5 HW5: due November 18

In the following problems concern an alternative definition of integral with respect to a measure. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $f : \Omega \to \mathbb{R}$ be a function, which may not be measurable. Let $\mathcal{P}$ be the collection of all finite $\mathcal{F}$-partition of Ω. Let

$$\int_* f d\mu = \sup_{\{A_i\}\in\mathcal{P}} \sum_i \left[\inf_{A_i} f(\omega)\right] \mu(A_i), \quad \int^* f d\mu = \inf_{\{A_i\}\in\mathcal{P}} \sum_i \left[\sup_{A_i} f(\omega)\right] \mu(A_i).$$

Problem 5.1. Suppose that f is measurable and nonnegative. Show that $\int^* f d\mu = \infty$ if $\mu(\{\omega : f(\omega) > 0\}) = \infty$.

Problem 5.2. Suppose that f is measurable and nonnegative. Show that $\int^* f d\mu = \infty$ if, for any $a > 0$, $\mu(\{\omega : f(\omega) > a\}) > 0$.

Problem 5.3. Let $\{A_i\}$ and $\{B_j\}$ be members of $\mathcal{P}$. We say that $\{B_j\}$ refines $\{A_i\}$ if for every $B_j \in \{B_j\}$ there exists an $A_i \in \{A_i\}$ such that $B_j \subseteq A_i$.

1. Show that for any $A_i \in \{A_i\}$, there is a $B_j \in \{B_j\}$ such that $A_i \supseteq B_j$; It is necessary to assume all A_i are nonempty.
2. Show that for each i,

$$A_i = \bigcup_{\{j:B_j\subseteq A_i\}} B_j.$$

Under the assumption {B_j} refines {A_i}.

Problem 5.4. Show that, if $\{B_j\}$ refines $\{A_i\}$, then

$$\sum_i \left[\inf_{\omega\in A_i} f(\omega)\right] \mu(A_i) \le \sum_j \left[\inf_{\omega\in B_j} f(\omega)\right] \mu(B_j)$$

Problem 5.5. Show that, if $\{B_j\}$ refines $\{A_i\}$, then

$$\sum_i \left[\sup_{\omega\in A_i} f(\omega)\right] \mu(A_i) \ge \sum_j \left[\sup_{\omega\in B_j} f(\omega)\right] \mu(B_j)$$

Problem 5.6. Show that, if $\{B_j\}$ refines $\{A_i\}$, then

This condition is not related to this problem.

$$\int_* f d\mu \le \int^* f d\mu.$$

Note that, in the above three problems, f is not required to be measurable.

Problem 5.7. Now suppose $\mu(\Omega) < \infty$, f is bounded; that is, there is an $M < \infty$ such that $|f(\omega)| \le M$ for all $\omega \in \Omega$, and f is measurable $\mathcal{F}/\mathcal{R}$. Consider the partition

$$A_i\{\omega : i\epsilon < f(\omega) \le (i+1)\epsilon\}, \quad i = -N, -N+1, \ldots, N-1, N,$$

where N is an integer such that $\epsilon N > M$. Show that

$$\sum_i \left[\sup_{\omega\in A_i} f(\omega)\right] \mu(A_i) - \sum_i \left[\inf_{\omega\in A_i} f(\omega)\right] \mu(A_i) \le \epsilon\mu(\Omega).$$

Conclude that

Equality sign missing.

$$\int_* f d\mu = \int^* f d\mu.$$

Where did you use the condition that f is measurable?

Problem 5.8. Define set functions $\mu^* : 2^\Omega \to \bar{\mathbb{R}}$ and $\mu_* : 2^\Omega \to \bar{\mathbb{R}}$ as follows: for any $A \in 2^\Omega$,

$$\underline{\mu^*(A)} = \inf\{\mu(B) : B \supseteq A,\ B \in \mathcal{F}\}$$
$$\underline{\mu_*(A)} = \sup\{\mu(B) : B \subseteq A,\ B \in \mathcal{F}\}.$$

1. Show that, for any $B \in \mathcal{F}$, $B \supseteq A$, there is $\{A_i\} \in \mathcal{P}$ such that

$$\sum_i \left[\sup_{A_i} I_A\right] \mu(A_i) \le \mu(B).$$

P is the collection of finite partitions of \Omega.

   Conclude that $\int^* I_A d\mu \le \mu(B)$, and hence that $\int^* I_A d\mu \le \mu^*(A)$.

2. Show that, for any $\{A_i\} \in \mathcal{P}$, there is $B \supseteq A$, $B \in \mathcal{F}$ such that

$$\sum_i \left[\sup_{A_i} I_A\right] \mu(A_i) = \mu(B).$$

   Conclude that $\sum_i [\sup_{A_i} I_A] \mu(A_i) \ge \mu^*(A)$, and hence that $\int^* I_A d\mu \ge \mu^*(A)$.

3. Show that, for any $B \subseteq A$, $B \in \mathcal{F}$, there is $\{A_i\} \in \mathcal{P}$ such that

This is I_A.

$$\mu(B) \le \sum_i \left[\inf_{A_i} I_A\right] \mu(A_i).$$

   Conclude that $\mu(B) \le \int_* f d\mu$, and hence that $\mu_*(A) \le \int_* I_A d\mu$.

4. Show that, for any $\{A_i\} \in \mathcal{P}$, there is $B \subseteq A$, $B \in \mathcal{F}$ such that

$$\mu(B) = \sum_i \left[\inf_{A_i} I_A\right] \mu(A_i).$$

   Conclude that $\mu_*(A) \ge \sum_i [\inf_{A_i} I_A] \mu(A_i)$, and hence that $\mu_*(A) \ge \int_* I_A d\mu$.