Homological Algebra.

$\boxed{1940}$ language.

① ↓

$\boxed{1956}$ Cartan & Eilenberg's book ← concerns all computational tools, std resolutions, spectral seq.

$\boxed{1957}$ derived functors, sheaves } Grothendieck's paper

②

③ Thesis Verdier 1963

Periods ① & ② : objects and their cohomology.

periods ③ : "think in terms of complexes".

Derived categories and triangulated categories.

Ref : ① Methods of Homological Algebra. - by Gelfand Manin.

② Notes of Piere Schapira

Categories & Sheaves. - by Schapira & Kashiwara.

③ Introduction to H.A. Weibel.

1.1 Sets & Maps.

X, Y two sets. $f: X \to Y$ a map. We'll say morphism.

$g: X \to Y$. bijection. We'll say "isomorphism", and write $X \simeq Y$.

$\mathrm{Hom}_{Set}(X,Y) := \{\text{morphisms } X \to Y\}$.

$X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow X \xrightarrow{g\circ f} Z$. Thus $\mathrm{Hom}_{Set}(X,Y) \xrightarrow{g\circ -} \mathrm{Hom}_{Set}(X,Z)$

$\mathrm{Hom}_{Set}(Y,Z) \xrightarrow{-\circ f} \mathrm{Hom}_{Set}(X,Z)$

a singleton

Given any set X. $\exists!$ morphism $X \to \{*\}$.
and $\exists!$ " $\emptyset \to X$.

The product of a family $\{X_i\}_{i\in I}$ of sets is the set

$$\prod_{i\in I} X_i = \{\{x_i\}_{i\in I} \mid x_i \in X_i, \forall i\}.$$

If $X_i = X$, $\forall i \in I$, then $\prod_{i\in I} X_i = X^I \cong \mathrm{Hom}(I, X)$.

Rmk:
$\exists$ natural iso. $\mathrm{Hom}(Y, \prod_{i\in I} X_i) \cong \prod_{i\in I} \mathrm{Hom}(Y, X_i)$.

Rmk:
$\exists$ natural iso. $\mathrm{Hom}(I \times X, Y) \cong \mathrm{Hom}(I, \mathrm{Hom}(X,Y)) \cong \mathrm{Hom}(X,Y)^I$.

Given a family $\{X_i\}_{i\in I}$ of sets, their disjoint union $\coprod_{i\in I} X_i$ is called the coproduct of the family.

If $I = \{1,2\}$, then $\coprod_{i\in I} X_i = X_1 \sqcup X_2$.

If $X_i = X$, $\forall i \in I$, then $\coprod_{i\in I} X_i = X^{(I)} \cong X \times I$.

Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$. Their equalizer $\mathrm{Ker}(f,g)$ is

$$\mathrm{Ker}(f,g) = \{x \in X \mid f(x) = g(x)\}.$$

Rmk:

$$\mathrm{Hom}(Z, \mathrm{Ker}(f,g)) \cong \mathrm{Ker}\left(\mathrm{Hom}(Z,X) \overset{f\circ -}{\underset{g\circ -}{\rightrightarrows}} \mathrm{Hom}(Z,Y)\right).$$

A relation R on a set X is a subset of $X \times X$.
xRy means $(x,y) \in R \subseteq X \times X$.
The opposite relation: $x R^{op} y$ iff yRx.
If xRx, $\forall x \in X$, we say that R is a reflexive relation.
If $xRy \Rightarrow yRx$, then we say that R is a symmetric relation.
If xRy and $yRx \Rightarrow x = y$, we say that R is anti-symmetric.

If xRy and $yRz \Rightarrow xRz$, we say that R is transitive.

A relation that is reflexive, symmetric, and transitive is called an equivalence relation.

e.g. ① "=" in $\mathbb{R}$ ② R = "//" on "straight lines in $\mathbb{R}^3$".

A relation is a pre-order if it is reflexive and transitive.

A preorder is often denoted "$\leq$".

A relation is an order if it is reflexive, transitive, and anti-sym.

A poset $(I, \leq)$ is directed if $I \neq \emptyset$ and $\forall i, j \in I$, $\exists k \in I$ s.t. $i \leq k$, $j \leq k$.

e.g.

X: topo. sp. $p \in X$, $I := \{$all open sets in X containing $p\}$.

$U \leq V$ means $U \supseteq V$.

Given a subset J of a directed set $(I, \leq)$.

We say that J is cofinal to I if $\forall i \in I$, $\exists j \in J$ s.t. $i \leq j$.

e.g.

J = set of all open balls centered at p.