Throughout this note,

M : cpt connected d-dim. submfd of $\mathbb{R}^n$

(boundary and boundary corners are permitted).

Def

For $x, y \in M$, define

$\boxed{d_M(x,y)} := \inf\{L(\gamma) \mid \gamma : \text{admissible from } x \text{ to } y.\}$

Def

$\mathcal{X} = \{x_i\} \subseteq M$, a finite set.

G : a graph on $\mathcal{X}$.

Define, $\forall\, x, y \in \mathcal{X}$,

$\boxed{d_G(x,y)} := \min\limits_P (\|x_0 - x_1\| + \cdots + \|x_{p-1} - x_p\|)$ and

$\boxed{d_S(x,y)} := \min\limits_P (d_M(x_0,x_1) + \cdots + d_M(x_{p-1},x_p))$,

where $P = (x_0, x_1, \cdots, x_p)$ is a path on G

w/ $x_0 = x$ and $x_p = y$.

Goal: We want to prove $d_G \approx d_M$.

Approach: $d_G \approx d_S$ and $d_S \approx d_M$.

Prop 1

$d_M(x,y) \le d_S(x,y)$ and

$d_G(x,y) \le d_S(x,y)$, $\forall\, x, y \in \mathcal{X}$.

<Pf>

The first is obvious.

The second comes from the fact that straight lines are shortest in $\mathbb{R}^n$. #

Thm 2 ($d_M \approx d_S$).

$\varepsilon, \delta > 0$ w/ $4\delta < \varepsilon$. Suppose

① G contains all edges xy w/ $d_M(x,y) \le \varepsilon$

② $\forall\, m \in M$, $\exists\, x_i \in \mathcal{X}$ s.t. $d_M(x_i, m) \le \delta$.

Then, $\forall\, x, y \in \mathcal{X}$,

$$d_M(x,y) \le d_S(x,y) \le (1 + 4\delta/\varepsilon)\, d_M(x,y).$$

Rmk:

Condition ② is called the $\boxed{\delta\text{-sampling condition}}$.

<Pf>

"$\le$" of LHS follows from Prop 1.

Now consider "$\le$" on the RHS.

It suffices to prove the following claim.

claim: $\forall$ piecewise C^∞ curve γ from x to y, w/ $\ell = L(\gamma)$, $\exists$ path $P = (x_0, \cdots, x_p)$ on G from x to y s.t. $d_M(x_0,x_1) + \cdots + d_M(x_{p-1},x_p) \le (1 + 4\delta/\varepsilon)\ell$

(then the result follows from taking inf. among all such γ).

<Pf of claim>.

case 1 $\ell \le \varepsilon - 2\delta$. (Then $d_M(x,y) \le \ell \le \varepsilon$).

By condition ①, $xy \in G$.

$\Rightarrow d_S(x,y) \le d_M(x,y) \le \ell \le (1 + 4\delta/\varepsilon)\,\ell$.

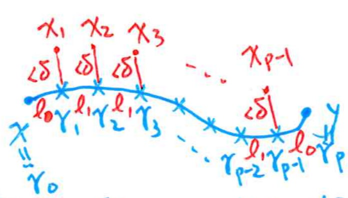
(x,y) is a path in G.

case 2 $\ell > \varepsilon - 2\delta$.

By division algorithm,

$\ell = (\varepsilon - 2\delta)\, q + r$, w/ $q \in \mathbb{N} \cup \{0\}$ and $r \in [\varepsilon - 2\delta, 2(\varepsilon - 2\delta)]$

$= \ell_0 + (\underbrace{\ell_1 + \cdots + \ell_1}_{q = p-2}) + \ell_0$, w/ $\ell_1 = \varepsilon - 2\delta$ and $\ell_0 = r/2 \in [\frac{\varepsilon - 2\delta}{2}, \varepsilon - 2\delta]$.



Divide γ according to this length subdivision as in the picture. (Denote the pts by $\gamma_1, \cdots, \gamma_{p-1}$).

By condition ②, for $1 \le i \le p-1$, $\exists\, x_i \in \mathcal{X}$ s.t. $d_M(x_i, \gamma_i) \le \delta$.

Then $d_M(x_i, x_{i+1}) \le \delta + \ell_1 + \delta = \varepsilon = \ell_1 \cdot \frac{\varepsilon}{\varepsilon - 2\delta}$.

$d_M(x, x_1) \le \ell_0 + \delta$ and $d_M(x_{p-1}, y) \le \delta + \ell_0$.

Notice that $\frac{1}{2} \le \frac{\ell_0}{\varepsilon - 2\delta} \Rightarrow \delta \le \ell_0 (2\delta / \varepsilon - 2\delta)$

$\Rightarrow l_0+\delta \le l_0\left(\frac{\varepsilon}{\varepsilon-2\delta}\right)$.

Thus $d_M(x,x_1) \le l_0 \cdot \left(\frac{\varepsilon}{\varepsilon-2\delta}\right)$ and

$d_M(x_{p-1},y) \le l_0 \cdot \left(\frac{\varepsilon}{\varepsilon-2\delta}\right)$.

Hence,

$$d_M(x,x_1)+d_M(x_1,x_2)+\cdots+d_M(x_{p-1},y)$$

$$\le \left(\frac{\varepsilon}{\varepsilon-2\delta}\right)(l_0+l_1+\cdots+l_1+l_0) \le \frac{\varepsilon}{\varepsilon-2\delta} \cdot l.$$

Notice that $\frac{1}{1-t} < 1+2t,\ \forall\, t \in (0, 1/2)$.

$$\frac{\varepsilon}{\varepsilon-2\delta} = \frac{1}{1-(2\delta/\varepsilon)}.$$

$2\delta/\varepsilon \in (0,1/2)$ (due to $4\delta < \varepsilon$).

Thus $\frac{\varepsilon}{\varepsilon-2\delta}\cdot l < (1+2\cdot(2\delta/\varepsilon))\cdot l = \left(1+\frac{4\delta}{\varepsilon}\right) l$

Therefore, the claim follows. #

Def

The $\boxed{\text{minimum radius of curvature}}$ $\boxed{r_0 = r_0(M)}$ is defined by $\frac{1}{r_0} = \max_{\gamma,t}\{\|\ddot\gamma(t)\|\}$, where

γ varies over all unit speed geod. and t is in the domain of γ.

Rmk: (Intuition)

Geod. in M "curl around less tightly" than circles of radii $r_0(M)$.

Def

The $\boxed{\text{minimum branch separation}}$ $\boxed{s_0 = s_0(M)}$ is

$$s_0 := \sup\{ s \mid d_M(x,y) \le \pi r_0\ \ \forall\, x,y \in M \text{ w/ } \|x-y\| < s\}$$

Rmk:

Existence and positivity of r_0 and s_0 are guaranteed by compactness of M.

(For s_0, we may need Lebesgue number lemma).

Now, we use r_0 and s_0 to give a technical lemma.

Lemma 3

γ: a geod. in M from x to y.

$l := L(\gamma) \le \pi r_0$.

Then $2r_0 \sin(l/2r_0) \le \|x-y\| \le l$.

Rmk:

① By calculus, $\sin(t) \ge t - t^3/6,\ \forall\, t \ge 0$.

Thus, we can weaken the ineq. as:

$$\left(1-\frac{l^2}{24r_0^2}\right) l \le \|x-y\| \le l.$$

Thus, for small l, $l \approx \|x-y\|$.

② For $l \le \pi r_0$, $l/2r_0 \le \frac{\pi}{2}$.

Treating $t := l/2r_0$, $t \in (0,\frac{\pi}{2})$ and $\sin t \le \frac{2}{\pi} t$.

$$\Rightarrow 2r_0 \sin(l/2r_0) = 2\cdot\frac{l}{2t}\sin t \ge 2\cdot\frac{l}{2t}\cdot\frac{2}{\pi}t = (2/\pi) l.$$

Thus, we may weaken the ineq. as

$$(2/\pi) l \le \|x-y\| \le l.$$

<Proof of L3 is postponed in the appendix>.

Cor 4 (Lemma for $d_S \approx d_G$)

$\lambda > 0$.

$x_i, x_{i+1} \in M$.

Assume (i) $\|x_i - x_{i+1}\| < s_0$,

(ii) $\|x_i - x_{i+1}\| \le (2/\pi) r_0 \sqrt{24\lambda}$, and

(iii) $\exists$ geod. on M of length $d_M(x_i,x_{i+1})$ connecting x_i and x_{i+1}.

Then

$$(1-\lambda)\, d_M(x_i,x_{i+1}) \le \|x_i - x_{i+1}\| \le d_M(x_i,x_{i+1}).$$

<Pf>

By (i) and def of s_0, $d_M(x_i,x_{i+1}) \le \pi r_0$.

By (iii), we can apply L3.

By ② in the Rmk above, $(2/\pi) l \le \|x_i - x_{i+1}\|$.

By (ii), we thus have $l \le r_0\sqrt{24\lambda}$.

$\Rightarrow 1-\lambda \le 1 - l^2/24r_0^2$. By Rmk ① above, we are done. #

Def
M is called geodesically convex if every $x, y \in M$ can be connected by a geod. of length $d_M(x,y)$.

Main Thm A.

M: cpt submfd of $\mathbb{R}^n$. (possibly w/ boundary)

$\mathcal{X} = \{x_i\}$: finite set of data pts in M.

G : a graph on $\mathcal{X}$.

$0 < \lambda_1, \lambda_2 < 1$.

$\varepsilon_{min}, \varepsilon_{max} > 0$, $\delta > 0$.

Assume

(i) G contains all edges xy w/ $\|x-y\| \leq \varepsilon_{min}$.

(ii) All edges of G have length $\|x-y\| \leq \varepsilon_{max}$.

(iii) $\mathcal{X}$ satisfies δ-sampling condition.

(iv) M is geod. convex.

and

(v) $\varepsilon_{max} < s_0$, where s_0 : minimum branch sep. of M.

(vi) $\varepsilon_{max} \leq (2/\pi) r_0 \sqrt{24\lambda_1}$,

where r_0 : minimum radius of curvature.

(vii) $\delta \leq \lambda_2 \varepsilon_{min}/4$. (hence $4\delta < \varepsilon_{min}$).

Then

$$(1-\lambda_1) d_M(x,y) \leq d_G(x,y) \leq (1+\lambda_2) d_M(x,y),$$

$\forall x, y \in \mathcal{X}$.

<Pf>

Condition (i), (iii), (vii) are for applying Thm 2.

Condition (ii), (v), (vi), (iv) are for applying Cor 4.

Thus, $d_M \leq d_S \leq (1 + 4\delta/\varepsilon_{min}) d_M \leq (1+\lambda_2) d_M$

and $(1-\lambda_1) d_S \leq d_G \leq d_S$.

$$\text{Hence, } \underbrace{(1-\lambda_1) d_M \leq (1-\lambda_1) d_S}_{\text{Thm 2}} \leq d_G \leq \underbrace{d_S \leq (1+\lambda_2) d_M}_{\text{Thm 2}}.$$

Cor 4

#