Sheet 6: More on representations

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Question 1.
Write out the action of $X, Y \in \mathfrak{sl}(2, \mathbf{C})$ on $\mathrm{Sym}^3(\mathbf{C}^2)$ explicitly.

Question 2.
Decompose the following representations of $\mathfrak{sl}(2, \mathbf{C})$ into irreducible summands:

(a) $\Lambda^2 \,\mathrm{Sym}^3\, \mathbf{C}^2$,

(b) $\mathrm{Sym}^2\,\mathrm{Sym}^2\,\mathbf{C}^2$,

(c) $\Lambda^3\,\mathrm{Sym}^4\,\mathbf{C}^2$,

(d) $\mathrm{Sym}^3\,\mathrm{Sym}^2\,\mathbf{C}^2$,

(e) $\mathrm{Sym}^2\,\mathrm{Sym}^3\,\mathbf{C}^2$,

(f) $\mathrm{Sym}^2\,\Lambda^2\,\mathrm{Sym}^3\,\mathbf{C}^2$,

(g) $\mathrm{Sym}^2\,\mathrm{Sym}^4\,\mathbf{C}^2$,

(h) $\mathrm{Sym}^3\,\mathrm{Sym}^4\,\mathbf{C}^2$,

From your computations in (g) and (h) deduce that there are quadratic and cubic invariants $g_2(a, b, c, d, e)$ and $g_3(a, b, c, d, e)$ for binary quartic polynomials $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ under the action of $SL(2, \mathbf{C})$.

Question 3. (Clebsch-Gordan theorem)
Let V denote the standard 2-dimensional representation of $\mathfrak{sl}(2, \mathbf{C})$. Prove that the tensor product $\mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(V)$ decomposes into irreducible representations

$$\bigoplus_{\substack{k=|m-n| \\ k\equiv m+n \mod 2}}^{m+n} \mathrm{Sym}^k(V).$$

Question 4.
Prove that if $\rho\colon \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{gl}(V)$ is a representation and X, Y, H denote the usual basis of $\mathfrak{sl}(2, \mathbf{C})$ satisfying the commutation relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$$

then

$$C := \rho(X)\rho(Y) + \rho(Y)\rho(X) + \frac{1}{2}\rho(H)^2$$

commutes with $\rho(X)$, $\rho(Y)$ and $\rho(H)$. Deduce that if $V = \bigoplus_\lambda V_\lambda$ is the decomposition of V into eigenspaces of C then each V_λ is a subrepresentation. If V is irreducible with highest weight m, deduce that C is the diagonal matrix $\left(m + \frac{1}{2}m^2\right) \mathrm{Id}$.

If $R\colon G \to GL(V)$ is a representation we say that:

- $M \in V$ is *R-invariant* if $R(g)M = M$ for all $g \in G$.
- a symmetric bilinear form $B\colon V \otimes V \to \mathbf{K}$ is *R-invariant* if

  $$B(R(g)v, R(g)w) = B(v, w)$$

  for all $v, w \in V$ and $g \in G$.

If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, we say that

- $M \in V$ is *ρ-invariant* if $\rho(X)M = 0$ for all $X \in \mathfrak{g}$.
- a symmetric bilinear form $B\colon V \times V \to \mathbf{K}$ is *ρ-invariant* if

  $$B(\rho(X)v, w) + B(v, \rho(X)w) = 0$$

  for all $v, w \in V$ and $X \in \mathfrak{g}$.

Question 5.
Let G be a connected Lie group and $R\colon G \to GL(V)$ be a representation. Let $\rho = R_*$ be the linearisation of R.

(a) Prove that $M \in V$ is G-invariant if and only if it is R_*-invariant.

(b) Prove that a symmetric bilinear form $B\colon V \times V \to \mathbf{K}$ is R-invariant if and only if it is R_*-invariant.

Hint: To show M or B is R-invariant it suffices to check it on an exponential chart because the group is connected and connected groups are generated by the image of an exponential chart.

(c) Let $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation $X \mapsto \mathrm{ad}_X$, $\mathrm{ad}_X Y = [X, Y]$. Define the symmetric bilinear form

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y)$$

where Tr denotes the trace. Using some form of the Jacobi identity, prove that B is ad-invariant. This is called the *Killing form*.

Hint: The trace of a commutator of matrices vanishes.

(d) Let X, H, Y be the usual basis for $\mathfrak{sl}(2, \mathbf{C})$. Check that with respect to this basis

$$\mathrm{ad}_X = \begin{pmatrix} 0 & -2 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \; \mathrm{ad}_H = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2 \end{pmatrix}, \; \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 2 & \cdots & 0 \end{pmatrix}$$

and hence compute the Killing form on all pairs $B(a, b)$, $a, b \in \{X, H, Y\}$.