Sheet 3: Lie algebras and the exponential map

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Question 1.

(a) Check that the commutator bracket $[X, Y] = XY - YX$ on matrices satisfies the Jacobi identity[1]
$$[[X, Y], C] + [[Y, C], X] + [[C, X], Y] = 0.$$

(b) Define ad_X to be the operator $\mathrm{ad}_X Y = [X, Y]$. Check that the Jacobi identity is equivalent to
$$\mathrm{ad}_{[X,Y]} Z = \mathrm{ad}_X \mathrm{ad}_Y Z - \mathrm{ad}_Y \mathrm{ad}_X Z.$$

(c) Expand the associativity relation
$$(\exp(A) \exp(B)) \exp(C) = \exp(A)(\exp(B) \exp(C))$$
using the Baker-Campbell-Hausdorff formula keeping all *cubic* terms involving precisely one A, one B and one C. Show that the Jacobi identity follows.

Question 2.

(a) Show that the tangent space to $O(n)$ at 1 is the vector space $\mathfrak{so}(n)$ of antisymmetric matrices.

(b) Using the Jacobi formula from Sheet 2, show that the tangent space to $SL(n, \mathbf{R})$ at 1 is the vector space $\mathfrak{sl}(n, \mathbf{R})$ of matrices with trace zero.

(c) Let $g \in U(n)$ (so $g^\dagger = g^{-1}$). By Taylor expanding the map $F(A) = A^\dagger A$ around g, show that $d_g F(B) = B^\dagger g + g^{-1} B$ and deduce that the tangent space of $U(n)$ at g is the space of matrices B such that $g^{-1} B$ is skew-Hermitian.

[1] Not to be confused the the Jacobi formula on the previous sheet!

Question 3.
Recall the $2n$-by-$2n$ matrix J from lectures (all you need to remember about it is $J^2 = -1$, $J^T = -J$). Inside $GL(2n, \mathbf{R})$ we defined subgroups

- $Sp(2n, \mathbf{R}) = \{A : \ A^T JA = J\}$,
- $O(2n) = \{A \ : \ A^T A = 1\}$,
- $GL(n, \mathbf{C}) = \{A \ : \ AJ = JA\}$.

Prove that $Sp(2n, \mathbf{R}) \cap O(2n) = GL(n, \mathbf{C}) \cap O(2n) = GL(n, \mathbf{C}) \cap Sp(2n, \mathbf{R}) = U(n)$. (Recall that conjugate-transpose on $A \in GL(n, \mathbf{C})$ is just transpose on $A \in GL(2n, \mathbf{R})$)

Question 4.

(a) (i) Show that $\exp\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} e^a & be^a \\ 0 & e^a \end{pmatrix}$ and hence find (complex) logarithms for the matrices

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \ (\lambda \neq 0).$$

(ii) If $A \in GL(2, \mathbf{C})$ let $N = P^{-1}AP$ be its Jordan normal form. Prove that if $N = \exp(X)$ then A is in the image of the exponential map. Deduce that $\exp\colon \mathfrak{gl}(2, \mathbf{C}) \to GL(2, \mathbf{C})$ is surjective.

(b) (i) Consider $B \in \mathfrak{sl}(2, \mathbf{R})$. Show that its Jordan normal form (considered as a complex matrix) is one of:

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \ (\lambda \in \mathbf{R} \text{ or } i\mathbf{R}) \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(ii) Deduce that there are three possibilities for the eigenvalues of $\exp(B)$: both are positive, both are unit complex numbers or both are equal to 1. By exhibiting a matrix in $SL(2, \mathbf{R})$ whose eigenvalues satisfy none of these, deduce that $\exp\colon \mathfrak{sl}(2, \mathbf{R}) \to SL(2, \mathbf{R})$ is not surjective.

Question 5.

(a) Let Q be a matrix. Show that the tangent space of the matrix group

$$G = \{A \in GL(n, \mathbf{R}) \ : \ A^T QA = Q\}$$

at the identity is $\mathfrak{g} = \{B \ : \ B^T Q + QB = 0\}$.

(b) Check that if $B_i^T Q + QB_i = 0$ for $i = 1, 2$ then $[B_1, B_2]^T Q + Q[B_1, B_2] = 0$.

(c) Let $n = 2$, $Q = j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (so that $G = Sp(2, \mathbf{R})$ and $\mathfrak{g} = \mathfrak{sp}(2, \mathbf{R})$). Prove that $\mathfrak{sp}(2, \mathbf{R}) = \mathfrak{sl}(2, \mathbf{R})$ (where $\mathfrak{sl}(2, \mathbf{R})$ is the space of tracefree matrices).

Question 6.

Let S be a neighbourhood of $1 \in G$. Let $\langle S \rangle \subset G$ denote the subgroup generated by S (i.e. $\langle S \rangle$ consists of all $g \in G$ that can be written as a product $s_1 \cdots s_n$ with each $s_i \in S$).

(a) Let $\gamma \colon [0,1] \to G$ be a path with $\gamma(0) = 1$. Prove that $\gamma(1) \in \langle S \rangle$.

(b) Assuming that any two points in S can be connected by a path, show that any two points in $\langle S \rangle$ can be connected by a path.

Hint for (a): Consider the cover of $[0,1]$ *by open sets* $U_t = \{r \in [0,1] \;:\; \gamma(r) = \gamma(t)s$ *for some* $s \in S\}_{t \in [0,1]}$. *Take a finite subcover* U_{t_i}, $0 = t_0 < t_1 < \cdots < t_N = 1$ *and show inductively that* $\gamma(t_i)$ *can be written as a product of elements in* S. *It might help to draw a picture to illustrate what's going on.*