Sheet 7: Representations of $SU(3)$ and beyond

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Question 1.
Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(3, \mathbf{C})$ (this will work more generally for any complex semisimple Lie algebra). Suppose that $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ and $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(W)$ are irreducible representations of $\mathfrak{g}$ both having highest weight λ.

(a) Let v be a highest weight vector in V and w a highest weight vector in W. Show that $v \oplus w$ is a highest weight vector in $V \oplus W$.

(b) Deduce that there is an irreducible subrepresentation $U \subset V \oplus W$ containing $v \oplus w$, and that the projections $U \to V$ and $U \to W$ are isomorphisms. Hence deduce that V is isomorphic to W. (*Hint: Use Schur's lemma.*)

Question 2.
Let $\rho\colon \mathfrak{su}(3) \to \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{su}(3)$ and v be a highest weight vector. Let D be the set of words in $\rho(E_{21}), \rho(E_{31}), \rho(E_{32})$ and let $W \subset V$ be the subspace spanned by the subset $\{wv \ : \ w \in D\}$. By induction on the length of w, show that W is preserved by $\rho(E_{12}), \rho(E_{13}), \rho(E_{23})$.

Question 3.
Let $\mathbf{C}^3$ denote the standard representation of $SU(3)$ and $\Gamma_{a,b}$ denote the unique irreducible representation with highest weight $aL_1 - bL_3$. We will see below that $\Gamma_{a,b}$ exists.

(a) Show that $\mathrm{Sym}^a \mathbf{C}^3$ is $\Gamma_{a,0}$ and $\mathrm{Sym}^b(\mathbf{C}^3)^*$ is $\Gamma_{0,b}$ and draw the weight diagrams.

(b) Prove that $\mathrm{Sym}^a \mathbf{C}^3 \otimes \mathrm{Sym}^b(\mathbf{C}^3)^*$ contains an irreducible summand isomorphic to $\Gamma_{a,b}$. This proves existence of an irreducible representation with given highest weight vector $aL_1 - bL_3$.

(c) Decompose $\mathbf{C}^3 \otimes (\mathbf{C}^3)^*$ into irreducible subrepresentations.

(d) Decompose $\mathrm{Sym}^2 \mathbf{C}^3 \otimes (\mathbf{C}^3)^*$ into irreducible subrepresentations.

(e) Decompose $\mathbf{C}^3 \otimes \Gamma_{2,1}$ into irreducible subrepresentations.

(f) Decompose $(\mathbf{C}^3)^{\otimes 3}$ into irreducible subrepresentations.

(g) Decompose $\Gamma_{1,1} \otimes \Gamma_{1,2}$ into irreducible subrepresentations.

Question 4. Let $\mathfrak{so}(5, \mathbf{C})$ be the Lie algebra of antisymmetric complex 5-by-5 matrices (the complexification of $\mathfrak{so}(5)$). What is its dimension?

You may use a computer algebra system for the rest of this question.

Consider the abelian Lie subalgebra $\mathfrak{t}$ spanned by elements

$$H_1 = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let L_1, L_2 be a $\mathbf{Z}$-basis for the weight lattice $\mathfrak{t}^*_{\mathbf{Z}}$ given by

$$L_i(H_j) = \delta_{ij}.$$

By considering the adjoint action of H_1 and H_2 on the eight matrices

$$K_1^{\pm} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \pm i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & \mp i & 0 & 0 & 0 \end{pmatrix}, \quad K_2^{\pm} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \pm i \\ 0 & 0 & 1 & \mp i & 0 \end{pmatrix}$$

$$L^{\pm} = \begin{pmatrix} 0 & 0 & -1 & \pm i & 0 \\ 0 & 0 & \pm i & 1 & 0 \\ 1 & \mp i & 0 & 0 & 0 \\ \mp i & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^{\pm} = \begin{pmatrix} 0 & 0 & \pm i & 1 & 0 \\ 0 & 0 & -1 & \pm i & 0 \\ \mp i & 1 & 0 & 0 & 0 \\ -1 & \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

find the roots in terms of L_1 and L_2 and draw the root diagram in $\mathfrak{t}^*_{\mathbf{Z}}$.

You can think of $\mathfrak{t}^*_{\mathbf{Z}}$ as the usual square lattice in $\mathbf{R}^2$.

Question 5. Let V denote the standard 4-dimensional complex representation of $SU(4)$.

(a) Decompose $V \otimes V$ into its irreducible pieces.

(b) Decompose $V \otimes V^*$ into its irreducible pieces.