Sheet 4: Lie's theorem

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Recall Lie's theorem:

Theorem 1 (Lie)**.** *Let G and H be two path-connected matrix groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Suppose moreover that G is simply-connected. For any Lie algebra homomorphism $f\colon \mathfrak{g} \to \mathfrak{h}$ there is a unique smooth homomorphism $F\colon G \to H$ with $F_* = f$.*

Question 1. Suppose that G and H are both path-connected, simply-connected matrix groups and that $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras. Prove that $G \cong H$. (You may assume Lie's theorem).

Answer 1. Let $f\colon \mathfrak{g} \to \mathfrak{h}$ be an isomorphism of the Lie algebras. Since G is connected and simply-connected and H is connected, this exponentiates (by Lie's theorem) to a homomorphism $F\colon G \to H$. The inverse f^{-1} also exponentiates to $\tilde{F}\colon H \to G$ since H is also simply-connected. The composition $\tilde{F} \circ F$ has differential $f^{-1} \circ f = \mathrm{id}$ and by the uniqueness part of Lie's theorem it must itself be the identity. Therefore $\tilde{F} = F^{-1}$ and F is an isomorphism.

The remaining exercises form a guided proof of Lie's theorem. The first ingredient we will need is a compatibility condition for a system of partial differential equations to have a solution.

Question 2 (Maurer-Cartan equation)**.** In this question, you can freely assume the existence[1] and uniqueness of solutions to *ordinary* differential equations. Let G be a matrix group with Lie algebra $\mathfrak{g}$. Let $\xi(s,t)$ and $\eta(s,t)$ be two smooth maps $[0,1] \times [0,1] \to \mathfrak{g}$. Prove that the following are equivalent:

(a) The *partial* differential equations

$$(\star) \quad \frac{\partial \phi}{\partial s}(s,t) = \phi(s,t)\xi(s,t), \qquad (\star\star) \quad \frac{\partial \phi}{\partial t}(s,t) = \phi(s,t)\eta(s,t)$$

have a solution $\phi\colon [0,1] \times [0,1] \to G$ with $\phi(0,0) = 1$.

(b) The *Maurer-Cartan equation* holds:

$$(\star\star\star) \quad \frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial s} = [\xi, \eta].$$

Hints:

[1] All of the ODEs we come across will have solutions for all time.

For (a) implies (b), try cross-differentiating ($\star$) and ($\star\star$) and seeing what turns up.

For (b) implies (a): This is tricky. Construct $\phi(s, 0)$ first by considering ($\star$) as an ODE at $t = 0$. Then use $\phi(s, 0)$ as the initial conditions for ($\star\star$), considered as an ODE for each fixed value of s. By construction this solves ($\star\star$) and it solves ($\star$) along $t = 0$; it remains to show that ($\star$) holds everywhere. For fixed s, consider $\beta(t) = \frac{\partial\phi}{\partial s}(s,t) - \phi(s,t)\xi(s,t)$ and show (using ($\star\star$) and ($\star\star\star$)) that $\frac{d\beta}{dt} = \beta\eta$; deduce that $\beta \equiv 0$ using uniqueness of solutions to ODEs and the initial condition ($\star$) along $t = 0$.

Answer 2. (a) implies (b): Given such a solution we differentiate to get

$$\frac{\partial^2\phi}{\partial s\partial t} = \frac{\partial\phi}{\partial t}\xi + \phi\frac{\partial\xi}{\partial t} = \frac{\partial\phi}{\partial s}\eta + \phi\frac{\partial\eta}{\partial s}$$

so using ($\star$) we get

$$\phi(\partial_t\xi - \partial_s\eta) = \phi[\xi, \eta]$$

which gives (b).

(b) implies (a): We first solve the ODE

$$\frac{d\phi}{ds}(s, 0) = \phi(s, 0)\xi(s, 0)$$

when $t = 0$. Now we use $\phi(s, 0)$ as an initial condition for the ODE

$$\frac{d\phi}{dt}(s, t) = \phi(s, t)\eta(s, t)$$

for each fixed s. This allows us to define $\phi(s, t)$ for all (s, t) and by construction it satisfies $\partial\phi/\partial t = \phi\eta$. It only remains to check that $\partial\phi/\partial s = \phi\xi$. Define $\beta(s,t) = \frac{\partial\phi}{\partial s}(s,t) - \phi(s,t)\xi(s,t)$ and note that by construction $\beta(s, 0) = 0$. We will show that $\frac{\partial\beta}{\partial t}(s,t) = \beta(s,t)\eta(s,t)$ so that β is identically zero (because of the zero initial condition). We have

$$\begin{aligned}\frac{\partial\beta}{\partial t} &= \frac{\partial^2\phi}{\partial s\partial t} - \frac{\partial\phi}{\partial t}\xi - \phi\frac{\partial\xi}{\partial t} \\ &= \frac{\partial\phi}{\partial s}\eta + \phi\frac{\partial\eta}{\partial s} - \phi\eta\xi - \phi\frac{\partial\xi}{\partial t}\end{aligned}$$

where we have used the equation $\partial\phi/\partial t = \phi\eta$ twice. Now we use the identity $\frac{\partial\xi}{\partial t} - \frac{\partial\eta}{\partial s} = [\xi, \eta]$ to get

$$\begin{aligned}\frac{\partial\beta}{\partial t} &= \phi[\eta, \xi] + \frac{\partial\phi}{\partial s}\eta - \frac{\partial\phi}{\partial t}\xi \\ &= \beta\eta\end{aligned}$$

since $[\eta, \xi] = \eta\xi - \xi\eta$.

Paths in G from paths in $\mathfrak{g}$:

Question 3. (a) Let $\gamma\colon [0, 1] \to G$ be a path. For each $s \in [0, 1]$, by considering the path $\gamma(s)^{-1}\gamma(t)$ show that $A(s) = \gamma^{-1}(s)\dot{\gamma}(s)$ is a tangent vector to G at 1. That is $A(s) \in \mathfrak{g}$.

(b) Conversely, let $X \in \mathfrak{g}$ and $g \in G$. Show that gX is a tangent vector to G at g. Hence $T_gG = g\mathfrak{g}$.

A consequence of this and the existence/uniqueness of solutions to ODEs is the following lemma which we will use to construct F.

Lemma 2. *Let $A\colon [0,1] \to \mathfrak{g}$ be a path in $\mathfrak{g}$. Show that if $\gamma\colon [0,1] \to GL(n, \mathbf{R})$ is a path satisfying $\dot{\gamma}(t) = \gamma(t)A(t)$ then $\gamma(t) \in G$ for all t.*

Answer 3. (a) For each s, the path $\delta(t) = \gamma(s)^{-1}\gamma(t)$ is a path in G with $\delta(s) = 1$. We have $\dot{\delta}(t) = \gamma(s)^{-1}\dot{\gamma}(t)$ so $\dot{\delta}(s) = \gamma(s)^{-1}\dot{\gamma}(s) = A(s)$ is a tangent vector to G at 1.

(b) Conversely if $X \in \mathfrak{g}$ and $g \in G$ then $\gamma(t) = g\exp(tX) \in G$ for all t so $\dot{\gamma}(0) = gX \in T_gG$.

Out of interest: To prove the Lemma, we will show that the interval inside $\mathbf{R}$ on which $\gamma(t) \in G$ is both open and closed, hence it's the whole of $\mathbf{R}$. Closedness is obvious because G is a closed subset and γ is a continuous path. To prove openness suppose that $\gamma(t) \in G$ - we will show that $\gamma(t+\epsilon) \in G$ for $|\epsilon|$ sufficiently small. Note that in local exponential coordinates $\gamma(t)\exp(w_1)\exp(w_2)$ near $\gamma(t)$, (where $w_1 \in \mathfrak{g}$ and $w_2 \in W_2$ for a complement W_2 of $\mathfrak{g}$) a neighbourhood of $\gamma(t) \in G$ is identified with a neighbourhood of $0 \in \mathfrak{g}$ and tangent vectors to G are identified with tangent vectors to $\mathfrak{g}$. We define the vector field $B(g, \epsilon) = gA(t+\epsilon)$ depending on $g \in GL(n, \mathbf{R})$ and a parameter ϵ. Along G this is tangent to G and hence in the exponential chart it becomes a vector field $\tilde{B}(w_1, w_2, \epsilon)$ which is tangent to $\mathfrak{g}$ along $\mathfrak{g}$ (that is $\tilde{B}(w_1, 0, \epsilon)$ has no W_2 component). By the existence/uniqueness theorems for ODEs there is a solution to $(\dot{\delta}(\epsilon), 0) = \tilde{B}(\delta(\epsilon), 0, \epsilon)$ starting at $\delta(0) = 0 \in \mathfrak{g}$ and the image of this under the exponential chart is a solution $\gamma(t+\epsilon)$ to the original ODE.

The construction of $F(g)$:

Let $\gamma\colon [0,1] \to G$ be a path with $\gamma(0) = 1$ and $\gamma(1) = g$. Define $A(t) = \gamma(t)^{-1}\dot{\gamma}(t) \in \mathfrak{g}$. Now consider the differential equation

$$\dot{\delta}(t) = \delta(t)f(A(t)), \quad \delta(0) = 1$$

for a path δ in H. We call δ the *path associated to* γ.

$$\boxed{\text{Define } F(g) = \delta(1).}$$

We will show that

- $\delta(1)$ does not depend on the choice γ of path from 1 to g;
- that $F\colon G \to H$ is a homomorphism;
- that $F_* = f$.

Path-independence of $\delta(1)$:

Suppose that γ_i, $i = 0, 1$ are two choices of path with $\gamma_i(0) = 1$ and $\gamma_i(1) = g$ and associated paths δ_i. Since G is simply-connected there is a map $\psi\colon [0,1] \times [0,1] \to G$ such that

$$\psi(0,t) = \gamma_0(t), \quad \psi(1,t) = \gamma_1(t), \quad \psi(s,0) = 1, \quad \psi(s,1) = g.$$

Question 4. (a) Show that ξ and η satisfy the Maurer-Cartan equation, where

$$\xi(s,t) = \psi^{-1}\frac{\partial \psi}{\partial s} \in \mathfrak{g}, \quad \eta(s,t) = \psi^{-1}\frac{\partial \psi}{\partial t} \in \mathfrak{g}.$$

(b) If $f\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism deduce that $f\xi$ and $f\eta$ also satisfy the Maurer-Cartan equation.

(c) Deduce that if f is a Lie algebra homomorphism then there exists a map $\phi\colon [0,1] \times [0,1] \to H$ such that $\phi(0,0) = 1$ and

$$\frac{\partial \phi}{\partial s} = \phi f(\xi), \quad \frac{\partial \phi}{\partial t} = \phi f(\eta).$$

(d) Why is $\xi(s,1) = 0$? Deduce that $\phi(s,1)$ is independent of s and hence that $\delta_0(1) = \delta_1(1)$.

Answer 4. (a) By the Maurer-Cartan theorem, the existence of ψ implies that if

$$\frac{\partial \psi}{\partial s}(s,t) = \psi(s,t)\xi(s,t), \quad \frac{\partial \psi}{\partial t}(s,t) = \psi(s,t)\eta(s,t)$$

then ξ and η solve the Maurer-Cartan equation.

(b) If f is a Lie algebra homomorphism then

$$\frac{\partial f\xi}{\partial t} - \frac{\partial f\eta}{\partial s} = f\left(\frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial s}\right) = f[\xi,\eta] = [f\xi, f\eta].$$

so $f\xi$ and $f\eta$ satisfy the Maurer-Cartan equation.

(c) By the Maurer-Cartan theorem if $f\xi$ and $f\eta$ satisfy the Maurer-Cartan equation then the existence of such a ϕ follows.

(d) Since $\psi(s,1) = g$ we have $\xi(s,1) = \frac{\partial \psi}{\partial s}(s,1) = 0$. Hence $f\xi(s,1) = 0$ and hence $\frac{\partial \phi}{\partial s}(s,1) = \phi(s,1)f\xi(s,1) = 0$ and so $\phi(s,1)$ is independent of s. Since δ_i is the associated path to γ_i ($i = 0,1$) it is equal to $\psi(i,t)$ and therefore we have proved that $\delta_0(1) = \delta_1(1)$.

F **is a homomorphism.**

Question 5. If $\delta(t)$ is a path in a group then, by differentiating $\delta(t)^{-1}\delta(t) = 1$, prove that

$$\frac{d(\delta(t)^{-1})}{dt} = -\delta(t)^{-1}\dot{\delta}(t)\delta(t)^{-1}.$$

Answer 5. Differentiating $\delta(t)^{-1}\delta(t) = 1$ with respect to t gives

$$\frac{d\delta^{-1}}{dt}(t)\delta(t) = -\delta(t)^{-1}\frac{d\delta}{dt}(t)$$

so

$$\frac{d\delta^{-1}}{dt}(t) = -\delta(t)^{-1}\frac{d\delta}{dt}(t)\delta(t)^{-1}.$$

Take a path γ in G with $\gamma(0) = 1$ and $\gamma(1) = g$ and let δ be the associated path in H (equivalently $\delta^{-1}\dot{\delta} = f(\gamma^{-1}\dot{\gamma})$) and recall that $F(g) = \delta(1)$.

Question 6. Show that $\delta(t)^{-1}f(X)\delta(t)$ and $f(\gamma(t)^{-1}X\gamma(t))$ are both solutions to the ODE

$$\dot{z}(t) = [z(t), \delta(t)^{-1}\dot{\delta}(t)]$$

with initial condition $z(0) = f(X)$. Deduce that they are equal for all t and hence prove that

$$\delta(1)^{-1}f(X)\delta(1) = f(\gamma^{-1}(1)X\gamma(1)),$$

or, equivalently,

$$F(g)^{-1}f(X)F(g) = f(g^{-1}Xg).$$

Answer 6.

$$\begin{aligned}\frac{d}{dt}\left(\delta(t)^{-1}f(X)\delta(t)\right) &= -\delta^{-1}\dot{\delta}\delta^{-1}f(X)\delta + \delta^{-1}f(X)\dot{\delta} \\ &= [\delta^{-1}f(X)\delta, \delta^{-1}\dot{\delta}].\end{aligned}$$

We also have

$$\begin{aligned}\frac{d}{dt}\left(f(\gamma^{-1}(t)X\gamma(t)\right) &= -f(\gamma^{-1}\dot{\gamma}\gamma^{-1}X\gamma) + f(\gamma^{-1}X\gamma\gamma^{-1}\dot{\gamma}) \\ &= f[\gamma^{-1}X\gamma, \gamma^{-1}\dot{\gamma}] \\ &= [f(\gamma^{-1}X\gamma), f(\gamma^{-1}\dot{\gamma})] \\ &= [f(\gamma^{-1}X\gamma), \delta^{-1}\dot{\delta}]\end{aligned}$$

Therefore $\delta^{-1}(t)f(X)\delta(t)$ and $f(\gamma(t)^{-1}X\gamma(t))$ are both solutions to the equation

$$\dot{z} = [z, \delta^{-1}\dot{\delta}]$$

with initial condition $\delta^{-1}(0)f(X)\delta(0) = f(\gamma(0)^{-1}X\gamma(0)) = f(X)$. By uniqueness we know they are equal for all t, in particular for $t = 1$, which implies the result.

Question 7. Suppose that α, β are paths in G with $\alpha(0) = \beta(0) = 1$ and $\alpha(1) = g$, $\beta(1) = h$. Define the path $\gamma(t) = \alpha(t)\beta(t)$ in G with $\gamma(1) = gh$ and associated paths u, v, w in H satisfying

$$\begin{aligned} \dot{u} &= uf(\alpha^{-1}\dot{\alpha}) \quad \text{so } F(g) = u(1) \\ \dot{v} &= vf(\beta^{-1}\dot{\beta}) \quad \text{so } F(h) = v(1) \\ \dot{w} &= wf(\gamma^{-1}\dot{\gamma}) \quad \text{so } F(gh) = w(1) \end{aligned}$$

Prove that $w = uv$ by showing that they both solve the same ODE

$$\dot{z} = zv^{-1}f(\alpha^{-1}\dot{\alpha})v + zf(\beta^{-1}\dot{\beta})$$

with the initial condition $z(0) = 1$.

Answer 7. We want to show that $w = uv$ and we do this by showing that they both solve the same ordinary differential equation with the same initial conditions. Differentiating:

$$\begin{aligned} \dot{w} &= wf(\gamma^{-1}\dot{\gamma}) \\ &= wf(\beta^{-1}\alpha^{-1}(\dot{\alpha}\beta + \alpha\dot{\beta}) \\ &= wf(\beta^{-1}\alpha^{-1}\dot{\alpha}\beta) + wf(\beta^{-1}\dot{\beta}) \\ &= wv^{-1}f(\alpha^{-1}\dot{\alpha})v + wf(\beta^{-1}\dot{\beta}) \end{aligned}$$

where we used Question 6 in this last line. We also have

$$\frac{d}{dt}(uv) = \dot{u}v + u\dot{v} = uf(\alpha^{-1}\dot{\alpha}v + uvf(\beta^{-1}\dot{\beta})$$

so we see that both uv and w solve

$$dz/dt = zv^{-1}f(\alpha^{-1}\dot{\alpha})v + zf(\beta^{-1}\dot{\beta})$$

with initial condition $z(0) = 1$.

$F_* = f$**:**

Question 8. Suppose that $\gamma(t) = \exp(tX)$. Prove that the associated path in H is given by $\delta(t) = \exp(tf(X))$. Deduce that $F(\exp X) = \exp(f(X))$ and hence that $f(X) = F_*(X)$.

Answer 8. Since $\dot{\gamma}(t) = \gamma(t)X$ we have $A(t) = X$ and hence the associated path is defined by the equation $\dot{\delta}(t) = \delta(t)f(X)$ and $\delta(0) = 1$. Thus $\delta(t) = \exp(tf(X))$ by uniqueness of solutions to ODEs. Therefore $F(\exp X) = \delta(1) = \exp(f(X))$ so $F_* = f$.

This completes the proof of Lie's theorem.