Sheet 5: Representations of Lie groups

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Question 1.
Let x, y be a basis for the standard representation $\mathbf{C}^2$ of $SU(2)$ and let $x\otimes x$, $\frac{1}{2}(x\otimes y+y\otimes x)$, $y\otimes y$ be a basis for $\mathrm{Sym}^2\,\mathbf{C}^2$. Write down a matrix for the action of $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$ on $\mathrm{Sym}^2\,\mathbf{C}^2$ in terms of this basis.

Answer 1. We have

$$\begin{aligned}
\mathrm{Sym}^2 \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} (x\otimes x) &= (\alpha x - \bar{\beta} y)\otimes(\alpha x - \bar{\beta} y) \\
&= \alpha^2 (x\otimes x) - 2\alpha\bar{\beta}\left(\frac{x\otimes y + y\otimes x}{2}\right) + \bar{\beta}^2 y\otimes y \\
\mathrm{Sym}^2 \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} (y\otimes y) &= (\beta x + \bar{\alpha} y)\otimes(\beta x + \bar{\alpha} y) \\
&= \beta^2 x\otimes x + 2\bar{\alpha}\beta\left(\frac{x\otimes y + y\otimes x}{2}\right) + \bar{\alpha}^2 y\otimes y
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Sym}^2 \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \left(\frac{x\otimes y + y\otimes x}{2}\right) &= \frac{1}{2}(\alpha x - \bar{\beta} y)\otimes(\beta x + \bar{\alpha} y) + \frac{1}{2}(\beta x + \bar{\alpha} y)\otimes(\alpha x - \bar{\beta} y) \\
&= \alpha\beta x\otimes x + (|\alpha|^2 - |\beta|^2)\left(\frac{x\otimes y + y\otimes x}{2}\right) - \bar{\alpha}\bar{\beta} y\otimes y
\end{aligned}$$

so the matrix is

$$\begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ -2\alpha\bar{\beta} & |\alpha|^2 - |\beta|^2 & 2\bar{\alpha}\beta \\ \bar{\beta}^2 & -\bar{\alpha}\bar{\beta} & \bar{\alpha}^2 \end{pmatrix}.$$

Question 2.

(a) Suppose that $R\colon G \to GL(V)$ is a representation. Show that $R^*\colon G \to GL(V^*)$ is a representation.

(b) Suppose that $R_1\colon G \to GL(V_1)$ and $R_2\colon G \to GL(V_2)$ are representations and that $L\colon V_1 \to V_2$ is a morphism of representations. Show that the kernel and image of L are subrepresentations of V_1 and V_2 respectively. If G is compact and L is surjective, deduce that $V_1 \cong \ker L \oplus V_2$ as representations.

(c) Prove that the trace map $\mathrm{Tr}\colon \mathfrak{gl}(n, \mathbf{R}) \to \mathbf{R}$ is a morphism from the adjoint representation of $GL(n, \mathbf{R})$ to the trivial one-dimensional representation. Deduce that $\mathfrak{sl}(n, \mathbf{R})$ is a subrepresentation.

(d) Let V denote the standard representation of $SU(2)$. Check that the map

$$\begin{gathered}\mathrm{Sym}^p V \otimes \mathrm{Sym}^q V \to \mathrm{Sym}^{p+q} V,\\ x_1 \cdots x_p \otimes x_{p+1} \cdots x_{p+q} \mapsto x_1 \cdots x_p x_{p+1} \cdots x_{p+q}\end{gathered}$$

is a morphism of $SU(2)$-representations. In the case that $p = 1$, $q = 2$:

(i) find the kernel;

(ii) show that the kernel is isomorphic to V;

(iii) deduce that $V \otimes \mathrm{Sym}^2 V \cong V \oplus \mathrm{Sym}^3 V$.

Answer 2. (a) We need to check that $R^*(gh) = R^*(g)R^*(h)$ and $R^*(1) = 1$. For $f \in V^*$ and $v \in V$ we have

$$\begin{aligned}(R^*(gh)f)(v) &= f((gh)^{-1}v)\\ &= f(h^{-1}g^{-1}v)\\ &= (R^*(h)f)(g^{-1}v)\\ &= (R^*(g)R^*(h)f)(v)\\ (R^*(1)f)(v) &= f(v)\end{aligned}$$

as required.

(b) Let F be a morphism of representations and consider the subspace $\ker F = \{v : F(v) = 0\}$. We must show that this is preserved by $R_1(g)$ for each g. We have $F(R_1(g)v) = R_2(F(v)) = 0$ if $F(v) = 0$ so $R_1(g)$ preserves $\ker F$. Similarly, if $w \in \mathrm{im}\, F$ then $R_2(g)w = R_2(g)F(v)$ for some v and so

$$R_2(g)w = R_2(g)F(v) = F(R_1(g)v) \in \mathrm{im}\, F$$

since F is a morphism of representations.

Consider an invariant Hermitian inner product on V_1 (possible because G is compact) and let $W \subset V_1$ be the orthogonal complement for $\ker F$ so that W is a subrepresentation and $V_1 \cong W \oplus \ker F$. The restriction $F|_W\colon W \to \mathrm{im}\, F$ is now an isomorphism of representations so if F is surjective we get $V_1 \cong V_2 \oplus \ker F$.

(c) The map Tr is a morphism if $\mathrm{Tr}(\mathrm{Ad}_g v) = \mathrm{Tr}(v)$ for all $g \in GL(n, \mathbf{C})$ and $v \in \mathfrak{gl}(n, \mathbf{C})$. Since $\mathrm{Ad}_g v = gvg^{-1}$ we have

$$\mathrm{Tr}(\mathrm{Ad}_g v) = \mathrm{Tr}(gvg^{-1}) = \mathrm{Tr}(v)$$

so Tr is a morphism. In particular, by the previous part of the question, its kernel (the subspace of tracefree matrices) is a subrepresentation.

(d) One can check that this map is a morphism by applying $R^{\otimes n}(g)$ to each side and checking they agree. In the case that $p = 1$ and $q = 2$:

(i) The effect of this map on a basis is $x \otimes x^2 \mapsto x^3$, $x \otimes xy, y \otimes x^2 \mapsto x^2y$, $x \otimes y^2, y \otimes xy \mapsto xy^2$, $y \otimes y^2 \mapsto y^3$. Therefore the kernel is generated by $x \otimes xy - y \otimes x^2$ and $y \otimes xy - x \otimes y^2$.

(ii) The kernel is a 2-dimensional representation. If we call this representation ρ then we get (after some work)

$$\rho \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} (x \otimes xy - y \otimes x^2) = \alpha(x \otimes xy - y \otimes x^2) + \beta(x \otimes y^2 - y \otimes xy)$$

and

$$\rho \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} (x \otimes y^2 - y \otimes xy) = -\bar{\beta}(x \otimes xy - y \otimes x^2) + \bar{\alpha}(x \otimes y^2 - y \otimes xy).$$

Therefore the kernel is isomorphic to the standard representation.

(iii) Part (b) tells us that $V \otimes \mathrm{Sym}^2 V$ is isomorphic to $\ker F \oplus \mathrm{Sym}^3 V$ since F is certainly surjective. Part (d.ii) tells us that $\ker F \cong V$.

Question 3.

(a) Suppose that $\mathbf{K}$ is a field of characteristic zero and $V = \mathbf{K}\langle e_1, e_2\rangle$. Show that

$$V^{\otimes 2} = \mathrm{Sym}^2(V) \oplus \Lambda^2(V).$$

(b) Let e_i be a basis of $\mathbf{R}^n$ and, for $i = 1, \ldots, n$, let a_i be the vector $\sum_{i=1}^n a_i^j e_j$. By inspecting the formula for the alternating map, show that $a_1 \wedge \cdots \wedge a_n = \det(a_i^j)(e_1 \wedge \cdots \wedge e_n)$.

Answer 3. (a) We have $V^{\otimes 2} = \mathbf{K}\langle e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\rangle$. The symmetric square is a 3-dimensional subspace containing $e_1 \otimes e_1$, $e_2 \otimes e_2$ and $e_1 \otimes e_2 + e_2 \otimes e_1$. A complementary subspace is spanned by $e_1 \otimes e_2 - e_2 \otimes e_1$ which also spans $\Lambda^2 V$.

(b) Let A denote the square matrix with entries a_i^j. We have

$$\begin{aligned} a_1 \wedge \cdots \wedge a_n &= \frac{1}{n!}\sum_{\sigma \in S_n} (-1)^{|\sigma|} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \\ &= \frac{1}{n!}\sum_{\sigma} (-1)^{|\sigma|} \sum_{i_1,\ldots,i_n} a_{\sigma(1)}^{i_1} e_{i_1} \otimes \cdots \otimes a_{\sigma(n)}^{i_n} e_{i_n} \end{aligned}$$

We ask: what is the coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_n}$? Clearly it is

$$\frac{1}{n!}\sum_{\sigma} (-1)^{|\sigma|} a_{\sigma(1)}^{i_1} \cdots a_{\sigma(n)}^{i_n}$$

which is $1/n!$ times the formula for the determinant of the matrix B whose entries are $b_\ell^k = a_{\sigma(\ell)}^{i_k}$. This is nonzero only if $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$ for otherwise two of the columns agree. Therefore B is the result of permuting the rows and columns of A and (remembering that a transposition of rows or columns changes the sign of the determinant) we get

$$a_1 \wedge \cdots \wedge a_n = \sum_{\tau \in S_n} (-1)^{|\tau|} (1/n!) \det(A) e_{\tau(1)} \otimes \cdots \otimes e_{\tau(n)} = \det(A) e_1 \wedge \cdots \wedge e_n.$$

Question 4.
Let $V = \mathbf{K}\langle e_1, e_2\rangle$ and $W = \mathbf{K}\langle f_1, f_2\rangle$ be vector spaces and consider an element $t = \sum_{i,j=1}^{2} t_{ij} e_i \otimes e_j \in V \otimes W$ in their tensor product. Show that there exist $v \in V$ and $w \in W$ such that $t = v \otimes w$ (i.e. t is a pure tensor) if and only if $t_{11}t_{22} = t_{12}t_{21}$.

This equation is called the Plücker relation. For tensor products of higher dimensional vector spaces there are many more Plücker relations $t_{ij}t_{k\ell} = t_{i\ell}t_{kj}$. This tells us that the pure tensors form a subvariety of $V \otimes W$ cut out by a collection of homogeneous polynomials of degree 2.

Answer 4. If $\sum t_{ij} e_i \otimes e_j$ is a pure tensor then it is equal to $\sum v_i e_i \otimes \sum w_j e_j$ and hence $t_{ij} = v_i w_j$. This means $t_{11}t_{22} = v_1 w_1 v_2 w_2 = v_1 w_2 v_2 w_1 = t_{12}t_{21}$.

Conversely, suppose the Plücker relation holds. If all $t_{ij} = 0$ then just take $v = w = 0$. If one of the $t_{ij} \neq 0$ then (renumbering the basis) we can assume it is t_{11}. Then set $v_1 = 1$, $w_1 = t_{11}$, $v_2 = t_{21}/t_{11}$ and $w_2 = t_{12}$. We get $v \otimes w = \sum t_{ij} e_i \otimes e_j$.

Question 5.
Suppose that $R\colon G \to GL(V)$ and $S\colon G \to GL(W)$ are representations and denote by T the representation of G on $\mathrm{Hom}(V, W)$ defined by

$$(T(g)F)(v) = S(g)F(R(g^{-1})v).$$

(a) Show that a vector $F \in \mathrm{Hom}(V, W)$ satisfies $T(g)F = F$ for all $g \in G$ if and only if F is a morphism of representations. If V and W are irreducible, show that either $F = 0$ or F is an isomorphism.

(b) Define a map $\Phi\colon V^* \otimes W \to \mathrm{Hom}(V, W)$ by defining it on pure tensors as

$$\Phi(f \otimes w)(v) = f(v)w$$

and extending linearly. Check that this is an isomorphism of representations $R^* \otimes S \cong T$.

Hint: Once you know it's a morphism of representations, to show it's an isomorphism, pick bases e_i of V and f_j of W and check that $\Phi(e_i^* \otimes f_j)$ is a basis for $\mathrm{Hom}(V, W)$. Why does this tell you it is an isomorphism?

Answer 5. (a) If $T(g)F = F$ for all $g \in G$ then $S(g)F(R(g^{-1})v) = F(v)$ and hence $F(R(h)v) = S(h)(F(v))$ where $h = g^{-1}$ runs over all of G. Therefore F is a morphism of representations. If V and W are irreducible then the kernel and image are subrepresentations of V and W respectively and so either $\ker F = 0$, $\mathrm{im}\, F = W$ or $\ker F = V$, $\mathrm{im}\, F = 0$. In the first case F is an isomorphism, in the second it is zero.

(b) To check it's a morphism of representations we need to show that

$$\Phi(R^*(g)f \otimes S(g)w)v = (T(g)\Phi(f \otimes w))(v)$$

The LHS is

$$f(R(g^{-1})v)S(g)w$$

while the RHS is

$$S(g)\Phi(f \otimes w)(R(g^{-1})v) = S(g)[f(R(g^{-1})v)w].$$

Since $f(R(g^{-1})v)$ is just a scalar, this equals $f(R(g^{-1})v)S(g)w$.

To see that it's an isomorphism we observe that $e_i^* \otimes f_j$ goes to the matrix with a one in the ith column and jth row. Φ therefore surjects onto the space of matrices as i and j. Since $\dim(V^* \otimes W) = \dim V \times \dim W = \dim \mathrm{Hom}(V, W)$ it is also an injection and hence an isomorphism.