Sheet 5: Representations of Lie groups

J. Evans

Question 1.
Let x, y be a basis for the standard representation $\mathbf{C}^2$ of $SU(2)$ and let $x \otimes x$, $\frac{1}{2}(x \otimes y + y \otimes x)$, $y \otimes y$ be a basis for $\mathrm{Sym}^2 \mathbf{C}^2$. Write down a matrix for the action of $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$ on $\mathrm{Sym}^2 \mathbf{C}^2$ in terms of this basis.

Question 2.

(a) Suppose that $R \colon G \to GL(V)$ is a representation. Show that $R^* \colon G \to GL(V^*)$ is a representation.

(b) Suppose that $R_1 \colon G \to GL(V_1)$ and $R_2 \colon G \to GL(V_2)$ are representations and that $L \colon V_1 \to V_2$ is a morphism of representations. Show that the kernel and image of L are subrepresentations of V_1 and V_2 respectively. If G is compact and L is surjective, deduce that $V_1 \cong \ker L \oplus V_2$ as representations.

(c) Prove that the trace map $\mathrm{Tr} \colon \mathfrak{gl}(n, \mathbf{R}) \to \mathbf{R}$ is a morphism from the adjoint representation of $GL(n, \mathbf{R})$ to the trivial one-dimensional representation. Deduce that $\mathfrak{sl}(n, \mathbf{R})$ is a subrepresentation.

(d) Let V denote the standard representation of $SU(2)$. Check that the map

$$\mathrm{Sym}^p V \otimes \mathrm{Sym}^q V \to \mathrm{Sym}^{p+q} V,$$
$$x_1 \cdots x_p \otimes x_{p+1} \cdots x_{p+q} \mapsto x_1 \cdots x_p x_{p+1} \cdots x_{p+q}$$

is a morphism of $SU(2)$-representations. In the case that $p = 1$, $q = 2$:

(i) find the kernel;

(ii) show that the kernel is isomorphic to V;

(iii) deduce that $V \otimes \mathrm{Sym}^2 V \cong V \oplus \mathrm{Sym}^3 V$.

Question 3.

(a) Suppose that $\mathbf{K}$ is a field of characteristic zero and $V = \mathbf{K}\langle e_1, e_2\rangle$. Show that

$$V^{\otimes 2} = \mathrm{Sym}^2(V) \oplus \Lambda^2(V).$$

(b) Let e_i be a basis of $\mathbf{R}^n$ and, for $i = 1, \ldots, n$, let a_i be the vector $\sum_{i=1}^n a_i^j e_j$. By inspecting the formula for the alternating map, show that $a_1 \wedge \cdots \wedge a_n = \det(a_i^j)(e_1 \wedge \cdots \wedge e_n)$.

Question 4.

Let $V = \mathbf{K}\langle e_1, e_2\rangle$ and $W = \mathbf{K}\langle f_1, f_2\rangle$ be vector spaces and consider an element $t = \sum_{i,j=1}^2 t_{ij} e_i \otimes e_j \in V \otimes W$ in their tensor product. Show that there exist $v \in V$ and $w \in W$ such that $t = v \otimes w$ (i.e. t is a pure tensor) if and only if $t_{11}t_{22} = t_{12}t_{21}$.

This equation is called the Plücker relation. For tensor products of higher dimensional vector spaces there are many more Plücker relations $t_{ij}t_{k\ell} = t_{i\ell}t_{kj}$. This tells us that the pure tensors form a subvariety of $V \otimes W$ cut out by a collection of homogeneous polynomials of degree 2.

Question 5.

Suppose that $R\colon G \to GL(V)$ and $S\colon G \to GL(W)$ are representations and denote by T the representation of G on $\mathrm{Hom}(V, W)$ defined by

$$(T(g)F)(v) = S(g)F(R(g^{-1})v).$$

(a) Show that a vector $F \in \mathrm{Hom}(V, W)$ satisfies $T(g)F = F$ for all $g \in G$ if and only if F is a morphism of representations. If V and W are irreducible, show that either $F = 0$ or F is an isomorphism.

(b) Define a map $\Phi\colon V^* \otimes W \to \mathrm{Hom}(V, W)$ by defining it on pure tensors as

$$\Phi(f \otimes w)(v) = f(v)w$$

and extending linearly. Check that this is an isomorphism of representations $R^* \otimes S \cong T$.

Hint: Once you know it's a morphism of representations, to show it's an isomorphism, pick bases e_i of V and f_j of W and check that $\Phi(e_i^* \otimes f_j)$ is a basis for $\mathrm{Hom}(V, W)$. Why does this tell you it is an isomorphism?