Example (14.22) (Cohomology of the complex projective space).

Regard $S^{2n+1} = \{(z_0, \cdots, z_n) \mid |z_0|^2 + \cdots + |z_n|^2 = 1\} \subseteq \mathbb{C}^{n+1}$.

Let S^1 act on S^{2n+1} by $\lambda \cdot (z_0, \cdots, z_n) = (\lambda z_0, \cdots, \lambda z_n)$.

The quotient of S^{2n+1} by this action (i.e. $(z_0, \cdots, z_n) \sim (z_0', \cdots, z_n')$ iff $(z_0', \cdots, z_n') = \lambda(z_0, \cdots, z_n)$, for some $\lambda \in S^1$).
is actually the complex projective space $\mathbb{C}P^n$.

With this, we have a circle bundle:

$$\begin{array}{ccc} S^1 \longrightarrow & S^{2n+1} \\ & \downarrow \\ & \mathbb{C}P^n. \end{array}$$

Assume now the fact (see (17.4)):
$\mathbb{C}P^n$ is simply connected.

Thus, $E_2^{p,q} = H^p(\mathbb{C}P^n) \otimes H^q(S^1)$.

$\because H^k(S^1) = \begin{cases} \mathbb{R}, & k=0,1 \\ 0, & \text{o.w.} \end{cases}$ $\therefore E_2$ has nonzero terms only on 0th and 1st row.

Moreover, since $\dim(\mathbb{C}P^n) = 2n$, $H^k(\mathbb{C}P^n) = 0, \ \forall k \geq 2n+1$.

$\Rightarrow E_2^{p,q} = 0, \ \forall p \geq 2n+1$.

Therefore, $E_2^{p,q} = 0, \ \forall p \geq 2n+1, q \geq 2$.

We now restrict temporily on $\mathbb{C}P^2$. (The same argument would be applicable for $\mathbb{C}P^n$).

Then

$$E_2 = \begin{array}{|cccccc} \mathbb{R} & A & B & C & D & 0 \\ \mathbb{R} & A & B & C & D & 0 \\ \hline 0 & 1 & 2 & 3 & 4 & 5 \end{array}.$$

Why the two rows are iso. is from $H^*(S^1) = \mathbb{R} \oplus \mathbb{R} \oplus 0 \oplus \cdots$
(In fact, due to this, the rows are iso. to $H^*(\mathbb{C}P^n)$).

$d_3 = 0$ since it moves down for two rows. $\Rightarrow d_4 = d_5 = d_6 = \cdots = 0$.

Moreover, $E_3 = E_4 = E_5 = \cdots = E_\infty$, which stabilizes at 3.

By the convergence of this spectral seq. (i.e. converging to the total cohomology),

$E_\infty = H^*(S^{2n+1}) \underset{\uparrow \text{ now } n=2}{=} H^*(S^5)$.
$(= E_3)$

$\because E_3 = H_{d_2}(E_2)$ $\therefore$ The zero terms in E_2 would remain zero in E_3.

Thus, the only possibility for E_3 is

$$E_3 = \begin{array}{|cccccc} 0 & 0 & 0 & 0 & \mathbb{R} & 0 \\ \mathbb{R} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 3 & 4 & 5 \end{array}.$$

Therefore, $d_2: 0 \to A, \mathbb{R} \to B, A \to C, B \to D, C \to 0$ are iso.
(since these terms become zero in E_3).

$\Rightarrow$ We may rewrite E_2 as:

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$$E_2 = \begin{array}{c|c|c|c|c|c|}
\mathbb{R} & 0 & \mathbb{R} & 0 & \mathbb{R} & 0 \\
\mathbb{R} & 0 & \mathbb{R} & 0 & \mathbb{R} & 0 \\
\hline
0 & 1 & 2 & 3 & 4 & 5
\end{array}$$

$\because E_2^{p,q} = H^p(\mathbb{C}P^2) \otimes H^q(S^1) \quad \therefore H^p(\mathbb{C}P^2) = \begin{cases} \mathbb{R}, & p = 0, 2, 4 \\ 0, & \text{o.w.} \end{cases}$ #

The same argument can be applied to $\mathbb{C}P^n$ and we'll have

$$H^k(\mathbb{C}P^n) = \begin{cases} \mathbb{R}, & k = 0, 2, 4, \cdots, 2n \\ 0, & \text{o.w.} \end{cases}$$ #