Sheet 6: More on representations

J. Evans

Question 1.
Write out the action of $X, Y \in \mathfrak{sl}(2, \mathbf{C})$ on $\mathrm{Sym}^3(\mathbf{C}^2)$ explicitly.

Answer 1. The $\mathrm{Sym}^3 \mathbf{C}^2$ action of X with respect to the basis

$$e_1^{\otimes 3},\ e_1^2 e_2 = \frac{1}{3}(e_1 \otimes e_1 \otimes e_2 + \text{cyclic permutations}),\ e_1 e_2^2 = \frac{1}{3}(e_1 \otimes e_2 \otimes e_2 + \text{cyclic permutations}),\ e_2^3$$

sends

$$\begin{aligned} e_1^{\otimes 3} &\mapsto 0 \\ e_1^2 e_2 &\mapsto e_1^{\otimes 3} \\ e_1 e_2^2 &\mapsto 2 e_1^2 e_2 \\ e_2^{\otimes 3} &\mapsto 3 e_1 e_2^2 \end{aligned}$$

and for Y we get

$$\begin{aligned} e_1^{\otimes 3} &\mapsto 3 e_1^2 e_2 \\ e_1^2 e_2 &\mapsto 2 e_1 e_2^2 \\ e_1 e_2^2 &\mapsto e_2^{\otimes 3} \\ e_2^{\otimes 3} &\mapsto 0. \end{aligned}$$

As matrices with respect to this polynomial basis, we have

$$\mathrm{Sym}^3(X) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathrm{Sym}^3(Y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Question 2.
Decompose the following representations of $\mathfrak{sl}(2, \mathbf{C})$ into irreducible summands:

(a) $\Lambda^2 \operatorname{Sym}^3 \mathbf{C}^2$,

(b) $\operatorname{Sym}^2 \operatorname{Sym}^2 \mathbf{C}^2$,

(c) $\Lambda^3 \operatorname{Sym}^4 \mathbf{C}^2$,

(d) $\operatorname{Sym}^3 \operatorname{Sym}^2 \mathbf{C}^2$,

(e) $\operatorname{Sym}^2 \operatorname{Sym}^3 \mathbf{C}^2$,

(f) $\operatorname{Sym}^2 \Lambda^2 \operatorname{Sym}^3 \mathbf{C}^2$,

(g) $\operatorname{Sym}^2 \operatorname{Sym}^4 \mathbf{C}^2$,

(h) $\operatorname{Sym}^3 \operatorname{Sym}^4 \mathbf{C}^2$,

From your computations in (g) and (h) deduce that there are quadratic and cubic invariants $g_2(a, b, c, d, e)$ and $g_3(a, b, c, d, e)$ for binary quartic polynomials $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ under the action of $SL(2, \mathbf{C})$.

Answer 2. In all cases the method is the same: write out a basis of vectors with fixed weights and then appeal to the classification of irreducible representations. In each case, v will denote a highest weight vector for $\operatorname{Sym}^m \mathbf{C}^2$ and e_n will denote $Y^n v$. Recall that $XY^n v = (m - n + 1)nY^{n-1}v$ so $Xe_n = (m - n + 1)ne_{n-1}$ and $Ye_n = e_{n+1}$.

(a) We have a basis e_0, e_1, e_2, e_3 for $\operatorname{Sym}^3 \mathbf{C}^2$ and a basis $e_0 \wedge e_1$, $e_0 \wedge e_2$, $e_0 \wedge e_3$, $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_2 \wedge e_3$ for $\Lambda^2 \operatorname{Sym}^3 \mathbf{C}^2$. Since e_n has weight $2n - 3$ and $e_i \wedge e_j$ has weight $2i + 2j - 6$, these terms each transform with weight $-4, -2, 0, 0, 2, 4$. There is therefore an irreducible subrepresentation with weight 4, isomorphic to $\operatorname{Sym}^4 \mathbf{C}^2$ by the classification of irreducible representations. This accounts for all but a one-dimensional subrepresentation which is therefore trivial, so $\Lambda^2 \operatorname{Sym}^3 \mathbf{C}^2 \cong \operatorname{Sym}^4 \mathbf{C}^2 \oplus \mathbf{C}$. Note that we can identify the trivial summand: it is spanned by a linear combination $\alpha = Ae_0 \wedge e_3 + Be_1 \wedge e_2$ which is annihilated by X (and Y). Since $Xe_3 = 3e_2$, $Xe_2 = 4e_1$, $Xe_1 = 3e_0$, $Xe_0 = 0$, we have

$$X\alpha = 3Ae_0 \wedge e_2 + 3Be_0 \wedge e_2$$

so the one-dimensional subrepresentation is spanned by $e_0 \wedge e_3 - e_1 \wedge e_2$.

(b) Taking a basis e_0, e_1, e_2 of $\operatorname{Sym}^2 \mathbf{C}^2$, forming all possible homogeneous quadratic polynomials in these basis elements and grouping them according to weights gives

$$e_0^2 \qquad e_0e_1, \qquad e_0e_2,\ e_1^2, \qquad e_1e_2,\ e_2^2$$

(note that we are writing e_ie_j for $\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$) so that, by the classification of irreducible representations, the representation $\operatorname{Sym}^2 \operatorname{Sym}^2 \mathbf{C}^2$ splits as $\operatorname{Sym}^4 \mathbf{C}^2 \oplus \mathbf{C}$. In this instance, the trivial subrepresentation is spanned by $e_1^2 - 4e_0e_2$. To see this, note that $X(e_1^2) = (Xe_1) \otimes e_1 + e_1 \otimes (Xe_1) = 2(e_0 \otimes e_1 + e_1 \otimes e_0) = 4e_0e_1$ while $X(e_0e_2) = e_0e_1$.

(c) By arguing similarly we get $\Lambda^3 \operatorname{Sym}^4 \mathbf{C}^2 \cong \operatorname{Sym}^6 \mathbf{C}^2 \oplus \operatorname{Sym}^2 \mathbf{C}^2$.

(d) By arguing similarly we get $\operatorname{Sym}^3 \operatorname{Sym}^2 \mathbf{C}^2 \cong \operatorname{Sym}^6 \mathbf{C}^2 \oplus \operatorname{Sym}^2 \mathbf{C}^2$.

(e) By arguing similarly we get $\operatorname{Sym}^2 \operatorname{Sym}^3 \mathbf{C}^2 \cong \operatorname{Sym}^6 \mathbf{C}^2 \oplus \operatorname{Sym}^2 \mathbf{C}^2$.

(f) In this example we already know from the first part that $\Lambda^2 \operatorname{Sym}^3 \mathbf{C}^2 = \operatorname{Sym}^4 \mathbf{C} \oplus \mathbf{C}$ so we take bases e_0, e_1, e_2, e_3, e_4 of $\operatorname{Sym}^4 \mathbf{C}^2$ and f of $\mathbf{C}$, so a basis for the symmetric-square is

$$e_ie_j,\ 0 \le i \le j \le 4, \qquad fe_i,\ 0 \le i \le 4, \qquad f^2$$

and we compute that the irreducible decomposition is $\mathrm{Sym}^8 \mathbf{C}^2 \oplus 2\,\mathrm{Sym}^4 \mathbf{C}^2 \oplus 2\mathbf{C}$.

(g) We get $\mathrm{Sym}^2 \mathrm{Sym}^4 \mathbf{C}^2 = \mathrm{Sym}^8 \mathbf{C}^2 \oplus \mathrm{Sym}^4 \mathbf{C}^2 \oplus \mathbf{C}$.

(h) We get $\mathrm{Sym}^3 \mathrm{Sym}^4 \mathbf{C}^2 = \mathrm{Sym}^{12} \mathbf{C}^2 \oplus \mathrm{Sym}^8 \mathbf{C}^2 \oplus \mathrm{Sym}^6 \mathbf{C}^2 \oplus \mathrm{Sym}^4 \mathbf{C}^2 \oplus \mathbf{C}$.

There is a unique quadratic invariant g_2 because $\mathrm{Sym}^2 \mathrm{Sym}^4 \mathbf{C}^2$ has a unique one-dimensional trivial subrepresentation. Similarly for g_3.

Question 3. (Clebsch-Gordan theorem)
Let V denote the standard 2-dimensional representation of $\mathfrak{sl}(2, \mathbf{C})$. Prove that the tensor product $\mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(V)$ decomposes into irreducible representations

$$\bigoplus_{\substack{k=|m-n| \\ k \equiv m+n \mod 2}}^{m+n} \mathrm{Sym}^k(V).$$

Answer 3. The representation $\mathrm{Sym}^n \mathbf{C}^2 = \bigoplus_{i=0}^n \mathbf{C} \cdot e_i$ is a direct sum of weight spaces $\mathbf{C} \cdot e_i$ with weight $n - 2i$ and $\mathrm{Sym}^m \mathbf{C}^2$ is a direct sum $\bigoplus_{j=0}^m \mathbf{C} \cdot f_j$. Suppose that $m \geq n$ so that $|m - n| = m - n$. If we list generators by their weight, we get a diagram like this:

$$\begin{array}{ccccccccccc}
e_0 \otimes f_0 & e_0 \otimes f_1 & e_0 \otimes f_2 & \cdots & e_0 \otimes f_n & e_0 \otimes f_{n+1} & \cdots & e_0 \otimes f_m & e_1 \otimes f_m & \cdots & e_n \otimes f_m \\
 & e_1 \otimes f_0 & e_1 \otimes f_1 & \cdots & e_1 \otimes f_{n-1} & e_1 \otimes f_n & \cdots & e_1 \otimes f_m & e_2 \otimes f_{m-1} & & \\
 & & e_2 \otimes f_0 & & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & & & & \vdots & \vdots & \vdots & \vdots & e_n \otimes f_{m-n+1} & & \\
 & & & & e_n \otimes f_0 & e_n \otimes f_1 & \cdots & e_n \otimes f_{m-n} & & &
\end{array}$$

i.e.

$$\begin{array}{ccccccccccc}
\bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet \\
 & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & \\
 & & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
 & & & \bullet & \vdots & \vdots & \vdots & \bullet & & & \\
 & & & & \bullet & \cdots & \bullet & & & &
\end{array}$$

The top row has length $n + m + 1$ and contains a highest weight vector with weight $n + m$ so we can peel off a copy of $\mathrm{Sym}^{n+m}(\mathbf{C}^2)$. The next row has length $n + m - 1$ and contains a highest weight vector with weight $n + m - 2$ so we can peel off a copy of $\mathrm{Sym}^{n+m-2}(\mathbf{C}^2)$. Continuing in this manner we reach the last row which has length $|m - n| + 1$ and we peel off a copy of $\mathrm{Sym}^{|m-n|}(\mathbf{C}^2)$. This gives the Clebsch-Gordan formula.

Question 4.
Prove that if $\rho\colon \mathfrak{sl}(2,\mathbf{C}) \to \mathfrak{gl}(V)$ is a representation and X, Y, H denote the usual basis of $\mathfrak{sl}(2,\mathbf{C})$ satisfying the commutation relations

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H$$

then

$$C := \rho(X)\rho(Y) + \rho(Y)\rho(X) + \frac{1}{2}\rho(H)^2$$

commutes with $\rho(X)$, $\rho(Y)$ and $\rho(H)$. Deduce that if $V = \bigoplus_\lambda V_\lambda$ is the decomposition of V into eigenspaces of C then each V_λ is a subrepresentation. If V is irreducible with highest weight m, deduce that C is the diagonal matrix $\left(m + \frac{1}{2}m^2\right)\mathrm{Id}$.

Answer 4. In the solution we will write X for $\rho(X)$, etc. and the fact that ρ is a homomorphism of Lie algebras means that $[X,Y]$ means both $[\rho(X),\rho(Y)]$ and $\rho([X,Y])$ so the notation is well-defined! We have

$$\begin{aligned}
CX &= (XY + YX + H^2/2)X \\
&= XYX + YXX + HHX/2 \\
&= XYX + [Y,X]X + XYX + H[H,X]/2 + HXH/2 \\
&= XYX - HX + X[Y,X] + XXY + HX + [H,X]H/2 + XHH/2 \\
&= XYX - HX - XH + XXY + HX + XH + XHH/2 \\
&= X(XY + YX + H^2/2) \\
&= XC.
\end{aligned}$$

Similar arguments work for Y and H.

Now since C commutes with $\rho(X)$, if $Cv = \lambda v$ then

$$C\rho(X)v = \rho(X)Cv = \rho(X)\lambda v = \lambda\rho(X)v$$

so V_λ is preserved by $\rho(X)$ (and $\rho(Y), \rho(H)$ by the same argument). Therefore it is a subrepresentation.

If V is irreducible then $V = V_\lambda$ for some eigenvalue λ and hence $Cv = \lambda v$ for all $v \in V$. To compute λ, assume that v is a highest weight vector with weight m. Then $\rho(X)v = 0$, $\rho(H)v = mv$ and $\rho(Y)v$ satisfies

$$\rho(X)\rho(Y)v = mv, \; \rho(H)\rho(Y)v = (m-2)\rho(Y)v$$

by the computations we did in the proof of the classification theorem for irreducible $\mathfrak{sl}(2,\mathbf{C})$-representations. Therefore

$$Cv = \rho(X)\rho(Y)v + \rho(Y)\rho(X)v + \frac{1}{2}\rho(H)^2 v = mv + m^2 v/2$$

and $\lambda = m + m^2/2$ as required.

If $R\colon G \to GL(V)$ is a representation we say that:

- $M \in V$ is *R-invariant* if $R(g)M = M$ for all $g \in G$.
- a symmetric bilinear form $B\colon V \otimes V \to \mathbf{K}$ is *R-invariant* if

  $$B(R(g)v, R(g)w) = B(v, w)$$

  for all $v, w \in V$ and $g \in G$.

If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, we say that

- $M \in V$ is *ρ-invariant* if $\rho(X)M = 0$ for all $X \in \mathfrak{g}$.
- a symmetric bilinear form $B\colon V \times V \to \mathbf{K}$ is *ρ-invariant* if

  $$B(\rho(X)v, w) + B(v, \rho(X)w) = 0$$

  for all $v, w \in V$ and $X \in \mathfrak{g}$.

Question 5.
Let G be a connected Lie group and $R\colon G \to GL(V)$ be a representation. Let $\rho = R_*$ be the linearisation of R.

(a) Prove that $M \in V$ is G-invariant if and only if it is R_*-invariant.

(b) Prove that a symmetric bilinear form $B\colon V \times V \to \mathbf{K}$ is R-invariant if and only if it is R_*-invariant.

Hint: To show M or B is R-invariant it suffices to check it on an exponential chart because the group is connected and connected groups are generated by the image of an exponential chart.

(c) Let $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation $X \mapsto \mathrm{ad}_X$, $\mathrm{ad}_X Y = [X, Y]$. Define the symmetric bilinear form

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y)$$

where Tr denotes the trace. Using some form of the Jacobi identity, prove that B is ad-invariant. This is called the *Killing form*.

Hint: The trace of a commutator of matrices vanishes.

(d) Let X, H, Y be the usual basis for $\mathfrak{sl}(2, \mathbf{C})$. Check that with respect to this basis

$$\mathrm{ad}_X = \begin{pmatrix} 0 & -2 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \ \mathrm{ad}_H = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2 \end{pmatrix}, \ \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 2 & \cdots & 0 \end{pmatrix}$$

and hence compute the Killing form on all pairs $B(a, b)$, $a, b \in \{X, H, Y\}$.

Answer 5. (a) If $\rho(X)M = 0$ then $\exp(\rho(X))M = (1 + \rho(X) + \frac{1}{2}\rho(X)^2 + \cdots)M = M$. But $R(\exp X)M = \exp(\rho(X))M$ so $R(\exp X)M = M$ for all X and hence $R(g)M = M$ for all g in a neighbourhood of 1 and hence for all $g \in G$ since G is generated by a neighbourhood of 1 by connectedness of G.

Conversely if $R(g)M = M$ for all g then $R(\exp tX)M = M$ for all t and differentiating with respect to t we get $\rho(X)M = 0$ as required.

(b) If B is R-invariant then differentiating $B(R(e^{tX})v, R(e^{tX})w) = B(v, w)$ with respect to t at $t = 0$ gives

$$B(\rho(X)v, w) + B(v, \rho(X)w) = 0.$$

Conversely if $B(\rho(X)v, w) = -B(v, \rho(X)w)$ then $B(\rho(X)^n v, \rho(X)^{k-n} w) = (-1)^n B(v, \rho(X)^k w)$ so

$$\begin{aligned}
B(R(\exp X)v, R(\exp X)w) &= B(\exp(\rho(X))v, \exp(\rho(X))w) \\
&= \sum_{n,m} \frac{1}{n!m!} B(\rho(X)^n v, \rho(X)^m w) \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{1}{k!} \binom{k}{n} B(\rho(X)^n v, \rho(X)^{k-n} w) \\
&= B(v, w) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{n=0}^{k} \binom{k}{n} (-1)^n \right) B(v, \rho(X)^k w) \\
&= B(v, w)
\end{aligned}$$

since all $k \geq 1$ terms contain $\sum_{n=0}^{k} \binom{k}{n} (-1)^n = (1-1)^k = 0$.

Alternatively you can argue as follows. The t-derivative of $B(R(e^{tX})v, R(e^{tX})w)$ is

$$B(\rho(X)R(e^{tX})v, R(e^{tX})w) + B(R(e^{tX})v, \rho(X)R(e^{tX})w) = 0$$

and certainly at $t = 0$ we have $B(R(e^{0X})v, R(e^{0X})w) = B(v, w)$, hence

$$B(R(e^{tX})v, R(e^{tX})w) = B(v, w) \text{ for all } t.$$

This proves invariance for a neighbourhood of the identity and hence for the whole group because G is connected and hence generated by a neighbourhood of the identity.

(c) The form is clearly bilinear and it is symmetric because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. To see ad-invariance we must check that

$$B(\mathrm{ad}_Z X, Y) + B(X, \mathrm{ad}_Z Y) = 0$$

i.e. that

$$\mathrm{Tr}\left(\mathrm{ad}_{[Z,X]} \mathrm{ad}_Y + \mathrm{ad}_X \mathrm{ad}_{[Z,Y]}\right) = 0.$$

The Jacobi identity allows us to rewrite the expression inside the trace as

$$\mathrm{ad}_Z \mathrm{ad}_X \mathrm{ad}_Y - ad_X \mathrm{ad}_Z \mathrm{ad}_Y + \mathrm{ad}_X \mathrm{ad}_Z \mathrm{ad}_Y - \mathrm{ad}_X \mathrm{ad}_Y \mathrm{ad}_Z$$

or

$$[\mathrm{ad}_Z, \mathrm{ad}_X \,\mathrm{ad}_Y].$$

Since the trace of a commutator vanishes, we get ad-invariance.

(d) For $\mathfrak{sl}(2, \mathbf{C})$ we use the basis H, X, Y to compute the Killing form. We have

$$\mathrm{ad}_X H = -2X,\ \mathrm{ad}_X Y = H,\ \mathrm{ad}_Y H = 2Y,\ \mathrm{ad}_Y X = -H,\ \mathrm{ad}_H X = 2X,\ \mathrm{ad}_H Y = -2Y$$

so:

- $\mathrm{ad}_H \mathrm{ad}_X$ sends X and Y to zero and H to $-4X$; $\mathrm{ad}_H \mathrm{ad}_Y$ sends X and Y to zero and H to $-4Y$. Both of these maps have trace zero.
- $\mathrm{ad}_X \mathrm{ad}_Y$ sends X to $2X$, Y to 0 and H to $2H$ which has trace 4.
- $\mathrm{ad}_H \mathrm{ad}_H$ sends X to $4X$, Y to $4Y$ and H to zero so the trace is 8.
- $\mathrm{ad}_X \mathrm{ad}_X$ sends Y to $-2X$ and X and H to zero so the trace is zero.
- $ad_Y \,\mathrm{ad}_Y$ sends X to $-2Y$ and Y and H to zero so the trace is zero.

Therefore the Killing form with respect to this basis is

$$B(H, H) = 8,\ B(X, X) = B(Y, Y) = 0,\ B(X, Y) = 4,\ B(H, X) = B(H, Y) = 0.$$