Sheet 2: Matrix groups

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Question 1.
Prove that $A^T = -A$ if and only if $\exp(tA) \in O(n)$ for all $t \in \mathbf{R}$. This implies that the Lie algebra of $O(n)$ is the space $\mathfrak{o}(n)$ of antisymmetric matrices.

Answer 1. If $A^T = -A$ then $(\exp(tA))^T = \left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n\right)^T = \sum_{n=0}^{\infty} \frac{1}{n!} (A^T)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n A^n = \exp(-tA)$ hence $\exp(tA) \left(\exp(tA)\right)^T = 1$ and $\exp(tA) \in O(n)$.

Conversely if $\exp(tA) \left(\exp(tA)\right)^T = 1$ for all t then differentiating with respect to t at $t = 0$ we get

$$A + A^T = 0$$

and hence $A^T = -A$.

Question 2. Show that the third order term in the Baker-Campbell-Hausdorff formula for $\log(\exp A \exp B)$ is

$$\frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]].$$

If you think this is getting messy, it just means you're on the right track. Everything will simplify beautifully.

Answer 2. Let

$$e^A e^B - 1 = X = \left(1 + A + A^2/2 + A^3/3! + \cdots\right)\left(1 + B + B^2/2 + B^3/3! + \cdots\right) - 1$$

and compute

$$\log(e^A e^B) = \log(1+X) = X - X^2/2 + X^3/3 + \cdots$$

to third order. Thereafter, good luck. Be careful to note that A and B do not commute.

Question 3.

(a) Suppose that W_1 and W_2 are complementary subspaces of $\mathfrak{gl}(n, \mathbf{R})$, so that $\mathfrak{gl}(n, \mathbf{R}) = W_1 \oplus W_2$. Consider the map

$$F\colon W_1 \oplus W_2 \to GL(n, \mathbf{R}), \quad F(w_1 \oplus w_2) = \exp(w_1)\exp(w_2).$$

By computing the Taylor expansion of F (plugging in the Taylor expansions of $\exp(w_1)$ and $\exp(w_2)$ and expanding), show that $d_{(0,0)}F(w_1 \oplus w_2) = w_1 + w_2$.

(b) Deduce that there are open neighbourhoods $0 \in U' \subset \mathfrak{gl}(n, \mathbf{R})$ and $1 \in V' \subset GL(n, \mathbf{R})$ such that $F|_{U'}\colon U' \to V'$ is a diffeomorphism.

(c) Let $G \subset GL(n, \mathbf{R})$ be a matrix group, set $W_1 = \mathfrak{g}$ and let W_2 be a complement for V. Given $g \in G$, define

$$F_g\colon \mathfrak{g} \oplus W_2 \to GL(n, \mathbf{R}), \quad F_g(w_1 \oplus w_2) = g\exp(w_1)\exp(w_2)$$

Show that there are open sets $0 \in U' \subset \mathfrak{gl}(n, \mathbf{R})$ and $g \in V' \subset GL(n, \mathbf{R})$ such that $F_g|_{U'}$ is a diffeomorphism.

Answer 3. (a) We have $F(w_1 \oplus w_2) = (1 + w_1 + \cdots)(1 + w_2 + \cdots) = 1 + w_1 + w_2 + \cdots$ so the first order term in the Taylor expansion is $w_1 + w_2$ (i.e. the differential $d_{(0,0)}F(w_1 \oplus w_2) = w_1 + w_2$ is the identity).

(b) Since the differential is the identity and in particular invertible, the inverse function theorem guarantees the existence of a local smooth inverse for F restricted to a sufficiently small neighbourhood of $0 \in W_1 \oplus W_2$.

(c) In this case the differential is $d_{(0,0)}F_g(w_1 \oplus w_2) = gw_1 + gw_2$ in other words it is multiplication by $g \in GL(n, \mathbf{R})$ which is invertible so the inverse function theorem guarantees that F_g is a local diffeomorphism. This provides exponential charts near every point $g \in G$.

The last two questions study the following formula and its applications.

Lemma 1 (Jacobi formula[1]). *Let $A(t)$ be a path of invertible matrices ($t \in \mathbf{R}$). Then*

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\,\mathrm{Tr}(A^{-1}\dot{A}(t)). \tag{1}$$

Here $\mathrm{Tr}(M)$ denotes the trace of M (the sum of its diagonal entries) and $\dot{A}(t)$ denotes the matrix whose entries are the t-derivatives of the entries of A.

Question 4. (Proof of Jacobi formula)

(a) Show that $\det(1+\epsilon H) = 1 + \epsilon\,\mathrm{Tr}(H) + \mathcal{O}(\epsilon^2)$. *Hint: It might help to write out*

$$\det(1+\epsilon H) = \det\begin{pmatrix} 1+\epsilon H_{11} & \epsilon H_{12} & \cdots & \epsilon H_{1n} \\ \epsilon H_{21} & 1+\epsilon H_{22} & \cdots & \epsilon H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon H_{n1} & \epsilon H_{n2} & \cdots & 1+\epsilon H_{nn} \end{pmatrix}.$$

(b) Show that $\det(A+\epsilon H) = \det(A) + \epsilon\det(A)\,\mathrm{Tr}(A^{-1}H) + \mathcal{O}(\epsilon^2)$.

(c) Suppose that $A(t+\epsilon) = A(t) + \epsilon\dot{A}(t) + \mathcal{O}(\epsilon^2)$ is the Taylor expansion of a path $A(t)$ of matrices. Deduce Jacobi's formula (1).

Answer 4. (a) We have

$$\begin{aligned}\det(1+\epsilon H) &= \det\begin{pmatrix} 1+\epsilon H_{11} & \epsilon H_{12} & \cdots & \epsilon H_{1n} \\ \epsilon H_{21} & 1+\epsilon H_{22} & \cdots & \epsilon H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon H_{n1} & \epsilon H_{n2} & \cdots & 1+\epsilon H_{nn} \end{pmatrix} \\ &= (1+\epsilon H_{11})(1+\epsilon H_{22})\cdots(1+\epsilon H_{nn}) + \mathcal{O}(\epsilon^2)\end{aligned}$$

where the product comes from the diagonal terms and all other products contributing to the determinant involve at least two off-diagonal terms and hence have a factor of ϵ^2. This expands to

$$\det(1+\epsilon H) = 1 + \epsilon\sum H_{ii} + \mathcal{O}(\epsilon^2)$$

and $\mathrm{Tr}(H) = \sum H_{ii}$.

(b) We have $\det(A+\epsilon H) = \det(A(1+\epsilon A^{-1}H)) = \det(A)\det(1+\epsilon A^{-1}H)$ so $\det(A+\epsilon H) = \det(A) + \epsilon\det(A)\,\mathrm{Tr}(A^{-1}H) + \mathcal{O}(\epsilon^2)$ by part (a).

(c) If $A(t)$ is a path of matrices then $\det(A(t+\epsilon)) = \det(A(t)+\epsilon\dot{A}(t)+\mathcal{O}(\epsilon^2)) = \det(A(t)) + \epsilon\det(A)\,\mathrm{Tr}(A^{-1}(t)\dot{A}(t)) + \mathcal{O}(\epsilon^2)$. Since the derivative is just the first order part of the Taylor series we have Jacobi's formula

$$\frac{d}{dt}\det(A(t)) = \det(A)\,\mathrm{Tr}(A^{-1}(t)\dot{A}(t)).$$

[1] Not to be confused with the Jacobi identity later in the course.

Question 5. (Application of Jacobi formula)
Given an n-by-n matrix H, let $\phi(t) = \det \exp(tH)$.

(a) Deduce from the Jacobi formula that $\dot{\phi}(t) = \phi(t)\operatorname{Tr}(H)$.

(b) Deduce from (a) that $\det(\exp H) = \exp \operatorname{Tr}(H)$.

Let $SL(n, \mathbf{R})$ denote the group of n-by-n matrices with determinant one and $\mathfrak{sl}(n, \mathbf{R})$ denote the space of n-by-n matrices with trace zero.

(c) Deduce that $H \in \mathfrak{sl}(n, \mathbf{R})$ if and only if $\exp(tH) \in SL(n, \mathbf{R})$ for all $t \in \mathbf{R}$. *This implies that $\mathfrak{sl}(n, \mathbf{R})$ is the Lie algebra of $SL(n, \mathbf{R})$.*

Answer 5. (a) We have $\frac{d}{dt}\exp(tH) = \exp(tH)H$ so the Jacobi formula gives $\frac{d}{dt}\det(\exp(tH)) = \det(\exp(tH))\operatorname{Tr}(\exp(-tH)\exp(tH)H)$ and if $\phi(t) = \det(\exp(tH))$ then $\dot{\phi}(t) = \phi(t)\operatorname{Tr}(H)$.

(b) We also know that $\phi(0) = 1$ so this tells us that $\phi(t)$ is a solution to the ODE $\dot{\phi}(t) = \phi(t)\operatorname{Tr}(H)$ with $\phi(0) = 1$. Another solution is $\exp(t\operatorname{Tr}(H))$. By uniqueness of solutions to ODEs with given initial conditions we get that $\phi(t) = \exp(t\operatorname{Tr}(H))$.

(c) If $H \in \mathfrak{sl}(n, \mathbf{R})$ then $\operatorname{Tr}(H) = 0$ so $\exp(t\operatorname{Tr}(H)) = 0$ so $\det(\exp(tH)) = \exp(0) = 1$ so $\exp(tH) \in SL(n, \mathbf{R})$. Conversely if $\exp(tH) \in SL(n, \mathbf{R})$ then $\det(\exp(tH)) = 1$ for all t so differentiating with respect to t at $t = 0$ we get $\operatorname{Tr}(H) = 0$.