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The Spectral Sequence of a Double Complex.

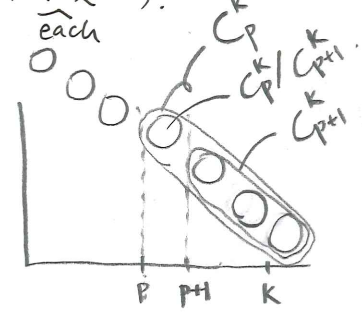
$D := \delta + (-1)^p d$

$D'' := (-1)^p d$. (on each $K^{p,q}$).

Given a double complex. $K = \bigoplus_{p,q\geq 0} K^{p,q}$. $d\uparrow$, $\overrightarrow{\delta}$

Then $K = K_0 = \bigoplus_{k=0}^{\infty} C_0^k$, $C_0^k := \bigoplus_{r+s=k} K^{r,s}$.

$K_p := \bigoplus_{\substack{r\geq p\\ s\geq 0}} K^{r,s}$ Then $K_p = \bigoplus_{k=0}^{\infty} C_p^k$, $C_p^k = \bigoplus_{\substack{r+s=k\\ r\geq p}} K^{r,s}$.



Now, $K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ is a filtered complex w/ differential D.
We follow the construction of the spectral seq. of a filtered complex: (as usual, $K_p := K$, $\forall p<0$).

$A := \bigoplus_{p\in\mathbb{Z}} K_p$. $B := \bigoplus_{p\in\mathbb{Z}} K_p/K_{p+1}$.

The short exact seq. $0 \to K_{p+1} \to K_p \to K_p/K_{p+1} \to 0$ induces long exact seq.

$$\begin{array}{ccccc} \to \cdots & & & & \\ \to H^{k+1}(K_{p+1}) & \to & H^{k+1}(K_p) & \to & H^{k+1}(K_p/K_{p+1}) \\ \to H^{k}(K_{p+1}) & \to & H^{k}(K_p) & \to & H^{k}(K_p/K_{p+1}) \\ & & & & \cdots \\ H(K_{p+1}) & & H(K_p) & & H(K_p/K_{p+1}) \end{array} \qquad (*)$$

$A_1 := H(A) = \bigoplus_{p\in\mathbb{Z}} H(K_p)$, $B_1 = E_1 := H(B) = \bigoplus_{p\in\mathbb{Z}} H(K_p/K_{p+1})$.

Putting $\bigoplus_{p\in\mathbb{Z}}$ on $(*)$, we have

$$\begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

, an exact couple.

By the construction of derived couple, $A_r \subseteq H(A)$, $\forall r$. Hence, we would denote elements in A_r simply by $[a]$.
However, $B_{r+1} = H_{d_r}(B_r)$. Thus, we would denote elements in B_r carefully by $[b]_r$.

Now, let's compute k_1 (so that d_1).

k_1 is the connecting hom. coming from $(*)$.

i.e.

$$\begin{tikzcd} & \uparrow & \uparrow & \uparrow & \\ 0 \arrow[r] & C_{p+1}^{k+1} \arrow[r, "(3)"'] \arrow[u] & C_p^{k+1} \arrow[r] \arrow[u] & C_p^{k+1}/C_{p+1}^{k+1} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & C_{p+1}^{k} \arrow[r] \arrow[u, "D"] & C_p^{k} \arrow[r, "(1)"'] \arrow[u, "D", "(2)"'] & C_p^{k}/C_{p+1}^{k} \arrow[r] \arrow[u] & 0 \\ & \uparrow & \uparrow & \uparrow & \end{tikzcd} \qquad (*)'$$

Note that $C_p^k/C_{p+1}^k = K^{p,k-p}$ and thus D on B is actually d.

Hence, $B_1 = H_D(B) = H_d(K)$.

Let $[b]_1 \in H^k(K_p/K_{p+1})$ ($\subseteq \bigoplus_{p\in\mathbb{Z}} (\bigoplus_{k=0}^{\infty} H^k(K_p/K_{p+1})) = \bigoplus_{p\in\mathbb{Z}} H(K_p/K_{p+1}) = H(\bigoplus_{p\in\mathbb{Z}} K_p/K_{p+1}) = H(B) = B_1$).

Choose $b \in K^{p,k-p} \subseteq C_p^k$ to "represent" $[b]_1$. (1).

$\because [b]_1 \in B_1$ $\therefore db = 0. \Rightarrow Db = (\delta + (-1)^p d)b = \delta b$. (2).

Choose δb (now in $K^{p+1,k-p} \subseteq C_{p+1}^{k+1}$) (actually, it's unique) mapping to δb. (3).

Thus, $k_1[b]_1 = [\delta b]$, where $b \in K^{p,k-p} \subseteq C_p^k$ represents $[b]_1$ in step (1).

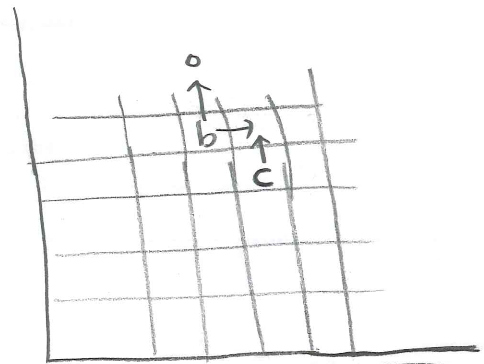
Now, $j_1[a] = [a]_1$ since it comes from $C^{k+1}_{p+1} \to C^{k+1}_{p+1}/C^{k+1}_{p+2}$.

$\Rightarrow d_1[b]_1 = [\delta b]_1$.

Therefore, $B_2 = H_{d_1}(B_1) = H_\delta(B_1) = H_\delta H_d(K)$.

$\boxed{\text{Next, we compute } d_2.}$

Before proceeding, let's see how elements look like in B_2.



element in B_2

Let $b \in K^{p,k-p}$ represent $[b]_2 \in B_2$.

Then $\boxed{db = 0}$ (since $[b]_1 \in B_1$).

$d_1[b]_1 = 0$ in B_1. i.e. $\delta b = dc$, for some c.

We can choose c s.t. $\boxed{\delta b = -D''c.}$

$d_2[b]_2 = j_2 k_2 [b]_2 = j_2 k_1 [b]_1$

$k_1[b]_1$ can be obtained by going through (1), (2), (3) in $(*)'$

Instead of b, we may choose $b+c$ in step (1).

Then $D(b+c) = (\delta + (-1)^p d)(b+c) = \delta b + \delta c + D''c = \delta c$ (step (2)).

δc is in C^{k+1}_{p+1}. (Step (3)). Thus, $k_2[b]_2 = [\delta c]$ in A_2.

Note also, $[\delta c] = i[\delta c]$, for $[\delta c]$ in A_1.

Therefore, $d_2[b]_2 = j_2[\delta c] = j_2 i[\delta c] = [j_1[\delta c]]_2 = [\delta c]_2$.

In conclusion, for an element $[b]_2 \in B_2$, represented by $b \in K^{p,k-p}$,

find c s.t. $\delta b = -D''c$. Then $\underline{d_2[b]_2 = [\delta c]_2}$.

$\boxed{\text{Next, we compute } d_3.}$

Let $b \in K^{p,k-p}$ represent $[b]_3 \in B_3 = H_{d_2} H_{d_1} H_d(K)$.

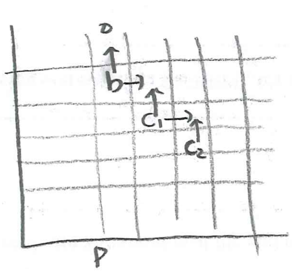
$\because [b]_1 \in B_1 = H_d(K)$ $\therefore \boxed{db = 0}$

$\because d_1[b]_1 = 0$ in B_1 $\therefore \boxed{\delta b = D''c,}$ for some c.

$\because d_2[b]_2 = 0$ in B_2 $\therefore [\delta c]_2 = 0$ in B_2. i.e. $[\delta c]_1 = d_1[c']_1 = [\delta c']_1$, some c'. Note $\boxed{dc' = 0}$ since $[c']_1 \in B_1$.

$\Rightarrow [\delta(c-c')]_1 = 0$ in B_1. $\Rightarrow \delta(c-c') = dc''$, some c''. We may choose c'' s.t. $\boxed{\delta(c-c') = -D''c''.}$

Define $c_1 = c - c'$ and $c_2 = c''$.



element in B_3

Then
$$db = 0$$
$$\delta b = -D''c_1$$
$$\delta c_1 = -D''c_2.$$

$d_3[b]_3 = j_3 k_3 [b]_3 = j_3 k_1 [b]_1$.

Use $b + c_1 + c_2$ in step (1) in $(*)'$. Then $D(b + c_1 + c_2) = (D'' + \delta)(b + c_1 + c_2) = \overbrace{\delta b + D''c_1}^{\text{cancel}} + \overbrace{\delta c_1 + D''c_2}^{\text{cancel}} + \delta c_2 = \delta c_2$. (step (2)).

δc_2 is already in C^{k+1}_{p+1}. $\Rightarrow k_1[b]_1 = [\delta c_2]$.

Similarly, $j_3[\delta c_2] = j_3 ii[\delta c_2] = [\delta c_2]_3$. Therefore, $\underline{d_3[b]_3 = [\delta c_2]_3}$.

The same arguments applies in each r.

In general, if $b \in K^{p,k-p}$ represent $[b]_r \in B_r$, then $\exists\, c_1, \cdots, c_{r-1}$ s.t.

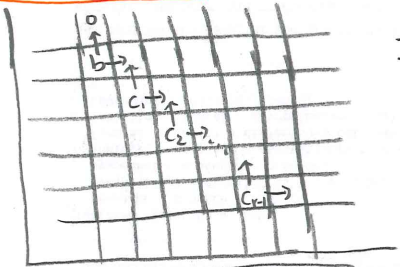
$$\boxed{\begin{aligned} db &= 0 \\ \delta b &= -D''c_1 \\ \delta c_1 &= -D''c_2 \\ &\vdots \\ \delta c_{r-2} &= -D''c_{r-1}. \end{aligned}}$$

Using $b+c_1+\cdots+c_{r-1}$ in step (1), $D(b+c_1+\cdots+c_{r-1}) = \underbrace{\delta b + D''c_1}_{\text{cancel}} + \underbrace{\delta c_1 + D''c_2}_{\text{cancel}} + \cdots + \overbrace{\delta c_{r-2} + D''c_{r-1}}^{\text{cancel}} + \delta c_{r-1}$

Choose δc_{r-1} in step (3). $= \delta c_{r-1}$. (step (2))

$\Rightarrow K_r[b]_r = K_1[b]_1 = [\delta c_{r-1}]$.

$\Rightarrow \underline{d_r[b]_r} = j_r K_r [b]_r = j_r[\delta c_{r-1}] = j_r i^{r-1}[\delta c_{r-1}] = \underline{[\delta c_{r-1}]_r}$.



element in B_r

In other words, $\underbrace{\text{element}}_{[b]_r}$ in B_r can be expressed as a zig-zag shown left and

$d_r[b]_r = [\delta c_{r-1}]_r$, the image of the tail of this zig-zag by δ.

This decompose B_r as $\bigoplus_{k=0}^{\infty} B_r^k$, where $B_r^k = \bigoplus_{p+q=k} B_r^{p,q}$ and

$d_r : B_r^{p,q} \longrightarrow B_r^{p+r,\, q-r+1}$ for each p, q.

Note that for fixed k (dimension), $C_p^k = 0$, $\forall\, p \geq k+1$.

i.e. the filtration of K, when fixing the dimension, vanishes after a finite position.

$\Rightarrow$ Theorem in the spectral seq. of filtered complex applies.

Denoting the stabilizing result of $\{B_r, d_r\}$ by $B_\infty = \bigoplus_{n\geq 0}\left(\bigoplus_{p+q=n} B_\infty^{p,q}\right)$, we have.

__Theorem__ (14.14).

$K = \bigoplus_{p,q\geq 0} K^{p,q}$ double complex.

Then there is a spectral seq. $\{B_r, d_r\}$ converging to $H_D(K)$ s.t.

$$B_r = \bigoplus_{p,q} B_r^{p,q} \quad \text{w/} \quad d_r : B_r^{p,q} \longrightarrow B_r^{p+r,\, q-r+1} \quad \text{and}$$

$$B_1^{p,q} = H_d^{p,q}(K), \quad B_2^{p,q} = H_\delta^{p,q} H_d^{p,q}(K); \text{ moreover,}$$

$$G H_D^n(K) = \bigoplus_{p+q=n} B_\infty^{p,q}(K).$$

__Rmk__:

We can choose vertical filtration instead of horizontal.

This gives a second spectral sequence $\{B_r', d_r'\}$ converging to $H_D(K)$.

But $B_1' = H_\delta(K)$, $B_2' = H_d H_\delta(K)$ and $d_r' : B_r'^{p,q} \longrightarrow B_r'^{p-r+1,\, q+r}$.

__Example__:

Use spectral seq. to prove Cech cohomology $\cong$ De Rham cohomology. $\cong$ total cohomology of $C^*(\mathcal{U}, \Omega^*)$.