Stat 517 homework, fall, 2016

1 HW1: due 9/9/2016

Problem 1.1. Suppose that $\Omega \in \mathcal{F}$ and that $A, B \in \mathcal{F}$ implies $AB^c \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.

Problem 1.2. Let $\mathcal{B}_0$ be the collection of all finite and disjoint unions of intervals in $(0, 1]$, as defined in class. Show that $\mathcal{B}_0$ is not a σ-field.

Problem 1.3. Prove by mathematical induction the inclusion-exclusion formula. That is, suppose $\mathcal{F}$ is a field, $A_1, \ldots, A_n$ are members of $\mathcal{F}$, and P is a probability measure on $\mathcal{F}$. Show that

$$P(A_1 \cup \cdots \cup A_n) = \textstyle\sum_{i=1}^n A_i - \sum_{i<j} P(A_i A_j) + \cdots + (-1)^{n+1} P(A_1 \cdots A_n).$$

Problem 1.4. Let $A_1, A_2, \ldots$ be a sequence of sets. Let $B_n = \cup_{i=1}^n A_n$. Show that $\cup_{n=1}^\infty B_n = \cup_{n=1}^\infty A_n$.

Problem 1.5. Let $A_1, A_2, \ldots$ be a sequence of sets. Let

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 A_1^c \\ &\vdots \\ B_n &= A_n A_1^c \cdots A_{n-1}^c \\ &\vdots \end{aligned}$$

Prove the following statements:

1. $B_i B_j = \varnothing$ for any $i \neq j$;
2. $\cup_{i=1}^n A_i = \cup_{i=1}^n B_i$ for any $n = 1, 2, \ldots$;
3. $\cup_{i=1}^\infty A_i = \cup_{i=1}^\infty B_i$.

Problem 1.6. Let Ω be a nonempty set and $\mathcal{A} \subseteq 2^\Omega$. Let

$$\mathbb{F}(\mathcal{A}) = \{\mathcal{B} \subseteq 2^\Omega : \mathcal{A} \subseteq \mathcal{B}, \mathcal{B} \text{ is a field}\}.$$

Prove the following statements:

1. $\mathbb{F}(\mathcal{A})$ is nonempty;
2. $\phi(\mathcal{A}) = \cap\{\mathcal{B} : \mathcal{B} \in \mathbb{F}(\mathcal{A})\}$ is a field;
3. $\phi(\mathcal{A}) \subseteq \sigma(\mathcal{A})$;
4. $\sigma(\phi(\mathcal{A})) = \sigma(\mathcal{A})$.

Problem 1.7. Suppose that P is a probability measure on a field $\mathcal{F}$, that $A_1, A_2, \ldots$ and $\cup_{n=1}^\infty A_n$ lie in $\mathcal{F}$, and that A_n are nearly disjoint in the sense that $P(A_i A_j) = 0$ for $i \neq j$. Show that $P(A) = \sum_{n=1}^\infty P(A_n)$.

Problem 1.8. Let P be a probability measure on a field $\mathcal{F}_0$ and for every subset A of Ω, let $P^*(A)$ be the outer measure defined in class. Let $\tilde{P}$ be the extension of P to $\sigma(\mathcal{F}_0)$. Show that

$$P^*(A) = \inf\{\tilde{P}(B) : \ A \subseteq B, \ B \in \mathcal{F}\}.$$

2 HW2: due 9/23/2016

Problem 2.1. Let Ω be the unit square $(0,1] \times (0,1]$, and let

$$\mathcal{F} = \{A \times (0,1] : \ A \in \mathcal{B}\},$$

where $\mathcal{B}$ is the Borel σ-field on $(0,1]$. For any member $A \times (0,1]$ of $\mathcal{F}$, define

$$P(A \times (0,1]) = \lambda(A),$$

where λ is the Lebesgue measure on $\mathcal{B}$. Show that $\mathcal{F}$ is a σ-field and P is a probability on $\mathcal{F}$.

Problem 2.2. Prove the following statements.

1. A λ-system satisfies the following conditions

   (λ_4) $A, B \in \mathcal{L}$ and $A \cap B = \varnothing$ imply $A \cup B \in \mathcal{L}$;

   (λ_5) $A_1, A_2, \ldots \in \mathcal{L}$ and $A_n \uparrow A$ imply $A \in \mathcal{L}$;

   (λ_6) $A_1, A_2, \ldots \in \mathcal{L}$ and $A_n \downarrow A$ imply $A \in \mathcal{L}$.

2. $\mathcal{L}$ is a λ-system if and only if it satisfies (λ_1), (λ_2') and (λ_5). Recall that (λ_2') means

   $A, B \in \mathcal{L}$ and $A \subseteq B$ imply $BA^c \in \mathcal{L}$.

Problem 2.3. Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of sets. Prove that

$$I_{\limsup_n A_n} = \limsup_n (I_{A_n}), \quad I_{\liminf_n A_n} = \liminf_n (I_{A_n}).$$

(Recall that, for a sequence of numbers a_n, $\limsup_n a_n$ is defined to be $\lim_n \sup_{k \ge n} a_k$; $\liminf_n a_n$ is defined to be $\lim_n \inf_{k \ge n} a_k$).

Problem 2.4. Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of subsets of Ω. Let

$$B_n = \cap_{k=n}^{\infty} A_k, \quad C_n = \cup_{k=n}^{\infty} A_k.$$

Show that

$$B_n \uparrow \liminf_n A_n, \quad C_n \downarrow \limsup_n A_n.$$

Problem 2.5. (a) Prove that

$$(\limsup_n A_n) \cap (\limsup_n B_n) \supseteq \limsup_n (A_n \cap B_n),$$
$$(\limsup_n A_n) \cup (\limsup_n B_n) = \limsup_n (A_n \cup B_n),$$
$$(\limsup_n A_n) \cap (\liminf_n B_n) = \liminf_n (A_n \cap B_n),$$
$$(\liminf_n A_n) \cup (\liminf_n B_n) \subseteq \liminf_n (A_n \cup A_n).$$

(b) Show that

$$\limsup_n A_n^c = (\liminf_n A_n)^c,$$
$$\liminf_n A_n^c = (\limsup_n A_n)^c,$$
$$\limsup_n A_n \setminus \liminf_n A_n = \limsup_n (A_n \cap A_{n+1}^c) = \limsup_n (A_n^c \cap A_{n+1}).$$

(c) Show that $A_n \to A$ and B_n together imply that $A_n \cup B_n \to A \cup B$ and $A_n \cap B_n \to A \cap B$.

Problem 2.6. For events $A_1, \ldots, A_n$, consider the 2^n equations

$$P(B_1 \cdots B_n) = P(B_1) \cdots P(B_n),$$

where $B_i = A_i$ or $B_i = A_i^c$ for each i. Show that $A_1, \ldots, A_n$ are independent if all these equations hold.

Problem 2.7. Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are π-systems and $\mathcal{A}_1 \perp\!\!\!\perp \cdots \perp\!\!\!\perp \mathcal{A}_n$. Let $\mathcal{B}_i = \mathcal{A}_i \cup \{\Omega\}$. Show that $B_1, \ldots, B_n$ are π-systems and $B_1 \perp\!\!\!\perp \cdots \perp\!\!\!\perp \mathcal{B}_n$.

Problem 2.8. Show that $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$.

Problem 2.9. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space.

1. Show that, for any sequence of independent $\mathcal{F}$-sets, say $\{B_n : n = 1, 2, \ldots\}$, we have

$$P(\cap_{n=1}^{\infty} B_n) = \prod_{n=1}^{\infty} P(B_n).$$

2. Use the above relation and the inequality in Problem 2.8 to prove the second Borel-Cantelli Lemma.

Problem 2.10. Show that a λ-system can be equivalently defined by these three conditions:

1. $\Omega \in \mathcal{L}$;
2. If $A \in \mathcal{L}$, $B \in \mathcal{L}$, and $A \subseteq B$, then $BA^c \in \mathcal{L}$;
3. If $A_1, A_2, \ldots$ are a disjoint sequence of members of $\mathcal{L}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{L}$.

3 HW3: due October 7, 2016

Problem 3.1. Show that, in the definition of measure on a field, if condition (i) and (iii) hold, and if $\mu(A) < \infty$ for some $A \in \mathcal{F}$, then condition (ii) holds.

Problem 3.2. On a σ-field of all subsets of $\Omega = \{1, 2, \ldots\}$, define the set function

$$\mu(A) = \begin{cases} \sum_{k \in A} 2^{-k} & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

Is μ finitely additive? Is μ countably additive?

Problem 3.3.

1. In connection with Theorem 10.2 (ii), show that if $A_n \downarrow A$ and $\mu(A_k) < \infty$ for some k, then $\mu(A_n) \downarrow \mu(A)$.
2. Find an example in which $A_n \downarrow A$, $\mu(A_n) = \infty$ for all n, and $A = \varnothing$.

Problem 3.4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. The following is a generalization of Theorem 4.1, part (i).

1. Show that

$$\mu\left(\liminf_n A_n\right) \le \liminf_n \mu(A_n)$$

2. If $\mu(\cup_{k\ge n} A_k) < \infty$ for some n then

$$\limsup_n \mu(A_n) \le \mu\left(\limsup_n A_n\right).$$

   Show that this equality can fail if $\mu(\cup_{k\ge n} A_k) = \infty$ for all n.

The next three problems give an alternative approach to extend a measure from a field to the σ-field generated by it.

Problem 3.5. Extend Theorem 3.1 to finite measure. That is, a finite measure on a field has a unique extension to the generated σ-field. Hint: a finite measure can always be re-scaled to a probability measure.

Problem 3.6. Suppose Ω is a nonempty set, $\mathcal{F}_0$ is a field on Ω, and μ is a measure on $\mathcal{F}_0$. Let A be a nonempty set in $\mathcal{F}_0$ and $\mu(A) < \infty$. Let μ_A be μ restricted on $\mathcal{F}_0 \cap A$; that is, μ_A is the set function

$$\mathcal{F}_0 \cap A \to [0, \infty], \quad BA \mapsto \mu(BA).$$

1. Show that $\mathcal{F}_0 \cap A$ is a field;
2. μ_A is a measure on $\mathcal{F}_0 \cap A$;
3. μ_A has an extension $\hat{\mu}_A$ on $\mathcal{F} \cap A$, where $\mathcal{F} = \sigma(\mathcal{F}_0)$, and $\hat{\mu}_A$ is also a finite measure.

Problem 3.7. Define a set function $\hat{\mu}$ on $\mathcal{F}$ as follows. For any $E \in \mathcal{F}$, if there exists a sequence of disjoint $\mathcal{F}_0$-sets A_n such that $E \subseteq \cup_n A_n$ and $\mu(A_n) < \infty$, then let

$$\hat{\mu}(E) = \textstyle\sum_n \hat{\mu}_{A_n}(E \cap A_n);$$

if there exists no such sequence then let $\hat{\mu}(E) = \infty$.

1. Show that this definition doesn't depend on the choice of sequence $\{A_n\}$.
2. Show that $\hat{\mu}$ is a measure on $\mathcal{F}$, and agrees with μ on $\mathcal{F}_0$.

4 HW4: due October 28, 2016

This week's homework problems are taken from Billingsley's book *Probability and Measure*. You can either use the second edition or the Anniversary edition (the problem numbers match but the page numbers don't match; so I only give problem numbers).

11.2 (a), 12.10, 12.11, 12.12, 13.2(a,c), 13.3, 13.5, 13.8, I may add more problems later on.

5 HW5: due November 18

In the following problems concern an alternative definition of integral with respect to a measure. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $f : \Omega \to \mathbb{R}$ be a function, which may not be measurable. Let $\mathcal{P}$ be the collection of all finite $\mathcal{F}$-partition of Ω. Let

$$\int_* f d\mu = \sup_{\{A_i\} \in \mathcal{P}} \sum_i \left[\inf_{A_i} f(\omega) \right] \mu(A_i), \qquad \int^* f d\mu = \inf_{\{A_i\} \in \mathcal{P}} \sum_i \left[\sup_{A_i} f(\omega) \right] \mu(A_i).$$

Problem 5.1. Suppose that f is measurable and nonnegative. Show that $\int^* f d\mu = \infty$ if $\mu(\{\omega : f(\omega) > 0\}) = \infty$.

Problem 5.2. Suppose that f is measurable and nonnegative. Show that $\int^* f d\mu = \infty$ if, for any $a > 0$, $\mu(\{\omega : f(\omega) > a\}) > 0$.

Problem 5.3. Let $\{A_i\}$ and $\{B_j\}$ be members of $\mathcal{P}$. We say that $\{B_j\}$ refines $\{A_i\}$ if for every $B_j \in \{B_j\}$ there exists an $A_i \in \{A_i\}$ such that $B_j \subseteq A_i$.

1. Show that for any $A_i \in \{A_i\}$, there is a $B_j \in \{B_j\}$ such that $A_i \supseteq B_j$;
2. Show that for each i,

$$A_i = \bigcup_{\{j : B_j \subseteq A_i\}} B_j.$$

Problem 5.4. Show that, if $\{B_j\}$ refines $\{A_i\}$, then

$$\sum_i \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i) \le \sum_j \left[\inf_{\omega \in B_j} f(\omega) \right] \mu(B_j)$$

Problem 5.5. Show that, if $\{B_j\}$ refines $\{A_i\}$, then

$$\sum_i \left[\sup_{\omega \in A_i} f(\omega) \right] \mu(A_i) \ge \sum_j \left[\sup_{\omega \in B_j} f(\omega) \right] \mu(B_j)$$

Problem 5.6. Show that, if $\{B_j\}$ refines $\{A_i\}$, then

$$\int_* f d\mu \le \int^* f d\mu.$$

Note that, in the above three problems, f is not required to be measurable.

Problem 5.7. Now suppose $\mu(\Omega) < \infty$, f is bounded; that is, there is an $M < \infty$ such that $|f(\omega)| \le M$ for all $\omega \in \Omega$, and f is measurable $\mathcal{F}/\mathcal{R}$. Consider the partition

$$A_i \{\omega : i\epsilon < f(\omega) \le (i+1)\epsilon\}, \quad i = -N, -N+1, \ldots, N-1, N,$$

where N is an integer such that $\epsilon N > M$. Show that

$$\sum_i \left[\sup_{\omega \in A_i} f(\omega) \right] \mu(A_i) - \sum_i \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i) \le \epsilon \mu(\Omega).$$

Conclude that

$$\int_* f d\mu = \int^* f d\mu.$$

Where did you use the condition that f is measurable?

Problem 5.8. Define set functions $\mu^* : 2^\Omega \to \bar{\mathbb{R}}$ and $\mu_* : 2^\Omega \to \bar{\mathbb{R}}$ as follows: for any $A \in 2^\Omega$,

$$\mu^*(A) = \inf\{\mu(B) : B \supseteq A,\ B \in \mathcal{F}\}$$
$$\mu_*(A) = \sup\{\mu(B) : B \subseteq A,\ B \in \mathcal{F}\}.$$

1. Show that, for any $B \in \mathcal{F}$, $B \supseteq A$, there is $\{A_i\} \in \mathcal{P}$ such that

$$\sum_i \left[\sup_{A_i} I_A\right] \mu(A_i) \le \mu(B).$$

   Conclude that $\int^* I_A d\mu \le \mu(B)$, and hence that $\int^* I_A d\mu \le \mu^*(A)$.

2. Show that, for any $\{A_i\} \in \mathcal{P}$, there is $B \supseteq A$, $B \in \mathcal{F}$ such that

$$\sum_i \left[\sup_{A_i} I_A\right] \mu(A_i) = \mu(B).$$

   Conclude that $\sum_i [\sup_{A_i} I_A]\mu(A_i) \ge \mu^*(A)$, and hence that $\int^* I_A d\mu \ge \mu^*(A)$.

3. Show that, for any $B \subseteq A$, $B \in \mathcal{F}$, there is $\{A_i\} \in \mathcal{P}$ such that

$$\mu(B) \le \sum_i \left[\inf_{A_i} I_A\right] \mu(A_i).$$

   Conclude that $\mu(B) \le \int_* f d\mu$, and hence that $\mu_*(A) \le \int_* I_A d\mu$.

4. Show that, for any $\{A_i\} \in \mathcal{P}$, there is $B \subseteq A$, $B \in \mathcal{F}$ such that

$$\mu(B) = \sum_i \left[\inf_{A_i} I_A\right] \mu(A_i).$$

   Conclude that $\mu_*(A) \ge \sum_i [\inf_{A_i} I_A]\mu(A_i)$, and hence that $\mu_*(A) \ge \int_* I_A d\mu$.

6 HW6: due 12/9/2016

Problem 6.1. Suppose that $\Omega = \{1, 2, \ldots\}$, $\mathcal{F} = 2^\Omega$. Let κ be the set function

$$\mu : 2^\Omega \to \mathbb{R}, \quad A \mapsto \#(A),$$

where $\#(A)$ is the number of elements in A if A is finite, and is infinity if A is infinite. Show that μ is a measure on $(\Omega, \mathcal{F})$.

Problem 6.2. Suppose, for each $n = 1, 2, \ldots$, $\{x_{nm} : m = 1, 2, \ldots\}$ is a nonnegative sequence.

1. Show that, if $0 \le x_{nm} \uparrow x_m$ for each m, then

$$\lim_n \textstyle\sum_m x_{nm} = \sum_m x_m,$$

   where, as usual, $\sum_k$ is a shorthand for $\sum_{k=1}^\infty$. Identify each components of $(\Omega, \mathcal{F}, \mu)$, as well as the integral $\int f d\mu$, in this setting.

2. Show that (without the monotone condition in part 1),

$$\sum_n \sum_m x_{nm} = \sum_m \sum_n x_{nm}.$$

Problem 6.3. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mu_n, n = 1, 2, \ldots$ be a sequence of measures on $(\Omega, \mathcal{F})$. Define the set function

$$\mu : \mathcal{F} \to \mathbb{R}, \quad A \mapsto \sum_n \mu_n(A).$$

1. Use part 2 of Problem 6.2 to show that μ is a measure on $(\Omega, \mathcal{F})$.

2. Show that, for any indicator function $f = I_A$, $A \in \mathcal{F}$, we have

$$\int f d\mu = \sum_n \int f d\mu_n \tag{1}$$

3. Show that (1) is satisfied if f is a nonnegative simple function.

4. Show that (1) is satisfied if f is a nonnegative measurable function.

Problem 6.4. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and f_n and f are measurable. Prove that if $0 \leq f_n \to f$ a.e. μ and $\int f_n d\mu \leq A$, for some $A < \infty$. Show that f is integrable μ and $\int f d\mu \leq A$.

Problem 6.5. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and f_n are measurable. Suppose that f_n are integrable μ and $\sup_n \int f_n d\mu < \infty$. Show that, if $f_n \uparrow f$, then f is integrable and $\int f_n d\mu \to \int f d\mu$.

Problem 6.6. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and f_n are measurable. Suppose that f_n are integrable μ and $\inf_n \int f_n d\mu > -\infty$. Show that, if $f_n \downarrow f$, then f is integrable and $\int f_n d\mu \to \int f d\mu$.

Problem 6.7. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and a_n, b_n, and f_n are measurable functions and a_n and b_n are integrable with respect to μ. Suppose that

$$a_n \to a, \quad b_n \to b, \quad f_n \to f, \quad a.e.\ \mu.$$

Furthermore, suppose that $\int a_n d\mu \to \int a d\mu$, $\int b_n d\mu \to \int b d\mu$ where a and b are integrable μ. Finally suppose $a_n \leq f_n \leq b_n$ a.e. μ.

1. Show that $\int f_n d\mu \to \int f d\mu$.

2. Deduce the Dominated Convergence Theorem from part 1.

Problem 6.8. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space. Suppose $\{f(\cdot, t) : t \in (a, b)\}$ is a class of measurable functions on Ω. Let $t_0 \in (a, b)$. Suppose that the class of functions is Lipschitz in the following sense: there exist an integrable $g(\omega)$ and a set $A \in \mathcal{F}$ with $\mu(A^c) = 0$ such that, for all distinct $t_1, t_2 \in (a, b)$

$$\left| \frac{f(\omega, t_2) - f(\omega, t_1)}{t_2 - t_1} \right| \leq g(\omega).$$

Show that, if the function $t \mapsto f(\omega, t)$ is differentiable at $t = t_0$ for each $\omega \in A$, then the function $t \mapsto \int f(\omega, t) d\mu(\omega)$ is differentiable at t_0 and

$$\frac{d}{dt} \int f(\omega, t_0) d\mu(\omega) = \int \left[\frac{\partial f(\omega, t_0)}{\partial t} \right] d\mu(\omega).$$

Problem 6.9. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f \geq 0$ is measurable. Show that the set function

$$\nu : \mathcal{F} \to \mathbb{R}, \quad A \mapsto \int_A f d\mu$$

is a measure on $(\Omega, \mathcal{F})$, and that $\nu(A) = 0$ whenever $A \in \mathcal{F}$, $\mu(A) = 0$.

Problem 6.10. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space, f is a measurable function, and $\mu(\Omega) < \infty$.

1. Use DCT to show that, if f is integrable with respect to μ, then

$$\lim_{\alpha \to \infty} \int_{|f| \geq \alpha} |f| d\mu = 0. \tag{2}$$

2. Show that (2) implies f is integrable with respect to μ.

Problem 6.11. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space, f_n and g are measurable functions.

1. Use Problem 6.10 to show that if $|f_n| \leq g$ where g is integrable with respect to μ, then $\{f_n\}$ is uniformly integrable. State which part of the assumptions in Theorem 16.4 is stronger than 16.14(i), and which part of the assumptions in Theorem 16.14(i) is stronger than 16.4.

2. Let $\Omega = (0, 1]$ and $\mathcal{F}$ be the σ-field of Borel sets in $(0, 1]$, and λ the Lebesgue measure on $(0, 1]$. Let

$$f_n = (n/\log n) I_{(0, n^{-1})}, \quad n = 3, 4, \ldots$$

   Show that $\{f_n\}$ are uniformly integrable with respect to $\cancel{\mu}$ although they are not dominated by any integrable g.

3. In the same setting as part 2, let

$$f_n = n I_{(0, n^{-1})} - n I_{(n^{-1}, 2n^{-1})}$$

   Show that $\lim_{n \to \infty} \int f_n d\lambda = \int \lim_{n \to \infty} f_n d\lambda$ even though $\{f_n\}$ are not uniformly integrable with respect to μ.

Problem 6.12. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and f is a measurable function. Show that, if f is integrable, then there is a $\delta > 0$ such that, for each $A \in \mathcal{F}$ and $\mu(A) < \delta$, we have $\int_A |f| d\mu < \epsilon$.

Problem 6.13. (Related to Problem 6.12) Suppose that $\mu(\Omega) < \infty$. Show that $\{f_n\}$ then the following statements hold true:

1. $\sup_n \int |f_n| d\mu < \infty$;

2. for each $\epsilon > 0$ there is a $\delta > 0$ such that, whenever $A \in \mathcal{F}$, $\mu(A) < \delta$, we have $\int_A |f_n| d\mu < \epsilon$ for all n.