Sheet 1: Examples and exponentials

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Question 1.
Prove that the following are equivalent:

(a) $A \in O(n)$ (recall that $A \in O(n)$ if and only if $A^T A = 1$),

(b) $(Av) \cdot (Aw) = v \cdot w$ for all $v, w \in \mathbf{R}^n$,

(c) $|Av|^2 = |v|^2$ for all $v \in \mathbf{R}^n$.

Answer 1. (c) is obvious given (b): just take $v = w$. To show (b) from (c), take $u = v + w$ and note that

$$|v|^2 + |w|^2 + 2v \cdot w = |u|^2 = |Au|^2 = |Av|^2 + |Aw|^2 + 2(Av) \cdot (Aw)$$

which implies (b) since we are in characteristic $0 \neq 2$.

Suppose that $A \in O(n)$. Then $A^T A = 1$ so

$$(Av) \cdot (Aw) = v^T A^T A w = v^T w = v \cdot w.$$

Conversely, (b) holds then $v^T A^T A w = v^T w$. If we let v and w run independently over an orthonormal basis $\{e_i\}$ then $e_i^T A^T A e_j = (A^T A)_{ij} = e_i^T e_j = \delta_{ij}$, so $A^T A = 1$.

Out of interest, here are two ways of going straight from (c) to (a):

- If $v = \sum_i x_i e_i$ then $|v|^2 = \sum_i x_i^2$ and $|Av|^2 = \sum_{j,k} \sum_i A_{ij} A_{ik} x_j x_k$. Thinking of these as polynomials in the x_n and comparing coefficients we get $A_{ij} A_{ik} = \delta_{ij}$ so $A^T A = 1$.
- $v^T v = |v|^2 = |Av|^2 = v^T A^T A v$ so $v^T (A^T A - 1) v = 0$. $A^T A - 1$ is symmetric and hence can be diagonalised in some orthonormal basis e_i. With respect to this basis, $e_i (A^T A - 1) e_i = (A^T A - 1)_{ii} = 0$ so the diagonal entries are all zero. Since this matrix is in diagonal form, it must vanish. Hence $A^T A = 1$.

Question 2. (a) Find the exponential of the matrix $H = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$.

(b) Given a matrix $K = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, find a matrix H such that $\exp(H) = K$.

(c) Compute $\exp \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$.

Answer 2. (a)

$$\begin{aligned}\exp(H) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

(b) We want to solve

$$\begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

for x, y, z so $x = a$, $y = b$ and $z = c - \frac{ab}{2}$, so we see

$$\exp \begin{pmatrix} 0 & a & c - \frac{ab}{2} \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) We have

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}^n = \begin{pmatrix} x^n & x^{n-1}y \\ 0 & 0 \end{pmatrix}$$

so

$$\exp \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{pmatrix} x^n & x^{n-1}y \\ 0 & 0 \end{pmatrix}$$

This gives

$$\exp \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^x & \frac{e^x - 1}{x} y \\ 0 & 1 \end{pmatrix}.$$

Question 3. Given $v = (x, y, z)$, consider the matrix

$$K_v := \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.$$

(a) Show that if $u \in \mathbf{R}^3$ then for any $v \in \mathbf{R}^3$, $K_u v = u \times v$.

(b) Hence or otherwise, show that if $|u|^2 = 1$ then $K_u^3 = -K_u$ (*Hint: Recall that* $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$.)

(c) Show that if $|u|^2 = 1$ then $\exp(\theta K_u) = 1 + K_u \sin\theta + (1 - \cos\theta)K_u^2$ and check that

$$(\star) \quad \exp(\theta K_u)v = v\cos\theta + (u \times v)\sin\theta + (1 - \cos\theta)(u \cdot v)u.$$

$(\star)$ is *Rodrigues's formula* for the rotation of v by an angle θ around u.

(d) Show by direct computation that $[K_u, K_v] := K_u K_v - K_v K_u = K_{u \times v}$ for any $u, v \in \mathbf{R}^3$.

Answer 3. (a) If $u = (x, y, z)$ and $v = (a, b, c)$ then we have

$$K_u v = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} cy - bz \\ az - cx \\ bx - ay \end{pmatrix} = u \times v.$$

(b) $K_u^3 v = K_u^2(u \times v) = K_u(u \times (u \times v)) = K_u(u(u \cdot v) - |u|^2 v) = -K_u v$ since $K_u u = u \times u = 0$ and $|u|^2 = 1$. Therefore $K_u^3 = -K_u$.

(c)

$$\begin{aligned} \exp(\theta K_u) &= 1 + \sum_{n=1}^{\infty} \frac{\theta^n}{n!} K_u^n \\ &= 1 + \sum_{n \equiv 1 \mod 2} \frac{\theta^n}{n!}(-1)^{(n-1)/2} K_u + \sum_{n \equiv 0 \mod 2} \frac{\theta^n}{n!}(-1)^{n/2+1} K_u^2 \\ &= 1 + K_u \sin\theta + K_u^2(1 - \cos\theta) \end{aligned}$$

Equation $(\star)$ now follows immediately from this formula, the fact that $K_u v = u \times v$ and the formula $u \times (u \times v) = (u \cdot v)u - v$ (using $|u|^2 = 1$)

(d) We have

$$\begin{aligned} K_u K_v w - K_v K_u w &= u \times (v \times w) - v \times (u \times w) \\ &= (u \cdot w)v - (u \cdot v)w - (v \cdot w)u + (u \cdot v)w \\ &= (u \cdot w)v - (v \cdot w)u \\ &= (u \times v) \times v \\ &= K_{u \times v} w. \end{aligned}$$

The last two questions concern the group $SU(2)$ of unitary 2-by-2 matrices with determinant 1 and its Lie algebra $\mathfrak{su}(2)$ of 2-by-2 skew-Hermitian matrices with trace zero.

For $v = (x, y, z) \in \mathbf{R}^3$ we define

$$M_v := \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \in \mathfrak{su}(2).$$

Question 4. Show that $M_u M_v = -(u \cdot v)1 + M_{u \times v}$. Deduce that if $|u|^2 = 1$ then $M_u^2 = -1$ and hence that

$$\exp(\theta M_u) = (\cos\theta)1 + \sin\theta M_u = \begin{pmatrix} \cos\theta + ix\sin\theta & y\sin\theta + iz\sin\theta \\ -y\sin\theta + iz\sin\theta & \cos\theta - ix\sin\theta \end{pmatrix} \in SU(2).$$

Answer 4. If $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then

$$\begin{aligned} \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \begin{pmatrix} ia & b + ic \\ -b + ic & -ia \end{pmatrix} &= \begin{pmatrix} -ax - by - cz + i(yc - bz) & az - cx + i(bx - ay) \\ cx - az + i(bx - ay) & -ax - by - cz + i(bz - cy) \end{pmatrix} \\ &= -(u \cdot v)1 + M_{u \times v} \end{aligned}$$

This implies immediately that if $|u|^2 = 1$ then $M_u^2 = -1$ so

$$\begin{aligned} \exp(\theta M_u) &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} M_u^n \\ &= \sum_{n \equiv 0 \mod 2} \frac{\theta^n}{n!}(-1)^{n/2} + \sum_{n \equiv 1 \mod 2} \frac{\theta^n}{n!}(-1)^{(n-1)/2} M_u \\ &= (\cos\theta)1 + M_u \sin\theta \end{aligned}$$

as required.

Question 5. Consider the action of $SU(2)$ on $\mathfrak{su}(2)$ given by

$$\tilde{\rho}\colon SU(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2), \quad \tilde{\rho}(g, m) = gmg^{-1}.$$

(a) Show that this defines a 3-dimensional real representation $\rho\colon SU(2) \to GL(\mathfrak{su}(2))$ of $SU(2)$.

(b) Show that if $\tilde{\rho}(g, M_v) = M_{v'}$ then $|v'|^2 = |v|^2$. (*Hint: Compute determinants.*)

(c) Recall from Question 4 that if $|u|^2 = 1$ then $\exp(\theta M_u) = (\cos\theta)1 + \sin\theta M_u$. Show that if u and v are vectors and $|u|^2 = 1$ then

$$\tilde{\rho}(\exp(\theta M_u), M_v) = M_{v'}$$

where

$$v' = v\cos 2\theta + (u \times v)\sin 2\theta + (1 - \cos 2\theta)(u \cdot v)u.$$

In other words (Rodrigues's formula), the matrix $\exp(\theta M_u)$ acts as a rotation around u by 2θ.

(d) Let $SO(3)$ denote the group of rotations of 3-dimensional space. Prove that the map $\rho\colon SU(2) \to SO(3)$ is 2-to-1.

The representation $\rho\colon SU(2) \to SO(3)$ is called the spin representation.

Answer 5. (a) We need to show that $\rho(g)$, the map sending m to gmg^{-1} is

- linear: this is easy to check because $g(k_1m_1 + k_2m_2)g^{-1} = k_1gm_1g^{-1} + k_2gm_2g^{-1}$ for $k_i \in \mathbf{R}$ and $m_i \in \mathbf{R}^3$;
- a homomorphism: $\rho(gh)m = ghm(gh)^{-1} = ghmh^{-1}g^{-1} = \rho(g)\rho(h)m$ hence $\rho(gh) = \rho(g)\rho(h)$.

(b) If $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then the determinant of M_v is $x^2 + y^2 + z^2$. Since $\det(gmg^{-1}) = \det(g)\det(m)\det(g)^{-1} = \det(m)$ this action preserves the determinant and hence preserves lengths of vectors.

(c) We need to show that

$$\exp(\theta M_u)M_v\exp(-\theta M_u) = M_{v'}.$$

Where v' is as given in the question. We have (using $M_uM_v = -(u \cdot v) + M_{u\times v}$)

$$\begin{aligned}
\exp(\theta M_u)M_v\exp(-\theta M_u) &= (\cos\theta + M_u\sin\theta)M_v(\cos\theta - M_u\sin\theta) \\
&= \cos^2\theta M_v + 2\sin\theta\cos\theta M_{u\times v} - \sin^2\theta M_v - 2\sin^2\theta(u \cdot v)M_u \\
&= M_{v'}
\end{aligned}$$

where $v' = v\cos 2\theta + (u \times v)\sin 2\theta + (1 - \cos 2\theta)(u \cdot v)u$.

(d) The map from $SU(2)$ to $SO(3)$ sends $\exp(\theta M_u)$ to the rotation by 2θ around u.

The map ρ is surjective because every element of $SO(3)$ is a rotation by some angle around some axis.

(To see this, it is sufficient to prove that $A \in SO(3)$ has an eigenvalue equal to 1 (the corresponding eigenvector will be the axis and the restriction of A to the orthogonal complement of the eigenvector will be an element of $SO(2)$ and hence specify the angle θ). The characteristic polynomial of A is cubic and real hence has a real root λ; since $Av = \lambda v$ implies $|Av|^2 = \lambda^2|v|^2 = |v|^2$ we see that $\lambda = \pm 1$. Assume that none of the eigenvalues were equal to 1. If all eigenvalues were real they'd have to be equal to -1 and then the determinant would be 1; if one was real and equal to -1 but the other two were complex conjugates then the determinant would be negative ($z\bar{z}(-1) < 0$). Therefore at least one eigenvalue has to be 1. Therefore we deduce that every $A \in SO(3)$ has an axis and hence sits in the image of our map $\rho\colon SU(2) \to SO(3)$.)

This map is a representation, hence a homomorphism, so the number of preimages of $A \in SO(3)$ is independent of A. In particular it is a $|\ker\rho|$-to-1 map since $\ker\rho$ is the set of preimages of the identity. Since $\exp(\theta M_u)$ acts by rotation by 2θ, we see that the kernel consists of $\exp(\theta M_u)$ for $2\theta \in 2\pi\mathbf{Z}$, i.e. $\theta \in \pi\mathbf{Z}$. But $\exp(n\pi M_u)\exp(\theta M_u) = \cos(n\pi) + M_u\sin(n\pi) = (-1)^n$, hence the kernel of the map comprises the two matrices ± 1. Hence we see it is 2-to-1.