Sheet 4: Lie's theorem

J. Evans

Recall Lie's theorem:

Theorem 1 (Lie)**.** *Let G and H be two path-connected matrix groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Suppose moreover that G is simply-connected. For any Lie algebra homomorphism $f\colon \mathfrak{g} \to \mathfrak{h}$ there is a unique smooth homomorphism $F\colon G \to H$ with $F_* = f$.*

Question 1. Suppose that G and H are both path-connected, simply-connected matrix groups and that $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras. Prove that $G \cong H$. (You may assume Lie's theorem).

The remaining exercises form a guided proof of Lie's theorem. The first ingredient we will need is a compatibility condition for a system of partial differential equations to have a solution.

Question 2 (Maurer-Cartan equation)**.** In this question, you can freely assume the existence[1] and uniqueness of solutions to *ordinary* differential equations. Let G be a matrix group with Lie algebra $\mathfrak{g}$. Let $\xi(s,t)$ and $\eta(s,t)$ be two smooth maps $[0,1]\times[0,1] \to \mathfrak{g}$. Prove that the following are equivalent:

(a) The *partial* differential equations

$$(\star) \quad \frac{\partial \phi}{\partial s}(s,t) = \phi(s,t)\xi(s,t), \quad (\star\star) \quad \frac{\partial \phi}{\partial t}(s,t) = \phi(s,t)\eta(s,t)$$

have a solution $\phi\colon [0,1]\times[0,1] \to G$ with $\phi(0,0) = 1$.

(b) The *Maurer-Cartan equation* holds:

$$(\star\star\star) \quad \frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial s} = [\xi, \eta].$$

Hints:

For (a) implies (b), try cross-differentiating $(\star)$ and $(\star\star)$ and seeing what turns up.

For (b) implies (a): This is tricky. Construct $\phi(s,0)$ first by considering $(\star)$ as an ODE at $t = 0$. Then use $\phi(s,0)$ as the initial conditions for $(\star\star)$, considered as an ODE for each fixed value of s. By construction this solves $(\star\star)$ and it solves $(\star)$ along $t = 0$; it remains to show that $(\star)$ holds everywhere. For fixed s, consider $\beta(t) = \frac{\partial \phi}{\partial s}(s,t) - \phi(s,t)\xi(s,t)$ and show (using $(\star\star)$ and $(\star\star\star)$) that $\frac{d\beta}{dt} = \beta\eta$; deduce that $\beta \equiv 0$ using uniqueness of solutions to ODEs and the initial condition $(\star)$ along $t = 0$.

[1] All of the ODEs we come across will have solutions for all time.

Paths in G from paths in $\mathfrak{g}$:

Question 3. (a) Let $\gamma\colon [0,1] \to G$ be a path. For each $s \in [0,1]$, by considering the path $\gamma(s)^{-1}\gamma(t)$ show that $A(s) = \gamma^{-1}(s)\dot{\gamma}(s)$ is a tangent vector to G at 1. That is $A(s) \in \mathfrak{g}$.

(b) Conversely, let $X \in \mathfrak{g}$ and $g \in G$. Show that gX is a tangent vector to G at g. Hence $T_g G = g\mathfrak{g}$.

A consequence of this and the existence/uniqueness of solutions to ODEs is the following lemma which we will use to construct F.

Lemma 2. *Let $A\colon [0,1] \to \mathfrak{g}$ be a path in $\mathfrak{g}$. Show that if $\gamma\colon [0,1] \to GL(n, \mathbf{R})$ is a path satisfying $\dot{\gamma}(t) = \gamma(t)A(t)$ then $\gamma(t) \in G$ for all t.*

The construction of $F(g)$:

Let $\gamma\colon [0,1] \to G$ be a path with $\gamma(0) = 1$ and $\gamma(1) = g$. Define $A(t) = \gamma(t)^{-1}\dot{\gamma}(t) \in \mathfrak{g}$. Now consider the differential equation

$$\dot{\delta}(t) = \delta(t) f(A(t)), \quad \delta(0) = 1$$

for a path δ in H. We call δ the *path associated to γ*.

$$\boxed{\text{Define } F(g) = \delta(1).}$$

We will show that

- $\delta(1)$ does not depend on the choice γ of path from 1 to g;
- that $F\colon G \to H$ is a homomorphism;
- that $F_* = f$.

Path-independence of $\delta(1)$:

Suppose that γ_i, $i = 0, 1$ are two choices of path with $\gamma_i(0) = 1$ and $\gamma_i(1) = g$ and associated paths δ_i. Since G is simply-connected there is a map $\psi\colon [0,1] \times [0,1] \to G$ such that

$$\psi(0,t) = \gamma_0(t), \quad \psi(1,t) = \gamma_1(t), \quad \psi(s,0) = 1, \quad \psi(s,1) = g.$$

Question 4. (a) Show that ξ and η satisfy the Maurer-Cartan equation, where

$$\xi(s,t) = \psi^{-1}\frac{\partial \psi}{\partial s} \in \mathfrak{g}, \quad \eta(s,t) = \psi^{-1}\frac{\partial \psi}{\partial t} \in \mathfrak{g}.$$

(b) If $f\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism deduce that $f\xi$ and $f\eta$ also satisfy the Maurer-Cartan equation.

(c) Deduce that if f is a Lie algebra homomorphism then there exists a map $\phi\colon [0,1] \times [0,1] \to H$ such that $\phi(0,0) = 1$ and

$$\frac{\partial \phi}{\partial s} = \phi f(\xi), \quad \frac{\partial \phi}{\partial t} = \phi f(\eta).$$

(d) Why is $\xi(s,1) = 0$? Deduce that $\phi(s,1)$ is independent of s and hence that $\delta_0(1) = \delta_1(1)$.

F is a homomorphism.

Question 5. If $\delta(t)$ is a path in a group then, by differentiating $\delta(t)^{-1}\delta(t) = 1$, prove that

$$\frac{d(\delta(t)^{-1})}{dt} = -\delta(t)^{-1}\dot{\delta}(t)\delta(t)^{-1}.$$

Take a path γ in G with $\gamma(0) = 1$ and $\gamma(1) = g$ and let δ be the associated path in H (equivalently $\delta^{-1}\dot{\delta} = f(\gamma^{-1}\dot{\gamma})$) and recall that $F(g) = \delta(1)$.

Question 6. Show that $\delta(t)^{-1}f(X)\delta(t)$ and $f(\gamma(t)^{-1}X\gamma(t))$ are both solutions to the ODE

$$\dot{z}(t) = [z(t), \delta(t)^{-1}\dot{\delta}(t)]$$

with initial condition $z(0) = f(X)$. Deduce that they are equal for all t and hence prove that

$$\delta(1)^{-1}f(X)\delta(1) = f(\gamma^{-1}(1)X\gamma(1)),$$

or, equivalently,

$$F(g)^{-1}f(X)F(g) = f(g^{-1}Xg).$$

Question 7. Suppose that α, β are paths in G with $\alpha(0) = \beta(0) = 1$ and $\alpha(1) = g$, $\beta(1) = h$. Define the path $\gamma(t) = \alpha(t)\beta(t)$ in G with $\gamma(1) = gh$ and associated paths u, v, w in H satisfying

$$\begin{aligned} \dot{u} &= uf(\alpha^{-1}\dot{\alpha}) \quad \text{so } F(g) = u(1) \\ \dot{v} &= vf(\beta^{-1}\dot{\beta}) \quad \text{so } F(h) = v(1) \\ \dot{w} &= wf(\gamma^{-1}\dot{\gamma}) \quad \text{so } F(gh) = w(1) \end{aligned}$$

Prove that $w = uv$ by showing that they both solve the same ODE

$$\dot{z} = zv^{-1}f(\alpha^{-1}\dot{\alpha})v + zf(\beta^{-1}\dot{\beta})$$

with the initial condition $z(0) = 1$.

$F_* = f$:

Question 8. Suppose that $\gamma(t) = \exp(tX)$. Prove that the associated path in H is given by $\delta(t) = \exp(tf(X))$. Deduce that $F(\exp X) = \exp(f(X))$ and hence that $f(X) = F_*(X)$.

This completes the proof of Lie's theorem.