Sheet 1: Examples and exponentials

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Question 1.
Prove that the following are equivalent:

(a) $A \in O(n)$ (recall that $A \in O(n)$ if and only if $A^T A = 1$),

(b) $(Av) \cdot (Aw) = v \cdot w$ for all $v, w \in \mathbf{R}^n$,

(c) $|Av|^2 = |v|^2$ for all $v \in \mathbf{R}^n$.

Question 2. (a) Find the exponential of the matrix $H = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$.

(b) Given a matrix $K = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, find a matrix H such that $\exp(H) = K$.

(c) Compute $\exp \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$.

Question 3. Given $v = (x, y, z)$, consider the matrix

$$K_v := \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.$$

(a) Show that if $u \in \mathbf{R}^3$ then for any $v \in \mathbf{R}^3$, $K_u v = u \times v$.

(b) Hence or otherwise, show that if $|u|^2 = 1$ then $K_u^3 = -K_u$ (*Hint: Recall that* $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$.)

(c) Show that if $|u|^2 = 1$ then $\exp(\theta K_u) = 1 + K_u \sin\theta + (1 - \cos\theta) K_u^2$ and check that

$$(\star) \quad \exp(\theta K_u) v = v \cos\theta + (u \times v) \sin\theta + (1 - \cos\theta)(u \cdot v) u.$$

$(\star)$ is *Rodrigues's formula* for the rotation of v by an angle θ around u.

(d) Show by direct computation that $[K_u, K_v] := K_u K_v - K_v K_u = K_{u \times v}$ for any $u, v \in \mathbf{R}^3$.

The last two questions concern the group $SU(2)$ of unitary 2-by-2 matrices with determinant 1 and its Lie algebra $\mathfrak{su}(2)$ of 2-by-2 skew-Hermitian matrices with trace zero.

For $v = (x, y, z) \in \mathbf{R}^3$ we define

$$M_v := \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \in \mathfrak{su}(2).$$

Question 4. Show that $M_u M_v = -(u \cdot v)1 + M_{u \times v}$. Deduce that if $|u|^2 = 1$ then $M_u^2 = -1$ and hence that

$$\exp(\theta M_u) = (\cos\theta)1 + \sin\theta M_u = \begin{pmatrix} \cos\theta + ix\sin\theta & y\sin\theta + iz\sin\theta \\ -y\sin\theta + iz\sin\theta & \cos\theta - ix\sin\theta \end{pmatrix} \in SU(2).$$

Question 5. Consider the action of $SU(2)$ on $\mathfrak{su}(2)$ given by

$$\tilde{\rho}\colon SU(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2), \quad \tilde{\rho}(g, m) = gmg^{-1}.$$

(a) Show that this defines a 3-dimensional real representation $\rho\colon SU(2) \to GL(\mathfrak{su}(2))$ of $SU(2)$.

(b) Show that if $\tilde{\rho}(g, M_v) = M_{v'}$ then $|v'|^2 = |v|^2$. (*Hint: Compute determinants.*)

(c) Recall from Question 4 that if $|u|^2 = 1$ then $\exp(\theta M_u) = (\cos\theta)1 + \sin\theta M_u$. Show that if u and v are vectors and $|u|^2 = 1$ then

$$\tilde{\rho}(\exp(\theta M_u), M_v) = M_{v'}$$

where

$$v' = v\cos 2\theta + (u \times v)\sin 2\theta + (1 - \cos 2\theta)(u \cdot v)u.$$

In other words (Rodrigues's formula), the matrix $\exp(\theta M_u)$ acts as a rotation around u by 2θ.

(d) Let $SO(3)$ denote the group of rotations of 3-dimensional space. Prove that the map $\rho\colon SU(2) \to SO(3)$ is 2-to-1.

The representation $\rho\colon SU(2) \to SO(3)$ is called the spin representation.