Sheet 7: Representations of $SU(3)$ and beyond

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Question 1.
Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(3, \mathbf{C})$ (this will work more generally for any complex semisimple Lie algebra). Suppose that $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ and $\sigma\colon \mathfrak{g} \to \mathfrak{gl}(W)$ are irreducible representations of $\mathfrak{g}$ both having highest weight λ.

(a) Let v be a highest weight vector in V and w a highest weight vector in W. Show that $v \oplus w$ is a highest weight vector in $V \oplus W$.

(b) Deduce that there is an irreducible subrepresentation $U \subset V \oplus W$ containing $v \oplus w$, and that the projections $U \to V$ and $U \to W$ are isomorphisms. Hence deduce that V is isomorphic to W. (*Hint: Use Schur's lemma.*)

Answer 1. Let $\mathfrak{h}$ be a Cartan subalgebra.

(a) For H in $\mathfrak{h}$ and $x \in V_{\lambda_1}$, $y \in W_{\lambda_2}$ we have

$$\rho(H)x = \lambda_1(H)x \text{ and } \sigma(H)y = \lambda_2(H)y$$

so

$$(\rho \oplus \sigma)(H)(x \oplus 0) = \rho(H)x \oplus 0 = \lambda_1(H)x \oplus 0$$

and

$$(\rho \oplus \sigma)(H)(0 \oplus y) = 0 \oplus \sigma(H)y = 0 \oplus \lambda_2(H)y$$

so the weights in the representation $V \oplus W$ are precisely the union of weights in V and in W. Therefore λ is the highest weight and $v \oplus w$ is one of the highest weight vectors.

(b) The images of $v \oplus w$ under words in the matrices $(\rho \oplus \sigma)(E)$ (where E is a negative root) generate an irreducible subrepresentation of highest weight λ which we call U. The projections of U to V and to W are nontrivial (in particular, their images contain v and w) but these projections are morphisms of irreducible representations and are therefore isomorphisms by Schur's lemma.

Question 2.
Let $\rho\colon \mathfrak{su}(3) \to \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{su}(3)$ and v be a highest weight vector. Let D be the set of words in $\rho(E_{21}), \rho(E_{31}), \rho(E_{32})$ and let $W \subset V$ be the subspace spanned by the subset $\{wv \ : \ w \in D\}$. By induction on the length of w, show that W is preserved by $\rho(E_{12}), \rho(E_{13}), \rho(E_{23})$.

Answer 2. It is certainly true for words of length zero: the matrices $\rho(E_{12}), \rho(E_{13}), \rho(E_{23})$ annihilate v and hence send it to $0 \in W$. Suppose it is true for all words of length $\leq k$. Let w be a word of length $k+1$ and suppose that its first letter is $\rho(E_{21})$. Then

$$wv = \rho(E_{21})\tilde{v}$$

and

$$\begin{aligned}\rho(E_{12})wv &= \rho(E_{12})\rho(E_{21})\tilde{v}\\ &= \rho([E_{12}, E_{21}])\tilde{v} + \rho(E_{21})\rho(E_{12})\tilde{v}\\ &= \rho(H_{12})\tilde{v} + \rho(E_{21})\rho(E_{12})\tilde{v}.\end{aligned}$$

Now $\tilde{v}$ is a weight vector so it is sent to a multiple of itself by $\rho(H_{12})$. The second term is $\rho(E_{21})$ applied to an element of W (by induction) so since W is closed under the action of $\rho(E_{21})$ we see that $\rho(E_{12})wv \in W$.

Similarly

$$\begin{aligned}\rho(E_{13})wv &= \rho(E_{12})\rho(E_{21})\tilde{v}\\ &= \rho([E_{13}, E_{21}])\tilde{v} + \rho(E_{21})\rho(E_{13})\tilde{v}\\ &= \rho(-E_{23})\tilde{v} + \rho(E_{21})\rho(E_{13})\tilde{v}.\end{aligned}$$

and by induction, both terms are in W.

Similarly

$$\begin{aligned}\rho(E_{23})wv &= \rho(E_{23})\rho(E_{21})\tilde{v}\\ &= \rho([E_{23}, E_{21}])\tilde{v} + \rho(E_{21})\rho(E_{23})\tilde{v}\\ &= \rho(E_{21})\rho(E_{23})\tilde{v}.\end{aligned}$$

which is in W by induction. Similarly for the other two possibilities of initial letter.

Question 3.
Let $\mathbf{C}^3$ denote the standard representation of $SU(3)$ and $\Gamma_{a,b}$ denote the unique irreducible representation with highest weight $aL_1 - bL_3$. We will see below that $\Gamma_{a,b}$ exists.

(a) Show that $\mathrm{Sym}^a \mathbf{C}^3$ is $\Gamma_{a,0}$ and $\mathrm{Sym}^b(\mathbf{C}^3)^*$ is $\Gamma_{0,b}$ and draw the weight diagrams.

(b) Prove that $\mathrm{Sym}^a \mathbf{C}^3 \otimes \mathrm{Sym}^b(\mathbf{C}^3)^*$ contains an irreducible summand isomorphic to $\Gamma_{a,b}$. This proves existence of an irreducible representation with given highest weight vector $aL_1 - bL_3$.

(c) Decompose $\mathbf{C}^3 \otimes (\mathbf{C}^3)^*$ into irreducible subrepresentations.

(d) Decompose $\mathrm{Sym}^2 \mathbf{C}^3 \otimes (\mathbf{C}^3)^*$ into irreducible subrepresentations.

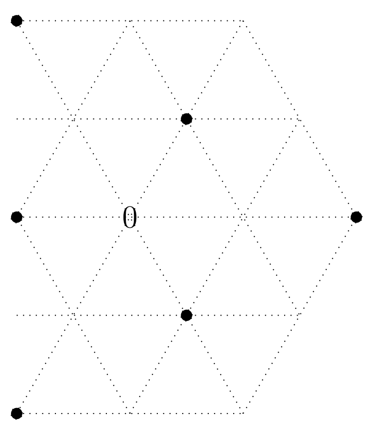
(e) Decompose $\mathbf{C}^3 \otimes \Gamma_{2,1}$ into irreducible subrepresentations.

(f) Decompose $(\mathbf{C}^3)^{\otimes 3}$ into irreducible subrepresentations.

(g) Decompose $\Gamma_{1,1} \otimes \Gamma_{1,2}$ into irreducible subrepresentations.

Answer 3. (a) In the standard representation e_1, e_2, e_3 is a basis of weight vectors with weights L_1, L_2, L_3. Consider the element $\mathrm{Av}(e_{i_1} \otimes \cdots \otimes e_{i_a}) \in \mathrm{Sym}^a \mathbf{C}^3$. This has weight $\sum_{k=1}^a L_{i_k}$. The element $\mathrm{Av}(e_1 \otimes \cdots \otimes e_1)$ has the highest weight aL_1 and generates an irreducible subrepresentation $\Gamma_{a,0}$. The weights of the representation $\mathrm{Sym}^a \mathbf{C}^3$ form a triangle with vertices at aL_1, aL_2, aL_3 and all weight spaces are one-dimensional. By the classification of irreducible representations of $\mathfrak{su}(3)$ this implies that it *is* $\Gamma_{a,0}$

Here is the diagram of the symmetric square:



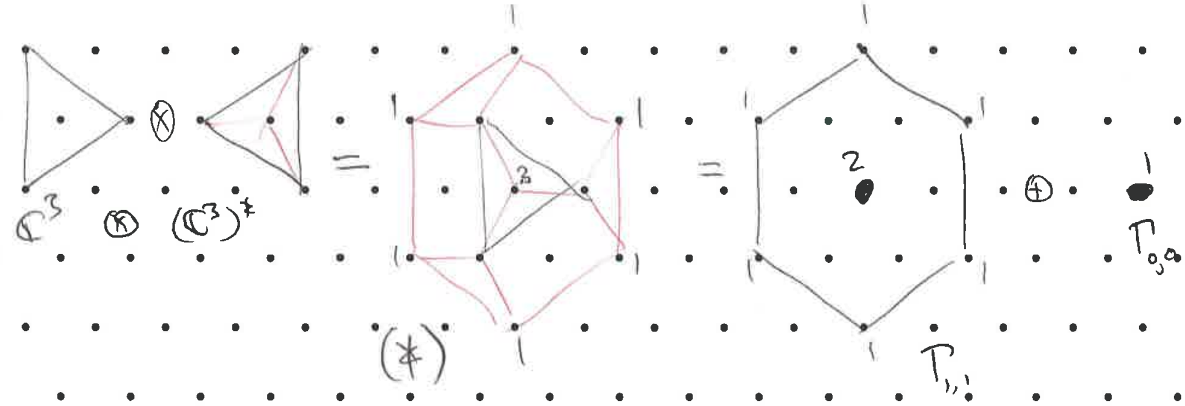
The bth symmetric power of the dual contains a vector of weight $-bL_3$ and the argument proceeds similarly.

(b) The representation $\mathrm{Sym}^a \mathbf{C}^3 \otimes \mathrm{Sym}^b(\mathbf{C}^3)^*$ contains a vector of weight $aL_1 - bL_3$ (namely $e_1^{\otimes a} \otimes (e_3^*)^{\otimes b}$). This is the highest weight that occurs, so it generates an irreducible subrepresentation $\Gamma_{a,b}$.

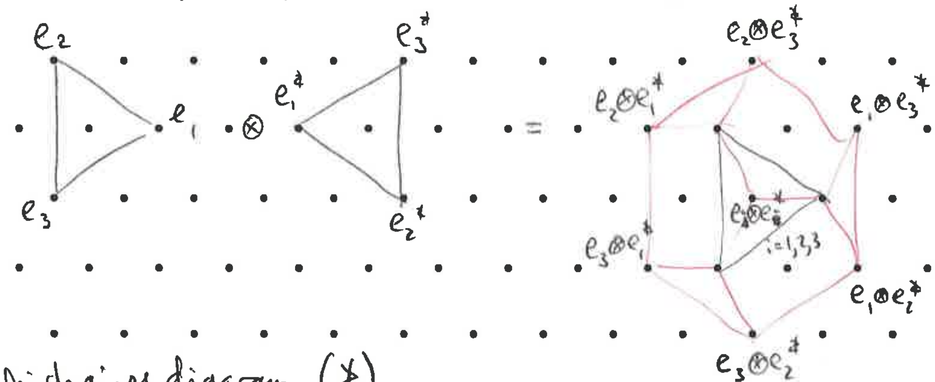
(c-f) See next few pages. Note (typo!) that in 3(e) it should be a $\Gamma_{2,0}$ not a $(\mathbf{C}^*)^3$.

(g) $\Gamma_{2,3} \oplus \Gamma_{0,4} \oplus \Gamma_{3,1} \oplus 2\Gamma_{1,2} \oplus \Gamma_{2,0} \oplus \Gamma_{0,1}$.

3.c)



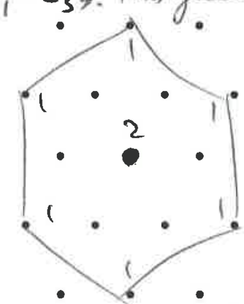
To find the weight diagram of $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ remember that weights add, so:



which gives diagram (*).

Now strip off irreducible subreps starting with the one generated by the highest weight vector (in this case $e_1\otimes e_3^*$ with weight $L_1 - L_3$). This generates a copy of $\Gamma_{1,1}$)

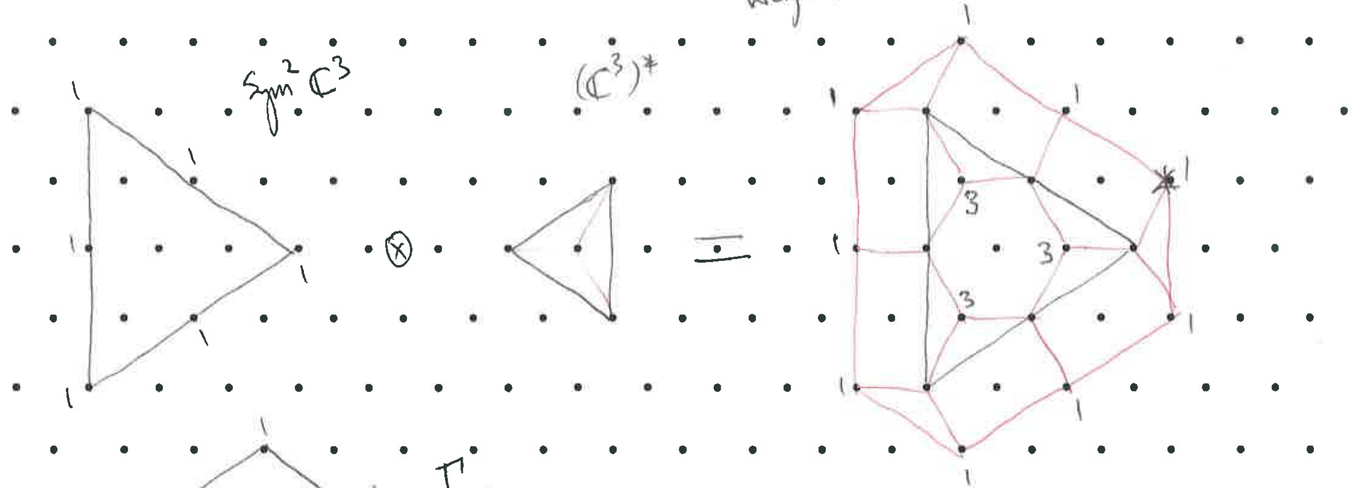
Since $\Gamma_{1,1}$ has weight diagram

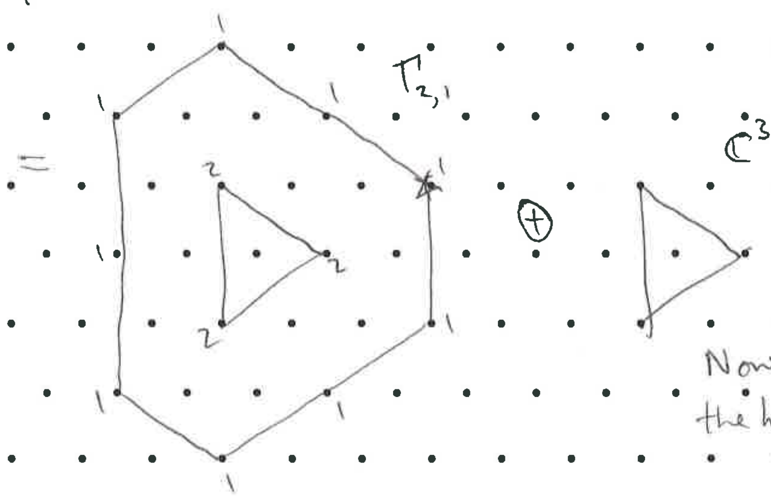


all that is left is a trivial one-dim. rep with weight 0. This is actually spanned by $\sum_{i=1}^{3} e_i \otimes e_i^*$.

3. d)

To find the weights of the tensor product add the weight diagrams by superimposing the diagram of $(\mathbb{C}^3)^*$ onto every vertex of $\mathrm{Sym}^2\mathbb{C}^3$ and adding weights.

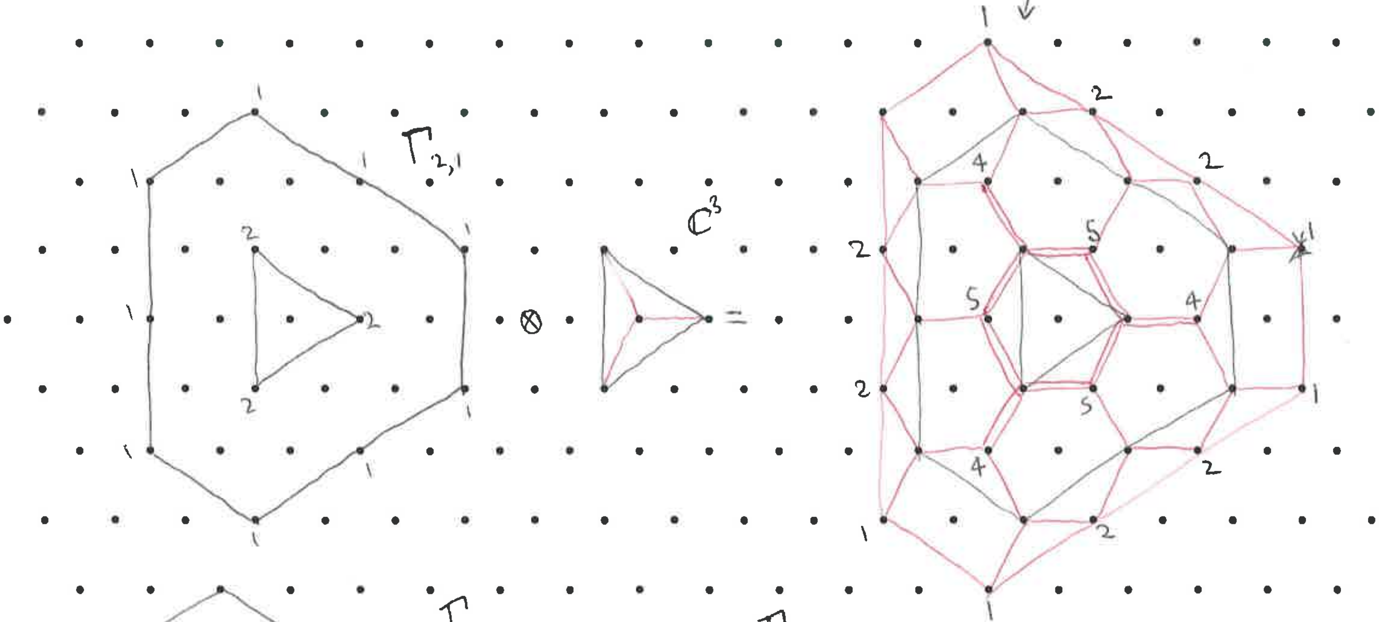


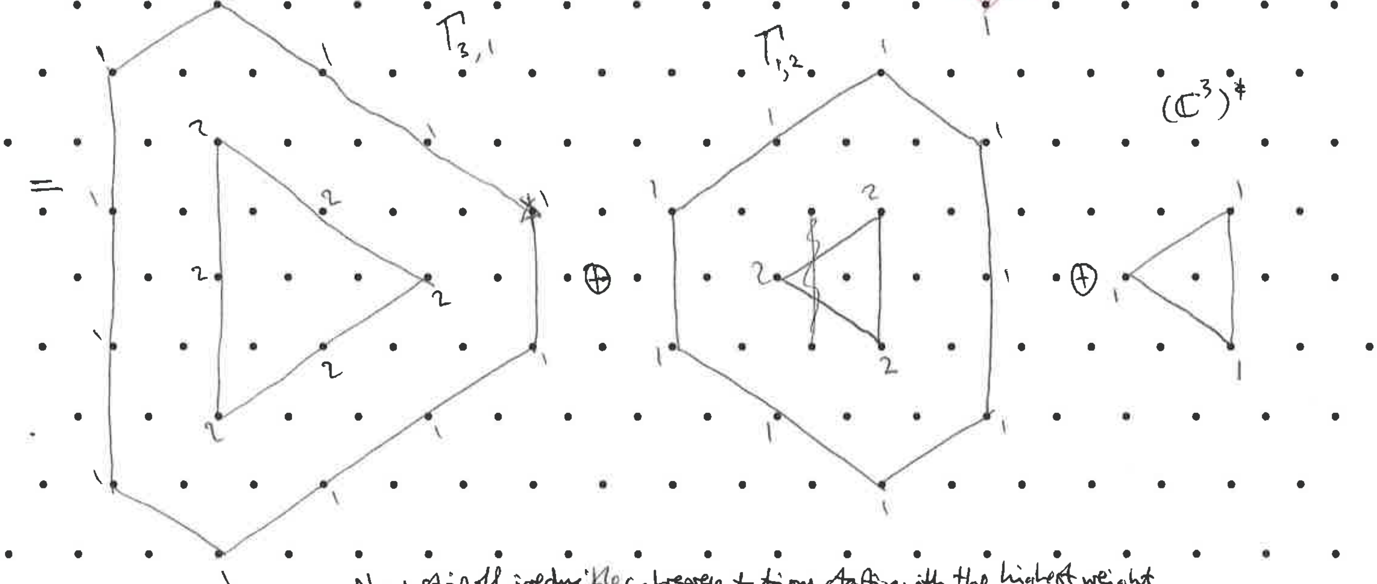


Now strip off subreps starting with the highest weight (in this case $2L_1 - L_3$ marked with a $*$).

3. e)

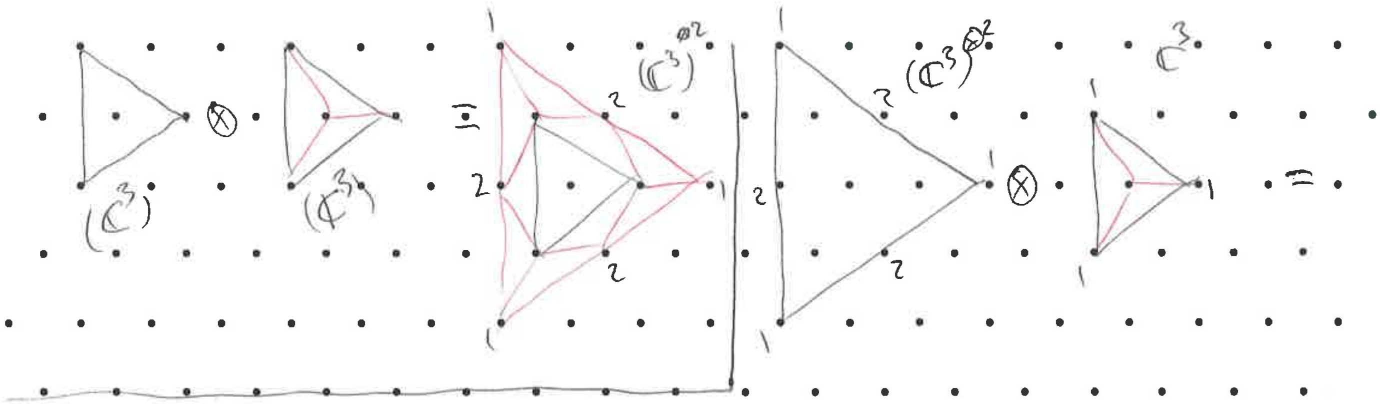
To find the weights of the tensor product, add the weight diagrams by superimposing the diagram for $\mathbb{C}^3$ onto every vertex) in $\Gamma_{2,1}$ and adding multiplicities

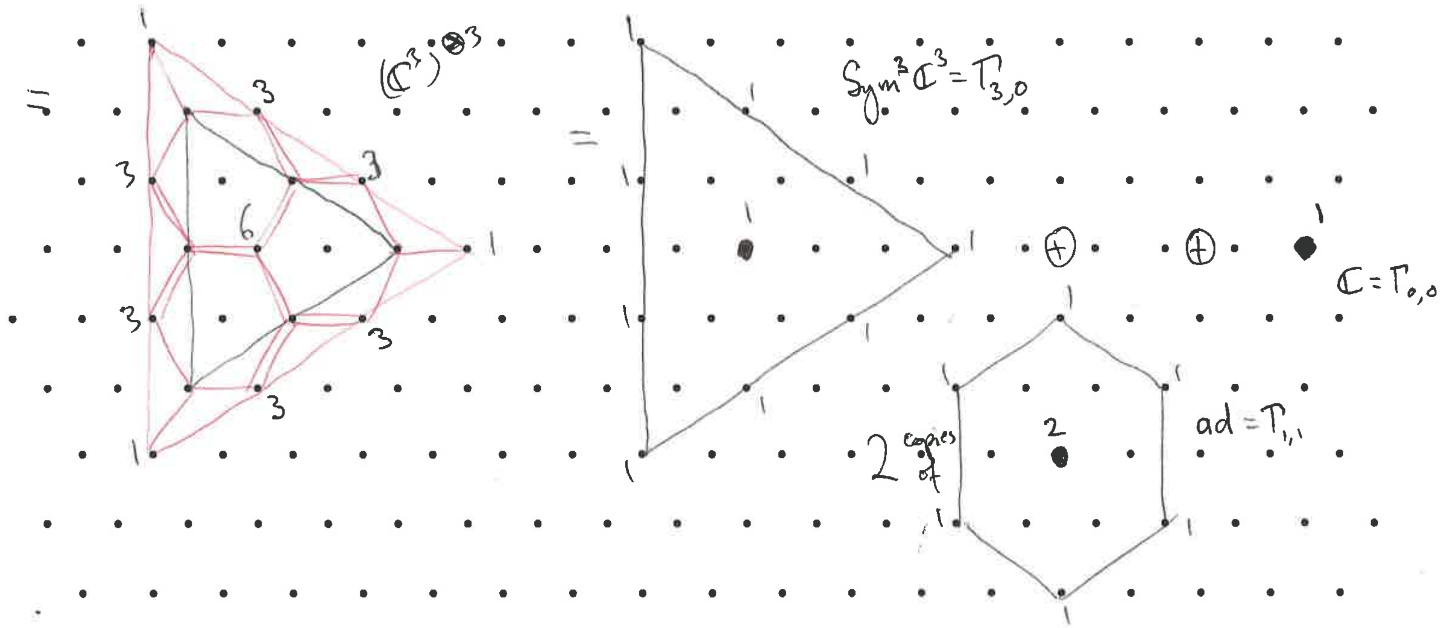




Now strip off irreducible subrepresentations starting with the highest weight (in this case $3L_1 - L_3$, marked with a $*$).

3.f)


$(\mathbb{C}^3)$
$\otimes$
$(\mathbb{C}^3)$
$=$
$(\mathbb{C}^3)^{\otimes 2}$
$(\mathbb{C}^3)^{\otimes 2}$
$\otimes$
$\mathbb{C}^3$
$=$


$=$
$(\mathbb{C}^3)^{\otimes 3}$
$=$
$\mathrm{Sym}^3\mathbb{C}^3 = \Gamma_{3,0}$
$\oplus$
$\oplus$
$\mathbb{C} = \Gamma_{0,0}$
2 copies of
$\mathrm{ad} = \Gamma_{1,1}$

Question 4. Let $\mathfrak{so}(5,\mathbf{C})$ be the Lie algebra of antisymmetric complex 5-by-5 matrices (the complexification of $\mathfrak{so}(5)$). What is its dimension?

You may use a computer algebra system for the rest of this question.

Consider the abelian Lie subalgebra $\mathfrak{t}$ spanned by elements

$$H_1 = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let L_1, L_2 be a $\mathbf{Z}$-basis for the weight lattice $\mathfrak{t}^*_{\mathbf{Z}}$ given by

$$L_i(H_j) = \delta_{ij}.$$

By considering the adjoint action of H_1 and H_2 on the eight matrices

$$K_1^\pm = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \pm i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & \mp i & 0 & 0 & 0 \end{pmatrix}, \quad K_2^\pm = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \pm i \\ 0 & 0 & 1 & \mp i & 0 \end{pmatrix}$$

$$L^\pm = \begin{pmatrix} 0 & 0 & -1 & \pm i & 0 \\ 0 & 0 & \pm i & 1 & 0 \\ 1 & \mp i & 0 & 0 & 0 \\ \mp i & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^\pm = \begin{pmatrix} 0 & 0 & \pm i & 1 & 0 \\ 0 & 0 & -1 & \pm i & 0 \\ \mp i & 1 & 0 & 0 & 0 \\ -1 & \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

find the roots in terms of L_1 and L_2 and draw the root diagram in $\mathfrak{t}^*_{\mathbf{Z}}$.

You can think of $\mathfrak{t}^*_{\mathbf{Z}}$ as the usual square lattice in $\mathbf{R}^2$.

Answer 4. The Lie algebra is 10 complex dimensional.

We have

$$\begin{aligned} [H_i, K_j^\pm] &= \pm\delta_{ij} K_j^\pm \\ [H_1, L^\pm] &= \pm L^\pm \\ [H_2, L^\pm] &= \pm L^\pm \\ [H_1, M^\pm] &= \pm M^\pm \\ [H_2, M^\pm] &= \mp M^\pm \end{aligned}$$

so the weights are $\pm L_i$ (with weight vectors $K_i^\pm$), $\pm(L_1 + L_2)$ (with weight vectors $L^\pm$) and $\pm(L_1 - L_2)$ (with weight vectors $M^\pm$). The root diagram is the set of points

$$\{\pm(1,1), \pm(1,-1), (\pm 1, 0), (0, \pm 1), 0\}$$

where all root spaces have multiplicity 1 except the zero root space, which has multiplicity 2.

Question 5. Let V denote the standard 4-dimensional complex representation of $SU(4)$.

(a) Decompose $V \otimes V$ into its irreducible pieces.

(b) Decompose $V \otimes V^*$ into its irreducible pieces.

Answer 5.

1. There is a tetrahedral weight diagram with ten vertices (all multiplicity 1), namely the symmetric square, and an octahedral diagram with its six corners (all multiplicity 1), namely the exterior square. These are the weight diagrams of the irreducible pieces.
2. This splits as the adjoint representation plus the trivial one-dimensional representation.