VHEGEN: A vibronic Hamiltonian expansion generator for trigonal and tetragonal polyatomic systems

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Thank you for using VHEGEN, the V-ibronic H-amiltonian E-xpansion GEN-erator for trigonal and tetragonal polyatomic systems. This is a VHEGEN output file compiled by pdflatex. If the VHEGEN package was used in research resulting in a publication, please reference the article in *Computer Physics Communications* which describes the program ([doi here]). Additional information regarding the matrix element expansion process, including the independent matrix element eigenvalues, their root formulas and constraints, and their transformation to the real basis (if applicable), can be found in the log output file. For questions, bugs, or comments, please contact robert.lang@mail.utoronto.ca.

Contents

1   Vibronic interaction

$E'' \otimes (e' + e')$ in D_{3h}

2   Vibronic Hamiltonian operator in the complex E basis

$$\hat{H} = \begin{pmatrix} |+\rangle & |-\rangle \end{pmatrix} \begin{pmatrix} H_{++} & H_{+-} \\ H_{-+} & H_{--} \end{pmatrix} \begin{pmatrix} \langle+| \\ \langle-| \end{pmatrix}$$

3   Matrix element expansions in the complex E basis

3.1   Order: 0

Number of fitting parameters: H_{++}: 1, H_{+-}: 0.

Polar e-coordinates:

$$H_{++}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{--}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{+-}^{(0)} = 0$$

$$H_{-+}^{(0)} = 0$$

Cartesian e-coordinates:

$$H_{++}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{--}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{+-}^{(0)} = 0$$

$$H_{-+}^{(0)} = 0$$

3.2   Order: 1

Number of fitting parameters: H_{++}: 0, H_{+-}: 2.

Polar e-coordinates:

$$H^{(1)}_{++} = 0$$

$$H^{(1)}_{--} = 0$$

$$H^{(1)}_{+-} = -ib^r_{0,0,-1,-1}\rho_\alpha \sin(\phi_\alpha) + b^r_{0,0,-1,-1}\rho_\alpha \cos(\phi_\alpha) - ib^r_{0,0,0,-1}\rho_\beta \sin(\phi_\beta) + b^r_{0,0,0,-1}\rho_\beta \cos(\phi_\beta)$$

$$H^{(1)}_{-+} = ib^r_{0,0,-1,-1}\rho_\alpha \sin(\phi_\alpha) + b^r_{0,0,-1,-1}\rho_\alpha \cos(\phi_\alpha) + ib^r_{0,0,0,-1}\rho_\beta \sin(\phi_\beta) + b^r_{0,0,0,-1}\rho_\beta \cos(\phi_\beta)$$

Cartesian e-coordinates:

$$H^{(1)}_{++} = 0$$

$$H^{(1)}_{--} = 0$$

$$H^{(1)}_{+-} = b^r_{0,0,-1,-1}x_\alpha - ib^r_{0,0,-1,-1}y_\alpha + b^r_{0,0,0,-1}x_\beta - ib^r_{0,0,0,-1}y_\beta$$

$$H^{(1)}_{-+} = b^r_{0,0,-1,-1}x_\alpha + ib^r_{0,0,-1,-1}y_\alpha + b^r_{0,0,0,-1}x_\beta + ib^r_{0,0,0,-1}y_\beta$$

3.3   Order: 2

Number of fitting parameters: H_{++}: 3, H_{+-}: 3.

Polar e-coordinates:

$$H^{(2)}_{++} = a^r_{0,0,1,0}\rho_\alpha\rho_\beta \cos(\phi_\alpha - \phi_\beta) + a^r_{0,2,0,0}\rho_\beta^2 + a^r_{2,0,0,0}\rho_\alpha^2$$

$$H^{(2)}_{--} = a^r_{0,0,1,0}\rho_\alpha\rho_\beta \cos(\phi_\alpha - \phi_\beta) + a^r_{0,2,0,0}\rho_\beta^2 + a^r_{2,0,0,0}\rho_\alpha^2$$

$$H^{(2)}_{+-} = ib^r_{0,0,0,2}\rho_\beta^2 \sin(2\phi_\beta) + b^r_{0,0,0,2}\rho_\beta^2 \cos(2\phi_\beta) + ib^r_{0,0,1,2}\rho_\alpha\rho_\beta \sin(\phi_\alpha + \phi_\beta) + b^r_{0,0,1,2}\rho_\alpha\rho_\beta \cos(\phi_\alpha + \phi_\beta)$$
$$+ ib^r_{0,0,2,2}\rho_\alpha^2 \sin(2\phi_\alpha) + b^r_{0,0,2,2}\rho_\alpha^2 \cos(2\phi_\alpha)$$

$$H^{(2)}_{-+} = -ib^r_{0,0,0,2}\rho_\beta^2 \sin(2\phi_\beta) + b^r_{0,0,0,2}\rho_\beta^2 \cos(2\phi_\beta) - ib^r_{0,0,1,2}\rho_\alpha\rho_\beta \sin(\phi_\alpha + \phi_\beta) + b^r_{0,0,1,2}\rho_\alpha\rho_\beta \cos(\phi_\alpha + \phi_\beta)$$
$$- ib^r_{0,0,2,2}\rho_\alpha^2 \sin(2\phi_\alpha) + b^r_{0,0,2,2}\rho_\alpha^2 \cos(2\phi_\alpha)$$

Cartesian e-coordinates:

$$H_{++}^{(2)} = a_{0,0,1,0}^{r} \left(x_\alpha x_\beta + y_\alpha y_\beta \right) + a_{0,2,0,0}^{r} \left(x_\beta^2 + y_\beta^2 \right) + a_{2,0,0,0}^{r} \left(x_\alpha^2 + y_\alpha^2 \right)$$

$$H_{--}^{(2)} = a_{0,0,1,0}^{r} \left(x_\alpha x_\beta + y_\alpha y_\beta \right) + a_{0,2,0,0}^{r} \left(x_\beta^2 + y_\beta^2 \right) + a_{2,0,0,0}^{r} \left(x_\alpha^2 + y_\alpha^2 \right)$$

$$H_{+-}^{(2)} = 2ib_{0,0,0,2}^{r} x_\beta y_\beta + b_{0,0,0,2}^{r} \left(x_\beta - y_\beta \right)\left(x_\beta + y_\beta \right) + b_{0,0,1,2}^{r} \left(x_\alpha x_\beta - y_\alpha y_\beta \right) + ib_{0,0,1,2}^{r} \left(x_\alpha y_\beta + x_\beta y_\alpha \right)$$
$$+ 2ib_{0,0,2,2}^{r} x_\alpha y_\alpha + b_{0,0,2,2}^{r} \left(x_\alpha - y_\alpha \right)\left(x_\alpha + y_\alpha \right)$$

$$H_{-+}^{(2)} = -2ib_{0,0,0,2}^{r} x_\beta y_\beta + b_{0,0,0,2}^{r} \left(x_\beta - y_\beta \right)\left(x_\beta + y_\beta \right) + b_{0,0,1,2}^{r} \left(x_\alpha x_\beta - y_\alpha y_\beta \right) - ib_{0,0,1,2}^{r} \left(x_\alpha y_\beta + x_\beta y_\alpha \right)$$
$$- 2ib_{0,0,2,2}^{r} x_\alpha y_\alpha + b_{0,0,2,2}^{r} \left(x_\alpha - y_\alpha \right)\left(x_\alpha + y_\alpha \right)$$

3.4 Order: 3

Number of fitting parameters: H_{++}: 4, H_{+-}: 6.

Polar e-coordinates:

$$H_{++}^{(3)} = a_{0,0,0,3}^{r} \rho_\beta^3 \cos\left(3\phi_\beta \right) + a_{0,0,1,3}^{r} \rho_\alpha \rho_\beta^2 \cos\left(\phi_\alpha + 2\phi_\beta \right) + a_{0,0,2,3}^{r} \rho_\alpha^2 \rho_\beta \cos\left(2\phi_\alpha + \phi_\beta \right) + a_{0,0,3,3}^{r} \rho_\alpha^3 \cos\left(3\phi_\alpha \right)$$

$$H_{--}^{(3)} = a_{0,0,0,3}^{r} \rho_\beta^3 \cos\left(3\phi_\beta \right) + a_{0,0,1,3}^{r} \rho_\alpha \rho_\beta^2 \cos\left(\phi_\alpha + 2\phi_\beta \right) + a_{0,0,2,3}^{r} \rho_\alpha^2 \rho_\beta \cos\left(2\phi_\alpha + \phi_\beta \right) + a_{0,0,3,3}^{r} \rho_\alpha^3 \cos\left(3\phi_\alpha \right)$$

$$H_{+-}^{(3)} = -ib_{0,0,-2,-1}^{r} \rho_\alpha^2 \rho_\beta \sin\left(2\phi_\alpha - \phi_\beta \right) + b_{0,0,-2,-1}^{r} \rho_\alpha^2 \rho_\beta \cos\left(2\phi_\alpha - \phi_\beta \right) + ib_{0,0,1,-1}^{r} \rho_\alpha \rho_\beta^2 \sin\left(\phi_\alpha - 2\phi_\beta \right)$$
$$+ b_{0,0,1,-1}^{r} \rho_\alpha \rho_\beta^2 \cos\left(\phi_\alpha - 2\phi_\beta \right) - ib_{0,2,-1,-1}^{r} \rho_\alpha \rho_\beta^2 \sin\left(\phi_\alpha \right) + b_{0,2,-1,-1}^{r} \rho_\alpha \rho_\beta^2 \cos\left(\phi_\alpha \right)$$
$$- ib_{0,2,0,-1}^{r} \rho_\beta^3 \sin\left(\phi_\beta \right) + b_{0,2,0,-1}^{r} \rho_\beta^3 \cos\left(\phi_\beta \right) - ib_{2,0,-1,-1}^{r} \rho_\alpha^3 \sin\left(\phi_\alpha \right) + b_{2,0,-1,-1}^{r} \rho_\alpha^3 \cos\left(\phi_\alpha \right)$$
$$- ib_{2,0,0,-1}^{r} \rho_\alpha^2 \rho_\beta \sin\left(\phi_\beta \right) + b_{2,0,0,-1}^{r} \rho_\alpha^2 \rho_\beta \cos\left(\phi_\beta \right)$$

$$H_{-+}^{(3)} = ib_{0,0,-2,-1}^{r} \rho_\alpha^2 \rho_\beta \sin\left(2\phi_\alpha - \phi_\beta \right) + b_{0,0,-2,-1}^{r} \rho_\alpha^2 \rho_\beta \cos\left(2\phi_\alpha - \phi_\beta \right) - ib_{0,0,1,-1}^{r} \rho_\alpha \rho_\beta^2 \sin\left(\phi_\alpha - 2\phi_\beta \right)$$
$$+ b_{0,0,1,-1}^{r} \rho_\alpha \rho_\beta^2 \cos\left(\phi_\alpha - 2\phi_\beta \right) + ib_{0,2,-1,-1}^{r} \rho_\alpha \rho_\beta^2 \sin\left(\phi_\alpha \right) + b_{0,2,-1,-1}^{r} \rho_\alpha \rho_\beta^2 \cos\left(\phi_\alpha \right)$$
$$+ ib_{0,2,0,-1}^{r} \rho_\beta^3 \sin\left(\phi_\beta \right) + b_{0,2,0,-1}^{r} \rho_\beta^3 \cos\left(\phi_\beta \right) + ib_{2,0,-1,-1}^{r} \rho_\alpha^3 \sin\left(\phi_\alpha \right) + b_{2,0,-1,-1}^{r} \rho_\alpha^3 \cos\left(\phi_\alpha \right)$$
$$+ ib_{2,0,0,-1}^{r} \rho_\alpha^2 \rho_\beta \sin\left(\phi_\beta \right) + b_{2,0,0,-1}^{r} \rho_\alpha^2 \rho_\beta \cos\left(\phi_\beta \right)$$

Cartesian e-coordinates:

$$H_{++}^{(3)} = a_{0,0,0,3}^{r} x_\beta \left(x_\beta^2 - 3y_\beta^2 \right) + a_{0,0,1,3}^{r} \left(x_\alpha \left(x_\beta^2 - y_\beta^2 \right) - 2x_\beta y_\alpha y_\beta \right)$$
$$+ a_{0,0,2,3}^{r} \left(-2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2 \right) \right) + a_{0,0,3,3}^{r} x_\alpha \left(x_\alpha^2 - 3y_\alpha^2 \right)$$

$$H_{--}^{(3)} = a_{0,0,0,3}^{r} x_\beta \left(x_\beta^2 - 3y_\beta^2 \right) + a_{0,0,1,3}^{r} \left(x_\alpha \left(x_\beta^2 - y_\beta^2 \right) - 2x_\beta y_\alpha y_\beta \right)$$
$$+ a_{0,0,2,3}^{r} \left(-2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2 \right) \right) + a_{0,0,3,3}^{r} x_\alpha \left(x_\alpha^2 - 3y_\alpha^2 \right)$$

$$H_{+-}^{(3)} = ib_{0,0,-2,-1}^r \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2 \right) \right) + b_{0,0,-2,-1}^r \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2 \right) \right)$$
$$+ b_{0,0,1,-1}^r \left(x_\alpha \left(x_\beta^2 - y_\beta^2 \right) + 2x_\beta y_\alpha y_\beta \right) - ib_{0,0,1,-1}^r \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2 \right) \right)$$
$$+ b_{0,2,-1,-1}^r x_\alpha \left(x_\beta^2 + y_\beta^2 \right) - ib_{0,2,-1,-1}^r y_\alpha \left(x_\beta^2 + y_\beta^2 \right) + b_{0,2,0,-1}^r x_\beta \left(x_\beta^2 + y_\beta^2 \right) - ib_{0,2,0,-1}^r y_\beta \left(x_\beta^2 + y_\beta^2 \right)$$
$$+ b_{2,0,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2 \right) - ib_{2,0,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2 \right) + b_{2,0,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2 \right) - ib_{2,0,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2 \right)$$

$$H_{-+}^{(3)} = -ib_{0,0,-2,-1}^r \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2 \right) \right) + b_{0,0,-2,-1}^r \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2 \right) \right)$$
$$+ b_{0,0,1,-1}^r \left(x_\alpha \left(x_\beta^2 - y_\beta^2 \right) + 2x_\beta y_\alpha y_\beta \right) + ib_{0,0,1,-1}^r \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2 \right) \right)$$
$$+ b_{0,2,-1,-1}^r x_\alpha \left(x_\beta^2 + y_\beta^2 \right) + ib_{0,2,-1,-1}^r y_\alpha \left(x_\beta^2 + y_\beta^2 \right) + b_{0,2,0,-1}^r x_\beta \left(x_\beta^2 + y_\beta^2 \right) + ib_{0,2,0,-1}^r y_\beta \left(x_\beta^2 + y_\beta^2 \right)$$
$$+ b_{2,0,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2 \right) + ib_{2,0,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2 \right) + b_{2,0,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2 \right) + ib_{2,0,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2 \right)$$

3.5 Order: 4

Number of fitting parameters: H_{++}: 6, H_{+-}: 13.

Polar e-coordinates:

$$H_{++}^{(4)} = a_{0,0,2,0}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\alpha - 2\phi_\beta \right) + a_{0,2,1,0}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha - \phi_\beta \right)$$
$$+ a_{0,4,0,0}^r \rho_\beta^4 + a_{2,0,1,0}^r \rho_\alpha^3 \rho_\beta \cos\left(\phi_\alpha - \phi_\beta \right) + a_{2,2,0,0}^r \rho_\alpha^2 \rho_\beta^2 + a_{4,0,0,0}^r \rho_\alpha^4$$

$$H_{--}^{(4)} = a_{0,0,2,0}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\alpha - 2\phi_\beta \right) + a_{0,2,1,0}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha - \phi_\beta \right)$$
$$+ a_{0,4,0,0}^r \rho_\beta^4 + a_{2,0,1,0}^r \rho_\alpha^3 \rho_\beta \cos\left(\phi_\alpha - \phi_\beta \right) + a_{2,2,0,0}^r \rho_\alpha^2 \rho_\beta^2 + a_{4,0,0,0}^r \rho_\alpha^4$$

$$H_{+-}^{(4)} = -ib_{0,0,-1,-4}^r \rho_\alpha \rho_\beta^3 \sin\left(\phi_\alpha + 3\phi_\beta \right) + b_{0,0,-1,-4}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha + 3\phi_\beta \right) - ib_{0,0,-1,2}^r \rho_\alpha \rho_\beta^3 \sin\left(\phi_\alpha - 3\phi_\beta \right)$$
$$+ b_{0,0,-1,2}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha - 3\phi_\beta \right) - ib_{0,0,-2,-4}^r \rho_\alpha^2 \rho_\beta^2 \sin\left(2\phi_\alpha + 2\phi_\beta \right) + b_{0,0,-2,-4}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\alpha + 2\phi_\beta \right)$$
$$- ib_{0,0,-3,-4}^r \rho_\alpha^3 \rho_\beta \sin\left(3\phi_\alpha + \phi_\beta \right) + b_{0,0,-3,-4}^r \rho_\alpha^3 \rho_\beta \cos\left(3\phi_\alpha + \phi_\beta \right) - ib_{0,0,-4,-4}^r \rho_\alpha^4 \sin\left(4\phi_\alpha \right) + b_{0,0,-4,-4}^r \rho_\alpha^4 \cos\left(4\phi_\alpha \right)$$
$$- ib_{0,0,0,-4}^r \rho_\beta^4 \sin\left(4\phi_\beta \right) + b_{0,0,0,-4}^r \rho_\beta^4 \cos\left(4\phi_\beta \right) + ib_{0,0,3,2}^r \rho_\alpha^3 \rho_\beta \sin\left(3\phi_\alpha - \phi_\beta \right) + b_{0,0,3,2}^r \rho_\alpha^3 \rho_\beta \cos\left(3\phi_\alpha - \phi_\beta \right)$$
$$+ ib_{0,2,0,2}^r \rho_\beta^4 \sin\left(2\phi_\beta \right) + b_{0,2,0,2}^r \rho_\beta^4 \cos\left(2\phi_\beta \right) + ib_{0,2,1,2}^r \rho_\alpha \rho_\beta^3 \sin\left(\phi_\alpha + \phi_\beta \right) + b_{0,2,1,2}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha + \phi_\beta \right)$$
$$+ ib_{0,2,2,2}^r \rho_\alpha^2 \rho_\beta^2 \sin\left(2\phi_\alpha \right) + b_{0,2,2,2}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\alpha \right) + ib_{2,0,0,2}^r \rho_\alpha^2 \rho_\beta^2 \sin\left(2\phi_\beta \right) + b_{2,0,0,2}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\beta \right)$$
$$+ ib_{2,0,1,2}^r \rho_\alpha^3 \rho_\beta \sin\left(\phi_\alpha + \phi_\beta \right) + b_{2,0,1,2}^r \rho_\alpha^3 \rho_\beta \cos\left(\phi_\alpha + \phi_\beta \right) + ib_{2,0,2,2}^r \rho_\alpha^4 \sin\left(2\phi_\alpha \right) + b_{2,0,2,2}^r \rho_\alpha^4 \cos\left(2\phi_\alpha \right)$$

$$H_{-+}^{(4)} = ib_{0,0,-1,-4}^r \rho_\alpha \rho_\beta^3 \sin\left(\phi_\alpha + 3\phi_\beta \right) + b_{0,0,-1,-4}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha + 3\phi_\beta \right) + ib_{0,0,-1,2}^r \rho_\alpha \rho_\beta^3 \sin\left(\phi_\alpha - 3\phi_\beta \right)$$
$$+ b_{0,0,-1,2}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha - 3\phi_\beta \right) + ib_{0,0,-2,-4}^r \rho_\alpha^2 \rho_\beta^2 \sin\left(2\phi_\alpha + 2\phi_\beta \right) + b_{0,0,-2,-4}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\alpha + 2\phi_\beta \right)$$
$$+ ib_{0,0,-3,-4}^r \rho_\alpha^3 \rho_\beta \sin\left(3\phi_\alpha + \phi_\beta \right) + b_{0,0,-3,-4}^r \rho_\alpha^3 \rho_\beta \cos\left(3\phi_\alpha + \phi_\beta \right) + ib_{0,0,-4,-4}^r \rho_\alpha^4 \sin\left(4\phi_\alpha \right) + b_{0,0,-4,-4}^r \rho_\alpha^4 \cos\left(4\phi_\alpha \right)$$
$$+ ib_{0,0,0,-4}^r \rho_\beta^4 \sin\left(4\phi_\beta \right) + b_{0,0,0,-4}^r \rho_\beta^4 \cos\left(4\phi_\beta \right) - ib_{0,0,3,2}^r \rho_\alpha^3 \rho_\beta \sin\left(3\phi_\alpha - \phi_\beta \right) + b_{0,0,3,2}^r \rho_\alpha^3 \rho_\beta \cos\left(3\phi_\alpha - \phi_\beta \right)$$
$$- ib_{0,2,0,2}^r \rho_\beta^4 \sin\left(2\phi_\beta \right) + b_{0,2,0,2}^r \rho_\beta^4 \cos\left(2\phi_\beta \right) - ib_{0,2,1,2}^r \rho_\alpha \rho_\beta^3 \sin\left(\phi_\alpha + \phi_\beta \right) + b_{0,2,1,2}^r \rho_\alpha \rho_\beta^3 \cos\left(\phi_\alpha + \phi_\beta \right)$$
$$- ib_{0,2,2,2}^r \rho_\alpha^2 \rho_\beta^2 \sin\left(2\phi_\alpha \right) + b_{0,2,2,2}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\alpha \right) - ib_{2,0,0,2}^r \rho_\alpha^2 \rho_\beta^2 \sin\left(2\phi_\beta \right) + b_{2,0,0,2}^r \rho_\alpha^2 \rho_\beta^2 \cos\left(2\phi_\beta \right)$$
$$- ib_{2,0,1,2}^r \rho_\alpha^3 \rho_\beta \sin\left(\phi_\alpha + \phi_\beta \right) + b_{2,0,1,2}^r \rho_\alpha^3 \rho_\beta \cos\left(\phi_\alpha + \phi_\beta \right) - ib_{2,0,2,2}^r \rho_\alpha^4 \sin\left(2\phi_\alpha \right) + b_{2,0,2,2}^r \rho_\alpha^4 \cos\left(2\phi_\alpha \right)$$

Cartesian e-coordinates:

$$H_{++}^{(4)} = a_{0,0,2,0}^r \left(x_\alpha \left(x_\beta - y_\beta\right) + y_\alpha \left(x_\beta + y_\beta\right)\right)\left(x_\alpha \left(x_\beta + y_\beta\right) + y_\alpha \left(-x_\beta + y_\beta\right)\right) + a_{0,2,1,0}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right)$$
$$+ a_{0,4,0,0}^r \left(x_\beta^2 + y_\beta^2\right)^2 + a_{2,0,1,0}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,2,0,0}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + a_{4,0,0,0}^r \left(x_\alpha^2 + y_\alpha^2\right)^2$$

$$H_{--}^{(4)} = a_{0,0,2,0}^r \left(x_\alpha \left(x_\beta - y_\beta\right) + y_\alpha \left(x_\beta + y_\beta\right)\right)\left(x_\alpha \left(x_\beta + y_\beta\right) + y_\alpha \left(-x_\beta + y_\beta\right)\right) + a_{0,2,1,0}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right)$$
$$+ a_{0,4,0,0}^r \left(x_\beta^2 + y_\beta^2\right)^2 + a_{2,0,1,0}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,2,0,0}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + a_{4,0,0,0}^r \left(x_\alpha^2 + y_\alpha^2\right)^2$$

$$H_{+-}^{(4)} = b_{0,0,-1,-4}^r \left(x_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha \left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) - ib_{0,0,-1,-4}^r \left(x_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right)$$
$$+ b_{0,0,-1,2}^r \left(x_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) + ib_{0,0,-1,2}^r \left(x_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha \left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right)$$
$$- 2ib_{0,0,-2,-4}^r \left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) + b_{0,0,-2,-4}^r \left(x_\alpha \left(x_\beta - y_\beta\right) + y_\alpha \left(-x_\beta - y_\beta\right)\right)\left(x_\alpha \left(x_\beta + y_\beta\right) + y_\alpha \left(x_\beta - y_\beta\right)\right)$$
$$+ b_{0,0,-3,-4}^r \left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) - ib_{0,0,-3,-4}^r \left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right)$$
$$- 4ib_{0,0,-4,-4}^r x_\alpha y_\alpha \left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right) + b_{0,0,-4,-4}^r \left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right)$$
$$- 4ib_{0,0,0,-4}^r x_\beta y_\beta \left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{0,0,0,-4}^r \left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right)$$
$$+ b_{0,0,3,2}^r \left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) - ib_{0,0,3,2}^r \left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right)$$
$$+ 2ib_{0,2,0,2}^r x_\beta y_\beta \left(x_\beta^2 + y_\beta^2\right) + b_{0,2,0,2}^r \left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) + b_{0,2,1,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)$$
$$+ ib_{0,2,1,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) + 2ib_{0,2,2,2}^r x_\alpha y_\alpha \left(x_\beta^2 + y_\beta^2\right) + b_{0,2,2,2}^r \left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right)$$
$$+ 2ib_{2,0,0,2}^r x_\beta y_\beta \left(x_\alpha^2 + y_\alpha^2\right) + b_{2,0,0,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{2,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)$$
$$+ ib_{2,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) + 2ib_{2,0,2,2}^r x_\alpha y_\alpha \left(x_\alpha^2 + y_\alpha^2\right) + b_{2,0,2,2}^r \left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)$$

$$H_{-+}^{(4)} = b_{0,0,-1,-4}^r \left(x_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha \left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) + ib_{0,0,-1,-4}^r \left(x_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right)$$
$$+ b_{0,0,-1,2}^r \left(x_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) - ib_{0,0,-1,2}^r \left(x_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha \left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right)$$
$$+ 2ib_{0,0,-2,-4}^r \left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) + b_{0,0,-2,-4}^r \left(x_\alpha \left(x_\beta - y_\beta\right) + y_\alpha \left(-x_\beta - y_\beta\right)\right)\left(x_\alpha \left(x_\beta + y_\beta\right) + y_\alpha \left(x_\beta - y_\beta\right)\right)$$
$$+ b_{0,0,-3,-4}^r \left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) + ib_{0,0,-3,-4}^r \left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right)$$
$$+ 4ib_{0,0,-4,-4}^r x_\alpha y_\alpha \left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right) + b_{0,0,-4,-4}^r \left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right)$$
$$+ 4ib_{0,0,0,-4}^r x_\beta y_\beta \left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{0,0,0,-4}^r \left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right)$$
$$+ b_{0,0,3,2}^r \left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) + ib_{0,0,3,2}^r \left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right)$$
$$- 2ib_{0,2,0,2}^r x_\beta y_\beta \left(x_\beta^2 + y_\beta^2\right) + b_{0,2,0,2}^r \left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) + b_{0,2,1,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)$$
$$- ib_{0,2,1,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2ib_{0,2,2,2}^r x_\alpha y_\alpha \left(x_\beta^2 + y_\beta^2\right) + b_{0,2,2,2}^r \left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right)$$
$$- 2ib_{2,0,0,2}^r x_\beta y_\beta \left(x_\alpha^2 + y_\alpha^2\right) + b_{2,0,0,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{2,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)$$
$$- ib_{2,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2ib_{2,0,2,2}^r x_\alpha y_\alpha \left(x_\alpha^2 + y_\alpha^2\right) + b_{2,0,2,2}^r \left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)$$

3.6 Order: 5

Number of fitting parameters: H_{++}: 10, H_{+-}: 18.

Polar e-coordinates:

$$H_{++}^{(5)} = a_{0,0,-1,3}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha - 4\phi_\beta\right) + a_{0,0,4,3}^r \rho_\alpha^4 \rho_\beta \cos\left(4\phi_\alpha - \phi_\beta\right) + a_{0,2,0,3}^r \rho_\beta^5 \cos\left(3\phi_\beta\right) + a_{0,2,1,3}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha + 2\phi_\beta\right)$$
$$+ a_{0,2,2,3}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha + \phi_\beta\right) + a_{0,2,3,3}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(3\phi_\alpha\right) + a_{2,0,0,3}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(3\phi_\beta\right) + a_{2,0,1,3}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(\phi_\alpha + 2\phi_\beta\right)$$
$$+ a_{2,0,2,3}^r \rho_\alpha^4 \rho_\beta \cos\left(2\phi_\alpha + \phi_\beta\right) + a_{2,0,3,3}^r \rho_\alpha^5 \cos\left(3\phi_\alpha\right)$$

$$H^{(5)}_{--} = a^r_{0,0,-1,3}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha - 4\phi_\beta\right) + a^r_{0,0,4,3}\rho_\alpha^4\rho_\beta\cos\left(4\phi_\alpha - \phi_\beta\right) + a^r_{0,2,0,3}\rho_\beta^5\cos\left(3\phi_\beta\right) + a^r_{0,2,1,3}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha + 2\phi_\beta\right)$$
$$+ a^r_{0,2,2,3}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha + \phi_\beta\right) + a^r_{0,2,3,3}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha\right) + a^r_{2,0,0,3}\rho_\alpha^2\rho_\beta^3\cos\left(3\phi_\beta\right) + a^r_{2,0,1,3}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha + 2\phi_\beta\right)$$
$$+ a^r_{2,0,2,3}\rho_\alpha^4\rho_\beta\cos\left(2\phi_\alpha + \phi_\beta\right) + a^r_{2,0,3,3}\rho_\alpha^5\cos\left(3\phi_\alpha\right)$$

$$H^{(5)}_{+-} = -ib^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha - 2\phi_\beta\right) + b^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha - 2\phi_\beta\right) + ib^r_{0,0,0,5}\rho_\beta^5\sin\left(5\phi_\beta\right) + b^r_{0,0,0,5}\rho_\beta^5\cos\left(5\phi_\beta\right)$$
$$+ ib^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha + 4\phi_\beta\right) + b^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha + 4\phi_\beta\right) + ib^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - 3\phi_\beta\right)$$
$$+ b^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha - 3\phi_\beta\right) + ib^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha + 3\phi_\beta\right) + b^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha + 3\phi_\beta\right)$$
$$+ ib^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha + 2\phi_\beta\right) + b^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha + 2\phi_\beta\right) + ib^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\sin\left(4\phi_\alpha + \phi_\beta\right)$$
$$+ b^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\cos\left(4\phi_\alpha + \phi_\beta\right) + ib^r_{0,0,5,5}\rho_\alpha^5\sin\left(5\phi_\alpha\right) + b^r_{0,0,5,5}\rho_\alpha^5\cos\left(5\phi_\alpha\right) - ib^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - \phi_\beta\right)$$
$$+ b^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha - \phi_\beta\right) + ib^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha - 2\phi_\beta\right) + b^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha - 2\phi_\beta\right)$$
$$- ib^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha\right) + b^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha\right) - ib^r_{0,4,0,-1}\rho_\beta^5\sin\left(\phi_\beta\right) + b^r_{0,4,0,-1}\rho_\beta^5\cos\left(\phi_\beta\right)$$
$$- ib^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\sin\left(2\phi_\alpha - \phi_\beta\right) + b^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\cos\left(2\phi_\alpha - \phi_\beta\right) + ib^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha - 2\phi_\beta\right)$$
$$+ b^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha - 2\phi_\beta\right) - ib^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha\right) + b^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha\right)$$
$$- ib^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\sin\left(\phi_\beta\right) + b^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\cos\left(\phi_\beta\right) - ib^r_{4,0,-1,-1}\rho_\alpha^5\sin\left(\phi_\alpha\right) + b^r_{4,0,-1,-1}\rho_\alpha^5\cos\left(\phi_\alpha\right)$$
$$- ib^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\sin\left(\phi_\beta\right) + b^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\cos\left(\phi_\beta\right)$$

$$H^{(5)}_{-+} = ib^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha - 2\phi_\beta\right) + b^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha - 2\phi_\beta\right) - ib^r_{0,0,0,5}\rho_\beta^5\sin\left(5\phi_\beta\right) + b^r_{0,0,0,5}\rho_\beta^5\cos\left(5\phi_\beta\right)$$
$$- ib^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha + 4\phi_\beta\right) + b^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha + 4\phi_\beta\right) - ib^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - 3\phi_\beta\right)$$
$$+ b^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha - 3\phi_\beta\right) - ib^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha + 3\phi_\beta\right) + b^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha + 3\phi_\beta\right)$$
$$- ib^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha + 2\phi_\beta\right) + b^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha + 2\phi_\beta\right) - ib^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\sin\left(4\phi_\alpha + \phi_\beta\right)$$
$$+ b^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\cos\left(4\phi_\alpha + \phi_\beta\right) - ib^r_{0,0,5,5}\rho_\alpha^5\sin\left(5\phi_\alpha\right) + b^r_{0,0,5,5}\rho_\alpha^5\cos\left(5\phi_\alpha\right) + ib^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - \phi_\beta\right)$$
$$+ b^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha - \phi_\beta\right) - ib^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha - 2\phi_\beta\right) + b^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha - 2\phi_\beta\right)$$
$$+ ib^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha\right) + b^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha\right) + ib^r_{0,4,0,-1}\rho_\beta^5\sin\left(\phi_\beta\right) + b^r_{0,4,0,-1}\rho_\beta^5\cos\left(\phi_\beta\right)$$
$$+ ib^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\sin\left(2\phi_\alpha - \phi_\beta\right) + b^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\cos\left(2\phi_\alpha - \phi_\beta\right) - ib^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha - 2\phi_\beta\right)$$
$$+ b^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha - 2\phi_\beta\right) + ib^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha\right) + b^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha\right)$$
$$+ ib^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\sin\left(\phi_\beta\right) + b^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\cos\left(\phi_\beta\right) + ib^r_{4,0,-1,-1}\rho_\alpha^5\sin\left(\phi_\alpha\right) + b^r_{4,0,-1,-1}\rho_\alpha^5\cos\left(\phi_\alpha\right)$$
$$+ ib^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\sin\left(\phi_\beta\right) + b^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\cos\left(\phi_\beta\right)$$

Cartesian e-coordinates:

$$H^{(5)}_{++} = a^r_{0,0,-1,3}\left(x_\alpha\left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha\left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right)\right) + a^r_{0,0,4,3}\left(4x_\alpha^3 y_\alpha y_\beta - 4x_\alpha y_\alpha^3 y_\beta + x_\beta\left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right)$$
$$+ a^r_{0,2,0,3}x_\beta\left(x_\beta^2 - 3y_\beta^2\right)\left(x_\beta^2 + y_\beta^2\right) + a^r_{0,2,1,3}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right)$$
$$+ a^r_{0,2,2,3}\left(x_\beta^2 + y_\beta^2\right)\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + a^r_{0,2,3,3}x_\alpha\left(x_\alpha^2 - 3y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)$$
$$+ a^r_{2,0,0,3}x_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 - 3y_\beta^2\right) + a^r_{2,0,1,3}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right)$$
$$+ a^r_{2,0,2,3}\left(x_\alpha^2 + y_\alpha^2\right)\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + a^r_{2,0,3,3}x_\alpha\left(x_\alpha^2 - 3y_\alpha^2\right)\left(x_\alpha^2 + y_\alpha^2\right)$$

$$H^{(5)}_{--} = a^r_{0,0,-1,3}\left(x_\alpha\left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha\left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right)\right) + a^r_{0,0,4,3}\left(4x_\alpha^3 y_\alpha y_\beta - 4x_\alpha y_\alpha^3 y_\beta + x_\beta\left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right)$$
$$+ a^r_{0,2,0,3}x_\beta\left(x_\beta^2 - 3y_\beta^2\right)\left(x_\beta^2 + y_\beta^2\right) + a^r_{0,2,1,3}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right)$$
$$+ a^r_{0,2,2,3}\left(x_\beta^2 + y_\beta^2\right)\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + a^r_{0,2,3,3}x_\alpha\left(x_\alpha^2 - 3y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)$$
$$+ a^r_{2,0,0,3}x_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 - 3y_\beta^2\right) + a^r_{2,0,1,3}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right)$$
$$+ a^r_{2,0,2,3}\left(x_\alpha^2 + y_\alpha^2\right)\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + a^r_{2,0,3,3}x_\alpha\left(x_\alpha^2 - 3y_\alpha^2\right)\left(x_\alpha^2 + y_\alpha^2\right)$$

$$H_{+-}^{(5)} = b_{0,0,-3,-1}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 + 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) - 2x_\beta y_\alpha^3 y_\beta\right)$$
$$+ ib_{0,0,-3,-1}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta + x_\beta^2 y_\alpha^3 - y_\alpha^3 y_\beta^2 + y_\alpha \left(-3x_\alpha^2 x_\beta^2 + 3x_\alpha^2 y_\beta^2\right)\right) + b_{0,0,0,5}^r x_\beta \left(x_\beta^4 - 10x_\beta^2 y_\beta^2 + 5y_\beta^4\right)$$
$$+ ib_{0,0,0,5}^r y_\beta \left(5x_\beta^4 - 10x_\beta^2 y_\beta^2 + y_\beta^4\right) + ib_{0,0,1,5}^r \left(x_\alpha \left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right) + y_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right)\right)$$
$$+ b_{0,0,1,5}^r \left(x_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha \left(-4x_\beta^3 y_\beta + 4x_\beta y_\beta^3\right)\right)$$
$$+ b_{0,0,2,-1}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(6x_\beta^2 y_\alpha y_\beta - 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$
$$- ib_{0,0,2,-1}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(-2x_\beta^3 y_\alpha + 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$
$$+ b_{0,0,2,5}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(-6x_\beta^2 y_\alpha y_\beta + 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$
$$+ ib_{0,0,2,5}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(2x_\beta^3 y_\alpha - 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$
$$+ b_{0,0,3,5}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 - 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) + 2x_\beta y_\alpha^3 y_\beta\right)$$
$$+ ib_{0,0,3,5}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta - x_\beta^2 y_\alpha^3 + y_\alpha^3 y_\beta^2 + y_\alpha \left(3x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2\right)\right)$$
$$+ ib_{0,0,4,5}^r \left(4x_\alpha^3 x_\beta y_\alpha - 4x_\alpha x_\beta y_\alpha^3 + y_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right)$$
$$+ b_{0,0,4,5}^r \left(-4x_\alpha^3 y_\alpha y_\beta + 4x_\alpha y_\alpha^3 y_\beta + x_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right) + b_{0,0,5,5}^r x_\alpha \left(x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + 5y_\alpha^4\right)$$
$$+ ib_{0,0,5,5}^r y_\alpha \left(5x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + y_\alpha^4\right) + ib_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$+ b_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + b_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)$$
$$- ib_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right) + b_{0,4,-1,-1}^r x_\alpha \left(x_\beta^2 + y_\beta^2\right)^2 - ib_{0,4,-1,-1}^r y_\alpha \left(x_\beta^2 + y_\beta^2\right)^2$$
$$+ b_{0,4,0,-1}^r x_\beta \left(x_\beta^2 + y_\beta^2\right)^2 - ib_{0,4,0,-1}^r y_\beta \left(x_\beta^2 + y_\beta^2\right)^2 + ib_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$+ b_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + b_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)$$
$$- ib_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right) + b_{2,2,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right)$$
$$- ib_{2,2,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right) + b_{2,2,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right) - ib_{2,2,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right)$$
$$+ b_{4,0,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 - ib_{4,0,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 + b_{4,0,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2 - ib_{4,0,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2$$

$$H_{-+}^{(5)} = b_{0,0,-3,-1}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 + 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) - 2x_\beta y_\alpha^3 y_\beta\right)$$
$$- ib_{0,0,-3,-1}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta + x_\beta^2 y_\alpha^3 - y_\alpha^3 y_\beta^2 + y_\alpha \left(-3x_\alpha^2 x_\beta^2 + 3x_\alpha^2 y_\beta^2\right)\right) + b_{0,0,0,5}^r x_\beta \left(x_\beta^4 - 10x_\beta^2 y_\beta^2 + 5y_\beta^4\right)$$
$$- ib_{0,0,0,5}^r y_\beta \left(5x_\beta^4 - 10x_\beta^2 y_\beta^2 + y_\beta^4\right) - ib_{0,0,1,5}^r \left(x_\alpha \left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right) + y_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right)\right)$$
$$+ b_{0,0,1,5}^r \left(x_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha \left(-4x_\beta^3 y_\beta + 4x_\beta y_\beta^3\right)\right)$$
$$+ b_{0,0,2,-1}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(6x_\beta^2 y_\alpha y_\beta - 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$
$$+ ib_{0,0,2,-1}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(-2x_\beta^3 y_\alpha + 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$
$$+ b_{0,0,2,5}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(-6x_\beta^2 y_\alpha y_\beta + 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$
$$- ib_{0,0,2,5}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(2x_\beta^3 y_\alpha - 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$
$$+ b_{0,0,3,5}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 - 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) + 2x_\beta y_\alpha^3 y_\beta\right)$$
$$- ib_{0,0,3,5}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta - x_\beta^2 y_\alpha^3 + y_\alpha^3 y_\beta^2 + y_\alpha \left(3x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2\right)\right)$$
$$- ib_{0,0,4,5}^r \left(4x_\alpha^3 x_\beta y_\alpha - 4x_\alpha x_\beta y_\alpha^3 + y_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right)$$
$$+ b_{0,0,4,5}^r \left(-4x_\alpha^3 y_\alpha y_\beta + 4x_\alpha y_\alpha^3 y_\beta + x_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right) + b_{0,0,5,5}^r x_\alpha \left(x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + 5y_\alpha^4\right)$$
$$- ib_{0,0,5,5}^r y_\alpha \left(5x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + y_\alpha^4\right) - ib_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$+ b_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + b_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)$$
$$+ ib_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right) + b_{0,4,-1,-1}^r x_\alpha \left(x_\beta^2 + y_\beta^2\right)^2 + ib_{0,4,-1,-1}^r y_\alpha \left(x_\beta^2 + y_\beta^2\right)^2$$
$$+ b_{0,4,0,-1}^r x_\beta \left(x_\beta^2 + y_\beta^2\right)^2 + ib_{0,4,0,-1}^r y_\beta \left(x_\beta^2 + y_\beta^2\right)^2 - ib_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$+ b_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + b_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)$$
$$+ ib_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right) + b_{2,2,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right)$$
$$+ ib_{2,2,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right) + b_{2,2,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right) + ib_{2,2,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right)$$
$$+ b_{4,0,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 + ib_{4,0,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 + b_{4,0,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2 + ib_{4,0,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2$$

3.7 Order: 6

Number of fitting parameters: H_{++}: 17, H_{+-}: 27.

Polar e-coordinates:

$$
\begin{aligned}
H_{++}^{(6)} =\; & a_{0,0,0,6}^{r}\rho_{\beta}^{6}\cos\left(6\phi_{\beta}\right) + a_{0,0,1,6}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}+5\phi_{\beta}\right) + a_{0,0,2,6}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}+4\phi_{\beta}\right) + a_{0,0,3,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}-3\phi_{\beta}\right) \\
& + a_{0,0,3,6}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}+3\phi_{\beta}\right) + a_{0,0,4,6}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}+2\phi_{\beta}\right) + a_{0,0,5,6}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(5\phi_{\alpha}+\phi_{\beta}\right) + a_{0,0,6,6}^{r}\rho_{\alpha}^{6}\cos\left(6\phi_{\alpha}\right) \\
& + a_{0,2,2,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}-2\phi_{\beta}\right) + a_{0,4,1,0}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{0,6,0,0}^{r}\rho_{\beta}^{6} + a_{2,0,2,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}-2\phi_{\beta}\right) \\
& + a_{2,2,1,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{2,4,0,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4} + a_{4,0,1,0}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{4,2,0,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2} + a_{6,0,0,0}^{r}\rho_{\alpha}^{6}
\end{aligned}
$$

$$
\begin{aligned}
H_{--}^{(6)} =\; & a_{0,0,0,6}^{r}\rho_{\beta}^{6}\cos\left(6\phi_{\beta}\right) + a_{0,0,1,6}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}+5\phi_{\beta}\right) + a_{0,0,2,6}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}+4\phi_{\beta}\right) + a_{0,0,3,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}-3\phi_{\beta}\right) \\
& + a_{0,0,3,6}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}+3\phi_{\beta}\right) + a_{0,0,4,6}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}+2\phi_{\beta}\right) + a_{0,0,5,6}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(5\phi_{\alpha}+\phi_{\beta}\right) + a_{0,0,6,6}^{r}\rho_{\alpha}^{6}\cos\left(6\phi_{\alpha}\right) \\
& + a_{0,2,2,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}-2\phi_{\beta}\right) + a_{0,4,1,0}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{0,6,0,0}^{r}\rho_{\beta}^{6} + a_{2,0,2,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}-2\phi_{\beta}\right) \\
& + a_{2,2,1,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{2,4,0,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4} + a_{4,0,1,0}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(\phi_{\alpha}-\phi_{\beta}\right) + a_{4,2,0,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2} + a_{6,0,0,0}^{r}\rho_{\alpha}^{6}
\end{aligned}
$$

$$
\begin{aligned}
H_{+-}^{(6)} =\; & -ib_{0,0,-2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\sin\left(2\phi_{\alpha}-4\phi_{\beta}\right) + b_{0,0,-2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}-4\phi_{\beta}\right) - ib_{0,0,-5,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\sin\left(5\phi_{\alpha}-\phi_{\beta}\right) \\
& + b_{0,0,-5,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(5\phi_{\alpha}-\phi_{\beta}\right) + ib_{0,0,1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\sin\left(\phi_{\alpha}-5\phi_{\beta}\right) + b_{0,0,1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}-5\phi_{\beta}\right) \\
& + ib_{0,0,4,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\sin\left(4\phi_{\alpha}-2\phi_{\beta}\right) + b_{0,0,4,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}-2\phi_{\beta}\right) - ib_{0,2,-1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\sin\left(\phi_{\alpha}+3\phi_{\beta}\right) \\
& + b_{0,2,-1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}+3\phi_{\beta}\right) - ib_{0,2,-1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\sin\left(\phi_{\alpha}-3\phi_{\beta}\right) + b_{0,2,-1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}-3\phi_{\beta}\right) \\
& - ib_{0,2,-2,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\sin\left(2\phi_{\alpha}+2\phi_{\beta}\right) + b_{0,2,-2,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}+2\phi_{\beta}\right) - ib_{0,2,-3,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\sin\left(3\phi_{\alpha}+\phi_{\beta}\right) \\
& + b_{0,2,-3,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}+\phi_{\beta}\right) - ib_{0,2,-4,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\sin\left(4\phi_{\alpha}\right) + b_{0,2,-4,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}\right) - ib_{0,2,0,-4}^{r}\rho_{\beta}^{6}\sin\left(4\phi_{\beta}\right) \\
& + b_{0,2,0,-4}^{r}\rho_{\beta}^{6}\cos\left(4\phi_{\beta}\right) + ib_{0,2,3,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\sin\left(3\phi_{\alpha}-\phi_{\beta}\right) + b_{0,2,3,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}-\phi_{\beta}\right) + ib_{0,4,0,2}^{r}\rho_{\beta}^{6}\sin\left(2\phi_{\beta}\right) \\
& + b_{0,4,0,2}^{r}\rho_{\beta}^{6}\cos\left(2\phi_{\beta}\right) + ib_{0,4,1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\sin\left(\phi_{\alpha}+\phi_{\beta}\right) + b_{0,4,1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}+\phi_{\beta}\right) + ib_{0,4,2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\sin\left(2\phi_{\alpha}\right) \\
& + b_{0,4,2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}\right) - ib_{2,0,-1,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\sin\left(\phi_{\alpha}+3\phi_{\beta}\right) + b_{2,0,-1,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}+3\phi_{\beta}\right) \\
& - ib_{2,0,-1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\sin\left(\phi_{\alpha}-3\phi_{\beta}\right) + b_{2,0,-1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}-3\phi_{\beta}\right) - ib_{2,0,-2,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\sin\left(2\phi_{\alpha}+2\phi_{\beta}\right) \\
& + b_{2,0,-2,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}+2\phi_{\beta}\right) - ib_{2,0,-3,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\sin\left(3\phi_{\alpha}+\phi_{\beta}\right) + b_{2,0,-3,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(3\phi_{\alpha}+\phi_{\beta}\right) \\
& - ib_{2,0,-4,-4}^{r}\rho_{\alpha}^{6}\sin\left(4\phi_{\alpha}\right) + b_{2,0,-4,-4}^{r}\rho_{\alpha}^{6}\cos\left(4\phi_{\alpha}\right) - ib_{2,0,0,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\sin\left(4\phi_{\beta}\right) + b_{2,0,0,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(4\phi_{\beta}\right) \\
& + ib_{2,0,3,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\sin\left(3\phi_{\alpha}-\phi_{\beta}\right) + b_{2,0,3,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(3\phi_{\alpha}-\phi_{\beta}\right) + ib_{2,2,0,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\sin\left(2\phi_{\beta}\right) + b_{2,2,0,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\beta}\right) \\
& + ib_{2,2,1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\sin\left(\phi_{\alpha}+\phi_{\beta}\right) + b_{2,2,1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}+\phi_{\beta}\right) + ib_{2,2,2,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\sin\left(2\phi_{\alpha}\right) + b_{2,2,2,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}\right) \\
& + ib_{4,0,0,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\sin\left(2\phi_{\beta}\right) + b_{4,0,0,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\beta}\right) + ib_{4,0,1,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\sin\left(\phi_{\alpha}+\phi_{\beta}\right) + b_{4,0,1,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(\phi_{\alpha}+\phi_{\beta}\right) \\
& + ib_{4,0,2,2}^{r}\rho_{\alpha}^{6}\sin\left(2\phi_{\alpha}\right) + b_{4,0,2,2}^{r}\rho_{\alpha}^{6}\cos\left(2\phi_{\alpha}\right)
\end{aligned}
$$

$$H_{-+}^{(6)} = ib_{0,0,-2,2}^r \rho_\alpha^2 \rho_\beta^4 \sin\left(2\phi_\alpha - 4\phi_\beta\right) + b_{0,0,-2,2}^r \rho_\alpha^2 \rho_\beta^4 \cos\left(2\phi_\alpha - 4\phi_\beta\right) + ib_{0,0,-5,-4}^r \rho_\alpha^5 \rho_\beta \sin\left(5\phi_\alpha - \phi_\beta\right)$$
$$+ b_{0,0,-5,-4}^r \rho_\alpha^5 \rho_\beta \cos\left(5\phi_\alpha - \phi_\beta\right) - ib_{0,0,1,-4}^r \rho_\alpha \rho_\beta^5 \sin\left(\phi_\alpha - 5\phi_\beta\right) + b_{0,0,1,-4}^r \rho_\alpha \rho_\beta^5 \cos\left(\phi_\alpha - 5\phi_\beta\right)$$
$$- ib_{0,0,4,2}^r \rho_\alpha^4 \rho_\beta^2 \sin\left(4\phi_\alpha - 2\phi_\beta\right) + b_{0,0,4,2}^r \rho_\alpha^4 \rho_\beta^2 \cos\left(4\phi_\alpha - 2\phi_\beta\right) + ib_{0,2,-1,-4}^r \rho_\alpha \rho_\beta^5 \sin\left(\phi_\alpha + 3\phi_\beta\right)$$
$$+ b_{0,2,-1,-4}^r \rho_\alpha \rho_\beta^5 \cos\left(\phi_\alpha + 3\phi_\beta\right) + ib_{0,2,-1,2}^r \rho_\alpha \rho_\beta^5 \sin\left(\phi_\alpha - 3\phi_\beta\right) + b_{0,2,-1,2}^r \rho_\alpha \rho_\beta^5 \cos\left(\phi_\alpha - 3\phi_\beta\right)$$
$$+ ib_{0,2,-2,-4}^r \rho_\alpha^2 \rho_\beta^4 \sin\left(2\phi_\alpha + 2\phi_\beta\right) + b_{0,2,-2,-4}^r \rho_\alpha^2 \rho_\beta^4 \cos\left(2\phi_\alpha + 2\phi_\beta\right) + ib_{0,2,-3,-4}^r \rho_\alpha^3 \rho_\beta^3 \sin\left(3\phi_\alpha + \phi_\beta\right)$$
$$+ b_{0,2,-3,-4}^r \rho_\alpha^3 \rho_\beta^3 \cos\left(3\phi_\alpha + \phi_\beta\right) + ib_{0,2,-4,-4}^r \rho_\alpha^4 \rho_\beta^2 \sin\left(4\phi_\alpha\right) + b_{0,2,-4,-4}^r \rho_\alpha^4 \rho_\beta^2 \cos\left(4\phi_\alpha\right) + ib_{0,2,0,-4}^r \rho_\beta^6 \sin\left(4\phi_\beta\right)$$
$$+ b_{0,2,0,-4}^r \rho_\beta^6 \cos\left(4\phi_\beta\right) - ib_{0,2,3,2}^r \rho_\alpha^3 \rho_\beta^3 \sin\left(3\phi_\alpha - \phi_\beta\right) + b_{0,2,3,2}^r \rho_\alpha^3 \rho_\beta^3 \cos\left(3\phi_\alpha - \phi_\beta\right) - ib_{0,4,0,2}^r \rho_\beta^6 \sin\left(2\phi_\beta\right)$$
$$+ b_{0,4,0,2}^r \rho_\beta^6 \cos\left(2\phi_\beta\right) - ib_{0,4,1,2}^r \rho_\alpha \rho_\beta^5 \sin\left(\phi_\alpha + \phi_\beta\right) + b_{0,4,1,2}^r \rho_\alpha \rho_\beta^5 \cos\left(\phi_\alpha + \phi_\beta\right) - ib_{0,4,2,2}^r \rho_\alpha^2 \rho_\beta^4 \sin\left(2\phi_\alpha\right)$$
$$+ b_{0,4,2,2}^r \rho_\alpha^2 \rho_\beta^4 \cos\left(2\phi_\alpha\right) + ib_{2,0,-1,-4}^r \rho_\alpha^3 \rho_\beta^3 \sin\left(\phi_\alpha + 3\phi_\beta\right) + b_{2,0,-1,-4}^r \rho_\alpha^3 \rho_\beta^3 \cos\left(\phi_\alpha + 3\phi_\beta\right)$$
$$+ ib_{2,0,-1,2}^r \rho_\alpha^3 \rho_\beta^3 \sin\left(\phi_\alpha - 3\phi_\beta\right) + b_{2,0,-1,2}^r \rho_\alpha^3 \rho_\beta^3 \cos\left(\phi_\alpha - 3\phi_\beta\right) + ib_{2,0,-2,-4}^r \rho_\alpha^4 \rho_\beta^2 \sin\left(2\phi_\alpha + 2\phi_\beta\right)$$
$$+ b_{2,0,-2,-4}^r \rho_\alpha^4 \rho_\beta^2 \cos\left(2\phi_\alpha + 2\phi_\beta\right) + ib_{2,0,-3,-4}^r \rho_\alpha^5 \rho_\beta \sin\left(3\phi_\alpha + \phi_\beta\right) + b_{2,0,-3,-4}^r \rho_\alpha^5 \rho_\beta \cos\left(3\phi_\alpha + \phi_\beta\right)$$
$$+ ib_{2,0,-4,-4}^r \rho_\alpha^6 \sin\left(4\phi_\alpha\right) + b_{2,0,-4,-4}^r \rho_\alpha^6 \cos\left(4\phi_\alpha\right) + ib_{2,0,0,-4}^r \rho_\alpha^2 \rho_\beta^4 \sin\left(4\phi_\beta\right) + b_{2,0,0,-4}^r \rho_\alpha^2 \rho_\beta^4 \cos\left(4\phi_\beta\right)$$
$$- ib_{2,0,3,2}^r \rho_\alpha^5 \rho_\beta \sin\left(3\phi_\alpha - \phi_\beta\right) + b_{2,0,3,2}^r \rho_\alpha^5 \rho_\beta \cos\left(3\phi_\alpha - \phi_\beta\right) - ib_{2,2,0,2}^r \rho_\alpha^2 \rho_\beta^4 \sin\left(2\phi_\beta\right) + b_{2,2,0,2}^r \rho_\alpha^2 \rho_\beta^4 \cos\left(2\phi_\beta\right)$$
$$- ib_{2,2,1,2}^r \rho_\alpha^3 \rho_\beta^3 \sin\left(\phi_\alpha + \phi_\beta\right) + b_{2,2,1,2}^r \rho_\alpha^3 \rho_\beta^3 \cos\left(\phi_\alpha + \phi_\beta\right) - ib_{2,2,2,2}^r \rho_\alpha^4 \rho_\beta^2 \sin\left(2\phi_\alpha\right) + b_{2,2,2,2}^r \rho_\alpha^4 \rho_\beta^2 \cos\left(2\phi_\alpha\right)$$
$$- ib_{4,0,0,2}^r \rho_\alpha^4 \rho_\beta^2 \sin\left(2\phi_\beta\right) + b_{4,0,0,2}^r \rho_\alpha^4 \rho_\beta^2 \cos\left(2\phi_\beta\right) - ib_{4,0,1,2}^r \rho_\alpha^5 \rho_\beta \sin\left(\phi_\alpha + \phi_\beta\right) + b_{4,0,1,2}^r \rho_\alpha^5 \rho_\beta \cos\left(\phi_\alpha + \phi_\beta\right)$$
$$- ib_{4,0,2,2}^r \rho_\alpha^6 \sin\left(2\phi_\alpha\right) + b_{4,0,2,2}^r \rho_\alpha^6 \cos\left(2\phi_\alpha\right)$$

Cartesian e-coordinates:

$$H_{++}^{(6)} = a_{0,0,0,6}^r \left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 - 4x_\beta y_\beta + y_\beta^2\right)\left(x_\beta^2 + 4x_\beta y_\beta + y_\beta^2\right)$$
$$+ a_{0,0,1,6}^r \left(x_\alpha \left(x_\beta^5 - 10x_\beta^3 y_\beta^2 + 5x_\beta y_\beta^4\right) + y_\alpha \left(-5x_\beta^4 y_\beta + 10x_\beta^2 y_\beta^3 - y_\beta^5\right)\right)$$
$$+ a_{0,0,2,6}^r \left(x_\alpha \left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha \left(-x_\beta^2 - 2x_\beta y_\beta + y_\beta^2\right)\right)\left(x_\alpha \left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha \left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\right)$$
$$+ a_{0,0,3,0}^r \left(x_\alpha x_\beta + y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 + 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right)$$
$$+ a_{0,0,3,6}^r \left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 - 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right) + a_{0,0,4,6}^r \left(x_\alpha \left(-2x_\beta y_\alpha - 2y_\alpha y_\beta\right)\right.$$
$$\left. + x_\beta \left(x_\alpha^2 - y_\alpha^2\right) + y_\beta \left(-x_\alpha^2 + y_\alpha^2\right)\right)\left(x_\alpha \left(2x_\beta y_\alpha - 2y_\alpha y_\beta\right) + x_\beta \left(x_\alpha^2 - y_\alpha^2\right) + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$+ a_{0,0,5,6}^r \left(-5x_\alpha^4 y_\alpha y_\beta + 5x_\alpha x_\beta y_\alpha^4 + x_\beta \left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right) + y_\beta \left(10x_\alpha^2 y_\alpha^3 - y_\alpha^5\right)\right)$$
$$+ a_{0,0,6,6}^r \left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 - 4x_\alpha y_\alpha + y_\alpha^2\right)\left(x_\alpha^2 + 4x_\alpha y_\alpha + y_\alpha^2\right)$$
$$+ a_{0,2,2,0}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha \left(x_\beta - y_\beta\right) + y_\alpha \left(x_\beta + y_\beta\right)\right)\left(x_\alpha \left(x_\beta + y_\beta\right) + y_\alpha \left(-x_\beta + y_\beta\right)\right) + a_{0,4,1,0}^r \left(x_\beta^2 + y_\beta^2\right)^2 \left(x_\alpha x_\beta + y_\alpha y_\beta\right)$$
$$+ a_{0,6,0,0}^r \left(x_\beta^2 + y_\beta^2\right)^3 + a_{2,0,2,0}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha \left(x_\beta - y_\beta\right) + y_\alpha \left(x_\beta + y_\beta\right)\right)\left(x_\alpha \left(x_\beta + y_\beta\right) + y_\alpha \left(-x_\beta + y_\beta\right)\right)$$
$$+ a_{2,2,1,0}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,4,0,0}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)^2 + a_{4,0,1,0}^r \left(x_\alpha^2 + y_\alpha^2\right)^2 \left(x_\alpha x_\beta + y_\alpha y_\beta\right)$$
$$+ a_{4,2,0,0}^r \left(x_\alpha^2 + y_\alpha^2\right)^2 \left(x_\beta^2 + y_\beta^2\right) + a_{6,0,0,0}^r \left(x_\alpha^2 + y_\alpha^2\right)^3$$

$$
\begin{aligned}
H_{--}^{(6)} = {}& a_{0,0,0,6}^{r}\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)\left(x_{\beta}^{2}-4x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\left(x_{\beta}^{2}+4x_{\beta}y_{\beta}+y_{\beta}^{2}\right) \\
& + a_{0,0,1,6}^{r}\left(x_{\alpha}\left(x_{\beta}^{5}-10x_{\beta}^{3}y_{\beta}^{2}+5x_{\beta}y_{\beta}^{4}\right)+y_{\alpha}\left(-5x_{\beta}^{4}y_{\beta}+10x_{\beta}^{2}y_{\beta}^{3}-y_{\beta}^{5}\right)\right) \\
& + a_{0,0,2,6}^{r}\left(x_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(-x_{\beta}^{2}-2x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\right)\left(x_{\alpha}\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\right) \\
& + a_{0,0,3,0}^{r}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)\left(x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}+8x_{\alpha}x_{\beta}y_{\alpha}y_{\beta}-3x_{\beta}^{2}y_{\alpha}^{2}+y_{\alpha}^{2}y_{\beta}^{2}\right) \\
& + a_{0,0,3,6}^{r}\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right)\left(x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}-8x_{\alpha}x_{\beta}y_{\alpha}y_{\beta}-3x_{\beta}^{2}y_{\alpha}^{2}+y_{\alpha}^{2}y_{\beta}^{2}\right)+a_{0,0,4,6}^{r}\left(x_{\alpha}\left(-2x_{\beta}y_{\alpha}-2y_{\alpha}y_{\beta}\right)\right. \\
& \qquad\qquad \left. + x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(-x_{\alpha}^{2}+y_{\alpha}^{2}\right)\right)\left(x_{\alpha}\left(2x_{\beta}y_{\alpha}-2y_{\alpha}y_{\beta}\right)+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right) \\
& + a_{0,0,5,6}^{r}\left(-5x_{\alpha}^{4}y_{\alpha}y_{\beta}+5x_{\alpha}x_{\beta}y_{\alpha}^{4}+x_{\beta}\left(x_{\alpha}^{5}-10x_{\alpha}^{3}y_{\alpha}^{2}\right)+y_{\beta}\left(10x_{\alpha}^{2}y_{\alpha}^{3}-y_{\alpha}^{5}\right)\right) \\
& + a_{0,0,6,6}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\alpha}^{2}-4x_{\alpha}y_{\alpha}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+4x_{\alpha}y_{\alpha}+y_{\alpha}^{2}\right) \\
& + a_{0,2,2,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(x_{\beta}+y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}+y_{\beta}\right)\right)+a_{0,4,1,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right) \\
& + a_{0,6,0,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{3}+a_{2,0,2,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(x_{\beta}+y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}+y_{\beta}\right)\right) \\
& + a_{2,2,1,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)+a_{2,4,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}+a_{4,0,1,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right) \\
& + a_{4,2,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+a_{6,0,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{3}
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{+-} = {}& 2ib^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)+2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta+y_\alpha\left(-x_\beta^2+y_\beta^2\right)\right) \\
&+ b^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(-x_\beta^2+2x_\beta y_\beta+y_\beta^2\right)\right) \\
&+ ib^r_{0,0,-5,-4}\left(-5x_\alpha^4 x_\beta y_\alpha+5x_\alpha y_\alpha^4 y_\beta+x_\beta\left(10x_\alpha^2 y_\alpha^3-y_\alpha^5\right)+y_\beta\left(x_\alpha^5-10x_\alpha^3 y_\alpha^2\right)\right) \\
&+ b^r_{0,0,-5,-4}\left(5x_\alpha^4 y_\alpha y_\beta+5x_\alpha x_\beta y_\alpha^4+x_\beta\left(x_\alpha^5-10x_\alpha^3 y_\alpha^2\right)+y_\beta\left(-10x_\alpha^2 y_\alpha^3+y_\alpha^5\right)\right) \\
&+ b^r_{0,0,1,-4}\left(x_\alpha\left(x_\beta^5-10x_\beta^3 y_\beta^2+5x_\beta y_\beta^4\right)+y_\alpha\left(5x_\beta^4 y_\beta-10x_\beta^2 y_\beta^3+y_\beta^5\right)\right) \\
&- ib^r_{0,0,1,-4}\left(x_\alpha\left(5x_\beta^4 y_\beta-10x_\beta^2 y_\beta^3+y_\beta^5\right)+y_\alpha\left(-x_\beta^5+10x_\beta^3 y_\beta^2-5x_\beta y_\beta^4\right)\right) \\
&- 2ib^r_{0,0,4,2}\left(-2x_\alpha x_\beta y_\alpha+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right) \\
&+ b^r_{0,0,4,2}\left(x_\alpha\left(-2x_\beta y_\alpha+2y_\alpha y_\beta\right)+x_\beta\left(x_\alpha^2-y_\alpha^2\right)+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha+2y_\alpha y_\beta\right)+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right. \\
&\qquad\qquad \left.+y_\beta\left(-x_\alpha^2+y_\alpha^2\right)\right)+b^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2 y_\beta+y_\beta^3\right)\right) \\
&- ib^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right) \\
&+ b^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2 y_\beta-y_\beta^3\right)\right) \\
&+ ib^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)-2ib^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right) \\
&+ b^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right) \\
&+ b^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3 x_\beta-3x_\alpha^2 y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3 y_\beta\right) \\
&- ib^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3 y_\beta+3x_\alpha^2 x_\beta y_\alpha-3x_\alpha y_\alpha^2 y_\beta-x_\beta y_\alpha^3\right)-4ib^r_{0,2,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right) \\
&+ b^r_{0,2,-4,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)-4ib^r_{0,2,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right) \\
&+ b^r_{0,2,0,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+b^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3 x_\beta+3x_\alpha^2 y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3 y_\beta\right) \\
&- ib^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3 y_\beta-3x_\alpha^2 x_\beta y_\alpha-3x_\alpha y_\alpha^2 y_\beta+x_\beta y_\alpha^3\right)+2ib^r_{0,4,0,2}x_\beta y_\beta\left(x_\beta^2+y_\beta^2\right)^2 \\
&+ b^r_{0,4,0,2}\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)^2+b^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha x_\beta-y_\alpha y_\beta\right) \\
&+ ib^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{0,4,2,2}x_\alpha y_\alpha\left(x_\beta^2+y_\beta^2\right)^2 \\
&+ b^r_{0,4,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)^2+b^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2 y_\beta+y_\beta^3\right)\right) \\
&- ib^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right) \\
&+ b^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2 y_\beta-y_\beta^3\right)\right) \\
&+ ib^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)-2ib^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right) \\
&+ b^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right) \\
&+ b^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3 x_\beta-3x_\alpha^2 y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3 y_\beta\right) \\
&- ib^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3 y_\beta+3x_\alpha^2 x_\beta y_\alpha-3x_\alpha y_\alpha^2 y_\beta-x_\beta y_\alpha^3\right)-4ib^r_{2,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right) \\
&+ b^r_{2,0,-4,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)-4ib^r_{2,0,0,-4}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right) \\
&+ b^r_{2,0,0,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+b^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3 x_\beta+3x_\alpha^2 y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3 y_\beta\right) \\
&- ib^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3 y_\beta-3x_\alpha^2 x_\beta y_\alpha-3x_\alpha y_\alpha^2 y_\beta+x_\beta y_\alpha^3\right)+2ib^r_{2,2,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right) \\
&+ b^r_{2,2,0,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)+b^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right) \\
&+ ib^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{2,2,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right) \\
&+ b^r_{2,2,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)+2ib^r_{4,0,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)^2 \\
&+ b^r_{4,0,0,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha x_\beta-y_\alpha y_\beta\right) \\
&+ ib^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{4,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)^2+b^r_{4,0,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)^2
\end{aligned}
$$

$$\begin{aligned}
H^{(6)}_{-+} = {}& -2ib^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) \\
& + b^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(-x_\beta^2 + 2x_\beta y_\beta + y_\beta^2\right)\right) \\
& - ib^r_{0,0,-5,-4}\left(-5x_\alpha^4 x_\beta y_\alpha + 5x_\alpha y_\alpha^4 y_\beta + x_\beta\left(10x_\alpha^2 y_\alpha^3 - y_\alpha^5\right) + y_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right)\right) \\
& + b^r_{0,0,-5,-4}\left(5x_\alpha^4 y_\alpha y_\beta + 5x_\alpha x_\beta y_\alpha^4 + x_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right) + y_\beta\left(-10x_\alpha^2 y_\alpha^3 + y_\alpha^5\right)\right) \\
& + b^r_{0,0,1,-4}\left(x_\alpha\left(x_\beta^5 - 10x_\beta^3 y_\beta^2 + 5x_\beta y_\beta^4\right) + y_\alpha\left(5x_\beta^4 y_\beta - 10x_\beta^2 y_\beta^3 + y_\beta^5\right)\right) \\
& + ib^r_{0,0,1,-4}\left(x_\alpha\left(5x_\beta^4 y_\beta - 10x_\beta^2 y_\beta^3 + y_\beta^5\right) + y_\alpha\left(-x_\beta^5 + 10x_\beta^3 y_\beta^2 - 5x_\beta y_\beta^4\right)\right) \\
& + 2ib^r_{0,0,4,2}\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + b^r_{0,0,4,2}\left(x_\alpha\left(-2x_\beta y_\alpha + 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha + 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right. \\
& \hspace{4cm}\left. + y_\beta\left(-x_\alpha^2 + y_\alpha^2\right)\right) + b^r_{0,2,-1,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) \\
& + ib^r_{0,2,-1,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) \\
& + b^r_{0,2,-1,2}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) \\
& - ib^r_{0,2,-1,2}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) + 2ib^r_{0,2,-2,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& + b^r_{0,2,-2,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& + b^r_{0,2,-3,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) \\
& + ib^r_{0,2,-3,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right) + 4ib^r_{0,2,-4,-4}x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + b^r_{0,2,-4,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) + 4ib^r_{0,2,0,-4}x_\beta y_\beta\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + b^r_{0,2,0,-4}\left(x_\beta^2 + y_\beta^2\right)\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + b^r_{0,2,3,2}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) \\
& + ib^r_{0,2,3,2}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right) - 2ib^r_{0,4,0,2}x_\beta y_\beta\left(x_\beta^2 + y_\beta^2\right)^2 \\
& + b^r_{0,4,0,2}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right)^2 + b^r_{0,4,1,2}\left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha x_\beta - y_\alpha y_\beta\right) \\
& - ib^r_{0,4,1,2}\left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2ib^r_{0,4,2,2}x_\alpha y_\alpha\left(x_\beta^2 + y_\beta^2\right)^2 \\
& + b^r_{0,4,2,2}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right)^2 + b^r_{2,0,-1,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) \\
& + ib^r_{2,0,-1,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) \\
& + b^r_{2,0,-1,2}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) \\
& - ib^r_{2,0,-1,2}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) + 2ib^r_{2,0,-2,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& + b^r_{2,0,-2,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& + b^r_{2,0,-3,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) \\
& + ib^r_{2,0,-3,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right) + 4ib^r_{2,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right) \\
& + b^r_{2,0,-4,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) + 4ib^r_{2,0,0,-4}x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) \\
& + b^r_{2,0,0,-4}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + b^r_{2,0,3,2}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) \\
& + ib^r_{2,0,3,2}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right) - 2ib^r_{2,2,0,2}x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + b^r_{2,2,0,2}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) + b^r_{2,2,1,2}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right) \\
& - ib^r_{2,2,1,2}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2ib^r_{2,2,2,2}x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + b^r_{2,2,2,2}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) - 2ib^r_{4,0,0,2}x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)^2 \\
& + b^r_{4,0,0,2}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b^r_{4,0,1,2}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha x_\beta - y_\alpha y_\beta\right) \\
& - ib^r_{4,0,1,2}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2ib^r_{4,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)^2 + b^r_{4,0,2,2}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)^2
\end{aligned}$$

4 Vibronic Hamiltonian operator in the real E basis

$$\hat{H} = \begin{pmatrix}|X\rangle & |Y\rangle\end{pmatrix}\begin{pmatrix}H_{XX} & H_{XY} \\ H_{YX} & H_{YY}\end{pmatrix}\begin{pmatrix}\langle X| \\ \langle Y|\end{pmatrix}$$

5 Matrix element expansions in the real E basis

5.1 Order: 0

Number of fitting parameters: H_{XX}: 1 (all from H_{++}), H_{XY}: 0, H_{YY}: 0.

Polar e-coordinates:

$$H_{XX}^{(0)} = a_{0,0,0,0}^r$$

$$H_{XY}^{(0)} = 0$$

$$H_{YX}^{(0)} = 0$$

$$H_{YY}^{(0)} = a_{0,0,0,0}^r$$

Cartesian e-coordinates:

$$H_{XX}^{(0)} = a_{0,0,0,0}^r$$

$$H_{XY}^{(0)} = 0$$

$$H_{YX}^{(0)} = 0$$

$$H_{YY}^{(0)} = a_{0,0,0,0}^r$$

5.2 Order: 1

Number of fitting parameters: H_{XX}: 2 (all from H_{+-}), H_{XY}: 2 (all from H_{+-}), H_{YY}: 2 (all from H_{+-}).

Polar e-coordinates:

$$H_{XX}^{(1)} = b_{0,0,-1,-1}^r \rho_\alpha \cos(\phi_\alpha) + b_{0,0,0,-1}^r \rho_\beta \cos(\phi_\beta)$$

$$H_{XY}^{(1)} = b_{0,0,-1,-1}^r \rho_\alpha \sin(\phi_\alpha) + b_{0,0,0,-1}^r \rho_\beta \sin(\phi_\beta)$$

$$H_{YX}^{(1)} = b_{0,0,-1,-1}^r \rho_\alpha \sin(\phi_\alpha) + b_{0,0,0,-1}^r \rho_\beta \sin(\phi_\beta)$$

$$H_{YY}^{(1)} = -b_{0,0,-1,-1}^r \rho_\alpha \cos(\phi_\alpha) - b_{0,0,0,-1}^r \rho_\beta \cos(\phi_\beta)$$

Cartesian e-coordinates:

$$H_{XX}^{(1)} = b_{0,0,-1,-1}^{r} x_{\alpha} + b_{0,0,0,-1}^{r} x_{\beta}$$

$$H_{XY}^{(1)} = b_{0,0,-1,-1}^{r} y_{\alpha} + b_{0,0,0,-1}^{r} y_{\beta}$$

$$H_{YX}^{(1)} = b_{0,0,-1,-1}^{r} y_{\alpha} + b_{0,0,0,-1}^{r} y_{\beta}$$

$$H_{YY}^{(1)} = -b_{0,0,-1,-1}^{r} x_{\alpha} - b_{0,0,0,-1}^{r} x_{\beta}$$

5.3 Order: 2

Number of fitting parameters: H_{XX}: 6 (3 from H_{++}, 3 from H_{+-}), H_{XY}: 3 (all from H_{+-}), H_{YY}: 3 (all from H_{+-}).

Polar e-coordinates:

$$H_{XX}^{(2)} = a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos(\phi_{\alpha} - \phi_{\beta}) + a_{0,2,0,0}^{r} \rho_{\beta}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} + b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos(2\phi_{\beta})$$
$$+ b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos(\phi_{\alpha} + \phi_{\beta}) + b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos(2\phi_{\alpha})$$

$$H_{XY}^{(2)} = -b_{0,0,0,2}^{r} \rho_{\beta}^{2} \sin(2\phi_{\beta}) - b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \sin(\phi_{\alpha} + \phi_{\beta}) - b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \sin(2\phi_{\alpha})$$

$$H_{YX}^{(2)} = -b_{0,0,0,2}^{r} \rho_{\beta}^{2} \sin(2\phi_{\beta}) - b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \sin(\phi_{\alpha} + \phi_{\beta}) - b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \sin(2\phi_{\alpha})$$

$$H_{YY}^{(2)} = a_{0,0,1,0}^{r} \rho_{\alpha} \rho_{\beta} \cos(\phi_{\alpha} - \phi_{\beta}) + a_{0,2,0,0}^{r} \rho_{\beta}^{2} + a_{2,0,0,0}^{r} \rho_{\alpha}^{2} - b_{0,0,0,2}^{r} \rho_{\beta}^{2} \cos(2\phi_{\beta})$$
$$- b_{0,0,1,2}^{r} \rho_{\alpha} \rho_{\beta} \cos(\phi_{\alpha} + \phi_{\beta}) - b_{0,0,2,2}^{r} \rho_{\alpha}^{2} \cos(2\phi_{\alpha})$$

Cartesian e-coordinates:

$$H_{XX}^{(2)} = a_{0,0,1,0}^{r} (x_{\alpha} x_{\beta} + y_{\alpha} y_{\beta}) + a_{0,2,0,0}^{r} (x_{\beta}^{2} + y_{\beta}^{2}) + a_{2,0,0,0}^{r} (x_{\alpha}^{2} + y_{\alpha}^{2}) + b_{0,0,0,2}^{r} (x_{\beta} - y_{\beta})(x_{\beta} + y_{\beta})$$
$$+ b_{0,0,1,2}^{r} (x_{\alpha} x_{\beta} - y_{\alpha} y_{\beta}) + b_{0,0,2,2}^{r} (x_{\alpha} - y_{\alpha})(x_{\alpha} + y_{\alpha})$$

$$H_{XY}^{(2)} = -2 b_{0,0,0,2}^{r} x_{\beta} y_{\beta} - b_{0,0,1,2}^{r} (x_{\alpha} y_{\beta} + x_{\beta} y_{\alpha}) - 2 b_{0,0,2,2}^{r} x_{\alpha} y_{\alpha}$$

$$H_{YX}^{(2)} = -2 b_{0,0,0,2}^{r} x_{\beta} y_{\beta} - b_{0,0,1,2}^{r} (x_{\alpha} y_{\beta} + x_{\beta} y_{\alpha}) - 2 b_{0,0,2,2}^{r} x_{\alpha} y_{\alpha}$$

$$H_{YY}^{(2)} = a_{0,0,1,0}^{r} (x_{\alpha} x_{\beta} + y_{\alpha} y_{\beta}) + a_{0,2,0,0}^{r} (x_{\beta}^{2} + y_{\beta}^{2}) + a_{2,0,0,0}^{r} (x_{\alpha}^{2} + y_{\alpha}^{2}) - b_{0,0,0,2}^{r} (x_{\beta} - y_{\beta})(x_{\beta} + y_{\beta})$$
$$- b_{0,0,1,2}^{r} (x_{\alpha} x_{\beta} - y_{\alpha} y_{\beta}) - b_{0,0,2,2}^{r} (x_{\alpha} - y_{\alpha})(x_{\alpha} + y_{\alpha})$$

5.4 Order: 3

Number of fitting parameters: H_{XX}: 10 (4 from H_{++}, 6 from H_{+-}), H_{XY}: 6 (all from H_{+-}), H_{YY}: 6 (all from H_{+-}).

Polar e-coordinates:

$$
\begin{aligned}
H_{XX}^{(3)} =\ & a_{0,0,0,3}^r \rho_\beta^3 \cos(3\phi_\beta) + a_{0,0,1,3}^r \rho_\alpha \rho_\beta^2 \cos(\phi_\alpha + 2\phi_\beta) + a_{0,0,2,3}^r \rho_\alpha^2 \rho_\beta \cos(2\phi_\alpha + \phi_\beta) + a_{0,0,3,3}^r \rho_\alpha^3 \cos(3\phi_\alpha) \\
& + b_{0,0,-2,-1}^r \rho_\alpha^2 \rho_\beta \cos(2\phi_\alpha - \phi_\beta) + b_{0,0,1,-1}^r \rho_\alpha \rho_\beta^2 \cos(\phi_\alpha - 2\phi_\beta) + b_{0,2,-1,-1}^r \rho_\alpha \rho_\beta^2 \cos(\phi_\alpha) + b_{0,2,0,-1}^r \rho_\beta^3 \cos(\phi_\beta) \\
& + b_{2,0,-1,-1}^r \rho_\alpha^3 \cos(\phi_\alpha) + b_{2,0,0,-1}^r \rho_\alpha^2 \rho_\beta \cos(\phi_\beta)
\end{aligned}
$$

$$
\begin{aligned}
H_{XY}^{(3)} =\ & b_{0,0,-2,-1}^r \rho_\alpha^2 \rho_\beta \sin(2\phi_\alpha - \phi_\beta) - b_{0,0,1,-1}^r \rho_\alpha \rho_\beta^2 \sin(\phi_\alpha - 2\phi_\beta) + b_{0,2,-1,-1}^r \rho_\alpha \rho_\beta^2 \sin(\phi_\alpha) + b_{0,2,0,-1}^r \rho_\beta^3 \sin(\phi_\beta) \\
& + b_{2,0,-1,-1}^r \rho_\alpha^3 \sin(\phi_\alpha) + b_{2,0,0,-1}^r \rho_\alpha^2 \rho_\beta \sin(\phi_\beta)
\end{aligned}
$$

$$
\begin{aligned}
H_{YX}^{(3)} =\ & b_{0,0,-2,-1}^r \rho_\alpha^2 \rho_\beta \sin(2\phi_\alpha - \phi_\beta) - b_{0,0,1,-1}^r \rho_\alpha \rho_\beta^2 \sin(\phi_\alpha - 2\phi_\beta) + b_{0,2,-1,-1}^r \rho_\alpha \rho_\beta^2 \sin(\phi_\alpha) + b_{0,2,0,-1}^r \rho_\beta^3 \sin(\phi_\beta) \\
& + b_{2,0,-1,-1}^r \rho_\alpha^3 \sin(\phi_\alpha) + b_{2,0,0,-1}^r \rho_\alpha^2 \rho_\beta \sin(\phi_\beta)
\end{aligned}
$$

$$
\begin{aligned}
H_{YY}^{(3)} =\ & a_{0,0,0,3}^r \rho_\beta^3 \cos(3\phi_\beta) + a_{0,0,1,3}^r \rho_\alpha \rho_\beta^2 \cos(\phi_\alpha + 2\phi_\beta) + a_{0,0,2,3}^r \rho_\alpha^2 \rho_\beta \cos(2\phi_\alpha + \phi_\beta) + a_{0,0,3,3}^r \rho_\alpha^3 \cos(3\phi_\alpha) \\
& - b_{0,0,-2,-1}^r \rho_\alpha^2 \rho_\beta \cos(2\phi_\alpha - \phi_\beta) - b_{0,0,1,-1}^r \rho_\alpha \rho_\beta^2 \cos(\phi_\alpha - 2\phi_\beta) - b_{0,2,-1,-1}^r \rho_\alpha \rho_\beta^2 \cos(\phi_\alpha) - b_{0,2,0,-1}^r \rho_\beta^3 \cos(\phi_\beta) \\
& - b_{2,0,-1,-1}^r \rho_\alpha^3 \cos(\phi_\alpha) - b_{2,0,0,-1}^r \rho_\alpha^2 \rho_\beta \cos(\phi_\beta)
\end{aligned}
$$

Cartesian e-coordinates:

$$
\begin{aligned}
H_{XX}^{(3)} =\ & a_{0,0,0,3}^r x_\beta(x_\beta^2 - 3y_\beta^2) + a_{0,0,1,3}^r \left(x_\alpha(x_\beta^2 - y_\beta^2) - 2x_\beta y_\alpha y_\beta\right) + a_{0,0,2,3}^r \left(-2x_\alpha y_\alpha y_\beta + x_\beta(x_\alpha^2 - y_\alpha^2)\right) \\
& + a_{0,0,3,3}^r x_\alpha(x_\alpha^2 - 3y_\alpha^2) + b_{0,0,-2,-1}^r \left(2x_\alpha y_\alpha y_\beta + x_\beta(x_\alpha^2 - y_\alpha^2)\right) + b_{0,0,1,-1}^r \left(x_\alpha(x_\beta^2 - y_\beta^2) + 2x_\beta y_\alpha y_\beta\right) \\
& + b_{0,2,-1,-1}^r x_\alpha(x_\beta^2 + y_\beta^2) + b_{0,2,0,-1}^r x_\beta(x_\beta^2 + y_\beta^2) + b_{2,0,-1,-1}^r x_\alpha(x_\alpha^2 + y_\alpha^2) + b_{2,0,0,-1}^r x_\beta(x_\alpha^2 + y_\alpha^2)
\end{aligned}
$$

$$
\begin{aligned}
H_{XY}^{(3)} =\ & -b_{0,0,-2,-1}^r \left(-2x_\alpha x_\beta y_\alpha + y_\beta(x_\alpha^2 - y_\alpha^2)\right) + b_{0,0,1,-1}^r \left(2x_\alpha x_\beta y_\beta + y_\alpha(-x_\beta^2 + y_\beta^2)\right) \\
& + b_{0,2,-1,-1}^r y_\alpha(x_\beta^2 + y_\beta^2) + b_{0,2,0,-1}^r y_\beta(x_\beta^2 + y_\beta^2) + b_{2,0,-1,-1}^r y_\alpha(x_\alpha^2 + y_\alpha^2) + b_{2,0,0,-1}^r y_\beta(x_\alpha^2 + y_\alpha^2)
\end{aligned}
$$

$$
\begin{aligned}
H_{YX}^{(3)} =\ & -b_{0,0,-2,-1}^r \left(-2x_\alpha x_\beta y_\alpha + y_\beta(x_\alpha^2 - y_\alpha^2)\right) + b_{0,0,1,-1}^r \left(2x_\alpha x_\beta y_\beta + y_\alpha(-x_\beta^2 + y_\beta^2)\right) \\
& + b_{0,2,-1,-1}^r y_\alpha(x_\beta^2 + y_\beta^2) + b_{0,2,0,-1}^r y_\beta(x_\beta^2 + y_\beta^2) + b_{2,0,-1,-1}^r y_\alpha(x_\alpha^2 + y_\alpha^2) + b_{2,0,0,-1}^r y_\beta(x_\alpha^2 + y_\alpha^2)
\end{aligned}
$$

$$
\begin{aligned}
H_{YY}^{(3)} =\ & a_{0,0,0,3}^r x_\beta(x_\beta^2 - 3y_\beta^2) + a_{0,0,1,3}^r \left(x_\alpha(x_\beta^2 - y_\beta^2) - 2x_\beta y_\alpha y_\beta\right) + a_{0,0,2,3}^r \left(-2x_\alpha y_\alpha y_\beta + x_\beta(x_\alpha^2 - y_\alpha^2)\right) \\
& + a_{0,0,3,3}^r x_\alpha(x_\alpha^2 - 3y_\alpha^2) - b_{0,0,-2,-1}^r \left(2x_\alpha y_\alpha y_\beta + x_\beta(x_\alpha^2 - y_\alpha^2)\right) - b_{0,0,1,-1}^r \left(x_\alpha(x_\beta^2 - y_\beta^2) + 2x_\beta y_\alpha y_\beta\right) \\
& - b_{0,2,-1,-1}^r x_\alpha(x_\beta^2 + y_\beta^2) - b_{0,2,0,-1}^r x_\beta(x_\beta^2 + y_\beta^2) - b_{2,0,-1,-1}^r x_\alpha(x_\alpha^2 + y_\alpha^2) - b_{2,0,0,-1}^r x_\beta(x_\alpha^2 + y_\alpha^2)
\end{aligned}
$$

5.5 Order: 4

Number of fitting parameters: H_{XX}: 19 (6 from H_{++}, 13 from H_{+-}), H_{XY}: 13 (all from H_{+-}), H_{YY}: 13 (all from H_{+-}).

Polar e-coordinates:

$$
\begin{aligned}
H_{XX}^{(4)} = {}& a_{0,0,2,0}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha - 2\phi_\beta\right) + a_{0,2,1,0}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - \phi_\beta\right) + a_{0,4,0,0}^{r}\rho_\beta^4 + a_{2,0,1,0}^{r}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha - \phi_\beta\right) + a_{2,2,0,0}^{r}\rho_\alpha^2\rho_\beta^2 \\
& + a_{4,0,0,0}^{r}\rho_\alpha^4 + b_{0,0,-1,-4}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + 3\phi_\beta\right) + b_{0,0,-1,2}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - 3\phi_\beta\right) + b_{0,0,-2,-4}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha + 2\phi_\beta\right) \\
& + b_{0,0,-3,-4}^{r}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha + \phi_\beta\right) + b_{0,0,-4,-4}^{r}\rho_\alpha^4\cos\left(4\phi_\alpha\right) + b_{0,0,0,-4}^{r}\rho_\beta^4\cos\left(4\phi_\beta\right) + b_{0,0,3,2}^{r}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha - \phi_\beta\right) \\
& + b_{0,2,0,2}^{r}\rho_\beta^4\cos\left(2\phi_\beta\right) + b_{0,2,1,2}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + \phi_\beta\right) + b_{0,2,2,2}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha\right) + b_{2,0,0,2}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\beta\right) \\
& + b_{2,0,1,2}^{r}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha + \phi_\beta\right) + b_{2,0,2,2}^{r}\rho_\alpha^4\cos\left(2\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{XY}^{(4)} = {}& b_{0,0,-1,-4}^{r}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + 3\phi_\beta\right) + b_{0,0,-1,2}^{r}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha - 3\phi_\beta\right) + b_{0,0,-2,-4}^{r}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha + 2\phi_\beta\right) \\
& + b_{0,0,-3,-4}^{r}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha + \phi_\beta\right) + b_{0,0,-4,-4}^{r}\rho_\alpha^4\sin\left(4\phi_\alpha\right) + b_{0,0,0,-4}^{r}\rho_\beta^4\sin\left(4\phi_\beta\right) - b_{0,0,3,2}^{r}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha - \phi_\beta\right) \\
& - b_{0,2,0,2}^{r}\rho_\beta^4\sin\left(2\phi_\beta\right) - b_{0,2,1,2}^{r}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + \phi_\beta\right) - b_{0,2,2,2}^{r}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha\right) - b_{2,0,0,2}^{r}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\beta\right) \\
& - b_{2,0,1,2}^{r}\rho_\alpha^3\rho_\beta\sin\left(\phi_\alpha + \phi_\beta\right) - b_{2,0,2,2}^{r}\rho_\alpha^4\sin\left(2\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{YX}^{(4)} = {}& b_{0,0,-1,-4}^{r}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + 3\phi_\beta\right) + b_{0,0,-1,2}^{r}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha - 3\phi_\beta\right) + b_{0,0,-2,-4}^{r}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha + 2\phi_\beta\right) \\
& + b_{0,0,-3,-4}^{r}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha + \phi_\beta\right) + b_{0,0,-4,-4}^{r}\rho_\alpha^4\sin\left(4\phi_\alpha\right) + b_{0,0,0,-4}^{r}\rho_\beta^4\sin\left(4\phi_\beta\right) - b_{0,0,3,2}^{r}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha - \phi_\beta\right) \\
& - b_{0,2,0,2}^{r}\rho_\beta^4\sin\left(2\phi_\beta\right) - b_{0,2,1,2}^{r}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + \phi_\beta\right) - b_{0,2,2,2}^{r}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha\right) - b_{2,0,0,2}^{r}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\beta\right) \\
& - b_{2,0,1,2}^{r}\rho_\alpha^3\rho_\beta\sin\left(\phi_\alpha + \phi_\beta\right) - b_{2,0,2,2}^{r}\rho_\alpha^4\sin\left(2\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{YY}^{(4)} = {}& a_{0,0,2,0}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha - 2\phi_\beta\right) + a_{0,2,1,0}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - \phi_\beta\right) + a_{0,4,0,0}^{r}\rho_\beta^4 + a_{2,0,1,0}^{r}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha - \phi_\beta\right) + a_{2,2,0,0}^{r}\rho_\alpha^2\rho_\beta^2 \\
& + a_{4,0,0,0}^{r}\rho_\alpha^4 - b_{0,0,-1,-4}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + 3\phi_\beta\right) - b_{0,0,-1,2}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - 3\phi_\beta\right) - b_{0,0,-2,-4}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha + 2\phi_\beta\right) \\
& - b_{0,0,-3,-4}^{r}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha + \phi_\beta\right) - b_{0,0,-4,-4}^{r}\rho_\alpha^4\cos\left(4\phi_\alpha\right) - b_{0,0,0,-4}^{r}\rho_\beta^4\cos\left(4\phi_\beta\right) - b_{0,0,3,2}^{r}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha - \phi_\beta\right) \\
& - b_{0,2,0,2}^{r}\rho_\beta^4\cos\left(2\phi_\beta\right) - b_{0,2,1,2}^{r}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + \phi_\beta\right) - b_{0,2,2,2}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha\right) - b_{2,0,0,2}^{r}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\beta\right) \\
& - b_{2,0,1,2}^{r}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha + \phi_\beta\right) - b_{2,0,2,2}^{r}\rho_\alpha^4\cos\left(2\phi_\alpha\right)
\end{aligned}
$$

Cartesian e-coordinates:

$$
\begin{aligned}
H_{XX}^{(4)} = {}& a_{0,0,2,0}^{r}\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) + a_{0,2,1,0}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) \\
& + a_{0,4,0,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^2 + a_{2,0,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,2,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + a_{4,0,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2 \\
& + b_{0,0,-1,-4}^{r}\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) + b_{0,0,-1,2}^{r}\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) \\
& + b_{0,0,-2,-4}^{r}\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& + b_{0,0,-3,-4}^{r}\left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) + b_{0,0,-4,-4}^{r}\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) \\
& + b_{0,0,0,-4}^{r}\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + b_{0,0,3,2}^{r}\left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) \\
& + b_{0,2,0,2}^{r}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) + b_{0,2,1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right) + b_{0,2,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + b_{2,0,0,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{2,0,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right) + b_{2,0,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{XY}^{(4)} = {}& b_{0,0,-1,-4}^{r}\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) - b_{0,0,-1,2}^{r}\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) \\
& + 2b_{0,0,-2,-4}^{r}\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) + b_{0,0,-3,-4}^{r}\left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right) \\
& + 4b_{0,0,-4,-4}^{r}x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right) + 4b_{0,0,0,-4}^{r}x_\beta y_\beta\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) \\
& + b_{0,0,3,2}^{r}\left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right) - 2b_{0,2,0,2}^{r}x_\beta y_\beta\left(x_\beta^2 + y_\beta^2\right) - b_{0,2,1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& - 2b_{0,2,2,2}^{r}x_\alpha y_\alpha\left(x_\beta^2 + y_\beta^2\right) - 2b_{2,0,0,2}^{r}x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right) - b_{2,0,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2b_{2,0,2,2}^{r}x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)
\end{aligned}
$$

$$H_{YX}^{(4)} = b_{0,0,-1,-4}^r \left(x_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3 \right) + y_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2 \right) \right) - b_{0,0,-1,2}^r \left(x_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3 \right) + y_\alpha \left(-x_\beta^3 + 3x_\beta y_\beta^2 \right) \right)$$
$$+ 2b_{0,0,-2,-4}^r \left(x_\alpha x_\beta - y_\alpha y_\beta \right) \left(x_\alpha y_\beta + x_\beta y_\alpha \right) + b_{0,0,-3,-4}^r \left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3 \right)$$
$$+ 4b_{0,0,-4,-4}^r x_\alpha y_\alpha \left(x_\alpha - y_\alpha \right) \left(x_\alpha + y_\alpha \right) + 4b_{0,0,0,-4}^r x_\beta y_\beta \left(x_\beta - y_\beta \right) \left(x_\beta + y_\beta \right)$$
$$+ b_{0,0,3,2}^r \left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3 \right) - 2b_{0,2,0,2}^r x_\beta y_\beta \left(x_\beta^2 + y_\beta^2 \right) - b_{0,2,1,2}^r \left(x_\beta^2 + y_\beta^2 \right) \left(x_\alpha y_\beta + x_\beta y_\alpha \right)$$
$$- 2b_{0,2,2,2}^r x_\alpha y_\alpha \left(x_\beta^2 + y_\beta^2 \right) - 2b_{2,0,0,2}^r x_\beta y_\beta \left(x_\alpha^2 + y_\alpha^2 \right) - b_{2,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2 \right) \left(x_\alpha y_\beta + x_\beta y_\alpha \right) - 2b_{2,0,2,2}^r x_\alpha y_\alpha \left(x_\alpha^2 + y_\alpha^2 \right)$$

$$H_{YY}^{(4)} = a_{0,0,2,0}^r \left(x_\alpha \left(x_\beta - y_\beta \right) + y_\alpha \left(x_\beta + y_\beta \right) \right) \left(x_\alpha \left(x_\beta + y_\beta \right) + y_\alpha \left(-x_\beta + y_\beta \right) \right) + a_{0,2,1,0}^r \left(x_\beta^2 + y_\beta^2 \right) \left(x_\alpha x_\beta + y_\alpha y_\beta \right)$$
$$+ a_{0,4,0,0}^r \left(x_\beta^2 + y_\beta^2 \right)^2 + a_{2,0,1,0}^r \left(x_\alpha^2 + y_\alpha^2 \right) \left(x_\alpha x_\beta + y_\alpha y_\beta \right) + a_{2,2,0,0}^r \left(x_\alpha^2 + y_\alpha^2 \right) \left(x_\beta^2 + y_\beta^2 \right) + a_{4,0,0,0}^r \left(x_\alpha^2 + y_\alpha^2 \right)^2$$
$$- b_{0,0,-1,-4}^r \left(x_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2 \right) + y_\alpha \left(-3x_\beta^2 y_\beta + y_\beta^3 \right) \right) - b_{0,0,-1,2}^r \left(x_\alpha \left(x_\beta^3 - 3x_\beta y_\beta^2 \right) + y_\alpha \left(3x_\beta^2 y_\beta - y_\beta^3 \right) \right)$$
$$- b_{0,0,-2,-4}^r \left(x_\alpha \left(x_\beta - y_\beta \right) + y_\alpha \left(-x_\beta - y_\beta \right) \right) \left(x_\alpha \left(x_\beta + y_\beta \right) + y_\alpha \left(x_\beta - y_\beta \right) \right)$$
$$- b_{0,0,-3,-4}^r \left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta \right) - b_{0,0,-4,-4}^r \left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2 \right) \left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2 \right)$$
$$- b_{0,0,0,-4}^r \left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2 \right) \left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2 \right) - b_{0,0,3,2}^r \left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta \right)$$
$$- b_{0,2,0,2}^r \left(x_\beta - y_\beta \right) \left(x_\beta + y_\beta \right) \left(x_\beta^2 + y_\beta^2 \right) - b_{0,2,1,2}^r \left(x_\beta^2 + y_\beta^2 \right) \left(x_\alpha x_\beta - y_\alpha y_\beta \right) - b_{0,2,2,2}^r \left(x_\alpha - y_\alpha \right) \left(x_\alpha + y_\alpha \right) \left(x_\beta^2 + y_\beta^2 \right)$$
$$- b_{2,0,0,2}^r \left(x_\alpha^2 + y_\alpha^2 \right) \left(x_\beta - y_\beta \right) \left(x_\beta + y_\beta \right) - b_{2,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2 \right) \left(x_\alpha x_\beta - y_\alpha y_\beta \right) - b_{2,0,2,2}^r \left(x_\alpha - y_\alpha \right) \left(x_\alpha + y_\alpha \right) \left(x_\alpha^2 + y_\alpha^2 \right)$$

5.6 Order: 5

Number of fitting parameters: H_{XX}: 28 (10 from H_{++}, 18 from H_{+-}), H_{XY}: 18 (all from H_{+-}), H_{YY}: 18 (all from H_{+-}).

Polar e-coordinates:

$$H_{XX}^{(5)} = a_{0,0,-1,3}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha - 4\phi_\beta \right) + a_{0,0,4,3}^r \rho_\alpha^4 \rho_\beta \cos\left(4\phi_\alpha - \phi_\beta \right) + a_{0,2,0,3}^r \rho_\beta^5 \cos\left(3\phi_\beta \right) + a_{0,2,1,3}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha + 2\phi_\beta \right)$$
$$+ a_{0,2,2,3}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha + \phi_\beta \right) + a_{0,2,3,3}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(3\phi_\alpha \right) + a_{2,0,0,3}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(3\phi_\beta \right) + a_{2,0,1,3}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(\phi_\alpha + 2\phi_\beta \right)$$
$$+ a_{2,0,2,3}^r \rho_\alpha^4 \rho_\beta \cos\left(2\phi_\alpha + \phi_\beta \right) + a_{2,0,3,3}^r \rho_\alpha^5 \cos\left(3\phi_\alpha \right) + b_{0,0,-3,-1}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(3\phi_\alpha - 2\phi_\beta \right) + b_{0,0,0,5}^r \rho_\beta^5 \cos\left(5\phi_\beta \right)$$
$$+ b_{0,0,1,5}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha + 4\phi_\beta \right) + b_{0,0,2,-1}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha - 3\phi_\beta \right) + b_{0,0,2,5}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha + 3\phi_\beta \right)$$
$$+ b_{0,0,3,5}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(3\phi_\alpha + 2\phi_\beta \right) + b_{0,0,4,5}^r \rho_\alpha^4 \rho_\beta \cos\left(4\phi_\alpha + \phi_\beta \right) + b_{0,0,5,5}^r \rho_\alpha^5 \cos\left(5\phi_\alpha \right)$$
$$+ b_{0,2,-2,-1}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha - \phi_\beta \right) + b_{0,2,1,-1}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha - 2\phi_\beta \right) + b_{0,4,-1,-1}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha \right)$$
$$+ b_{0,4,0,-1}^r \rho_\beta^5 \cos\left(\phi_\beta \right) + b_{2,0,-2,-1}^r \rho_\alpha^4 \rho_\beta \cos\left(2\phi_\alpha - \phi_\beta \right) + b_{2,0,1,-1}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(\phi_\alpha - 2\phi_\beta \right)$$
$$+ b_{2,2,-1,-1}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(\phi_\alpha \right) + b_{2,2,0,-1}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(\phi_\beta \right) + b_{4,0,-1,-1}^r \rho_\alpha^5 \cos\left(\phi_\alpha \right) + b_{4,0,0,-1}^r \rho_\alpha^4 \rho_\beta \cos\left(\phi_\beta \right)$$

$$H_{XY}^{(5)} = b_{0,0,-3,-1}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(3\phi_\alpha - 2\phi_\beta \right) - b_{0,0,0,5}^r \rho_\beta^5 \sin\left(5\phi_\beta \right) - b_{0,0,1,5}^r \rho_\alpha \rho_\beta^4 \sin\left(\phi_\alpha + 4\phi_\beta \right) - b_{0,0,2,-1}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(2\phi_\alpha - 3\phi_\beta \right)$$
$$- b_{0,0,2,5}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(2\phi_\alpha + 3\phi_\beta \right) - b_{0,0,3,5}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(3\phi_\alpha + 2\phi_\beta \right) - b_{0,0,4,5}^r \rho_\alpha^4 \rho_\beta \sin\left(4\phi_\alpha + \phi_\beta \right) - b_{0,0,5,5}^r \rho_\alpha^5 \sin\left(5\phi_\alpha \right)$$
$$+ b_{0,2,-2,-1}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(2\phi_\alpha - \phi_\beta \right) - b_{0,2,1,-1}^r \rho_\alpha \rho_\beta^4 \sin\left(\phi_\alpha - 2\phi_\beta \right) + b_{0,4,-1,-1}^r \rho_\alpha \rho_\beta^4 \sin\left(\phi_\alpha \right) + b_{0,4,0,-1}^r \rho_\beta^5 \sin\left(\phi_\beta \right)$$
$$+ b_{2,0,-2,-1}^r \rho_\alpha^4 \rho_\beta \sin\left(2\phi_\alpha - \phi_\beta \right) - b_{2,0,1,-1}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(\phi_\alpha - 2\phi_\beta \right) + b_{2,2,-1,-1}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(\phi_\alpha \right) + b_{2,2,0,-1}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(\phi_\beta \right)$$
$$+ b_{4,0,-1,-1}^r \rho_\alpha^5 \sin\left(\phi_\alpha \right) + b_{4,0,0,-1}^r \rho_\alpha^4 \rho_\beta \sin\left(\phi_\beta \right)$$

$$H_{YX}^{(5)} = b_{0,0,-3,-1}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(3\phi_\alpha - 2\phi_\beta \right) - b_{0,0,0,5}^r \rho_\beta^5 \sin\left(5\phi_\beta \right) - b_{0,0,1,5}^r \rho_\alpha \rho_\beta^4 \sin\left(\phi_\alpha + 4\phi_\beta \right) - b_{0,0,2,-1}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(2\phi_\alpha - 3\phi_\beta \right)$$
$$- b_{0,0,2,5}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(2\phi_\alpha + 3\phi_\beta \right) - b_{0,0,3,5}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(3\phi_\alpha + 2\phi_\beta \right) - b_{0,0,4,5}^r \rho_\alpha^4 \rho_\beta \sin\left(4\phi_\alpha + \phi_\beta \right) - b_{0,0,5,5}^r \rho_\alpha^5 \sin\left(5\phi_\alpha \right)$$
$$+ b_{0,2,-2,-1}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(2\phi_\alpha - \phi_\beta \right) - b_{0,2,1,-1}^r \rho_\alpha \rho_\beta^4 \sin\left(\phi_\alpha - 2\phi_\beta \right) + b_{0,4,-1,-1}^r \rho_\alpha \rho_\beta^4 \sin\left(\phi_\alpha \right) + b_{0,4,0,-1}^r \rho_\beta^5 \sin\left(\phi_\beta \right)$$
$$+ b_{2,0,-2,-1}^r \rho_\alpha^4 \rho_\beta \sin\left(2\phi_\alpha - \phi_\beta \right) - b_{2,0,1,-1}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(\phi_\alpha - 2\phi_\beta \right) + b_{2,2,-1,-1}^r \rho_\alpha^3 \rho_\beta^2 \sin\left(\phi_\alpha \right) + b_{2,2,0,-1}^r \rho_\alpha^2 \rho_\beta^3 \sin\left(\phi_\beta \right)$$
$$+ b_{4,0,-1,-1}^r \rho_\alpha^5 \sin\left(\phi_\alpha \right) + b_{4,0,0,-1}^r \rho_\alpha^4 \rho_\beta \sin\left(\phi_\beta \right)$$

$$H_{YY}^{(5)} = a_{0,0,-1,3}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha - 4\phi_\beta\right) + a_{0,0,4,3}^r \rho_\alpha^4 \rho_\beta \cos\left(4\phi_\alpha - \phi_\beta\right) + a_{0,2,0,3}^r \rho_\beta^5 \cos\left(3\phi_\beta\right) + a_{0,2,1,3}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha + 2\phi_\beta\right)$$

$$+ a_{0,2,2,3}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha + \phi_\beta\right) + a_{0,2,3,3}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(3\phi_\alpha\right) + a_{2,0,0,3}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(3\phi_\beta\right) + a_{2,0,1,3}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(\phi_\alpha + 2\phi_\beta\right)$$

$$+ a_{2,0,2,3}^r \rho_\alpha^4 \rho_\beta \cos\left(2\phi_\alpha + \phi_\beta\right) + a_{2,0,3,3}^r \rho_\alpha^5 \cos\left(3\phi_\alpha\right) - b_{0,0,-3,-1}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(3\phi_\alpha - 2\phi_\beta\right) - b_{0,0,0,5}^r \rho_\beta^5 \cos\left(5\phi_\beta\right)$$

$$- b_{0,0,1,5}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha + 4\phi_\beta\right) - b_{0,0,2,-1}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha - 3\phi_\beta\right) - b_{0,0,2,5}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha + 3\phi_\beta\right)$$

$$- b_{0,0,3,5}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(3\phi_\alpha + 2\phi_\beta\right) - b_{0,0,4,5}^r \rho_\alpha^4 \rho_\beta \cos\left(4\phi_\alpha + \phi_\beta\right) - b_{0,0,5,5}^r \rho_\alpha^5 \cos\left(5\phi_\alpha\right)$$

$$- b_{0,2,-2,-1}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(2\phi_\alpha - \phi_\beta\right) - b_{0,2,1,-1}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha - 2\phi_\beta\right) - b_{0,4,-1,-1}^r \rho_\alpha \rho_\beta^4 \cos\left(\phi_\alpha\right) - b_{0,4,0,-1}^r \rho_\beta^5 \cos\left(\phi_\beta\right)$$

$$- b_{2,0,-2,-1}^r \rho_\alpha^4 \rho_\beta \cos\left(2\phi_\alpha - \phi_\beta\right) - b_{2,0,1,-1}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(\phi_\alpha - 2\phi_\beta\right) - b_{2,2,-1,-1}^r \rho_\alpha^3 \rho_\beta^2 \cos\left(\phi_\alpha\right) - b_{2,2,0,-1}^r \rho_\alpha^2 \rho_\beta^3 \cos\left(\phi_\beta\right)$$

$$- b_{4,0,-1,-1}^r \rho_\alpha^5 \cos\left(\phi_\alpha\right) - b_{4,0,0,-1}^r \rho_\alpha^4 \rho_\beta \cos\left(\phi_\beta\right)$$

Cartesian e-coordinates:

$$H_{XX}^{(5)} = a_{0,0,-1,3}^r \left(x_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha \left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right)\right) + a_{0,0,4,3}^r \left(4x_\alpha^3 y_\alpha y_\beta - 4x_\alpha y_\alpha^3 y_\beta + x_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right)$$

$$+ a_{0,2,0,3}^r x_\beta \left(x_\beta^2 - 3y_\beta^2\right)\left(x_\beta^2 + y_\beta^2\right) + a_{0,2,1,3}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right)$$

$$+ a_{0,2,2,3}^r \left(x_\beta^2 + y_\beta^2\right)\left(-2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + a_{0,2,3,3}^r x_\alpha \left(x_\alpha^2 - 3y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + a_{2,0,0,3}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 - 3y_\beta^2\right)$$

$$+ a_{2,0,1,3}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right) + a_{2,0,2,3}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(-2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$

$$+ a_{2,0,3,3}^r x_\alpha \left(x_\alpha^2 - 3y_\alpha^2\right)\left(x_\alpha^2 + y_\alpha^2\right) + b_{0,0,-3,-1}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 + 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) - 2x_\beta y_\alpha^3 y_\beta\right)$$

$$+ b_{0,0,0,5}^r x_\beta \left(x_\beta^4 - 10x_\beta^2 y_\beta^2 + 5y_\beta^4\right) + b_{0,0,1,5}^r \left(x_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha \left(-4x_\beta^3 y_\beta + 4x_\beta y_\beta^3\right)\right)$$

$$+ b_{0,0,2,-1}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(6x_\beta^2 y_\alpha y_\beta - 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$

$$+ b_{0,0,2,5}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(-6x_\beta^2 y_\alpha y_\beta + 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$

$$+ b_{0,0,3,5}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 - 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) + 2x_\beta y_\alpha^3 y_\beta\right)$$

$$+ b_{0,0,4,5}^r \left(-4x_\alpha^3 y_\alpha y_\beta + 4x_\alpha y_\alpha^3 y_\beta + x_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right) + b_{0,0,5,5}^r x_\alpha \left(x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + 5y_\alpha^4\right)$$

$$+ b_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + b_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)$$

$$+ b_{0,4,-1,-1}^r x_\alpha \left(x_\beta^2 + y_\beta^2\right)^2 + b_{0,4,0,-1}^r x_\beta \left(x_\beta^2 + y_\beta^2\right)^2 + b_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$

$$+ b_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right) + b_{2,2,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)$$

$$+ b_{2,2,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + b_{4,0,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 + b_{4,0,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2$$

$$H_{XY}^{(5)} = -b_{0,0,-3,-1}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta + x_\beta^2 y_\alpha^3 - y_\alpha^3 y_\beta^2 + y_\alpha \left(-3x_\alpha^2 x_\beta^2 + 3x_\alpha^2 y_\beta^2\right)\right)$$

$$- b_{0,0,0,5}^r y_\beta \left(5x_\beta^4 - 10x_\beta^2 y_\beta^2 + y_\beta^4\right) - b_{0,0,1,5}^r \left(x_\alpha \left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right) + y_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right)\right)$$

$$+ b_{0,0,2,-1}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(-2x_\beta^3 y_\alpha + 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$

$$- b_{0,0,2,5}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(2x_\beta^3 y_\alpha - 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$

$$- b_{0,0,3,5}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta - x_\beta^2 y_\alpha^3 + y_\alpha^3 y_\beta^2 + y_\alpha \left(3x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2\right)\right)$$

$$- b_{0,0,4,5}^r \left(4x_\alpha^3 x_\beta y_\alpha - 4x_\alpha x_\beta y_\alpha^3 + y_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right) - b_{0,0,5,5}^r y_\alpha \left(5x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)$$

$$- b_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right)\left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + b_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right)$$

$$+ b_{0,4,-1,-1}^r y_\alpha \left(x_\beta^2 + y_\beta^2\right)^2 + b_{0,4,0,-1}^r y_\beta \left(x_\beta^2 + y_\beta^2\right)^2 - b_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$

$$+ b_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right) + b_{2,2,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)$$

$$+ b_{2,2,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + b_{4,0,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 + b_{4,0,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2$$

$$H_{YX}^{(5)} = -b_{0,0,-3,-1}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta + x_\beta^2 y_\alpha^3 - y_\alpha^3 y_\beta^2 + y_\alpha \left(-3x_\alpha^2 x_\beta^2 + 3x_\alpha^2 y_\beta^2\right)\right)$$
$$- b_{0,0,0,5}^r y_\beta \left(5x_\beta^4 - 10x_\beta^2 y_\beta^2 + y_\beta^4\right) - b_{0,0,1,5}^r \left(x_\alpha \left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right) + y_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right)\right)$$
$$+ b_{0,0,2,-1}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(-2x_\beta^3 y_\alpha + 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$
$$- b_{0,0,2,5}^r \left(-x_\alpha^2 y_\beta^3 + x_\alpha \left(2x_\beta^3 y_\alpha - 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2 y_\beta^3 + y_\beta \left(3x_\alpha^2 x_\beta^2 - 3x_\beta^2 y_\alpha^2\right)\right)$$
$$- b_{0,0,3,5}^r \left(2x_\alpha^3 x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2 y_\beta - x_\beta^2 y_\alpha^3 + y_\alpha^3 y_\beta^2 + y_\alpha \left(3x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2\right)\right)$$
$$- b_{0,0,4,5}^r \left(4x_\alpha^3 x_\beta y_\alpha - 4x_\alpha x_\beta y_\alpha^3 + y_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right) - b_{0,0,5,5}^r y_\alpha \left(5x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)$$
$$- b_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + b_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right)$$
$$+ b_{0,4,-1,-1}^r y_\alpha \left(x_\beta^2 + y_\beta^2\right)^2 + b_{0,4,0,-1}^r y_\beta \left(x_\beta^2 + y_\beta^2\right)^2 - b_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(-2x_\alpha x_\beta y_\alpha + y_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$+ b_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(2x_\alpha x_\beta y_\beta + y_\alpha \left(-x_\beta^2 + y_\beta^2\right)\right) + b_{2,2,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right)$$
$$+ b_{2,2,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right) + b_{4,0,-1,-1}^r y_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 + b_{4,0,0,-1}^r y_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2$$

$$H_{YY}^{(5)} = a_{0,0,-1,3}^r \left(x_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha \left(4x_\beta^3 y_\beta - 4x_\beta y_\beta^3\right)\right) + a_{0,0,4,3}^r \left(4x_\alpha^3 y_\alpha y_\beta - 4x_\alpha y_\alpha^3 y_\beta + x_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right)$$
$$+ a_{0,2,0,3}^r x_\beta \left(x_\beta^2 - 3y_\beta^2\right) \left(x_\beta^2 + y_\beta^2\right) + a_{0,2,1,3}^r \left(x_\beta^2 + y_\beta^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right)$$
$$+ a_{0,2,2,3}^r \left(x_\beta^2 + y_\beta^2\right) \left(-2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) + a_{0,2,3,3}^r x_\alpha \left(x_\alpha^2 - 3y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right) + a_{2,0,0,3}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 - 3y_\beta^2\right)$$
$$+ a_{2,0,1,3}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) - 2x_\beta y_\alpha y_\beta\right) + a_{2,0,2,3}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(-2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$+ a_{2,0,3,3}^r x_\alpha \left(x_\alpha^2 - 3y_\alpha^2\right) \left(x_\alpha^2 + y_\alpha^2\right) - b_{0,0,-3,-1}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 + 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) - 2x_\beta y_\alpha^3 y_\beta\right)$$
$$- b_{0,0,0,5}^r x_\beta \left(x_\beta^4 - 10x_\beta^2 y_\beta^2 + 5y_\beta^4\right) - b_{0,0,1,5}^r \left(x_\alpha \left(x_\beta^4 - 6x_\beta^2 y_\beta^2 + y_\beta^4\right) + y_\alpha \left(-4x_\beta^3 y_\beta + 4x_\beta y_\beta^3\right)\right)$$
$$- b_{0,0,2,-1}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(6x_\beta^2 y_\alpha y_\beta - 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$
$$- b_{0,0,2,5}^r \left(x_\alpha^2 x_\beta^3 + x_\alpha \left(-6x_\beta^2 y_\alpha y_\beta + 2y_\alpha y_\beta^3\right) - x_\beta^3 y_\alpha^2 + x_\beta \left(-3x_\alpha^2 y_\beta^2 + 3y_\alpha^2 y_\beta^2\right)\right)$$
$$- b_{0,0,3,5}^r \left(x_\alpha^3 x_\beta^2 - x_\alpha^3 y_\beta^2 - 6x_\alpha^2 x_\beta y_\alpha y_\beta + x_\alpha \left(-3x_\beta^2 y_\alpha^2 + 3y_\alpha^2 y_\beta^2\right) + 2x_\beta y_\alpha^3 y_\beta\right)$$
$$- b_{0,0,4,5}^r \left(-4x_\alpha^3 y_\alpha y_\beta + 4x_\alpha y_\alpha^3 y_\beta + x_\beta \left(x_\alpha^4 - 6x_\alpha^2 y_\alpha^2 + y_\alpha^4\right)\right) - b_{0,0,5,5}^r x_\alpha \left(x_\alpha^4 - 10x_\alpha^2 y_\alpha^2 + 5y_\alpha^4\right)$$
$$- b_{0,2,-2,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right) - b_{0,2,1,-1}^r \left(x_\beta^2 + y_\beta^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)$$
$$- b_{0,4,-1,-1}^r x_\alpha \left(x_\beta^2 + y_\beta^2\right)^2 - b_{0,4,0,-1}^r x_\beta \left(x_\beta^2 + y_\beta^2\right)^2 - b_{2,0,-2,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(2x_\alpha y_\alpha y_\beta + x_\beta \left(x_\alpha^2 - y_\alpha^2\right)\right)$$
$$- b_{2,0,1,-1}^r \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right) - b_{2,2,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right)$$
$$- b_{2,2,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right) \left(x_\beta^2 + y_\beta^2\right) - b_{4,0,-1,-1}^r x_\alpha \left(x_\alpha^2 + y_\alpha^2\right)^2 - b_{4,0,0,-1}^r x_\beta \left(x_\alpha^2 + y_\alpha^2\right)^2$$

5.7 Order: 6

Number of fitting parameters: H_{XX}: 44 (17 from H_{++}, 27 from H_{+-}), H_{XY}: 27 (all from H_{+-}), H_{YY}: 27 (all from H_{+-}).

Polar e-coordinates:

$$
\begin{aligned}
H_{XX}^{(6)} = {}& a_{0,0,0,6}^{r}\rho_\beta^6 \cos(6\phi_\beta) + a_{0,0,1,6}^{r}\rho_\alpha\rho_\beta^5 \cos(\phi_\alpha + 5\phi_\beta) + a_{0,0,2,6}^{r}\rho_\alpha^2\rho_\beta^4 \cos(2\phi_\alpha + 4\phi_\beta) + a_{0,0,3,0}^{r}\rho_\alpha^3\rho_\beta^3 \cos(3\phi_\alpha - 3\phi_\beta) \\
& + a_{0,0,3,6}^{r}\rho_\alpha^3\rho_\beta^3 \cos(3\phi_\alpha + 3\phi_\beta) + a_{0,0,4,6}^{r}\rho_\alpha^4\rho_\beta^2 \cos(4\phi_\alpha + 2\phi_\beta) + a_{0,0,5,6}^{r}\rho_\alpha^5\rho_\beta \cos(5\phi_\alpha + \phi_\beta) + a_{0,0,6,6}^{r}\rho_\alpha^6 \cos(6\phi_\alpha) \\
& + a_{0,2,2,0}^{r}\rho_\alpha^2\rho_\beta^4 \cos(2\phi_\alpha - 2\phi_\beta) + a_{0,4,1,0}^{r}\rho_\alpha\rho_\beta^5 \cos(\phi_\alpha - \phi_\beta) + a_{0,6,0,0}^{r}\rho_\beta^6 + a_{2,0,2,0}^{r}\rho_\alpha^4\rho_\beta^2 \cos(2\phi_\alpha - 2\phi_\beta) \\
& + a_{2,2,1,0}^{r}\rho_\alpha^3\rho_\beta^3 \cos(\phi_\alpha - \phi_\beta) + a_{2,4,0,0}^{r}\rho_\alpha^2\rho_\beta^4 + a_{4,0,1,0}^{r}\rho_\alpha^5\rho_\beta \cos(\phi_\alpha - \phi_\beta) + a_{4,2,0,0}^{r}\rho_\alpha^4\rho_\beta^2 + a_{6,0,0,0}^{r}\rho_\alpha^6 \\
& + b_{0,0,-2,2}^{r}\rho_\alpha^2\rho_\beta^4 \cos(2\phi_\alpha - 4\phi_\beta) + b_{0,0,-5,-4}^{r}\rho_\alpha^5\rho_\beta \cos(5\phi_\alpha - \phi_\beta) + b_{0,0,1,-4}^{r}\rho_\alpha\rho_\beta^5 \cos(\phi_\alpha - 5\phi_\beta) \\
& + b_{0,0,4,2}^{r}\rho_\alpha^4\rho_\beta^2 \cos(4\phi_\alpha - 2\phi_\beta) + b_{0,2,-1,-4}^{r}\rho_\alpha\rho_\beta^5 \cos(\phi_\alpha + 3\phi_\beta) + b_{0,2,-1,2}^{r}\rho_\alpha\rho_\beta^5 \cos(\phi_\alpha - 3\phi_\beta) \\
& + b_{0,2,-2,-4}^{r}\rho_\alpha^2\rho_\beta^4 \cos(2\phi_\alpha + 2\phi_\beta) + b_{0,2,-3,-4}^{r}\rho_\alpha^3\rho_\beta^3 \cos(3\phi_\alpha + \phi_\beta) + b_{0,2,-4,-4}^{r}\rho_\alpha^4\rho_\beta^2 \cos(4\phi_\alpha) \\
& + b_{0,2,0,-4}^{r}\rho_\beta^6 \cos(4\phi_\beta) + b_{0,2,3,2}^{r}\rho_\alpha^3\rho_\beta^3 \cos(3\phi_\alpha - \phi_\beta) + b_{0,4,0,2}^{r}\rho_\beta^6 \cos(2\phi_\beta) + b_{0,4,1,2}^{r}\rho_\alpha\rho_\beta^5 \cos(\phi_\alpha + \phi_\beta) \\
& + b_{0,4,2,2}^{r}\rho_\alpha^2\rho_\beta^4 \cos(2\phi_\alpha) + b_{2,0,-1,-4}^{r}\rho_\alpha^3\rho_\beta^3 \cos(\phi_\alpha + 3\phi_\beta) + b_{2,0,-1,2}^{r}\rho_\alpha^3\rho_\beta^3 \cos(\phi_\alpha - 3\phi_\beta) \\
& + b_{2,0,-2,-4}^{r}\rho_\alpha^4\rho_\beta^2 \cos(2\phi_\alpha + 2\phi_\beta) + b_{2,0,-3,-4}^{r}\rho_\alpha^5\rho_\beta \cos(3\phi_\alpha + \phi_\beta) + b_{2,0,-4,-4}^{r}\rho_\alpha^6 \cos(4\phi_\alpha) \\
& + b_{2,0,0,-4}^{r}\rho_\alpha^2\rho_\beta^4 \cos(4\phi_\beta) + b_{2,0,3,2}^{r}\rho_\alpha^5\rho_\beta \cos(3\phi_\alpha - \phi_\beta) + b_{2,2,0,2}^{r}\rho_\alpha^2\rho_\beta^4 \cos(2\phi_\beta) + b_{2,2,1,2}^{r}\rho_\alpha^3\rho_\beta^3 \cos(\phi_\alpha + \phi_\beta) \\
& + b_{2,2,2,2}^{r}\rho_\alpha^4\rho_\beta^2 \cos(2\phi_\alpha) + b_{4,0,0,2}^{r}\rho_\alpha^4\rho_\beta^2 \cos(2\phi_\beta) + b_{4,0,1,2}^{r}\rho_\alpha^5\rho_\beta \cos(\phi_\alpha + \phi_\beta) + b_{4,0,2,2}^{r}\rho_\alpha^6 \cos(2\phi_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
H_{XY}^{(6)} = {}& b_{0,0,-2,2}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\alpha - 4\phi_\beta) + b_{0,0,-5,-4}^{r}\rho_\alpha^5\rho_\beta \sin(5\phi_\alpha - \phi_\beta) - b_{0,0,1,-4}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha - 5\phi_\beta) \\
& - b_{0,0,4,2}^{r}\rho_\alpha^4\rho_\beta^2 \sin(4\phi_\alpha - 2\phi_\beta) + b_{0,2,-1,-4}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha + 3\phi_\beta) + b_{0,2,-1,2}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha - 3\phi_\beta) \\
& + b_{0,2,-2,-4}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\alpha + 2\phi_\beta) + b_{0,2,-3,-4}^{r}\rho_\alpha^3\rho_\beta^3 \sin(3\phi_\alpha + \phi_\beta) + b_{0,2,-4,-4}^{r}\rho_\alpha^4\rho_\beta^2 \sin(4\phi_\alpha) \\
& + b_{0,2,0,-4}^{r}\rho_\beta^6 \sin(4\phi_\beta) - b_{0,2,3,2}^{r}\rho_\alpha^3\rho_\beta^3 \sin(3\phi_\alpha - \phi_\beta) - b_{0,4,0,2}^{r}\rho_\beta^6 \sin(2\phi_\beta) - b_{0,4,1,2}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha + \phi_\beta) \\
& - b_{0,4,2,2}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\alpha) + b_{2,0,-1,-4}^{r}\rho_\alpha^3\rho_\beta^3 \sin(\phi_\alpha + 3\phi_\beta) + b_{2,0,-1,2}^{r}\rho_\alpha^3\rho_\beta^3 \sin(\phi_\alpha - 3\phi_\beta) \\
& + b_{2,0,-2,-4}^{r}\rho_\alpha^4\rho_\beta^2 \sin(2\phi_\alpha + 2\phi_\beta) + b_{2,0,-3,-4}^{r}\rho_\alpha^5\rho_\beta \sin(3\phi_\alpha + \phi_\beta) + b_{2,0,-4,-4}^{r}\rho_\alpha^6 \sin(4\phi_\alpha) \\
& + b_{2,0,0,-4}^{r}\rho_\alpha^2\rho_\beta^4 \sin(4\phi_\beta) - b_{2,0,3,2}^{r}\rho_\alpha^5\rho_\beta \sin(3\phi_\alpha - \phi_\beta) - b_{2,2,0,2}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\beta) - b_{2,2,1,2}^{r}\rho_\alpha^3\rho_\beta^3 \sin(\phi_\alpha + \phi_\beta) \\
& - b_{2,2,2,2}^{r}\rho_\alpha^4\rho_\beta^2 \sin(2\phi_\alpha) - b_{4,0,0,2}^{r}\rho_\alpha^4\rho_\beta^2 \sin(2\phi_\beta) - b_{4,0,1,2}^{r}\rho_\alpha^5\rho_\beta \sin(\phi_\alpha + \phi_\beta) - b_{4,0,2,2}^{r}\rho_\alpha^6 \sin(2\phi_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
H_{YX}^{(6)} = {}& b_{0,0,-2,2}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\alpha - 4\phi_\beta) + b_{0,0,-5,-4}^{r}\rho_\alpha^5\rho_\beta \sin(5\phi_\alpha - \phi_\beta) - b_{0,0,1,-4}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha - 5\phi_\beta) \\
& - b_{0,0,4,2}^{r}\rho_\alpha^4\rho_\beta^2 \sin(4\phi_\alpha - 2\phi_\beta) + b_{0,2,-1,-4}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha + 3\phi_\beta) + b_{0,2,-1,2}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha - 3\phi_\beta) \\
& + b_{0,2,-2,-4}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\alpha + 2\phi_\beta) + b_{0,2,-3,-4}^{r}\rho_\alpha^3\rho_\beta^3 \sin(3\phi_\alpha + \phi_\beta) + b_{0,2,-4,-4}^{r}\rho_\alpha^4\rho_\beta^2 \sin(4\phi_\alpha) \\
& + b_{0,2,0,-4}^{r}\rho_\beta^6 \sin(4\phi_\beta) - b_{0,2,3,2}^{r}\rho_\alpha^3\rho_\beta^3 \sin(3\phi_\alpha - \phi_\beta) - b_{0,4,0,2}^{r}\rho_\beta^6 \sin(2\phi_\beta) - b_{0,4,1,2}^{r}\rho_\alpha\rho_\beta^5 \sin(\phi_\alpha + \phi_\beta) \\
& - b_{0,4,2,2}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\alpha) + b_{2,0,-1,-4}^{r}\rho_\alpha^3\rho_\beta^3 \sin(\phi_\alpha + 3\phi_\beta) + b_{2,0,-1,2}^{r}\rho_\alpha^3\rho_\beta^3 \sin(\phi_\alpha - 3\phi_\beta) \\
& + b_{2,0,-2,-4}^{r}\rho_\alpha^4\rho_\beta^2 \sin(2\phi_\alpha + 2\phi_\beta) + b_{2,0,-3,-4}^{r}\rho_\alpha^5\rho_\beta \sin(3\phi_\alpha + \phi_\beta) + b_{2,0,-4,-4}^{r}\rho_\alpha^6 \sin(4\phi_\alpha) \\
& + b_{2,0,0,-4}^{r}\rho_\alpha^2\rho_\beta^4 \sin(4\phi_\beta) - b_{2,0,3,2}^{r}\rho_\alpha^5\rho_\beta \sin(3\phi_\alpha - \phi_\beta) - b_{2,2,0,2}^{r}\rho_\alpha^2\rho_\beta^4 \sin(2\phi_\beta) - b_{2,2,1,2}^{r}\rho_\alpha^3\rho_\beta^3 \sin(\phi_\alpha + \phi_\beta) \\
& - b_{2,2,2,2}^{r}\rho_\alpha^4\rho_\beta^2 \sin(2\phi_\alpha) - b_{4,0,0,2}^{r}\rho_\alpha^4\rho_\beta^2 \sin(2\phi_\beta) - b_{4,0,1,2}^{r}\rho_\alpha^5\rho_\beta \sin(\phi_\alpha + \phi_\beta) - b_{4,0,2,2}^{r}\rho_\alpha^6 \sin(2\phi_\alpha)
\end{aligned}
$$

$$\begin{aligned}
H_{YY}^{(6)} = {}& a_{0,0,0,6}^{r}\rho_{\beta}^{6}\cos\left(6\phi_{\beta}\right) + a_{0,0,1,6}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha} + 5\phi_{\beta}\right) + a_{0,0,2,6}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha} + 4\phi_{\beta}\right) + a_{0,0,3,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha} - 3\phi_{\beta}\right) \\
&+ a_{0,0,3,6}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha} + 3\phi_{\beta}\right) + a_{0,0,4,6}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha} + 2\phi_{\beta}\right) + a_{0,0,5,6}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(5\phi_{\alpha} + \phi_{\beta}\right) + a_{0,0,6,6}^{r}\rho_{\alpha}^{6}\cos\left(6\phi_{\alpha}\right) \\
&+ a_{0,2,2,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha} - 2\phi_{\beta}\right) + a_{0,4,1,0}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha} - \phi_{\beta}\right) + a_{0,6,0,0}^{r}\rho_{\beta}^{6} + a_{2,0,2,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha} - 2\phi_{\beta}\right) \\
&+ a_{2,2,1,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha} - \phi_{\beta}\right) + a_{2,4,0,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4} + a_{4,0,1,0}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(\phi_{\alpha} - \phi_{\beta}\right) + a_{4,2,0,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2} + a_{6,0,0,0}^{r}\rho_{\alpha}^{6} \\
&- b_{0,0,-2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha} - 4\phi_{\beta}\right) - b_{0,0,-5,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(5\phi_{\alpha} - \phi_{\beta}\right) - b_{0,0,1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha} - 5\phi_{\beta}\right) \\
&- b_{0,0,4,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha} - 2\phi_{\beta}\right) - b_{0,2,-1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha} + 3\phi_{\beta}\right) - b_{0,2,-1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha} - 3\phi_{\beta}\right) \\
&- b_{0,2,-2,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha} + 2\phi_{\beta}\right) - b_{0,2,-3,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha} + \phi_{\beta}\right) - b_{0,2,-4,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}\right) \\
&- b_{0,2,0,-4}^{r}\rho_{\beta}^{6}\cos\left(4\phi_{\beta}\right) - b_{0,2,3,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha} - \phi_{\beta}\right) - b_{0,4,0,2}^{r}\rho_{\beta}^{6}\cos\left(2\phi_{\beta}\right) - b_{0,4,1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha} + \phi_{\beta}\right) \\
&- b_{0,4,2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}\right) - b_{2,0,-1,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha} + 3\phi_{\beta}\right) - b_{2,0,-1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha} - 3\phi_{\beta}\right) \\
&- b_{2,0,-2,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha} + 2\phi_{\beta}\right) - b_{2,0,-3,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(3\phi_{\alpha} + \phi_{\beta}\right) - b_{2,0,-4,-4}^{r}\rho_{\alpha}^{6}\cos\left(4\phi_{\alpha}\right) \\
&- b_{2,0,0,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(4\phi_{\beta}\right) - b_{2,0,3,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(3\phi_{\alpha} - \phi_{\beta}\right) - b_{2,2,0,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\beta}\right) - b_{2,2,1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha} + \phi_{\beta}\right) \\
&- b_{2,2,2,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}\right) - b_{4,0,0,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\beta}\right) - b_{4,0,1,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(\phi_{\alpha} + \phi_{\beta}\right) - b_{4,0,2,2}^{r}\rho_{\alpha}^{6}\cos\left(2\phi_{\alpha}\right)
\end{aligned}$$

Cartesian e-coordinates:

$$
\begin{aligned}
H_{XX}^{(6)} =\ & a_{0,0,0,6}^{r}\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)\left(x_{\beta}^{2}-4x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\left(x_{\beta}^{2}+4x_{\beta}y_{\beta}+y_{\beta}^{2}\right) \\
&+a_{0,0,1,6}^{r}\left(x_{\alpha}\left(x_{\beta}^{5}-10x_{\beta}^{3}y_{\beta}^{2}+5x_{\beta}y_{\beta}^{4}\right)+y_{\alpha}\left(-5x_{\beta}^{4}y_{\beta}+10x_{\beta}^{2}y_{\beta}^{3}-y_{\beta}^{5}\right)\right) \\
&+a_{0,0,2,6}^{r}\left(x_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(-x_{\beta}^{2}-2x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\right)\left(x_{\alpha}\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\right) \\
&+a_{0,0,3,0}^{r}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)\left(x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}+8x_{\alpha}x_{\beta}y_{\alpha}y_{\beta}-3x_{\beta}^{2}y_{\alpha}^{2}+y_{\alpha}^{2}y_{\beta}^{2}\right) \\
&+a_{0,0,3,6}^{r}\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right)\left(x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}-8x_{\alpha}x_{\beta}y_{\alpha}y_{\beta}-3x_{\beta}^{2}y_{\alpha}^{2}+y_{\alpha}^{2}y_{\beta}^{2}\right)+a_{0,0,4,6}^{r}\left(x_{\alpha}\left(-2x_{\beta}y_{\alpha}-2y_{\alpha}y_{\beta}\right)\right. \\
&\qquad\qquad\left.+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(-x_{\alpha}^{2}+y_{\alpha}^{2}\right)\right)\left(x_{\alpha}\left(2x_{\beta}y_{\alpha}-2y_{\alpha}y_{\beta}\right)+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right) \\
&+a_{0,0,5,6}^{r}\left(-5x_{\alpha}^{4}y_{\alpha}y_{\beta}+5x_{\alpha}x_{\beta}y_{\alpha}^{4}+x_{\beta}\left(x_{\alpha}^{5}-10x_{\alpha}^{3}y_{\alpha}^{2}\right)+y_{\beta}\left(10x_{\alpha}^{2}y_{\alpha}^{3}-y_{\alpha}^{5}\right)\right) \\
&+a_{0,0,6,6}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\alpha}^{2}-4x_{\alpha}y_{\alpha}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+4x_{\alpha}y_{\alpha}+y_{\alpha}^{2}\right) \\
&+a_{0,2,2,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(x_{\beta}+y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}+y_{\beta}\right)\right)+a_{0,4,1,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right) \\
&+a_{0,6,0,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{3}+a_{2,0,2,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(x_{\beta}+y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}+y_{\beta}\right)\right) \\
&+a_{2,2,1,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)+a_{2,4,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2} \\
&+a_{4,0,1,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)+a_{4,2,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+a_{6,0,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{3} \\
&+b_{0,0,-2,2}^{r}\left(x_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\right)\left(x_{\alpha}\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(-x_{\beta}^{2}+2x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\right) \\
&+b_{0,0,-5,-4}^{r}\left(5x_{\alpha}^{4}y_{\alpha}y_{\beta}+5x_{\alpha}x_{\beta}y_{\alpha}^{4}+x_{\beta}\left(x_{\alpha}^{5}-10x_{\alpha}^{3}y_{\alpha}^{2}\right)+y_{\beta}\left(-10x_{\alpha}^{2}y_{\alpha}^{3}+y_{\alpha}^{5}\right)\right) \\
&+b_{0,0,1,-4}^{r}\left(x_{\alpha}\left(x_{\beta}^{5}-10x_{\beta}^{3}y_{\beta}^{2}+5x_{\beta}y_{\beta}^{4}\right)+y_{\alpha}\left(5x_{\beta}^{4}y_{\beta}-10x_{\beta}^{2}y_{\beta}^{3}+y_{\beta}^{5}\right)\right) \\
&+b_{0,0,4,2}^{r}\left(x_{\alpha}\left(-2x_{\beta}y_{\alpha}+2y_{\alpha}y_{\beta}\right)+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\left(x_{\alpha}\left(2x_{\beta}y_{\alpha}+2y_{\alpha}y_{\beta}\right)+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right. \\
&\qquad\qquad\left.+y_{\beta}\left(-x_{\alpha}^{2}+y_{\alpha}^{2}\right)\right)+b_{0,2,-1,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(-3x_{\beta}^{2}y_{\beta}+y_{\beta}^{3}\right)\right) \\
&+b_{0,2,-1,2}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(3x_{\beta}^{2}y_{\beta}-y_{\beta}^{3}\right)\right) \\
&+b_{0,2,-2,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}-y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(x_{\beta}-y_{\beta}\right)\right) \\
&+b_{0,2,-3,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}-3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}+y_{\alpha}^{3}y_{\beta}\right) \\
&+b_{0,2,-4,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}^{2}-2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right) \\
&+b_{0,2,0,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+b_{0,2,3,2}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}+3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}-y_{\alpha}^{3}y_{\beta}\right) \\
&+b_{0,4,0,2}^{r}\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}+b_{0,4,1,2}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right) \\
&+b_{0,4,2,2}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}+b_{2,0,-1,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(-3x_{\beta}^{2}y_{\beta}+y_{\beta}^{3}\right)\right) \\
&+b_{2,0,-1,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(3x_{\beta}^{2}y_{\beta}-y_{\beta}^{3}\right)\right) \\
&+b_{2,0,-2,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}-y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(x_{\beta}-y_{\beta}\right)\right) \\
&+b_{2,0,-3,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}-3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}+y_{\alpha}^{3}y_{\beta}\right) \\
&+b_{2,0,-4,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}-2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right) \\
&+b_{2,0,0,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right) \\
&+b_{2,0,3,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}+3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}-y_{\alpha}^{3}y_{\beta}\right)+b_{2,2,0,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right) \\
&+b_{2,2,1,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right)+b_{2,2,2,2}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right) \\
&+b_{4,0,0,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)+b_{4,0,1,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right)+b_{4,0,2,2}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}
\end{aligned}
$$

$$\begin{aligned}
H_{XY}^{(6)} = {} & -2b_{0,0,-2,2}^r \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) \\
& - b_{0,0,-5,-4}^r \left(-5x_\alpha^4 x_\beta y_\alpha + 5x_\alpha y_\alpha^4 y_\beta + x_\beta\left(10x_\alpha^2 y_\alpha^3 - y_\alpha^5\right) + y_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right)\right) \\
& + b_{0,0,1,-4}^r \left(x_\alpha\left(5x_\beta^4 y_\beta - 10x_\beta^2 y_\beta^3 + y_\beta^5\right) + y_\alpha\left(-x_\beta^5 + 10x_\beta^3 y_\beta^2 - 5x_\beta y_\beta^4\right)\right) \\
& + 2b_{0,0,4,2}^r \left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + b_{0,2,-1,-4}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) \\
& - b_{0,2,-1,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) + 2b_{0,2,-2,-4}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& + b_{0,2,-3,-4}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right) + 4b_{0,2,-4,-4}^r x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + 4b_{0,2,0,-4}^r x_\beta y_\beta\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) + b_{0,2,3,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right) \\
& - 2b_{0,4,0,2}^r x_\beta y_\beta\left(x_\beta^2 + y_\beta^2\right)^2 - b_{0,4,1,2}^r \left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& - 2b_{0,4,2,2}^r x_\alpha y_\alpha\left(x_\beta^2 + y_\beta^2\right)^2 + b_{2,0,-1,-4}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) \\
& - b_{2,0,-1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) + 2b_{2,0,-2,-4}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& + b_{2,0,-3,-4}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right) + 4b_{2,0,-4,-4}^r x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right) \\
& + 4b_{2,0,0,-4}^r x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{2,0,3,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right) \\
& - 2b_{2,2,0,2}^r x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) - b_{2,2,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2b_{2,2,2,2}^r x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) \\
& - 2b_{4,0,0,2}^r x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)^2 - b_{4,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2b_{4,0,2,2}^r x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)^2
\end{aligned}$$

$$\begin{aligned}
H_{YX}^{(6)} = {} & -2b_{0,0,-2,2}^r \left(x_\alpha \left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) \\
& - b_{0,0,-5,-4}^r \left(-5x_\alpha^4 x_\beta y_\alpha + 5x_\alpha y_\alpha^4 y_\beta + x_\beta\left(10x_\alpha^2 y_\alpha^3 - y_\alpha^5\right) + y_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right)\right) \\
& + b_{0,0,1,-4}^r \left(x_\alpha\left(5x_\beta^4 y_\beta - 10x_\beta^2 y_\beta^3 + y_\beta^5\right) + y_\alpha\left(-x_\beta^5 + 10x_\beta^3 y_\beta^2 - 5x_\beta y_\beta^4\right)\right) \\
& + 2b_{0,0,4,2}^r \left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + b_{0,2,-1,-4}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) \\
& - b_{0,2,-1,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) + 2b_{0,2,-2,-4}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& + b_{0,2,-3,-4}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right) + 4b_{0,2,-4,-4}^r x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + 4b_{0,2,0,-4}^r x_\beta y_\beta\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) + b_{0,2,3,2}^r \left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right) \\
& - 2b_{0,4,0,2}^r x_\beta y_\beta\left(x_\beta^2 + y_\beta^2\right)^2 - b_{0,4,1,2}^r \left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& - 2b_{0,4,2,2}^r x_\alpha y_\alpha\left(x_\beta^2 + y_\beta^2\right)^2 + b_{2,0,-1,-4}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) \\
& - b_{2,0,-1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) + 2b_{2,0,-2,-4}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& + b_{2,0,-3,-4}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 y_\beta + 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta - x_\beta y_\alpha^3\right) + 4b_{2,0,-4,-4}^r x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right) \\
& + 4b_{2,0,0,-4}^r x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{2,0,3,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 y_\beta - 3x_\alpha^2 x_\beta y_\alpha - 3x_\alpha y_\alpha^2 y_\beta + x_\beta y_\alpha^3\right) \\
& - 2b_{2,2,0,2}^r x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) - b_{2,2,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2b_{2,2,2,2}^r x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) \\
& - 2b_{4,0,0,2}^r x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right)^2 - b_{4,0,1,2}^r \left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2b_{4,0,2,2}^r x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)^2
\end{aligned}$$

$$\begin{aligned}
H_{YY}^{(6)} = {} & a_{0,0,0,6}^{r}\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)\left(x_{\beta}^{2}-4x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\left(x_{\beta}^{2}+4x_{\beta}y_{\beta}+y_{\beta}^{2}\right) \\
& + a_{0,0,1,6}^{r}\left(x_{\alpha}\left(x_{\beta}^{5}-10x_{\beta}^{3}y_{\beta}^{2}+5x_{\beta}y_{\beta}^{4}\right)+y_{\alpha}\left(-5x_{\beta}^{4}y_{\beta}+10x_{\beta}^{2}y_{\beta}^{3}-y_{\beta}^{5}\right)\right) \\
& + a_{0,0,2,6}^{r}\left(x_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(-x_{\beta}^{2}-2x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\right)\left(x_{\alpha}\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\right) \\
& + a_{0,0,3,0}^{r}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)\left(x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}+8x_{\alpha}x_{\beta}y_{\alpha}y_{\beta}-3x_{\beta}^{2}y_{\alpha}^{2}+y_{\alpha}^{2}y_{\beta}^{2}\right) \\
& + a_{0,0,3,6}^{r}\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right)\left(x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}-8x_{\alpha}x_{\beta}y_{\alpha}y_{\beta}-3x_{\beta}^{2}y_{\alpha}^{2}+y_{\alpha}^{2}y_{\beta}^{2}\right)+a_{0,0,4,6}^{r}\left(x_{\alpha}\left(-2x_{\beta}y_{\alpha}-2y_{\alpha}y_{\beta}\right)\right. \\
& \qquad\qquad \left. + x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(-x_{\alpha}^{2}+y_{\alpha}^{2}\right)\right)\left(x_{\alpha}\left(2x_{\beta}y_{\alpha}-2y_{\alpha}y_{\beta}\right)+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right) \\
& + a_{0,0,5,6}^{r}\left(-5x_{\alpha}^{4}y_{\alpha}y_{\beta}+5x_{\alpha}x_{\beta}y_{\alpha}^{4}+x_{\beta}\left(x_{\alpha}^{5}-10x_{\alpha}^{3}y_{\alpha}^{2}\right)+y_{\beta}\left(10x_{\alpha}^{2}y_{\alpha}^{3}-y_{\alpha}^{5}\right)\right) \\
& + a_{0,0,6,6}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\alpha}^{2}-4x_{\alpha}y_{\alpha}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+4x_{\alpha}y_{\alpha}+y_{\alpha}^{2}\right) \\
& + a_{0,2,2,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(x_{\beta}+y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}+y_{\beta}\right)\right)+a_{0,4,1,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right) \\
& + a_{0,6,0,0}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{3}+a_{2,0,2,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(x_{\beta}+y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}+y_{\beta}\right)\right) \\
& + a_{2,2,1,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)+a_{2,4,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2} \\
& + a_{4,0,1,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}+y_{\alpha}y_{\beta}\right)+a_{4,2,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+a_{6,0,0,0}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{3} \\
& - b_{0,0,-2,2}^{r}\left(x_{\alpha}\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\right)\left(x_{\alpha}\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)+y_{\alpha}\left(-x_{\beta}^{2}+2x_{\beta}y_{\beta}+y_{\beta}^{2}\right)\right) \\
& - b_{0,0,-5,-4}^{r}\left(5x_{\alpha}^{4}y_{\alpha}y_{\beta}+5x_{\alpha}x_{\beta}y_{\alpha}^{4}+x_{\beta}\left(x_{\alpha}^{5}-10x_{\alpha}^{3}y_{\alpha}^{2}\right)+y_{\beta}\left(-10x_{\alpha}^{2}y_{\alpha}^{3}+y_{\alpha}^{5}\right)\right) \\
& - b_{0,0,1,-4}^{r}\left(x_{\alpha}\left(x_{\beta}^{5}-10x_{\beta}^{3}y_{\beta}^{2}+5x_{\beta}y_{\beta}^{4}\right)+y_{\alpha}\left(5x_{\beta}^{4}y_{\beta}-10x_{\beta}^{2}y_{\beta}^{3}+y_{\beta}^{5}\right)\right) \\
& - b_{0,0,4,2}^{r}\left(x_{\alpha}\left(-2x_{\beta}y_{\alpha}+2y_{\alpha}y_{\beta}\right)+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\left(x_{\alpha}\left(2x_{\beta}y_{\alpha}+2y_{\alpha}y_{\beta}\right)+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right. \\
& \qquad\qquad \left. + y_{\beta}\left(-x_{\alpha}^{2}+y_{\alpha}^{2}\right)\right)-b_{0,2,-1,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(-3x_{\beta}^{2}y_{\beta}+y_{\beta}^{3}\right)\right) \\
& - b_{0,2,-1,2}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(3x_{\beta}^{2}y_{\beta}-y_{\beta}^{3}\right)\right) \\
& - b_{0,2,-2,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}-y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(x_{\beta}-y_{\beta}\right)\right) \\
& - b_{0,2,-3,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}-3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}+y_{\alpha}^{3}y_{\beta}\right) \\
& - b_{0,2,-4,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}^{2}-2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right) \\
& - b_{0,2,0,-4}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)-b_{0,2,3,2}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}+3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}-y_{\alpha}^{3}y_{\beta}\right) \\
& - b_{0,4,0,2}^{r}\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}-b_{0,4,1,2}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right) \\
& - b_{0,4,2,2}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}-b_{2,0,-1,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(-3x_{\beta}^{2}y_{\beta}+y_{\beta}^{3}\right)\right) \\
& - b_{2,0,-1,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{3}-3x_{\beta}y_{\beta}^{2}\right)+y_{\alpha}\left(3x_{\beta}^{2}y_{\beta}-y_{\beta}^{3}\right)\right) \\
& - b_{2,0,-2,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}-y_{\beta}\right)+y_{\alpha}\left(-x_{\beta}-y_{\beta}\right)\right)\left(x_{\alpha}\left(x_{\beta}+y_{\beta}\right)+y_{\alpha}\left(x_{\beta}-y_{\beta}\right)\right) \\
& - b_{2,0,-3,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}-3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}+y_{\alpha}^{3}y_{\beta}\right) \\
& - b_{2,0,-4,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}-2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+2x_{\alpha}y_{\alpha}-y_{\alpha}^{2}\right) \\
& - b_{2,0,0,-4}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}-2x_{\beta}y_{\beta}-y_{\beta}^{2}\right)\left(x_{\beta}^{2}+2x_{\beta}y_{\beta}-y_{\beta}^{2}\right) \\
& - b_{2,0,3,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}^{3}x_{\beta}+3x_{\alpha}^{2}y_{\alpha}y_{\beta}-3x_{\alpha}x_{\beta}y_{\alpha}^{2}-y_{\alpha}^{3}y_{\beta}\right)-b_{2,2,0,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right) \\
& - b_{2,2,1,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right)-b_{2,2,2,2}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right) \\
& - b_{4,0,0,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\beta}-y_{\beta}\right)\left(x_{\beta}+y_{\beta}\right)-b_{4,0,1,2}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}\left(x_{\alpha}x_{\beta}-y_{\alpha}y_{\beta}\right)-b_{4,0,2,2}^{r}\left(x_{\alpha}-y_{\alpha}\right)\left(x_{\alpha}+y_{\alpha}\right)\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}
\end{aligned}$$