# §8. The Generalized Mayer-Vietoris Principle.

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## Generalization to Countably many Open sets and Applications

$\mathcal{U} = \{U_\alpha\}_{\alpha\in J}$ : open cover of $M$, $J$ : index set of countable ordered set.

$U_{\alpha\beta} := U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, etc.

Consider the inclusions:

$$M \leftarrow \coprod U_{\alpha_0} \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} \coprod_{\alpha_0<\alpha_1} U_{\alpha_0\alpha_1} \overset{\partial_0}{\underset{\partial_2}{\overset{\partial_1}{\leftleftarrows\!\!\leftarrow}}} \coprod_{\alpha_0<\alpha_1<\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \leftleftarrows\!\!\leftleftarrows \cdots$$

where $\partial_i$ : inclusion ignoring the $i$th open set.

e.g. $\partial_0 : U_{\alpha_0\alpha_1\alpha_2} \hookrightarrow U_{\alpha_1\alpha_2}$.

Taking $\Omega^*$, we have

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \prod_{\alpha_0<\alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\rightrightarrows\!\!\rightarrow}}} \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \rightrightarrows\!\!\rightrightarrows \cdots$$

e.g. $\partial_0 : \coprod_\alpha U_{\alpha\beta\gamma} \to U_{\beta\gamma}$ induces the restriction $\delta_0 : \Omega^*(U_{\beta\gamma}) \to \prod_\alpha \Omega^*(U_{\alpha\beta\gamma})$.

Define $\delta : \prod \Omega^*(U_{\alpha_0}) \to \prod_{\alpha_0<\alpha_1} \Omega^*(U_{\alpha_0\alpha_1})$ by $\delta = \delta_0 - \delta_1$

$\delta : \prod_{\alpha_0<\alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \to \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2})$ by $\delta = \delta_0 - \delta_1 + \delta_2$.

In general, $\delta : \prod_{\alpha_0<\cdots<\alpha_p} \Omega^*(U_{\alpha_0\cdots\alpha_p}) \to \prod_{\alpha_0<\cdots<\alpha_{p+1}} \Omega^*(U_{\alpha_0\cdots\alpha_{p+1}})$, $\delta = \delta_0 - \delta_1 + \cdots + (-1)^{p+1}\delta_{p+1}$
$= \sum_{i=0}^{p+1} (-1)^i \delta_i$.

More explicitly, for $\omega \in \prod_{\alpha_0<\cdots<\alpha_p} \Omega^q(U_{\alpha_0\cdots\alpha_p})$, expressing $\delta\omega = \sum_{\alpha_0<\cdots<\alpha_{p+1}} (\delta\omega)_{\alpha_0\cdots\alpha_{p+1}}$, we have

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}$$, where $\hat{}$ means omit and $\omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}$ on RHS is actually $\omega_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}\big|_{U_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}}$, where $\omega = \sum_{\alpha_0<\cdots<\alpha_p} \omega_{\alpha_0\cdots\alpha_p}$.

**Prop (8.3)** $\delta^2 = 0$.

**Prop (8.5)** (The Generalized Mayer-Vietoris)

The seq.

$$0 \to \Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \cdots$$

is exact. i.e. this complex has zero cohomology.

$K^{p,q} := C^p(\mathcal{U}, \Omega^q) = \prod \Omega^q(U_{\alpha_0 \cdots \alpha_p})$

$$\begin{array}{l} q \uparrow \\ d \uparrow \ \lfloor \longrightarrow p \\ \quad \overrightarrow{\delta} \end{array}$$

$\delta$ : difference operator.

$d$ : exterior derivative.

$D := D' + D''$, where $D' = \delta$ and $D'' = (-1)^p d$ on $K^{p,q}$.

$C^*(\mathcal{U}, \Omega^*) := \bigoplus_{p,q \geq 0} C^p(\mathcal{U}, \Omega^q)$ is a double complex, called the Čech-de Rham complex.

Elements in $C^*(\mathcal{U}, \Omega^*)$ are called Čech-de Rham cochains, or D-cochains.

**Prop (8.8)** (Generalized Mayer-Vietoris Principle).

$H^*_{DR}(M) \cong H_D\{C^*(\mathcal{U}, \Omega^*)\}$. More precisely, the restriction $r : \Omega^*(M) \to C^*(\mathcal{U}, \Omega^*)$ induces an iso. $r^* : H^*_{DR}(M) \to H_D\{C^*(\mathcal{U}, \Omega^*)\}$.

Rmk: The statement can be generalized as:

If all rows of the augmented double complex are exact, then the D-cohomology of the complex is iso. to the cohomology of the augmented initial column.

By taking the kernel of the first "d" on each column, we can augment each column and get another bottom row:

$C^0(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathbb{R}) \to \cdots$, which is another differential complex, whose homology is called the Čech cohomology of the cover $\mathcal{U}$, denoted by $H^*(\mathcal{U}, \mathbb{R})$.

Rmk: ① $C^p(\mathcal{U}, \mathbb{R}) = \{$ locally constant functions on the $(p+1)$-fold intersections $U_{\alpha_0 \cdots \alpha_p}\}$.

② This complex is purely combinatorial.

③ This is no longer exact (since the domains and codomains are changed) but we still have $\delta^2 = 0$ and it is a complex.

**Thm. (8.9)**

$\mathcal{U}$ : $\boxed{\text{good}}$ cover of a mfd $M$.

$\Rightarrow H^*_{DR}(M) \cong H^*(\mathcal{U}, \mathbb{R})$.

**Cor (8.9.1)**

The Čech cohomology $H^*(\mathcal{U}, \mathbb{R})$ is the same for all good cover $\mathcal{U}$ of $M$.

**Cor (8.9.2)+(8.9.3)**

Whenever $M$ has a finite good cover, $H^*_{DR}(M)$ is finite-dimensional.

In particular, if $M$ is cpt, then $H^*_{DR}(M)$ is always finite-dimensional.