# VHEGEN: A vibronic Hamiltonian expansion generator for trigonal and tetragonal polyatomic systems

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Thank you for using `VHEGEN`, the `V`-ibronic `H`-amiltonian `E`-xpansion `GEN`-erator for trigonal and tetragonal polyatomic systems. This is a `VHEGEN` output file compiled by `pdflatex`. If the `VHEGEN` package was used in research resulting in a publication, please reference the article in *Computer Physics Communications* which describes the program ([doi here]). Additional information regarding the matrix element expansion process, including the independent matrix element eigenvalues, their root formulas and constraints, and their transformation to the real basis (if applicable), can be found in the `log` output file. For questions, bugs, or comments, please contact robert.lang@mail.utoronto.ca.

## Contents

# 1 Vibronic interaction

$E'' \otimes (e' + e')$ in $D_{3h}$

# 2 Vibronic Hamiltonian operator in the complex $E$ basis

$$\hat{H} = \begin{pmatrix} |+\rangle & |-\rangle \end{pmatrix} \begin{pmatrix} H_{++} & H_{+-} \\ H_{-+} & H_{--} \end{pmatrix} \begin{pmatrix} \langle +| \\ \langle -| \end{pmatrix}$$

# 3 Matrix element expansions in the complex $E$ basis

## 3.1 Order: 0

Number of terms: $H_{++}$: 1, $H_{--}$: 1, $H_{+-}$: 0.

**Polar e-coordinates:**

$$H_{++}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{--}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{+-}^{(0)} = 0$$

$$H_{-+}^{(0)} = 0$$

**Cartesian e-coordinates:**

$$H_{++}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{--}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{+-}^{(0)} = 0$$

$$H_{-+}^{(0)} = 0$$

## 3.2 Order: 1

Number of terms: $H_{++}$: 0, $H_{--}$: 0, $H_{+-}$: 4.

**Polar e-coordinates:**

$$H^{(1)}_{++} = 0$$

$$H^{(1)}_{--} = 0$$

$$H^{(1)}_{+-} = -ib^r_{0,0,-1,-1}\rho_\alpha \sin(\phi_\alpha) + b^r_{0,0,-1,-1}\rho_\alpha \cos(\phi_\alpha) - ib^r_{0,0,0,-1}\rho_\beta \sin(\phi_\beta) + b^r_{0,0,0,-1}\rho_\beta \cos(\phi_\beta)$$

$$H^{(1)}_{-+} = ib^r_{0,0,-1,-1}\rho_\alpha \sin(\phi_\alpha) + b^r_{0,0,-1,-1}\rho_\alpha \cos(\phi_\alpha) + ib^r_{0,0,0,-1}\rho_\beta \sin(\phi_\beta) + b^r_{0,0,0,-1}\rho_\beta \cos(\phi_\beta)$$

**Cartesian e-coordinates:**

$$H^{(1)}_{++} = 0$$

$$H^{(1)}_{--} = 0$$

$$H^{(1)}_{+-} = b^r_{0,0,-1,-1}x_\alpha - ib^r_{0,0,-1,-1}y_\alpha + b^r_{0,0,0,-1}x_\beta - ib^r_{0,0,0,-1}y_\beta$$

$$H^{(1)}_{-+} = b^r_{0,0,-1,-1}x_\alpha + ib^r_{0,0,-1,-1}y_\alpha + b^r_{0,0,0,-1}x_\beta + ib^r_{0,0,0,-1}y_\beta$$

## 3.3 Order: 2

Number of terms: $H_{++}$: 3, $H_{--}$: 3, $H_{+-}$: 6.

**Polar e-coordinates:**

$$H^{(2)}_{++} = a^r_{0,0,1,0}\rho_\alpha\rho_\beta \cos(\phi_\alpha - \phi_\beta) + a^r_{0,2,0,0}\rho^2_\beta + a^r_{2,0,0,0}\rho^2_\alpha$$

$$H^{(2)}_{--} = a^r_{0,0,1,0}\rho_\alpha\rho_\beta \cos(\phi_\alpha - \phi_\beta) + a^r_{0,2,0,0}\rho^2_\beta + a^r_{2,0,0,0}\rho^2_\alpha$$

$$\begin{aligned} H^{(2)}_{+-} = {} & ib^r_{0,0,0,2}\rho^2_\beta \sin(2\phi_\beta) + b^r_{0,0,0,2}\rho^2_\beta \cos(2\phi_\beta) + ib^r_{0,0,1,2}\rho_\alpha\rho_\beta \sin(\phi_\alpha + \phi_\beta) + b^r_{0,0,1,2}\rho_\alpha\rho_\beta \cos(\phi_\alpha + \phi_\beta) \\ & + ib^r_{0,0,2,2}\rho^2_\alpha \sin(2\phi_\alpha) + b^r_{0,0,2,2}\rho^2_\alpha \cos(2\phi_\alpha) \end{aligned}$$

$$\begin{aligned} H^{(2)}_{-+} = {} & -ib^r_{0,0,0,2}\rho^2_\beta \sin(2\phi_\beta) + b^r_{0,0,0,2}\rho^2_\beta \cos(2\phi_\beta) - ib^r_{0,0,1,2}\rho_\alpha\rho_\beta \sin(\phi_\alpha + \phi_\beta) + b^r_{0,0,1,2}\rho_\alpha\rho_\beta \cos(\phi_\alpha + \phi_\beta) \\ & - ib^r_{0,0,2,2}\rho^2_\alpha \sin(2\phi_\alpha) + b^r_{0,0,2,2}\rho^2_\alpha \cos(2\phi_\alpha) \end{aligned}$$

**Cartesian e-coordinates:**

$$H^{(2)}_{++} = a^r_{0,0,1,0}\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a^r_{0,2,0,0}\left(x^2_\beta + y^2_\beta\right) + a^r_{2,0,0,0}\left(x^2_\alpha + y^2_\alpha\right)$$

$$H^{(2)}_{--} = a^r_{0,0,1,0}\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a^r_{0,2,0,0}\left(x^2_\beta + y^2_\beta\right) + a^r_{2,0,0,0}\left(x^2_\alpha + y^2_\alpha\right)$$

$$\begin{aligned} H^{(2)}_{+-} = {} & 2ib^r_{0,0,0,2}x_\beta y_\beta + b^r_{0,0,0,2}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b^r_{0,0,1,2}\left(x_\alpha x_\beta - y_\alpha y_\beta\right) + ib^r_{0,0,1,2}\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\ & + 2ib^r_{0,0,2,2}x_\alpha y_\alpha + b^r_{0,0,2,2}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right) \end{aligned}$$

$$\begin{aligned} H^{(2)}_{-+} = {} & -2ib^r_{0,0,0,2}x_\beta y_\beta + b^r_{0,0,0,2}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b^r_{0,0,1,2}\left(x_\alpha x_\beta - y_\alpha y_\beta\right) - ib^r_{0,0,1,2}\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\ & - 2ib^r_{0,0,2,2}x_\alpha y_\alpha + b^r_{0,0,2,2}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right) \end{aligned}$$

## 3.4 Order: 3

Number of terms: $H_{++}$: 4, $H_{--}$: 4, $H_{+-}$: 12.

**Polar e-coordinates:**

$$H^{(3)}_{++} = a^r_{0,0,0,3}\rho^3_\beta\cos\left(3\phi_\beta\right) + a^r_{0,0,1,3}\rho_\alpha\rho^2_\beta\cos\left(\phi_\alpha + 2\phi_\beta\right) + a^r_{0,0,2,3}\rho^2_\alpha\rho_\beta\cos\left(2\phi_\alpha + \phi_\beta\right) + a^r_{0,0,3,3}\rho^3_\alpha\cos\left(3\phi_\alpha\right)$$

$$H^{(3)}_{--} = a^r_{0,0,0,3}\rho^3_\beta\cos\left(3\phi_\beta\right) + a^r_{0,0,1,3}\rho_\alpha\rho^2_\beta\cos\left(\phi_\alpha + 2\phi_\beta\right) + a^r_{0,0,2,3}\rho^2_\alpha\rho_\beta\cos\left(2\phi_\alpha + \phi_\beta\right) + a^r_{0,0,3,3}\rho^3_\alpha\cos\left(3\phi_\alpha\right)$$

$$\begin{aligned} H^{(3)}_{+-} = {} & -ib^r_{0,0,-2,-1}\rho^2_\alpha\rho_\beta\sin\left(2\phi_\alpha - \phi_\beta\right) + b^r_{0,0,-2,-1}\rho^2_\alpha\rho_\beta\cos\left(2\phi_\alpha - \phi_\beta\right) + ib^r_{0,0,1,-1}\rho_\alpha\rho^2_\beta\sin\left(\phi_\alpha - 2\phi_\beta\right) \\ & + b^r_{0,0,1,-1}\rho_\alpha\rho^2_\beta\cos\left(\phi_\alpha - 2\phi_\beta\right) - ib^r_{0,2,-1,-1}\rho_\alpha\rho^2_\beta\sin\left(\phi_\alpha\right) + b^r_{0,2,-1,-1}\rho_\alpha\rho^2_\beta\cos\left(\phi_\alpha\right) \\ & - ib^r_{0,2,0,-1}\rho^3_\beta\sin\left(\phi_\beta\right) + b^r_{0,2,0,-1}\rho^3_\beta\cos\left(\phi_\beta\right) - ib^r_{2,0,-1,-1}\rho^3_\alpha\sin\left(\phi_\alpha\right) + b^r_{2,0,-1,-1}\rho^3_\alpha\cos\left(\phi_\alpha\right) \\ & - ib^r_{2,0,0,-1}\rho^2_\alpha\rho_\beta\sin\left(\phi_\beta\right) + b^r_{2,0,0,-1}\rho^2_\alpha\rho_\beta\cos\left(\phi_\beta\right) \end{aligned}$$

$$\begin{aligned} H^{(3)}_{-+} = {} & ib^r_{0,0,-2,-1}\rho^2_\alpha\rho_\beta\sin\left(2\phi_\alpha - \phi_\beta\right) + b^r_{0,0,-2,-1}\rho^2_\alpha\rho_\beta\cos\left(2\phi_\alpha - \phi_\beta\right) - ib^r_{0,0,1,-1}\rho_\alpha\rho^2_\beta\sin\left(\phi_\alpha - 2\phi_\beta\right) \\ & + b^r_{0,0,1,-1}\rho_\alpha\rho^2_\beta\cos\left(\phi_\alpha - 2\phi_\beta\right) + ib^r_{0,2,-1,-1}\rho_\alpha\rho^2_\beta\sin\left(\phi_\alpha\right) + b^r_{0,2,-1,-1}\rho_\alpha\rho^2_\beta\cos\left(\phi_\alpha\right) \\ & + ib^r_{0,2,0,-1}\rho^3_\beta\sin\left(\phi_\beta\right) + b^r_{0,2,0,-1}\rho^3_\beta\cos\left(\phi_\beta\right) + ib^r_{2,0,-1,-1}\rho^3_\alpha\sin\left(\phi_\alpha\right) + b^r_{2,0,-1,-1}\rho^3_\alpha\cos\left(\phi_\alpha\right) \\ & + ib^r_{2,0,0,-1}\rho^2_\alpha\rho_\beta\sin\left(\phi_\beta\right) + b^r_{2,0,0,-1}\rho^2_\alpha\rho_\beta\cos\left(\phi_\beta\right) \end{aligned}$$

**Cartesian e-coordinates:**

$$\begin{aligned} H^{(3)}_{++} = {} & a^r_{0,0,0,3}x_\beta\left(x^2_\beta - 3y^2_\beta\right) + a^r_{0,0,1,3}\left(x_\alpha\left(x^2_\beta - y^2_\beta\right) - 2x_\beta y_\alpha y_\beta\right) \\ & + a^r_{0,0,2,3}\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x^2_\alpha - y^2_\alpha\right)\right) + a^r_{0,0,3,3}x_\alpha\left(x^2_\alpha - 3y^2_\alpha\right) \end{aligned}$$

$$\begin{aligned} H^{(3)}_{--} = {} & a^r_{0,0,0,3}x_\beta\left(x^2_\beta - 3y^2_\beta\right) + a^r_{0,0,1,3}\left(x_\alpha\left(x^2_\beta - y^2_\beta\right) - 2x_\beta y_\alpha y_\beta\right) \\ & + a^r_{0,0,2,3}\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x^2_\alpha - y^2_\alpha\right)\right) + a^r_{0,0,3,3}x_\alpha\left(x^2_\alpha - 3y^2_\alpha\right) \end{aligned}$$

$$
\begin{aligned}
H^{(3)}_{+-} = & ib^r_{0,0,-2,-1}\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + b^r_{0,0,-2,-1}\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + b^r_{0,0,1,-1}\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right) - ib^r_{0,0,1,-1}\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) \\
& + b^r_{0,2,-1,-1}x_\alpha\left(x_\beta^2 + y_\beta^2\right) - ib^r_{0,2,-1,-1}y_\alpha\left(x_\beta^2 + y_\beta^2\right) + b^r_{0,2,0,-1}x_\beta\left(x_\beta^2 + y_\beta^2\right) - ib^r_{0,2,0,-1}y_\beta\left(x_\beta^2 + y_\beta^2\right) \\
& + b^r_{2,0,-1,-1}x_\alpha\left(x_\alpha^2 + y_\alpha^2\right) - ib^r_{2,0,-1,-1}y_\alpha\left(x_\alpha^2 + y_\alpha^2\right) + b^r_{2,0,0,-1}x_\beta\left(x_\alpha^2 + y_\alpha^2\right) - ib^r_{2,0,0,-1}y_\beta\left(x_\alpha^2 + y_\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(3)}_{-+} = & -ib^r_{0,0,-2,-1}\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + b^r_{0,0,-2,-1}\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + b^r_{0,0,1,-1}\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right) + ib^r_{0,0,1,-1}\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) \\
& + b^r_{0,2,-1,-1}x_\alpha\left(x_\beta^2 + y_\beta^2\right) + ib^r_{0,2,-1,-1}y_\alpha\left(x_\beta^2 + y_\beta^2\right) + b^r_{0,2,0,-1}x_\beta\left(x_\beta^2 + y_\beta^2\right) + ib^r_{0,2,0,-1}y_\beta\left(x_\beta^2 + y_\beta^2\right) \\
& + b^r_{2,0,-1,-1}x_\alpha\left(x_\alpha^2 + y_\alpha^2\right) + ib^r_{2,0,-1,-1}y_\alpha\left(x_\alpha^2 + y_\alpha^2\right) + b^r_{2,0,0,-1}x_\beta\left(x_\alpha^2 + y_\alpha^2\right) + ib^r_{2,0,0,-1}y_\beta\left(x_\alpha^2 + y_\alpha^2\right)
\end{aligned}
$$

## 3.5 Order: 4

Number of terms: $H_{++}$: 6, $H_{--}$: 6, $H_{+-}$: 26.

**Polar e-coordinates:**

$$
\begin{aligned}
H^{(4)}_{++} = & a^r_{0,0,2,0}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha - 2\phi_\beta\right) + a^r_{0,2,1,0}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - \phi_\beta\right) \\
& + a^r_{0,4,0,0}\rho_\beta^4 + a^r_{2,0,1,0}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha - \phi_\beta\right) + a^r_{2,2,0,0}\rho_\alpha^2\rho_\beta^2 + a^r_{4,0,0,0}\rho_\alpha^4
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{--} = & a^r_{0,0,2,0}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha - 2\phi_\beta\right) + a^r_{0,2,1,0}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - \phi_\beta\right) \\
& + a^r_{0,4,0,0}\rho_\beta^4 + a^r_{2,0,1,0}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha - \phi_\beta\right) + a^r_{2,2,0,0}\rho_\alpha^2\rho_\beta^2 + a^r_{4,0,0,0}\rho_\alpha^4
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{+-} = & -ib^r_{0,0,-1,-4}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + 3\phi_\beta\right) + b^r_{0,0,-1,-4}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + 3\phi_\beta\right) - ib^r_{0,0,-1,2}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha - 3\phi_\beta\right) \\
& + b^r_{0,0,-1,2}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - 3\phi_\beta\right) - ib^r_{0,0,-2,-4}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha + 2\phi_\beta\right) + b^r_{0,0,-2,-4}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha + 2\phi_\beta\right) \\
& - ib^r_{0,0,-3,-4}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha + \phi_\beta\right) + b^r_{0,0,-3,-4}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha + \phi_\beta\right) - ib^r_{0,0,-4,-4}\rho_\alpha^4\sin\left(4\phi_\alpha\right) + b^r_{0,0,-4,-4}\rho_\alpha^4\cos\left(4\phi_\alpha\right) \\
& - ib^r_{0,0,0,-4}\rho_\beta^4\sin\left(4\phi_\beta\right) + b^r_{0,0,0,-4}\rho_\beta^4\cos\left(4\phi_\beta\right) + ib^r_{0,0,3,2}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha - \phi_\beta\right) + b^r_{0,0,3,2}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha - \phi_\beta\right) \\
& + ib^r_{0,2,0,2}\rho_\beta^4\sin\left(2\phi_\beta\right) + b^r_{0,2,0,2}\rho_\beta^4\cos\left(2\phi_\beta\right) + ib^r_{0,2,1,2}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + \phi_\beta\right) + b^r_{0,2,1,2}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + \phi_\beta\right) \\
& + ib^r_{0,2,2,2}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha\right) + b^r_{0,2,2,2}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha\right) + ib^r_{2,0,0,2}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\beta\right) + b^r_{2,0,0,2}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\beta\right) \\
& + ib^r_{2,0,1,2}\rho_\alpha^3\rho_\beta\sin\left(\phi_\alpha + \phi_\beta\right) + b^r_{2,0,1,2}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha + \phi_\beta\right) + ib^r_{2,0,2,2}\rho_\alpha^4\sin\left(2\phi_\alpha\right) + b^r_{2,0,2,2}\rho_\alpha^4\cos\left(2\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{-+} = & ib^r_{0,0,-1,-4}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + 3\phi_\beta\right) + b^r_{0,0,-1,-4}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + 3\phi_\beta\right) + ib^r_{0,0,-1,2}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha - 3\phi_\beta\right) \\
& + b^r_{0,0,-1,2}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha - 3\phi_\beta\right) + ib^r_{0,0,-2,-4}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha + 2\phi_\beta\right) + b^r_{0,0,-2,-4}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha + 2\phi_\beta\right) \\
& + ib^r_{0,0,-3,-4}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha + \phi_\beta\right) + b^r_{0,0,-3,-4}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha + \phi_\beta\right) + ib^r_{0,0,-4,-4}\rho_\alpha^4\sin\left(4\phi_\alpha\right) + b^r_{0,0,-4,-4}\rho_\alpha^4\cos\left(4\phi_\alpha\right) \\
& + ib^r_{0,0,0,-4}\rho_\beta^4\sin\left(4\phi_\beta\right) + b^r_{0,0,0,-4}\rho_\beta^4\cos\left(4\phi_\beta\right) - ib^r_{0,0,3,2}\rho_\alpha^3\rho_\beta\sin\left(3\phi_\alpha - \phi_\beta\right) + b^r_{0,0,3,2}\rho_\alpha^3\rho_\beta\cos\left(3\phi_\alpha - \phi_\beta\right) \\
& - ib^r_{0,2,0,2}\rho_\beta^4\sin\left(2\phi_\beta\right) + b^r_{0,2,0,2}\rho_\beta^4\cos\left(2\phi_\beta\right) - ib^r_{0,2,1,2}\rho_\alpha\rho_\beta^3\sin\left(\phi_\alpha + \phi_\beta\right) + b^r_{0,2,1,2}\rho_\alpha\rho_\beta^3\cos\left(\phi_\alpha + \phi_\beta\right) \\
& - ib^r_{0,2,2,2}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\alpha\right) + b^r_{0,2,2,2}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\alpha\right) - ib^r_{2,0,0,2}\rho_\alpha^2\rho_\beta^2\sin\left(2\phi_\beta\right) + b^r_{2,0,0,2}\rho_\alpha^2\rho_\beta^2\cos\left(2\phi_\beta\right) \\
& - ib^r_{2,0,1,2}\rho_\alpha^3\rho_\beta\sin\left(\phi_\alpha + \phi_\beta\right) + b^r_{2,0,1,2}\rho_\alpha^3\rho_\beta\cos\left(\phi_\alpha + \phi_\beta\right) - ib^r_{2,0,2,2}\rho_\alpha^4\sin\left(2\phi_\alpha\right) + b^r_{2,0,2,2}\rho_\alpha^4\cos\left(2\phi_\alpha\right)
\end{aligned}
$$

**Cartesian e-coordinates:**

$$
\begin{aligned}
H^{(4)}_{++} = {} & a^r_{0,0,2,0}\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(x_\beta+y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(-x_\beta+y_\beta\right)\right)+a^r_{0,2,1,0}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta+y_\alpha y_\beta\right)\\
& +a^r_{0,4,0,0}\left(x_\beta^2+y_\beta^2\right)^2+a^r_{2,0,1,0}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta+y_\alpha y_\beta\right)+a^r_{2,2,0,0}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)+a^r_{4,0,0,0}\left(x_\alpha^2+y_\alpha^2\right)^2
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{--} = {} & a^r_{0,0,2,0}\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(x_\beta+y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(-x_\beta+y_\beta\right)\right)+a^r_{0,2,1,0}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta+y_\alpha y_\beta\right)\\
& +a^r_{0,4,0,0}\left(x_\beta^2+y_\beta^2\right)^2+a^r_{2,0,1,0}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta+y_\alpha y_\beta\right)+a^r_{2,2,0,0}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)+a^r_{4,0,0,0}\left(x_\alpha^2+y_\alpha^2\right)^2
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{+-} = {} & b^r_{0,0,-1,-4}\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2y_\beta+y_\beta^3\right)\right)-ib^r_{0,0,-1,-4}\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& +b^r_{0,0,-1,2}\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)\right)+ib^r_{0,0,-1,2}\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)\\
& -2ib^r_{0,0,-2,-4}\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+b^r_{0,0,-2,-4}\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right)\\
& +b^r_{0,0,-3,-4}\left(x_\alpha^3x_\beta-3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3y_\beta\right)-ib^r_{0,0,-3,-4}\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)\\
& -4ib^r_{0,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)+b^r_{0,0,-4,-4}\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)\\
& -4ib^r_{0,0,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{0,0,0,-4}\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)\\
& +b^r_{0,0,3,2}\left(x_\alpha^3x_\beta+3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3y_\beta\right)-ib^r_{0,0,3,2}\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)\\
& +2ib^r_{0,2,0,2}x_\beta y_\beta\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,0,2}\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& +ib^r_{0,2,1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{0,2,2,2}x_\alpha y_\alpha\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)\\
& +2ib^r_{2,0,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)+b^r_{2,0,0,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{2,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& +ib^r_{2,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{2,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)+b^r_{2,0,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{-+} = {} & b^r_{0,0,-1,-4}\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2y_\beta+y_\beta^3\right)\right)+ib^r_{0,0,-1,-4}\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& +b^r_{0,0,-1,2}\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)\right)-ib^r_{0,0,-1,2}\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)\\
& +2ib^r_{0,0,-2,-4}\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+b^r_{0,0,-2,-4}\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right)\\
& +b^r_{0,0,-3,-4}\left(x_\alpha^3x_\beta-3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3y_\beta\right)+ib^r_{0,0,-3,-4}\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)\\
& +4ib^r_{0,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)+b^r_{0,0,-4,-4}\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)\\
& +4ib^r_{0,0,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{0,0,0,-4}\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)\\
& +b^r_{0,0,3,2}\left(x_\alpha^3x_\beta+3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3y_\beta\right)+ib^r_{0,0,3,2}\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)\\
& -2ib^r_{0,2,0,2}x_\beta y_\beta\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,0,2}\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& -ib^r_{0,2,1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2ib^r_{0,2,2,2}x_\alpha y_\alpha\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)\\
& -2ib^r_{2,0,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)+b^r_{2,0,0,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{2,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& -ib^r_{2,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2ib^r_{2,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)+b^r_{2,0,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)
\end{aligned}
$$

## 3.6 Order: 5

Number of terms: $H_{++}$: 10, $H_{--}$: 10, $H_{+-}$: 36.

**Polar e-coordinates:**

$$
\begin{aligned}
H^{(5)}_{++} = {} & a^r_{0,0,-1,3}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha-4\phi_\beta\right)+a^r_{0,0,4,3}\rho_\alpha^4\rho_\beta\cos\left(4\phi_\alpha-\phi_\beta\right)+a^r_{0,2,0,3}\rho_\beta^5\cos\left(3\phi_\beta\right)+a^r_{0,2,1,3}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha+2\phi_\beta\right)\\
& +a^r_{0,2,2,3}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha+\phi_\beta\right)+a^r_{0,2,3,3}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha\right)+a^r_{2,0,0,3}\rho_\alpha^2\rho_\beta^3\cos\left(3\phi_\beta\right)+a^r_{2,0,1,3}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha+2\phi_\beta\right)\\
& +a^r_{2,0,2,3}\rho_\alpha^4\rho_\beta\cos\left(2\phi_\alpha+\phi_\beta\right)+a^r_{2,0,3,3}\rho_\alpha^5\cos\left(3\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(5)}_{--} = {} & a^r_{0,0,-1,3}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha-4\phi_\beta)+a^r_{0,0,4,3}\rho_\alpha^4\rho_\beta\cos(4\phi_\alpha-\phi_\beta)+a^r_{0,2,0,3}\rho_\beta^5\cos(3\phi_\beta)+a^r_{0,2,1,3}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha+2\phi_\beta)\\
& +a^r_{0,2,2,3}\rho_\alpha^2\rho_\beta^3\cos(2\phi_\alpha+\phi_\beta)+a^r_{0,2,3,3}\rho_\alpha^3\rho_\beta^2\cos(3\phi_\alpha)+a^r_{2,0,0,3}\rho_\alpha^2\rho_\beta^3\cos(3\phi_\beta)+a^r_{2,0,1,3}\rho_\alpha^3\rho_\beta^2\cos(\phi_\alpha+2\phi_\beta)\\
& +a^r_{2,0,2,3}\rho_\alpha^4\rho_\beta\cos(2\phi_\alpha+\phi_\beta)+a^r_{2,0,3,3}\rho_\alpha^5\cos(3\phi_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
H^{(5)}_{+-} = {} & -ib^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\sin(3\phi_\alpha-2\phi_\beta)+b^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\cos(3\phi_\alpha-2\phi_\beta)+ib^r_{0,0,0,5}\rho_\beta^5\sin(5\phi_\beta)+b^r_{0,0,0,5}\rho_\beta^5\cos(5\phi_\beta)\\
& +ib^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\sin(\phi_\alpha+4\phi_\beta)+b^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha+4\phi_\beta)+ib^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\sin(2\phi_\alpha-3\phi_\beta)\\
& +b^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\cos(2\phi_\alpha-3\phi_\beta)+ib^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\sin(2\phi_\alpha+3\phi_\beta)+b^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\cos(2\phi_\alpha+3\phi_\beta)\\
& +ib^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\sin(3\phi_\alpha+2\phi_\beta)+b^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\cos(3\phi_\alpha+2\phi_\beta)+ib^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\sin(4\phi_\alpha+\phi_\beta)\\
& +b^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\cos(4\phi_\alpha+\phi_\beta)+ib^r_{0,0,5,5}\rho_\alpha^5\sin(5\phi_\alpha)+b^r_{0,0,5,5}\rho_\alpha^5\cos(5\phi_\alpha)-ib^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\sin(2\phi_\alpha-\phi_\beta)\\
& +b^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\cos(2\phi_\alpha-\phi_\beta)+ib^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\sin(\phi_\alpha-2\phi_\beta)+b^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha-2\phi_\beta)\\
& -ib^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\sin(\phi_\alpha)+b^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha)-ib^r_{0,4,0,-1}\rho_\beta^5\sin(\phi_\beta)+b^r_{0,4,0,-1}\rho_\beta^5\cos(\phi_\beta)\\
& -ib^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\sin(2\phi_\alpha-\phi_\beta)+b^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\cos(2\phi_\alpha-\phi_\beta)+ib^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\sin(\phi_\alpha-2\phi_\beta)\\
& +b^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\cos(\phi_\alpha-2\phi_\beta)-ib^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\sin(\phi_\alpha)+b^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\cos(\phi_\alpha)\\
& -ib^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\sin(\phi_\beta)+b^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\cos(\phi_\beta)-ib^r_{4,0,-1,-1}\rho_\alpha^5\sin(\phi_\alpha)+b^r_{4,0,-1,-1}\rho_\alpha^5\cos(\phi_\alpha)\\
& -ib^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\sin(\phi_\beta)+b^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\cos(\phi_\beta)
\end{aligned}
$$

$$
\begin{aligned}
H^{(5)}_{-+} = {} & ib^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\sin(3\phi_\alpha-2\phi_\beta)+b^r_{0,0,-3,-1}\rho_\alpha^3\rho_\beta^2\cos(3\phi_\alpha-2\phi_\beta)-ib^r_{0,0,0,5}\rho_\beta^5\sin(5\phi_\beta)+b^r_{0,0,0,5}\rho_\beta^5\cos(5\phi_\beta)\\
& -ib^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\sin(\phi_\alpha+4\phi_\beta)+b^r_{0,0,1,5}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha+4\phi_\beta)-ib^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\sin(2\phi_\alpha-3\phi_\beta)\\
& +b^r_{0,0,2,-1}\rho_\alpha^2\rho_\beta^3\cos(2\phi_\alpha-3\phi_\beta)-ib^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\sin(2\phi_\alpha+3\phi_\beta)+b^r_{0,0,2,5}\rho_\alpha^2\rho_\beta^3\cos(2\phi_\alpha+3\phi_\beta)\\
& -ib^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\sin(3\phi_\alpha+2\phi_\beta)+b^r_{0,0,3,5}\rho_\alpha^3\rho_\beta^2\cos(3\phi_\alpha+2\phi_\beta)-ib^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\sin(4\phi_\alpha+\phi_\beta)\\
& +b^r_{0,0,4,5}\rho_\alpha^4\rho_\beta\cos(4\phi_\alpha+\phi_\beta)-ib^r_{0,0,5,5}\rho_\alpha^5\sin(5\phi_\alpha)+b^r_{0,0,5,5}\rho_\alpha^5\cos(5\phi_\alpha)+ib^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\sin(2\phi_\alpha-\phi_\beta)\\
& +b^r_{0,2,-2,-1}\rho_\alpha^2\rho_\beta^3\cos(2\phi_\alpha-\phi_\beta)-ib^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\sin(\phi_\alpha-2\phi_\beta)+b^r_{0,2,1,-1}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha-2\phi_\beta)\\
& +ib^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\sin(\phi_\alpha)+b^r_{0,4,-1,-1}\rho_\alpha\rho_\beta^4\cos(\phi_\alpha)+ib^r_{0,4,0,-1}\rho_\beta^5\sin(\phi_\beta)+b^r_{0,4,0,-1}\rho_\beta^5\cos(\phi_\beta)\\
& +ib^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\sin(2\phi_\alpha-\phi_\beta)+b^r_{2,0,-2,-1}\rho_\alpha^4\rho_\beta\cos(2\phi_\alpha-\phi_\beta)-ib^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\sin(\phi_\alpha-2\phi_\beta)\\
& +b^r_{2,0,1,-1}\rho_\alpha^3\rho_\beta^2\cos(\phi_\alpha-2\phi_\beta)+ib^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\sin(\phi_\alpha)+b^r_{2,2,-1,-1}\rho_\alpha^3\rho_\beta^2\cos(\phi_\alpha)\\
& +ib^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\sin(\phi_\beta)+b^r_{2,2,0,-1}\rho_\alpha^2\rho_\beta^3\cos(\phi_\beta)+ib^r_{4,0,-1,-1}\rho_\alpha^5\sin(\phi_\alpha)+b^r_{4,0,-1,-1}\rho_\alpha^5\cos(\phi_\alpha)\\
& +ib^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\sin(\phi_\beta)+b^r_{4,0,0,-1}\rho_\alpha^4\rho_\beta\cos(\phi_\beta)
\end{aligned}
$$

**Cartesian e-coordinates:**

$$
\begin{aligned}
H^{(5)}_{++} = {} & a^r_{0,0,-1,3}\left(x_\alpha\left(x_\beta^4-6x_\beta^2y_\beta^2+y_\beta^4\right)+y_\alpha\left(4x_\beta^3y_\beta-4x_\beta y_\beta^3\right)\right)+a^r_{0,0,4,3}\left(4x_\alpha^3y_\alpha y_\beta-4x_\alpha y_\alpha^3y_\beta+x_\beta\left(x_\alpha^4-6x_\alpha^2y_\alpha^2+y_\alpha^4\right)\right)\\
& +a^r_{0,2,0,3}x_\beta\left(x_\beta^2-3y_\beta^2\right)\left(x_\beta^2+y_\beta^2\right)+a^r_{0,2,1,3}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)-2x_\beta y_\alpha y_\beta\right)\\
& +a^r_{0,2,2,3}\left(x_\beta^2+y_\beta^2\right)\left(-2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)+a^r_{0,2,3,3}x_\alpha\left(x_\alpha^2-3y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& +a^r_{2,0,0,3}x_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2-3y_\beta^2\right)+a^r_{2,0,1,3}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)-2x_\beta y_\alpha y_\beta\right)\\
& +a^r_{2,0,2,3}\left(x_\alpha^2+y_\alpha^2\right)\left(-2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)+a^r_{2,0,3,3}x_\alpha\left(x_\alpha^2-3y_\alpha^2\right)\left(x_\alpha^2+y_\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(5)}_{--} = {} & a^r_{0,0,-1,3}\left(x_\alpha\left(x_\beta^4-6x_\beta^2y_\beta^2+y_\beta^4\right)+y_\alpha\left(4x_\beta^3y_\beta-4x_\beta y_\beta^3\right)\right)+a^r_{0,0,4,3}\left(4x_\alpha^3y_\alpha y_\beta-4x_\alpha y_\alpha^3y_\beta+x_\beta\left(x_\alpha^4-6x_\alpha^2y_\alpha^2+y_\alpha^4\right)\right)\\
& +a^r_{0,2,0,3}x_\beta\left(x_\beta^2-3y_\beta^2\right)\left(x_\beta^2+y_\beta^2\right)+a^r_{0,2,1,3}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)-2x_\beta y_\alpha y_\beta\right)\\
& +a^r_{0,2,2,3}\left(x_\beta^2+y_\beta^2\right)\left(-2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)+a^r_{0,2,3,3}x_\alpha\left(x_\alpha^2-3y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& +a^r_{2,0,0,3}x_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2-3y_\beta^2\right)+a^r_{2,0,1,3}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)-2x_\beta y_\alpha y_\beta\right)\\
& +a^r_{2,0,2,3}\left(x_\alpha^2+y_\alpha^2\right)\left(-2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)+a^r_{2,0,3,3}x_\alpha\left(x_\alpha^2-3y_\alpha^2\right)\left(x_\alpha^2+y_\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(5)}_{+-} = {} & b^r_{0,0,-3,-1}\left(x_\alpha^3x_\beta^2 - x_\alpha^3y_\beta^2 + 6x_\alpha^2x_\beta y_\alpha y_\beta + x_\alpha\left(-3x_\beta^2y_\alpha^2 + 3y_\alpha^2y_\beta^2\right) - 2x_\beta y_\alpha^3y_\beta\right)\\
& + ib^r_{0,0,-3,-1}\left(2x_\alpha^3x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2y_\beta + x_\beta^2y_\alpha^3 - y_\alpha^3y_\beta^2 + y_\alpha\left(-3x_\alpha^2x_\beta^2 + 3x_\alpha^2y_\beta^2\right)\right) + b^r_{0,0,0,0,5}x_\beta\left(x_\beta^4 - 10x_\beta^2y_\beta^2 + 5y_\beta^4\right)\\
& + ib^r_{0,0,0,0,5}y_\beta\left(5x_\beta^4 - 10x_\beta^2y_\beta^2 + y_\beta^4\right) + ib^r_{0,0,1,5}\left(x_\alpha\left(4x_\beta^3y_\beta - 4x_\beta y_\beta^3\right) + y_\alpha\left(x_\beta^4 - 6x_\beta^2y_\beta^2 + y_\beta^4\right)\right)\\
& + b^r_{0,0,1,5}\left(x_\alpha\left(x_\beta^4 - 6x_\beta^2y_\beta^2 + y_\beta^4\right) + y_\alpha\left(-4x_\beta^3y_\beta + 4x_\beta y_\beta^3\right)\right)\\
& + b^r_{0,0,2,-1}\left(x_\alpha^2x_\beta^3 + x_\alpha\left(6x_\beta^2y_\alpha y_\beta - 2y_\alpha y_\beta^3\right) - x_\beta^3y_\alpha^2 + x_\beta\left(-3x_\alpha^2y_\beta^2 + 3y_\alpha^2y_\beta^2\right)\right)\\
& - ib^r_{0,0,2,-1}\left(-x_\alpha^2y_\beta^3 + x_\alpha\left(-2x_\beta^3y_\alpha + 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2y_\beta^3 + y_\beta\left(3x_\alpha^2x_\beta^2 - 3x_\beta^2y_\alpha^2\right)\right)\\
& + b^r_{0,0,2,5}\left(x_\alpha^2x_\beta^3 + x_\alpha\left(-6x_\beta^2y_\alpha y_\beta + 2y_\alpha y_\beta^3\right) - x_\beta^3y_\alpha^2 + x_\beta\left(-3x_\alpha^2y_\beta^2 + 3y_\alpha^2y_\beta^2\right)\right)\\
& + ib^r_{0,0,2,5}\left(-x_\alpha^2y_\beta^3 + x_\alpha\left(2x_\beta^3y_\alpha - 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2y_\beta^3 + y_\beta\left(3x_\alpha^2x_\beta^2 - 3x_\beta^2y_\alpha^2\right)\right)\\
& + b^r_{0,0,3,5}\left(x_\alpha^3x_\beta^2 - x_\alpha^3y_\beta^2 - 6x_\alpha^2x_\beta y_\alpha y_\beta + x_\alpha\left(-3x_\beta^2y_\alpha^2 + 3y_\alpha^2y_\beta^2\right) + 2x_\beta y_\alpha^3y_\beta\right)\\
& + ib^r_{0,0,3,5}\left(2x_\alpha^3x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2y_\beta - x_\beta^2y_\alpha^3 + y_\alpha^3y_\beta^2 + y_\alpha\left(3x_\alpha^2x_\beta^2 - 3x_\alpha^2y_\beta^2\right)\right)\\
& + ib^r_{0,0,4,5}\left(4x_\alpha^3x_\beta y_\alpha - 4x_\alpha x_\beta y_\alpha^3 + y_\beta\left(x_\alpha^4 - 6x_\alpha^2y_\alpha^2 + y_\alpha^4\right)\right)\\
& + b^r_{0,0,4,5}\left(-4x_\alpha^3y_\alpha y_\beta + 4x_\alpha y_\alpha^3y_\beta + x_\beta\left(x_\alpha^4 - 6x_\alpha^2y_\alpha^2 + y_\alpha^4\right)\right) + b^r_{0,0,5,5}x_\alpha\left(x_\alpha^4 - 10x_\alpha^2y_\alpha^2 + 5y_\alpha^4\right)\\
& + ib^r_{0,0,5,5}y_\alpha\left(5x_\alpha^4 - 10x_\alpha^2y_\alpha^2 + y_\alpha^4\right) + ib^r_{0,2,-2,-1}\left(x_\beta^2 + y_\beta^2\right)\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\\
& + b^r_{0,2,-2,-1}\left(x_\beta^2 + y_\beta^2\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + b^r_{0,2,1,-1}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)\\
& - ib^r_{0,2,1,-1}\left(x_\beta^2 + y_\beta^2\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) + b^r_{0,4,-1,-1}x_\alpha\left(x_\beta^2 + y_\beta^2\right)^2 - ib^r_{0,4,-1,-1}y_\alpha\left(x_\beta^2 + y_\beta^2\right)^2\\
& + b^r_{0,4,0,-1}x_\beta\left(x_\beta^2 + y_\beta^2\right)^2 - ib^r_{0,4,0,-1}y_\beta\left(x_\beta^2 + y_\beta^2\right)^2 + ib^r_{2,0,-2,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\\
& + b^r_{2,0,-2,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + b^r_{2,0,1,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)\\
& - ib^r_{2,0,1,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) + b^r_{2,2,-1,-1}x_\alpha\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\\
& - ib^r_{2,2,-1,-1}y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + b^r_{2,2,0,-1}x_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) - ib^r_{2,2,0,-1}y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\\
& + b^r_{4,0,-1,-1}x_\alpha\left(x_\alpha^2 + y_\alpha^2\right)^2 - ib^r_{4,0,-1,-1}y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)^2 + b^r_{4,0,0,-1}x_\beta\left(x_\alpha^2 + y_\alpha^2\right)^2 - ib^r_{4,0,0,-1}y_\beta\left(x_\alpha^2 + y_\alpha^2\right)^2
\end{aligned}
$$

$$
\begin{aligned}
H^{(5)}_{-+} = {} & b^r_{0,0,-3,-1}\left(x_\alpha^3x_\beta^2 - x_\alpha^3y_\beta^2 + 6x_\alpha^2x_\beta y_\alpha y_\beta + x_\alpha\left(-3x_\beta^2y_\alpha^2 + 3y_\alpha^2y_\beta^2\right) - 2x_\beta y_\alpha^3y_\beta\right)\\
& - ib^r_{0,0,-3,-1}\left(2x_\alpha^3x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2y_\beta + x_\beta^2y_\alpha^3 - y_\alpha^3y_\beta^2 + y_\alpha\left(-3x_\alpha^2x_\beta^2 + 3x_\alpha^2y_\beta^2\right)\right) + b^r_{0,0,0,0,5}x_\beta\left(x_\beta^4 - 10x_\beta^2y_\beta^2 + 5y_\beta^4\right)\\
& - ib^r_{0,0,0,0,5}y_\beta\left(5x_\beta^4 - 10x_\beta^2y_\beta^2 + y_\beta^4\right) - ib^r_{0,0,1,5}\left(x_\alpha\left(4x_\beta^3y_\beta - 4x_\beta y_\beta^3\right) + y_\alpha\left(x_\beta^4 - 6x_\beta^2y_\beta^2 + y_\beta^4\right)\right)\\
& + b^r_{0,0,1,5}\left(x_\alpha\left(x_\beta^4 - 6x_\beta^2y_\beta^2 + y_\beta^4\right) + y_\alpha\left(-4x_\beta^3y_\beta + 4x_\beta y_\beta^3\right)\right)\\
& + b^r_{0,0,2,-1}\left(x_\alpha^2x_\beta^3 + x_\alpha\left(6x_\beta^2y_\alpha y_\beta - 2y_\alpha y_\beta^3\right) - x_\beta^3y_\alpha^2 + x_\beta\left(-3x_\alpha^2y_\beta^2 + 3y_\alpha^2y_\beta^2\right)\right)\\
& + ib^r_{0,0,2,-1}\left(-x_\alpha^2y_\beta^3 + x_\alpha\left(-2x_\beta^3y_\alpha + 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2y_\beta^3 + y_\beta\left(3x_\alpha^2x_\beta^2 - 3x_\beta^2y_\alpha^2\right)\right)\\
& + b^r_{0,0,2,5}\left(x_\alpha^2x_\beta^3 + x_\alpha\left(-6x_\beta^2y_\alpha y_\beta + 2y_\alpha y_\beta^3\right) - x_\beta^3y_\alpha^2 + x_\beta\left(-3x_\alpha^2y_\beta^2 + 3y_\alpha^2y_\beta^2\right)\right)\\
& - ib^r_{0,0,2,5}\left(-x_\alpha^2y_\beta^3 + x_\alpha\left(2x_\beta^3y_\alpha - 6x_\beta y_\alpha y_\beta^2\right) + y_\alpha^2y_\beta^3 + y_\beta\left(3x_\alpha^2x_\beta^2 - 3x_\beta^2y_\alpha^2\right)\right)\\
& + b^r_{0,0,3,5}\left(x_\alpha^3x_\beta^2 - x_\alpha^3y_\beta^2 - 6x_\alpha^2x_\beta y_\alpha y_\beta + x_\alpha\left(-3x_\beta^2y_\alpha^2 + 3y_\alpha^2y_\beta^2\right) + 2x_\beta y_\alpha^3y_\beta\right)\\
& - ib^r_{0,0,3,5}\left(2x_\alpha^3x_\beta y_\beta - 6x_\alpha x_\beta y_\alpha^2y_\beta - x_\beta^2y_\alpha^3 + y_\alpha^3y_\beta^2 + y_\alpha\left(3x_\alpha^2x_\beta^2 - 3x_\alpha^2y_\beta^2\right)\right)\\
& - ib^r_{0,0,4,5}\left(4x_\alpha^3x_\beta y_\alpha - 4x_\alpha x_\beta y_\alpha^3 + y_\beta\left(x_\alpha^4 - 6x_\alpha^2y_\alpha^2 + y_\alpha^4\right)\right)\\
& + b^r_{0,0,4,5}\left(-4x_\alpha^3y_\alpha y_\beta + 4x_\alpha y_\alpha^3y_\beta + x_\beta\left(x_\alpha^4 - 6x_\alpha^2y_\alpha^2 + y_\alpha^4\right)\right) + b^r_{0,0,5,5}x_\alpha\left(x_\alpha^4 - 10x_\alpha^2y_\alpha^2 + 5y_\alpha^4\right)\\
& - ib^r_{0,0,5,5}y_\alpha\left(5x_\alpha^4 - 10x_\alpha^2y_\alpha^2 + y_\alpha^4\right) - ib^r_{0,2,-2,-1}\left(x_\beta^2 + y_\beta^2\right)\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\\
& + b^r_{0,2,-2,-1}\left(x_\beta^2 + y_\beta^2\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + b^r_{0,2,1,-1}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)\\
& + ib^r_{0,2,1,-1}\left(x_\beta^2 + y_\beta^2\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) + b^r_{0,4,-1,-1}x_\alpha\left(x_\beta^2 + y_\beta^2\right)^2 + ib^r_{0,4,-1,-1}y_\alpha\left(x_\beta^2 + y_\beta^2\right)^2\\
& + b^r_{0,4,0,-1}x_\beta\left(x_\beta^2 + y_\beta^2\right)^2 + ib^r_{0,4,0,-1}y_\beta\left(x_\beta^2 + y_\beta^2\right)^2 - ib^r_{2,0,-2,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\\
& + b^r_{2,0,-2,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) + b^r_{2,0,1,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^2 - y_\beta^2\right) + 2x_\beta y_\alpha y_\beta\right)\\
& + ib^r_{2,0,1,-1}\left(x_\alpha^2 + y_\alpha^2\right)\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2 + y_\beta^2\right)\right) + b^r_{2,2,-1,-1}x_\alpha\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\\
& + ib^r_{2,2,-1,-1}y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + b^r_{2,2,0,-1}x_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + ib^r_{2,2,0,-1}y_\beta\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\\
& + b^r_{4,0,-1,-1}x_\alpha\left(x_\alpha^2 + y_\alpha^2\right)^2 + ib^r_{4,0,-1,-1}y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)^2 + b^r_{4,0,0,-1}x_\beta\left(x_\alpha^2 + y_\alpha^2\right)^2 + ib^r_{4,0,0,-1}y_\beta\left(x_\alpha^2 + y_\alpha^2\right)^2
\end{aligned}
$$

## 3.7 Order: 6

Number of terms: $H_{++}$: 17, $H_{--}$: 17, $H_{+-}$: 54.

**Polar e-coordinates:**

$$
\begin{aligned}
H^{(6)}_{++} = {} & a^r_{0,0,0,6}\rho_\beta^6\cos(6\phi_\beta) + a^r_{0,0,1,6}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha+5\phi_\beta) + a^r_{0,0,2,6}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha+4\phi_\beta) + a^r_{0,0,3,0}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha-3\phi_\beta)\\
& + a^r_{0,0,3,6}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha+3\phi_\beta) + a^r_{0,0,4,6}\rho_\alpha^4\rho_\beta^2\cos(4\phi_\alpha+2\phi_\beta) + a^r_{0,0,5,6}\rho_\alpha^5\rho_\beta\cos(5\phi_\alpha+\phi_\beta) + a^r_{0,0,6,6}\rho_\alpha^6\cos(6\phi_\alpha)\\
& + a^r_{0,2,2,0}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha-2\phi_\beta) + a^r_{0,4,1,0}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha-\phi_\beta) + a^r_{0,6,0,0}\rho_\beta^6 + a^r_{2,0,2,0}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\alpha-2\phi_\beta)\\
& + a^r_{2,2,1,0}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha-\phi_\beta) + a^r_{2,4,0,0}\rho_\alpha^2\rho_\beta^4 + a^r_{4,0,1,0}\rho_\alpha^5\rho_\beta\cos(\phi_\alpha-\phi_\beta) + a^r_{4,2,0,0}\rho_\alpha^4\rho_\beta^2 + a^r_{6,0,0,0}\rho_\alpha^6
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{--} = {} & a^r_{0,0,0,6}\rho_\beta^6\cos(6\phi_\beta) + a^r_{0,0,1,6}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha+5\phi_\beta) + a^r_{0,0,2,6}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha+4\phi_\beta) + a^r_{0,0,3,0}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha-3\phi_\beta)\\
& + a^r_{0,0,3,6}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha+3\phi_\beta) + a^r_{0,0,4,6}\rho_\alpha^4\rho_\beta^2\cos(4\phi_\alpha+2\phi_\beta) + a^r_{0,0,5,6}\rho_\alpha^5\rho_\beta\cos(5\phi_\alpha+\phi_\beta) + a^r_{0,0,6,6}\rho_\alpha^6\cos(6\phi_\alpha)\\
& + a^r_{0,2,2,0}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha-2\phi_\beta) + a^r_{0,4,1,0}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha-\phi_\beta) + a^r_{0,6,0,0}\rho_\beta^6 + a^r_{2,0,2,0}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\alpha-2\phi_\beta)\\
& + a^r_{2,2,1,0}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha-\phi_\beta) + a^r_{2,4,0,0}\rho_\alpha^2\rho_\beta^4 + a^r_{4,0,1,0}\rho_\alpha^5\rho_\beta\cos(\phi_\alpha-\phi_\beta) + a^r_{4,2,0,0}\rho_\alpha^4\rho_\beta^2 + a^r_{6,0,0,0}\rho_\alpha^6
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{+-} = {} & -ib^r_{0,0,-2,2}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\alpha-4\phi_\beta) + b^r_{0,0,-2,2}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha-4\phi_\beta) - ib^r_{0,0,-5,-4}\rho_\alpha^5\rho_\beta\sin(5\phi_\alpha-\phi_\beta)\\
& + b^r_{0,0,-5,-4}\rho_\alpha^5\rho_\beta\cos(5\phi_\alpha-\phi_\beta) + ib^r_{0,0,1,-4}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha-5\phi_\beta) + b^r_{0,0,1,-4}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha-5\phi_\beta)\\
& + ib^r_{0,0,4,2}\rho_\alpha^4\rho_\beta^2\sin(4\phi_\alpha-2\phi_\beta) + b^r_{0,0,4,2}\rho_\alpha^4\rho_\beta^2\cos(4\phi_\alpha-2\phi_\beta) - ib^r_{0,2,-1,-4}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha+3\phi_\beta)\\
& + b^r_{0,2,-1,-4}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha+3\phi_\beta) - ib^r_{0,2,-1,2}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha-3\phi_\beta) + b^r_{0,2,-1,2}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha-3\phi_\beta)\\
& - ib^r_{0,2,-2,-4}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\alpha+2\phi_\beta) + b^r_{0,2,-2,-4}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha+2\phi_\beta) - ib^r_{0,2,-3,-4}\rho_\alpha^3\rho_\beta^3\sin(3\phi_\alpha+\phi_\beta)\\
& + b^r_{0,2,-3,-4}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha+\phi_\beta) - ib^r_{0,2,-4,-4}\rho_\alpha^4\rho_\beta^2\sin(4\phi_\alpha) + b^r_{0,2,-4,-4}\rho_\alpha^4\rho_\beta^2\cos(4\phi_\alpha) - ib^r_{0,2,0,-4}\rho_\beta^6\sin(4\phi_\beta)\\
& + b^r_{0,2,0,-4}\rho_\beta^6\cos(4\phi_\beta) + ib^r_{0,2,3,2}\rho_\alpha^3\rho_\beta^3\sin(3\phi_\alpha-\phi_\beta) + b^r_{0,2,3,2}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha-\phi_\beta) + ib^r_{0,4,0,2}\rho_\beta^6\sin(2\phi_\beta)\\
& + b^r_{0,4,0,2}\rho_\beta^6\cos(2\phi_\beta) + ib^r_{0,4,1,2}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha+\phi_\beta) + b^r_{0,4,1,2}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha+\phi_\beta) + ib^r_{0,4,2,2}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\alpha)\\
& + b^r_{0,4,2,2}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha) - ib^r_{2,0,-1,-4}\rho_\alpha^3\rho_\beta^3\sin(\phi_\alpha+3\phi_\beta) + b^r_{2,0,-1,-4}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha+3\phi_\beta)\\
& - ib^r_{2,0,-1,2}\rho_\alpha^3\rho_\beta^3\sin(\phi_\alpha-3\phi_\beta) + b^r_{2,0,-1,2}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha-3\phi_\beta) - ib^r_{2,0,-2,-4}\rho_\alpha^4\rho_\beta^2\sin(2\phi_\alpha+2\phi_\beta)\\
& + b^r_{2,0,-2,-4}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\alpha+2\phi_\beta) - ib^r_{2,0,-3,-4}\rho_\alpha^5\rho_\beta\sin(3\phi_\alpha+\phi_\beta) + b^r_{2,0,-3,-4}\rho_\alpha^5\rho_\beta\cos(3\phi_\alpha+\phi_\beta)\\
& - ib^r_{2,0,-4,-4}\rho_\alpha^6\sin(4\phi_\alpha) + b^r_{2,0,-4,-4}\rho_\alpha^6\cos(4\phi_\alpha) - ib^r_{2,0,0,-4}\rho_\alpha^2\rho_\beta^4\sin(4\phi_\beta) + b^r_{2,0,0,-4}\rho_\alpha^2\rho_\beta^4\cos(4\phi_\beta)\\
& + ib^r_{2,0,3,2}\rho_\alpha^5\rho_\beta\sin(3\phi_\alpha-\phi_\beta) + b^r_{2,0,3,2}\rho_\alpha^5\rho_\beta\cos(3\phi_\alpha-\phi_\beta) + ib^r_{2,2,0,2}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\beta) + b^r_{2,2,0,2}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\beta)\\
& + ib^r_{2,2,1,2}\rho_\alpha^3\rho_\beta^3\sin(\phi_\alpha+\phi_\beta) + b^r_{2,2,1,2}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha+\phi_\beta) + ib^r_{2,2,2,2}\rho_\alpha^4\rho_\beta^2\sin(2\phi_\alpha) + b^r_{2,2,2,2}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\alpha)\\
& + ib^r_{4,0,0,2}\rho_\alpha^4\rho_\beta^2\sin(2\phi_\beta) + b^r_{4,0,0,2}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\beta) + ib^r_{4,0,1,2}\rho_\alpha^5\rho_\beta\sin(\phi_\alpha+\phi_\beta) + b^r_{4,0,1,2}\rho_\alpha^5\rho_\beta\cos(\phi_\alpha+\phi_\beta)\\
& + ib^r_{4,0,2,2}\rho_\alpha^6\sin(2\phi_\alpha) + b^r_{4,0,2,2}\rho_\alpha^6\cos(2\phi_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{-+} = {} & ib^r_{0,0,-2,2}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\alpha-4\phi_\beta)+b^r_{0,0,-2,2}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha-4\phi_\beta)+ib^r_{0,0,-5,-4}\rho_\alpha^5\rho_\beta\sin(5\phi_\alpha-\phi_\beta)\\
& +b^r_{0,0,-5,-4}\rho_\alpha^5\rho_\beta\cos(5\phi_\alpha-\phi_\beta)-ib^r_{0,0,1,-4}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha-5\phi_\beta)+b^r_{0,0,1,-4}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha-5\phi_\beta)\\
& -ib^r_{0,0,4,2}\rho_\alpha^4\rho_\beta^2\sin(4\phi_\alpha-2\phi_\beta)+b^r_{0,0,4,2}\rho_\alpha^4\rho_\beta^2\cos(4\phi_\alpha-2\phi_\beta)+ib^r_{0,2,-1,-4}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha+3\phi_\beta)\\
& +b^r_{0,2,-1,-4}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha+3\phi_\beta)+ib^r_{0,2,-1,2}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha-3\phi_\beta)+b^r_{0,2,-1,2}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha-3\phi_\beta)\\
& +ib^r_{0,2,-2,-4}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\alpha+2\phi_\beta)+b^r_{0,2,-2,-4}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha+2\phi_\beta)+ib^r_{0,2,-3,-4}\rho_\alpha^3\rho_\beta^3\sin(3\phi_\alpha+\phi_\beta)\\
& +b^r_{0,2,-3,-4}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha+\phi_\beta)+ib^r_{0,2,-4,-4}\rho_\alpha^4\rho_\beta^2\sin(4\phi_\alpha)+b^r_{0,2,-4,-4}\rho_\alpha^4\rho_\beta^2\cos(4\phi_\alpha)+ib^r_{0,2,0,-4}\rho_\beta^6\sin(4\phi_\beta)\\
& +b^r_{0,2,0,-4}\rho_\beta^6\cos(4\phi_\beta)-ib^r_{0,2,3,2}\rho_\alpha^3\rho_\beta^3\sin(3\phi_\alpha-\phi_\beta)+b^r_{0,2,3,2}\rho_\alpha^3\rho_\beta^3\cos(3\phi_\alpha-\phi_\beta)-ib^r_{0,4,0,2}\rho_\beta^6\sin(2\phi_\beta)\\
& +b^r_{0,4,0,2}\rho_\beta^6\cos(2\phi_\beta)-ib^r_{0,4,1,2}\rho_\alpha\rho_\beta^5\sin(\phi_\alpha+\phi_\beta)+b^r_{0,4,1,2}\rho_\alpha\rho_\beta^5\cos(\phi_\alpha+\phi_\beta)-ib^r_{0,4,2,2}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\alpha)\\
& +b^r_{0,4,2,2}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\alpha)+ib^r_{2,0,-1,-4}\rho_\alpha^3\rho_\beta^3\sin(\phi_\alpha+3\phi_\beta)+b^r_{2,0,-1,-4}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha+3\phi_\beta)\\
& +ib^r_{2,0,-1,2}\rho_\alpha^3\rho_\beta^3\sin(\phi_\alpha-3\phi_\beta)+b^r_{2,0,-1,2}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha-3\phi_\beta)+ib^r_{2,0,-2,-4}\rho_\alpha^4\rho_\beta^2\sin(2\phi_\alpha+2\phi_\beta)\\
& +b^r_{2,0,-2,-4}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\alpha+2\phi_\beta)+ib^r_{2,0,-3,-4}\rho_\alpha^5\rho_\beta\sin(3\phi_\alpha+\phi_\beta)+b^r_{2,0,-3,-4}\rho_\alpha^5\rho_\beta\cos(3\phi_\alpha+\phi_\beta)\\
& +ib^r_{2,0,-4,-4}\rho_\alpha^6\sin(4\phi_\alpha)+b^r_{2,0,-4,-4}\rho_\alpha^6\cos(4\phi_\alpha)+ib^r_{2,0,0,-4}\rho_\alpha^2\rho_\beta^4\sin(4\phi_\beta)+b^r_{2,0,0,-4}\rho_\alpha^2\rho_\beta^4\cos(4\phi_\beta)\\
& -ib^r_{2,0,3,2}\rho_\alpha^5\rho_\beta\sin(3\phi_\alpha-\phi_\beta)+b^r_{2,0,3,2}\rho_\alpha^5\rho_\beta\cos(3\phi_\alpha-\phi_\beta)-ib^r_{2,2,0,2}\rho_\alpha^2\rho_\beta^4\sin(2\phi_\beta)+b^r_{2,2,0,2}\rho_\alpha^2\rho_\beta^4\cos(2\phi_\beta)\\
& -ib^r_{2,2,1,2}\rho_\alpha^3\rho_\beta^3\sin(\phi_\alpha+\phi_\beta)+b^r_{2,2,1,2}\rho_\alpha^3\rho_\beta^3\cos(\phi_\alpha+\phi_\beta)-ib^r_{2,2,2,2}\rho_\alpha^4\rho_\beta^2\sin(2\phi_\alpha)+b^r_{2,2,2,2}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\alpha)\\
& -ib^r_{4,0,0,2}\rho_\alpha^4\rho_\beta^2\sin(2\phi_\beta)+b^r_{4,0,0,2}\rho_\alpha^4\rho_\beta^2\cos(2\phi_\beta)-ib^r_{4,0,1,2}\rho_\alpha^5\rho_\beta\sin(\phi_\alpha+\phi_\beta)+b^r_{4,0,1,2}\rho_\alpha^5\rho_\beta\cos(\phi_\alpha+\phi_\beta)\\
& -ib^r_{4,0,2,2}\rho_\alpha^6\sin(2\phi_\alpha)+b^r_{4,0,2,2}\rho_\alpha^6\cos(2\phi_\alpha)
\end{aligned}
$$

**Cartesian e-coordinates:**

$$
\begin{aligned}
H^{(6)}_{++} = {} & a^r_{0,0,0,6}(x_\beta-y_\beta)(x_\beta+y_\beta)\left(x_\beta^2-4x_\beta y_\beta+y_\beta^2\right)\left(x_\beta^2+4x_\beta y_\beta+y_\beta^2\right)\\
& +a^r_{0,0,1,6}\left(x_\alpha\left(x_\beta^5-10x_\beta^3y_\beta^2+5x_\beta y_\beta^4\right)+y_\alpha\left(-5x_\beta^4y_\beta+10x_\beta^2y_\beta^3-y_\beta^5\right)\right)\\
& +a^r_{0,0,2,6}\left(x_\alpha\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(-x_\beta^2-2x_\beta y_\beta+y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\right)\\
& +a^r_{0,0,3,0}(x_\alpha x_\beta+y_\alpha y_\beta)\left(x_\alpha^2x_\beta^2-3x_\alpha^2y_\beta^2+8x_\alpha x_\beta y_\alpha y_\beta-3x_\beta^2y_\alpha^2+y_\alpha^2y_\beta^2\right)\\
& +a^r_{0,0,3,6}(x_\alpha x_\beta-y_\alpha y_\beta)\left(x_\alpha^2x_\beta^2-3x_\alpha^2y_\beta^2-8x_\alpha x_\beta y_\alpha y_\beta-3x_\beta^2y_\alpha^2+y_\alpha^2y_\beta^2\right)+a^r_{0,0,4,6}\left(x_\alpha\left(-2x_\beta y_\alpha-2y_\alpha y_\beta\right)\right.\\
& \qquad\left.+x_\beta\left(x_\alpha^2-y_\alpha^2\right)+y_\beta\left(-x_\alpha^2+y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha-2y_\alpha y_\beta\right)+x_\beta\left(x_\alpha^2-y_\alpha^2\right)+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\\
& +a^r_{0,0,5,6}\left(-5x_\alpha^4y_\alpha y_\beta+5x_\alpha x_\beta y_\alpha^4+x_\beta\left(x_\alpha^5-10x_\alpha^3y_\alpha^2\right)+y_\beta\left(10x_\alpha^2y_\alpha^3-y_\alpha^5\right)\right)\\
& +a^r_{0,0,6,6}(x_\alpha-y_\alpha)(x_\alpha+y_\alpha)\left(x_\alpha^2-4x_\alpha y_\alpha+y_\alpha^2\right)\left(x_\alpha^2+4x_\alpha y_\alpha+y_\alpha^2\right)\\
& +a^r_{0,2,2,0}\left(x_\beta^2+y_\beta^2\right)(x_\alpha(x_\beta-y_\beta)+y_\alpha(x_\beta+y_\beta))(x_\alpha(x_\beta+y_\beta)+y_\alpha(-x_\beta+y_\beta))+a^r_{0,4,1,0}\left(x_\beta^2+y_\beta^2\right)^2(x_\alpha x_\beta+y_\alpha y_\beta)\\
& +a^r_{0,6,0,0}\left(x_\beta^2+y_\beta^2\right)^3+a^r_{2,0,2,0}\left(x_\alpha^2+y_\alpha^2\right)(x_\alpha(x_\beta-y_\beta)+y_\alpha(x_\beta+y_\beta))(x_\alpha(x_\beta+y_\beta)+y_\alpha(-x_\beta+y_\beta))\\
& +a^r_{2,2,1,0}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)(x_\alpha x_\beta+y_\alpha y_\beta)+a^r_{2,4,0,0}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)^2+a^r_{4,0,1,0}\left(x_\alpha^2+y_\alpha^2\right)^2(x_\alpha x_\beta+y_\alpha y_\beta)\\
& +a^r_{4,2,0,0}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\beta^2+y_\beta^2\right)+a^r_{6,0,0,0}\left(x_\alpha^2+y_\alpha^2\right)^3
\end{aligned}
$$

$$
\begin{aligned}
H_{--}^{(6)} = {} & a_{0,0,0,6}^{r}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 - 4x_\beta y_\beta + y_\beta^2\right)\left(x_\beta^2 + 4x_\beta y_\beta + y_\beta^2\right) \\
& + a_{0,0,1,6}^{r}\left(x_\alpha\left(x_\beta^5 - 10x_\beta^3 y_\beta^2 + 5x_\beta y_\beta^4\right) + y_\alpha\left(-5x_\beta^4 y_\beta + 10x_\beta^2 y_\beta^3 - y_\beta^5\right)\right) \\
& + a_{0,0,2,6}^{r}\left(x_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(-x_\beta^2 - 2x_\beta y_\beta + y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\right) \\
& + a_{0,0,3,0}^{r}\left(x_\alpha x_\beta + y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 + 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right) \\
& + a_{0,0,3,6}^{r}\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 - 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right) + a_{0,0,4,6}^{r}\left(x_\alpha\left(-2x_\beta y_\alpha - 2y_\alpha y_\beta\right)\right. \\
& \qquad \left. + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(-x_\alpha^2 + y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha - 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + a_{0,0,5,6}^{r}\left(-5x_\alpha^4 y_\alpha y_\beta + 5x_\alpha x_\beta y_\alpha^4 + x_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right) + y_\beta\left(10x_\alpha^2 y_\alpha^3 - y_\alpha^5\right)\right) \\
& + a_{0,0,6,6}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 - 4x_\alpha y_\alpha + y_\alpha^2\right)\left(x_\alpha^2 + 4x_\alpha y_\alpha + y_\alpha^2\right) \\
& + a_{0,2,2,0}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) + a_{0,4,1,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha x_\beta + y_\alpha y_\beta\right) \\
& + a_{0,6,0,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^3 + a_{2,0,2,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) \\
& + a_{2,2,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,4,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)^2 + a_{4,0,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha x_\beta + y_\alpha y_\beta\right) \\
& + a_{4,2,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\beta^2 + y_\beta^2\right) + a_{6,0,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^3
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{+-} = {} & 2ib^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)+2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta+y_\alpha\left(-x_\beta^2+y_\beta^2\right)\right)\\
& + b^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(-x_\beta^2+2x_\beta y_\beta+y_\beta^2\right)\right)\\
& + ib^r_{0,0,-5,-4}\left(-5x_\alpha^4x_\beta y_\alpha+5x_\alpha y_\alpha^4y_\beta+x_\beta\left(10x_\alpha^2y_\alpha^3-y_\alpha^5\right)+y_\beta\left(x_\alpha^5-10x_\alpha^3y_\alpha^2\right)\right)\\
& + b^r_{0,0,-5,-4}\left(5x_\alpha^4y_\alpha y_\beta+5x_\alpha x_\beta y_\alpha^4+x_\beta\left(x_\alpha^5-10x_\alpha^3y_\alpha^2\right)+y_\beta\left(-10x_\alpha^2y_\alpha^3+y_\alpha^5\right)\right)\\
& + b^r_{0,0,1,-4}\left(x_\alpha\left(x_\beta^5-10x_\beta^3y_\beta^2+5x_\beta y_\beta^4\right)+y_\alpha\left(5x_\beta^4y_\beta-10x_\beta^2y_\beta^3+y_\beta^5\right)\right)\\
& - ib^r_{0,0,1,-4}\left(x_\alpha\left(5x_\beta^4y_\beta-10x_\beta^2y_\beta^3+y_\beta^5\right)+y_\alpha\left(-x_\beta^5+10x_\beta^3y_\beta^2-5x_\beta y_\beta^4\right)\right)\\
& - 2ib^r_{0,0,4,2}\left(-2x_\alpha x_\beta y_\alpha+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\\
& + b^r_{0,0,4,2}\left(x_\alpha\left(-2x_\beta y_\alpha+2y_\alpha y_\beta\right)+x_\beta\left(x_\alpha^2-y_\alpha^2\right)+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha+2y_\alpha y_\beta\right)+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right.\\
& \qquad \left.+y_\beta\left(-x_\alpha^2+y_\alpha^2\right)\right)+b^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2y_\beta+y_\beta^3\right)\right)\\
& - ib^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& + b^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)\right)\\
& + ib^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)-2ib^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& + b^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right)\\
& + b^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3x_\beta-3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3y_\beta\right)\\
& - ib^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)-4ib^r_{0,2,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{0,2,-4,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)-4ib^r_{0,2,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{0,2,0,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+b^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3x_\beta+3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3y_\beta\right)\\
& - ib^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)+2ib^r_{0,4,0,2}x_\beta y_\beta\left(x_\beta^2+y_\beta^2\right)^2\\
& + b^r_{0,4,0,2}\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)^2+b^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& + ib^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{0,4,2,2}x_\alpha y_\alpha\left(x_\beta^2+y_\beta^2\right)^2\\
& + b^r_{0,4,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)^2+b^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2y_\beta+y_\beta^3\right)\right)\\
& - ib^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& + b^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)\right)\\
& + ib^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)-2ib^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& + b^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right)\\
& + b^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3x_\beta-3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3y_\beta\right)\\
& - ib^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)-4ib^r_{2,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)\\
& + b^r_{2,0,-4,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)-4ib^r_{2,0,0,-4}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\\
& + b^r_{2,0,0,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+b^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3x_\beta+3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3y_\beta\right)\\
& - ib^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)+2ib^r_{2,2,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{2,2,0,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)+b^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& + ib^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{2,2,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{2,2,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)+2ib^r_{4,0,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)^2\\
& + b^r_{4,0,0,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& + ib^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+2ib^r_{4,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)^2+b^r_{4,0,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)^2
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{-+} = & -2ib^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)+2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta+y_\alpha\left(-x_\beta^2+y_\beta^2\right)\right)\\
& + b^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+y_\alpha\left(-x_\beta^2+2x_\beta y_\beta+y_\beta^2\right)\right)\\
& - ib^r_{0,0,-5,-4}\left(-5x_\alpha^4x_\beta y_\alpha+5x_\alpha y_\alpha^4y_\beta+x_\beta\left(10x_\alpha^2y_\alpha^3-y_\alpha^5\right)+y_\beta\left(x_\alpha^5-10x_\alpha^3y_\alpha^2\right)\right)\\
& + b^r_{0,0,-5,-4}\left(5x_\alpha^4y_\alpha y_\beta+5x_\alpha x_\beta y_\alpha^4+x_\beta\left(x_\alpha^5-10x_\alpha^3y_\alpha^2\right)+y_\beta\left(-10x_\alpha^2y_\alpha^3+y_\alpha^5\right)\right)\\
& + b^r_{0,0,1,-4}\left(x_\alpha\left(x_\beta^5-10x_\beta^3y_\beta^2+5x_\beta y_\beta^4\right)+y_\alpha\left(5x_\beta^4y_\beta-10x_\beta^2y_\beta^3+y_\beta^5\right)\right)\\
& + ib^r_{0,0,1,-4}\left(x_\alpha\left(5x_\beta^4y_\beta-10x_\beta^2y_\beta^3+y_\beta^5\right)+y_\alpha\left(-x_\beta^5+10x_\beta^3y_\beta^2-5x_\beta y_\beta^4\right)\right)\\
& + 2ib^r_{0,0,4,2}\left(-2x_\alpha x_\beta y_\alpha+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\\
& + b^r_{0,0,4,2}\left(x_\alpha\left(-2x_\beta y_\alpha+2y_\alpha y_\beta\right)+x_\beta\left(x_\alpha^2-y_\alpha^2\right)+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha+2y_\alpha y_\beta\right)+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right.\\
& \qquad \left.+y_\beta\left(-x_\alpha^2+y_\alpha^2\right)\right)+b^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2y_\beta+y_\beta^3\right)\right)\\
& + ib^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& + b^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)\right)\\
& - ib^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)+2ib^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& + b^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right)\\
& + b^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3x_\beta-3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3y_\beta\right)\\
& + ib^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)+4ib^r_{0,2,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{0,2,-4,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)+4ib^r_{0,2,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{0,2,0,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+b^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3x_\beta+3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3y_\beta\right)\\
& + ib^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)-2ib^r_{0,4,0,2}x_\beta y_\beta\left(x_\beta^2+y_\beta^2\right)^2\\
& + b^r_{0,4,0,2}\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)^2+b^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& - ib^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2ib^r_{0,4,2,2}x_\alpha y_\alpha\left(x_\beta^2+y_\beta^2\right)^2\\
& + b^r_{0,4,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)^2+b^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(-3x_\beta^2y_\beta+y_\beta^3\right)\right)\\
& + ib^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& + b^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)+y_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)\right)\\
& - ib^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)+2ib^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& + b^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right)\\
& + b^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3x_\beta-3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2+y_\alpha^3y_\beta\right)\\
& + ib^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)+4ib^r_{2,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)\\
& + b^r_{2,0,-4,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^2-2x_\alpha y_\alpha-y_\alpha^2\right)\left(x_\alpha^2+2x_\alpha y_\alpha-y_\alpha^2\right)+4ib^r_{2,0,0,-4}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\\
& + b^r_{2,0,0,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2-2x_\beta y_\beta-y_\beta^2\right)\left(x_\beta^2+2x_\beta y_\beta-y_\beta^2\right)+b^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3x_\beta+3x_\alpha^2y_\alpha y_\beta-3x_\alpha x_\beta y_\alpha^2-y_\alpha^3y_\beta\right)\\
& + ib^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)-2ib^r_{2,2,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{2,2,0,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)+b^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& - ib^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2ib^r_{2,2,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& + b^r_{2,2,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)-2ib^r_{4,0,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)^2\\
& + b^r_{4,0,0,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\\
& - ib^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2ib^r_{4,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)^2+b^r_{4,0,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)^2
\end{aligned}
$$

# 4 Vibronic Hamiltonian operator in the real $E$ basis

$$
\hat{H} = \begin{pmatrix} |X\rangle & |Y\rangle \end{pmatrix} \begin{pmatrix} H_{XX} & H_{XY} \\ H_{YX} & H_{YY} \end{pmatrix} \begin{pmatrix} \langle X| \\ \langle Y| \end{pmatrix}
$$

# 5 Matrix element expansions in the real $E$ basis

## 5.1 Order: 0

Number of terms: $H_{XX}$: 1 (all from $H_{++}$), $H_{XY}$: 0, $H_{YY}$: 0.

**Polar e-coordinates:**

$$H_{XX}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{XY}^{(0)} = 0$$

$$H_{YX}^{(0)} = 0$$

$$H_{YY}^{(0)} = a_{0,0,0,0}^{r}$$

**Cartesian e-coordinates:**

$$H_{XX}^{(0)} = a_{0,0,0,0}^{r}$$

$$H_{XY}^{(0)} = 0$$

$$H_{YX}^{(0)} = 0$$

$$H_{YY}^{(0)} = a_{0,0,0,0}^{r}$$

## 5.2 Order: 1

Number of terms: $H_{XX}$: 2 (all from $H_{+-}$), $H_{XY}$: 2 (all from $H_{+-}$), $H_{YY}$: 2 (all from $H_{+-}$).

**Polar e-coordinates:**

$$H_{XX}^{(1)} = b_{0,0,-1,-1}^{r}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\cos\left(\phi_{\beta}\right)$$

$$H_{XY}^{(1)} = b_{0,0,-1,-1}^{r}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\sin\left(\phi_{\beta}\right)$$

$$H_{YX}^{(1)} = b_{0,0,-1,-1}^{r}\rho_{\alpha}\sin\left(\phi_{\alpha}\right) + b_{0,0,0,-1}^{r}\rho_{\beta}\sin\left(\phi_{\beta}\right)$$

$$H_{YY}^{(1)} = -b_{0,0,-1,-1}^{r}\rho_{\alpha}\cos\left(\phi_{\alpha}\right) - b_{0,0,0,-1}^{r}\rho_{\beta}\cos\left(\phi_{\beta}\right)$$

**Cartesian e-coordinates:**

$$H^{(1)}_{XX} = b^r_{0,0,-1,-1}x_\alpha + b^r_{0,0,0,-1}x_\beta$$

$$H^{(1)}_{XY} = b^r_{0,0,-1,-1}y_\alpha + b^r_{0,0,0,-1}y_\beta$$

$$H^{(1)}_{YX} = b^r_{0,0,-1,-1}y_\alpha + b^r_{0,0,0,-1}y_\beta$$

$$H^{(1)}_{YY} = -b^r_{0,0,-1,-1}x_\alpha - b^r_{0,0,0,-1}x_\beta$$

## 5.3 Order: 2

Number of terms: $H_{XX}$: 6 (3 from $H_{++}$, 3 from $H_{+-}$), $H_{XY}$: 3 (all from $H_{+-}$), $H_{YY}$: 3 (all from $H_{+-}$).

**Polar e-coordinates:**

$$\begin{aligned}H^{(2)}_{XX} = {} & a^r_{0,0,1,0}\rho_\alpha\rho_\beta\cos(\phi_\alpha-\phi_\beta) + a^r_{0,2,0,0}\rho^2_\beta + a^r_{2,0,0,0}\rho^2_\alpha + b^r_{0,0,0,2}\rho^2_\beta\cos(2\phi_\beta)\\ & + b^r_{0,0,1,2}\rho_\alpha\rho_\beta\cos(\phi_\alpha+\phi_\beta) + b^r_{0,0,2,2}\rho^2_\alpha\cos(2\phi_\alpha)\end{aligned}$$

$$H^{(2)}_{XY} = -b^r_{0,0,0,2}\rho^2_\beta\sin(2\phi_\beta) - b^r_{0,0,1,2}\rho_\alpha\rho_\beta\sin(\phi_\alpha+\phi_\beta) - b^r_{0,0,2,2}\rho^2_\alpha\sin(2\phi_\alpha)$$

$$H^{(2)}_{YX} = -b^r_{0,0,0,2}\rho^2_\beta\sin(2\phi_\beta) - b^r_{0,0,1,2}\rho_\alpha\rho_\beta\sin(\phi_\alpha+\phi_\beta) - b^r_{0,0,2,2}\rho^2_\alpha\sin(2\phi_\alpha)$$

$$\begin{aligned}H^{(2)}_{YY} = {} & a^r_{0,0,1,0}\rho_\alpha\rho_\beta\cos(\phi_\alpha-\phi_\beta) + a^r_{0,2,0,0}\rho^2_\beta + a^r_{2,0,0,0}\rho^2_\alpha - b^r_{0,0,0,2}\rho^2_\beta\cos(2\phi_\beta)\\ & - b^r_{0,0,1,2}\rho_\alpha\rho_\beta\cos(\phi_\alpha+\phi_\beta) - b^r_{0,0,2,2}\rho^2_\alpha\cos(2\phi_\alpha)\end{aligned}$$

**Cartesian e-coordinates:**

$$\begin{aligned}H^{(2)}_{XX} = {} & a^r_{0,0,1,0}(x_\alpha x_\beta + y_\alpha y_\beta) + a^r_{0,2,0,0}\left(x^2_\beta + y^2_\beta\right) + a^r_{2,0,0,0}\left(x^2_\alpha + y^2_\alpha\right) + b^r_{0,0,0,2}(x_\beta - y_\beta)(x_\beta + y_\beta)\\ & + b^r_{0,0,1,2}(x_\alpha x_\beta - y_\alpha y_\beta) + b^r_{0,0,2,2}(x_\alpha - y_\alpha)(x_\alpha + y_\alpha)\end{aligned}$$

$$H^{(2)}_{XY} = -2b^r_{0,0,0,2}x_\beta y_\beta - b^r_{0,0,1,2}(x_\alpha y_\beta + x_\beta y_\alpha) - 2b^r_{0,0,2,2}x_\alpha y_\alpha$$

$$H^{(2)}_{YX} = -2b^r_{0,0,0,2}x_\beta y_\beta - b^r_{0,0,1,2}(x_\alpha y_\beta + x_\beta y_\alpha) - 2b^r_{0,0,2,2}x_\alpha y_\alpha$$

$$\begin{aligned}H^{(2)}_{YY} = {} & a^r_{0,0,1,0}(x_\alpha x_\beta + y_\alpha y_\beta) + a^r_{0,2,0,0}\left(x^2_\beta + y^2_\beta\right) + a^r_{2,0,0,0}\left(x^2_\alpha + y^2_\alpha\right) - b^r_{0,0,0,2}(x_\beta - y_\beta)(x_\beta + y_\beta)\\ & - b^r_{0,0,1,2}(x_\alpha x_\beta - y_\alpha y_\beta) - b^r_{0,0,2,2}(x_\alpha - y_\alpha)(x_\alpha + y_\alpha)\end{aligned}$$

## 5.4 Order: 3

Number of terms: $H_{XX}$: 10 (4 from $H_{++}$, 6 from $H_{+-}$), $H_{XY}$: 6 (all from $H_{+-}$), $H_{YY}$: 6 (all from $H_{+-}$).

**Polar e-coordinates:**

$$\begin{aligned}H^{(3)}_{XX} = {} & a^r_{0,0,0,3}\rho_\beta^3\cos(3\phi_\beta) + a^r_{0,0,1,3}\rho_\alpha\rho_\beta^2\cos(\phi_\alpha+2\phi_\beta) + a^r_{0,0,2,3}\rho_\alpha^2\rho_\beta\cos(2\phi_\alpha+\phi_\beta) + a^r_{0,0,3,3}\rho_\alpha^3\cos(3\phi_\alpha)\\ & + b^r_{0,0,-2,-1}\rho_\alpha^2\rho_\beta\cos(2\phi_\alpha-\phi_\beta) + b^r_{0,0,1,-1}\rho_\alpha\rho_\beta^2\cos(\phi_\alpha-2\phi_\beta) + b^r_{0,2,-1,-1}\rho_\alpha\rho_\beta^2\cos(\phi_\alpha) + b^r_{0,2,0,-1}\rho_\beta^3\cos(\phi_\beta)\\ & + b^r_{2,0,-1,-1}\rho_\alpha^3\cos(\phi_\alpha) + b^r_{2,0,0,-1}\rho_\alpha^2\rho_\beta\cos(\phi_\beta)\end{aligned}$$

$$\begin{aligned}H^{(3)}_{XY} = {} & b^r_{0,0,-2,-1}\rho_\alpha^2\rho_\beta\sin(2\phi_\alpha-\phi_\beta) - b^r_{0,0,1,-1}\rho_\alpha\rho_\beta^2\sin(\phi_\alpha-2\phi_\beta) + b^r_{0,2,-1,-1}\rho_\alpha\rho_\beta^2\sin(\phi_\alpha) + b^r_{0,2,0,-1}\rho_\beta^3\sin(\phi_\beta)\\ & + b^r_{2,0,-1,-1}\rho_\alpha^3\sin(\phi_\alpha) + b^r_{2,0,0,-1}\rho_\alpha^2\rho_\beta\sin(\phi_\beta)\end{aligned}$$

$$\begin{aligned}H^{(3)}_{YX} = {} & b^r_{0,0,-2,-1}\rho_\alpha^2\rho_\beta\sin(2\phi_\alpha-\phi_\beta) - b^r_{0,0,1,-1}\rho_\alpha\rho_\beta^2\sin(\phi_\alpha-2\phi_\beta) + b^r_{0,2,-1,-1}\rho_\alpha\rho_\beta^2\sin(\phi_\alpha) + b^r_{0,2,0,-1}\rho_\beta^3\sin(\phi_\beta)\\ & + b^r_{2,0,-1,-1}\rho_\alpha^3\sin(\phi_\alpha) + b^r_{2,0,0,-1}\rho_\alpha^2\rho_\beta\sin(\phi_\beta)\end{aligned}$$

$$\begin{aligned}H^{(3)}_{YY} = {} & a^r_{0,0,0,3}\rho_\beta^3\cos(3\phi_\beta) + a^r_{0,0,1,3}\rho_\alpha\rho_\beta^2\cos(\phi_\alpha+2\phi_\beta) + a^r_{0,0,2,3}\rho_\alpha^2\rho_\beta\cos(2\phi_\alpha+\phi_\beta) + a^r_{0,0,3,3}\rho_\alpha^3\cos(3\phi_\alpha)\\ & - b^r_{0,0,-2,-1}\rho_\alpha^2\rho_\beta\cos(2\phi_\alpha-\phi_\beta) - b^r_{0,0,1,-1}\rho_\alpha\rho_\beta^2\cos(\phi_\alpha-2\phi_\beta) - b^r_{0,2,-1,-1}\rho_\alpha\rho_\beta^2\cos(\phi_\alpha) - b^r_{0,2,0,-1}\rho_\beta^3\cos(\phi_\beta)\\ & - b^r_{2,0,-1,-1}\rho_\alpha^3\cos(\phi_\alpha) - b^r_{2,0,0,-1}\rho_\alpha^2\rho_\beta\cos(\phi_\beta)\end{aligned}$$

**Cartesian e-coordinates:**

$$\begin{aligned}H^{(3)}_{XX} = {} & a^r_{0,0,0,3}x_\beta\left(x_\beta^2-3y_\beta^2\right) + a^r_{0,0,1,3}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)-2x_\beta y_\alpha y_\beta\right) + a^r_{0,0,2,3}\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\\ & + a^r_{0,0,3,3}x_\alpha\left(x_\alpha^2-3y_\alpha^2\right) + b^r_{0,0,-2,-1}\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right) + b^r_{0,0,1,-1}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)+2x_\beta y_\alpha y_\beta\right)\\ & + b^r_{0,2,-1,-1}x_\alpha\left(x_\beta^2+y_\beta^2\right) + b^r_{0,2,0,-1}x_\beta\left(x_\beta^2+y_\beta^2\right) + b^r_{2,0,-1,-1}x_\alpha\left(x_\alpha^2+y_\alpha^2\right) + b^r_{2,0,0,-1}x_\beta\left(x_\alpha^2+y_\alpha^2\right)\end{aligned}$$

$$\begin{aligned}H^{(3)}_{XY} = {} & -b^r_{0,0,-2,-1}\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right) + b^r_{0,0,1,-1}\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2+y_\beta^2\right)\right)\\ & + b^r_{0,2,-1,-1}y_\alpha\left(x_\beta^2+y_\beta^2\right) + b^r_{0,2,0,-1}y_\beta\left(x_\beta^2+y_\beta^2\right) + b^r_{2,0,-1,-1}y_\alpha\left(x_\alpha^2+y_\alpha^2\right) + b^r_{2,0,0,-1}y_\beta\left(x_\alpha^2+y_\alpha^2\right)\end{aligned}$$

$$\begin{aligned}H^{(3)}_{YX} = {} & -b^r_{0,0,-2,-1}\left(-2x_\alpha x_\beta y_\alpha + y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right) + b^r_{0,0,1,-1}\left(2x_\alpha x_\beta y_\beta + y_\alpha\left(-x_\beta^2+y_\beta^2\right)\right)\\ & + b^r_{0,2,-1,-1}y_\alpha\left(x_\beta^2+y_\beta^2\right) + b^r_{0,2,0,-1}y_\beta\left(x_\beta^2+y_\beta^2\right) + b^r_{2,0,-1,-1}y_\alpha\left(x_\alpha^2+y_\alpha^2\right) + b^r_{2,0,0,-1}y_\beta\left(x_\alpha^2+y_\alpha^2\right)\end{aligned}$$

$$\begin{aligned}H^{(3)}_{YY} = {} & a^r_{0,0,0,3}x_\beta\left(x_\beta^2-3y_\beta^2\right) + a^r_{0,0,1,3}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)-2x_\beta y_\alpha y_\beta\right) + a^r_{0,0,2,3}\left(-2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\\ & + a^r_{0,0,3,3}x_\alpha\left(x_\alpha^2-3y_\alpha^2\right) - b^r_{0,0,-2,-1}\left(2x_\alpha y_\alpha y_\beta + x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right) - b^r_{0,0,1,-1}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)+2x_\beta y_\alpha y_\beta\right)\\ & - b^r_{0,2,-1,-1}x_\alpha\left(x_\beta^2+y_\beta^2\right) - b^r_{0,2,0,-1}x_\beta\left(x_\beta^2+y_\beta^2\right) - b^r_{2,0,-1,-1}x_\alpha\left(x_\alpha^2+y_\alpha^2\right) - b^r_{2,0,0,-1}x_\beta\left(x_\alpha^2+y_\alpha^2\right)\end{aligned}$$

## 5.5 Order: 4

Number of terms: $H_{XX}$: 19 (6 from $H_{++}$, 13 from $H_{+-}$), $H_{XY}$: 13 (all from $H_{+-}$), $H_{YY}$: 13 (all from $H_{+-}$).

**Polar e-coordinates:**

$$
\begin{aligned}
H^{(4)}_{XX} = {} & a^r_{0,0,2,0}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\alpha-2\phi_\beta\right)+a^r_{0,2,1,0}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha-\phi_\beta\right)+a^r_{0,4,0,0}\rho^4_\beta+a^r_{2,0,1,0}\rho^3_\alpha\rho_\beta\cos\left(\phi_\alpha-\phi_\beta\right)+a^r_{2,2,0,0}\rho^2_\alpha\rho^2_\beta\\
& +a^r_{4,0,0,0}\rho^4_\alpha+b^r_{0,0,-1,-4}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha+3\phi_\beta\right)+b^r_{0,0,-1,2}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha-3\phi_\beta\right)+b^r_{0,0,-2,-4}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\alpha+2\phi_\beta\right)\\
& +b^r_{0,0,-3,-4}\rho^3_\alpha\rho_\beta\cos\left(3\phi_\alpha+\phi_\beta\right)+b^r_{0,0,-4,-4}\rho^4_\alpha\cos\left(4\phi_\alpha\right)+b^r_{0,0,0,-4}\rho^4_\beta\cos\left(4\phi_\beta\right)+b^r_{0,0,3,2}\rho^3_\alpha\rho_\beta\cos\left(3\phi_\alpha-\phi_\beta\right)\\
& +b^r_{0,2,0,2}\rho^4_\beta\cos\left(2\phi_\beta\right)+b^r_{0,2,1,2}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha+\phi_\beta\right)+b^r_{0,2,2,2}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\alpha\right)+b^r_{2,0,0,2}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\beta\right)\\
& +b^r_{2,0,1,2}\rho^3_\alpha\rho_\beta\cos\left(\phi_\alpha+\phi_\beta\right)+b^r_{2,0,2,2}\rho^4_\alpha\cos\left(2\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{XY} = {} & b^r_{0,0,-1,-4}\rho_\alpha\rho^3_\beta\sin\left(\phi_\alpha+3\phi_\beta\right)+b^r_{0,0,-1,2}\rho_\alpha\rho^3_\beta\sin\left(\phi_\alpha-3\phi_\beta\right)+b^r_{0,0,-2,-4}\rho^2_\alpha\rho^2_\beta\sin\left(2\phi_\alpha+2\phi_\beta\right)\\
& +b^r_{0,0,-3,-4}\rho^3_\alpha\rho_\beta\sin\left(3\phi_\alpha+\phi_\beta\right)+b^r_{0,0,-4,-4}\rho^4_\alpha\sin\left(4\phi_\alpha\right)+b^r_{0,0,0,-4}\rho^4_\beta\sin\left(4\phi_\beta\right)-b^r_{0,0,3,2}\rho^3_\alpha\rho_\beta\sin\left(3\phi_\alpha-\phi_\beta\right)\\
& -b^r_{0,2,0,2}\rho^4_\beta\sin\left(2\phi_\beta\right)-b^r_{0,2,1,2}\rho_\alpha\rho^3_\beta\sin\left(\phi_\alpha+\phi_\beta\right)-b^r_{0,2,2,2}\rho^2_\alpha\rho^2_\beta\sin\left(2\phi_\alpha\right)-b^r_{2,0,0,2}\rho^2_\alpha\rho^2_\beta\sin\left(2\phi_\beta\right)\\
& -b^r_{2,0,1,2}\rho^3_\alpha\rho_\beta\sin\left(\phi_\alpha+\phi_\beta\right)-b^r_{2,0,2,2}\rho^4_\alpha\sin\left(2\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{YX} = {} & b^r_{0,0,-1,-4}\rho_\alpha\rho^3_\beta\sin\left(\phi_\alpha+3\phi_\beta\right)+b^r_{0,0,-1,2}\rho_\alpha\rho^3_\beta\sin\left(\phi_\alpha-3\phi_\beta\right)+b^r_{0,0,-2,-4}\rho^2_\alpha\rho^2_\beta\sin\left(2\phi_\alpha+2\phi_\beta\right)\\
& +b^r_{0,0,-3,-4}\rho^3_\alpha\rho_\beta\sin\left(3\phi_\alpha+\phi_\beta\right)+b^r_{0,0,-4,-4}\rho^4_\alpha\sin\left(4\phi_\alpha\right)+b^r_{0,0,0,-4}\rho^4_\beta\sin\left(4\phi_\beta\right)-b^r_{0,0,3,2}\rho^3_\alpha\rho_\beta\sin\left(3\phi_\alpha-\phi_\beta\right)\\
& -b^r_{0,2,0,2}\rho^4_\beta\sin\left(2\phi_\beta\right)-b^r_{0,2,1,2}\rho_\alpha\rho^3_\beta\sin\left(\phi_\alpha+\phi_\beta\right)-b^r_{0,2,2,2}\rho^2_\alpha\rho^2_\beta\sin\left(2\phi_\alpha\right)-b^r_{2,0,0,2}\rho^2_\alpha\rho^2_\beta\sin\left(2\phi_\beta\right)\\
& -b^r_{2,0,1,2}\rho^3_\alpha\rho_\beta\sin\left(\phi_\alpha+\phi_\beta\right)-b^r_{2,0,2,2}\rho^4_\alpha\sin\left(2\phi_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{YY} = {} & a^r_{0,0,2,0}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\alpha-2\phi_\beta\right)+a^r_{0,2,1,0}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha-\phi_\beta\right)+a^r_{0,4,0,0}\rho^4_\beta+a^r_{2,0,1,0}\rho^3_\alpha\rho_\beta\cos\left(\phi_\alpha-\phi_\beta\right)+a^r_{2,2,0,0}\rho^2_\alpha\rho^2_\beta\\
& +a^r_{4,0,0,0}\rho^4_\alpha-b^r_{0,0,-1,-4}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha+3\phi_\beta\right)-b^r_{0,0,-1,2}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha-3\phi_\beta\right)-b^r_{0,0,-2,-4}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\alpha+2\phi_\beta\right)\\
& -b^r_{0,0,-3,-4}\rho^3_\alpha\rho_\beta\cos\left(3\phi_\alpha+\phi_\beta\right)-b^r_{0,0,-4,-4}\rho^4_\alpha\cos\left(4\phi_\alpha\right)-b^r_{0,0,0,-4}\rho^4_\beta\cos\left(4\phi_\beta\right)-b^r_{0,0,3,2}\rho^3_\alpha\rho_\beta\cos\left(3\phi_\alpha-\phi_\beta\right)\\
& -b^r_{0,2,0,2}\rho^4_\beta\cos\left(2\phi_\beta\right)-b^r_{0,2,1,2}\rho_\alpha\rho^3_\beta\cos\left(\phi_\alpha+\phi_\beta\right)-b^r_{0,2,2,2}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\alpha\right)-b^r_{2,0,0,2}\rho^2_\alpha\rho^2_\beta\cos\left(2\phi_\beta\right)\\
& -b^r_{2,0,1,2}\rho^3_\alpha\rho_\beta\cos\left(\phi_\alpha+\phi_\beta\right)-b^r_{2,0,2,2}\rho^4_\alpha\cos\left(2\phi_\alpha\right)
\end{aligned}
$$

**Cartesian e-coordinates:**

$$
\begin{aligned}
H^{(4)}_{XX} = {} & a^r_{0,0,2,0}\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(x_\beta+y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(-x_\beta+y_\beta\right)\right)+a^r_{0,2,1,0}\left(x^2_\beta+y^2_\beta\right)\left(x_\alpha x_\beta+y_\alpha y_\beta\right)\\
& +a^r_{0,4,0,0}\left(x^2_\beta+y^2_\beta\right)^2+a^r_{2,0,1,0}\left(x^2_\alpha+y^2_\alpha\right)\left(x_\alpha x_\beta+y_\alpha y_\beta\right)+a^r_{2,2,0,0}\left(x^2_\alpha+y^2_\alpha\right)\left(x^2_\beta+y^2_\beta\right)+a^r_{4,0,0,0}\left(x^2_\alpha+y^2_\alpha\right)^2\\
& +b^r_{0,0,-1,-4}\left(x_\alpha\left(x^3_\beta-3x_\beta y^2_\beta\right)+y_\alpha\left(-3x^2_\beta y_\beta+y^3_\beta\right)\right)+b^r_{0,0,-1,2}\left(x_\alpha\left(x^3_\beta-3x_\beta y^2_\beta\right)+y_\alpha\left(3x^2_\beta y_\beta-y^3_\beta\right)\right)\\
& +b^r_{0,0,-2,-4}\left(x_\alpha\left(x_\beta-y_\beta\right)+y_\alpha\left(-x_\beta-y_\beta\right)\right)\left(x_\alpha\left(x_\beta+y_\beta\right)+y_\alpha\left(x_\beta-y_\beta\right)\right)\\
& +b^r_{0,0,-3,-4}\left(x^3_\alpha x_\beta-3x^2_\alpha y_\alpha y_\beta-3x_\alpha x_\beta y^2_\alpha+y^3_\alpha y_\beta\right)+b^r_{0,0,-4,-4}\left(x^2_\alpha-2x_\alpha y_\alpha-y^2_\alpha\right)\left(x^2_\alpha+2x_\alpha y_\alpha-y^2_\alpha\right)\\
& +b^r_{0,0,0,-4}\left(x^2_\beta-2x_\beta y_\beta-y^2_\beta\right)\left(x^2_\beta+2x_\beta y_\beta-y^2_\beta\right)+b^r_{0,0,3,2}\left(x^3_\alpha x_\beta+3x^2_\alpha y_\alpha y_\beta-3x_\alpha x_\beta y^2_\alpha-y^3_\alpha y_\beta\right)\\
& +b^r_{0,2,0,2}\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x^2_\beta+y^2_\beta\right)+b^r_{0,2,1,2}\left(x^2_\beta+y^2_\beta\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)+b^r_{0,2,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x^2_\beta+y^2_\beta\right)\\
& +b^r_{2,0,0,2}\left(x^2_\alpha+y^2_\alpha\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{2,0,1,2}\left(x^2_\alpha+y^2_\alpha\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)+b^r_{2,0,2,2}\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x^2_\alpha+y^2_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(4)}_{XY} = {} & b^r_{0,0,-1,-4}\left(x_\alpha\left(3x^2_\beta y_\beta-y^3_\beta\right)+y_\alpha\left(x^3_\beta-3x_\beta y^2_\beta\right)\right)-b^r_{0,0,-1,2}\left(x_\alpha\left(3x^2_\beta y_\beta-y^3_\beta\right)+y_\alpha\left(-x^3_\beta+3x_\beta y^2_\beta\right)\right)\\
& +2b^r_{0,0,-2,-4}\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)+b^r_{0,0,-3,-4}\left(x^3_\alpha y_\beta+3x^2_\alpha x_\beta y_\alpha-3x_\alpha y^2_\alpha y_\beta-x_\beta y^3_\alpha\right)\\
& +4b^r_{0,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)+4b^r_{0,0,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\\
& +b^r_{0,0,3,2}\left(x^3_\alpha y_\beta-3x^2_\alpha x_\beta y_\alpha-3x_\alpha y^2_\alpha y_\beta+x_\beta y^3_\alpha\right)-2b^r_{0,2,0,2}x_\beta y_\beta\left(x^2_\beta+y^2_\beta\right)-b^r_{0,2,1,2}\left(x^2_\beta+y^2_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& -2b^r_{0,2,2,2}x_\alpha y_\alpha\left(x^2_\beta+y^2_\beta\right)-2b^r_{2,0,0,2}x_\beta y_\beta\left(x^2_\alpha+y^2_\alpha\right)-b^r_{2,0,1,2}\left(x^2_\alpha+y^2_\alpha\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2b^r_{2,0,2,2}x_\alpha y_\alpha\left(x^2_\alpha+y^2_\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{YX}^{(4)} = {} & b_{0,0,-1,-4}^{r}\left(x_\alpha\left(3x_\beta^2y_\beta - y_\beta^3\right) + y_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right)\right) - b_{0,0,-1,2}^{r}\left(x_\alpha\left(3x_\beta^2y_\beta - y_\beta^3\right) + y_\alpha\left(-x_\beta^3 + 3x_\beta y_\beta^2\right)\right) \\
& + 2b_{0,0,-2,-4}^{r}\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) + b_{0,0,-3,-4}^{r}\left(x_\alpha^3y_\beta + 3x_\alpha^2x_\beta y_\alpha - 3x_\alpha y_\alpha^2y_\beta - x_\beta y_\alpha^3\right) \\
& + 4b_{0,0,-4,-4}^{r}x_\alpha y_\alpha\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right) + 4b_{0,0,0,-4}^{r}x_\beta y_\beta\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) \\
& + b_{0,0,3,2}^{r}\left(x_\alpha^3y_\beta - 3x_\alpha^2x_\beta y_\alpha - 3x_\alpha y_\alpha^2y_\beta + x_\beta y_\alpha^3\right) - 2b_{0,2,0,2}^{r}x_\beta y_\beta\left(x_\beta^2 + y_\beta^2\right) - b_{0,2,1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) \\
& - 2b_{0,2,2,2}^{r}x_\alpha y_\alpha\left(x_\beta^2 + y_\beta^2\right) - 2b_{2,0,0,2}^{r}x_\beta y_\beta\left(x_\alpha^2 + y_\alpha^2\right) - b_{2,0,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha y_\beta + x_\beta y_\alpha\right) - 2b_{2,0,2,2}^{r}x_\alpha y_\alpha\left(x_\alpha^2 + y_\alpha^2\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{YY}^{(4)} = {} & a_{0,0,2,0}^{r}\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) + a_{0,2,1,0}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) \\
& + a_{0,4,0,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^2 + a_{2,0,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,2,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) + a_{4,0,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2 \\
& - b_{0,0,-1,-4}^{r}\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2y_\beta + y_\beta^3\right)\right) - b_{0,0,-1,2}^{r}\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2y_\beta - y_\beta^3\right)\right) \\
& - b_{0,0,-2,-4}^{r}\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& - b_{0,0,-3,-4}^{r}\left(x_\alpha^3x_\beta - 3x_\alpha^2y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3y_\beta\right) - b_{0,0,-4,-4}^{r}\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) \\
& - b_{0,0,0,-4}^{r}\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) - b_{0,0,3,2}^{r}\left(x_\alpha^3x_\beta + 3x_\alpha^2y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3y_\beta\right) \\
& - b_{0,2,0,2}^{r}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) - b_{0,2,1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right) - b_{0,2,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right) \\
& - b_{2,0,0,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) - b_{2,0,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right) - b_{2,0,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)
\end{aligned}
$$

## 5.6 Order: 5

Number of terms: $H_{XX}$: 28 (10 from $H_{++}$, 18 from $H_{+-}$), $H_{XY}$: 18 (all from $H_{+-}$), $H_{YY}$: 18 (all from $H_{+-}$).

**Polar e-coordinates:**

$$
\begin{aligned}
H_{XX}^{(5)} = {} & a_{0,0,-1,3}^{r}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha - 4\phi_\beta\right) + a_{0,0,4,3}^{r}\rho_\alpha^4\rho_\beta\cos\left(4\phi_\alpha - \phi_\beta\right) + a_{0,2,0,3}^{r}\rho_\beta^5\cos\left(3\phi_\beta\right) + a_{0,2,1,3}^{r}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha + 2\phi_\beta\right) \\
& + a_{0,2,2,3}^{r}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha + \phi_\beta\right) + a_{0,2,3,3}^{r}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha\right) + a_{2,0,0,3}^{r}\rho_\alpha^2\rho_\beta^3\cos\left(3\phi_\beta\right) + a_{2,0,1,3}^{r}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha + 2\phi_\beta\right) \\
& + a_{2,0,2,3}^{r}\rho_\alpha^4\rho_\beta\cos\left(2\phi_\alpha + \phi_\beta\right) + a_{2,0,3,3}^{r}\rho_\alpha^5\cos\left(3\phi_\alpha\right) + b_{0,0,-3,-1}^{r}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha - 2\phi_\beta\right) + b_{0,0,0,5}^{r}\rho_\beta^5\cos\left(5\phi_\beta\right) \\
& + b_{0,0,1,5}^{r}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha + 4\phi_\beta\right) + b_{0,0,2,-1}^{r}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha - 3\phi_\beta\right) + b_{0,0,2,5}^{r}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha + 3\phi_\beta\right) \\
& + b_{0,0,3,5}^{r}\rho_\alpha^3\rho_\beta^2\cos\left(3\phi_\alpha + 2\phi_\beta\right) + b_{0,0,4,5}^{r}\rho_\alpha^4\rho_\beta\cos\left(4\phi_\alpha + \phi_\beta\right) + b_{0,0,5,5}^{r}\rho_\alpha^5\cos\left(5\phi_\alpha\right) \\
& + b_{0,2,-2,-1}^{r}\rho_\alpha^2\rho_\beta^3\cos\left(2\phi_\alpha - \phi_\beta\right) + b_{0,2,1,-1}^{r}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha - 2\phi_\beta\right) + b_{0,4,-1,-1}^{r}\rho_\alpha\rho_\beta^4\cos\left(\phi_\alpha\right) \\
& + b_{0,4,0,-1}^{r}\rho_\beta^5\cos\left(\phi_\beta\right) + b_{2,0,-2,-1}^{r}\rho_\alpha^4\rho_\beta\cos\left(2\phi_\alpha - \phi_\beta\right) + b_{2,0,1,-1}^{r}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha - 2\phi_\beta\right) \\
& + b_{2,2,-1,-1}^{r}\rho_\alpha^3\rho_\beta^2\cos\left(\phi_\alpha\right) + b_{2,2,0,-1}^{r}\rho_\alpha^2\rho_\beta^3\cos\left(\phi_\beta\right) + b_{4,0,-1,-1}^{r}\rho_\alpha^5\cos\left(\phi_\alpha\right) + b_{4,0,0,-1}^{r}\rho_\alpha^4\rho_\beta\cos\left(\phi_\beta\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{XY}^{(5)} = {} & b_{0,0,-3,-1}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha - 2\phi_\beta\right) - b_{0,0,0,5}^{r}\rho_\beta^5\sin\left(5\phi_\beta\right) - b_{0,0,1,5}^{r}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha + 4\phi_\beta\right) - b_{0,0,2,-1}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - 3\phi_\beta\right) \\
& - b_{0,0,2,5}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha + 3\phi_\beta\right) - b_{0,0,3,5}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha + 2\phi_\beta\right) - b_{0,0,4,5}^{r}\rho_\alpha^4\rho_\beta\sin\left(4\phi_\alpha + \phi_\beta\right) - b_{0,0,5,5}^{r}\rho_\alpha^5\sin\left(5\phi_\alpha\right) \\
& + b_{0,2,-2,-1}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - \phi_\beta\right) - b_{0,2,1,-1}^{r}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha - 2\phi_\beta\right) + b_{0,4,-1,-1}^{r}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha\right) + b_{0,4,0,-1}^{r}\rho_\beta^5\sin\left(\phi_\beta\right) \\
& + b_{2,0,-2,-1}^{r}\rho_\alpha^4\rho_\beta\sin\left(2\phi_\alpha - \phi_\beta\right) - b_{2,0,1,-1}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha - 2\phi_\beta\right) + b_{2,2,-1,-1}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha\right) + b_{2,2,0,-1}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(\phi_\beta\right) \\
& + b_{4,0,-1,-1}^{r}\rho_\alpha^5\sin\left(\phi_\alpha\right) + b_{4,0,0,-1}^{r}\rho_\alpha^4\rho_\beta\sin\left(\phi_\beta\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{YX}^{(5)} = {} & b_{0,0,-3,-1}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha - 2\phi_\beta\right) - b_{0,0,0,5}^{r}\rho_\beta^5\sin\left(5\phi_\beta\right) - b_{0,0,1,5}^{r}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha + 4\phi_\beta\right) - b_{0,0,2,-1}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - 3\phi_\beta\right) \\
& - b_{0,0,2,5}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha + 3\phi_\beta\right) - b_{0,0,3,5}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(3\phi_\alpha + 2\phi_\beta\right) - b_{0,0,4,5}^{r}\rho_\alpha^4\rho_\beta\sin\left(4\phi_\alpha + \phi_\beta\right) - b_{0,0,5,5}^{r}\rho_\alpha^5\sin\left(5\phi_\alpha\right) \\
& + b_{0,2,-2,-1}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(2\phi_\alpha - \phi_\beta\right) - b_{0,2,1,-1}^{r}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha - 2\phi_\beta\right) + b_{0,4,-1,-1}^{r}\rho_\alpha\rho_\beta^4\sin\left(\phi_\alpha\right) + b_{0,4,0,-1}^{r}\rho_\beta^5\sin\left(\phi_\beta\right) \\
& + b_{2,0,-2,-1}^{r}\rho_\alpha^4\rho_\beta\sin\left(2\phi_\alpha - \phi_\beta\right) - b_{2,0,1,-1}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha - 2\phi_\beta\right) + b_{2,2,-1,-1}^{r}\rho_\alpha^3\rho_\beta^2\sin\left(\phi_\alpha\right) + b_{2,2,0,-1}^{r}\rho_\alpha^2\rho_\beta^3\sin\left(\phi_\beta\right) \\
& + b_{4,0,-1,-1}^{r}\rho_\alpha^5\sin\left(\phi_\alpha\right) + b_{4,0,0,-1}^{r}\rho_\alpha^4\rho_\beta\sin\left(\phi_\beta\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{YY}^{(5)} = {} & a_{0,0,-1,3}^{r}\rho_{\alpha}\rho_{\beta}^{4}\cos\left(\phi_{\alpha}-4\phi_{\beta}\right)+a_{0,0,4,3}^{r}\rho_{\alpha}^{4}\rho_{\beta}\cos\left(4\phi_{\alpha}-\phi_{\beta}\right)+a_{0,2,0,3}^{r}\rho_{\beta}^{5}\cos\left(3\phi_{\beta}\right)+a_{0,2,1,3}^{r}\rho_{\alpha}\rho_{\beta}^{4}\cos\left(\phi_{\alpha}+2\phi_{\beta}\right)\\
& +a_{0,2,2,3}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{3}\cos\left(2\phi_{\alpha}+\phi_{\beta}\right)+a_{0,2,3,3}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{2}\cos\left(3\phi_{\alpha}\right)+a_{2,0,0,3}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{3}\cos\left(3\phi_{\beta}\right)+a_{2,0,1,3}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{2}\cos\left(\phi_{\alpha}+2\phi_{\beta}\right)\\
& +a_{2,0,2,3}^{r}\rho_{\alpha}^{4}\rho_{\beta}\cos\left(2\phi_{\alpha}+\phi_{\beta}\right)+a_{2,0,3,3}^{r}\rho_{\alpha}^{5}\cos\left(3\phi_{\alpha}\right)-b_{0,0,-3,-1}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{2}\cos\left(3\phi_{\alpha}-2\phi_{\beta}\right)-b_{0,0,0,5}^{r}\rho_{\beta}^{5}\cos\left(5\phi_{\beta}\right)\\
& -b_{0,0,1,5}^{r}\rho_{\alpha}\rho_{\beta}^{4}\cos\left(\phi_{\alpha}+4\phi_{\beta}\right)-b_{0,0,2,-1}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{3}\cos\left(2\phi_{\alpha}-3\phi_{\beta}\right)-b_{0,0,2,5}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{3}\cos\left(2\phi_{\alpha}+3\phi_{\beta}\right)\\
& -b_{0,0,3,5}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{2}\cos\left(3\phi_{\alpha}+2\phi_{\beta}\right)-b_{0,0,4,5}^{r}\rho_{\alpha}^{4}\rho_{\beta}\cos\left(4\phi_{\alpha}+\phi_{\beta}\right)-b_{0,0,5,5}^{r}\rho_{\alpha}^{5}\cos\left(5\phi_{\alpha}\right)\\
& -b_{0,2,-2,-1}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{3}\cos\left(2\phi_{\alpha}-\phi_{\beta}\right)-b_{0,2,1,-1}^{r}\rho_{\alpha}\rho_{\beta}^{4}\cos\left(\phi_{\alpha}-2\phi_{\beta}\right)-b_{0,4,-1,-1}^{r}\rho_{\alpha}\rho_{\beta}^{4}\cos\left(\phi_{\alpha}\right)-b_{0,4,0,-1}^{r}\rho_{\beta}^{5}\cos\left(\phi_{\beta}\right)\\
& -b_{2,0,-2,-1}^{r}\rho_{\alpha}^{4}\rho_{\beta}\cos\left(2\phi_{\alpha}-\phi_{\beta}\right)-b_{2,0,1,-1}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{2}\cos\left(\phi_{\alpha}-2\phi_{\beta}\right)-b_{2,2,-1,-1}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{2}\cos\left(\phi_{\alpha}\right)-b_{2,2,0,-1}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{3}\cos\left(\phi_{\beta}\right)\\
& -b_{4,0,-1,-1}^{r}\rho_{\alpha}^{5}\cos\left(\phi_{\alpha}\right)-b_{4,0,0,-1}^{r}\rho_{\alpha}^{4}\rho_{\beta}\cos\left(\phi_{\beta}\right)
\end{aligned}
$$

**Cartesian e-coordinates:**

$$
\begin{aligned}
H_{XX}^{(5)} = {} & a_{0,0,-1,3}^{r}\left(x_{\alpha}\left(x_{\beta}^{4}-6x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)+y_{\alpha}\left(4x_{\beta}^{3}y_{\beta}-4x_{\beta}y_{\beta}^{3}\right)\right)+a_{0,0,4,3}^{r}\left(4x_{\alpha}^{3}y_{\alpha}y_{\beta}-4x_{\alpha}y_{\alpha}^{3}y_{\beta}+x_{\beta}\left(x_{\alpha}^{4}-6x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\right)\\
& +a_{0,2,0,3}^{r}x_{\beta}\left(x_{\beta}^{2}-3y_{\beta}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+a_{0,2,1,3}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)-2x_{\beta}y_{\alpha}y_{\beta}\right)\\
& +a_{0,2,2,3}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(-2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)+a_{0,2,3,3}^{r}x_{\alpha}\left(x_{\alpha}^{2}-3y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+a_{2,0,0,3}^{r}x_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}-3y_{\beta}^{2}\right)\\
& +a_{2,0,1,3}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)-2x_{\beta}y_{\alpha}y_{\beta}\right)+a_{2,0,2,3}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(-2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\\
& +a_{2,0,3,3}^{r}x_{\alpha}\left(x_{\alpha}^{2}-3y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)+b_{0,0,-3,-1}^{r}\left(x_{\alpha}^{3}x_{\beta}^{2}-x_{\alpha}^{3}y_{\beta}^{2}+6x_{\alpha}^{2}x_{\beta}y_{\alpha}y_{\beta}+x_{\alpha}\left(-3x_{\beta}^{2}y_{\alpha}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)-2x_{\beta}y_{\alpha}^{3}y_{\beta}\right)\\
& +b_{0,0,0,5}^{r}x_{\beta}\left(x_{\beta}^{4}-10x_{\beta}^{2}y_{\beta}^{2}+5y_{\beta}^{4}\right)+b_{0,0,1,5}^{r}\left(x_{\alpha}\left(x_{\beta}^{4}-6x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)+y_{\alpha}\left(-4x_{\beta}^{3}y_{\beta}+4x_{\beta}y_{\beta}^{3}\right)\right)\\
& +b_{0,0,2,-1}^{r}\left(x_{\alpha}^{2}x_{\beta}^{3}+x_{\alpha}\left(6x_{\beta}^{2}y_{\alpha}y_{\beta}-2y_{\alpha}y_{\beta}^{3}\right)-x_{\beta}^{3}y_{\alpha}^{2}+x_{\beta}\left(-3x_{\alpha}^{2}y_{\beta}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& +b_{0,0,2,5}^{r}\left(x_{\alpha}^{2}x_{\beta}^{3}+x_{\alpha}\left(-6x_{\beta}^{2}y_{\alpha}y_{\beta}+2y_{\alpha}y_{\beta}^{3}\right)-x_{\beta}^{3}y_{\alpha}^{2}+x_{\beta}\left(-3x_{\alpha}^{2}y_{\beta}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& +b_{0,0,3,5}^{r}\left(x_{\alpha}^{3}x_{\beta}^{2}-x_{\alpha}^{3}y_{\beta}^{2}-6x_{\alpha}^{2}x_{\beta}y_{\alpha}y_{\beta}+x_{\alpha}\left(-3x_{\beta}^{2}y_{\alpha}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)+2x_{\beta}y_{\alpha}^{3}y_{\beta}\right)\\
& +b_{0,0,4,5}^{r}\left(-4x_{\alpha}^{3}y_{\alpha}y_{\beta}+4x_{\alpha}y_{\alpha}^{3}y_{\beta}+x_{\beta}\left(x_{\alpha}^{4}-6x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\right)+b_{0,0,5,5}^{r}x_{\alpha}\left(x_{\alpha}^{4}-10x_{\alpha}^{2}y_{\alpha}^{2}+5y_{\alpha}^{4}\right)\\
& +b_{0,2,-2,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)+b_{0,2,1,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)+2x_{\beta}y_{\alpha}y_{\beta}\right)\\
& +b_{0,4,-1,-1}^{r}x_{\alpha}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}+b_{0,4,0,-1}^{r}x_{\beta}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}+b_{2,0,-2,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\\
& +b_{2,0,1,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)+2x_{\beta}y_{\alpha}y_{\beta}\right)+b_{2,2,-1,-1}^{r}x_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\\
& +b_{2,2,0,-1}^{r}x_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+b_{4,0,-1,-1}^{r}x_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}+b_{4,0,0,-1}^{r}x_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
H_{XY}^{(5)} = {} & -b_{0,0,-3,-1}^{r}\left(2x_{\alpha}^{3}x_{\beta}y_{\beta}-6x_{\alpha}x_{\beta}y_{\alpha}^{2}y_{\beta}+x_{\beta}^{2}y_{\alpha}^{3}-y_{\alpha}^{3}y_{\beta}^{2}+y_{\alpha}\left(-3x_{\alpha}^{2}x_{\beta}^{2}+3x_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& -b_{0,0,0,5}^{r}y_{\beta}\left(5x_{\beta}^{4}-10x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)-b_{0,0,1,5}^{r}\left(x_{\alpha}\left(4x_{\beta}^{3}y_{\beta}-4x_{\beta}y_{\beta}^{3}\right)+y_{\alpha}\left(x_{\beta}^{4}-6x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)\right)\\
& +b_{0,0,2,-1}^{r}\left(-x_{\alpha}^{2}y_{\beta}^{3}+x_{\alpha}\left(-2x_{\beta}^{3}y_{\alpha}+6x_{\beta}y_{\alpha}y_{\beta}^{2}\right)+y_{\alpha}^{2}y_{\beta}^{3}+y_{\beta}\left(3x_{\alpha}^{2}x_{\beta}^{2}-3x_{\beta}^{2}y_{\alpha}^{2}\right)\right)\\
& -b_{0,0,2,5}^{r}\left(-x_{\alpha}^{2}y_{\beta}^{3}+x_{\alpha}\left(2x_{\beta}^{3}y_{\alpha}-6x_{\beta}y_{\alpha}y_{\beta}^{2}\right)+y_{\alpha}^{2}y_{\beta}^{3}+y_{\beta}\left(3x_{\alpha}^{2}x_{\beta}^{2}-3x_{\beta}^{2}y_{\alpha}^{2}\right)\right)\\
& -b_{0,0,3,5}^{r}\left(2x_{\alpha}^{3}x_{\beta}y_{\beta}-6x_{\alpha}x_{\beta}y_{\alpha}^{2}y_{\beta}-x_{\beta}^{2}y_{\alpha}^{3}+y_{\alpha}^{3}y_{\beta}^{2}+y_{\alpha}\left(3x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& -b_{0,0,4,5}^{r}\left(4x_{\alpha}^{3}x_{\beta}y_{\alpha}-4x_{\alpha}x_{\beta}y_{\alpha}^{3}+y_{\beta}\left(x_{\alpha}^{4}-6x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\right)-b_{0,0,5,5}^{r}y_{\alpha}\left(5x_{\alpha}^{4}-10x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\\
& -b_{0,2,-2,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(-2x_{\alpha}x_{\beta}y_{\alpha}+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)+b_{0,2,1,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(2x_{\alpha}x_{\beta}y_{\beta}+y_{\alpha}\left(-x_{\beta}^{2}+y_{\beta}^{2}\right)\right)\\
& +b_{0,4,-1,-1}^{r}y_{\alpha}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}+b_{0,4,0,-1}^{r}y_{\beta}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}-b_{2,0,-2,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(-2x_{\alpha}x_{\beta}y_{\alpha}+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\\
& +b_{2,0,1,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(2x_{\alpha}x_{\beta}y_{\beta}+y_{\alpha}\left(-x_{\beta}^{2}+y_{\beta}^{2}\right)\right)+b_{2,2,-1,-1}^{r}y_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\\
& +b_{2,2,0,-1}^{r}y_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+b_{4,0,-1,-1}^{r}y_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}+b_{4,0,0,-1}^{r}y_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
H_{YX}^{(5)} = & -b_{0,0,-3,-1}^{r}\left(2x_{\alpha}^{3}x_{\beta}y_{\beta}-6x_{\alpha}x_{\beta}y_{\alpha}^{2}y_{\beta}+x_{\beta}^{2}y_{\alpha}^{3}-y_{\alpha}^{3}y_{\beta}^{2}+y_{\alpha}\left(-3x_{\alpha}^{2}x_{\beta}^{2}+3x_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& -b_{0,0,0,5}^{r}y_{\beta}\left(5x_{\beta}^{4}-10x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)-b_{0,0,1,5}^{r}\left(x_{\alpha}\left(4x_{\beta}^{3}y_{\beta}-4x_{\beta}y_{\beta}^{3}\right)+y_{\alpha}\left(x_{\beta}^{4}-6x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)\right)\\
& +b_{0,0,2,-1}^{r}\left(-x_{\alpha}^{2}y_{\beta}^{3}+x_{\alpha}\left(-2x_{\beta}^{3}y_{\alpha}+6x_{\beta}y_{\alpha}y_{\beta}^{2}\right)+y_{\alpha}^{2}y_{\beta}^{3}+y_{\beta}\left(3x_{\alpha}^{2}x_{\beta}^{2}-3x_{\beta}^{2}y_{\alpha}^{2}\right)\right)\\
& -b_{0,0,2,5}^{r}\left(-x_{\alpha}^{2}y_{\beta}^{3}+x_{\alpha}\left(2x_{\beta}^{3}y_{\alpha}-6x_{\beta}y_{\alpha}y_{\beta}^{2}\right)+y_{\alpha}^{2}y_{\beta}^{3}+y_{\beta}\left(3x_{\alpha}^{2}x_{\beta}^{2}-3x_{\beta}^{2}y_{\alpha}^{2}\right)\right)\\
& -b_{0,0,3,5}^{r}\left(2x_{\alpha}^{3}x_{\beta}y_{\beta}-6x_{\alpha}x_{\beta}y_{\alpha}^{2}y_{\beta}-x_{\beta}^{2}y_{\alpha}^{3}+y_{\alpha}^{3}y_{\beta}^{2}+y_{\alpha}\left(3x_{\alpha}^{2}x_{\beta}^{2}-3x_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& -b_{0,0,4,5}^{r}\left(4x_{\alpha}^{3}x_{\beta}y_{\alpha}-4x_{\alpha}x_{\beta}y_{\alpha}^{3}+y_{\beta}\left(x_{\alpha}^{4}-6x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\right)-b_{0,0,5,5}^{r}y_{\alpha}\left(5x_{\alpha}^{4}-10x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\\
& -b_{0,2,-2,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(-2x_{\alpha}x_{\beta}y_{\alpha}+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)+b_{0,2,1,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(2x_{\alpha}x_{\beta}y_{\beta}+y_{\alpha}\left(-x_{\beta}^{2}+y_{\beta}^{2}\right)\right)\\
& +b_{0,4,-1,-1}^{r}y_{\alpha}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}+b_{0,4,0,-1}^{r}y_{\beta}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}-b_{2,0,-2,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(-2x_{\alpha}x_{\beta}y_{\alpha}+y_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\\
& +b_{2,0,1,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(2x_{\alpha}x_{\beta}y_{\beta}+y_{\alpha}\left(-x_{\beta}^{2}+y_{\beta}^{2}\right)\right)+b_{2,2,-1,-1}^{r}y_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\\
& +b_{2,2,0,-1}^{r}y_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+b_{4,0,-1,-1}^{r}y_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}+b_{4,0,0,-1}^{r}y_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
H_{YY}^{(5)} = & a_{0,0,-1,3}^{r}\left(x_{\alpha}\left(x_{\beta}^{4}-6x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)+y_{\alpha}\left(4x_{\beta}^{3}y_{\beta}-4x_{\beta}y_{\beta}^{3}\right)\right)+a_{0,0,4,3}^{r}\left(4x_{\alpha}^{3}y_{\alpha}y_{\beta}-4x_{\alpha}y_{\alpha}^{3}y_{\beta}+x_{\beta}\left(x_{\alpha}^{4}-6x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\right)\\
& +a_{0,2,0,3}^{r}x_{\beta}\left(x_{\beta}^{2}-3y_{\beta}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+a_{0,2,1,3}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)-2x_{\beta}y_{\alpha}y_{\beta}\right)\\
& +a_{0,2,2,3}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(-2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)+a_{0,2,3,3}^{r}x_{\alpha}\left(x_{\alpha}^{2}-3y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)+a_{2,0,0,3}^{r}x_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}-3y_{\beta}^{2}\right)\\
& +a_{2,0,1,3}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)-2x_{\beta}y_{\alpha}y_{\beta}\right)+a_{2,0,2,3}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(-2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\\
& +a_{2,0,3,3}^{r}x_{\alpha}\left(x_{\alpha}^{2}-3y_{\alpha}^{2}\right)\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)-b_{0,0,-3,-1}^{r}\left(x_{\alpha}^{3}x_{\beta}^{2}-x_{\alpha}^{3}y_{\beta}^{2}+6x_{\alpha}^{2}x_{\beta}y_{\alpha}y_{\beta}+x_{\alpha}\left(-3x_{\beta}^{2}y_{\alpha}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)-2x_{\beta}y_{\alpha}^{3}y_{\beta}\right)\\
& -b_{0,0,0,5}^{r}x_{\beta}\left(x_{\beta}^{4}-10x_{\beta}^{2}y_{\beta}^{2}+5y_{\beta}^{4}\right)-b_{0,0,1,5}^{r}\left(x_{\alpha}\left(x_{\beta}^{4}-6x_{\beta}^{2}y_{\beta}^{2}+y_{\beta}^{4}\right)+y_{\alpha}\left(-4x_{\beta}^{3}y_{\beta}+4x_{\beta}y_{\beta}^{3}\right)\right)\\
& -b_{0,0,2,-1}^{r}\left(x_{\alpha}^{2}x_{\beta}^{3}+x_{\alpha}\left(6x_{\beta}^{2}y_{\alpha}y_{\beta}-2y_{\alpha}y_{\beta}^{3}\right)-x_{\beta}^{3}y_{\alpha}^{2}+x_{\beta}\left(-3x_{\alpha}^{2}y_{\beta}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& -b_{0,0,2,5}^{r}\left(x_{\alpha}^{2}x_{\beta}^{3}+x_{\alpha}\left(-6x_{\beta}^{2}y_{\alpha}y_{\beta}+2y_{\alpha}y_{\beta}^{3}\right)-x_{\beta}^{3}y_{\alpha}^{2}+x_{\beta}\left(-3x_{\alpha}^{2}y_{\beta}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)\right)\\
& -b_{0,0,3,5}^{r}\left(x_{\alpha}^{3}x_{\beta}^{2}-x_{\alpha}^{3}y_{\beta}^{2}-6x_{\alpha}^{2}x_{\beta}y_{\alpha}y_{\beta}+x_{\alpha}\left(-3x_{\beta}^{2}y_{\alpha}^{2}+3y_{\alpha}^{2}y_{\beta}^{2}\right)+2x_{\beta}y_{\alpha}^{3}y_{\beta}\right)\\
& -b_{0,0,4,5}^{r}\left(-4x_{\alpha}^{3}y_{\alpha}y_{\beta}+4x_{\alpha}y_{\alpha}^{3}y_{\beta}+x_{\beta}\left(x_{\alpha}^{4}-6x_{\alpha}^{2}y_{\alpha}^{2}+y_{\alpha}^{4}\right)\right)-b_{0,0,5,5}^{r}x_{\alpha}\left(x_{\alpha}^{4}-10x_{\alpha}^{2}y_{\alpha}^{2}+5y_{\alpha}^{4}\right)\\
& -b_{0,2,-2,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)-b_{0,2,1,-1}^{r}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)+2x_{\beta}y_{\alpha}y_{\beta}\right)\\
& -b_{0,4,-1,-1}^{r}x_{\alpha}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}-b_{0,4,0,-1}^{r}x_{\beta}\left(x_{\beta}^{2}+y_{\beta}^{2}\right)^{2}-b_{2,0,-2,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(2x_{\alpha}y_{\alpha}y_{\beta}+x_{\beta}\left(x_{\alpha}^{2}-y_{\alpha}^{2}\right)\right)\\
& -b_{2,0,1,-1}^{r}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\alpha}\left(x_{\beta}^{2}-y_{\beta}^{2}\right)+2x_{\beta}y_{\alpha}y_{\beta}\right)-b_{2,2,-1,-1}^{r}x_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)\\
& -b_{2,2,0,-1}^{r}x_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)\left(x_{\beta}^{2}+y_{\beta}^{2}\right)-b_{4,0,-1,-1}^{r}x_{\alpha}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}-b_{4,0,0,-1}^{r}x_{\beta}\left(x_{\alpha}^{2}+y_{\alpha}^{2}\right)^{2}
\end{aligned}
$$

## 5.7 Order: 6

Number of terms: $H_{XX}$: 44 (17 from $H_{++}$, 27 from $H_{+-}$), $H_{XY}$: 27 (all from $H_{+-}$), $H_{YY}$: 27 (all from $H_{+-}$).

**Polar e-coordinates:**

$$
\begin{aligned}
H_{XX}^{(6)} = & a_{0,0,0,6}^{r}\rho_{\beta}^{6}\cos\left(6\phi_{\beta}\right)+a_{0,0,1,6}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}+5\phi_{\beta}\right)+a_{0,0,2,6}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}+4\phi_{\beta}\right)+a_{0,0,3,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}-3\phi_{\beta}\right)\\
& +a_{0,0,3,6}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}+3\phi_{\beta}\right)+a_{0,0,4,6}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}+2\phi_{\beta}\right)+a_{0,0,5,6}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(5\phi_{\alpha}+\phi_{\beta}\right)+a_{0,0,6,6}^{r}\rho_{\alpha}^{6}\cos\left(6\phi_{\alpha}\right)\\
& +a_{0,2,2,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}-2\phi_{\beta}\right)+a_{0,4,1,0}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}-\phi_{\beta}\right)+a_{0,6,0,0}^{r}\rho_{\beta}^{6}+a_{2,0,2,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}-2\phi_{\beta}\right)\\
& +a_{2,2,1,0}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}-\phi_{\beta}\right)+a_{2,4,0,0}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}+a_{4,0,1,0}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(\phi_{\alpha}-\phi_{\beta}\right)+a_{4,2,0,0}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}+a_{6,0,0,0}^{r}\rho_{\alpha}^{6}\\
& +b_{0,0,-2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}-4\phi_{\beta}\right)+b_{0,0,-5,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(5\phi_{\alpha}-\phi_{\beta}\right)+b_{0,0,1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}-5\phi_{\beta}\right)\\
& +b_{0,0,4,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}-2\phi_{\beta}\right)+b_{0,2,-1,-4}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}+3\phi_{\beta}\right)+b_{0,2,-1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}-3\phi_{\beta}\right)\\
& +b_{0,2,-2,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}+2\phi_{\beta}\right)+b_{0,2,-3,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}+\phi_{\beta}\right)+b_{0,2,-4,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(4\phi_{\alpha}\right)\\
& +b_{0,2,0,-4}^{r}\rho_{\beta}^{6}\cos\left(4\phi_{\beta}\right)+b_{0,2,3,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(3\phi_{\alpha}-\phi_{\beta}\right)+b_{0,4,0,2}^{r}\rho_{\beta}^{6}\cos\left(2\phi_{\beta}\right)+b_{0,4,1,2}^{r}\rho_{\alpha}\rho_{\beta}^{5}\cos\left(\phi_{\alpha}+\phi_{\beta}\right)\\
& +b_{0,4,2,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\alpha}\right)+b_{2,0,-1,-4}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}+3\phi_{\beta}\right)+b_{2,0,-1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}-3\phi_{\beta}\right)\\
& +b_{2,0,-2,-4}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}+2\phi_{\beta}\right)+b_{2,0,-3,-4}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(3\phi_{\alpha}+\phi_{\beta}\right)+b_{2,0,-4,-4}^{r}\rho_{\alpha}^{6}\cos\left(4\phi_{\alpha}\right)\\
& +b_{2,0,0,-4}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(4\phi_{\beta}\right)+b_{2,0,3,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(3\phi_{\alpha}-\phi_{\beta}\right)+b_{2,2,0,2}^{r}\rho_{\alpha}^{2}\rho_{\beta}^{4}\cos\left(2\phi_{\beta}\right)+b_{2,2,1,2}^{r}\rho_{\alpha}^{3}\rho_{\beta}^{3}\cos\left(\phi_{\alpha}+\phi_{\beta}\right)\\
& +b_{2,2,2,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\alpha}\right)+b_{4,0,0,2}^{r}\rho_{\alpha}^{4}\rho_{\beta}^{2}\cos\left(2\phi_{\beta}\right)+b_{4,0,1,2}^{r}\rho_{\alpha}^{5}\rho_{\beta}\cos\left(\phi_{\alpha}+\phi_{\beta}\right)+b_{4,0,2,2}^{r}\rho_{\alpha}^{6}\cos\left(2\phi_{\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{XY} = {} & b^r_{0,0,-2,2}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\alpha-4\phi_\beta)+b^r_{0,0,-5,-4}\rho^5_\alpha\rho_\beta\sin(5\phi_\alpha-\phi_\beta)-b^r_{0,0,1,-4}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha-5\phi_\beta)\\
& -b^r_{0,0,4,2}\rho^4_\alpha\rho^2_\beta\sin(4\phi_\alpha-2\phi_\beta)+b^r_{0,2,-1,-4}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha+3\phi_\beta)+b^r_{0,2,-1,2}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha-3\phi_\beta)\\
& +b^r_{0,2,-2,-4}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\alpha+2\phi_\beta)+b^r_{0,2,-3,-4}\rho^3_\alpha\rho^3_\beta\sin(3\phi_\alpha+\phi_\beta)+b^r_{0,2,-4,-4}\rho^4_\alpha\rho^2_\beta\sin(4\phi_\alpha)\\
& +b^r_{0,2,0,-4}\rho^6_\beta\sin(4\phi_\beta)-b^r_{0,2,3,2}\rho^3_\alpha\rho^3_\beta\sin(3\phi_\alpha-\phi_\beta)-b^r_{0,4,0,2}\rho^6_\beta\sin(2\phi_\beta)-b^r_{0,4,1,2}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha+\phi_\beta)\\
& -b^r_{0,4,2,2}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\alpha)+b^r_{2,0,-1,-4}\rho^3_\alpha\rho^3_\beta\sin(\phi_\alpha+3\phi_\beta)+b^r_{2,0,-1,2}\rho^3_\alpha\rho^3_\beta\sin(\phi_\alpha-3\phi_\beta)\\
& +b^r_{2,0,-2,-4}\rho^4_\alpha\rho^2_\beta\sin(2\phi_\alpha+2\phi_\beta)+b^r_{2,0,-3,-4}\rho^5_\alpha\rho_\beta\sin(3\phi_\alpha+\phi_\beta)+b^r_{2,0,-4,-4}\rho^6_\alpha\sin(4\phi_\alpha)\\
& +b^r_{2,0,0,-4}\rho^2_\alpha\rho^4_\beta\sin(4\phi_\beta)-b^r_{2,0,3,2}\rho^5_\alpha\rho_\beta\sin(3\phi_\alpha-\phi_\beta)-b^r_{2,2,0,2}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\beta)-b^r_{2,2,1,2}\rho^3_\alpha\rho^3_\beta\sin(\phi_\alpha+\phi_\beta)\\
& -b^r_{2,2,2,2}\rho^4_\alpha\rho^2_\beta\sin(2\phi_\alpha)-b^r_{4,0,0,2}\rho^4_\alpha\rho^2_\beta\sin(2\phi_\beta)-b^r_{4,0,1,2}\rho^5_\alpha\rho_\beta\sin(\phi_\alpha+\phi_\beta)-b^r_{4,0,2,2}\rho^6_\alpha\sin(2\phi_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{YX} = {} & b^r_{0,0,-2,2}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\alpha-4\phi_\beta)+b^r_{0,0,-5,-4}\rho^5_\alpha\rho_\beta\sin(5\phi_\alpha-\phi_\beta)-b^r_{0,0,1,-4}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha-5\phi_\beta)\\
& -b^r_{0,0,4,2}\rho^4_\alpha\rho^2_\beta\sin(4\phi_\alpha-2\phi_\beta)+b^r_{0,2,-1,-4}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha+3\phi_\beta)+b^r_{0,2,-1,2}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha-3\phi_\beta)\\
& +b^r_{0,2,-2,-4}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\alpha+2\phi_\beta)+b^r_{0,2,-3,-4}\rho^3_\alpha\rho^3_\beta\sin(3\phi_\alpha+\phi_\beta)+b^r_{0,2,-4,-4}\rho^4_\alpha\rho^2_\beta\sin(4\phi_\alpha)\\
& +b^r_{0,2,0,-4}\rho^6_\beta\sin(4\phi_\beta)-b^r_{0,2,3,2}\rho^3_\alpha\rho^3_\beta\sin(3\phi_\alpha-\phi_\beta)-b^r_{0,4,0,2}\rho^6_\beta\sin(2\phi_\beta)-b^r_{0,4,1,2}\rho_\alpha\rho^5_\beta\sin(\phi_\alpha+\phi_\beta)\\
& -b^r_{0,4,2,2}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\alpha)+b^r_{2,0,-1,-4}\rho^3_\alpha\rho^3_\beta\sin(\phi_\alpha+3\phi_\beta)+b^r_{2,0,-1,2}\rho^3_\alpha\rho^3_\beta\sin(\phi_\alpha-3\phi_\beta)\\
& +b^r_{2,0,-2,-4}\rho^4_\alpha\rho^2_\beta\sin(2\phi_\alpha+2\phi_\beta)+b^r_{2,0,-3,-4}\rho^5_\alpha\rho_\beta\sin(3\phi_\alpha+\phi_\beta)+b^r_{2,0,-4,-4}\rho^6_\alpha\sin(4\phi_\alpha)\\
& +b^r_{2,0,0,-4}\rho^2_\alpha\rho^4_\beta\sin(4\phi_\beta)-b^r_{2,0,3,2}\rho^5_\alpha\rho_\beta\sin(3\phi_\alpha-\phi_\beta)-b^r_{2,2,0,2}\rho^2_\alpha\rho^4_\beta\sin(2\phi_\beta)-b^r_{2,2,1,2}\rho^3_\alpha\rho^3_\beta\sin(\phi_\alpha+\phi_\beta)\\
& -b^r_{2,2,2,2}\rho^4_\alpha\rho^2_\beta\sin(2\phi_\alpha)-b^r_{4,0,0,2}\rho^4_\alpha\rho^2_\beta\sin(2\phi_\beta)-b^r_{4,0,1,2}\rho^5_\alpha\rho_\beta\sin(\phi_\alpha+\phi_\beta)-b^r_{4,0,2,2}\rho^6_\alpha\sin(2\phi_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
H^{(6)}_{YY} = {} & a^r_{0,0,0,6}\rho^6_\beta\cos(6\phi_\beta)+a^r_{0,0,1,6}\rho_\alpha\rho^5_\beta\cos(\phi_\alpha+5\phi_\beta)+a^r_{0,0,2,6}\rho^2_\alpha\rho^4_\beta\cos(2\phi_\alpha+4\phi_\beta)+a^r_{0,0,3,0}\rho^3_\alpha\rho^3_\beta\cos(3\phi_\alpha-3\phi_\beta)\\
& +a^r_{0,0,3,6}\rho^3_\alpha\rho^3_\beta\cos(3\phi_\alpha+3\phi_\beta)+a^r_{0,0,4,6}\rho^4_\alpha\rho^2_\beta\cos(4\phi_\alpha+2\phi_\beta)+a^r_{0,0,5,6}\rho^5_\alpha\rho_\beta\cos(5\phi_\alpha+\phi_\beta)+a^r_{0,0,6,6}\rho^6_\alpha\cos(6\phi_\alpha)\\
& +a^r_{0,2,2,0}\rho^2_\alpha\rho^4_\beta\cos(2\phi_\alpha-2\phi_\beta)+a^r_{0,4,1,0}\rho_\alpha\rho^5_\beta\cos(\phi_\alpha-\phi_\beta)+a^r_{0,6,0,0}\rho^6_\beta+a^r_{2,0,2,0}\rho^4_\alpha\rho^2_\beta\cos(2\phi_\alpha-2\phi_\beta)\\
& +a^r_{2,2,1,0}\rho^3_\alpha\rho^3_\beta\cos(\phi_\alpha-\phi_\beta)+a^r_{2,4,0,0}\rho^2_\alpha\rho^4_\beta+a^r_{4,0,1,0}\rho^5_\alpha\rho_\beta\cos(\phi_\alpha-\phi_\beta)+a^r_{4,2,0,0}\rho^4_\alpha\rho^2_\beta+a^r_{6,0,0,0}\rho^6_\alpha\\
& -b^r_{0,0,-2,2}\rho^2_\alpha\rho^4_\beta\cos(2\phi_\alpha-4\phi_\beta)-b^r_{0,0,-5,-4}\rho^5_\alpha\rho_\beta\cos(5\phi_\alpha-\phi_\beta)-b^r_{0,0,1,-4}\rho_\alpha\rho^5_\beta\cos(\phi_\alpha-5\phi_\beta)\\
& -b^r_{0,0,4,2}\rho^4_\alpha\rho^2_\beta\cos(4\phi_\alpha-2\phi_\beta)-b^r_{0,2,-1,-4}\rho_\alpha\rho^5_\beta\cos(\phi_\alpha+3\phi_\beta)-b^r_{0,2,-1,2}\rho_\alpha\rho^5_\beta\cos(\phi_\alpha-3\phi_\beta)\\
& -b^r_{0,2,-2,-4}\rho^2_\alpha\rho^4_\beta\cos(2\phi_\alpha+2\phi_\beta)-b^r_{0,2,-3,-4}\rho^3_\alpha\rho^3_\beta\cos(3\phi_\alpha+\phi_\beta)-b^r_{0,2,-4,-4}\rho^4_\alpha\rho^2_\beta\cos(4\phi_\alpha)\\
& -b^r_{0,2,0,-4}\rho^6_\beta\cos(4\phi_\beta)-b^r_{0,2,3,2}\rho^3_\alpha\rho^3_\beta\cos(3\phi_\alpha-\phi_\beta)-b^r_{0,4,0,2}\rho^6_\beta\cos(2\phi_\beta)-b^r_{0,4,1,2}\rho_\alpha\rho^5_\beta\cos(\phi_\alpha+\phi_\beta)\\
& -b^r_{0,4,2,2}\rho^2_\alpha\rho^4_\beta\cos(2\phi_\alpha)-b^r_{2,0,-1,-4}\rho^3_\alpha\rho^3_\beta\cos(\phi_\alpha+3\phi_\beta)-b^r_{2,0,-1,2}\rho^3_\alpha\rho^3_\beta\cos(\phi_\alpha-3\phi_\beta)\\
& -b^r_{2,0,-2,-4}\rho^4_\alpha\rho^2_\beta\cos(2\phi_\alpha+2\phi_\beta)-b^r_{2,0,-3,-4}\rho^5_\alpha\rho_\beta\cos(3\phi_\alpha+\phi_\beta)-b^r_{2,0,-4,-4}\rho^6_\alpha\cos(4\phi_\alpha)\\
& -b^r_{2,0,0,-4}\rho^2_\alpha\rho^4_\beta\cos(4\phi_\beta)-b^r_{2,0,3,2}\rho^5_\alpha\rho_\beta\cos(3\phi_\alpha-\phi_\beta)-b^r_{2,2,0,2}\rho^2_\alpha\rho^4_\beta\cos(2\phi_\beta)-b^r_{2,2,1,2}\rho^3_\alpha\rho^3_\beta\cos(\phi_\alpha+\phi_\beta)\\
& -b^r_{2,2,2,2}\rho^4_\alpha\rho^2_\beta\cos(2\phi_\alpha)-b^r_{4,0,0,2}\rho^4_\alpha\rho^2_\beta\cos(2\phi_\beta)-b^r_{4,0,1,2}\rho^5_\alpha\rho_\beta\cos(\phi_\alpha+\phi_\beta)-b^r_{4,0,2,2}\rho^6_\alpha\cos(2\phi_\alpha)
\end{aligned}
$$

**Cartesian e-coordinates:**

$$
\begin{aligned}
H_{XX}^{(6)} = {} & a_{0,0,0,6}^{r}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 - 4x_\beta y_\beta + y_\beta^2\right)\left(x_\beta^2 + 4x_\beta y_\beta + y_\beta^2\right) \\
& + a_{0,0,1,6}^{r}\left(x_\alpha\left(x_\beta^5 - 10x_\beta^3 y_\beta^2 + 5x_\beta y_\beta^4\right) + y_\alpha\left(-5x_\beta^4 y_\beta + 10x_\beta^2 y_\beta^3 - y_\beta^5\right)\right) \\
& + a_{0,0,2,6}^{r}\left(x_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(-x_\beta^2 - 2x_\beta y_\beta + y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\right) \\
& + a_{0,0,3,0}^{r}\left(x_\alpha x_\beta + y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 + 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right) \\
& + a_{0,0,3,6}^{r}\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 - 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right) + a_{0,0,4,6}^{r}\left(x_\alpha\left(-2x_\beta y_\alpha - 2y_\alpha y_\beta\right)\right. \\
& \qquad \left. + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(-x_\alpha^2 + y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha - 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + a_{0,0,5,6}^{r}\left(-5x_\alpha^4 y_\alpha y_\beta + 5x_\alpha x_\beta y_\alpha^4 + x_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right) + y_\beta\left(10x_\alpha^2 y_\alpha^3 - y_\alpha^5\right)\right) \\
& + a_{0,0,6,6}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 - 4x_\alpha y_\alpha + y_\alpha^2\right)\left(x_\alpha^2 + 4x_\alpha y_\alpha + y_\alpha^2\right) \\
& + a_{0,2,2,0}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) + a_{0,4,1,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha x_\beta + y_\alpha y_\beta\right) \\
& + a_{0,6,0,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^3 + a_{2,0,2,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) \\
& + a_{2,2,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,4,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)^2 \\
& + a_{4,0,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{4,2,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\beta^2 + y_\beta^2\right) + a_{6,0,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^3 \\
& + b_{0,0,-2,2}^{r}\left(x_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(-x_\beta^2 + 2x_\beta y_\beta + y_\beta^2\right)\right) \\
& + b_{0,0,-5,-4}^{r}\left(5x_\alpha^4 y_\alpha y_\beta + 5x_\alpha x_\beta y_\alpha^4 + x_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right) + y_\beta\left(-10x_\alpha^2 y_\alpha^3 + y_\alpha^5\right)\right) \\
& + b_{0,0,1,-4}^{r}\left(x_\alpha\left(x_\beta^5 - 10x_\beta^3 y_\beta^2 + 5x_\beta y_\beta^4\right) + y_\alpha\left(5x_\beta^4 y_\beta - 10x_\beta^2 y_\beta^3 + y_\beta^5\right)\right) \\
& + b_{0,0,4,2}^{r}\left(x_\alpha\left(-2x_\beta y_\alpha + 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha + 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right. \\
& \qquad \left. + y_\beta\left(-x_\alpha^2 + y_\alpha^2\right)\right) + b_{0,2,-1,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) \\
& + b_{0,2,-1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) \\
& + b_{0,2,-2,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& + b_{0,2,-3,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) \\
& + b_{0,2,-4,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) \\
& + b_{0,2,0,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + b_{0,2,3,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) \\
& + b_{0,4,0,2}^{r}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right)^2 + b_{0,4,1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha x_\beta - y_\alpha y_\beta\right) \\
& + b_{0,4,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right)^2 + b_{2,0,-1,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) \\
& + b_{2,0,-1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) \\
& + b_{2,0,-2,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& + b_{2,0,-3,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) \\
& + b_{2,0,-4,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) \\
& + b_{2,0,0,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) \\
& + b_{2,0,3,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) + b_{2,2,0,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + b_{2,2,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right) + b_{2,2,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) \\
& + b_{4,0,0,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) + b_{4,0,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha x_\beta - y_\alpha y_\beta\right) + b_{4,0,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)^2
\end{aligned}
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\begin{aligned}
H^{(6)}_{XY} = & -2b^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)+2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta+y_\alpha\left(-x_\beta^2+y_\beta^2\right)\right)\\
& -b^r_{0,0,-5,-4}\left(-5x_\alpha^4x_\beta y_\alpha+5x_\alpha y_\alpha^4y_\beta+x_\beta\left(10x_\alpha^2y_\alpha^3-y_\alpha^5\right)+y_\beta\left(x_\alpha^5-10x_\alpha^3y_\alpha^2\right)\right)\\
& +b^r_{0,0,1,-4}\left(x_\alpha\left(5x_\beta^4y_\beta-10x_\beta^2y_\beta^3+y_\beta^5\right)+y_\alpha\left(-x_\beta^5+10x_\beta^3y_\beta^2-5x_\beta y_\beta^4\right)\right)\\
& +2b^r_{0,0,4,2}\left(-2x_\alpha x_\beta y_\alpha+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\\
& +b^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& -b^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)+2b^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& +b^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)+4b^r_{0,2,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)\\
& +4b^r_{0,2,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)\\
& -2b^r_{0,4,0,2}x_\beta y_\beta\left(x_\beta^2+y_\beta^2\right)^2-b^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& -2b^r_{0,4,2,2}x_\alpha y_\alpha\left(x_\beta^2+y_\beta^2\right)^2+b^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& -b^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)+2b^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& +b^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)+4b^r_{2,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)\\
& +4b^r_{2,0,0,-4}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)\\
& -2b^r_{2,2,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)-b^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2b^r_{2,2,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& -2b^r_{4,0,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)^2-b^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2b^r_{4,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)^2
\end{aligned}
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\begin{aligned}
H^{(6)}_{YX} = & -2b^r_{0,0,-2,2}\left(x_\alpha\left(x_\beta^2-y_\beta^2\right)+2x_\beta y_\alpha y_\beta\right)\left(2x_\alpha x_\beta y_\beta+y_\alpha\left(-x_\beta^2+y_\beta^2\right)\right)\\
& -b^r_{0,0,-5,-4}\left(-5x_\alpha^4x_\beta y_\alpha+5x_\alpha y_\alpha^4y_\beta+x_\beta\left(10x_\alpha^2y_\alpha^3-y_\alpha^5\right)+y_\beta\left(x_\alpha^5-10x_\alpha^3y_\alpha^2\right)\right)\\
& +b^r_{0,0,1,-4}\left(x_\alpha\left(5x_\beta^4y_\beta-10x_\beta^2y_\beta^3+y_\beta^5\right)+y_\alpha\left(-x_\beta^5+10x_\beta^3y_\beta^2-5x_\beta y_\beta^4\right)\right)\\
& +2b^r_{0,0,4,2}\left(-2x_\alpha x_\beta y_\alpha+y_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\left(2x_\alpha y_\alpha y_\beta+x_\beta\left(x_\alpha^2-y_\alpha^2\right)\right)\\
& +b^r_{0,2,-1,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& -b^r_{0,2,-1,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)+2b^r_{0,2,-2,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& +b^r_{0,2,-3,-4}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)+4b^r_{0,2,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\beta^2+y_\beta^2\right)\\
& +4b^r_{0,2,0,-4}x_\beta y_\beta\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)\left(x_\beta^2+y_\beta^2\right)+b^r_{0,2,3,2}\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)\\
& -2b^r_{0,4,0,2}x_\beta y_\beta\left(x_\beta^2+y_\beta^2\right)^2-b^r_{0,4,1,2}\left(x_\beta^2+y_\beta^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& -2b^r_{0,4,2,2}x_\alpha y_\alpha\left(x_\beta^2+y_\beta^2\right)^2+b^r_{2,0,-1,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(x_\beta^3-3x_\beta y_\beta^2\right)\right)\\
& -b^r_{2,0,-1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha\left(3x_\beta^2y_\beta-y_\beta^3\right)+y_\alpha\left(-x_\beta^3+3x_\beta y_\beta^2\right)\right)+2b^r_{2,0,-2,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha x_\beta-y_\alpha y_\beta\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)\\
& +b^r_{2,0,-3,-4}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta+3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta-x_\beta y_\alpha^3\right)+4b^r_{2,0,-4,-4}x_\alpha y_\alpha\left(x_\alpha-y_\alpha\right)\left(x_\alpha+y_\alpha\right)\left(x_\alpha^2+y_\alpha^2\right)\\
& +4b^r_{2,0,0,-4}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta-y_\beta\right)\left(x_\beta+y_\beta\right)+b^r_{2,0,3,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\alpha^3y_\beta-3x_\alpha^2x_\beta y_\alpha-3x_\alpha y_\alpha^2y_\beta+x_\beta y_\alpha^3\right)\\
& -2b^r_{2,2,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)-b^r_{2,2,1,2}\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2b^r_{2,2,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)\left(x_\beta^2+y_\beta^2\right)\\
& -2b^r_{4,0,0,2}x_\beta y_\beta\left(x_\alpha^2+y_\alpha^2\right)^2-b^r_{4,0,1,2}\left(x_\alpha^2+y_\alpha^2\right)^2\left(x_\alpha y_\beta+x_\beta y_\alpha\right)-2b^r_{4,0,2,2}x_\alpha y_\alpha\left(x_\alpha^2+y_\alpha^2\right)^2
\end{aligned}
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\begin{aligned}
H_{YY}^{(6)} = {} & a_{0,0,0,6}^{r}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 - 4x_\beta y_\beta + y_\beta^2\right)\left(x_\beta^2 + 4x_\beta y_\beta + y_\beta^2\right) \\
& + a_{0,0,1,6}^{r}\left(x_\alpha\left(x_\beta^5 - 10x_\beta^3 y_\beta^2 + 5x_\beta y_\beta^4\right) + y_\alpha\left(-5x_\beta^4 y_\beta + 10x_\beta^2 y_\beta^3 - y_\beta^5\right)\right) \\
& + a_{0,0,2,6}^{r}\left(x_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(-x_\beta^2 - 2x_\beta y_\beta + y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\right) \\
& + a_{0,0,3,0}^{r}\left(x_\alpha x_\beta + y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 + 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right) \\
& + a_{0,0,3,6}^{r}\left(x_\alpha x_\beta - y_\alpha y_\beta\right)\left(x_\alpha^2 x_\beta^2 - 3x_\alpha^2 y_\beta^2 - 8x_\alpha x_\beta y_\alpha y_\beta - 3x_\beta^2 y_\alpha^2 + y_\alpha^2 y_\beta^2\right) + a_{0,0,4,6}^{r}\left(x_\alpha\left(-2x_\beta y_\alpha - 2y_\alpha y_\beta\right)\right. \\
& \qquad \left. + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(-x_\alpha^2 + y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha - 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right) \\
& + a_{0,0,5,6}^{r}\left(-5x_\alpha^4 y_\alpha y_\beta + 5x_\alpha x_\beta y_\alpha^4 + x_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right) + y_\beta\left(10x_\alpha^2 y_\alpha^3 - y_\alpha^5\right)\right) \\
& + a_{0,0,6,6}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 - 4x_\alpha y_\alpha + y_\alpha^2\right)\left(x_\alpha^2 + 4x_\alpha y_\alpha + y_\alpha^2\right) \\
& + a_{0,2,2,0}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) + a_{0,4,1,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha x_\beta + y_\alpha y_\beta\right) \\
& + a_{0,6,0,0}^{r}\left(x_\beta^2 + y_\beta^2\right)^3 + a_{2,0,2,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(x_\beta + y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(-x_\beta + y_\beta\right)\right) \\
& + a_{2,2,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{2,4,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)^2 \\
& + a_{4,0,1,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha x_\beta + y_\alpha y_\beta\right) + a_{4,2,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\beta^2 + y_\beta^2\right) + a_{6,0,0,0}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^3 \\
& - b_{0,0,-2,2}^{r}\left(x_\alpha\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right)\right)\left(x_\alpha\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) + y_\alpha\left(-x_\beta^2 + 2x_\beta y_\beta + y_\beta^2\right)\right) \\
& - b_{0,0,-5,-4}^{r}\left(5x_\alpha^4 y_\alpha y_\beta + 5x_\alpha x_\beta y_\alpha^4 + x_\beta\left(x_\alpha^5 - 10x_\alpha^3 y_\alpha^2\right) + y_\beta\left(-10x_\alpha^2 y_\alpha^3 + y_\alpha^5\right)\right) \\
& - b_{0,0,1,-4}^{r}\left(x_\alpha\left(x_\beta^5 - 10x_\beta^3 y_\beta^2 + 5x_\beta y_\beta^4\right) + y_\alpha\left(5x_\beta^4 y_\beta - 10x_\beta^2 y_\beta^3 + y_\beta^5\right)\right) \\
& - b_{0,0,4,2}^{r}\left(x_\alpha\left(-2x_\beta y_\alpha + 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right) + y_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right)\left(x_\alpha\left(2x_\beta y_\alpha + 2y_\alpha y_\beta\right) + x_\beta\left(x_\alpha^2 - y_\alpha^2\right)\right. \\
& \qquad \left. + y_\beta\left(-x_\alpha^2 + y_\alpha^2\right)\right) - b_{0,2,-1,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) \\
& - b_{0,2,-1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) \\
& - b_{0,2,-2,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& - b_{0,2,-3,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) \\
& - b_{0,2,-4,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) \\
& - b_{0,2,0,-4}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) - b_{0,2,3,2}^{r}\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) \\
& - b_{0,4,0,2}^{r}\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right)^2 - b_{0,4,1,2}^{r}\left(x_\beta^2 + y_\beta^2\right)^2\left(x_\alpha x_\beta - y_\alpha y_\beta\right) \\
& - b_{0,4,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\beta^2 + y_\beta^2\right)^2 - b_{2,0,-1,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(-3x_\beta^2 y_\beta + y_\beta^3\right)\right) \\
& - b_{2,0,-1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta^3 - 3x_\beta y_\beta^2\right) + y_\alpha\left(3x_\beta^2 y_\beta - y_\beta^3\right)\right) \\
& - b_{2,0,-2,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha\left(x_\beta - y_\beta\right) + y_\alpha\left(-x_\beta - y_\beta\right)\right)\left(x_\alpha\left(x_\beta + y_\beta\right) + y_\alpha\left(x_\beta - y_\beta\right)\right) \\
& - b_{2,0,-3,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 x_\beta - 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 + y_\alpha^3 y_\beta\right) \\
& - b_{2,0,-4,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^2 - 2x_\alpha y_\alpha - y_\alpha^2\right)\left(x_\alpha^2 + 2x_\alpha y_\alpha - y_\alpha^2\right) \\
& - b_{2,0,0,-4}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 - 2x_\beta y_\beta - y_\beta^2\right)\left(x_\beta^2 + 2x_\beta y_\beta - y_\beta^2\right) \\
& - b_{2,0,3,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\alpha^3 x_\beta + 3x_\alpha^2 y_\alpha y_\beta - 3x_\alpha x_\beta y_\alpha^2 - y_\alpha^3 y_\beta\right) - b_{2,2,0,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right)\left(x_\beta^2 + y_\beta^2\right) \\
& - b_{2,2,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right)\left(x_\alpha x_\beta - y_\alpha y_\beta\right) - b_{2,2,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)\left(x_\beta^2 + y_\beta^2\right) \\
& - b_{4,0,0,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\beta - y_\beta\right)\left(x_\beta + y_\beta\right) - b_{4,0,1,2}^{r}\left(x_\alpha^2 + y_\alpha^2\right)^2\left(x_\alpha x_\beta - y_\alpha y_\beta\right) - b_{4,0,2,2}^{r}\left(x_\alpha - y_\alpha\right)\left(x_\alpha + y_\alpha\right)\left(x_\alpha^2 + y_\alpha^2\right)^2
\end{aligned}
$$