Let's start with $\mathbb{R}^n$ with Euclidean connection $\bar{\nabla}$

Given $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$\bar{\nabla}_X \bar{\nabla}_Y Z = \bar{\nabla}_X (Y Z^k \partial_k) = XYZ^k \partial_k$$

$$\bar{\nabla}_Y \bar{\nabla}_X Z = \bar{\nabla}_Y (X Z^k \partial_k) = YX Z^k \partial_k$$

So $\bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z = (XYZ^k - YXZ^k)\partial_k = \bar{\nabla}_{[X,Y]} Z$

meaning

$$\bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z = 0$$

$\boxed{\text{Def}}$ Given (M, g), we define $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Rmk $R(X,Y)Z = -R(Y,X)Z$

This is due to $[X,Y] = -[Y,X]$ and $\nabla_V W$ is $C^\infty(M)$-linear in V

$\boxed{\text{Prop}}$ R is a $\binom{3}{1}$-tensor field.

Pf: Take $f \in C^\infty(M)$, $X, Y, Z \in \mathfrak{X}(M)$

- $R(X, fY)Z = \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X, fY]} Z$

$$= \nabla_X (f \nabla_Y Z) - f \nabla_Y \nabla_X Z - \nabla_{f[X,Y] + (Xf)Y} Z$$

$$= \cancel{Xf \nabla_Y Z} + f \nabla_X \nabla_Y Z - f \nabla_Y \nabla_X Z$$

$$- f \nabla_{[X,Y]} Z \cancel{- (Xf) \nabla_Y Z}$$

$$= f R(X,Y)Z$$

- $R(fX, Y)Z = -R(Y, fX)Z = -f R(Y,X)Z = f R(X,Y)Z$

- $R(X,Y)(fZ) = \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]} fZ$

$$= f \nabla_X \nabla_Y Z + (Xf) \nabla_Y Z + (Yf) \nabla_X Z + (XYf) Z$$

$$- f \nabla_Y \nabla_X Z - (Yf) \nabla_X Z - (Xf) \nabla_Y Z - (YXf) Z$$

$$- f \nabla_{[X,Y]} Z - [X,Y] f Z$$

Due to $[X, Y]f := XYf - YXf$, we have

$$R(X,Y)(fZ) = f R(X,Y,Z)$$

Rmk. We shall call R "the (Riemann) curvature endomorphism" from now on

Rmk. In local coordinates,

$$R = R_{ijk}{}^{l}\, dx^i \otimes dx^j \otimes dx^k \otimes \partial_l$$

where $$R_{ijk}{}^{l}\, \partial_l = R(\partial_i, \partial_j)\, \partial_k$$

Def We define "the (Riemann) curvature tensor" by lowering the last index of R, denoted by $Rm = R^\flat$, or

$$Rm(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle$$

and in coordinates, we have

$$Rm = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where $R_{ijkl} = g_{lm} R_{ijk}{}^{m}$

Lem Suppose $\varphi : (M, g) \to (\tilde{M}, \tilde{g})$ is a local isometry, then (1) $\varphi^* \tilde{Rm} = Rm$ (2) $\tilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z = \varphi_*(R(X,Y)Z)$ for all $p \in M$, and $X, Y, Z \in T_pM$.

Symmetries of the Curvature tensor

Prop The curvature tensor Rm has following symmetry properties: for any $W, X, Y, Z \in \mathfrak{X}(M)$

(a) $Rm(W,X,Y,Z) = -Rm(X,W,Y,Z)$

(b) $Rm(W,X,Y,Z) = -Rm(W,X,Z,Y)$

(c) $Rm(W,X,Y,Z) = Rm(Y,Z,W,X)$

(d) First Bianchi identity

$$Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) = 0$$

(e) Second Bianchi identity

$$\nabla Rm(X,Y,Z,V,W) + \nabla Rm(X,Y,V,W,Z) + \nabla Rm(X,Y,W,Z,V) = 0$$

in components:

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0$$

Pf: take $W, X, Y, Z \in \mathfrak{X}(M)$

(a) due to $R(W,X)Y = -R(X,W)Y$

(b) Note that due to compatibility of ∇ w.r.t g,

$$X|Y|^2 = \nabla_X |Y|^2 = \nabla_X \langle Y, Y\rangle = 2\langle \nabla_X Y, Y\rangle$$

So $WX|Y|^2 = W(2\langle \nabla_X Y, Y\rangle) = 2\langle \nabla_W \nabla_X Y, Y\rangle$
$+ 2\langle \nabla_X Y, \nabla_W Y\rangle$ (1)

Similarly,

$$XW|Y|^2 = 2\langle \nabla_X \nabla_W Y, Y\rangle + 2\langle \nabla_W Y, \nabla_X Y\rangle \quad (2)$$

Also we have $[W,X]|Y|^2 = 2\langle \nabla_{[W,X]} Y, Y\rangle$ (3)

Now, compute (1)-(2)-(3), we obtain

$$\begin{aligned} 0 &= 2\langle \nabla_W \nabla_X Y, Y\rangle - 2\langle \nabla_X \nabla_W Y, Y\rangle - 2\langle \nabla_{[W,X]} Y, Y\rangle \\ &= 2\langle R(W,X)Y, Y\rangle \\ &= 2Rm(W,X,Y,Y) \end{aligned}$$

So $Rm(W,X,Y,Z) + Rm(W,X,Z,Y) = Rm(W,X,Y+Z,Y+Z)$
$= 0$

(d) Recall that due to symmetry of ∇, we have $\nabla_V W - \nabla_W V = [V,W]$ for any $V, W \in \mathfrak{X}(M)$. Now for any $W, X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned} &R(W,X)Y + R(X,Y)W + R(Y,W)X \\ &= (\nabla_W \nabla_X Y - \nabla_X \nabla_W Y - \nabla_{[W,X]} Y) + (\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \\ &\quad \nabla_{[X,Y]} W) + (\nabla_Y \nabla_W X - \nabla_W \nabla_Y X - \nabla_{[Y,W]} X) \\ &= \nabla_W(\nabla_X Y - \nabla_Y X) + \nabla_X(\nabla_Y W - \nabla_W Y) + \nabla_Y(\nabla_W X - \nabla_X W) \\ &\quad - \nabla_{[W,X]} Y - \nabla_{[X,Y]} W - \nabla_{[Y,W]} X \\ &= \nabla_W [X,Y] - \nabla_{[X,Y]} W + \nabla_X [Y,W] - \nabla_{[Y,W]} X \\ &\quad + \nabla_Y [W,X] - \nabla_{[W,X]} Y \\ &= [W,[X,Y]] + [X,[Y,W]] + [Y,[W,X]] = 0 \end{aligned}$$

The last step follows from <u>prop 8.28(c)</u> from Lee's Smooth manifold book (which we call "Jacobi identity")

(e) Suppose (c) is true, then we have (e) is equivalent to

$$\nabla Rm(Z,V,X,Y,W) + \nabla Rm(V,W,X,Y,Z) + \nabla Rm(W,Z,X,Y,V) = 0$$

Pick $p \in M$, and (x^i) be the normal coordinates around p. Note that we only need to prove the case when X, Y, Z, V, W are just coordinate basis vectors ∂_i (due to multilinearity).

Recall that ① $[\partial_i, \partial_j] \equiv 0$ ② $\Gamma^k_{ij} = 0$ at p, $\forall i,j,k$.

$$\begin{aligned}\nabla_W Rm(Z,V,X,Y) &= \nabla_W \langle R(Z,V)X, Y\rangle \\ &= \langle \nabla_W R(Z,V)X, Y\rangle + \langle R(Z,V)X, \cancelto{\text{vanish}}{\nabla_W Y}\rangle \\ &= \langle \nabla_W \nabla_Z \nabla_V X - \nabla_W \nabla_V \nabla_Z X, Y\rangle\end{aligned}$$

So we get

$$\begin{aligned}&\nabla_W Rm(Z,V,X,Y) + \nabla_Z Rm(V,W,X,Y) + \nabla_V Rm(W,Z,X,Y)\Big|_p \\ &= \langle \nabla_W \nabla_Z \nabla_V X - \nabla_W \nabla_V \nabla_Z X + \nabla_Z \nabla_V \nabla_W X - \nabla_Z \nabla_W \nabla_V X \\ &\quad + \nabla_V \nabla_W \nabla_Z X - \nabla_V \nabla_Z \nabla_W X, Y\rangle \\ &= \langle R(W,Z)\nabla_V X + R(Z,V)\nabla_W V + R(V,W)\nabla_Z X, Y\rangle\Big|_p \\ &= 0\end{aligned}$$

due to $\nabla_V X = \nabla_W X = \nabla_Z X = 0$ at p.

More curvatures:

Def The <u>Ricci curvature</u> Rc (or Ric) is defined by taking contraction on 1st and last indices of Rm. In components, we have $Rc = R_{ij}\, dx^i \otimes dx^j$ with

$$R_{ij} = g^{km} R_{kijm}$$

Def The <u>scalar curvature</u> S is the smooth function defined by $S = tr_g(Ric) = g^{ij} R_{ij}$

Def Take $p \in M$, say Π is any two-dimensional subspace of T_pM. Say $\mathcal{V} \subseteq T_pM$ contains $0 \in T_pM$ and $\exp_p|_{\mathcal{V}}$ is a diffeomorphism. Then $S_\Pi := \exp_p(\Pi \cap \mathcal{V})$, which is a 2-dim submanifold of M. We call S_Π the plane section determined by Π.

Def We define the <u>sectional curvature</u> of M associated with Π, denoted as $K(\Pi)$, by

$$K_\Pi(X,Y) = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X, Y\rangle^2}$$

where X, Y is any basis for Π.

(check: it does not depend on the choice of X, Y).

Jacobi Field and Jacobi Equation:

Def Given $\gamma: [a,b] \to M$ as geodesic. A variation Γ: $(-\varepsilon,\varepsilon)\times[a,b] \to M$ of γ is called a <u>variation through geodesic</u> if Γ_s is geodesic for all $s \in (-\varepsilon, \varepsilon)$. (Recall $\Gamma_s(t) = \Gamma(s,t)$).

Lem If Γ is any admissible family of curves, and V is a vector field along Γ, then

$$D_s D_t V - D_t D_s V = R(S,T)V$$

Pf: Let $T(s,t) = \partial_t \Gamma(s,t) = \partial_t \Gamma_s$, $S(s,t) = \partial_s \Gamma(s,t) = \partial_s \Gamma_t$

$V(s,t) = V^i(s,t)\partial_i$. By (4.10), we have

$$D_t V = \frac{\partial V^i}{\partial t}\partial_i + V^i D_t \partial_i, \quad D_s V = \frac{\partial V^i}{\partial s}\partial_i + V^i D_s \partial_i$$

$$\Longrightarrow \begin{cases} D_s D_t V = \dfrac{\partial^2 V^i}{\partial s \partial t}\partial_i + \dfrac{\partial V^i}{\partial t} D_s \partial_i + \dfrac{\partial V^i}{\partial s} D_t \partial_i + V^i D_s D_t \partial_i \\ D_t D_s V = \dfrac{\partial^2 V^i}{\partial t \partial s}\partial_i + \dfrac{\partial V^i}{\partial s} D_t \partial_i + \dfrac{\partial V^i}{\partial t} D_s \partial_i + V^i D_t D_s \partial_i \end{cases}$$

$$D_s D_t V - D_t D_s V = V^i (D_s D_t \partial_i - D_t D_s \partial_i)$$

Note that $S = \frac{\partial x^k}{\partial s} \partial_k$, $T = \frac{\partial x^j}{\partial t} \partial_j$, and because

∂_i is extendible, $D_t \partial_i = \nabla_T \partial_i = \frac{\partial x^j}{\partial t} \nabla_{\partial_j} \partial_i$

Because $\nabla_{\partial_j} \partial_i$ is also extendible, so

$$\begin{aligned} D_s D_t \partial_i &= D_s \left(\frac{\partial x^j}{\partial t} \nabla_{\partial_j} \partial_i \right) \\ &= \frac{\partial^2 x^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial x^j}{\partial t} \nabla_S (\nabla_{\partial_j} \partial_i) \\ &= \frac{\partial^2 x^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i \end{aligned}$$

Similarly

$$D_t D_s \partial_i = \frac{\partial^2 x^j}{\partial t \partial s} \nabla_{\partial_j} \partial_i + \frac{\partial x^k}{\partial s} \frac{\partial x^j}{\partial t} \nabla_{\partial_j} \nabla_{\partial_k} \partial_i$$

So

$$\begin{aligned} D_s D_t \partial_i - D_t D_s \partial_i &= \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} (\nabla_{\partial_k} \nabla_{\partial_j} \partial_i - \nabla_{\partial_j} \nabla_{\partial_k} \partial_i) \\ &= \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} R(\partial_k, \partial_j) \partial_i \\ &= R(S, T) \partial_i \end{aligned}$$

$\boxtimes$

$\boxed{\text{Thm}}$ (The Jacobi Equation)

Let γ be a geodesic and V a vector field along γ.
If V is the variation field of a variation through geodesic, then

$$(*) \qquad D_t D_t V + R(V, \dot\gamma)\dot\gamma = 0$$

Pf. because Γ_s is geodesic for all $s \in (-\varepsilon, \varepsilon)$, then

$D_t T = D_t (\partial_t \Gamma_s) = 0$ by geodesic equation. Then,

$$\begin{aligned} 0 = D_s D_t T &= D_t D_s T + R(S, T) T \\ &= D_t D_t S + R(S, T) T \qquad (\forall s, \forall t) \end{aligned}$$

by preceding lemma. and symmetry Lemma (lemma 6.3).

Then, recall $V(t) = \partial_s \Gamma_t(0) = S(0, t)$, $\dot\gamma(t) = \partial_t \Gamma_0(t) = T(0, t)$

So $D_t D_t V + R(V, \dot\gamma)\dot\gamma = 0$

$\boxed{\text{Def}}$ Any vector field along a geodesic satisfying $(*)$ is called a Jacobi field.

Rmk / Examples: Given geodesic γ.

(1) $J_0(t) = \dot\gamma(t)$ satisfies Jacobi Equation with initial conditions $J_0(0) = \dot\gamma(0)$, $D_t J_0(0) = 0$

(2) $J_1(t) = t\dot\gamma(t)$ satisfies Jacobi Equation with initial conditions $J_1(0) = 0$, $D_t J_1(0) = \dot\gamma(0)$.

Second Variation Formula

$\boxed{\text{Thm}}$ (Second Variation Formula)

Let $\gamma: [a,b] \to M$ be a unit-speed geodesic, Γ is a proper variation of γ with V being its variation field. Then

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\Gamma_s) = \int_a^b \left(|D_t V^\perp|^2 - Rm(V^\perp, \dot\gamma, \dot\gamma, V^\perp) \right) dt$$

where $V^\perp$ is the normal component of V

Pf: set again $T(s,t) = \partial_t \Gamma(s,t)$ $S(s,t) = \partial_s \Gamma(s,t)$.

choose rectangle $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ where Γ is smooth,

$$\frac{d}{ds} L\left(\Gamma_s\big|_{[a_{i-1}, a_i]}\right) = \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial s} \langle T, T\rangle^{1/2} \, dt$$

$$= \int_{a_{i-1}}^{a_i} \frac{\langle D_t S, T\rangle}{\langle T, T\rangle^{\frac{1}{2}}} \, dt$$

(chain rule, symmetry lemma, compactibility of ∇ are used)

$$\frac{d^2}{ds^2} L\left(\Gamma_s\big|_{[a_{i-1}, a_i]}\right)$$

$$= \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial s}\left(\frac{\langle D_t S, T\rangle}{\langle T, T\rangle^{\frac{1}{2}}} \right) dt$$

$$= \int_{a_{i-1}}^{a_i} \frac{\langle D_s D_t S, T\rangle + \langle D_t S, D_s T\rangle}{\langle T, T\rangle^{\frac{1}{2}}} - \frac{1}{2} \cdot \frac{\langle D_t S, T\rangle 2 \langle D_s T, T\rangle}{\langle T, T\rangle^{3/2}} \, dt$$

(use compactibility again)

$$= \int_{a_{i-1}}^{a_i} \frac{\langle D_t D_s S + R(S,T)S, T\rangle}{|T|} + \frac{\langle D_t S, D_t S\rangle}{|T|} - \frac{\langle D_t S, T\rangle^2}{|T|^3} \, dt$$

Let $S=0$, $|T|=1$.

$$\frac{d^2}{ds^2}\Big|_{s=0} L\left(\Gamma_s\big|_{[a_{i-1},a_i]}\right)$$

$$= \int_{a_{i-1}}^{a_i} \left(\langle D_t D_s S, T\rangle - Rm(S,T,T,S) + |D_t S|^2 - \langle D_t S, T\rangle^2 \right)\Big|_{s=0} dt$$

Now, because $D_t T\big|_{s=0} = D_t \dot\gamma = 0$,

$$\int_{a_{i-1}}^{a_i} \langle D_t D_s S, T\rangle\, dt = \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial t}\langle D_s S, T\rangle\, dt$$

b/c $D_t T = 0$ when $s=0$

$$= \langle D_s S, T\rangle\Big|_{t=a_{i-1}}^{t=a_i} \quad (s=0)$$

① when $t=a_0$ or $t=a_k=b$, $D_s S = 0$ for all $s\in(-\varepsilon,\varepsilon)$.
since Γ is a proper variation.

② $D_s S = D_s(\partial_s \Gamma)$ is continuous for all (s,t),
namely, there is no "jump" across a_i,

So $\sum_i \langle D_s S, T\rangle\Big|_{t=a_{i-1}}^{t=a_i}$ vanishes ~~for all i~~

So we get

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\Gamma_s) = \int_a^b \left(|D_t V|^2 - \langle D_t V, \dot\gamma\rangle^2 - Rm(V,\dot\gamma,\dot\gamma,V)\right) dt \quad (**)$$

For geodesic γ, any vector field V along γ can be uniquely written as $V = V^\perp + V^T$, where

$$V^T = \langle V, \dot\gamma\rangle\dot\gamma, \quad V^\perp = V - V^T$$

Because $D_t\dot\gamma = 0$,

$$D_t V^T = \langle D_t V, \dot\gamma\rangle\dot\gamma = (D_t V)^T$$

$$D_t V^\perp = D_t V - D_t V^T = D_t V - (D_t V)^T = (D_t V)^\perp$$

Therefore,

$$|D_t V|^2 = |(D_t V)^T|^2 + |(D_t V)^\perp|^2$$

$$= \langle D_t V, \dot\gamma\rangle^2 + |D_t V^\perp|^2$$

Also, $Rm(V,\dot\gamma,\dot\gamma,V) = Rm(V^\perp,\dot\gamma,\dot\gamma,V^\perp)$,

So by (**) we are done ∎