VHEGEN: A vibronic Hamiltonian expansion generator for trigonal and tetragonal polyatomic systems

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Thank you for using VHEGEN, the V-ibronic H-amiltonian E-xpansion GEN-erator for trigonal and tetragonal polyatomic systems. This is a VHEGEN output file compiled by pdflatex. If the VHEGEN package was used in research resulting in a publication, please reference the article in *Computer Physics Communications* which describes the program ([doi here]). Additional information regarding the matrix element expansion process, including the independent matrix element eigenvalues, their root formulas and constraints, and their transformation to the real basis (if applicable), can be found in the log output file. For questions, bugs, or comments, please contact robert.lang@mail.utoronto.ca.

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1 Vibronic interaction

$$(E+E)\otimes (e+e)$$
 in C_3

2 Vibronic Hamiltonian operator in the complex E basis

$$\begin{split} \hat{H} &= \left(\left| +_{\alpha} \right\rangle \ \left| -_{\alpha} \right\rangle \ \left| +_{\beta} \right\rangle \ \left| -_{\beta} \right\rangle \right) \begin{pmatrix} 0 & 0 & H_{+\alpha+\beta} & H_{+\alpha-\beta} \\ 0 & 0 & H_{-\alpha+\beta} & H_{-\alpha-\beta} \\ H_{+\beta+\alpha} & H_{+\beta-\alpha} & 0 & 0 \\ H_{-\beta+\alpha} & H_{-\beta-\alpha} & 0 & 0 \end{pmatrix} \begin{pmatrix} \left\langle +_{\alpha} \right| \\ \left\langle -_{\alpha} \right| \\ \left\langle +_{\beta} \right| \\ \left\langle -_{\beta} \right| \end{pmatrix} \\ H_{+\alpha+\beta} &= H_{+\alpha+\beta} = H_{-\alpha-\beta} \stackrel{*}{=} H_{+\beta+\alpha} \stackrel{*}{=} H_{-\beta-\alpha} \\ H_{+\alpha-\beta} &= H_{+\alpha-\beta} = H_{-\alpha+\beta} \stackrel{*}{=} H_{+\beta-\alpha} = H_{-\beta+\alpha} \stackrel{*}{=} \end{split}$$

3 Matrix element expansions in the complex E basis

3.1 Order: 0

Number of fitting parameters: $H_{+_{\alpha}+_{\beta}}$: 2, $H_{+_{\alpha}-_{\beta}}$: 0.

$$H^{(0)}_{+_{\alpha}+_{\beta}} = ia^{i}_{0,0,0,0} + a^{r}_{0,0,0,0}$$

$$H_{-\alpha-\beta}^{(0)} = -ia_{0,0,0,0}^i + a_{0,0,0,0}^r$$

$$H^{(0)}_{+_{\beta}+_{\alpha}} = -ia^{i}_{0,0,0,0} + a^{r}_{0,0,0,0}$$

$$H^{(0)}_{-_{\beta}-_{\alpha}}=ia^{i}_{0,0,0,0}+a^{r}_{0,0,0,0}$$

$$H_{+_{\alpha}-_{\beta}}^{(0)} = 0$$

$$H_{-\alpha+\beta}^{(0)} = 0$$

$$H_{+\beta-\alpha}^{(0)} = 0$$

$$H_{-\beta+\alpha}^{(0)}=0$$

$$H^{(0)}_{+_{\alpha}+_{\beta}}=ia^i_{0,0,0,0}+a^r_{0,0,0,0}$$

$$H_{-\alpha-\beta}^{(0)} = -ia_{0,0,0,0}^i + a_{0,0,0,0}^r$$

$$H^{(0)}_{+_{\beta}+_{\alpha}} = -ia^{i}_{0,0,0,0} + a^{r}_{0,0,0,0}$$

$$H_{-\beta-\alpha}^{(0)} = ia_{0,0,0,0}^i + a_{0,0,0,0}^r$$

$$H_{+_{\alpha}-_{\beta}}^{(0)} = 0$$

$$H_{-\alpha+\beta}^{(0)} = 0$$

$$H_{+\beta-\alpha}^{(0)} = 0$$

$$H_{-\beta+\alpha}^{(0)} = 0$$

3.2 Order: 1

Number of fitting parameters: $H_{+\alpha+\beta}$: 0, $H_{+\alpha-\beta}$: 4.

$$H_{+\alpha+\beta}^{(1)} = 0$$

$$H_{-\alpha-\beta}^{(1)} = 0$$

$$H_{+_{\beta}+_{\alpha}}^{(1)} = 0$$

$$H_{-\beta-\alpha}^{(1)} = 0$$

$$H_{+\alpha-\beta}^{(1)} = b_{0,0,-1,-1}^{i} \rho_{\alpha} \sin{(\phi_{\alpha})} + i b_{0,0,-1,-1}^{i} \rho_{\alpha} \cos{(\phi_{\alpha})} + b_{0,0,0,-1}^{i} \rho_{\beta} \sin{(\phi_{\beta})} + i b_{0,0,0,-1}^{i} \rho_{\beta} \cos{(\phi_{\beta})} - i b_{0,0,-1,-1}^{r} \rho_{\alpha} \sin{(\phi_{\alpha})} + b_{0,0,-1,-1}^{r} \rho_{\alpha} \cos{(\phi_{\alpha})} - i b_{0,0,0,-1}^{r} \rho_{\beta} \sin{(\phi_{\beta})} + b_{0,0,0,-1}^{r} \rho_{\beta} \cos{(\phi_{\beta})}$$

$$\begin{split} H_{-\alpha+\beta}^{(1)} &= b_{0,0,-1,-1}^{i} \rho_{\alpha} \sin{(\phi_{\alpha})} - i b_{0,0,-1,-1}^{i} \rho_{\alpha} \cos{(\phi_{\alpha})} + b_{0,0,0,-1}^{i} \rho_{\beta} \sin{(\phi_{\beta})} - i b_{0,0,0,-1}^{i} \rho_{\beta} \cos{(\phi_{\beta})} \\ &+ i b_{0,0,-1,-1}^{r} \rho_{\alpha} \sin{(\phi_{\alpha})} + b_{0,0,-1,-1}^{r} \rho_{\alpha} \cos{(\phi_{\alpha})} + i b_{0,0,0,-1}^{r} \rho_{\beta} \sin{(\phi_{\beta})} + b_{0,0,0,-1}^{r} \rho_{\beta} \cos{(\phi_{\beta})} \end{split}$$

$$H_{+\beta-\alpha}^{(1)} = b_{0,0,-1,-1}^{i} \rho_{\alpha} \sin{(\phi_{\alpha})} + i b_{0,0,-1,-1}^{i} \rho_{\alpha} \cos{(\phi_{\alpha})} + b_{0,0,0,-1}^{i} \rho_{\beta} \sin{(\phi_{\beta})} + i b_{0,0,0,-1}^{i} \rho_{\beta} \cos{(\phi_{\beta})} - i b_{0,0,-1,-1}^{r} \rho_{\alpha} \sin{(\phi_{\alpha})} + b_{0,0,-1,-1}^{r} \rho_{\alpha} \cos{(\phi_{\alpha})} - i b_{0,0,0,-1}^{r} \rho_{\beta} \sin{(\phi_{\beta})} + b_{0,0,0,-1}^{r} \rho_{\beta} \cos{(\phi_{\beta})}$$

$$\begin{split} H_{-_{\beta}+_{\alpha}}^{(1)} &= b_{0,0,-1,-1}^{i} \rho_{\alpha} \sin{(\phi_{\alpha})} - i b_{0,0,-1,-1}^{i} \rho_{\alpha} \cos{(\phi_{\alpha})} + b_{0,0,0,-1}^{i} \rho_{\beta} \sin{(\phi_{\beta})} - i b_{0,0,0,-1}^{i} \rho_{\beta} \cos{(\phi_{\beta})} \\ &+ i b_{0,0,-1,-1}^{r} \rho_{\alpha} \sin{(\phi_{\alpha})} + b_{0,0,-1,-1}^{r} \rho_{\alpha} \cos{(\phi_{\alpha})} + i b_{0,0,0,-1}^{r} \rho_{\beta} \sin{(\phi_{\beta})} + b_{0,0,0,-1}^{r} \rho_{\beta} \cos{(\phi_{\beta})} \end{split}$$

$$H_{+\alpha+\beta}^{(1)} = 0$$

$$H_{-\alpha^{-\beta}}^{(1)} = 0$$

$$H_{+\beta+\alpha}^{(1)} = 0$$

$$H_{-\beta-\alpha}^{(1)}=0$$

$$H_{+\alpha-\beta}^{(1)}=ib_{0,0,-1,-1}^{i}x_{\alpha}+b_{0,0,-1,-1}^{i}y_{\alpha}+ib_{0,0,0,-1}^{i}x_{\beta}+b_{0,0,0,-1}^{i}y_{\beta}+b_{0,0,-1,-1}^{r}x_{\alpha}-ib_{0,0,-1,-1}^{r}y_{\alpha}+b_{0,0,0,-1}^{r}x_{\beta}-ib_{0,0,0,-1}^{r}y_{\beta}+b_{0,0,0,-1,-1}^{r}x_{\alpha}-ib_{0,0,0,-1}^{r}x_{\beta}-ib_{0,0,0,-1}^{r}x_{\beta}+b_{0,0,0,-1}^{$$

$$H_{-\alpha+\beta}^{(1)} = -ib_{0,0,-1,-1}^{i}x_{\alpha} + b_{0,0,-1,-1}^{i}y_{\alpha} - ib_{0,0,0,-1}^{i}x_{\beta} + b_{0,0,0,-1}^{i}y_{\beta} + b_{0,0,-1,-1}^{r}x_{\alpha} + ib_{0,0,-1,-1}^{r}y_{\alpha} + b_{0,0,0,-1}^{r}x_{\beta} + ib_{0,0,0,-1}^{r}y_{\beta} + b_{0,0,0,-1}^{r}x_{\beta} + ib_{0,0,0,-1}^{r}x_{\beta} + ib$$

$$H^{(1)}_{+_{\beta}-\alpha}=ib^{i}_{0,0,-1,-1}x_{\alpha}+b^{i}_{0,0,-1,-1}y_{\alpha}+ib^{i}_{0,0,0,-1}x_{\beta}+b^{i}_{0,0,0,-1}y_{\beta}+b^{r}_{0,0,-1,-1}x_{\alpha}-ib^{r}_{0,0,-1,-1}y_{\alpha}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}y_{\beta}+b^{r}_{0,0,0,-1,-1}x_{\alpha}-ib^{r}_{0,0,0,-1,-1}y_{\alpha}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}y_{\beta}+b^{r}_{0,0,0,-1,-1}x_{\alpha}-ib^{r}_{0,0,0,-1,-1}y_{\alpha}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1,-1}x_{\alpha}-ib^{r}_{0,0,0,-1,-1}y_{\alpha}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1,-1}x_{\alpha}-ib^{r}_{0,0,0,-1,-1}y_{\alpha}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{0,0,0,-1}x_{\beta}+b^{r}_{0,0,0,-1}x_{\beta}-ib^{r}_{$$

$$H_{-\beta+\alpha}^{(1)} = -ib_{0,0,-1,-1}^{i}x_{\alpha} + b_{0,0,-1,-1}^{i}y_{\alpha} - ib_{0,0,0,-1}^{i}x_{\beta} + b_{0,0,0,-1}^{i}y_{\beta} + b_{0,0,-1,-1}^{r}x_{\alpha} + ib_{0,0,-1,-1}^{r}y_{\alpha} + b_{0,0,0,-1}^{r}x_{\beta} + ib_{0,0,0,-1}^{r}y_{\beta} + b_{0,0,0,-1}^{r}x_{\beta} + ib_{0,0,0,-1}^{r}x_{\beta} + ib$$

3.3 Order: 2

Number of fitting parameters: $H_{+_{\alpha}+_{\beta}}$: 8, $H_{+_{\alpha}-_{\beta}}$: 6.

$$\begin{split} H_{+\alpha+\beta}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + i a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ i a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + i a_{0,2,0,0}^{i} \rho_{\beta}^{2} + i a_{2,0,0,0}^{i} \rho_{\alpha}^{2} - i a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + i a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\alpha}^{2} \end{split}$$

$$\begin{split} H_{-\alpha-\beta}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) - i a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &- i a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - i a_{0,2,0,0}^{i} \rho_{\beta}^{2} - i a_{2,0,0,0}^{i} \rho_{\alpha}^{2} + i a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - i a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\alpha}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} \end{split}$$

$$\begin{split} H_{+\beta+\alpha}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) - i a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &- i a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - i a_{0,2,0,0}^{i} \rho_{\beta}^{2} - i a_{2,0,0,0}^{i} \rho_{\alpha}^{2} + i a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - i a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\alpha}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} \end{split}$$

$$\begin{split} H_{-\beta-\alpha}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + i a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ i a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + i a_{0,2,0,0}^{i} \rho_{\beta}^{2} + i a_{2,0,0,0}^{i} \rho_{\alpha}^{2} - i a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + i a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\alpha}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} \end{split}$$

$$\begin{split} H_{+\alpha-\beta}^{(2)} &= -b_{0,0,0,2}^{i} \rho_{\beta}^{2} \sin{(2\phi_{\beta})} + i b_{0,0,0,2}^{i} \rho_{\beta}^{2} \cos{(2\phi_{\beta})} - b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \sin{(\phi_{\alpha} + \phi_{\beta})} + i b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \cos{(\phi_{\alpha} + \phi_{\beta})} \\ &- b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \sin{(2\phi_{\alpha})} + i b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \cos{(2\phi_{\alpha})} + i b_{0,0,0,2}^{r} \rho_{\beta}^{2} \sin{(2\phi_{\beta})} + b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos{(2\phi_{\beta})} \\ &+ i b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \sin{(\phi_{\alpha} + \phi_{\beta})} + b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos{(\phi_{\alpha} + \phi_{\beta})} + i b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \sin{(2\phi_{\alpha})} + b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos{(2\phi_{\alpha})} \end{split}$$

$$\begin{split} H_{-\alpha+\beta}^{(2)} &= -b_{0,0,0,2}^{i} \rho_{\beta}^{2} \sin{(2\phi_{\beta})} - ib_{0,0,0,2}^{i} \rho_{\beta}^{2} \cos{(2\phi_{\beta})} - b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \sin{(\phi_{\alpha}+\phi_{\beta})} - ib_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \cos{(\phi_{\alpha}+\phi_{\beta})} \\ &- b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \sin{(2\phi_{\alpha})} - ib_{0,0,2,2}^{i} \rho_{\alpha}^{2} \cos{(2\phi_{\alpha})} - ib_{0,0,0,2}^{r} \rho_{\beta}^{2} \sin{(2\phi_{\beta})} + b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos{(2\phi_{\beta})} \\ &- ib_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \sin{(\phi_{\alpha}+\phi_{\beta})} + b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos{(\phi_{\alpha}+\phi_{\beta})} - ib_{0,0,2,2}^{r} \rho_{\alpha}^{2} \sin{(2\phi_{\alpha})} + b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos{(2\phi_{\alpha})} \end{split}$$

$$\begin{split} H_{+\beta-\alpha}^{(2)} &= -b_{0,0,0,2}^{i} \rho_{\beta}^{2} \sin{(2\phi_{\beta})} + i b_{0,0,0,2}^{i} \rho_{\beta}^{2} \cos{(2\phi_{\beta})} - b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \sin{(\phi_{\alpha} + \phi_{\beta})} + i b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \cos{(\phi_{\alpha} + \phi_{\beta})} \\ &- b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \sin{(2\phi_{\alpha})} + i b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \cos{(2\phi_{\alpha})} + i b_{0,0,0,2}^{r} \rho_{\beta}^{2} \sin{(2\phi_{\beta})} + b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos{(2\phi_{\beta})} \\ &+ i b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \sin{(\phi_{\alpha} + \phi_{\beta})} + b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos{(\phi_{\alpha} + \phi_{\beta})} + i b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \sin{(2\phi_{\alpha})} + b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos{(2\phi_{\alpha})} \end{split}$$

$$\begin{split} H^{(2)}_{-\beta+\alpha} &= -b^i_{0,0,0,2} \rho^2_\beta \sin{(2\phi_\beta)} - i b^i_{0,0,0,2} \rho^2_\beta \cos{(2\phi_\beta)} - b^i_{0,0,1,2} \rho_\alpha \rho_\beta \sin{(\phi_\alpha+\phi_\beta)} - i b^i_{0,0,1,2} \rho_\alpha \rho_\beta \cos{(\phi_\alpha+\phi_\beta)} \\ &\quad - b^i_{0,0,2,2} \rho^2_\alpha \sin{(2\phi_\alpha)} - i b^i_{0,0,2,2} \rho^2_\alpha \cos{(2\phi_\alpha)} - i b^r_{0,0,0,2} \rho^2_\beta \sin{(2\phi_\beta)} + b^r_{0,0,0,2} \rho^2_\beta \cos{(2\phi_\beta)} \\ &\quad - i b^r_{0,0,1,2} \rho_\alpha \rho_\beta \sin{(\phi_\alpha+\phi_\beta)} + b^r_{0,0,1,2} \rho_\alpha \rho_\beta \cos{(\phi_\alpha+\phi_\beta)} - i b^r_{0,0,2,2} \rho^2_\alpha \sin{(2\phi_\alpha)} + b^r_{0,0,2,2} \rho^2_\alpha \cos{(2\phi_\alpha)} \end{split}$$

$$\begin{split} H_{+\alpha+\beta}^{(2)} &= ia_{0,0,-1,0}^{i}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,0,-1,0}^{i}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + ia_{0,0,1,0}^{i}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + a_{0,0,1,0}^{i}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &+ ia_{0,2,0,0}^{i}\left(x_{\beta}^{2} + y_{\beta}^{2}\right) + ia_{2,0,0,0}^{i}\left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) + a_{0,0,-1,0}^{r}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + ia_{0,0,-1,0}^{r}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &+ a_{0,0,1,0}^{r}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - ia_{0,0,1,0}^{r}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,2,0,0}^{r}\left(x_{\beta}^{2} + y_{\beta}^{2}\right) + a_{2,0,0,0}^{r}\left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) \end{split}$$

$$\begin{split} H_{-\alpha-\beta}^{(2)} &= -ia_{0,0,-1,0}^{i}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,0,-1,0}^{i}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) - ia_{0,0,1,0}^{i}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + a_{0,0,1,0}^{i}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &\quad - ia_{0,2,0,0}^{i}\left(x_{\beta}^{2} + y_{\beta}^{2}\right) - ia_{2,0,0,0}^{i}\left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) + a_{0,0,-1,0}^{r}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - ia_{0,0,-1,0}^{r}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &\quad + a_{0,0,1,0}^{r}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + ia_{0,0,1,0}^{r}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,2,0,0}^{r}\left(x_{\beta}^{2} + y_{\beta}^{2}\right) + a_{2,0,0,0}^{r}\left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) \end{split}$$

$$\begin{split} H_{+\beta+\alpha}^{(2)} &= -ia_{0,0,-1,0}^{i}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,0,-1,0}^{i}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) - ia_{0,0,1,0}^{i}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + a_{0,0,1,0}^{i}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &\quad - ia_{0,2,0,0}^{i}\left(x_{\beta}^{2} + y_{\beta}^{2}\right) - ia_{2,0,0,0}^{i}\left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) + a_{0,0,-1,0}^{r}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - ia_{0,0,-1,0}^{r}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &\quad + a_{0,0,1,0}^{r}\left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + ia_{0,0,1,0}^{r}\left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,2,0,0}^{r}\left(x_{\beta}^{2} + y_{\beta}^{2}\right) + a_{2,0,0,0}^{r}\left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) \end{split}$$

$$\begin{split} H_{-\beta-\alpha}^{(2)} &= ia_{0,0,-1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,0,-1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + ia_{0,0,1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + a_{0,0,1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &+ ia_{0,2,0,0}^{i} \left(x_{\beta}^{2} + y_{\beta}^{2}\right) + ia_{2,0,0,0}^{i} \left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) + a_{0,0,-1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + ia_{0,0,-1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) \\ &+ a_{0,0,1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - ia_{0,0,1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,2,0,0}^{r} \left(x_{\beta}^{2} + y_{\beta}^{2}\right) + a_{2,0,0,0}^{r} \left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) \end{split}$$

$$\begin{split} H_{+_{\alpha}-_{\beta}}^{(2)} &= -2b_{0,0,0,2}^{i}x_{\beta}y_{\beta} + ib_{0,0,0,2}^{i}\left(x_{\beta} - y_{\beta}\right)\left(x_{\beta} + y_{\beta}\right) + ib_{0,0,1,2}^{i}\left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) - b_{0,0,1,2}^{i}\left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) \\ &\quad - 2b_{0,0,2,2}^{i}x_{\alpha}y_{\alpha} + ib_{0,0,2,2}^{i}\left(x_{\alpha} - y_{\alpha}\right)\left(x_{\alpha} + y_{\alpha}\right) + 2ib_{0,0,0,2}^{r}x_{\beta}y_{\beta} + b_{0,0,0,2}^{r}\left(x_{\beta} - y_{\beta}\right)\left(x_{\beta} + y_{\beta}\right) \\ &\quad + b_{0,0,1,2}^{r}\left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) + ib_{0,0,1,2}^{r}\left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) + 2ib_{0,0,2,2}^{r}x_{\alpha}y_{\alpha} + b_{0,0,2,2}^{r}\left(x_{\alpha} - y_{\alpha}\right)\left(x_{\alpha} + y_{\alpha}\right) \end{split}$$

$$\begin{split} H_{-\alpha+\beta}^{(2)} &= -2b_{0,0,0,2}^{i}x_{\beta}y_{\beta} - ib_{0,0,0,2}^{i}\left(x_{\beta} - y_{\beta}\right)\left(x_{\beta} + y_{\beta}\right) - ib_{0,0,1,2}^{i}\left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) - b_{0,0,1,2}^{i}\left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) \\ &\quad - 2b_{0,0,2,2}^{i}x_{\alpha}y_{\alpha} - ib_{0,0,2,2}^{i}\left(x_{\alpha} - y_{\alpha}\right)\left(x_{\alpha} + y_{\alpha}\right) - 2ib_{0,0,0,2}^{r}x_{\beta}y_{\beta} + b_{0,0,0,2}^{r}\left(x_{\beta} - y_{\beta}\right)\left(x_{\beta} + y_{\beta}\right) \\ &\quad + b_{0,0,1,2}^{r}\left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) - ib_{0,0,1,2}^{r}\left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) - 2ib_{0,0,2,2}^{r}x_{\alpha}y_{\alpha} + b_{0,0,2,2}^{r}\left(x_{\alpha} - y_{\alpha}\right)\left(x_{\alpha} + y_{\alpha}\right) \end{split}$$

$$H_{+\beta-\alpha}^{(2)} = -2b_{0,0,0,2}^{i}x_{\beta}y_{\beta} + ib_{0,0,0,2}^{i}(x_{\beta} - y_{\beta})(x_{\beta} + y_{\beta}) + ib_{0,0,1,2}^{i}(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}) - b_{0,0,1,2}^{i}(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}) - 2b_{0,0,2,2}^{i}x_{\alpha}y_{\alpha} + ib_{0,0,2,2}^{i}(x_{\alpha} - y_{\alpha})(x_{\alpha} + y_{\alpha}) + 2ib_{0,0,0,2}^{r}x_{\beta}y_{\beta} + b_{0,0,0,2}^{r}(x_{\beta} - y_{\beta})(x_{\beta} + y_{\beta}) + b_{0,0,1,2}^{r}(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}) + ib_{0,0,1,2}^{r}(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}) + 2ib_{0,0,2,2}^{r}x_{\alpha}y_{\alpha} + b_{0,0,2,2}^{r}(x_{\alpha} - y_{\alpha})(x_{\alpha} + y_{\alpha})$$

$$\begin{split} H_{-\beta+\alpha}^{(2)} &= -2b_{0,0,0,2}^{i}x_{\beta}y_{\beta} - ib_{0,0,0,2}^{i}\left(x_{\beta} - y_{\beta}\right)\left(x_{\beta} + y_{\beta}\right) - ib_{0,0,1,2}^{i}\left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) - b_{0,0,1,2}^{i}\left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) \\ &\quad - 2b_{0,0,2,2}^{i}x_{\alpha}y_{\alpha} - ib_{0,0,2,2}^{i}\left(x_{\alpha} - y_{\alpha}\right)\left(x_{\alpha} + y_{\alpha}\right) - 2ib_{0,0,0,2}^{r}x_{\beta}y_{\beta} + b_{0,0,0,2}^{r}\left(x_{\beta} - y_{\beta}\right)\left(x_{\beta} + y_{\beta}\right) \\ &\quad + b_{0,0,1,2}^{r}\left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) - ib_{0,0,1,2}^{r}\left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) - 2ib_{0,0,2,2}^{r}x_{\alpha}y_{\alpha} + b_{0,0,2,2}^{r}\left(x_{\alpha} - y_{\alpha}\right)\left(x_{\alpha} + y_{\alpha}\right) \end{split}$$

4 Vibronic Hamiltonian operator in the real E basis

$$\begin{split} \hat{H} &= \begin{pmatrix} |X_{\alpha}\rangle & |Y_{\alpha}\rangle & |X_{\beta}\rangle & |Y_{\beta}\rangle \end{pmatrix} \begin{pmatrix} 0 & 0 & H_{X_{\alpha}X_{\beta}} & H_{X_{\alpha}Y_{\beta}} \\ 0 & 0 & H_{Y_{\alpha}X_{\beta}} & H_{Y_{\alpha}Y_{\beta}} \\ H_{X_{\beta}X_{\alpha}} & H_{X_{\beta}Y_{\alpha}} & 0 & 0 \\ H_{Y_{\beta}X_{\alpha}} & H_{Y_{\beta}Y_{\alpha}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle X_{\alpha}| \\ \langle Y_{\alpha}| \\ \langle X_{\beta}| \\ \langle Y_{\beta}| \end{pmatrix} \\ H_{+\alpha+\beta} &= H_{+\alpha+\beta} = H_{-\alpha-\beta} \stackrel{*}{=} H_{+\beta+\alpha} \stackrel{*}{=} H_{-\beta-\alpha} \\ H_{+\alpha-\beta} &= H_{+\alpha-\beta} = H_{-\alpha+\beta} \stackrel{*}{=} H_{+\beta-\alpha} = H_{-\beta+\alpha} \end{split}$$

5 Matrix element expansions in the real E basis

5.1 Order: 0

Number of fitting parameters: $H_{X_{\alpha}X_{\beta}}$: 1 (all from $H_{+_{\alpha}+_{\beta}}$), $H_{X_{\alpha}Y_{\beta}}$: 1 (all from $H_{+_{\alpha}+_{\beta}}$), $H_{Y_{\alpha}X_{\beta}}$: 1 (all from $H_{+_{\alpha}+_{\beta}}$), $H_{Y_{\alpha}Y_{\beta}}$: 1 (all from $H_{+_{\alpha}+_{\beta}}$), $H_{X_{\beta}X_{\alpha}}$: 1 (all from $H_{+_{\alpha}+_{\beta}}$), $H_{Y_{\beta}X_{\alpha}}$: 1 (all from $H_{+_{\alpha}+_{\beta}}$), $H_{Y_{\beta}X_{\alpha}}$: 1 (all from $H_{+_{\alpha}+_{\beta}}$).

Polar e-coordinates:

$$H^{(0)}_{X_{\alpha}X_{\beta}} = a^r_{0,0,0,0}$$

$$H^{(0)}_{X_{\alpha}Y_{\beta}}=a^i_{0,0,0,0}$$

$$H^{(0)}_{Y_{\alpha}X_{\beta}} = -a^i_{0,0,0,0}$$

$$H^{(0)}_{Y_{\alpha}Y_{\beta}} = a^r_{0,0,0,0}$$

$$H^{(0)}_{X_{\beta}X_{\alpha}} = a^r_{0,0,0,0}$$

$$H^{(0)}_{X_{\beta}Y_{\alpha}} = -a^i_{0,0,0,0}$$

$$H^{(0)}_{Y_{\beta}X_{\alpha}} = a^i_{0,0,0,0}$$

$$H_{Y_{\beta}Y_{\alpha}}^{(0)} = a_{0,0,0,0}^r$$

Cartesian e-coordinates:

$$H^{(0)}_{X_{\alpha}X_{\beta}} = a^r_{0,0,0,0}$$

$$H^{(0)}_{X_\alpha Y_\beta} = a^i_{0,0,0,0}$$

$$H_{Y_{\alpha}X_{\beta}}^{(0)} = -a_{0,0,0,0}^{i}$$

$$H^{(0)}_{Y_{\alpha}Y_{\beta}}=a^r_{0,0,0,0}$$

$$H^{(0)}_{X_{\beta}X_{\alpha}} = a^r_{0,0,0,0}$$

$$H^{(0)}_{X_{\beta}Y_{\alpha}} = -a^{i}_{0,0,0,0}$$

$$H^{(0)}_{Y_{\beta}X_{\alpha}}=a^i_{0,0,0,0}$$

$$H^{(0)}_{Y_{\beta}Y_{\alpha}} = a^r_{0,0,0,0}$$

5.2 Order: 1

Number of fitting parameters: $H_{X_{\alpha}X_{\beta}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$), $H_{X_{\alpha}Y_{\beta}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$), $H_{Y_{\alpha}X_{\beta}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$), $H_{Y_{\alpha}Y_{\alpha}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$), $H_{X_{\beta}X_{\alpha}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$), $H_{Y_{\beta}X_{\alpha}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$), $H_{Y_{\beta}X_{\alpha}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$), $H_{Y_{\beta}X_{\alpha}}$: 4 (all from $H_{+_{\alpha}-_{\beta}}$).

Polar e-coordinates:

$$H_{X_{\alpha}X_{\beta}}^{(1)} = b_{0,0,-1,-1}^{i}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{i}\rho_{\beta}\sin\left(\phi_{\beta}\right) + b_{0,0,-1,-1}^{r}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\cos\left(\phi_{\beta}\right)$$

$$H_{X_{\alpha}Y_{\beta}}^{(1)} = -b_{0,0,-1,-1}^{i}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{i}\rho_{\beta}\cos\left(\phi_{\beta}\right) + b_{0,0,-1,-1}^{r}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\sin\left(\phi_{\beta}\right)$$

$$H_{Y_{\alpha}X_{\beta}}^{(1)} = -b_{0,0,-1,-1}^{i}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{i}\rho_{\beta}\cos\left(\phi_{\beta}\right) + b_{0,0,-1,-1}^{r}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\sin\left(\phi_{\beta}\right)$$

$$H_{Y_{\alpha}Y_{\beta}}^{(1)} = -b_{0,0,-1,-1}^{i}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{i}\rho_{\beta}\sin\left(\phi_{\beta}\right) - b_{0,0,-1,-1}^{r}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{r}\rho_{\beta}\cos\left(\phi_{\beta}\right)$$

$$H_{X_{\beta}X_{\alpha}}^{(1)} = b_{0,0,-1,-1}^{i}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{i}\rho_{\beta}\sin\left(\phi_{\beta}\right) + b_{0,0,-1,-1}^{r}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\cos\left(\phi_{\beta}\right)$$

$$H_{X_{\beta}Y_{\alpha}}^{(1)} = -b_{0,0,-1,-1}^{i}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{i}\rho_{\beta}\cos\left(\phi_{\beta}\right) + b_{0,0,-1,-1}^{r}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\sin\left(\phi_{\beta}\right)$$

$$H_{Y_{\beta}X_{\alpha}}^{(1)} = -b_{0,0,-1,-1}^{i}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{i}\rho_{\beta}\cos\left(\phi_{\beta}\right) + b_{0,0,-1,-1}^{r}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\sin\left(\phi_{\beta}\right)$$

$$H_{Y_{\beta}Y_{\alpha}}^{(1)} = -b_{0,0,-1,-1}^{i}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{i}\rho_{\beta}\sin\left(\phi_{\beta}\right) - b_{0,0,-1,-1}^{r}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{r}\rho_{\beta}\cos\left(\phi_{\beta}\right)$$

Cartesian e-coordinates:

$$H^{(1)}_{X_{\alpha}X_{\beta}} = b^{i}_{0,0,-1,-1}y_{\alpha} + b^{i}_{0,0,0,-1}y_{\beta} + b^{r}_{0,0,-1,-1}x_{\alpha} + b^{r}_{0,0,0,-1}x_{\beta}$$

$$H_{X_{\alpha}Y_{\beta}}^{(1)} = -b_{0,0,-1,-1}^{i} x_{\alpha} - b_{0,0,0,-1}^{i} x_{\beta} + b_{0,0,-1,-1}^{r} y_{\alpha} + b_{0,0,0,-1}^{r} y_{\beta}$$

$$H^{(1)}_{Y_{\alpha}X_{\beta}} = -b^{i}_{0,0,-1,-1}x_{\alpha} - b^{i}_{0,0,0,-1}x_{\beta} + b^{r}_{0,0,-1,-1}y_{\alpha} + b^{r}_{0,0,0,-1}y_{\beta}$$

$$H_{Y_{\alpha}Y_{\beta}}^{(1)} = -b_{0,0,-1,-1}^{i} y_{\alpha} - b_{0,0,0,-1}^{i} y_{\beta} - b_{0,0,-1,-1}^{r} x_{\alpha} - b_{0,0,0,-1}^{r} x_{\beta}$$

$$H_{X_{\beta}X_{\alpha}}^{(1)} = b_{0,0,-1,-1}^{i} y_{\alpha} + b_{0,0,0,-1}^{i} y_{\beta} + b_{0,0,-1,-1}^{r} x_{\alpha} + b_{0,0,0,-1}^{r} x_{\beta}$$

$$H^{(1)}_{X_{\beta}Y_{\alpha}} = -b^{i}_{0,0,-1,-1}x_{\alpha} - b^{i}_{0,0,0,-1}x_{\beta} + b^{r}_{0,0,-1,-1}y_{\alpha} + b^{r}_{0,0,0,-1}y_{\beta}$$

$$H_{Y_{\beta}X_{\alpha}}^{(1)} = -b_{0,0,-1,-1}^{i} x_{\alpha} - b_{0,0,0,-1}^{i} x_{\beta} + b_{0,0,-1,-1}^{r} y_{\alpha} + b_{0,0,0,-1}^{r} y_{\beta}$$

$$H^{(1)}_{Y_{\beta}Y_{\alpha}} = -b^{i}_{0,0,-1,-1}y_{\alpha} - b^{i}_{0,0,0,-1}y_{\beta} - b^{r}_{0,0,-1,-1}x_{\alpha} - b^{r}_{0,0,0,-1}x_{\beta}$$

5.3 Order: 2

Number of fitting parameters: $H_{X_{\alpha}X_{\beta}}$: 12 (6 from $H_{+_{\alpha}+_{\beta}}$, 6 from $H_{+_{\alpha}-_{\beta}}$), $H_{X_{\alpha}Y_{\beta}}$: 12 (6 from $H_{+_{\alpha}+_{\beta}}$, 6 from $H_{+_{\alpha}-_{\beta}}$), $H_{Y_{\alpha}X_{\beta}}$: 12 (6 from $H_{+_{\alpha}+_{\beta}}$, 6 from $H_{+_{\alpha}-_{\beta}}$), $H_{X_{\beta}X_{\alpha}}$: 12 (6 from $H_{+_{\alpha}+_{\beta}}$, 6 from $H_{+_{\alpha}+_{\beta}}$), $H_{X_{\beta}X_{\alpha}}$: 12 (6 from $H_{+_{\alpha}+_{\beta}}$), $H_{X_{\beta}X_{\alpha}}$: 12 (6 from $H_{+_{\alpha}+_{\beta}}$), $H_{Y_{\beta}Y_{\alpha}}$: 12 (6 from $H_{+_{\alpha}+_{\beta}}$), 6 from $H_{+_{\alpha}+_{\beta}}$).

$$\begin{split} H_{X_{\alpha}X_{\beta}}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\beta}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} - b_{0,0,0,2}^{i} \rho_{\beta}^{2} \sin \left(2\phi_{\beta}\right) - b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} + \phi_{\beta}\right) \\ &- b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \sin \left(2\phi_{\alpha}\right) + b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos \left(2\phi_{\beta}\right) + b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} + \phi_{\beta}\right) + b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos \left(2\phi_{\alpha}\right) \end{split}$$

$$\begin{split} H_{X_{\alpha}Y_{\beta}}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{i} \rho_{\beta}^{2} + a_{2,0,0,0}^{i} \rho_{\alpha}^{2} - a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) - b_{0,0,0,2}^{i} \rho_{\beta}^{2} \cos \left(2\phi_{\beta}\right) - b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} + \phi_{\beta}\right) \\ &- b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \cos \left(2\phi_{\alpha}\right) - b_{0,0,0,2}^{r} \rho_{\beta}^{2} \sin \left(2\phi_{\beta}\right) - b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} + \phi_{\beta}\right) - b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \sin \left(2\phi_{\alpha}\right) \end{split}$$

$$\begin{split} H_{Y_{\alpha}X_{\beta}}^{(2)} &= -a_{0,0,-1,0}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,2,0,0}^{i}\rho_{\beta}^{2} - a_{2,0,0,0}^{i}\rho_{\alpha}^{2} \\ &+ a_{0,0,-1,0}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha} - \phi_{\beta}\right) - b_{0,0,0,2}^{i}\rho_{\beta}^{2}\cos\left(2\phi_{\beta}\right) - b_{0,0,1,2}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha} + \phi_{\beta}\right) \\ &- b_{0,0,2,2}^{i}\rho_{\alpha}^{2}\cos\left(2\phi_{\alpha}\right) - b_{0,0,0,2}^{r}\rho_{\beta}^{2}\sin\left(2\phi_{\beta}\right) - b_{0,0,1,2}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha} + \phi_{\beta}\right) - b_{0,0,2,2}^{r}\rho_{\alpha}^{2}\sin\left(2\phi_{\alpha}\right) \end{split}$$

$$\begin{split} H_{Y_{\alpha}Y_{\beta}}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\beta}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} + b_{0,0,0,2}^{i} \rho_{\beta}^{2} \sin \left(2\phi_{\beta}\right) + b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} + \phi_{\beta}\right) \\ &+ b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \sin \left(2\phi_{\alpha}\right) - b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos \left(2\phi_{\beta}\right) - b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} + \phi_{\beta}\right) - b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos \left(2\phi_{\alpha}\right) \end{split}$$

$$\begin{split} H_{X_{\beta}X_{\alpha}}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\beta}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} - b_{0,0,0,2}^{i} \rho_{\beta}^{2} \sin \left(2\phi_{\beta}\right) - b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} + \phi_{\beta}\right) \\ &- b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \sin \left(2\phi_{\alpha}\right) + b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos \left(2\phi_{\beta}\right) + b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} + \phi_{\beta}\right) + b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos \left(2\phi_{\alpha}\right) \end{split}$$

$$\begin{split} H_{X_{\beta}Y_{\alpha}}^{(2)} &= -a_{0,0,-1,0}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,2,0,0}^{i}\rho_{\beta}^{2} - a_{2,0,0,0}^{i}\rho_{\alpha}^{2} \\ &+ a_{0,0,-1,0}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha} - \phi_{\beta}\right) - b_{0,0,2,2}^{i}\rho_{\beta}^{2}\cos\left(2\phi_{\beta}\right) - b_{0,0,1,2}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha} + \phi_{\beta}\right) \\ &- b_{0,0,2,2}^{i}\rho_{\alpha}^{2}\cos\left(2\phi_{\alpha}\right) - b_{0,0,0,2}^{r}\rho_{\beta}^{2}\sin\left(2\phi_{\beta}\right) - b_{0,0,1,2}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha} + \phi_{\beta}\right) - b_{0,0,2,2}^{r}\rho_{\alpha}^{2}\sin\left(2\phi_{\alpha}\right) \end{split}$$

$$\begin{split} H_{Y_{\beta}X_{\alpha}}^{(2)} &= a_{0,0,-1,0}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{0,0,1,0}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{0,2,0,0}^{i}\rho_{\beta}^{2} + a_{2,0,0,0}^{i}\rho_{\alpha}^{2} - a_{0,0,-1,0}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha}-\phi_{\beta}\right) \\ &+ a_{0,0,1,0}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha}-\phi_{\beta}\right) - b_{0,0,0,2}^{i}\rho_{\beta}^{2}\cos\left(2\phi_{\beta}\right) - b_{0,0,1,2}^{i}\rho_{\alpha}\rho_{\beta}\cos\left(\phi_{\alpha}+\phi_{\beta}\right) \\ &- b_{0,0,2,2}^{i}\rho_{\alpha}^{2}\cos\left(2\phi_{\alpha}\right) - b_{0,0,0,2}^{r}\rho_{\beta}^{2}\sin\left(2\phi_{\beta}\right) - b_{0,0,1,2}^{r}\rho_{\alpha}\rho_{\beta}\sin\left(\phi_{\alpha}+\phi_{\beta}\right) - b_{0,0,2,2}^{r}\rho_{\alpha}^{2}\sin\left(2\phi_{\alpha}\right) \end{split}$$

$$\begin{split} H_{Y_{\beta}Y_{\alpha}}^{(2)} &= a_{0,0,-1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) - a_{0,0,1,0}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,0,-1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) \\ &+ a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,2,0,0}^{r} \rho_{\beta}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} + b_{0,0,0,2}^{i} \rho_{\beta}^{2} \sin \left(2\phi_{\beta}\right) + b_{0,0,1,2}^{i} \rho_{\alpha} \rho_{\beta} \sin \left(\phi_{\alpha} + \phi_{\beta}\right) \\ &+ b_{0,0,2,2}^{i} \rho_{\alpha}^{2} \sin \left(2\phi_{\alpha}\right) - b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos \left(2\phi_{\beta}\right) - b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos \left(\phi_{\alpha} + \phi_{\beta}\right) - b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos \left(2\phi_{\alpha}\right) \end{split}$$

$$H_{X_{\alpha}X_{\beta}}^{(2)} = -a_{0,0,-1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha} \right) + a_{0,0,1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha} \right) + a_{0,0,-1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta} \right) + a_{0,0,1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta} \right) + a_{0,0,0,0}^{r} \left(x_{\alpha}^{2} + y_{\beta}^{2} \right) + a_{2,0,0,0}^{r} \left(x_{\alpha}^{2} + y_{\alpha}^{2} \right) - 2b_{0,0,0,2}^{i} x_{\beta}y_{\beta} - b_{0,0,1,2}^{i} \left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha} \right) - 2b_{0,0,2,2}^{i} x_{\alpha}y_{\alpha} + b_{0,0,0,2}^{r} \left(x_{\beta} - y_{\beta} \right) \left(x_{\beta} + y_{\beta} \right) + b_{0,0,1,2}^{r} \left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta} \right) + b_{0,0,2,2}^{r} \left(x_{\alpha} - y_{\alpha} \right) \left(x_{\alpha} + y_{\alpha} \right)$$

$$\begin{split} H_{X_{\alpha}Y_{\beta}}^{(2)} &= a_{0,0,-1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta} \right) + a_{0,0,1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta} \right) + a_{0,2,0,0}^{i} \left(x_{\beta}^{2} + y_{\beta}^{2} \right) + a_{2,0,0,0}^{i} \left(x_{\alpha}^{2} + y_{\alpha}^{2} \right) \\ &+ a_{0,0,-1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha} \right) - a_{0,0,1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha} \right) - b_{0,0,0,2}^{i} \left(x_{\beta} - y_{\beta} \right) \left(x_{\beta} + y_{\beta} \right) - b_{0,0,1,2}^{i} \left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta} \right) \\ &- b_{0,0,2,2}^{i} \left(x_{\alpha} - y_{\alpha} \right) \left(x_{\alpha} + y_{\alpha} \right) - 2b_{0,0,0,2}^{r} x_{\beta}y_{\beta} - b_{0,0,1,2}^{r} \left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha} \right) - 2b_{0,0,2,2}^{r} x_{\alpha}y_{\alpha} \end{split}$$

$$\begin{split} H_{Y_{\alpha}X_{\beta}}^{(2)} &= -a_{0,0,-1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,0,1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,2,0,0}^{i} \left(x_{\beta}^{2} + y_{\beta}^{2}\right) - a_{2,0,0,0}^{i} \left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) \\ &- a_{0,0,-1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,0,1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) - b_{0,0,0,2}^{i} \left(x_{\beta} - y_{\beta}\right) \left(x_{\beta} + y_{\beta}\right) - b_{0,0,1,2}^{i} \left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) \\ &- b_{0,0,2,2}^{i} \left(x_{\alpha} - y_{\alpha}\right) \left(x_{\alpha} + y_{\alpha}\right) - 2b_{0,0,0,2}^{r} x_{\beta}y_{\beta} - b_{0,0,1,2}^{r} \left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) - 2b_{0,0,2,2}^{r} x_{\alpha}y_{\alpha} \end{split}$$

$$\begin{split} H_{Y_{\alpha}Y_{\beta}}^{(2)} &= -a_{0,0,-1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,0,1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,0,-1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + a_{0,0,1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) \\ &+ a_{0,2,0,0}^{r} \left(x_{\beta}^{2} + y_{\beta}^{2}\right) + a_{2,0,0,0}^{r} \left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) + 2b_{0,0,0,2}^{i} x_{\beta}y_{\beta} + b_{0,0,1,2}^{i} \left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) + 2b_{0,0,2,2}^{i} x_{\alpha}y_{\alpha} \\ &- b_{0,0,0,2}^{r} \left(x_{\beta} - y_{\beta}\right) \left(x_{\beta} + y_{\beta}\right) - b_{0,0,1,2}^{r} \left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) - b_{0,0,2,2}^{r} \left(x_{\alpha} - y_{\alpha}\right) \left(x_{\alpha} + y_{\alpha}\right) \end{split}$$

$$\begin{split} H_{X_{\beta}X_{\alpha}}^{(2)} &= -a_{0,0,-1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,0,1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,0,-1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) + a_{0,0,1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) \\ &+ a_{0,2,0,0}^{r} \left(x_{\beta}^{2} + y_{\beta}^{2}\right) + a_{2,0,0,0}^{r} \left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) - 2b_{0,0,2}^{i} x_{\beta}y_{\beta} - b_{0,0,1,2}^{i} \left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) - 2b_{0,0,2,2}^{i} x_{\alpha}y_{\alpha} \\ &+ b_{0,0,0,2}^{r} \left(x_{\beta} - y_{\beta}\right) \left(x_{\beta} + y_{\beta}\right) + b_{0,0,1,2}^{r} \left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) + b_{0,0,2,2}^{r} \left(x_{\alpha} - y_{\alpha}\right) \left(x_{\alpha} + y_{\alpha}\right) \end{split}$$

$$\begin{split} H_{X_{\beta}Y_{\alpha}}^{(2)} &= -a_{0,0,-1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,0,1,0}^{i} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta}\right) - a_{0,2,0,0}^{i} \left(x_{\beta}^{2} + y_{\beta}^{2}\right) - a_{2,0,0,0}^{i} \left(x_{\alpha}^{2} + y_{\alpha}^{2}\right) \\ &- a_{0,0,-1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) + a_{0,0,1,0}^{r} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha}\right) - b_{0,0,2}^{i} \left(x_{\beta} - y_{\beta}\right) \left(x_{\beta} + y_{\beta}\right) - b_{0,0,1,2}^{i} \left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta}\right) \\ &- b_{0,0,2,2}^{i} \left(x_{\alpha} - y_{\alpha}\right) \left(x_{\alpha} + y_{\alpha}\right) - 2b_{0,0,0,2}^{r} x_{\beta}y_{\beta} - b_{0,0,1,2}^{r} \left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha}\right) - 2b_{0,0,2,2}^{r} x_{\alpha}y_{\alpha} \end{split}$$

$$\begin{split} H_{Y_{\beta}X_{\alpha}}^{(2)} &= a_{0,0,-1,0}^{i} \left(x_{\alpha} x_{\beta} + y_{\alpha} y_{\beta} \right) + a_{0,0,1,0}^{i} \left(x_{\alpha} x_{\beta} + y_{\alpha} y_{\beta} \right) + a_{0,2,0,0}^{i} \left(x_{\beta}^{2} + y_{\beta}^{2} \right) + a_{2,0,0,0}^{i} \left(x_{\alpha}^{2} + y_{\alpha}^{2} \right) \\ &+ a_{0,0,-1,0}^{r} \left(x_{\alpha} y_{\beta} - x_{\beta} y_{\alpha} \right) - a_{0,0,1,0}^{r} \left(x_{\alpha} y_{\beta} - x_{\beta} y_{\alpha} \right) - b_{0,0,0,2}^{i} \left(x_{\beta} - y_{\beta} \right) \left(x_{\beta} + y_{\beta} \right) - b_{0,0,1,2}^{i} \left(x_{\alpha} x_{\beta} - y_{\alpha} y_{\beta} \right) \\ &- b_{0,0,2,2}^{i} \left(x_{\alpha} - y_{\alpha} \right) \left(x_{\alpha} + y_{\alpha} \right) - 2b_{0,0,0,2}^{r} x_{\beta} y_{\beta} - b_{0,0,1,2}^{r} \left(x_{\alpha} y_{\beta} + x_{\beta} y_{\alpha} \right) - 2b_{0,0,2,2}^{r} x_{\alpha} y_{\alpha} \end{split}$$

$$\begin{split} H_{Y_{\beta}Y_{\alpha}}^{(2)} &= -a_{0,0,-1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha} \right) + a_{0,0,1,0}^{i} \left(x_{\alpha}y_{\beta} - x_{\beta}y_{\alpha} \right) + a_{0,0,-1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta} \right) + a_{0,0,1,0}^{r} \left(x_{\alpha}x_{\beta} + y_{\alpha}y_{\beta} \right) \\ &+ a_{0,2,0,0}^{r} \left(x_{\beta}^{2} + y_{\beta}^{2} \right) + a_{2,0,0,0}^{r} \left(x_{\alpha}^{2} + y_{\alpha}^{2} \right) + 2b_{0,0,2,2}^{i} x_{\beta}y_{\beta} + b_{0,0,1,2}^{i} \left(x_{\alpha}y_{\beta} + x_{\beta}y_{\alpha} \right) + 2b_{0,0,2,2}^{i} x_{\alpha}y_{\alpha} \\ &- b_{0,0,0,2}^{r} \left(x_{\beta} - y_{\beta} \right) \left(x_{\beta} + y_{\beta} \right) - b_{0,0,1,2}^{r} \left(x_{\alpha}x_{\beta} - y_{\alpha}y_{\beta} \right) - b_{0,0,2,2}^{r} \left(x_{\alpha} - y_{\alpha} \right) \left(x_{\alpha} + y_{\alpha} \right) \end{split}$$