VHEGEN: A vibronic Hamiltonian expansion generator for trigonal and tetragonal polyatomic systems

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Thank you for using `VHEGEN`, the `V`-ibronic `H`-amiltonian `E`-xpansion `GEN`-erator for trigonal and tetragonal polyatomic systems. This is a `VHEGEN` output file compiled by `pdflatex`. If the `VHEGEN` package was used in research resulting in a publication, please reference the article in *Computer Physics Communications* which describes the program ([doi here]). Additional information regarding the matrix element expansion process, including the independent matrix element eigenvalues, their root formulas and constraints, and their transformation to the real basis (if applicable), can be found in the `log` output file. For questions, bugs, or comments, please contact robert.lang@mail.utoronto.ca.

Contents

1 Vibronic interaction

$(E_g + A_{1u}) \otimes (e_g + b_{2u})$ in D_{4h}

2 Vibronic Hamiltonian operator in the complex E basis

$$\hat{H} = \begin{pmatrix} |+\rangle & |-\rangle & |A\rangle \end{pmatrix} \begin{pmatrix} 0 & 0 & H_{+A} \\ 0 & 0 & H_{-A} \\ H_{A+} & H_{A-} & 0 \end{pmatrix} \begin{pmatrix} \langle +| \\ \langle -| \\ \langle A| \end{pmatrix}$$

3 Matrix element expansions in the complex E basis

3.1 Order: 0

Number of fitting parameters: H_{+A}: 0.

Polar e-coordinates:

$$H^{(0)}_{+A} = 0$$

$$H^{(0)}_{-A} = 0$$

$$H^{(0)}_{A+} = 0$$

$$H^{(0)}_{A-} = 0$$

Cartesian e-coordinates:

$$H^{(0)}_{+A} = 0$$

$$H^{(0)}_{-A} = 0$$

$$H^{(0)}_{A+} = 0$$

$$H^{(0)}_{A-} = 0$$

3.2 Order: 1

Number of fitting parameters: H_{+A}: 0.

Polar e-coordinates:

$$H_{+A}^{(1)} = 0$$

$$H_{-A}^{(1)} = 0$$

$$H_{A+}^{(1)} = 0$$

$$H_{A-}^{(1)} = 0$$

Cartesian e-coordinates:

$$H_{+A}^{(1)} = 0$$

$$H_{-A}^{(1)} = 0$$

$$H_{A+}^{(1)} = 0$$

$$H_{A-}^{(1)} = 0$$

3.3 Order: 2

Number of fitting parameters: H_{+A}: 1.

Polar e-coordinates:

$$H_{+A}^{(2)} = -c_{0,0,1}^{i} \rho w \sin(\phi) + i c_{0,0,1}^{i} \rho w \cos(\phi)$$

$$H_{-A}^{(2)} = -c_{0,0,1}^{i} \rho w \sin(\phi) - i c_{0,0,1}^{i} \rho w \cos(\phi)$$

$$H_{A+}^{(2)} = -c_{0,0,1}^{i} \rho w \sin(\phi) - i c_{0,0,1}^{i} \rho w \cos(\phi)$$

$$H_{A-}^{(2)} = -c_{0,0,1}^{i} \rho w \sin(\phi) + i c_{0,0,1}^{i} \rho w \cos(\phi)$$

Cartesian e-coordinates:

$$H^{(2)}_{+A} = ic^i_{0,0,1}wx - c^i_{0,0,1}wy$$

$$H^{(2)}_{-A} = -ic^i_{0,0,1}wx - c^i_{0,0,1}wy$$

$$H^{(2)}_{A+} = -ic^i_{0,0,1}wx - c^i_{0,0,1}wy$$

$$H^{(2)}_{A-} = ic^i_{0,0,1}wx - c^i_{0,0,1}wy$$

3.4 Order: 3

Number of fitting parameters: H_{+A}: 0.

Polar e-coordinates:

$$H^{(3)}_{+A} = 0$$

$$H^{(3)}_{-A} = 0$$

$$H^{(3)}_{A+} = 0$$

$$H^{(3)}_{A-} = 0$$

Cartesian e-coordinates:

$$H^{(3)}_{+A} = 0$$

$$H^{(3)}_{-A} = 0$$

$$H^{(3)}_{A+} = 0$$

$$H^{(3)}_{A-} = 0$$

3.5 Order: 4

Number of fitting parameters: H_{+A}: 3.

Polar e-coordinates:

$$H_{+A}^{(4)} = c_{0,0,-3}^{i}\rho^3 w\sin(3\phi) + ic_{0,0,-3}^{i}\rho^3 w\cos(3\phi) - c_{0,2,1}^{i}\rho^3 w\sin(\phi) + ic_{0,2,1}^{i}\rho^3 w\cos(\phi) - c_{2,0,1}^{i}\rho w^3\sin(\phi) + ic_{2,0,1}^{i}\rho w^3\cos(\phi)$$

$$H_{-A}^{(4)} = c_{0,0,-3}^{i}\rho^3 w\sin(3\phi) - ic_{0,0,-3}^{i}\rho^3 w\cos(3\phi) - c_{0,2,1}^{i}\rho^3 w\sin(\phi) - ic_{0,2,1}^{i}\rho^3 w\cos(\phi) - c_{2,0,1}^{i}\rho w^3\sin(\phi) - ic_{2,0,1}^{i}\rho w^3\cos(\phi)$$

$$H_{A+}^{(4)} = c_{0,0,-3}^{i}\rho^3 w\sin(3\phi) - ic_{0,0,-3}^{i}\rho^3 w\cos(3\phi) - c_{0,2,1}^{i}\rho^3 w\sin(\phi) - ic_{0,2,1}^{i}\rho^3 w\cos(\phi) - c_{2,0,1}^{i}\rho w^3\sin(\phi) - ic_{2,0,1}^{i}\rho w^3\cos(\phi)$$

$$H_{A-}^{(4)} = c_{0,0,-3}^{i}\rho^3 w\sin(3\phi) + ic_{0,0,-3}^{i}\rho^3 w\cos(3\phi) - c_{0,2,1}^{i}\rho^3 w\sin(\phi) + ic_{0,2,1}^{i}\rho^3 w\cos(\phi) - c_{2,0,1}^{i}\rho w^3\sin(\phi) + ic_{2,0,1}^{i}\rho w^3\cos(\phi)$$

Cartesian e-coordinates:

$$H_{+A}^{(4)} = ic_{0,0,-3}^{i}wx\left(x^2-3y^2\right) + c_{0,0,-3}^{i}wy\left(3x^2-y^2\right) + ic_{0,2,1}^{i}wx\left(x^2+y^2\right) - c_{0,2,1}^{i}wy\left(x^2+y^2\right) + ic_{2,0,1}^{i}w^3x - c_{2,0,1}^{i}w^3y$$

$$H_{-A}^{(4)} = -ic_{0,0,-3}^{i}wx\left(x^2-3y^2\right) + c_{0,0,-3}^{i}wy\left(3x^2-y^2\right) - ic_{0,2,1}^{i}wx\left(x^2+y^2\right) - c_{0,2,1}^{i}wy\left(x^2+y^2\right) - ic_{2,0,1}^{i}w^3x - c_{2,0,1}^{i}w^3y$$

$$H_{A+}^{(4)} = -ic_{0,0,-3}^{i}wx\left(x^2-3y^2\right) + c_{0,0,-3}^{i}wy\left(3x^2-y^2\right) - ic_{0,2,1}^{i}wx\left(x^2+y^2\right) - c_{0,2,1}^{i}wy\left(x^2+y^2\right) - ic_{2,0,1}^{i}w^3x - c_{2,0,1}^{i}w^3y$$

$$H_{A-}^{(4)} = ic_{0,0,-3}^{i}wx\left(x^2-3y^2\right) + c_{0,0,-3}^{i}wy\left(3x^2-y^2\right) + ic_{0,2,1}^{i}wx\left(x^2+y^2\right) - c_{0,2,1}^{i}wy\left(x^2+y^2\right) + ic_{2,0,1}^{i}w^3x - c_{2,0,1}^{i}w^3y$$

3.6 Order: 5

Number of fitting parameters: H_{+A}: 0.

Polar e-coordinates:

$$H_{+A}^{(5)} = 0$$

$$H_{-A}^{(5)} = 0$$

$$H_{A+}^{(5)} = 0$$

$$H_{A-}^{(5)} = 0$$

Cartesian e-coordinates:

$$H_{+A}^{(5)} = 0$$

$$H_{-A}^{(5)} = 0$$

$$H_{A+}^{(5)} = 0$$

$$H_{A-}^{(5)} = 0$$

3.7 Order: 6

Number of fitting parameters: H_{+A}: 6.

Polar e-coordinates:

$$\begin{aligned} H_{+A}^{(6)} = & -c_{0,0,5}^{i}\rho^5 w\sin(5\phi) + ic_{0,0,5}^{i}\rho^5 w\cos(5\phi) + c_{0,2,-3}^{i}\rho^5 w\sin(3\phi) + ic_{0,2,-3}^{i}\rho^5 w\cos(3\phi) - c_{0,4,1}^{i}\rho^5 w\sin(\phi) \\ & + ic_{0,4,1}^{i}\rho^5 w\cos(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\sin(3\phi) + ic_{2,0,-3}^{i}\rho^3 w^3\cos(3\phi) - c_{2,2,1}^{i}\rho^3 w^3\sin(\phi) + ic_{2,2,1}^{i}\rho^3 w^3\cos(\phi) \\ & - c_{4,0,1}^{i}\rho w^5\sin(\phi) + ic_{4,0,1}^{i}\rho w^5\cos(\phi) \end{aligned}$$

$$\begin{aligned} H_{-A}^{(6)} = & -c_{0,0,5}^{i}\rho^5 w\sin(5\phi) - ic_{0,0,5}^{i}\rho^5 w\cos(5\phi) + c_{0,2,-3}^{i}\rho^5 w\sin(3\phi) - ic_{0,2,-3}^{i}\rho^5 w\cos(3\phi) - c_{0,4,1}^{i}\rho^5 w\sin(\phi) \\ & - ic_{0,4,1}^{i}\rho^5 w\cos(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\sin(3\phi) - ic_{2,0,-3}^{i}\rho^3 w^3\cos(3\phi) - c_{2,2,1}^{i}\rho^3 w^3\sin(\phi) - ic_{2,2,1}^{i}\rho^3 w^3\cos(\phi) \\ & - c_{4,0,1}^{i}\rho w^5\sin(\phi) - ic_{4,0,1}^{i}\rho w^5\cos(\phi) \end{aligned}$$

$$\begin{aligned} H_{A+}^{(6)} = & -c_{0,0,5}^{i}\rho^5 w\sin(5\phi) - ic_{0,0,5}^{i}\rho^5 w\cos(5\phi) + c_{0,2,-3}^{i}\rho^5 w\sin(3\phi) - ic_{0,2,-3}^{i}\rho^5 w\cos(3\phi) - c_{0,4,1}^{i}\rho^5 w\sin(\phi) \\ & - ic_{0,4,1}^{i}\rho^5 w\cos(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\sin(3\phi) - ic_{2,0,-3}^{i}\rho^3 w^3\cos(3\phi) - c_{2,2,1}^{i}\rho^3 w^3\sin(\phi) - ic_{2,2,1}^{i}\rho^3 w^3\cos(\phi) \\ & - c_{4,0,1}^{i}\rho w^5\sin(\phi) - ic_{4,0,1}^{i}\rho w^5\cos(\phi) \end{aligned}$$

$$\begin{aligned} H_{A-}^{(6)} = & -c_{0,0,5}^{i}\rho^5 w\sin(5\phi) + ic_{0,0,5}^{i}\rho^5 w\cos(5\phi) + c_{0,2,-3}^{i}\rho^5 w\sin(3\phi) + ic_{0,2,-3}^{i}\rho^5 w\cos(3\phi) - c_{0,4,1}^{i}\rho^5 w\sin(\phi) \\ & + ic_{0,4,1}^{i}\rho^5 w\cos(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\sin(3\phi) + ic_{2,0,-3}^{i}\rho^3 w^3\cos(3\phi) - c_{2,2,1}^{i}\rho^3 w^3\sin(\phi) + ic_{2,2,1}^{i}\rho^3 w^3\cos(\phi) \\ & - c_{4,0,1}^{i}\rho w^5\sin(\phi) + ic_{4,0,1}^{i}\rho w^5\cos(\phi) \end{aligned}$$

Cartesian e-coordinates:

$$\begin{aligned} H_{+A}^{(6)} = & ic_{0,0,5}^{i}wx\left(x^4 - 10x^2y^2 + 5y^4\right) - c_{0,0,5}^{i}wy\left(5x^4 - 10x^2y^2 + y^4\right) + ic_{0,2,-3}^{i}wx\left(x^2 - 3y^2\right)\left(x^2 + y^2\right) \\ & + c_{0,2,-3}^{i}wy\left(x^2 + y^2\right)\left(3x^2 - y^2\right) + ic_{0,4,1}^{i}wx\left(x^2 + y^2\right)^2 - c_{0,4,1}^{i}wy\left(x^2 + y^2\right)^2 + ic_{2,0,-3}^{i}w^3x\left(x^2 - 3y^2\right) \\ & + c_{2,0,-3}^{i}w^3y\left(3x^2 - y^2\right) + ic_{2,2,1}^{i}w^3x\left(x^2 + y^2\right) - c_{2,2,1}^{i}w^3y\left(x^2 + y^2\right) + ic_{4,0,1}^{i}w^5x - c_{4,0,1}^{i}w^5y \end{aligned}$$

$$H_{-A}^{(6)} = -ic_{0,0,5}^{i} wx\left(x^4 - 10x^2y^2 + 5y^4\right) - c_{0,0,5}^{i} wy\left(5x^4 - 10x^2y^2 + y^4\right) - ic_{0,2,-3}^{i} wx\left(x^2 - 3y^2\right)\left(x^2 + y^2\right)$$
$$+ c_{0,2,-3}^{i} wy\left(x^2 + y^2\right)\left(3x^2 - y^2\right) - ic_{0,4,1}^{i} wx\left(x^2 + y^2\right)^2 - c_{0,4,1}^{i} wy\left(x^2 + y^2\right)^2 - ic_{2,0,-3}^{i} w^3 x\left(x^2 - 3y^2\right)$$
$$+ c_{2,0,-3}^{i} w^3 y\left(3x^2 - y^2\right) - ic_{2,2,1}^{i} w^3 x\left(x^2 + y^2\right) - c_{2,2,1}^{i} w^3 y\left(x^2 + y^2\right) - ic_{4,0,1}^{i} w^5 x - c_{4,0,1}^{i} w^5 y$$

$$H_{A+}^{(6)} = -ic_{0,0,5}^{i} wx\left(x^4 - 10x^2y^2 + 5y^4\right) - c_{0,0,5}^{i} wy\left(5x^4 - 10x^2y^2 + y^4\right) - ic_{0,2,-3}^{i} wx\left(x^2 - 3y^2\right)\left(x^2 + y^2\right)$$
$$+ c_{0,2,-3}^{i} wy\left(x^2 + y^2\right)\left(3x^2 - y^2\right) - ic_{0,4,1}^{i} wx\left(x^2 + y^2\right)^2 - c_{0,4,1}^{i} wy\left(x^2 + y^2\right)^2 - ic_{2,0,-3}^{i} w^3 x\left(x^2 - 3y^2\right)$$
$$+ c_{2,0,-3}^{i} w^3 y\left(3x^2 - y^2\right) - ic_{2,2,1}^{i} w^3 x\left(x^2 + y^2\right) - c_{2,2,1}^{i} w^3 y\left(x^2 + y^2\right) - ic_{4,0,1}^{i} w^5 x - c_{4,0,1}^{i} w^5 y$$

$$H_{A-}^{(6)} = ic_{0,0,5}^{i} wx\left(x^4 - 10x^2y^2 + 5y^4\right) - c_{0,0,5}^{i} wy\left(5x^4 - 10x^2y^2 + y^4\right) + ic_{0,2,-3}^{i} wx\left(x^2 - 3y^2\right)\left(x^2 + y^2\right)$$
$$+ c_{0,2,-3}^{i} wy\left(x^2 + y^2\right)\left(3x^2 - y^2\right) + ic_{0,4,1}^{i} wx\left(x^2 + y^2\right)^2 - c_{0,4,1}^{i} wy\left(x^2 + y^2\right)^2 + ic_{2,0,-3}^{i} w^3 x\left(x^2 - 3y^2\right)$$
$$+ c_{2,0,-3}^{i} w^3 y\left(3x^2 - y^2\right) + ic_{2,2,1}^{i} w^3 x\left(x^2 + y^2\right) - c_{2,2,1}^{i} w^3 y\left(x^2 + y^2\right) + ic_{4,0,1}^{i} w^5 x - c_{4,0,1}^{i} w^5 y$$

3.8 Order: 7

Number of fitting parameters: H_{+A}: 0.

Polar e-coordinates:

$$H_{+A}^{(7)} = 0$$

$$H_{-A}^{(7)} = 0$$

$$H_{A+}^{(7)} = 0$$

$$H_{A-}^{(7)} = 0$$

Cartesian e-coordinates:

$$H_{+A}^{(7)} = 0$$

$$H_{-A}^{(7)} = 0$$

$$H_{A+}^{(7)} = 0$$

$$H_{A-}^{(7)} = 0$$

3.9 Order: 8

Number of fitting parameters: H_{+A}: 10.

Polar e-coordinates:

$$
\begin{aligned}
H^{(8)}_{+A} = {} & c^i_{0,0,-7}\rho^7 w\sin(7\phi) + ic^i_{0,0,-7}\rho^7 w\cos(7\phi) - c^i_{0,2,5}\rho^7 w\sin(5\phi) + ic^i_{0,2,5}\rho^7 w\cos(5\phi) + c^i_{0,4,-3}\rho^7 w\sin(3\phi) \\
& + ic^i_{0,4,-3}\rho^7 w\cos(3\phi) - c^i_{0,6,1}\rho^7 w\sin(\phi) + ic^i_{0,6,1}\rho^7 w\cos(\phi) - c^i_{2,0,5}\rho^5 w^3\sin(5\phi) + ic^i_{2,0,5}\rho^5 w^3\cos(5\phi) \\
& + c^i_{2,2,-3}\rho^5 w^3\sin(3\phi) + ic^i_{2,2,-3}\rho^5 w^3\cos(3\phi) - c^i_{2,4,1}\rho^5 w^3\sin(\phi) + ic^i_{2,4,1}\rho^5 w^3\cos(\phi) + c^i_{4,0,-3}\rho^3 w^5\sin(3\phi) \\
& + ic^i_{4,0,-3}\rho^3 w^5\cos(3\phi) - c^i_{4,2,1}\rho^3 w^5\sin(\phi) + ic^i_{4,2,1}\rho^3 w^5\cos(\phi) - c^i_{6,0,1}\rho w^7\sin(\phi) + ic^i_{6,0,1}\rho w^7\cos(\phi)
\end{aligned}
$$

$$
\begin{aligned}
H^{(8)}_{-A} = {} & c^i_{0,0,-7}\rho^7 w\sin(7\phi) - ic^i_{0,0,-7}\rho^7 w\cos(7\phi) - c^i_{0,2,5}\rho^7 w\sin(5\phi) - ic^i_{0,2,5}\rho^7 w\cos(5\phi) + c^i_{0,4,-3}\rho^7 w\sin(3\phi) \\
& - ic^i_{0,4,-3}\rho^7 w\cos(3\phi) - c^i_{0,6,1}\rho^7 w\sin(\phi) - ic^i_{0,6,1}\rho^7 w\cos(\phi) - c^i_{2,0,5}\rho^5 w^3\sin(5\phi) - ic^i_{2,0,5}\rho^5 w^3\cos(5\phi) \\
& + c^i_{2,2,-3}\rho^5 w^3\sin(3\phi) - ic^i_{2,2,-3}\rho^5 w^3\cos(3\phi) - c^i_{2,4,1}\rho^5 w^3\sin(\phi) - ic^i_{2,4,1}\rho^5 w^3\cos(\phi) + c^i_{4,0,-3}\rho^3 w^5\sin(3\phi) \\
& - ic^i_{4,0,-3}\rho^3 w^5\cos(3\phi) - c^i_{4,2,1}\rho^3 w^5\sin(\phi) - ic^i_{4,2,1}\rho^3 w^5\cos(\phi) - c^i_{6,0,1}\rho w^7\sin(\phi) - ic^i_{6,0,1}\rho w^7\cos(\phi)
\end{aligned}
$$

$$
\begin{aligned}
H^{(8)}_{A+} = {} & c^i_{0,0,-7}\rho^7 w\sin(7\phi) - ic^i_{0,0,-7}\rho^7 w\cos(7\phi) - c^i_{0,2,5}\rho^7 w\sin(5\phi) - ic^i_{0,2,5}\rho^7 w\cos(5\phi) + c^i_{0,4,-3}\rho^7 w\sin(3\phi) \\
& - ic^i_{0,4,-3}\rho^7 w\cos(3\phi) - c^i_{0,6,1}\rho^7 w\sin(\phi) - ic^i_{0,6,1}\rho^7 w\cos(\phi) - c^i_{2,0,5}\rho^5 w^3\sin(5\phi) - ic^i_{2,0,5}\rho^5 w^3\cos(5\phi) \\
& + c^i_{2,2,-3}\rho^5 w^3\sin(3\phi) - ic^i_{2,2,-3}\rho^5 w^3\cos(3\phi) - c^i_{2,4,1}\rho^5 w^3\sin(\phi) - ic^i_{2,4,1}\rho^5 w^3\cos(\phi) + c^i_{4,0,-3}\rho^3 w^5\sin(3\phi) \\
& - ic^i_{4,0,-3}\rho^3 w^5\cos(3\phi) - c^i_{4,2,1}\rho^3 w^5\sin(\phi) - ic^i_{4,2,1}\rho^3 w^5\cos(\phi) - c^i_{6,0,1}\rho w^7\sin(\phi) - ic^i_{6,0,1}\rho w^7\cos(\phi)
\end{aligned}
$$

$$
\begin{aligned}
H^{(8)}_{A-} = {} & c^i_{0,0,-7}\rho^7 w\sin(7\phi) + ic^i_{0,0,-7}\rho^7 w\cos(7\phi) - c^i_{0,2,5}\rho^7 w\sin(5\phi) + ic^i_{0,2,5}\rho^7 w\cos(5\phi) + c^i_{0,4,-3}\rho^7 w\sin(3\phi) \\
& + ic^i_{0,4,-3}\rho^7 w\cos(3\phi) - c^i_{0,6,1}\rho^7 w\sin(\phi) + ic^i_{0,6,1}\rho^7 w\cos(\phi) - c^i_{2,0,5}\rho^5 w^3\sin(5\phi) + ic^i_{2,0,5}\rho^5 w^3\cos(5\phi) \\
& + c^i_{2,2,-3}\rho^5 w^3\sin(3\phi) + ic^i_{2,2,-3}\rho^5 w^3\cos(3\phi) - c^i_{2,4,1}\rho^5 w^3\sin(\phi) + ic^i_{2,4,1}\rho^5 w^3\cos(\phi) + c^i_{4,0,-3}\rho^3 w^5\sin(3\phi) \\
& + ic^i_{4,0,-3}\rho^3 w^5\cos(3\phi) - c^i_{4,2,1}\rho^3 w^5\sin(\phi) + ic^i_{4,2,1}\rho^3 w^5\cos(\phi) - c^i_{6,0,1}\rho w^7\sin(\phi) + ic^i_{6,0,1}\rho w^7\cos(\phi)
\end{aligned}
$$

Cartesian e-coordinates:

$$
\begin{aligned}
H^{(8)}_{+A} = {} & ic^i_{0,0,-7}wx\left(x^6 - 21x^4y^2 + 35x^2y^4 - 7y^6\right) + c^i_{0,0,-7}wy\left(7x^6 - 35x^4y^2 + 21x^2y^4 - y^6\right) \\
& + ic^i_{0,2,5}wx\left(x^2+y^2\right)\left(x^4 - 10x^2y^2 + 5y^4\right) - c^i_{0,2,5}wy\left(x^2+y^2\right)\left(5x^4 - 10x^2y^2 + y^4\right) \\
& + ic^i_{0,4,-3}wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^2 + c^i_{0,4,-3}wy\left(x^2+y^2\right)^2\left(3x^2-y^2\right) + ic^i_{0,6,1}wx\left(x^2+y^2\right)^3 - c^i_{0,6,1}wy\left(x^2+y^2\right)^3 \\
& + ic^i_{2,0,5}w^3x\left(x^4 - 10x^2y^2 + 5y^4\right) - c^i_{2,0,5}w^3y\left(5x^4 - 10x^2y^2 + y^4\right) + ic^i_{2,2,-3}w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right) \\
& + c^i_{2,2,-3}w^3y\left(x^2+y^2\right)\left(3x^2-y^2\right) + ic^i_{2,4,1}w^3x\left(x^2+y^2\right)^2 - c^i_{2,4,1}w^3y\left(x^2+y^2\right)^2 + ic^i_{4,0,-3}w^5x\left(x^2-3y^2\right) \\
& + c^i_{4,0,-3}w^5y\left(3x^2-y^2\right) + ic^i_{4,2,1}w^5x\left(x^2+y^2\right) - c^i_{4,2,1}w^5y\left(x^2+y^2\right) + ic^i_{6,0,1}w^7x - c^i_{6,0,1}w^7y
\end{aligned}
$$

$$
\begin{aligned}
H^{(8)}_{-A} = {} & -ic^i_{0,0,-7}wx\left(x^6 - 21x^4y^2 + 35x^2y^4 - 7y^6\right) + c^i_{0,0,-7}wy\left(7x^6 - 35x^4y^2 + 21x^2y^4 - y^6\right) \\
& - ic^i_{0,2,5}wx\left(x^2+y^2\right)\left(x^4 - 10x^2y^2 + 5y^4\right) - c^i_{0,2,5}wy\left(x^2+y^2\right)\left(5x^4 - 10x^2y^2 + y^4\right) \\
& - ic^i_{0,4,-3}wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^2 + c^i_{0,4,-3}wy\left(x^2+y^2\right)^2\left(3x^2-y^2\right) - ic^i_{0,6,1}wx\left(x^2+y^2\right)^3 - c^i_{0,6,1}wy\left(x^2+y^2\right)^3 \\
& - ic^i_{2,0,5}w^3x\left(x^4 - 10x^2y^2 + 5y^4\right) - c^i_{2,0,5}w^3y\left(5x^4 - 10x^2y^2 + y^4\right) - ic^i_{2,2,-3}w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right) \\
& + c^i_{2,2,-3}w^3y\left(x^2+y^2\right)\left(3x^2-y^2\right) - ic^i_{2,4,1}w^3x\left(x^2+y^2\right)^2 - c^i_{2,4,1}w^3y\left(x^2+y^2\right)^2 - ic^i_{4,0,-3}w^5x\left(x^2-3y^2\right) \\
& + c^i_{4,0,-3}w^5y\left(3x^2-y^2\right) - ic^i_{4,2,1}w^5x\left(x^2+y^2\right) - c^i_{4,2,1}w^5y\left(x^2+y^2\right) - ic^i_{6,0,1}w^7x - c^i_{6,0,1}w^7y
\end{aligned}
$$

$$
\begin{aligned}
H_{A+}^{(8)} = & -ic_{0,0,-7}^{i} wx\left(x^6-21x^4y^2+35x^2y^4-7y^6\right)+c_{0,0,-7}^{i} wy\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& -ic_{0,2,5}^{i} wx\left(x^2+y^2\right)\left(x^4-10x^2y^2+5y^4\right)-c_{0,2,5}^{i} wy\left(x^2+y^2\right)\left(5x^4-10x^2y^2+y^4\right) \\
& -ic_{0,4,-3}^{i} wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^2+c_{0,4,-3}^{i} wy\left(x^2+y^2\right)^2\left(3x^2-y^2\right)-ic_{0,6,1}^{i} wx\left(x^2+y^2\right)^3-c_{0,6,1}^{i} wy\left(x^2+y^2\right)^3 \\
& -ic_{2,0,5}^{i} w^3x\left(x^4-10x^2y^2+5y^4\right)-c_{2,0,5}^{i} w^3y\left(5x^4-10x^2y^2+y^4\right)-ic_{2,2,-3}^{i} w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right) \\
& +c_{2,2,-3}^{i} w^3y\left(x^2+y^2\right)\left(3x^2-y^2\right)-ic_{2,4,1}^{i} w^3x\left(x^2+y^2\right)^2-c_{2,4,1}^{i} w^3y\left(x^2+y^2\right)^2-ic_{4,0,-3}^{i} w^5x\left(x^2-3y^2\right) \\
& +c_{4,0,-3}^{i} w^5y\left(3x^2-y^2\right)-ic_{4,2,1}^{i} w^5x\left(x^2+y^2\right)-c_{4,2,1}^{i} w^5y\left(x^2+y^2\right)-ic_{6,0,1}^{i} w^7x-c_{6,0,1}^{i} w^7y
\end{aligned}
$$

$$
\begin{aligned}
H_{A-}^{(8)} = & ic_{0,0,-7}^{i} wx\left(x^6-21x^4y^2+35x^2y^4-7y^6\right)+c_{0,0,-7}^{i} wy\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& +ic_{0,2,5}^{i} wx\left(x^2+y^2\right)\left(x^4-10x^2y^2+5y^4\right)-c_{0,2,5}^{i} wy\left(x^2+y^2\right)\left(5x^4-10x^2y^2+y^4\right) \\
& +ic_{0,4,-3}^{i} wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^2+c_{0,4,-3}^{i} wy\left(x^2+y^2\right)^2\left(3x^2-y^2\right)+ic_{0,6,1}^{i} wx\left(x^2+y^2\right)^3-c_{0,6,1}^{i} wy\left(x^2+y^2\right)^3 \\
& +ic_{2,0,5}^{i} w^3x\left(x^4-10x^2y^2+5y^4\right)-c_{2,0,5}^{i} w^3y\left(5x^4-10x^2y^2+y^4\right)+ic_{2,2,-3}^{i} w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right) \\
& +c_{2,2,-3}^{i} w^3y\left(x^2+y^2\right)\left(3x^2-y^2\right)+ic_{2,4,1}^{i} w^3x\left(x^2+y^2\right)^2-c_{2,4,1}^{i} w^3y\left(x^2+y^2\right)^2+ic_{4,0,-3}^{i} w^5x\left(x^2-3y^2\right) \\
& +c_{4,0,-3}^{i} w^5y\left(3x^2-y^2\right)+ic_{4,2,1}^{i} w^5x\left(x^2+y^2\right)-c_{4,2,1}^{i} w^5y\left(x^2+y^2\right)+ic_{6,0,1}^{i} w^7x-c_{6,0,1}^{i} w^7y
\end{aligned}
$$

3.10 Order: 9

Number of fitting parameters: H_{+A}: 0.

Polar e-coordinates:

$$H_{+A}^{(9)} = 0$$

$$H_{-A}^{(9)} = 0$$

$$H_{A+}^{(9)} = 0$$

$$H_{A-}^{(9)} = 0$$

Cartesian e-coordinates:

$$H_{+A}^{(9)} = 0$$

$$H_{-A}^{(9)} = 0$$

$$H_{A+}^{(9)} = 0$$

$$H_{A-}^{(9)} = 0$$

3.11 Order: 10

Number of fitting parameters: H_{+A}: 15.

Polar e-coordinates:

$$
\begin{aligned}
H^{(10)}_{+A} = & -c^i_{0,0,9}\rho^9 w\sin(9\phi) + ic^i_{0,0,9}\rho^9 w\cos(9\phi) + c^i_{0,2,-7}\rho^9 w\sin(7\phi) + ic^i_{0,2,-7}\rho^9 w\cos(7\phi) - c^i_{0,4,5}\rho^9 w\sin(5\phi) \\
& + ic^i_{0,4,5}\rho^9 w\cos(5\phi) + c^i_{0,6,-3}\rho^9 w\sin(3\phi) + ic^i_{0,6,-3}\rho^9 w\cos(3\phi) - c^i_{0,8,1}\rho^9 w\sin(\phi) + ic^i_{0,8,1}\rho^9 w\cos(\phi) \\
& + c^i_{2,0,-7}\rho^7 w^3\sin(7\phi) + ic^i_{2,0,-7}\rho^7 w^3\cos(7\phi) - c^i_{2,2,5}\rho^7 w^3\sin(5\phi) + ic^i_{2,2,5}\rho^7 w^3\cos(5\phi) + c^i_{2,4,-3}\rho^7 w^3\sin(3\phi) \\
& + ic^i_{2,4,-3}\rho^7 w^3\cos(3\phi) - c^i_{2,6,1}\rho^7 w^3\sin(\phi) + ic^i_{2,6,1}\rho^7 w^3\cos(\phi) - c^i_{4,0,5}\rho^5 w^5\sin(5\phi) + ic^i_{4,0,5}\rho^5 w^5\cos(5\phi) \\
& + c^i_{4,2,-3}\rho^5 w^5\sin(3\phi) + ic^i_{4,2,-3}\rho^5 w^5\cos(3\phi) - c^i_{4,4,1}\rho^5 w^5\sin(\phi) + ic^i_{4,4,1}\rho^5 w^5\cos(\phi) + c^i_{6,0,-3}\rho^3 w^7\sin(3\phi) \\
& + ic^i_{6,0,-3}\rho^3 w^7\cos(3\phi) - c^i_{6,2,1}\rho^3 w^7\sin(\phi) + ic^i_{6,2,1}\rho^3 w^7\cos(\phi) - c^i_{8,0,1}\rho w^9\sin(\phi) + ic^i_{8,0,1}\rho w^9\cos(\phi)
\end{aligned}
$$

$$
\begin{aligned}
H^{(10)}_{-A} = & -c^i_{0,0,9}\rho^9 w\sin(9\phi) - ic^i_{0,0,9}\rho^9 w\cos(9\phi) + c^i_{0,2,-7}\rho^9 w\sin(7\phi) - ic^i_{0,2,-7}\rho^9 w\cos(7\phi) - c^i_{0,4,5}\rho^9 w\sin(5\phi) \\
& - ic^i_{0,4,5}\rho^9 w\cos(5\phi) + c^i_{0,6,-3}\rho^9 w\sin(3\phi) - ic^i_{0,6,-3}\rho^9 w\cos(3\phi) - c^i_{0,8,1}\rho^9 w\sin(\phi) - ic^i_{0,8,1}\rho^9 w\cos(\phi) \\
& + c^i_{2,0,-7}\rho^7 w^3\sin(7\phi) - ic^i_{2,0,-7}\rho^7 w^3\cos(7\phi) - c^i_{2,2,5}\rho^7 w^3\sin(5\phi) - ic^i_{2,2,5}\rho^7 w^3\cos(5\phi) + c^i_{2,4,-3}\rho^7 w^3\sin(3\phi) \\
& - ic^i_{2,4,-3}\rho^7 w^3\cos(3\phi) - c^i_{2,6,1}\rho^7 w^3\sin(\phi) - ic^i_{2,6,1}\rho^7 w^3\cos(\phi) - c^i_{4,0,5}\rho^5 w^5\sin(5\phi) - ic^i_{4,0,5}\rho^5 w^5\cos(5\phi) \\
& + c^i_{4,2,-3}\rho^5 w^5\sin(3\phi) - ic^i_{4,2,-3}\rho^5 w^5\cos(3\phi) - c^i_{4,4,1}\rho^5 w^5\sin(\phi) - ic^i_{4,4,1}\rho^5 w^5\cos(\phi) + c^i_{6,0,-3}\rho^3 w^7\sin(3\phi) \\
& - ic^i_{6,0,-3}\rho^3 w^7\cos(3\phi) - c^i_{6,2,1}\rho^3 w^7\sin(\phi) - ic^i_{6,2,1}\rho^3 w^7\cos(\phi) - c^i_{8,0,1}\rho w^9\sin(\phi) - ic^i_{8,0,1}\rho w^9\cos(\phi)
\end{aligned}
$$

$$
\begin{aligned}
H^{(10)}_{A+} = & -c^i_{0,0,9}\rho^9 w\sin(9\phi) - ic^i_{0,0,9}\rho^9 w\cos(9\phi) + c^i_{0,2,-7}\rho^9 w\sin(7\phi) - ic^i_{0,2,-7}\rho^9 w\cos(7\phi) - c^i_{0,4,5}\rho^9 w\sin(5\phi) \\
& - ic^i_{0,4,5}\rho^9 w\cos(5\phi) + c^i_{0,6,-3}\rho^9 w\sin(3\phi) - ic^i_{0,6,-3}\rho^9 w\cos(3\phi) - c^i_{0,8,1}\rho^9 w\sin(\phi) - ic^i_{0,8,1}\rho^9 w\cos(\phi) \\
& + c^i_{2,0,-7}\rho^7 w^3\sin(7\phi) - ic^i_{2,0,-7}\rho^7 w^3\cos(7\phi) - c^i_{2,2,5}\rho^7 w^3\sin(5\phi) - ic^i_{2,2,5}\rho^7 w^3\cos(5\phi) + c^i_{2,4,-3}\rho^7 w^3\sin(3\phi) \\
& - ic^i_{2,4,-3}\rho^7 w^3\cos(3\phi) - c^i_{2,6,1}\rho^7 w^3\sin(\phi) - ic^i_{2,6,1}\rho^7 w^3\cos(\phi) - c^i_{4,0,5}\rho^5 w^5\sin(5\phi) - ic^i_{4,0,5}\rho^5 w^5\cos(5\phi) \\
& + c^i_{4,2,-3}\rho^5 w^5\sin(3\phi) - ic^i_{4,2,-3}\rho^5 w^5\cos(3\phi) - c^i_{4,4,1}\rho^5 w^5\sin(\phi) - ic^i_{4,4,1}\rho^5 w^5\cos(\phi) + c^i_{6,0,-3}\rho^3 w^7\sin(3\phi) \\
& - ic^i_{6,0,-3}\rho^3 w^7\cos(3\phi) - c^i_{6,2,1}\rho^3 w^7\sin(\phi) - ic^i_{6,2,1}\rho^3 w^7\cos(\phi) - c^i_{8,0,1}\rho w^9\sin(\phi) - ic^i_{8,0,1}\rho w^9\cos(\phi)
\end{aligned}
$$

$$
\begin{aligned}
H^{(10)}_{A-} = & -c^i_{0,0,9}\rho^9 w\sin(9\phi) + ic^i_{0,0,9}\rho^9 w\cos(9\phi) + c^i_{0,2,-7}\rho^9 w\sin(7\phi) + ic^i_{0,2,-7}\rho^9 w\cos(7\phi) - c^i_{0,4,5}\rho^9 w\sin(5\phi) \\
& + ic^i_{0,4,5}\rho^9 w\cos(5\phi) + c^i_{0,6,-3}\rho^9 w\sin(3\phi) + ic^i_{0,6,-3}\rho^9 w\cos(3\phi) - c^i_{0,8,1}\rho^9 w\sin(\phi) + ic^i_{0,8,1}\rho^9 w\cos(\phi) \\
& + c^i_{2,0,-7}\rho^7 w^3\sin(7\phi) + ic^i_{2,0,-7}\rho^7 w^3\cos(7\phi) - c^i_{2,2,5}\rho^7 w^3\sin(5\phi) + ic^i_{2,2,5}\rho^7 w^3\cos(5\phi) + c^i_{2,4,-3}\rho^7 w^3\sin(3\phi) \\
& + ic^i_{2,4,-3}\rho^7 w^3\cos(3\phi) - c^i_{2,6,1}\rho^7 w^3\sin(\phi) + ic^i_{2,6,1}\rho^7 w^3\cos(\phi) - c^i_{4,0,5}\rho^5 w^5\sin(5\phi) + ic^i_{4,0,5}\rho^5 w^5\cos(5\phi) \\
& + c^i_{4,2,-3}\rho^5 w^5\sin(3\phi) + ic^i_{4,2,-3}\rho^5 w^5\cos(3\phi) - c^i_{4,4,1}\rho^5 w^5\sin(\phi) + ic^i_{4,4,1}\rho^5 w^5\cos(\phi) + c^i_{6,0,-3}\rho^3 w^7\sin(3\phi) \\
& + ic^i_{6,0,-3}\rho^3 w^7\cos(3\phi) - c^i_{6,2,1}\rho^3 w^7\sin(\phi) + ic^i_{6,2,1}\rho^3 w^7\cos(\phi) - c^i_{8,0,1}\rho w^9\sin(\phi) + ic^i_{8,0,1}\rho w^9\cos(\phi)
\end{aligned}
$$

Cartesian e-coordinates:

$$
\begin{aligned}
H^{(10)}_{+A} = {} & ic^i_{0,0,9} wx\left(x^2-3y^2\right)\left(x^6-33x^4y^2+27x^2y^4-3y^6\right) - c^i_{0,0,9} wy\left(3x^2-y^2\right)\left(3x^6-27x^4y^2+33x^2y^4-y^6\right) \\
& + ic^i_{0,2,-7} wx\left(x^2+y^2\right)\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c^i_{0,2,-7} wy\left(x^2+y^2\right)\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& + ic^i_{0,4,5} wx\left(x^2+y^2\right)^2\left(x^4-10x^2y^2+5y^4\right) - c^i_{0,4,5} wy\left(x^2+y^2\right)^2\left(5x^4-10x^2y^2+y^4\right) \\
& + ic^i_{0,6,-3} wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^3 + c^i_{0,6,-3} wy\left(x^2+y^2\right)^3\left(3x^2-y^2\right) + ic^i_{0,8,1} wx\left(x^2+y^2\right)^4 \\
& - c^i_{0,8,1} wy\left(x^2+y^2\right)^4 + ic^i_{2,0,-7} w^3x\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c^i_{2,0,-7} w^3y\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& + ic^i_{2,2,5} w^3x\left(x^2+y^2\right)\left(x^4-10x^2y^2+5y^4\right) - c^i_{2,2,5} w^3y\left(x^2+y^2\right)\left(5x^4-10x^2y^2+y^4\right) \\
& + ic^i_{2,4,-3} w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right)^2 + c^i_{2,4,-3} w^3y\left(x^2+y^2\right)^2\left(3x^2-y^2\right) + ic^i_{2,6,1} w^3x\left(x^2+y^2\right)^3 \\
& - c^i_{2,6,1} w^3y\left(x^2+y^2\right)^3 + ic^i_{4,0,5} w^5x\left(x^4-10x^2y^2+5y^4\right) - c^i_{4,0,5} w^5y\left(5x^4-10x^2y^2+y^4\right) \\
& + ic^i_{4,2,-3} w^5x\left(x^2-3y^2\right)\left(x^2+y^2\right) + c^i_{4,2,-3} w^5y\left(x^2+y^2\right)\left(3x^2-y^2\right) + ic^i_{4,4,1} w^5x\left(x^2+y^2\right)^2 \\
& - c^i_{4,4,1} w^5y\left(x^2+y^2\right)^2 + ic^i_{6,0,-3} w^7x\left(x^2-3y^2\right) + c^i_{6,0,-3} w^7y\left(3x^2-y^2\right) + ic^i_{6,2,1} w^7x\left(x^2+y^2\right) \\
& - c^i_{6,2,1} w^7y\left(x^2+y^2\right) + ic^i_{8,0,1} w^9x - c^i_{8,0,1} w^9y
\end{aligned}
$$

$$
\begin{aligned}
H^{(10)}_{-A} = {} & -ic^i_{0,0,9} wx\left(x^2-3y^2\right)\left(x^6-33x^4y^2+27x^2y^4-3y^6\right) - c^i_{0,0,9} wy\left(3x^2-y^2\right)\left(3x^6-27x^4y^2+33x^2y^4-y^6\right) \\
& - ic^i_{0,2,-7} wx\left(x^2+y^2\right)\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c^i_{0,2,-7} wy\left(x^2+y^2\right)\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& - ic^i_{0,4,5} wx\left(x^2+y^2\right)^2\left(x^4-10x^2y^2+5y^4\right) - c^i_{0,4,5} wy\left(x^2+y^2\right)^2\left(5x^4-10x^2y^2+y^4\right) \\
& - ic^i_{0,6,-3} wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^3 + c^i_{0,6,-3} wy\left(x^2+y^2\right)^3\left(3x^2-y^2\right) - ic^i_{0,8,1} wx\left(x^2+y^2\right)^4 \\
& - c^i_{0,8,1} wy\left(x^2+y^2\right)^4 - ic^i_{2,0,-7} w^3x\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c^i_{2,0,-7} w^3y\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& - ic^i_{2,2,5} w^3x\left(x^2+y^2\right)\left(x^4-10x^2y^2+5y^4\right) - c^i_{2,2,5} w^3y\left(x^2+y^2\right)\left(5x^4-10x^2y^2+y^4\right) \\
& - ic^i_{2,4,-3} w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right)^2 + c^i_{2,4,-3} w^3y\left(x^2+y^2\right)^2\left(3x^2-y^2\right) - ic^i_{2,6,1} w^3x\left(x^2+y^2\right)^3 \\
& - c^i_{2,6,1} w^3y\left(x^2+y^2\right)^3 - ic^i_{4,0,5} w^5x\left(x^4-10x^2y^2+5y^4\right) - c^i_{4,0,5} w^5y\left(5x^4-10x^2y^2+y^4\right) \\
& - ic^i_{4,2,-3} w^5x\left(x^2-3y^2\right)\left(x^2+y^2\right) + c^i_{4,2,-3} w^5y\left(x^2+y^2\right)\left(3x^2-y^2\right) - ic^i_{4,4,1} w^5x\left(x^2+y^2\right)^2 \\
& - c^i_{4,4,1} w^5y\left(x^2+y^2\right)^2 - ic^i_{6,0,-3} w^7x\left(x^2-3y^2\right) + c^i_{6,0,-3} w^7y\left(3x^2-y^2\right) - ic^i_{6,2,1} w^7x\left(x^2+y^2\right) \\
& - c^i_{6,2,1} w^7y\left(x^2+y^2\right) - ic^i_{8,0,1} w^9x - c^i_{8,0,1} w^9y
\end{aligned}
$$

$$
\begin{aligned}
H^{(10)}_{A+} = {} & -ic^i_{0,0,9} wx\left(x^2-3y^2\right)\left(x^6-33x^4y^2+27x^2y^4-3y^6\right) - c^i_{0,0,9} wy\left(3x^2-y^2\right)\left(3x^6-27x^4y^2+33x^2y^4-y^6\right) \\
& - ic^i_{0,2,-7} wx\left(x^2+y^2\right)\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c^i_{0,2,-7} wy\left(x^2+y^2\right)\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& - ic^i_{0,4,5} wx\left(x^2+y^2\right)^2\left(x^4-10x^2y^2+5y^4\right) - c^i_{0,4,5} wy\left(x^2+y^2\right)^2\left(5x^4-10x^2y^2+y^4\right) \\
& - ic^i_{0,6,-3} wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^3 + c^i_{0,6,-3} wy\left(x^2+y^2\right)^3\left(3x^2-y^2\right) - ic^i_{0,8,1} wx\left(x^2+y^2\right)^4 \\
& - c^i_{0,8,1} wy\left(x^2+y^2\right)^4 - ic^i_{2,0,-7} w^3x\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c^i_{2,0,-7} w^3y\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& - ic^i_{2,2,5} w^3x\left(x^2+y^2\right)\left(x^4-10x^2y^2+5y^4\right) - c^i_{2,2,5} w^3y\left(x^2+y^2\right)\left(5x^4-10x^2y^2+y^4\right) \\
& - ic^i_{2,4,-3} w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right)^2 + c^i_{2,4,-3} w^3y\left(x^2+y^2\right)^2\left(3x^2-y^2\right) - ic^i_{2,6,1} w^3x\left(x^2+y^2\right)^3 \\
& - c^i_{2,6,1} w^3y\left(x^2+y^2\right)^3 - ic^i_{4,0,5} w^5x\left(x^4-10x^2y^2+5y^4\right) - c^i_{4,0,5} w^5y\left(5x^4-10x^2y^2+y^4\right) \\
& - ic^i_{4,2,-3} w^5x\left(x^2-3y^2\right)\left(x^2+y^2\right) + c^i_{4,2,-3} w^5y\left(x^2+y^2\right)\left(3x^2-y^2\right) - ic^i_{4,4,1} w^5x\left(x^2+y^2\right)^2 \\
& - c^i_{4,4,1} w^5y\left(x^2+y^2\right)^2 - ic^i_{6,0,-3} w^7x\left(x^2-3y^2\right) + c^i_{6,0,-3} w^7y\left(3x^2-y^2\right) - ic^i_{6,2,1} w^7x\left(x^2+y^2\right) \\
& - c^i_{6,2,1} w^7y\left(x^2+y^2\right) - ic^i_{8,0,1} w^9x - c^i_{8,0,1} w^9y
\end{aligned}
$$

$$
\begin{aligned}
H_{A-}^{(10)} = {} & ic_{0,0,9}^{i} wx\left(x^2-3y^2\right)\left(x^6-33x^4y^2+27x^2y^4-3y^6\right) - c_{0,0,9}^{i} wy\left(3x^2-y^2\right)\left(3x^6-27x^4y^2+33x^2y^4-y^6\right) \\
& + ic_{0,2,-7}^{i} wx\left(x^2+y^2\right)\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c_{0,2,-7}^{i} wy\left(x^2+y^2\right)\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& + ic_{0,4,5}^{i} wx\left(x^2+y^2\right)^2\left(x^4-10x^2y^2+5y^4\right) - c_{0,4,5}^{i} wy\left(x^2+y^2\right)^2\left(5x^4-10x^2y^2+y^4\right) \\
& + ic_{0,6,-3}^{i} wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^3 + c_{0,6,-3}^{i} wy\left(x^2+y^2\right)^3\left(3x^2-y^2\right) + ic_{0,8,1}^{i} wx\left(x^2+y^2\right)^4 \\
& - c_{0,8,1}^{i} wy\left(x^2+y^2\right)^4 + ic_{2,0,-7}^{i} w^3x\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) + c_{2,0,-7}^{i} w^3y\left(7x^6-35x^4y^2+21x^2y^4-y^6\right) \\
& + ic_{2,2,5}^{i} w^3x\left(x^2+y^2\right)\left(x^4-10x^2y^2+5y^4\right) - c_{2,2,5}^{i} w^3y\left(x^2+y^2\right)\left(5x^4-10x^2y^2+y^4\right) \\
& + ic_{2,4,-3}^{i} w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right)^2 + c_{2,4,-3}^{i} w^3y\left(x^2+y^2\right)^2\left(3x^2-y^2\right) + ic_{2,6,1}^{i} w^3x\left(x^2+y^2\right)^3 \\
& - c_{2,6,1}^{i} w^3y\left(x^2+y^2\right)^3 + ic_{4,0,5}^{i} w^5x\left(x^4-10x^2y^2+5y^4\right) - c_{4,0,5}^{i} w^5y\left(5x^4-10x^2y^2+y^4\right) \\
& + ic_{4,2,-3}^{i} w^5x\left(x^2-3y^2\right)\left(x^2+y^2\right) + c_{4,2,-3}^{i} w^5y\left(x^2+y^2\right)\left(3x^2-y^2\right) + ic_{4,4,1}^{i} w^5x\left(x^2+y^2\right)^2 \\
& - c_{4,4,1}^{i} w^5y\left(x^2+y^2\right)^2 + ic_{6,0,-3}^{i} w^7x\left(x^2-3y^2\right) + c_{6,0,-3}^{i} w^7y\left(3x^2-y^2\right) + ic_{6,2,1}^{i} w^7x\left(x^2+y^2\right) \\
& - c_{6,2,1}^{i} w^7y\left(x^2+y^2\right) + ic_{8,0,1}^{i} w^9x - c_{8,0,1}^{i} w^9y
\end{aligned}
$$

4 Vibronic Hamiltonian operator in the real E basis

$$
\hat{H} = \begin{pmatrix} |X\rangle & |Y\rangle & |A\rangle \end{pmatrix} \begin{pmatrix} 0 & 0 & H_{XA} \\ 0 & 0 & H_{YA} \\ H_{AX} & H_{AY} & 0 \end{pmatrix} \begin{pmatrix} \langle X| \\ \langle Y| \\ \langle A| \end{pmatrix}
$$

5 Matrix element expansions in the real E basis

5.1 Order: 0

Number of fitting parameters: H_{XA}: 0, H_{YA}: 0.

Polar e-coordinates:

$$H_{XA}^{(0)} = 0$$

$$H_{YA}^{(0)} = 0$$

$$H_{AX}^{(0)} = 0$$

$$H_{AY}^{(0)} = 0$$

Cartesian e-coordinates:

$$H_{XA}^{(0)} = 0$$

$$H_{YA}^{(0)} = 0$$

$$H_{AX}^{(0)} = 0$$

$$H_{AY}^{(0)} = 0$$

5.2 Order: 1

Number of fitting parameters: H_{XA}: 0, H_{YA}: 0.

Polar e-coordinates:

$$H_{XA}^{(1)} = 0$$

$$H_{YA}^{(1)} = 0$$

$$H_{AX}^{(1)} = 0$$

$$H_{AY}^{(1)} = 0$$

Cartesian e-coordinates:

$$H_{XA}^{(1)} = 0$$

$$H_{YA}^{(1)} = 0$$

$$H_{AX}^{(1)} = 0$$

$$H_{AY}^{(1)} = 0$$

5.3 Order: 2

Number of fitting parameters: H_{XA}: 1 (all from H_{+A}), H_{YA}: 1 (all from H_{+A}).

Polar e-coordinates:

$$H_{XA}^{(2)} = -\sqrt{2}c_{0,0,1}^{i}\rho w \sin(\phi)$$

$$H_{YA}^{(2)} = -\sqrt{2}c_{0,0,1}^{i}\rho w \cos(\phi)$$

$$H_{AX}^{(2)} = -\sqrt{2}c_{0,0,1}^{i}\rho w \sin(\phi)$$

$$H_{AY}^{(2)} = -\sqrt{2}c_{0,0,1}^{i}\rho w \cos(\phi)$$

Cartesian e-coordinates:

$H^{(2)}_{XA} = -\sqrt{2}c^i_{0,0,1}wy$

$H^{(2)}_{YA} = -\sqrt{2}c^i_{0,0,1}wx$

$H^{(2)}_{AX} = -\sqrt{2}c^i_{0,0,1}wy$

$H^{(2)}_{AY} = -\sqrt{2}c^i_{0,0,1}wx$

5.4 Order: 3

Number of fitting parameters: H_{XA}: 0, H_{YA}: 0.

Polar e-coordinates:

$H^{(3)}_{XA} = 0$

$H^{(3)}_{YA} = 0$

$H^{(3)}_{AX} = 0$

$H^{(3)}_{AY} = 0$

Cartesian e-coordinates:

$H^{(3)}_{XA} = 0$

$H^{(3)}_{YA} = 0$

$H^{(3)}_{AX} = 0$

$H^{(3)}_{AY} = 0$

5.5 Order: 4

Number of fitting parameters: H_{XA}: 3 (all from H_{+A}), H_{YA}: 3 (all from H_{+A}).

Polar e-coordinates:

$$H_{XA}^{(4)} = \sqrt{2}\left(c_{0,0,-3}^{i}\rho^3 w\sin\left(3\phi\right) - c_{0,2,1}^{i}\rho^3 w\sin\left(\phi\right) - c_{2,0,1}^{i}\rho w^3\sin\left(\phi\right)\right)$$

$$H_{YA}^{(4)} = -\sqrt{2}\left(c_{0,0,-3}^{i}\rho^3 w\cos\left(3\phi\right) + c_{0,2,1}^{i}\rho^3 w\cos\left(\phi\right) + c_{2,0,1}^{i}\rho w^3\cos\left(\phi\right)\right)$$

$$H_{AX}^{(4)} = \sqrt{2}\left(c_{0,0,-3}^{i}\rho^3 w\sin\left(3\phi\right) - c_{0,2,1}^{i}\rho^3 w\sin\left(\phi\right) - c_{2,0,1}^{i}\rho w^3\sin\left(\phi\right)\right)$$

$$H_{AY}^{(4)} = -\sqrt{2}\left(c_{0,0,-3}^{i}\rho^3 w\cos\left(3\phi\right) + c_{0,2,1}^{i}\rho^3 w\cos\left(\phi\right) + c_{2,0,1}^{i}\rho w^3\cos\left(\phi\right)\right)$$

Cartesian e-coordinates:

$$H_{XA}^{(4)} = \sqrt{2}\left(c_{0,0,-3}^{i}wy\left(3x^2 - y^2\right) - c_{0,2,1}^{i}wy\left(x^2 + y^2\right) - c_{2,0,1}^{i}w^3 y\right)$$

$$H_{YA}^{(4)} = -\sqrt{2}\left(c_{0,0,-3}^{i}wx\left(x^2 - 3y^2\right) + c_{0,2,1}^{i}wx\left(x^2 + y^2\right) + c_{2,0,1}^{i}w^3 x\right)$$

$$H_{AX}^{(4)} = \sqrt{2}\left(c_{0,0,-3}^{i}wy\left(3x^2 - y^2\right) - c_{0,2,1}^{i}wy\left(x^2 + y^2\right) - c_{2,0,1}^{i}w^3 y\right)$$

$$H_{AY}^{(4)} = -\sqrt{2}\left(c_{0,0,-3}^{i}wx\left(x^2 - 3y^2\right) + c_{0,2,1}^{i}wx\left(x^2 + y^2\right) + c_{2,0,1}^{i}w^3 x\right)$$

5.6 Order: 5

Number of fitting parameters: H_{XA}: 0, H_{YA}: 0.

Polar e-coordinates:

$$H_{XA}^{(5)} = 0$$

$$H_{YA}^{(5)} = 0$$

$$H_{AX}^{(5)} = 0$$

$$H_{AY}^{(5)} = 0$$

Cartesian e-coordinates:

$$H_{XA}^{(5)} = 0$$

$$H_{YA}^{(5)} = 0$$

$$H_{AX}^{(5)} = 0$$

$$H_{AY}^{(5)} = 0$$

5.7 Order: 6

Number of fitting parameters: H_{XA}: 6 (all from H_{+A}), H_{YA}: 6 (all from H_{+A}).

Polar e-coordinates:

$$H_{XA}^{(6)} = \sqrt{2}\left(-c_{0,0,5}^{i}\rho^5 w\sin(5\phi) + c_{0,2,-3}^{i}\rho^5 w\sin(3\phi) - c_{0,4,1}^{i}\rho^5 w\sin(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\sin(3\phi) - c_{2,2,1}^{i}\rho^3 w^3\sin(\phi) - c_{4,0,1}^{i}\rho w^5\sin(\phi)\right)$$

$$H_{YA}^{(6)} = -\sqrt{2}\left(c_{0,0,5}^{i}\rho^5 w\cos(5\phi) + c_{0,2,-3}^{i}\rho^5 w\cos(3\phi) + c_{0,4,1}^{i}\rho^5 w\cos(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\cos(3\phi) + c_{2,2,1}^{i}\rho^3 w^3\cos(\phi) + c_{4,0,1}^{i}\rho w^5\cos(\phi)\right)$$

$$H_{AX}^{(6)} = \sqrt{2}\left(-c_{0,0,5}^{i}\rho^5 w\sin(5\phi) + c_{0,2,-3}^{i}\rho^5 w\sin(3\phi) - c_{0,4,1}^{i}\rho^5 w\sin(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\sin(3\phi) - c_{2,2,1}^{i}\rho^3 w^3\sin(\phi) - c_{4,0,1}^{i}\rho w^5\sin(\phi)\right)$$

$$H_{AY}^{(6)} = -\sqrt{2}\left(c_{0,0,5}^{i}\rho^5 w\cos(5\phi) + c_{0,2,-3}^{i}\rho^5 w\cos(3\phi) + c_{0,4,1}^{i}\rho^5 w\cos(\phi) + c_{2,0,-3}^{i}\rho^3 w^3\cos(3\phi) + c_{2,2,1}^{i}\rho^3 w^3\cos(\phi) + c_{4,0,1}^{i}\rho w^5\cos(\phi)\right)$$

Cartesian e-coordinates:

$$H_{XA}^{(6)} = \sqrt{2}\left(-c_{0,0,5}^{i}wy\left(5x^4 - 10x^2y^2 + y^4\right) + c_{0,2,-3}^{i}wy\left(x^2+y^2\right)\left(3x^2-y^2\right) - c_{0,4,1}^{i}wy\left(x^2+y^2\right)^2 + c_{2,0,-3}^{i}w^3y\left(3x^2-y^2\right) - c_{2,2,1}^{i}w^3y\left(x^2+y^2\right) - c_{4,0,1}^{i}w^5y\right)$$

$$H_{YA}^{(6)} = -\sqrt{2}\left(c_{0,0,5}^{i}wx\left(x^4 - 10x^2y^2 + 5y^4\right) + c_{0,2,-3}^{i}wx\left(x^2-3y^2\right)\left(x^2+y^2\right) + c_{0,4,1}^{i}wx\left(x^2+y^2\right)^2 + c_{2,0,-3}^{i}w^3x\left(x^2-3y^2\right) + c_{2,2,1}^{i}w^3x\left(x^2+y^2\right) + c_{4,0,1}^{i}w^5x\right)$$

$$H_{AX}^{(6)} = \sqrt{2}\left(-c_{0,0,5}^{i}wy\left(5x^4 - 10x^2y^2 + y^4\right) + c_{0,2,-3}^{i}wy\left(x^2+y^2\right)\left(3x^2-y^2\right) - c_{0,4,1}^{i}wy\left(x^2+y^2\right)^2 + c_{2,0,-3}^{i}w^3y\left(3x^2-y^2\right) - c_{2,2,1}^{i}w^3y\left(x^2+y^2\right) - c_{4,0,1}^{i}w^5y\right)$$

$$H_{AY}^{(6)} = -\sqrt{2}\left(c_{0,0,5}^{i}wx\left(x^4 - 10x^2y^2 + 5y^4\right) + c_{0,2,-3}^{i}wx\left(x^2-3y^2\right)\left(x^2+y^2\right) + c_{0,4,1}^{i}wx\left(x^2+y^2\right)^2 + c_{2,0,-3}^{i}w^3x\left(x^2-3y^2\right) + c_{2,2,1}^{i}w^3x\left(x^2+y^2\right) + c_{4,0,1}^{i}w^5x\right)$$

5.8 Order: 7

Number of fitting parameters: H_{XA}: 0, H_{YA}: 0.

Polar e-coordinates:

$$H_{XA}^{(7)} = 0$$

$$H_{YA}^{(7)} = 0$$

$$H_{AX}^{(7)} = 0$$

$$H_{AY}^{(7)} = 0$$

Cartesian e-coordinates:

$$H_{XA}^{(7)} = 0$$

$$H_{YA}^{(7)} = 0$$

$$H_{AX}^{(7)} = 0$$

$$H_{AY}^{(7)} = 0$$

5.9 Order: 8

Number of fitting parameters: H_{XA}: 10 (all from H_{+A}), H_{YA}: 10 (all from H_{+A}).

Polar e-coordinates:

$$\begin{aligned} H_{XA}^{(8)} = \sqrt{2}\big(&c_{0,0,-7}^{i}\rho^{7}w\sin(7\phi) - c_{0,2,5}^{i}\rho^{7}w\sin(5\phi) + c_{0,4,-3}^{i}\rho^{7}w\sin(3\phi) - c_{0,6,1}^{i}\rho^{7}w\sin(\phi) - c_{2,0,5}^{i}\rho^{5}w^{3}\sin(5\phi) \\ &+ c_{2,2,-3}^{i}\rho^{5}w^{3}\sin(3\phi) - c_{2,4,1}^{i}\rho^{5}w^{3}\sin(\phi) + c_{4,0,-3}^{i}\rho^{3}w^{5}\sin(3\phi) - c_{4,2,1}^{i}\rho^{3}w^{5}\sin(\phi) - c_{6,0,1}^{i}\rho w^{7}\sin(\phi)\big) \end{aligned}$$

$$\begin{aligned} H_{YA}^{(8)} = -\sqrt{2}\big(&c_{0,0,-7}^{i}\rho^{7}w\cos(7\phi) + c_{0,2,5}^{i}\rho^{7}w\cos(5\phi) + c_{0,4,-3}^{i}\rho^{7}w\cos(3\phi) + c_{0,6,1}^{i}\rho^{7}w\cos(\phi) + c_{2,0,5}^{i}\rho^{5}w^{3}\cos(5\phi) \\ &+ c_{2,2,-3}^{i}\rho^{5}w^{3}\cos(3\phi) + c_{2,4,1}^{i}\rho^{5}w^{3}\cos(\phi) + c_{4,0,-3}^{i}\rho^{3}w^{5}\cos(3\phi) + c_{4,2,1}^{i}\rho^{3}w^{5}\cos(\phi) + c_{6,0,1}^{i}\rho w^{7}\cos(\phi)\big) \end{aligned}$$

$$\begin{aligned} H_{AX}^{(8)} = \sqrt{2}\big(&c_{0,0,-7}^{i}\rho^{7}w\sin(7\phi) - c_{0,2,5}^{i}\rho^{7}w\sin(5\phi) + c_{0,4,-3}^{i}\rho^{7}w\sin(3\phi) - c_{0,6,1}^{i}\rho^{7}w\sin(\phi) - c_{2,0,5}^{i}\rho^{5}w^{3}\sin(5\phi) \\ &+ c_{2,2,-3}^{i}\rho^{5}w^{3}\sin(3\phi) - c_{2,4,1}^{i}\rho^{5}w^{3}\sin(\phi) + c_{4,0,-3}^{i}\rho^{3}w^{5}\sin(3\phi) - c_{4,2,1}^{i}\rho^{3}w^{5}\sin(\phi) - c_{6,0,1}^{i}\rho w^{7}\sin(\phi)\big) \end{aligned}$$

$$\begin{aligned} H_{AY}^{(8)} = -\sqrt{2}\big(&c_{0,0,-7}^{i}\rho^{7}w\cos(7\phi) + c_{0,2,5}^{i}\rho^{7}w\cos(5\phi) + c_{0,4,-3}^{i}\rho^{7}w\cos(3\phi) + c_{0,6,1}^{i}\rho^{7}w\cos(\phi) + c_{2,0,5}^{i}\rho^{5}w^{3}\cos(5\phi) \\ &+ c_{2,2,-3}^{i}\rho^{5}w^{3}\cos(3\phi) + c_{2,4,1}^{i}\rho^{5}w^{3}\cos(\phi) + c_{4,0,-3}^{i}\rho^{3}w^{5}\cos(3\phi) + c_{4,2,1}^{i}\rho^{3}w^{5}\cos(\phi) + c_{6,0,1}^{i}\rho w^{7}\cos(\phi)\big) \end{aligned}$$

Cartesian e-coordinates:

$$H_{XA}^{(8)} = \sqrt{2}\Big(c_{0,0,-7}^{i} wy\left(7x^6 - 35x^4y^2 + 21x^2y^4 - y^6\right) - c_{0,2,5}^{i} wy\left(x^2+y^2\right)\left(5x^4 - 10x^2y^2 + y^4\right) \\ + c_{0,4,-3}^{i} wy\left(x^2+y^2\right)^2\left(3x^2 - y^2\right) - c_{0,6,1}^{i} wy\left(x^2+y^2\right)^3 - c_{2,0,5}^{i} w^3 y\left(5x^4 - 10x^2y^2 + y^4\right) \\ + c_{2,2,-3}^{i} w^3 y\left(x^2+y^2\right)\left(3x^2 - y^2\right) - c_{2,4,1}^{i} w^3 y\left(x^2+y^2\right)^2 + c_{4,0,-3}^{i} w^5 y\left(3x^2 - y^2\right) - c_{4,2,1}^{i} w^5 y\left(x^2+y^2\right) - c_{6,0,1}^{i} w^7 y\Big)$$

$$H_{YA}^{(8)} = -\sqrt{2}\Big(c_{0,0,-7}^{i} wx\left(x^6 - 21x^4y^2 + 35x^2y^4 - 7y^6\right) + c_{0,2,5}^{i} wx\left(x^2+y^2\right)\left(x^4 - 10x^2y^2 + 5y^4\right) \\ + c_{0,4,-3}^{i} wx\left(x^2 - 3y^2\right)\left(x^2+y^2\right)^2 + c_{0,6,1}^{i} wx\left(x^2+y^2\right)^3 + c_{2,0,5}^{i} w^3 x\left(x^4 - 10x^2y^2 + 5y^4\right) \\ + c_{2,2,-3}^{i} w^3 x\left(x^2 - 3y^2\right)\left(x^2+y^2\right) + c_{2,4,1}^{i} w^3 x\left(x^2+y^2\right)^2 + c_{4,0,-3}^{i} w^5 x\left(x^2 - 3y^2\right) + c_{4,2,1}^{i} w^5 x\left(x^2+y^2\right) + c_{6,0,1}^{i} w^7 x\Big)$$

$$H_{AX}^{(8)} = \sqrt{2}\Big(c_{0,0,-7}^{i} wy\left(7x^6 - 35x^4y^2 + 21x^2y^4 - y^6\right) - c_{0,2,5}^{i} wy\left(x^2+y^2\right)\left(5x^4 - 10x^2y^2 + y^4\right) \\ + c_{0,4,-3}^{i} wy\left(x^2+y^2\right)^2\left(3x^2 - y^2\right) - c_{0,6,1}^{i} wy\left(x^2+y^2\right)^3 - c_{2,0,5}^{i} w^3 y\left(5x^4 - 10x^2y^2 + y^4\right) \\ + c_{2,2,-3}^{i} w^3 y\left(x^2+y^2\right)\left(3x^2 - y^2\right) - c_{2,4,1}^{i} w^3 y\left(x^2+y^2\right)^2 + c_{4,0,-3}^{i} w^5 y\left(3x^2 - y^2\right) - c_{4,2,1}^{i} w^5 y\left(x^2+y^2\right) - c_{6,0,1}^{i} w^7 y\Big)$$

$$H_{AY}^{(8)} = -\sqrt{2}\Big(c_{0,0,-7}^{i} wx\left(x^6 - 21x^4y^2 + 35x^2y^4 - 7y^6\right) + c_{0,2,5}^{i} wx\left(x^2+y^2\right)\left(x^4 - 10x^2y^2 + 5y^4\right) \\ + c_{0,4,-3}^{i} wx\left(x^2 - 3y^2\right)\left(x^2+y^2\right)^2 + c_{0,6,1}^{i} wx\left(x^2+y^2\right)^3 + c_{2,0,5}^{i} w^3 x\left(x^4 - 10x^2y^2 + 5y^4\right) \\ + c_{2,2,-3}^{i} w^3 x\left(x^2 - 3y^2\right)\left(x^2+y^2\right) + c_{2,4,1}^{i} w^3 x\left(x^2+y^2\right)^2 + c_{4,0,-3}^{i} w^5 x\left(x^2 - 3y^2\right) + c_{4,2,1}^{i} w^5 x\left(x^2+y^2\right) + c_{6,0,1}^{i} w^7 x\Big)$$

5.10 Order: 9

Number of fitting parameters: H_{XA}: 0, H_{YA}: 0.

Polar e-coordinates:

$$H_{XA}^{(9)} = 0$$

$$H_{YA}^{(9)} = 0$$

$$H_{AX}^{(9)} = 0$$

$$H_{AY}^{(9)} = 0$$

Cartesian e-coordinates:

$$H_{XA}^{(9)} = 0$$

$$H_{YA}^{(9)} = 0$$

$$H_{AX}^{(9)} = 0$$

$$H_{AY}^{(9)} = 0$$

5.11 Order: 10

Number of fitting parameters: H_{XA}: 15 (all from H_{+A}), H_{YA}: 15 (all from H_{+A}).

Polar e-coordinates:

$$\begin{aligned}H_{XA}^{(10)} = \sqrt{2}\,\big(&-c_{0,0,9}^{i}\rho^{9}w\sin(9\phi) + c_{0,2,-7}^{i}\rho^{9}w\sin(7\phi) - c_{0,4,5}^{i}\rho^{9}w\sin(5\phi) + c_{0,6,-3}^{i}\rho^{9}w\sin(3\phi) - c_{0,8,1}^{i}\rho^{9}w\sin(\phi)\\ &+ c_{2,0,-7}^{i}\rho^{7}w^{3}\sin(7\phi) - c_{2,2,5}^{i}\rho^{7}w^{3}\sin(5\phi) + c_{2,4,-3}^{i}\rho^{7}w^{3}\sin(3\phi) - c_{2,6,1}^{i}\rho^{7}w^{3}\sin(\phi) - c_{4,0,5}^{i}\rho^{5}w^{5}\sin(5\phi)\\ &+ c_{4,2,-3}^{i}\rho^{5}w^{5}\sin(3\phi) - c_{4,4,1}^{i}\rho^{5}w^{5}\sin(\phi) + c_{6,0,-3}^{i}\rho^{3}w^{7}\sin(3\phi) - c_{6,2,1}^{i}\rho^{3}w^{7}\sin(\phi) - c_{8,0,1}^{i}\rho w^{9}\sin(\phi)\big)\end{aligned}$$

$$\begin{aligned}H_{YA}^{(10)} = -\sqrt{2}\,\big(&c_{0,0,9}^{i}\rho^{9}w\cos(9\phi) + c_{0,2,-7}^{i}\rho^{9}w\cos(7\phi) + c_{0,4,5}^{i}\rho^{9}w\cos(5\phi) + c_{0,6,-3}^{i}\rho^{9}w\cos(3\phi) + c_{0,8,1}^{i}\rho^{9}w\cos(\phi)\\ &+ c_{2,0,-7}^{i}\rho^{7}w^{3}\cos(7\phi) + c_{2,2,5}^{i}\rho^{7}w^{3}\cos(5\phi) + c_{2,4,-3}^{i}\rho^{7}w^{3}\cos(3\phi) + c_{2,6,1}^{i}\rho^{7}w^{3}\cos(\phi) + c_{4,0,5}^{i}\rho^{5}w^{5}\cos(5\phi)\\ &+ c_{4,2,-3}^{i}\rho^{5}w^{5}\cos(3\phi) + c_{4,4,1}^{i}\rho^{5}w^{5}\cos(\phi) + c_{6,0,-3}^{i}\rho^{3}w^{7}\cos(3\phi) + c_{6,2,1}^{i}\rho^{3}w^{7}\cos(\phi) + c_{8,0,1}^{i}\rho w^{9}\cos(\phi)\big)\end{aligned}$$

$$\begin{aligned}H_{AX}^{(10)} = \sqrt{2}\,\big(&-c_{0,0,9}^{i}\rho^{9}w\sin(9\phi) + c_{0,2,-7}^{i}\rho^{9}w\sin(7\phi) - c_{0,4,5}^{i}\rho^{9}w\sin(5\phi) + c_{0,6,-3}^{i}\rho^{9}w\sin(3\phi) - c_{0,8,1}^{i}\rho^{9}w\sin(\phi)\\ &+ c_{2,0,-7}^{i}\rho^{7}w^{3}\sin(7\phi) - c_{2,2,5}^{i}\rho^{7}w^{3}\sin(5\phi) + c_{2,4,-3}^{i}\rho^{7}w^{3}\sin(3\phi) - c_{2,6,1}^{i}\rho^{7}w^{3}\sin(\phi) - c_{4,0,5}^{i}\rho^{5}w^{5}\sin(5\phi)\\ &+ c_{4,2,-3}^{i}\rho^{5}w^{5}\sin(3\phi) - c_{4,4,1}^{i}\rho^{5}w^{5}\sin(\phi) + c_{6,0,-3}^{i}\rho^{3}w^{7}\sin(3\phi) - c_{6,2,1}^{i}\rho^{3}w^{7}\sin(\phi) - c_{8,0,1}^{i}\rho w^{9}\sin(\phi)\big)\end{aligned}$$

$$\begin{aligned}H_{AY}^{(10)} = -\sqrt{2}\,\big(&c_{0,0,9}^{i}\rho^{9}w\cos(9\phi) + c_{0,2,-7}^{i}\rho^{9}w\cos(7\phi) + c_{0,4,5}^{i}\rho^{9}w\cos(5\phi) + c_{0,6,-3}^{i}\rho^{9}w\cos(3\phi) + c_{0,8,1}^{i}\rho^{9}w\cos(\phi)\\ &+ c_{2,0,-7}^{i}\rho^{7}w^{3}\cos(7\phi) + c_{2,2,5}^{i}\rho^{7}w^{3}\cos(5\phi) + c_{2,4,-3}^{i}\rho^{7}w^{3}\cos(3\phi) + c_{2,6,1}^{i}\rho^{7}w^{3}\cos(\phi) + c_{4,0,5}^{i}\rho^{5}w^{5}\cos(5\phi)\\ &+ c_{4,2,-3}^{i}\rho^{5}w^{5}\cos(3\phi) + c_{4,4,1}^{i}\rho^{5}w^{5}\cos(\phi) + c_{6,0,-3}^{i}\rho^{3}w^{7}\cos(3\phi) + c_{6,2,1}^{i}\rho^{3}w^{7}\cos(\phi) + c_{8,0,1}^{i}\rho w^{9}\cos(\phi)\big)\end{aligned}$$

Cartesian e-coordinates:

$$\begin{aligned}H_{XA}^{(10)} = \sqrt{2}\,\Big(&-c_{0,0,9}^{i}wy\left(3x^{2}-y^{2}\right)\left(3x^{6}-27x^{4}y^{2}+33x^{2}y^{4}-y^{6}\right) + c_{0,2,-7}^{i}wy\left(x^{2}+y^{2}\right)\left(7x^{6}-35x^{4}y^{2}+21x^{2}y^{4}-y^{6}\right)\\ &- c_{0,4,5}^{i}wy\left(x^{2}+y^{2}\right)^{2}\left(5x^{4}-10x^{2}y^{2}+y^{4}\right) + c_{0,6,-3}^{i}wy\left(x^{2}+y^{2}\right)^{3}\left(3x^{2}-y^{2}\right) - c_{0,8,1}^{i}wy\left(x^{2}+y^{2}\right)^{4}\\ &+ c_{2,0,-7}^{i}w^{3}y\left(7x^{6}-35x^{4}y^{2}+21x^{2}y^{4}-y^{6}\right) - c_{2,2,5}^{i}w^{3}y\left(x^{2}+y^{2}\right)\left(5x^{4}-10x^{2}y^{2}+y^{4}\right)\\ &+ c_{2,4,-3}^{i}w^{3}y\left(x^{2}+y^{2}\right)^{2}\left(3x^{2}-y^{2}\right) - c_{2,6,1}^{i}w^{3}y\left(x^{2}+y^{2}\right)^{3} - c_{4,0,5}^{i}w^{5}y\left(5x^{4}-10x^{2}y^{2}+y^{4}\right)\\ &+ c_{4,2,-3}^{i}w^{5}y\left(x^{2}+y^{2}\right)\left(3x^{2}-y^{2}\right) - c_{4,4,1}^{i}w^{5}y\left(x^{2}+y^{2}\right)^{2} + c_{6,0,-3}^{i}w^{7}y\left(3x^{2}-y^{2}\right) - c_{6,2,1}^{i}w^{7}y\left(x^{2}+y^{2}\right) - c_{8,0,1}^{i}w^{9}y\Big)\end{aligned}$$

$$\begin{aligned}H_{YA}^{(10)} = -\sqrt{2}\,\Big(&c_{0,0,9}^{i}wx\left(x^{2}-3y^{2}\right)\left(x^{6}-33x^{4}y^{2}+27x^{2}y^{4}-3y^{6}\right) + c_{0,2,-7}^{i}wx\left(x^{2}+y^{2}\right)\left(x^{6}-21x^{4}y^{2}+35x^{2}y^{4}-7y^{6}\right)\\ &+ c_{0,4,5}^{i}wx\left(x^{2}+y^{2}\right)^{2}\left(x^{4}-10x^{2}y^{2}+5y^{4}\right) + c_{0,6,-3}^{i}wx\left(x^{2}-3y^{2}\right)\left(x^{2}+y^{2}\right)^{3} + c_{0,8,1}^{i}wx\left(x^{2}+y^{2}\right)^{4}\\ &+ c_{2,0,-7}^{i}w^{3}x\left(x^{6}-21x^{4}y^{2}+35x^{2}y^{4}-7y^{6}\right) + c_{2,2,5}^{i}w^{3}x\left(x^{2}+y^{2}\right)\left(x^{4}-10x^{2}y^{2}+5y^{4}\right)\\ &+ c_{2,4,-3}^{i}w^{3}x\left(x^{2}-3y^{2}\right)\left(x^{2}+y^{2}\right)^{2} + c_{2,6,1}^{i}w^{3}x\left(x^{2}+y^{2}\right)^{3} + c_{4,0,5}^{i}w^{5}x\left(x^{4}-10x^{2}y^{2}+5y^{4}\right)\\ &+ c_{4,2,-3}^{i}w^{5}x\left(x^{2}-3y^{2}\right)\left(x^{2}+y^{2}\right) + c_{4,4,1}^{i}w^{5}x\left(x^{2}+y^{2}\right)^{2} + c_{6,0,-3}^{i}w^{7}x\left(x^{2}-3y^{2}\right) + c_{6,2,1}^{i}w^{7}x\left(x^{2}+y^{2}\right) + c_{8,0,1}^{i}w^{9}x\Big)\end{aligned}$$

$$\begin{aligned}H_{AX}^{(10)} = \sqrt{2}\,\Big(&-c_{0,0,9}^{i}wy\left(3x^{2}-y^{2}\right)\left(3x^{6}-27x^{4}y^{2}+33x^{2}y^{4}-y^{6}\right) + c_{0,2,-7}^{i}wy\left(x^{2}+y^{2}\right)\left(7x^{6}-35x^{4}y^{2}+21x^{2}y^{4}-y^{6}\right)\\ &- c_{0,4,5}^{i}wy\left(x^{2}+y^{2}\right)^{2}\left(5x^{4}-10x^{2}y^{2}+y^{4}\right) + c_{0,6,-3}^{i}wy\left(x^{2}+y^{2}\right)^{3}\left(3x^{2}-y^{2}\right) - c_{0,8,1}^{i}wy\left(x^{2}+y^{2}\right)^{4}\\ &+ c_{2,0,-7}^{i}w^{3}y\left(7x^{6}-35x^{4}y^{2}+21x^{2}y^{4}-y^{6}\right) - c_{2,2,5}^{i}w^{3}y\left(x^{2}+y^{2}\right)\left(5x^{4}-10x^{2}y^{2}+y^{4}\right)\\ &+ c_{2,4,-3}^{i}w^{3}y\left(x^{2}+y^{2}\right)^{2}\left(3x^{2}-y^{2}\right) - c_{2,6,1}^{i}w^{3}y\left(x^{2}+y^{2}\right)^{3} - c_{4,0,5}^{i}w^{5}y\left(5x^{4}-10x^{2}y^{2}+y^{4}\right)\\ &+ c_{4,2,-3}^{i}w^{5}y\left(x^{2}+y^{2}\right)\left(3x^{2}-y^{2}\right) - c_{4,4,1}^{i}w^{5}y\left(x^{2}+y^{2}\right)^{2} + c_{6,0,-3}^{i}w^{7}y\left(3x^{2}-y^{2}\right) - c_{6,2,1}^{i}w^{7}y\left(x^{2}+y^{2}\right) - c_{8,0,1}^{i}w^{9}y\Big)\end{aligned}$$

$$
\begin{aligned}
H_{AY}^{(10)} = -\sqrt{2}\Big(& c_{0,0,9}^{i} wx\left(x^2-3y^2\right)\left(x^6-33x^4y^2+27x^2y^4-3y^6\right)+c_{0,2,-7}^{i} wx\left(x^2+y^2\right)\left(x^6-21x^4y^2+35x^2y^4-7y^6\right) \\
& + c_{0,4,5}^{i} wx\left(x^2+y^2\right)^2\left(x^4-10x^2y^2+5y^4\right)+c_{0,6,-3}^{i} wx\left(x^2-3y^2\right)\left(x^2+y^2\right)^3+c_{0,8,1}^{i} wx\left(x^2+y^2\right)^4 \\
& + c_{2,0,-7}^{i} w^3x\left(x^6-21x^4y^2+35x^2y^4-7y^6\right)+c_{2,2,5}^{i} w^3x\left(x^2+y^2\right)\left(x^4-10x^2y^2+5y^4\right) \\
& + c_{2,4,-3}^{i} w^3x\left(x^2-3y^2\right)\left(x^2+y^2\right)^2+c_{2,6,1}^{i} w^3x\left(x^2+y^2\right)^3+c_{4,0,5}^{i} w^5x\left(x^4-10x^2y^2+5y^4\right) \\
& + c_{4,2,-3}^{i} w^5x\left(x^2-3y^2\right)\left(x^2+y^2\right)+c_{4,4,1}^{i} w^5x\left(x^2+y^2\right)^2+c_{6,0,-3}^{i} w^7x\left(x^2-3y^2\right)+c_{6,2,1}^{i} w^7x\left(x^2+y^2\right)+c_{8,0,1}^{i} w^9x\Big)
\end{aligned}
$$