Ref : [2004][S. Boyd, etal] Convex Optimization.

Chap 2 Convex Sets.

2.1 Affine and convex sets.

Def $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$.

The $\boxed{\text{line}}$ passing through x_1 and x_2:

$$\{ y \mid y = \theta x_1 + (1-\theta) x_2,\ \theta \in \mathbb{R} \}.$$

The $\boxed{\text{line segment}}$ b/w x_1 and x_2:

$$\{ y \mid y = \theta x_1 + (1-\theta) x_2,\ \theta \in [0,1] \}.$$

Def $C \subseteq \mathbb{R}^n$.

① C is $\boxed{\text{affine}}$ if every line through any two distinct pts in C lies in C.

i.e. $\forall\, x_1 \neq x_2$ in C, $\theta x_1 + (1-\theta) x_2 \in C$, $\forall\, \theta \in \mathbb{R}$.

② $x_1, \cdots, x_k \in \mathbb{R}^n$.

A pt of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ w/ $\theta_1 + \cdots + \theta_k = 1$ is called an $\boxed{\text{affine combination}}$ of $x_1, \cdots, x_k$.

Prop

① An affine set contains every affine combination of its pts.

② C: affine set, $x_0 \in C$.

$\Rightarrow V := C - x_0 = \{ x - x_0 \mid x \in C \}$ is a subsp., indep. of choice of $x_0 \in C$.

Def C: affine.

The $\boxed{\text{dimension}}$ of C is defined as $\dim(V)$, where $V = C - x_0$, $x_0 \in C$.

Rmk:

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

Then $C = \{ x \in \mathbb{R}^n \mid Ax = b \}$ is an affine set.

Conversely, any affine set can be expressed like this.

Def $C \subseteq \mathbb{R}^n$.

$\boxed{\operatorname{aff}(C)} := \{ \theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1 \}$

is called the $\boxed{\text{affine hull}}$ of C.

Rmk:

$\operatorname{aff}(C)$ is the smallest affine set containing C.

Def $C \subseteq \mathbb{R}^n$.

① The $\boxed{\text{affine dimension}}$ of C is defined to be the dimension of $\operatorname{aff} C$.

② The $\boxed{\text{relative interior}}$ of C, denoted $\boxed{\operatorname{relint}(C)}$, is the interior of C relative to $\operatorname{aff} C$.

i.e. $\operatorname{relint}(C) := \{ x \in C \mid B(x,r) \cap \operatorname{aff}(C) \subseteq C$ for some $r > 0 \}$.

$B(x,r) := \{ y \mid \|y - x\| \leq r \}$.

③ The $\boxed{\text{relative boundary}}$ of C is defined as $\operatorname{cl}(C) \setminus \operatorname{relint}(C)$.

Def

① A set $C \subseteq \mathbb{R}^n$ is $\boxed{\text{convex}}$ if the line segment b/w any pair of pts in C lies in C.

② $x_1, \cdots, x_k \in \mathbb{R}^n$.

A $\boxed{\text{convex combination}}$ of $x_1, \cdots, x_k$ is a pt of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ w/ $\theta_i \geq 0, \forall i$ and $\theta_1 + \cdots + \theta_k = 1$.

Rmk:

A set is convex iff it contains all convex comb. of its pts.

Def $C \subseteq \mathbb{R}^n$.

The $\boxed{\text{convex hull}}$ of C, denoted $\boxed{\operatorname{conv}(C)}$, is

$$\operatorname{conv}(C) = \{ \theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C,\ \theta_i \geq 0,\ \theta_1 + \cdots + \theta_k = 1 \}.$$

Rmk:

① $\operatorname{conv}(C)$ is always convex.

② The idea of convex combinations can be generalized to infinite sum or integral (over pdf).

End of W1

Def

① A set $C \subseteq \mathbb{R}^n$ is a $\boxed{\text{cone}}$ or $\boxed{\text{nonnegative homogeneous}}$ if $\forall\, x \in C$ and $\theta \geq 0$, $\theta x \in C$.

② A set $C \subseteq \mathbb{R}^n$ is a $\boxed{\text{convex cone}}$ if it is both convex and a cone.

Rmk:

$C \subseteq \mathbb{R}^n$ is a convex cone iff

$\forall\, x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$, $\theta_1 x_1 + \theta_2 x_2 \in C$.

Def $x_1, \cdots, x_k \in \mathbb{R}^n$.

A pt of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ w/ $\theta_1, \cdots, \theta_k \geq 0$ is called a $\boxed{\text{conic combination}}$ (or $\boxed{\text{nonnegative linear comb.}}$)

Rmk:

① A set C is a convex cone iff it contains all conic combinations of its elements.

② We can generalize the idea of conic combination to infinite sum or integral (over nonnegative measure.)

Def $C \subseteq \mathbb{R}^n$.

The $\boxed{\text{conic hull}}$ of C is

$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C, \theta_i \geq 0, i = 1, \cdots, k\}$,

the smallest convex cone containing C.

2.2 Some important examples.

2.2.1 Hyperplanes and halfspaces.

Def

A set of the form $\{x \in \mathbb{R}^n \mid a^T x = b\}$, where $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$, is called a hyperplane.

Rmk:

a above is the normal vector. Indeed, w/ $x_0 \in \mathbb{R}^n$ s.t. $a^T x_0 = b$, $\{x \mid a^T x = b\} = x_0 + a^\perp$.

Def

A (closed) halfspace is a set of the form

$\{x \mid a^T x \leq b\}$, where $a \neq 0$.

The interior of a closed halfspace is called an open halfspace.

2.2.2 Euclidean balls and ellipsoids.

Def

$B(x_c, r) := \{x \mid \|x - x_c\| \leq r\}$ is called an (Euclidean) ball w/ center x_c and radius r.

Prop $B(x_c, r)$ is convex.

Def

A set of the form $\mathcal{E} = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$ is called an ellipsoid, where $P = P^T \succ 0$.

x_c: center of $\mathcal{E}$.

The lengths of the semi-axes of $\mathcal{E}$ are $\sqrt{\lambda_i}$, where λ_i : e-values of P.

A set of the above form but w/ $P = P^T \succeq 0$ and $P \nsucc 0$ is called a degenerate ellipsoid.

Prop

An ellipsoid can be expressed as

$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \leq 1\}$, where

A: square and nonsingular.

2.2.3 Norm balls and norm cones.

Def $\|\cdot\|$: any norm on $\mathbb{R}^n$.

A norm ball is a set of the form

$\{x \mid \|x - x_c\| \leq r\}$, where $x_c \in \mathbb{R}^n$ and $r > 0$.

The norm cone associated w/ $\|\cdot\|$ is the set

$C = \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$.

Prop

The norm cone is a convex cone.

Ex

The norm cone associated w/ the Euclidean norm is called the second-order cone, quadratic cone, Lorentz cone, or ice-cream cone.

To see why the name,

$$C = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq t\}$$
$$= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq 0, t \geq 0 \right\}.$$

2.2.4 Polyhedra.

Def

A polyhedron is the sol'n set of a finite number of linear ineq. and linear eq. :

$P = \{x \mid a_j^T x \leq b_j, j = 1, \cdots, m, c_j^T x = d_j, j = 1, \cdots, p\}$.

A bdd polyhedron is called a polytope.

(Note: Some authors use the opposite convention.)

Compact Notation: $P = \{x \mid Ax \preceq b, Cx = d\}$.

Ex: nonnegative orthant.

It is actually a polyhedral cone. (i.e. both a polyhedron and a cone.)

Def simplex.

Ex: unit simplex $:= \underbrace{\operatorname{conv}(0, e_1, \cdots, e_n)}_{n\text{-simplex}} \subseteq \mathbb{R}^n$.

Probability simplex $:= \underbrace{\operatorname{conv}(e_1, \cdots, e_n)}_{(n-1)\text{-simplex}} \subseteq \mathbb{R}^n$.

Describing a simplex as a polyhedron. :

Given $v_0, \cdots, v_k$ affinely indep. (i.e. $v_1 - v_0, \cdots, v_k - v_0$ is l.i.)

$C := \operatorname{conv}\{v_0, \cdots, v_k\}$, the simplex defined by them.

We want to express C as a polyhedron.

Define $B = [\, v_1 - v_0 \ \cdots \ v_k - v_0 \,]$ $(n \times k)$.

Then $x \in C$

$\iff x = v_0 + By$, for some $y \succeq 0$, $1^T y \le 1$. $(*)$

$\because \operatorname{rank}(B) = k$

$\therefore \exists$ nonsingular $n \times n$ $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ s.t.

$$AB = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Thus

since A is invertible.

$$(*) \iff \begin{pmatrix} A_1 x \\ A_2 x \end{pmatrix} = Ax = Av_0 + ABy$$

$$= \begin{pmatrix} A_1 v_0 \\ A_2 v_0 \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} y, \quad \text{for some } y \succeq 0,\ 1^T y \le 1.$$

$\iff$ $A_1 x = A_1 v_0 + y$, $A_2 x = A_2 v_0$, for some $y \succeq 0$, $1^T y \le 1$

$\iff$ $A_1 x \succeq A_1 v_0$, $A_2 x = A_2 v_0$, $1^T A_1 x \le 1^T A_1 v_0 + 1$.

by defining $y = A_1 x - A_1 v_0$.

i.e. two linear ineq. and one linear eq., a polyhedron. #

2.2.5 The positive semidefinite cone.

Def

$S^n := \{X \in \mathbb{R}^{n \times n} : X = X^T\}$.

$S^n_+ := \{$ symmetric positive semidef. $n \times n$ matrices $\}$

$S^n_{++} := \{$ 〃 〃 def. 〃 〃 $\}$

Prop

S^n_+ is a convex cone.

<Pf>

Check directly. #

2.3 Operations that preserve convexity.

2.3.1 Intersection.

Prop Convexity is preserved under intersection (of any num).

Ex :

Let us prove S^n_+ is convex via the Prop above.

Note that

$$S^n_+ = \bigcap_{z \neq 0} \{X \in S^n \mid z^T X z \ge 0\}.$$

$\because z^T X z$ is linear in X

$\therefore \{X \in S^n : z^T X z \ge 0\}$ is a halfspace in S^n, whence convex.

Thus S^n_+, as an intersect'n of convex sets, is convex. #

Rmk : (Proved in 2.5.1) (A converse of above).

Every closed convex set S is a (usually infinite) intersect'n of halfspaces.

(Indeed, $S = \bigcap \{\mathcal{H} \mid \mathcal{H}$: halfspace w/ $S \subseteq \mathcal{H}\}$.)

2.3.2 Affine functions.

Def

$f : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if $f(x) = Ax + b$ for some matrix A and vector b.

Prop

$f : \mathbb{R}^n \to \mathbb{R}^m$ affine.

(1) If $S \subseteq \mathbb{R}^n$ is convex, then $f(S)$ is convex.

(2) If $S \subseteq \mathbb{R}^m$ 〃 〃, 〃 $f^{-1}(S)$ 〃 〃.

<Pf>

Direct check! #

Ex : (via Prop above).

① scaling and translat'n.

i.e. If $S \subseteq \mathbb{R}^n$: convex, $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^n$, then αS and $S + a$ are convex. $\boxed{\mathbb{R}^n \to \mathbb{R}^n : x \mapsto \alpha x,\ x \mapsto x + a}$

② project'n onto some coordinate.

i.e. If $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then $\{x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S$, for some $x_2\}$ is convex. $\boxed{\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m,\ (x,y) \mapsto x.}$

③ sum of two sets.

i.e. If S_1 and S_2 are convex, then $S_1 + S_2 := \{x + y \mid x \in S_1, y \in S_2\}$ is convex. $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto x+y$.

④ partial sum.

i.e. If $S_1, S_2 \subseteq \mathbb{R}^m \times \mathbb{R}^n$ are convex, then

$S := \{(x, y_1 + y_2) \mid (x, y_1) \in S_1, (x, y_2) \in S_2\}$ is convex.

Consider the ~~fun.~~ image of the $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$, $(x, y_1, y_2) \mapsto (x, y_1 + y_2)$ under appropriate convex set.

⑤ The polyhedron $\{x \mid Ax \preceq b, Cx = d\}$ can be expressed as $f^{-1}(\mathbb{R}^m_+ \times \{0\})$, where $f(x) = (b - Ax, d - Cx)$.
Thus it is convex.

⑥ $A_1, \cdots, A_n \in S^m$, $B \in S^m$.
The condition $A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B$ is called a linear matrix inequality (LMI) in x.
The sol'n set $\{x \mid A(x) \preceq B\}$ is convex since it is the inverse image under $f(x) = B - A(x)$ of the psd cone. $\boxed{f: \mathbb{R}^n \to S^m}$

End of W2