot))^2]\,dt$. (By ②, it's actually $\mathrm{var}(I[f])$.) $\boxed{\text{Itô isometry}}$

④ $I[\alpha f+\beta g]=\alpha I[f]+\beta I[g]$, w.p. 1.

$\boxed{\text{Lem 3.5}}$ (3.2.1 in [KP92]).

$\mathcal{S}_T^2$ is dense in $\mathcal{L}_T^2$, where the norm used in $\mathcal{L}_T^2$ is

$$\|f\|_{2,T}:=\left(\int_0^T\mathbb{E}[(f(t,\cdot))^2]\,dt\right)^{1/2}.$$

Rmk: Given $f\in\mathcal{L}_T^2$ and a seq. $\{h^{(n)}\}_{n=1}^{\infty}\subseteq\mathcal{S}_T^2$, s.t. $\|h^{(n)}-f\|_{2,T}\to 0$ as $n\to\infty$.

i.e. $\int_0^T \mathbb{E}[|h^{(n)}(t,\cdot) - f(t,\cdot)|^2]\,dt \xrightarrow[(n\to\infty)]{} 0.$

Consider the seq. $\{I[h^{(n)}]\}_{n=1}^{\infty} \subseteq L^2(\Omega, \mathcal{A}, \mathbb{P})$.

Then $\int_\Omega |I[h^{(n)}](\omega) - I[h^{(m)}](\omega)|^2 \,d\omega$

$= \mathbb{E}[|I[h^{(n)} - h^{(m)}]|^2]$ (linearity of I)

$= \int_0^T \mathbb{E}[(h^{(n)}(t,\cdot) - h^{(m)}(t,\cdot))^2]\,dt$ (Itô isometry)

$\leq 2\int_0^T \mathbb{E}[(h^{(n)}(t,\cdot) - f(t,\cdot))^2]\,dt$
$+ 2\int_0^T \mathbb{E}[(h^{(m)}(t,\cdot) - f(t,\cdot))^2]\,dt$ $\left((a+b)^2 \leq 2a^2 + 2b^2 \right)$

$\to 0$ as $m, n \to \infty$.

Thus $\{I[h^{(n)}]\}_{n=1}^{\infty}$ is Cauchy in $L^2(\Omega, \mathcal{A}, \mathbb{P})$

$\because L^2(\Omega, \mathcal{A}, \mathbb{P})$ is complete

$\therefore \exists!$ (up to a measure zero set) L^2 fun.,
denoted $I[f]$, s.t. $I[h^{(n)}] \xrightarrow{L^2} I[f]$.

<u>Def</u> The $\boxed{\text{Itô integral}}$ of $f \in \mathcal{L}_T^2$ is defined by

$$\boxed{I[f]}(\omega) = \int_0^T f(t,\omega)\,dW_t(\omega)$$

$$\boxed{L^2 \text{ limit}} := \lim_{n\to\infty} \int_0^T h^{(n)}(t,\omega)\,dW_t(\omega)$$

where $\{h^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{S}_T^2$ w/ $\|h^{(n)} - f\|_{2,T} \to 0$ as $n \to \infty$.

Rmk:

① By limitting procedure, results in <u>L3.4</u> hold for Itô integrals in $\mathcal{L}_T^2$ as well.

② <u>Ex 3.7</u> demonstrates how to compute $\int_0^T W_s(\omega)\,dW_s(\omega)$ $\left(= \frac{1}{2}(W_T(\omega))^2 - \frac{1}{2}T\right)$ from definition. It's quite messy and omitted here.

③ The result in ② above shows ordinary calculus rules do NOT hold for Itô integral. The rule to be followed is the Itô's formula. (Sec 3.4).

3.3 Martingales.

Given $f \in \mathcal{L}_T^2$ and $t_0 \in [0,T)$.

Define a stochastic pr. $\{Z_t\}_{t_0 \le t \le T}$ by

$$\boxed{Z_t(\omega)} := \int_{t_0}^{t} f(s,\omega)\, dW_s(\omega) \quad (3.44).$$

Q: What "good properties" does Z_t have?

Def

M: stochastic pr. adapted to $\{\mathcal{M}_t\}$.

M is called a **martingale** w.r.t. $\{\mathcal{M}_t\}$ if

① $\mathbb{E}[|M_t|] < \infty$, $\forall t$, and

② $\mathbb{E}[M_t | \mathcal{M}_s] = M_s$, $\forall s \le t$.

Ex:

The Wiener pr. W_t is a martingale as shown in the Rmk after Def 3.1.

i.e. $\mathbb{E}[W_t | \mathcal{A}_s] = W_s$, $\forall s \le t$.

Prop 3.9 (3.2.5 in [KP92]).

Let $f \in \mathcal{L}_T^2$.

Then the stoch. pr. $Z = \{Z_t\}_{t \in [t_0, T]}$ is a martingale w.r.t. $\{\mathcal{A}_t\}$.

Being a martingale has at least the following two advantages:

Prop 3.10 (Martingale inequality). (Sub)

Y: a nonnegative (sub) martingale.

i.e. ① $\mathbb{E}[|M_t|] < \infty$ ② $\mathbb{E}[M_t | \mathcal{M}_s] \ge M_s$ a.s. $\forall s \le t$

Then, $\forall \gamma > 0$ and $p \ge 1$,

$$\mathbb{P}\left(\left(\sup_{0 \le t \le T} Y_t \right) \ge \gamma \right) \le \frac{1}{\gamma^p} \mathbb{E}[|Y_T|^p].$$

Thus, for Z defined by (3.44), we have

$$\mathbb{P}\left(\sup_{t_0 \le s \le t} |Z_s - \underbrace{Z_{t_0}}_{=0}| \ge \gamma \right) \le \frac{1}{\gamma^2} \mathbb{E}[|Z_t - \underbrace{Z_{t_0}}_{=0}|^2] = \frac{1}{\gamma^2} \int_{t_0}^{t} \mathbb{E}[(f(s,\cdot))^2]\, ds.$$

$\int_{t_0}^{t_0} f(s,\omega)\, dW_s(\omega) = 0$

(may be proved by def.)

Prop 3.11 (Doob inequality).

Y: a nonnegative (sub)martingale.

Then, $\forall\, r > 1$,

$$\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^r\right] \le \left(\frac{r}{r-1}\right)^r \mathbb{E}\left[|Y_T|^r\right].$$

Thus, for Z defined by (3.44), we have

$$\mathbb{E}\left[\sup_{t_0\le s\le t} |Z_s|^2\right] \le \left(\frac{2}{2-1}\right)^2 \cdot \mathbb{E}\left[|Z_t|^2\right]$$

$$= 4\int_{t_0}^{t} \mathbb{E}\left[(f(s,\cdot))^2\right] ds.$$

Rmk:

In Prop 3.10 and Prop 3.11, we use the fact: If X is a martingale and $\phi: \mathbb{R}\to\mathbb{R}$ is a convex fun. w/ $\mathbb{E}[\phi(X_t)] < \infty$, then $(\phi(X_t))_{t\ge 0}$ is a submartingale.

This fact may be proved by conditional Jensen's ineq.

Taking $\phi(x) = |x|$, $|Z|$ is a submartingale.

Thm 3.12 (3.2.6 in [KP92]).

Z: stoch. pr. defined by (3.44).

Then $\exists$ cont. version of Z.

3.4 Itô's formula.

Def 3.13

An **Itô process** is a sto. pr. $(X_t)_{t\ge 0}$ s.t.

$$X_t(\omega) = X_s(\omega) + \int_s^t e_u(\omega)\,du + \int_s^t f_u(\omega)\,dW_u(\omega) \tag{3.58}$$

w.p.1, $\forall\, 0\le s\le t\le T$, where

① e and f are $\mathcal{L}\times\mathcal{A}$-mea., $\mathcal{A}_t$-adapted,

② $\int_s^t |e_u(\omega)|\,du < \infty$, w.p.1, and

③ $\int_s^t |f_u(\omega)|^2\,du < \infty$, w.p.1.

(3.58) may be abbreviated as

$dX_t = e_t\,dt + f_t\,dW_t$, called a stochastic differential.