Sheet 2: Matrix groups

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Question 1.
Prove that $A^T = -A$ if and only if $\exp(tA) \in O(n)$ for all $t \in \mathbf{R}$. This implies that the Lie algebra of $O(n)$ is the space $\mathfrak{o}(n)$ of antisymmetric matrices.

Question 2. Show that the third order term in the Baker-Campbell-Hausdorff formula for $\log(\exp A \exp B)$ is

$$\frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]].$$

If you think this is getting messy, it just means you're on the right track. Everything will simplify beautifully.

Question 3.

(a) Suppose that W_1 and W_2 are complementary subspaces of $\mathfrak{gl}(n,\mathbf{R})$, so that $\mathfrak{gl}(n,\mathbf{R}) = W_1 \oplus W_2$. Consider the map

$$F \colon W_1 \oplus W_2 \to GL(n,\mathbf{R}), \quad F(w_1 \oplus w_2) = \exp(w_1)\exp(w_2).$$

By computing the Taylor expansion of F (plugging in the Taylor expansions of $\exp(w_1)$ and $\exp(w_2)$ and expanding), show that $d_{(0,0)}F(w_1 \oplus w_2) = w_1 + w_2$.

(b) Deduce that there are open neighbourhoods $0 \in U' \subset \mathfrak{gl}(n,\mathbf{R})$ and $1 \in V' \subset GL(n,\mathbf{R})$ such that $F|_{U'} \colon U' \to V'$ is a diffeomorphism.

(c) Let $G \subset GL(n,\mathbf{R})$ be a matrix group, set $W_1 = \mathfrak{g}$ and let W_2 be a complement for V. Given $g \in G$, define

$$F_g \colon \mathfrak{g} \oplus W_2 \to GL(n,\mathbf{R}), \quad F_g(w_1 \oplus w_2) = g\exp(w_1)\exp(w_2)$$

Show that there are open sets $0 \in U' \subset \mathfrak{gl}(n,\mathbf{R})$ and $g \in V' \subset GL(n,\mathbf{R})$ such that $F_g|_{U'}$ is a diffeomorphism.

The last two questions study the following formula and its applications.

Lemma 1 (Jacobi formula[1])**.** *Let $A(t)$ be a path of invertible matrices ($t \in \mathbf{R}$). Then*

$$\frac{d}{dt}\det(A(t)) = \det(A(t))\,\mathrm{Tr}(A^{-1}\dot{A}(t)). \tag{1}$$

Here $\mathrm{Tr}(M)$ denotes the trace of M (the sum of its diagonal entries) and $\dot{A}(t)$ denotes the matrix whose entries are the t-derivatives of the entries of A.

Question 4. (Proof of Jacobi formula)

(a) Show that $\det(1+\epsilon H) = 1 + \epsilon\,\mathrm{Tr}(H) + \mathcal{O}(\epsilon^2)$. *Hint: It might help to write out*

$$\det(1+\epsilon H) = \det\begin{pmatrix} 1+\epsilon H_{11} & \epsilon H_{12} & \cdots & \epsilon H_{1n} \\ \epsilon H_{21} & 1+\epsilon H_{22} & \cdots & \epsilon H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon H_{n1} & \epsilon H_{n2} & \cdots & 1+\epsilon H_{nn} \end{pmatrix}.$$

(b) Show that $\det(A+\epsilon H) = \det(A) + \epsilon\det(A)\,\mathrm{Tr}(A^{-1}H) + \mathcal{O}(\epsilon^2)$.

(c) Suppose that $A(t+\epsilon) = A(t) + \epsilon\dot{A}(t) + \mathcal{O}(\epsilon^2)$ is the Taylor expansion of a path $A(t)$ of matrices. Deduce Jacobi's formula (1).

Question 5. (Application of Jacobi formula)
Given an n-by-n matrix H, let $\phi(t) = \det\exp(tH)$.

(a) Deduce from the Jacobi formula that $\dot{\phi}(t) = \phi(t)\,\mathrm{Tr}(H)$.

(b) Deduce from (a) that $\det(\exp H) = \exp\mathrm{Tr}(H)$.

Let $SL(n,\mathbf{R})$ denote the group of n-by-n matrices with determinant one and $\mathfrak{sl}(n,\mathbf{R})$ denote the space of n-by-n matrices with trace zero.

(c) Deduce that $H \in \mathfrak{sl}(n,\mathbf{R})$ if and only if $\exp(tH) \in SL(n,\mathbf{R})$ for all $t \in \mathbf{R}$. *This implies that $\mathfrak{sl}(n,\mathbf{R})$ is the Lie algebra of $SL(n,\mathbf{R})$.*

[1] Not to be confused with the Jacobi identity later in the course.