Sheet 8: (a) The Lie algebra $\mathfrak{g}_2$ and (b) Pentaquarks

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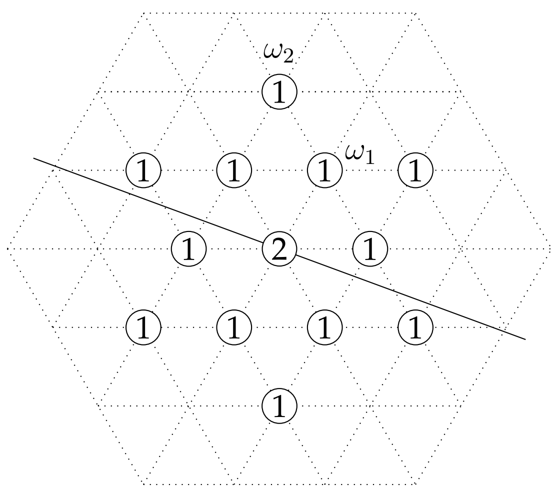
In the first part of this question sheet we will consider a simple[1] Lie algebra called $\mathfrak{g}_2$. This can be defined explicitly as a Lie algebra of 7-by-7 matrices, but thanks to the general theory we have developed, the only information you will need is its root diagram, which looks like:



(The solid line is there to demarcate positive/negative roots). Denote by $\Gamma_{a,b}$ the irreducible representation with highest weight $a\omega_1 + b\omega_2$ and by $V_{a,b}$ the underlying vector space of this representation. Note that $\|\omega_1\|^2 = 1$, $\|\omega_2\|^2 = 3$ and $\langle \omega_1, \omega_2 \rangle = 3/2$.

Question 1. (a) Identify the Weyl group of this Lie algebra.

(b) Sketch a Weyl chamber on the root diagram.

(c) Identify the set of positive roots relative to the solid line in the diagram explicitly as linear combinations of ω_1 and ω_2.

Question 2. (a) Draw the weight diagrams for the irreducible representations

$$\text{(i) } \Gamma_{1,0}, \qquad \text{(ii) } \Gamma_{0,1}, \qquad \text{(iii) } \Gamma_{2,0},$$

and calculate the multiplicities of the weights that occur using the Freudenthal multiplicity formula or otherwise.

[1]...in the technical sense of having no nonzero ideals...

(b) In each case, what is the dimension of the representation?

Note that since $\mathfrak{g}_2$ is simple, the kernel of any nontrivial representation is zero. In particular, $\mathfrak{g}_2$ injects into $\mathfrak{gl}(V_{1,0})$. This yields the explicit description of $\mathfrak{g}_2$ as a Lie algebra of matrices alluded to at the beginning of the sheet.

Question 3. Decompose the following representations into their irreducible parts:

$$\text{(a) } \mathrm{Sym}^2(\Gamma_{1,0}), \qquad \text{(b) } \Lambda^3\Gamma_{1,0}.$$

Deduce that there are: (a) a unique (up to rescaling) quadratic form and (b) a unique (up to rescaling) skew-symmetric trilinear form on the vector space $V_{1,0}$ which are invariant under the action of $\mathfrak{g}_2$.

In fact: (a) This quadratic form turns out to be nondegenerate so this tells us that the image of this 7-dimensional representation lands in $SO(7, \mathbf{C}) \subset GL(7, \mathbf{C})$. (b) This trilinear form is related to the algebra of octonions; one way to define $\mathfrak{g}_2$ is as the Lie algebra of derivations of the octonion algebra $\mathbf{O}$. In this sheet we have effectively proved that $\mathfrak{g}_2 \subset \mathfrak{der}(\mathbf{O})$.

Question 4 (Pentaquarks)**.** A pentaquark is a hypothetical particle which is a bound state of four quarks and an antiquark. These are believed to be so unstable that, even if it were possible to produce them, they decay too quickly to observe (an all-too-convenient explanation for why we haven't seen them yet...). Classify pentaquarks made from u, d, s quarks/antiquarks into multiplets according to their transformation properties under the group $SU(3)$.