Chapter 5 Measuring cosmological parameters

In the previous chapter you learned how to describe the evolution of the scale factor $a(t)$ for different model universes, which can have different contents and different intrinsic properties, like curvature. In this chapter you will learn how cosmologists can determine the properties of the *real* Universe by measuring the different parameters that we find in the Friedmann equations.

Our strategy will involve carefully measuring the distances to different celestial objects, which will let us determine how $a(t)$ has evolved over time. Once we know $a(t)$, we can use the Friedmann equations to constrain the energy densities of matter, radiation and the cosmological constant (Λ), as well as the overall curvature of the Universe.

Objectives

Working through this chapter will enable you to:

- list and describe four ways to define the distance to objects in an expanding universe
- discuss how the apparent distances to objects depend on the parameters of the Friedmann equation
- explain how cosmologists use several types of celestial objects and phenomena to measure distances in the Universe
- summarise the different techniques that cosmologists apply to infer distances to these objects
- explain how cosmologists use observations of distant objects to measure the Hubble constant, and thereby determine the *current* expansion rate of the Universe
- explain how cosmologists use observations of distant objects to measure density parameters in the Friedmann equation, which determine the expansion *history* of the Universe.

5.1 Defining distance

In this chapter you will learn how measurements of the distances to celestial objects allow cosmologists to determine the values of cosmological parameters like the Hubble parameter, H_0, and the density parameters $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$. However, in an expanding universe, there are several ways that the distances to remote objects like distant galaxies can be defined, and which definition to choose often depends on the scientific question that you are trying to answer.

You have already learned about co-moving coordinates, which measure distances that remain unchanged as the Universe expands. In this section you will also learn about the proper distance – which measures the length of spatial geodesics – as well as the luminosity distance and the angular diameter distance, which are both related to the proper distance but based on observable properties of the objects whose distance they quantify.

5.1.1 Proper distance

The formal definition of the **proper distance** between two objects is the length of a spatial geodesic connecting their positions at a *particular* cosmic time, t, in the history of the Universe, when its scale factor was $a(t)$. An observer located at the origin at cosmic time t can compute the proper distance to a distant galaxy (or any distant object) by integrating the component of the Robertson–Walker metric in Equation 3.10 that depends directly on the radial coordinate r. If distance to the galaxy is measured in *radial coordinates* as R, then the corresponding proper distance d_p is defined by Equation 5.1.

Proper distance

$$d_\mathrm{p}(t) = a(t) \int_0^R \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} \tag{5.1}$$

The proper distance is closely related to the **co-moving distance**, d_c. Specifically, at cosmic time t the two are related via the scale factor, such that:

$$d_\mathrm{p}(t) = a(t)d_\mathrm{c} \tag{5.2}$$

Note that d_c itself does *not* depend on cosmic time. This is because it is defined in terms of the co-moving coordinate grid that you read about in Section 3.3.1. In that earlier discussion, the scale of the co-moving coordinate grid – in other words, the numerical coordinate values associated with each of the grid lines – was left unspecified. In *principle* we could choose to label the lines in the evenly spaced coordinate grid with any set of evenly spaced numerical values.

However, in *practice*, cosmologists conventionally fix the co-moving coordinate scale by defining the present value of the scale factor to be $a(t_0) = 1$, which implies that:

$$d_\mathrm{p}(t_0) = a(t_0)d_\mathrm{c} = d_\mathrm{c} \tag{5.3}$$

Equation 5.3 reveals that an important consequence of choosing $a(t_0) = 1$ is that the *present* proper distance between two objects equals the co-moving distance d_c between them, and vice-versa.

The proper distance turns out to be a very useful quantity for theoretically understanding the Universe, but unfortunately it cannot be measured in any practical way. Cosmologists must rely on observations of light from distant objects to infer how far away they are. Bearing this in mind, let's find an expression for the present proper distance in terms of quantities that are more closely related to observable properties of distant objects that *can* actually be measured.

We start by considering a beam of light propagating from a distant galaxy, at radial coordinate R, to a detector on the Earth. The photons in the beam travel along null geodesics in the radial direction (i.e. $\mathrm{d}s^2 = 0$), and so the Robertson–Walker metric can be written as:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)^2 \frac{\mathrm{d}r^2}{1-kr^2} = 0 \tag{5.4}$$

We can collect together the terms that depend on t and those that depend on r as follows:

$$\frac{c^2\,\mathrm{d}t^2}{a(t)^2} = \frac{\mathrm{d}r^2}{1-kr^2} \tag{5.5}$$

If we assume that the photons from the galaxy were emitted at time t_{em} and detected some later at time t_{obs}, and that they travelled from $r = R$ to $r = 0$, then we can write that:

$$c\int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)} = \int_0^R \frac{\mathrm{d}r}{\sqrt{1-kr^2}} \tag{5.6}$$

The right-hand side of this equation is just the definition of d_{c}, or $d_{\mathrm{p}}(t)/a(t)$ (compare with Equation 5.1). In the situation where $t_{\mathrm{obs}} = t_0$, and therefore $a(t) = a(t_0) = 1$, we arrive at an expression for the proper distance that relates directly to measurements of cosmic time:

$$d_{\mathrm{p}}(t_0) = c\int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)} \tag{5.7}$$

In the past, when $a(t) < 1$, then $d_{\mathrm{p}} < d_{\mathrm{c}}$. In the future the opposite will be true (i.e. $d_{\mathrm{p}} > d_{\mathrm{c}}$) for as long as the Universe continues to expand. Later in this chapter, you will learn that Equation 5.7 is more straightforwardly linked to quantities that can actually be observed than the original definition of proper distance (Equation 5.1), and is therefore practically useful. The following example shows how to calculate an important quantity called the **horizon distance**, which is equal to the maximum proper distance that photons could have travelled since the big bang.

Example 5.1

The horizon distance d_{hor} defines the proper radius of a sphere (i.e. the proper distance from its centre to its outer surface), which is also called the **particle horizon**. This sphere surrounds an observer at a particular cosmic time and encompasses their entire observable universe.

Find an expression for d_{hor} in a universe that is spatially flat and is matter-dominated.

Solution

To find d_{hor} we set $t_{\text{em}} = 0$ in Equation 5.7, which lets us compute the proper distance that photons have been travelling since the Universe began. We cannot observe objects at greater distances than this, because light from them has not yet had time to reach us. In other words, they are outside of our observable universe.

In Section 4.2.2 we showed that $a(t) = (t/t_0)^{2/3}$ in a matter-dominated universe, and $t = 0$ at the big bang, so Equation 5.7 becomes:

$$d_{\text{hor}}(t_0) = ct_0^{2/3} \int_0^{t_0} t^{-2/3} \, \mathrm{d}t \tag{5.8}$$

Now, we simply evaluate the integral to find:

$$\begin{aligned} d_{\text{hor}}(t_0) &= ct_0^{2/3} \left[3t^{1/3}\right]_0^{t_0} \\ &= 3ct_0^{2/3} t_0^{1/3} \\ &= 3ct_0 \end{aligned}$$

This result shows that the present-day horizon distance in a flat, matter-dominated universe is three times further than the distance light could have travelled since the big bang!

Example 5.1 demonstrates the remarkable result that the Friedman equations can model universes in which the horizon distance exceeds ct_0. The example assumed a simple matter-only model, but if we performed a similar calculation for the constituents of the real Universe, we would still find that $d_{\text{hor}}(t_0) > ct_0$.

- Does this assertion about the real Universe mean that photons from the particle horizon have travelled with superluminal velocity (i.e. faster than the speed of light) to reach us? If not, how are the findings of Example 5.1 possible?

- No, these photons have not travelled faster than c. Remember that $d_{\text{hor}}(t_0)$ specifies the distance to the particle horizon *today*. Light that is now reaching us from the horizon completed most of its journey in the past, when the Universe – and therefore the horizon distance – was much smaller.

Equation 5.7 in its current form is still not directly useful. If we knew t_{em} and the functional form for $a(t)$, we could use Equation 5.7 to compute the proper distance that photons arriving at Earth today have travelled since the time when they were emitted. However, even if we assume a model for $a(t)$, we do not yet have any way of inferring a value for the time t_{em}.

Fortunately, we can rewrite Equation 5.7 in terms of a quantity that we *can* actually measure – the redshift z of the photons arriving from the distant galaxy. The following exercise asks you to derive an expression that explicitly links d_{p} and z.

Exercise 5.1

By using the Friedmann equation in the form shown in Equation 4.41 and adopting the standard convention that $a(t_0) = 1$, show that

$$d_\mathrm{p}(t_0) = \frac{c}{H_0} \int_0^{z_\mathrm{em}} \frac{\mathrm{d}z}{E(z)} \tag{5.9}$$

is equivalent to Equation 5.7. In Equation 5.9, z_em is the redshift of the photons we observe at time t_0 from a source at a proper distance $d_\mathrm{p}(t_0)$, and

$$E(z) = \sqrt{\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2}$$

where $\Omega_{k,0}$ is the present-day value of Ω_k. (*Hint*: you may need to refer back to an equation in Chapter 3 that relates a and z in the context of emitted and observed light.)

If we can measure the redshift of photons from a distant object then Equation 5.9 becomes very useful, because it directly relates $d_\mathrm{p}(t_0)$ to the values of H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{r},0}$, $\Omega_{\Lambda,0}$ and Ω_k. In Sections 5.1.2 and 5.1.3 you will learn about two methods that cosmologists use to estimate $d_\mathrm{p}(t_0)$ for distant objects, based on their observable properties. By using such estimates, obtained for a large number of distant objects, we can use Equation 5.9 to constrain the parameters in the Friedmann equation.

Different sources of redshift

At low redshifts, Equation 5.9 reduces to the Hubble–Lemaître law that you learned about in Chapter 1. Now we see that the distance that appears in Equation 1.2 is in fact the proper distance evaluated at the present time, t_0:

$$z = \frac{H_0 d_\mathrm{p}(t_0)}{c} \tag{5.10}$$

It is important to remember that the redshift that appears in Equation 5.10 is the *cosmological redshift*. This is the redshift that we would measure for a galaxy that was static with respect to the Hubble flow, with fixed co-moving coordinates. This distinction has implications for how accurately we can actually measure H_0, because objects in the Universe are not static. Every galaxy has its own **peculiar velocity**, which adds a Doppler-shift component to the redshift that we measure.

There is no way to determine what proportion of a galaxy's *measured redshift* results from its peculiar velocity. If we tried to estimate H_0 using observations of a *single* galaxy, then our answer would be shifted by an unknown amount from the true value. However, if we apply the cosmological principle that the Universe is homogeneous and isotropic, then we can expect the *average* peculiar velocity of *many* galaxies to be very close to zero. Therefore, if we can accurately estimate the proper distances to a large number of galaxies with known redshifts, then we can compute an average value of H_0 that should be very close to the true value.

Measuring galaxies' redshifts using spectroscopy is relatively straightforward with modern astronomical instruments. However, collecting sufficiently precise distance measurements for a large sample of distant galaxies is a significant technical challenge. Later in this chapter you will learn about the sophisticated techniques that cosmologists use to measure distances on cosmological scales.

5.1.2 Luminosity distance

Suppose we somehow *know* that the luminosity of a distant object is L. Objects with known luminosity are often referred to as **standard candles**, and you will see some examples later in this chapter. If we observe one of these standard candles and measure its flux to be F, then we can calculate an estimate for its distance from us using a quantity called the **luminosity distance**.

Luminosity distance

$$d_\mathrm{L} = \sqrt{\frac{L}{4\pi F}} \tag{5.11}$$

It is important to recognise that the luminosity distance for an object at a particular redshift is not necessarily the same as its (present-day) proper distance, $d_\mathrm{p}(t_0)$. Consider the scenario shown in Figure 5.1.

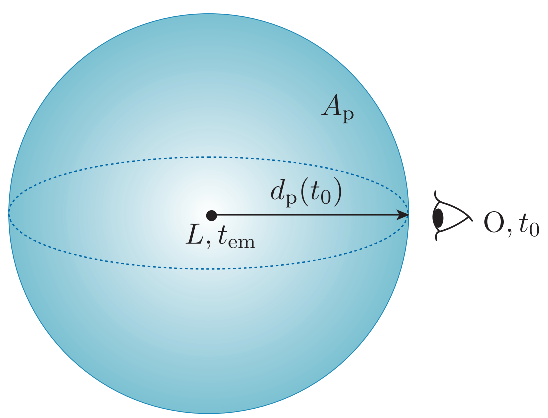


Figure 5.1 At time t_0 an observer located at point O, at a proper distance $d_\mathrm{p}(t_0)$, measures the light from an standard candle of luminosity L. Photons that were emitted earlier, at time t_em, are distributed over a sphere with proper surface area $A_\mathrm{p} = 4\pi d_\mathrm{p}(t_0)^2$.

A present-day observer at point O studies a standard candle with luminosity L, at a co-moving distance of $d_\mathrm{p}(t_0)$. They measure at time t_0

the photons that were emitted isotropically at time $t_{\rm em}$. If the current proper distance between the observer and the light source is $d_{\rm p}(t_0)$, then photons that were emitted at $t_{\rm em}$ will be distributed over the surface of a sphere with radius $d_{\rm p}(t_0)$ and proper surface area $A_{\rm p} = 4\pi d_{\rm p}(t_0)^2$. For a static universe this means that the observed flux at O will be $F = L/A_{\rm p}$.

- Does the value of F depend on the value of the curvature parameter k? Briefly explain your answer.

- Yes it does, because k appears in the definition of the proper distance $d_{\rm p}$ (Equation 5.1), which itself appears in the expression for $A_{\rm p}$.

In a static, spatially flat universe, $d_{\rm p}(t_0)$ is equal to the luminosity distance. However, $d_{\rm L}$ and $d_{\rm p}(t_0)$ are *not* equal in the real Universe, where both cosmic expansion and curvature can modify the flux that we observe.

For simplicity we will start by considering the impact of cosmic expansion in a spatially flat universe. In such a universe, two related effects act to reduce the observed flux of distant luminous objects relative to what would be expected in a static universe.

To understand these two effects, it is important to recognise that both flux and luminosity are defined as *rates* of energy transfer: they describe the amount of energy that is emitted or received within a particular time interval. In Chapter 3 you saw how cosmic expansion produces a cosmological redshift that reduces the frequency, and therefore the *energy*, of photons by a factor of $1 + z$ as they propagate through the Universe.

Recall that the energy E and frequency ν of a photon are related via $E = h\nu$, where h is the Planck constant.

A similar effect decreases the *rate* at which photons arrive compared to the static universe case. The following example explores this effect.

Example 5.2

Consider two photons that follow identical spatial trajectories and arrive at O in Figure 5.1 when the scale factor is $a(t_0)$.

Assuming that the photons were *emitted* at time $t_{\rm em}$, when the scale factor was $a(t_{\rm em})$, show that the interval between the photons' arrivals, δt_0, is longer than the interval between their emissions, $\delta t_{\rm em}$, by a factor $(1 + z)$, where z is the redshift of the standard candle.

Solution

To show that this is the case, we can use a very similar argument to the steps used to derive the relation between a and z in Section 3.3.3.

Let the arrival times of the photons be t_0 and $t_0 + \delta t_0$, and assume that their corresponding emission times are $t_{\rm em}$ and $t_{\rm em} + \delta t_{\rm em}$. We can use Equation 5.7 to relate the two photons' emission and reception times to the corresponding scale factors:

$$c\int_{t_{\rm em}}^{t_0} \frac{{\rm d}t}{a(t)} = c\int_{t_{\rm em}+\delta t_{\rm em}}^{t_0+\delta t_0} \frac{{\rm d}t}{a(t)}$$

To separate terms that depend on t_0 and $t_{\rm em}$ we can subtract a third integral from both sides, following steps (i) to (iv) below.

(i) $c\displaystyle\int_{t_{\rm em}}^{t_0}\frac{\mathrm{d}t}{a(t)} - c\int_{t_{\rm em}+\delta t_{\rm em}}^{t_0}\frac{\mathrm{d}t}{a(t)} = c\int_{t_{\rm em}+\delta t_{\rm em}}^{t_0+\delta t_0}\frac{\mathrm{d}t}{a(t)} - c\int_{t_{\rm em}+\delta t_{\rm em}}^{t_0}\frac{\mathrm{d}t}{a(t)}$

(ii) $c\displaystyle\int_{t_{\rm em}}^{t_{\rm em}+\delta t_{\rm em}}\frac{\mathrm{d}t}{a(t)} = c\int_{t_0}^{t_0+\delta t_0}\frac{\mathrm{d}t}{a(t)}$

(iii) $\displaystyle\frac{1}{a(t_{\rm em})}\int_{t_{\rm em}}^{t_{\rm em}+\delta t_{\rm em}}\mathrm{d}t = \frac{1}{a(t_0)}\int_{t_0}^{t_0+\delta t_0}\mathrm{d}t$

(iv) $\displaystyle\frac{\delta t_{\rm em}}{a(t_{\rm em})} = \frac{\delta t_0}{a(t_0)}$

We can now use Equation S1 to show:

$$\frac{\delta t_0}{\delta t_{\rm em}} = 1 + z$$

Therefore we have shown that the interval between photon arrival from an object at redshift z will be a factor of $1 + z$ larger in an expanding universe than in a static one.

Accounting for both effects of the Universe's expansion on the observed flux, the relationship between the flux and luminosity of an object at redshift z in a universe where $k = 0$ is:

$$F = \frac{L}{4\pi(1+z)^2\, d_{\rm p}(t_0)^2}$$

By comparison with Equation 5.11, we find that an object's luminosity distance, its cosmological redshift and its current proper distance from Earth are related such that

$$d_{\rm L} = (1+z)\, d_{\rm p}(t_0) \qquad (5.12)$$

In curved universes this relationship becomes more mathematically complex, but with a similar increasing divergence between the two distance measures at large redshifts.

Figure 5.2 compares curves of $d_{\rm L}$ and $d_{\rm p}(t_0)$ versus redshift for three model universes with very different contents. Panel (a) shows $d_{\rm L}$ for a large range of redshifts between 0 and 10. In panel (b) $d_{\rm L}$ is plotted for much smaller redshifts, corresponding to much closer light sources. For comparison, both panels also show grey curves that represent the present day proper distance $d_{\rm p}(t_0)$ as a function of redshift. The grey and coloured curves with the same line styles correspond to the same model universe assumptions. Note that $d_{\rm L}$ is always larger than the corresponding proper distance.

In all three cases, the luminosity distance can be used to accurately estimate the proper distance for nearby objects with $z \lesssim 0.04$. However, for more distant objects $d_{\rm L}$ consistently overestimates $d_{\rm p}(t_0)$. In other words, objects in the high-redshift universe appear to be further away than they really are if we only use their apparent brightness to estimate their distance from us.

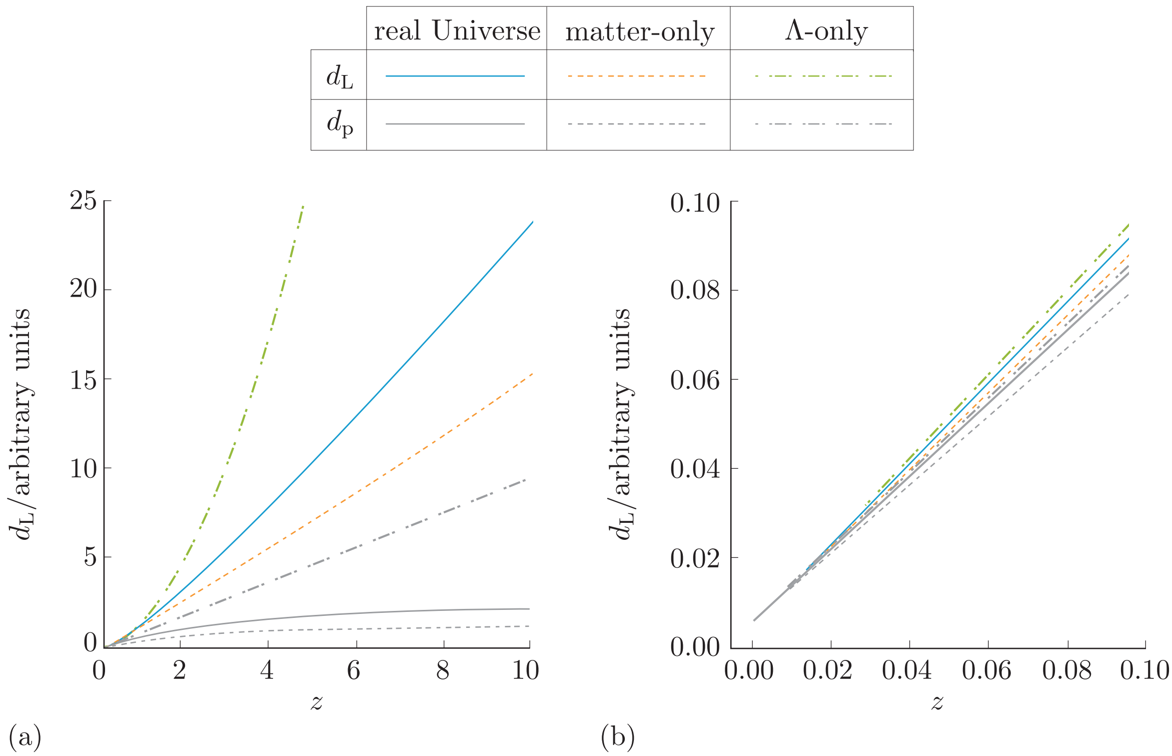


Figure 5.2 d_L (coloured lines) and d_p (grey lines) versus redshift for three model universes. Panel (a) shows a large range of redshifts out to $z = 10$, whereas panel (b) focuses on the nearby Universe.

Later in this chapter you will learn how measurements of the luminosity distance to standard candles can be used to constrain the cosmological parameters. However, this is not the only way that d_L can be useful. Astrophysicists and astronomers often want to know the intrinsic luminosities of distant celestial objects that are *not* standard candles. If they can measure an object's redshift z and its flux F, then Equations 5.9, 5.11 and 5.12 can be used to determine L.

- What extra assumptions must we make in order to calculate L once F and z have been measured?

- We need to assume a particular set of cosmological parameters so that we can evaluate $d_p(t_0)$ using Equation 5.9.

The luminosity distances to standard candles can be used to measure the cosmological parameters, but once those parameters have been measured, the luminosity distance allows the luminosities of *any* celestial light source to be inferred using its measured flux.

5.1.3 Angular diameter distance

Objects or phenomena in the Universe that are known to have a fixed physical size or length scale are called **standard rods**. You may also see the terms 'standard ruler' and 'standard yardstick' used to describe such objects.

Cosmologists define a quantity called the **angular diameter distance**, d_A, that relates the length l of a standard rod to its apparent angular size θ when we observe it. For the distant objects that we are focused on in this chapter, $\theta \ll 1$, so we can use the small angle approximation and define d_A as follows:

Angular diameter distance

$$d_\mathrm{A} = \frac{l}{\sin\theta} \approx \frac{l}{\theta} \quad \text{for } \theta \text{ expressed in radians.} \tag{5.13}$$

The angular diameter distance would be equal to the current proper distance in a static, spatially flat universe. In the *real* Universe d_A and $d_\mathrm{p}(t_0)$ are *not* equal, but they are related to each other such that

$$d_\mathrm{A} = \frac{d_\mathrm{p}(t_0)}{1+z} \tag{5.14}$$

To derive this relationship, we will consider the scenario shown in Figure 5.3. We will again assume a flat universe for simplicity.

The two ends of a standard rod are defined in terms of metric coordinates by the points $\mathrm{A} = (t_\mathrm{em}, r, \theta_\mathrm{A}, \phi)$ and $\mathrm{B} = (t_\mathrm{em}, r, \theta_\mathrm{B}, \phi)$. Photons emitted from A and B at time t_em propagate along null geodesics to reach an observer at the point O at time t_0. The observed angle between the photons is θ_obs.

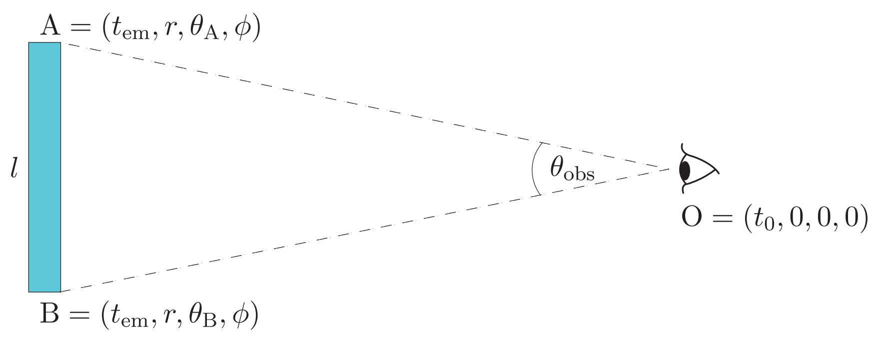


Figure 5.3 An observer O at the origin observes the light from points A and B at either end of a standard rod of length l. The observed angle between A and B is θ_obs.

Now, let's consider another observer who was located at O at time $t_{\rm em}$, when the photons were *emitted* and the proper distance between O and the rod was $d_{\rm p}(t_{\rm em})$. Let $\theta_{\rm AB}$ represent the angular size of the rod as seen by our second observer. If we assume that the length of the rod is constant and that $d_{\rm p}(t_{\rm em})$ is large enough that $\theta_{\rm AB} \ll 1$, then we can write:

$$\theta_{\rm AB} = \frac{l}{d_{\rm p}(t_{\rm em})} \tag{5.15}$$

The key insight is that because the expansion of the Universe is homogeneous and isotropic, the angle between the photons' trajectories must remain unchanged throughout their journey. Therefore, it must be true that $\theta_{\rm AB} = \theta_{\rm obs}$. Using this fact, we can rearrange Equation 5.15 to find an expression for l in terms of $\theta_{\rm obs}$, and use the relationships between $d_{\rm p}(t)$, $d_{\rm p}(t_0)$ and $d_{\rm c}$ from Equations 5.2 and 5.3 to find that:

$$\begin{aligned} l &= d_{\rm p}(t_{\rm em})\,\theta_{\rm AB} = d_{\rm p}(t_{\rm em})\,\theta_{\rm obs} \\ &= a(t_{\rm em})\,d_{\rm c}\,\theta_{\rm obs} = a(t_{\rm em})\,d_{\rm p}(t_0)\,\theta_{\rm obs} = \frac{1}{1+z}\,d_{\rm p}(t_0)\,\theta_{\rm obs} \end{aligned} \tag{5.16}$$

Finally, combining Equations 5.13 and 5.16 we recover the result stated in Equation 5.14:

$$d_{\rm A} = \frac{l}{\theta_{\rm obs}} = \frac{d_{\rm p}(t_0)}{1+z}$$

As with Equation 5.12, this relation applies only to flat ($k = 0$) geometries, but a similar redshift dependence is present in more complex curved geometries.

Figure 5.4 compares curves of $d_{\rm A}$ and $d_{\rm p}(t_0)$ versus redshift for three model universes with very different contents. The panels are analogous to those shown for the luminosity distance in Figure 5.2. Panel (a) shows $d_{\rm A}$ for a large range redshifts between 0 and 10; in panel (b), $d_{\rm A}$ is plotted for much smaller redshifts. Note that unlike the luminosity distance (which was always larger than $d_{\rm p}(t_0)$) the angular diameter distance is always smaller than the corresponding proper distance.

For objects at low redshift ($z \lesssim 0.04$) the angular diameter distance provides a good approximation for the proper distance. For more distant objects, $d_{\rm A}$ consistently *underestimates* $d_{\rm p}(t_0)$. To see why this is the case, remember that the co-moving distance r to the rod is constant so the integral in Equation 5.1 is also constant. Bearing this in mind, we can define $z_{\rm em}$ to be the observed redshift of photons that were emitted at time $t_{\rm em}$ and use Equation S1 to write

$$d_{\rm p}(t_{\rm em}) = \frac{a(t_{\rm em})}{a(t_0)} d_{\rm p}(t_0) = a(t_{\rm em}) d_{\rm p}(t_0) = \frac{d_{\rm p}(t_0)}{1+z_{\rm em}} = d_{\rm A} \tag{5.17}$$

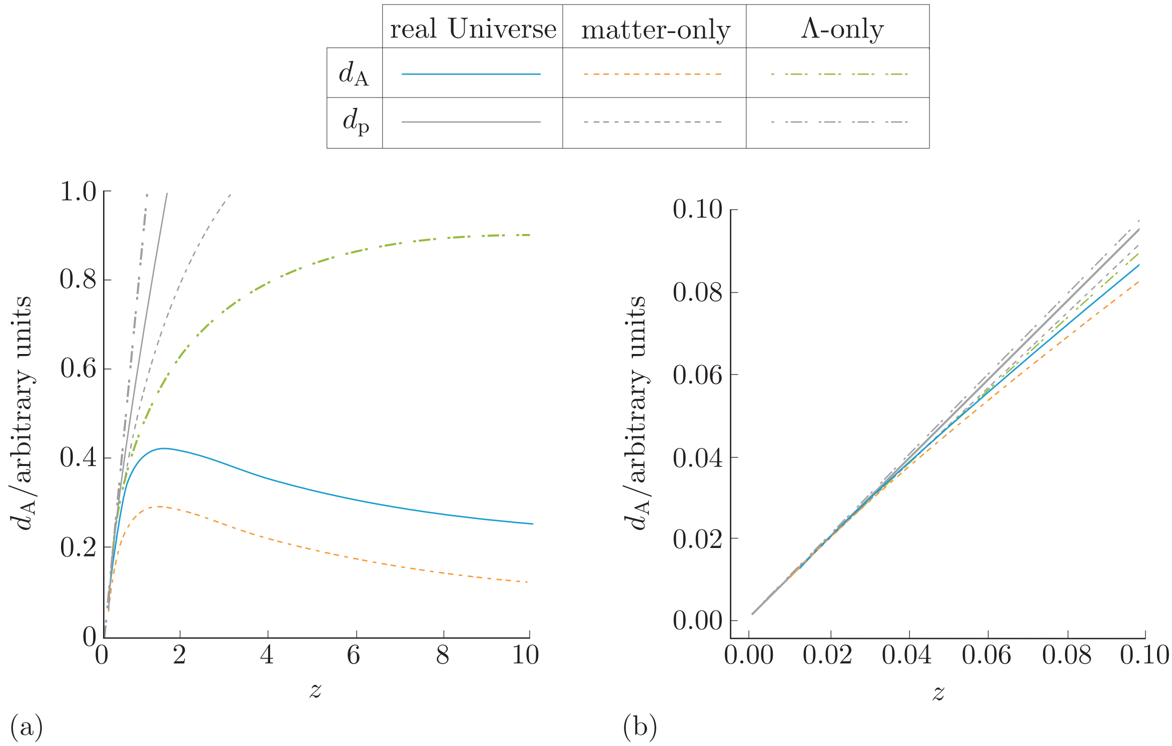


Figure 5.4 d_A (coloured lines) and d_p (grey lines) versus redshift across two ranges for three model universes. Panel (a) shows a large range of redshifts out to $z = 10$, whereas panel (b) focuses on the nearby Universe.

This shows that the present angular diameter distance is equal to the proper distance between the observer and the standard rod at the time t_{em}, when the observed photons were emitted and the scale factor was smaller.

Most remarkably, if a universe contains any component of matter or radiation, then the angular diameter distance only increases up to a certain critical redshift before objects start to appear *larger* as they move further away. If we use d_A as our distance estimate, then objects in the very distant Universe do not just appear closer than they really are: they also appear closer than other objects that actually *are* closer!

Exercise 5.2

Consider a population of objects that are simultaneously standard candles with known luminosity L *and* standard rods with known physical extent l. Show that the *observed* surface brightness Σ of these objects decreases rapidly with increasing redshift according to:

$$\Sigma \propto (1 + z)^{-4} \tag{5.18}$$

(*Hint*: the surface brightness Σ is equivalent to the observed flux of the object, F_{obs}, divided by the solid angle it subtends on the sky.)

The result of Exercise 5.2 also has implications when observing celestial objects that are neither standard candles *nor* standard rods. The surface brightness of these objects is one of the key factors that determines whether or not they can be detected by a particular telescope, with a particular sensitivity. The fact that the surface brightness decreases so rapidly with redshift means that very powerful telescopes are required to detect and study high-redshift objects.

Just like luminosity distance, the angular diameter distance can be used to determine the properties of distant celestial objects.

- Which physical property of distant objects do you think d_{A} can be used to measure?

- It can be used to measure the physical *size* of distant objects.

If astronomers can measure the redshift and the *angular* size of an object, then Equations 5.9, 5.13 and 5.14 can be used to infer l. Again, this requires that we *assume* a particular set of cosmological parameters in order to calculate $d_{\mathrm{p}}(t_0)$. Knowing the physical size of extended structures can help astrophysicists to understand the physical processes that produced them. For example, measuring the lengths of the jets launched by distant radio galaxies and the sizes of their radio lobes allows us to investigate the acceleration of charged particles close to the supermassive black holes that power distant active galactic nuclei.

5.2 Measuring distances

In the previous section you read about the different ways that cosmologists can define distances in the Universe. In this section we will introduce and explore some of the observational techniques that can be used to *measure* the distances to remote celestial objects. We will focus primarily on the ways that astronomers measure the distances to standard candles in the nearby and distant Universe. Then, in Section 5.3, you will learn how these distance measurements have allowed cosmologists to constrain the cosmological parameters.

5.2.1 Stellar parallax

It is possible to measure the distance to stars in the Milky Way and its satellite galaxies using very precise measurements of those stars' positions.

Consider a scenario in which an astronomer observes the apparent position of a nearby star relative to the positions of *much* more distant background objects, as illustrated schematically in Figure 5.5. Six months later, the Earth has completed half of its orbit around the Sun and its position has changed by a distance $2d_{\mathrm{b}}$. The apparent position of the nearby star will also have shifted slightly when it is observed at this point, but the background objects will not seem to have moved. (In fact, the apparent positions of the distant objects will have shifted *very* slightly as well, but they are so far away that this shift is negligible.)

The shift in a nearby star's position relative to a fixed background is called **stellar parallax**, and can be used to determine the proper distance d_p to that star.

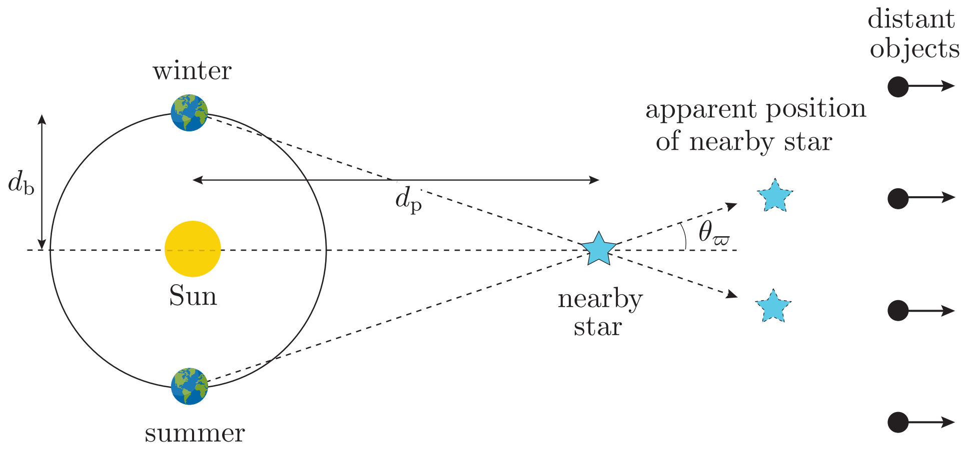


Figure 5.5 The apparent position of a nearby star (at distance d_p) is measured twice, six months apart, relative to much more distant background objects. During that interval the Earth's position has changed by $2d_b$. The small but measurable shift in the nearby star's apparent position is called its stellar parallax.

- If you were choosing a set of background sources to measure the stellar parallax of a star against, what properties would you want those sources to have?

- The most obvious property that the objects must have is that they are much further away than the target star. The fact that they must be so distant means that they should also be very luminous, so that we can actually detect them.

If we want to use the background sources to define a fixed frame of reference, then we need to be able to measure *their* positions very precisely without having to make any subjective decisions about them, for example where their centres are. For that reason, the ideal background sources are point-like, with no measurable angular extension. Distant quasars are celestial objects that fulfil all of these criteria and are frequently used when measuring distances using stellar parallax.

The angle between the two measured positions ($2\theta_\varpi$) is related in the following way to the distance to the star (d_p) and the 'baseline' distance between the Earth's location when the measurements were made ($2d_b$).

Calculating distances using stellar parallax

$$d_p = \frac{d_b}{\theta_\varpi} \tag{5.19}$$

If the units of $d_{\rm p}$ and $d_{\rm b}$ are the same in Equation 5.19, then θ_{ϖ} has units of radians. However, lengths and angles in astronomy are often specified using a variety of different units. The following exercise will give you some practice using Equation 5.19 when a mixture of non-SI units are used.

Exercise 5.3

The Large Magellanic Cloud (LMC) is a satellite galaxy of the Milky Way that lies approximately $d_{\rm p} = 163\,000$ light-years (ly) away. Assuming the nearby star illustrated in Figure 5.5 is somewhere in the LMC, estimate the value of θ_{ϖ} in arcseconds as seen from Earth.

The parsec unit

As you have seen in previous chapters, astronomers often state measured or estimated distances in units called parsecs, where $1\,\text{pc} \approx 3.1 \times 10^{16}\,\text{m}$. The parsec unit is defined, using Equation 5.19, as the value of $d_{\rm p}$ when $d_{\rm b}$ equals the radius of the Earth's orbit around the Sun and θ_{ϖ} is one arcsecond. The word parsec is actually a contraction of the term 'parallax second'.

The maximum distance that can be measured using stellar parallax is limited by the precision with which the positions of distant objects can be measured. Figure 5.6 shows an artist's impression of the *Gaia* satellite, which was launched in 2013 by the European Space Agency. *Gaia* has since been used to measure the positions and parallaxes of over 1 billion stars with an accuracy reaching $\sim$24 microarcseconds!

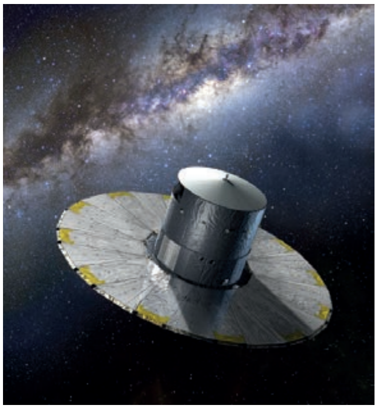


Figure 5.6 An artist's impression of the *Gaia* satellite.

Even with this astonishing precision, Equation 5.19 tells us that *Gaia* can only measure parallaxes for objects closer than $\sim 42\,\text{kpc}$ (equivalent to $z \sim 10^{-5}$ via the Hubble-Lemaître law). Now look at the following example, which considers whether distances derived using stellar parallax – even using measurements made by *Gaia* – can be used to accurately determine the expansion rate of the Universe.

Example 5.3

(a) The recession velocity of a star at time t is just the rate of change of proper radial distance:

$$v_{\rm r}(t) = \frac{\rm d}{{\rm d}t} d_{\rm p}(t)$$

Estimate the radial recession velocity caused by the expansion of the Universe for a star at the maximum distance that can be measured by *Gaia* using stellar parallax.

(b) Compare this value with the estimated peculiar velocity of the Milky Way, which is $\sim 600\,\text{km}\,\text{s}^{-1}$, and briefly comment on the implications for using nearby objects to estimate H_0.

Solution

(a) We are only interested in the component of the recession velocity that is related to cosmic expansion, so we can write v_r in terms of the *co-moving* distance to the star, d_c, and relate this to the proper radial distance using the scale factor a and its time derivative:

$$v_r(t) = \dot{a}(t) d_c = \frac{\dot{a}(t)}{a(t)} d_p(t) = H(t) d_p(t) \tag{5.20}$$

$H(t)$ is the time-dependent Hubble parameter that appears in the Friedmann equation.

Now, we will consider a photon that is emitted by the star at time t_{em} and then detected by *Gaia* at time t_0. We start by rearranging the relation between a and z (e.g. Equation S1) to isolate z:

$$z = \frac{a(t_0)}{a(t_{em})} - 1 \tag{5.21}$$

We have seen that *Gaia* can only measure parallaxes for very nearby objects, so we can assume $t_{em} \approx t_0$ and replace $a(t_{em})$ with its Taylor expansion around t_0:

$$a(t_{em}) \approx a(t_0) + \dot{a}(t_0)(t_{em} - t_0) = a(t_0)\left[1 + H_0(t_{em} - t_0)\right]$$

Substituting for $a(t_{em})$ in Equation 5.21 we find:

$$z \approx \frac{a(t_0)}{a(t_0)\left[1 + H_0(t_{em} - t_0)\right]} - 1 = \frac{1}{1 - H_0(t_0 - t_{em})} - 1 \tag{5.22}$$

We can use a series expansion to further simplify our approximate expression for z.

$$\frac{1}{1-x} \approx 1 + x + \mathcal{O}(x^2)$$

Here, we use the notation $\mathcal{O}(x^n)$ as shorthand to refer to terms that depend on powers of x greater than or equal to n. In this case, we expect that terms depending on powers of x greater than or equal to 2 will be negligible compared to terms that depend on smaller powers of x. It follows that:

$$z \approx \frac{1}{1 - H_0(t_0 - t_{em})} - 1 \approx H_0(t_0 - t_{em})$$

Our photon propagates at the speed of light, so if the current proper distance to the star is $d_p(t_0)$ then we can write:

$$c = \frac{d_p(t_0)}{t_0 - t_{em}}$$

So finally, our approximate expression for z becomes:

$$z \approx H_0(t_0 - t_{em}) \approx \frac{H_0 d_p(t_0)}{c}$$

We have therefore derived a form of the Hubble–Lemaître Law (Equation 1.2). Now we can use Equation 5.20 to relate z to the current recession velocity due to cosmic expansion:

$$z \approx \frac{v_\mathrm{r}(t_0)}{c}$$

This expression allows apparent superluminal recession velocities, but this does not violate special relativity. Remember that this recession velocity represents the expansion of spacetime, not the peculiar motions of objects within it, which *are* constrained to be less than c.

We know that the most distant objects for which *Gaia* can measure parallaxes would have cosmological redshifts as small as $z \sim 10^{-5}$. Using this we compute a recession velocity:

$$v_\mathrm{r}(t_0) \approx cz \approx 3\,\mathrm{km\,s^{-1}} \qquad (5.23)$$

(b) This recession velocity value is just 0.5% of the Milky Way's peculiar velocity, which implies that any redshifts we measure for nearby objects are likely to be completely dominated by their peculiar motions. To measure velocities caused by the cosmic expansion, we need to measure distances to objects that are much further away.

5.2.2 Standard candles

Earlier in this chapter we described standard candles in the context of the luminosity distance. In this section you will learn about two types of celestial object that cosmologists use as standard candles, and which allow the distances to those objects and their host galaxies to be estimated.

Classical Cepheid stars

Classical Cepheids (CCs) are a group of periodic variable stars with luminosities that rise and fall on a regular timescale. They are named after a star called δ Cephei, which was the first example to be discovered. CCs are bright ($10^3\,\mathrm{L_\odot} \lesssim L \lesssim 10^4\,\mathrm{L_\odot}$) and massive ($4\,\mathrm{M_\odot} \lesssim M \lesssim 20\,\mathrm{M_\odot}$). They lie in a region of the Hertzsprung–Russell (H–R) diagram known as the **instability strip**, which is populated with several types of pulsating stars.

The variability of CCs is driven by instabilities in their outer atmospheres, which expand and contract in regular cycles that typically last between 1 and 100 days. As CCs expand, their brightness increases by more than a factor of 10 (up to two magnitudes), before beginning to fade again as the star shrinks.

Online resources: the magnitude system

You may not be familiar with with the concept of defining the brightness of astrophysical objects using the magnitude system. In that case, the online resources for this chapter provide some additional material from a Stage 2 astronomy module that you can refer to.

In 1907, the American astronomer Henrietta Swan Leavitt observed a large number of CCs in the LMC and discovered that the periods of CCs' regular brightness variations are correlated with their mean *luminosities*.

- How do you think Henrietta Swan Leavitt was able to determine the intrinsic luminosities of the CCs she observed?

- She was able to measure their distances using stellar parallax! In Exercise 5.3 you computed the stellar parallax of a star in the LMC. This *is* a small angle, but it *was* measurable using the technology that was available in 1907. Once the distances to the CCs were known, she could use their observed brightnesses to compute their luminosities.

The absolute magnitude, M, quantifies a celestial object's intrinsic brightness. It is defined as the apparent magnitude m that would be measured by an observer at a distance of 10 pc.

The **period–luminosity relation** discovered by Leavitt has since been corroborated by numerous observations using more modern telescopes, including the *Hubble Space Telescope* (*HST*). The relation is normally expressed in terms of the mean **absolute magnitude** $\langle M \rangle$ of the star, the period P of the brightness variation in *days*, and two empirically determined parameters, A and B.

Classical Cepheid period–luminosity relation

$$\langle M \rangle = A\,[\log_{10}(P) - 1] - B \tag{5.24}$$

Equation 5.24 defines a linear relationship between $\langle M \rangle$ and $[\log_{10}(P) - 1]$, and B represents the intercept value of $\langle M \rangle$ when $[\log_{10}(P) - 1] = 0$. Physically, this means that B is the mean absolute magnitude of a 'standard' CC that has a period of exactly 10 days. Using *HST* observations, astronomers have derived values for $A = -2.43 \pm 0.12$ and $B = 4.05 \pm 0.02$ using V-band optical observations of nine CCs in the Milky Way (Benedict *et al.*, 2007).

The V band describes the range of photon wavelengths within the visible part of the electromagnetic spectrum.

Equation 5.24 allows cosmologists to infer the intrinsic brightness of a CC by measuring the period of its variability cycle. In the next example you will see how to use this information to work out how far away the CC is, without needing to be able to measure its parallax.

- When cosmologists use the period–luminosity relation to estimate the distance to CCs, which of the distance definitions discussed in Section 5.1 are they actually measuring?

- They are estimating the distance to the CCs using their apparent brightness, so they are measuring the *luminosity distance*, $d_{\rm L}$.

Example 5.4

Using *HST* observations, astronomers have measured the period and mean V-band apparent magnitude of δ Cephei to be $P = 5.36627$ days and $\langle m_{\rm V} \rangle = 3.960$ magnitudes, respectively. Use this information and the A and B parameters derived by Benedict *et al.* (2007) to compute the distance to δ Cephei in parsecs.

Solution

The difference between the apparent magnitude m of an object and its absolute magnitude M is related to the object's luminosity distance d_L *in parsecs* by a quantity called the **distance modulus**, which is typically represented using the symbol μ.

Distance modulus

$$\mu = m - M = 5\log_{10}(d_L/\mathrm{pc}) - 5 \tag{5.25}$$

For the specific case of the V band, we can write:

$$m_V - M_V = 5\log_{10}(d_L) - 5$$

We substitute for M_V using Equation 5.24:

$$\langle m_V \rangle - A\left[\log_{10}(P) - 1\right] + B = 5\log_{10}(d_L) - 5$$

We have been given the period in days and we are told that our mean magnitude measurement is for the V band, so we can assume that $A = -2.43$ and $B = 4.05$ using the result from Benedict *et al.* (2007).

Now it is straightforward to compute:

$$\begin{aligned}
\log_{10} d_L &= \frac{5 + m_V - A\left[\log_{10}(P) - 1\right] + B}{5} \\
&= \frac{5 + 3.960 + 2.43 \times \left[\log_{10}(5.36627) - 1\right] + 4.05}{5} \\
&= 2.47 \\
\implies d_L &= 295\,\mathrm{pc}
\end{aligned}$$

The brightness of classical Cepheids means that they can be detected at much larger distances than those that can be measured using stellar parallax. The unprecedented sensitivity of the *HST* allowed Newman *et al.* (1999) to detect CCs and measure their periods in the galaxy NGC 4603, which is 33 Mpc away!

- What is the redshift of NGC 4603? How fast is it receding due to cosmic expansion, and how does this value compare with the peculiar velocity of the Milky Way ($\sim 600\,\mathrm{km\,s^{-1}}$)?

- According to the Hubble–Lemaître law (Equation 1.2) the redshift of NGC 4603 is $z \approx H_0 D/c \approx 0.007$ for $D = 33\,\mathrm{Mpc}$. This corresponds to a cosmological recession velocity $v_r \approx cz \approx 2230\,\mathrm{km\,s^{-1}}$. This is approximately four times faster than the peculiar velocity of the Milky Way, which means that the influence of cosmic expansion is clearly detectable, albeit not completely dominant, at this distance.

Type Ia supernovae

Earlier in Section 5.2.2 you read that classical Cepheids are only bright enough to enable measurement of the luminosity distance to galaxies that are ~ 30 Mpc away. At this distance galaxies' peculiar velocities are comparable to their cosmological recession velocities. To measure the luminosity distances to objects with motions that are *dominated* by the Hubble flow, cosmologists must find standard candles that are thousands of times brighter than CCs.

Figure 5.7 A Type Ia supernova (SN 1994D) in the galaxy NGC 4526. The supernova is the bright star-like object in the bottom left. For a short time it outshines all other stars in the galaxy combined!

The vast majority of stars end their lives as **white dwarfs** (WDs). WDs are the remnants of stellar cores after all nuclear fusion reactions within them have stopped. They are extremely dense, with masses in the range $\sim 0.1\,\mathrm{M}_\odot$ to $1\,\mathrm{M}_\odot$ compressed into a sphere roughly the size of the Earth!

The luminosities of WDs are typically $< 10^{-2}\,\mathrm{L}_\odot$ and they vary over a large range, so they cannot be used by themselves as standard candles. However, sometimes a WD forms in a binary system with another star that is still burning nuclear fuel. In some of these systems, the WD starts to accrete gas and plasma from its binary companion and its mass slowly increases.

Eventually, the mass of the WD approaches a theoretical threshold called the **Chandrasekhar limit** $M_\mathrm{C} \approx 1.39\,\mathrm{M}_\odot$, above which the star's internal pressure cannot counteract the force of its self-gravity. Just before it gets to this limit, carbon fusion ignites in its outer layers and starts a runaway thermonuclear reaction that completely destroys the remnant. This detonation is called a **Type Ia supernova** (Figure 5.7), and releases so much energy ($\sim 10^{44}$ J) that the exploding WD briefly outshines all of the stars in its host galaxy combined. This astonishing luminosity means that Type Ia supernovae can be detected at very large distances.

The scenario we just described – with a single WD accreting material from a stellar companion – is actually quite rare. Only around 20% of Type Ia supernovae are believed to originate this way. More frequently, the stellar companion itself also runs out of fuel, and a binary WD system forms. Subsequent tidal interactions between the two WDs slowly remove angular momentum from the system, and the stellar remnants orbit closer and closer to each other. Eventually, the orbital separation becomes so small that the more massive WD is able to tidally disrupt its smaller companion and accrete material from it. The accretion happens much more rapidly than it did in the single WD scenario, but the ultimate end state is very similar. As the more massive WD approaches the Chandrasekhar limit, a thermonuclear detonation is ignited and the remnant is destroyed, forming a Type Ia supernova.

Regardless of their origin, the fact that all Type Ia supernovae occur when accreting white dwarfs reach a *specific* mass means that the amount of energy they release on detonation is almost identical. Theoretically, this means that the peak luminosity of every Type Ia supernova in the Universe is almost exactly the same, making them excellent standard candles.

In reality the situation is slightly more complicated. The actual peak luminosity depends somewhat on the details of the WD's exact mass and composition at the time of the explosion, and the immediate environment surrounding the supernova can also modify and attenuate the radiation that we eventually observe. Fortunately, astronomers have discovered that the shapes of the supernova's **light curves** – time-ordered series of measurements of the object's apparent brightness obtained at different times – when measured in different wavelength bands can be used to infer how these different effects have impacted the maximum brightness that the supernova actually reached.

To derive a quantitative relationship between light curve shapes and intrinsic luminosities of Type Ia supernovae in distant galaxies, cosmologists use an approach that is conceptually similar to the one used to calibrate the Cepheid period–luminosity relation. By measuring the periods and inferring the luminosities of CCs in the same galaxies as the supernovae, they can work out how far away those supernovae really are. Then, by observing the apparent brightnesses of the supernovae, cosmologists can infer how intrinsically luminous they are. Finally, they can measure the light curve shapes for the different supernovae and calibrate a brightness–shape relation. CCs are only bright enough to be detectable at relatively low redshifts (33 Mpc corresponds to $z \sim 0.01$), so cosmologists must assume that the supernovae in the local Universe behave in the same way as their more distant counterparts.

- Why is this assumption about the similarity of Type Ia supernovae throughout the Universe valid?

- This is a good assumption because the behaviour of Type Ia supernovae is governed by the basic laws of physics that govern star formation, evolution and destruction. We do not expect these laws to vary according to spatial location or as the Universe ages.

The light curves in Figure 5.8a show how supernovae that are more intrinsically luminous also tend to fade more slowly over time. While these curves all look generally similar in shape, their peak luminosities are clearly not similar enough for them to be used as standard candles.

To use observed Type Ia supernovae as standard candles, cosmologists define an idealised 'standard' supernova with a specific light curve shape and corresponding intrinsic luminosity (Figure 5.8b). The observed peak magnitudes of real supernovae are 'corrected' to represent the peak apparent magnitude that *would* have been observed for the standard supernovae, if it was placed at the same luminosity distances as the real ones. To reflect the fact that differences in their intrinsic brightness must be corrected in order to use their light curve shapes, Type Ia supernovae are often referred to as '*standardisable* candles'.

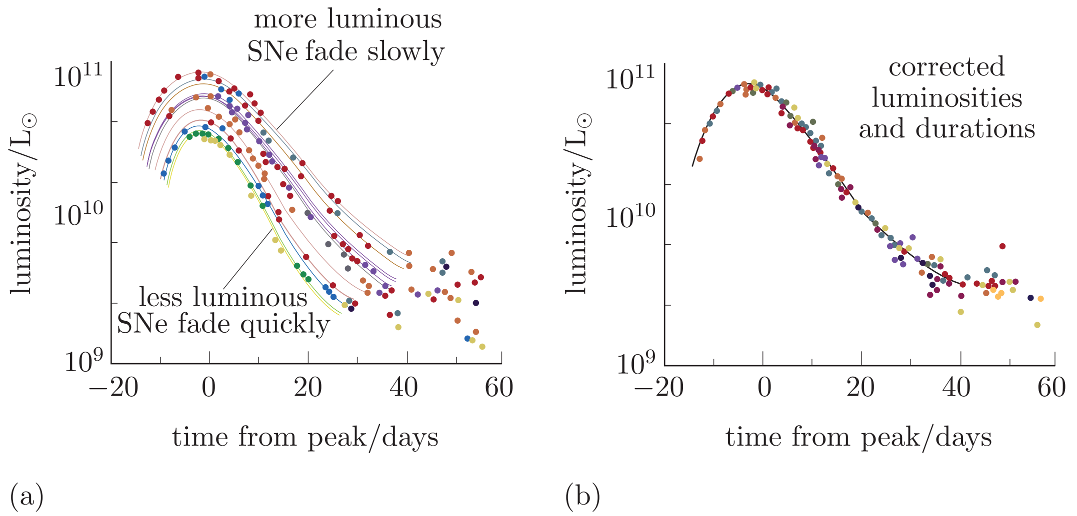


Figure 5.8 (a) Selection of observed Type Ia supernovae (or 'SNe') light curves, showing that more luminous objects tend to fade more slowly than less luminous ones. (b) The same light curves after standardisation, where the data have been calibrated so that the curves appear to lie on top of each other.

Type Ia supernova are very rare events. We would expect approximately one to occur per century in a galaxy like the Milky Way.* Nonetheless, there so many galaxies in the Universe that astronomers have detected thousands of Type Ia supernovae during the last 80 years. As the sensitivity of telescopes has improved, supernovae have been discovered at larger and larger distances. At the time of writing (2023), the most distant Type Ia supernova that has been observed is SN UDS10Wil, which has a measured redshift $z = 1.914$.

The 2011 Nobel Prize in Physics was awarded to astronomers from two independent teams – the Supernova Cosmology Project (SCP) and High-z Supernova Search Team – who used observations of Type Ia supernovae at high redshift to demonstrate a completely unexpected result.
They discovered that that $\Omega_\Lambda > 0$, and the expansion of the Universe is accelerating. The next section describes how this discovery was made.

5.3 Measuring H_0 and density parameters

In Chapter 4 you saw that the Friedmann equations and the values of H_0, $\Omega_{\rm m,0}$, $\Omega_{\rm r,0}$, $\Omega_{\Lambda,0}$ and k completely determine the expansion history of the Universe, its future evolution and its ultimate fate.

Measuring these parameters allows cosmologists to determine how physical properties of the Universe – like temperature and density – have evolved in

*The closest recent Type Ia supernova was observed in the Large Magellanic Cloud in 1987. It was named SN 1987A to reflect the fact that it was the first Type Ia supernova to be observed that year.

the past, and to predict how they will continue to evolve in the future. Their values also tell cosmologists about the fundamental properties of the Universe like its spatial curvature, its age and its physical scale.

So far in this chapter we have introduced the concepts and some of the techniques that cosmologists use in order to constrain these values using observational data. Next we will see how these techniques are actually used in practice. You will learn that, by measuring d_L for large numbers of distant objects, cosmologists are able to *constrain* H_0 and two of the four density parameters.

This is not as simple as it might sound. Figures 5.2b and 5.4b, in Sections 5.1.2 and 5.1.3 respectively, show that the predicted values of d_L (and d_A) for $z \lesssim 0.1$ are very similar for model universes with very different cosmological parameters. To precisely constrain the values of the cosmological parameters of the real Universe, cosmologists need to distinguish between potential models that are much more similar than those illustrated in Figures 5.2 and 5.4. Their goal is to precisely measure contributions of matter, radiation and Λ to the energy density of the Universe. This means that they need very precise measurements of d_L (or d_A) for galaxies at redshifts $z \gg 0.1$, which is a technical challenge that has only recently been overcome.

5.3.1 Measuring the Hubble constant

There are many reasons that cosmologists would *like* to know the value of H_0, beyond the simple fact that it measures the current expansion rate of the Universe. For example, it can be used to define a **Hubble distance**, d_H. This common cosmological unit has dimensions of length, and sets an approximate size scale for the observable Universe.

The Hubble distance

$$d_H = c/H_0 \tag{5.26}$$

Cosmologists also define a quantity called the **Hubble time**, t_H, which provides an approximate time scale for the age of the Universe.

Note that in general the Hubble time and the true age of the Universe are *not* equal.

The Hubble time

$$t_H = H_0^{-1} \tag{5.27}$$

Later in this chapter you will see that the *overall* matter density of the Universe can be constrained without knowing H_0. However, for reasons that we will touch on in a later chapter, the fractions of baryons and dark matter can often only be *measured* as multiples of a quantity denoted as h^2 and defined as

$$h^2 = \left(\frac{H_0}{100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}\right)^2 \tag{5.28}$$

This means that measuring the density of baryons in the Universe requires knowledge of the true value of H_0.

Exercise 5.4

Estimate a value for the Hubble time in units of My (millions of years); use the central value of H_0 listed in the table of constants.

One way to estimate the Hubble constant is to *assume* a cosmological model for the Universe and simultaneously find ranges of that model's parameters – like H_0 and the density parameters – that are consistent with actual observations. In fact, that is how the value of H_0 provided in this book was derived. That value *assumes* a flat Universe with $\Omega_k = 0$ and that Λ truly is a cosmological constant, with equation of state parameter $w_\Lambda = -1$. Neither of these assumptions is derived from fundamental physical principles, and neither has to be true.

It would be much better if we could find some way of measuring H_0 that is completely independent of any assumptions about the Universe or the values of the other cosmological parameters. As we discussed earlier this chapter, it should be possible to infer H_0 directly if we can measure the proper distances and redshifts for a large sample of objects whose motions are dominated by the Hubble flow.

We have identified Type Ia supernovae as a population of standardisable candles that are bright enough to be detectable in galaxies as far away as $z \sim 1.5$. It turns out that if we know the *true* peak absolute magnitude M_{true} of a 'standard' Type Ia supernova, then we can use the standardised ('corrected') apparent peak magnitudes m_{corr} of many distant supernovae to directly measure H_0. The following exercise asks you to derive an expression that will be very useful for the remainder of this section, and again in Section 5.3.3.

Exercise 5.5

Use the definition of the distance modulus (Equation 5.25) to show that the brightest apparent magnitude m of a Type Ia supernova can be expressed as:

$$m = M - 5\log_{10} H_0 + 5\log_{10}(H_0 d_{\mathrm{L}}) + 25 \tag{5.29}$$

where M is the supernova's brightest absolute magnitude, d_{L} is the luminosity distance to the supernova in Mpc and H_0 is the present Hubble constant expressed in $\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

By rearranging the definition of the distance modulus in Equation 5.29 we can write

$$\log_{10} H_0 = \log_{10}(H_0 d_{\mathrm{L}}) - \frac{m_{\mathrm{corr}}}{5} + \frac{M_{\mathrm{true}}}{5} + 5 \tag{5.30}$$

For $z \ll 1$ we know $H_0 d_{\mathrm{L}} = H_0(1+z)d_{\mathrm{p}}(t_0) \approx cz$, and we can define the variable

$$\mathcal{A} = \log_{10}(H_0 d_{\mathrm{L}}) - \frac{m_{\mathrm{corr}}}{5} \approx \log_{10}(cz) - \frac{m_{\mathrm{corr}}}{5} \tag{5.31}$$

The approximation $H_0 d_{\mathrm{L}} \approx cz$ holds at low redshifts, and the value of $\mathcal{A}$ can be directly measured using observations of a sample of Type Ia supernovae. It is just the intercept of a linear fit to the logarithms of the supernova redshifts and corrected apparent peak magnitudes. However, at high redshifts the situation is more complicated. Exercise 5.6 explores how to estimate the redshift at which the values $H_0 d_{\mathrm{L}}$ and cz become significantly different.

Exercise 5.6

The Hubble–Lemaître law, $H_0 d_{\mathrm{L}} \approx cz$, provides a good approximation for the relationship between the measured redshifts and luminosity distances of nearby objects, for which $z \ll 1$. However, for more distant objects, this approximation becomes increasingly inaccurate. An alternative expression, which is accurate to within 1% for values of $z < 0.2$, is:

$$H_0 d_{\mathrm{L}} = cz\left(1 + \frac{1-q_0}{2}z\right) \tag{5.32}$$

where q_0 is the deceleration parameter from Equation 4.43.

Assuming that the Universe is flat and that $\Omega_{\mathrm{r},0} \approx 0$, use the cosmological parameters listed in the table of constants to show that by $z = 0.13$, the result of Equation 5.32 is already 10% greater than the $H_0 d_{\mathrm{L}} \approx cz$ approximation that applies for $z \ll 1$.

Once $\mathcal{A}$ has been measured, we just need to find the true peak absolute magnitude (M_{true}) of a Type Ia supernova, and then we can use Equation 5.30 to compute H_0 directly. In the next section you will learn how cosmologists construct a cosmological distance ladder that ultimately allows M_{true} to be determined.

5.3.2 Cosmological distance ladders

To determine the value of M needed to compute H_0 in Equation 5.30, cosmologists need to measure the corrected apparent magnitude (M_{corr}) of at least one Type Ia supernova for which the proper distance is already known. If a Type Ia supernova exploded in the Milky Way we could measure its distance directly using parallax, and immediately compute its absolute magnitude. If astronomers could accurately measure the supernova's light curve then we could also compute what its standardised absolute magnitude would be.

However, as mentioned in Section 5.2.2, Type Ia supernovae are very rare events: with an occurrence rate of roughly one per decade within 20 Mpc of the Sun, we would be very lucky to detect a single instance within our galaxy within the next century. Even if we did observe one, that supernova might be unusual in some way and then our single measurement of M would be misleading.

To accurately measure the distances to large numbers of Type Ia supernovae, cosmologists start by measuring the distances to fainter standard candles like classical Cepheids, which exist in all galaxies and are therefore much more likely to be found nearby. The goal is to find examples of Type Ia supernovae in galaxies that also contain detectable CCs. The distances to those CCs, and by extension their host galaxies, can then be inferred using the period–luminosity relation. The following exercise asks you to estimate the rate at which Type Ia supernovae occur in galaxies that are close enough for the CCs they contain to be detected.

Exercise 5.7

In Section 5.2.2 you read that CCs can be detected out to redshifts $z \sim 0.01$. Using the value of H_0 listed in the table of constants, calculate how many Type Ia supernovae would you expect to find per year in galaxies that contain *detectable* CCs?

The distances to galaxies containing observable CCs may be relatively small on the scale of the observable Universe, but they are still very large when compared to the size of a single galaxy (typically $\sim$10 kpc). Therefore, we can assume that the distances to the CCs and a Type Ia supernova within a single galaxy are effectively identical. If we could directly measure the distance to the CCs, then we would also know the true distance to the Type Ia supernovae in their host galaxies. With enough examples, we could then straightforwardly calibrate the light-curve-based standardisation procedure that we need in order to accurately infer the distances to supernovae at much larger redshifts.

However, as you read in Section 5.2.2, the distances to CCs in distant galaxies cannot be measured directly either: they must be inferred from their apparent magnitudes and their variability periods using Leavitt's period–luminosity relation. To calibrate this relation, astronomers need to find CCs that are close enough for their distances to be measured directly using parallax.

Figure 5.9 shows how cosmologists measure the value of H_0 by constructing a **cosmological distance ladder**. To 'anchor' the distance ladder, distances measured using stellar parallax for CCs in the Milky Way and its satellite galaxies are used to calibrate the CC period–luminosity relation (Equation 5.24). The next rung on the distance ladder uses more-distant CCs to calibrate the peak absolute magnitude M of a 'standard' Type Ia supernova. Finally, by assuming this value for M and measuring the apparent magnitudes of more distant supernovae, cosmologists use Equation 5.30 to measure the value of H_0. In 2022, cosmologists using this distance ladder method reported a directly measured value of H_0 to be $73.04 \pm 1.04\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

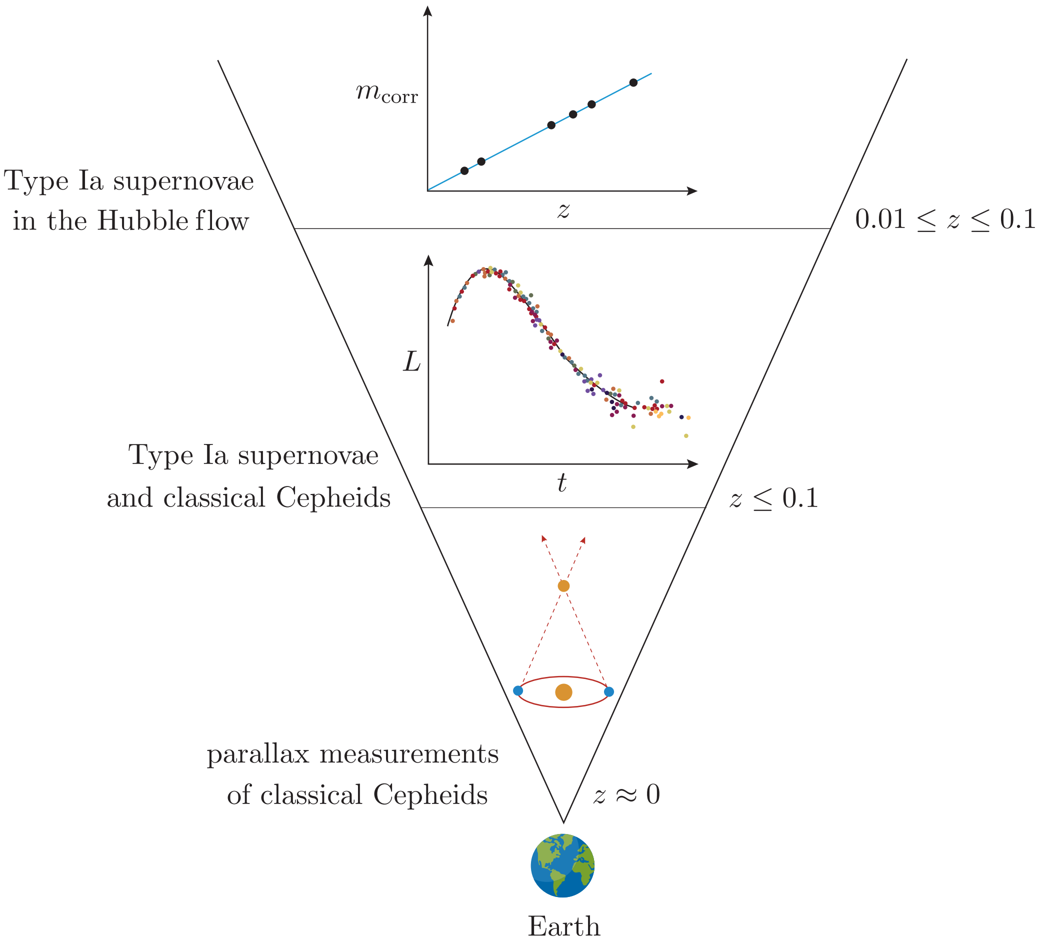


Figure 5.9 A schematic of the cosmological distance ladder. The ladder is anchored by measuring distances to nearby CCs using stellar parallax. The distances to these anchored CCs are used to calibrate the period–luminosity relation Equation 5.24 which allows their intrinsic luminosities to be inferred from their periodic variability timescales. On the second rung of the ladder CCs are used to measure the distances to distant galaxies that host Type Ia supernovae. The light curves and apparent magnitudes of these supernovae are used to calibrate the standardisation procedure that allows them to be used as standard candles. On the third rung, a sample of more distant Type Ia supernovae whose motions are dominated by the Hubble flow are identified. Their apparent magnitudes are standardised and used together with Equation 5.30 to measure H_0.

5.3.3 Measuring the density parameters for matter and Λ

In this section you will learn how observations of distant Type Ia supernovae can also be used to constrain the present-day density parameters of matter and Λ. For simplicity, we will assume a flat universe in which $\Omega_{\rm r}$ is negligible, but the techniques being explored can also be used to constrain more complicated models for the Universe.

We can start by noting that the version of the distance modulus expression shown in Equation 5.29 is just the equation of a straight line. It expresses a linear relationship between m and $\log_{10} H_0 d_{\rm L}$ with intercept $\mathcal{M}$, which can be written as:

$$\mathcal{M} \equiv M - 5\log_{10} H_0 + 25 \qquad (5.33)$$

At low redshift, $\mathcal{M}$ can be measured if we can identify a sample of Type Ia supernovae that are close enough to use the approximation $d_{\rm L} \approx cz/H_0$. For any one of these supernovae, we could rewrite Equation 5.29 as

$$m = M - 5\log_{10} H_0 + 5\log_{10}(cz) + 25 = \mathcal{M} + 5\log_{10}(cz) \qquad (5.34)$$

To measure $d_{\rm L}$ in Equation 5.29 we would need to know the intrinsic peak luminosity of the Type Ia supernovae in our sample but, unlike $d_{\rm L}$, the quantity z in Equation 5.34 is *directly* measurable. To estimate $\mathcal{M}$, all we need to do is fit a straight line to measured values of m and $\log_{10}(cz)$ and read off the intercept.

Now that we know how to calculate $\mathcal{M}$, we could solve Equation 5.29 to determine $d_{\rm L}$ for a sample of high-redshift supernovae without needing to know their absolute magnitudes. We just need to assume that the low- and high-redshift supernovae have the *same* absolute magnitude so that $\mathcal{M}$ is equal for both samples. In Section 5.2.2 you read that the peak apparent magnitudes of Type Ia supernovae can be corrected based on their light curve shapes to represent a 'standard' Type Ia supernova with a *specific* absolute magnitude. Therefore, if we use corrected apparent magnitudes in Equation 5.29, then we *can* assume that $\mathcal{M}$ is completely independent of redshift. You will apply these ideas in Examples 5.5 and 5.6.

Example 5.5

In this example you will perform calculations that will let you determine whether two small sets of real observational data are consistent with a flat, matter-only model for the Universe.

Tables 5.1 and 5.2 list the redshifts and corrected apparent magnitudes for two samples of Type Ia supernovae. Values in the 'σ_z' and '$\sigma_{m_{\rm corr}}$' columns represent the 1σ uncertainty on these properties. Use the tables as directed to answer the following questions.

Table 5.1 Redshifts (z) and corrected apparent peak magnitudes (m_{corr}) for a sample of low-redshift supernovae, in order of date observed, as measured by the Calán/Tololo Supernova Survey.

Supernova	z	σ_z	m_{corr}	$\sigma_{m_{\text{corr}}}$
1990O	0.030	0.002	16.26	0.20
1992ae	0.075	0.002	18.43	0.20
1992aq	0.101	0.002	19.16	0.23
1992bo	0.018	0.002	15.61	0.21
1992bp	0.079	0.002	18.27	0.18

(Adapted from Hamuy *et al.*, 1993)

Table 5.2 Redshifts (z) and corrected apparent peak magnitudes (m_{corr}) for a sample of high-redshift supernovae, in order of date observed, as measured by the Supernova Cosmology Project.

Supernova	z	σ_z	m_{corr}	$\sigma_{m_{\text{corr}}}$
1995aw	0.400	0.030	22.36	0.19
1995ax	0.615	0.001	23.19	0.25
1997L	0.550	0.010	23.51	0.25
1997S	0.612	0.001	23.69	0.21
1997aj	0.581	0.001	23.09	0.22

(Adapted from Perlmutter *et al.*, 1997)

(a) Estimate $\mathcal{M}$ by using the low-redshift data in Table 5.1 to evaluate Equation 5.33 for each supernova, and then averaging the results.

(b) Use your estimate for $\mathcal{M}$ to calculate the luminosity distance to each of the high-redshift supernovae listed in Table 5.2, in units of the Hubble distance, $d_{\text{H}} = c/H_0$.

(c) Now, use Equation 5.9 and Equation 5.12 to compute a *theoretical* prediction for the luminosity distance, in d_{H} units, to each high-redshift supernova, assuming $k = 0$ and $\Omega_{\text{m}} = 1$.

Solution

(a) If we label the five low-redshift supernovae in Table 5.1 with an index $i = 1, 2, \ldots, 5$, then we can compute $\mathcal{M}_i$ for each supernova by rearranging equation Equation 5.34.

$$\mathcal{M}_i = m_{\text{corr},i} - 5\log_{10} z_i - 5\log_{10} c$$

Table 5.3 lists the result of this computation for each supernova. Note that the numerical results throughout this example are calculated using c in units of kilometres per second.

Table 5.3 Computed results for the low-redshift supernova sample.

Supernova	$\mathcal{M}_i$
1990O	−3.510
1992ae	−3.329
1992aq	−3.246
1992bo	−3.051
1992bp	−3.602

Now we can combine the results for the individual supernovae and compute a mean value for $\mathcal{M}$ to use as our final estimate:

$$\mathcal{M} = \langle \mathcal{M}_i \rangle = \frac{1}{5}\sum_{i=1}^{5} \mathcal{M}_i = -3.348$$

(b) For brevity will use the symbol $\mathcal{D}$ to denote d_L in units of d_H. To calculate $\mathcal{D}$ for the high-redshift supernovae listed in Table 5.2 we can label each of objects with a index $j = 1, 2, \ldots, 5$, then rearrange Equation 5.29 to give:

$$\mathcal{D}_j = \frac{d_\mathrm{L,j}}{d_\mathrm{H}} = \frac{d_\mathrm{L,j} H_0}{c} = \frac{1}{c} 10^{\alpha_j}$$

where we have defined

$$\alpha_j = \frac{m_{\mathrm{corr},j} - \mathcal{M}}{5}$$

Table 5.4 lists computed values of $\mathcal{D}$ for the high-redshift sample.

Table 5.4 Computed values of $\mathcal{D}$ for the high-redshift supernovae.

Supernova	$\mathcal{D}_j$
1995aw	0.462
1995ax	0.677
1997L	0.785
1997S	0.852
1997aj	0.647

(c) To determine whether these results are consistent with a flat matter-only universe with $\Omega_\mathrm{m} = 1$ we need to compare them with a theoretical prediction that we can compute using Equation 5.9 and Equation 5.12. Combining these equations, assuming that $\Omega_\mathrm{r} = \Omega_\Lambda = \Omega_\mathrm{k} = 0$ and $\Omega_\mathrm{m} = 1$, we can write

$$\begin{aligned}\mathcal{D}_{j,\mathrm{predicted}} &= (1+z_j)\int_0^{z_j} \frac{\mathrm{d}z'}{\sqrt{(1+z')^3}} \\ &= (1+z_j)\left[-\frac{2(1+z')}{\sqrt{(1+z')^3}}\right]_0^{z_j} \\ &= 2(1+z_j)\left[1 - \frac{(1+z_j)}{\sqrt{(1+z_j)^3}}\right]\end{aligned}$$

Table 5.5 lists the predicted values of $\mathcal{D}$ for the high-redshift sample.

Table 5.5 Computed and predicted values of $\mathcal{D}$ for the high-redshift supernovae.

Supernova	$\mathcal{D}_j$	$\mathcal{D}_{j,\mathrm{predicted}}$
1995aw	0.462	0.434
1995ax	0.677	0.688
1997L	0.785	0.610
1997S	0.852	0.685
1997aj	0.647	0.647

To determine whether the observational data are consistent with the corresponding theoretical predictions, it is important to correctly account for the measurement uncertainties. To do this you will need to use two standard results related to the calculation and propagation of experimental uncertainties.

First, if the quantity x is measured with an associated uncertainty σ_x^2, then the result of applying a unary function f to x is also uncertain, such that:

A unary function is one that operates on a single variable.

$$\sigma^2_{f(x)} = \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2 \sigma_x^2 \tag{5.35}$$

Second, if a and b are two *uncorrelated* measurements with uncertainties σ_a^2 and σ_b^2, respectively, then the result of applying a binary function f to a and b is also uncertain, such that:

A binary function is one that operates on two variables.

$$\sigma^2_{f(a,b)} = \left(\frac{\partial f(a,b)}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial f(a,b)}{\partial b}\right)^2 \sigma_b^2 \tag{5.36}$$

Here we have used the standard notation $\partial y/\partial x$ to denote the partial derivative of y with respect to x.

The following example shows how you can use these two results to estimate the uncertainties associated with the quantities you calculated in Example 5.5. Note that this example reuses some of the notation and results from Example 5.5.

Example 5.6

Use standard techniques for propagation of measurement uncertainties and the values listed in Tables 5.1 and 5.2 to do the following.

(a) Propagate the observational uncertainties listed in Table 5.1 to calculate the 1σ uncertainty in your estimate for $\mathcal{M}$ from Example 5.5.

(b) Use this estimate for the uncertainty in $\mathcal{M}$ and the observational uncertainties listed in Table 5.2 to calculate 1σ uncertainties for each of your estimated luminosity distances from part (b) of Example 5.5.

Solution

(a) To estimate the uncertainty on $\mathcal{M}_i$ for each separate supernova we can use Equation 5.35 (with $f = \log_{10}$) to propagate the 1σ errors listed in Table 5.1.

$$\begin{aligned}\sigma^2_{\mathcal{M}_i} &= \sigma^2_{m_{\mathrm{corr},i}} + 5\,\sigma^2_{z_i}\left[\frac{\mathrm{d}}{\mathrm{d}z_i}\log_{10} z_i\right]^2 \\ &= \sigma^2_{m_{\mathrm{corr},i}} + 5\left[\frac{\sigma_{z_i}}{z_i \ln 10}\right]^2\end{aligned}$$

The results of this calculation for each low-redshift supernova are listed in Table 5.6.

Table 5.6 Computed uncertainties and their squares for the low-redshift supernova sample.

Supernova	$\sigma_{\mathcal{M}_i}$	$\sigma^2_{\mathcal{M}_i}$
1990O	0.210	0.0441
1992ae	0.202	0.0408
1992aq	0.231	0.0534
1992bo	0.236	0.0557
1992bp	0.182	0.0331

Summing the individual results gives the overall uncertainty on $\mathcal{M}$:

$$\sigma^2_{\mathcal{M}} = \frac{1}{5}\sum_{i=1}^{5}\sigma^2_{\mathcal{M}_i} = 0.045$$

The final value of $\sigma_{\mathcal{M}}$ is calculated by taking the square root of $\sigma^2_{\mathcal{M}}$:

$$\sigma_{\mathcal{M}} = 0.213$$

(b) We can assume that $m_{\text{corr},j}$ and $\mathcal{M}$ are uncorrelated, so the uncertainties for each luminosity distance estimate $\mathcal{D}_j$ can be computed using Equation 5.36 repeatedly (with $f = 10^g$ and $g = \alpha_j$).

$$\sigma^2_{\mathcal{D}_j} = \sigma^2_{m_{\text{corr},j}}\left[\frac{\mathrm{d}}{\mathrm{d}m_{\text{corr},j}}\frac{10^{\alpha_j}}{c}\right]^2 + \sigma^2_{\mathcal{M}}\left[\frac{\mathrm{d}}{\mathrm{d}\mathcal{M}}\frac{10^{\alpha_j}}{c}\right]^2 \tag{5.37}$$

The derivatives can be evaluated using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}m_{\text{corr},j}}10^{\alpha_j} = \frac{\mathrm{d}}{\mathrm{d}\alpha_j}10^{\alpha_j}\cdot\frac{\mathrm{d}\alpha_j}{\mathrm{d}m_{\text{corr},j}} = \ln(10)10^{\alpha_j}\cdot\frac{1}{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{M}}10^{\alpha_j} = \frac{\mathrm{d}}{\mathrm{d}\alpha_j}10^{\alpha_j}\cdot\frac{\mathrm{d}\alpha_j}{\mathrm{d}\mathcal{M}} = \ln(10)10^{\alpha_j}\cdot -\frac{1}{5}$$

Substituting these two results into Equation 5.37, our uncertainty estimates for $\mathcal{D}$ become

$$\sigma^2_{\mathcal{D}_j} = \frac{1}{25c^2}\left\{\sigma^2_{m_{\text{corr},j}}\left[\ln(10)\cdot 10^{\alpha_j}\right]^2 + \sigma^2_{\mathcal{M}}\left[\ln(10)\cdot 10^{\alpha_j}\right]^2\right\}$$

Table 5.7 lists computed values of $\sigma_{\mathcal{D},j}$ for the high-redshift sample.

Table 5.7 Computed uncertainties on the luminosity distance for the high-redshift supernova sample.

Supernova	$\sigma_{\mathcal{D}}$
1995aw	0.061
1995ax	0.102
1997L	0.119
1997S	0.117
1997aj	0.091

Tables of data like Tables 5.4 and 5.7 can be difficult to interpret. Instead, look at Figure 5.10, which presents the results from Examples 5.5 and 5.6 in graphical form.

Now we can straightforwardly observe that three of our $\mathcal{D}$ estimates *are* consistent with a flat, matter-only universe. However, one of the other estimates is consistent with a very different universe that only contains a cosmological constant, and another appears to suggest an intermediate model with matter and a Λ component. This example illustrates just how difficult it is to constrain the cosmological parameters once experimental uncertainties are taken into account.

The approach taken in Example 5.5 is very similar to the way that professional cosmologists use Type Ia supernova observations to constrain the cosmological parameters. They also use a low-redshift sample of supernovae to constrain $\mathcal{M}$, and compare their results for high-redshift supernovae with model predictions, to determine what combinations of cosmological parameters are consistent with their observations. The differences lie primarily in the sophistication of the statistical techniques that are used, and the number of different supernovae that are considered.

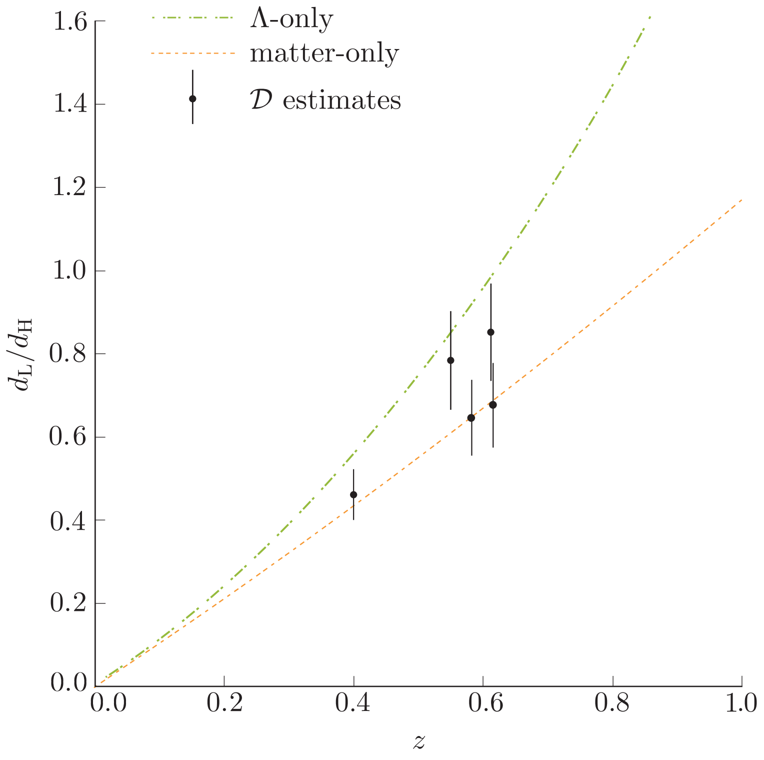


Figure 5.10 Estimates of $\mathcal{D}$ and its uncertainty for the high-redshift supernova sample referenced in Tables 5.2 and 5.4. Predicted curves of $\mathcal{D}(z)$ are shown for flat universes that contain only matter (orange dot-dashes) or only a cosmological constant (short green dashes). Different subsets of the estimated supernova $\mathcal{D}$ values are consistent with both universe models at the 1σ level.

A noteworthy feature of this method is that we can constrain plausible values for Ω_m and Ω_Λ without needing to know the value of H_0. The expression for $\mathcal{M}$ contains H_0, but we were able to measure $\mathcal{M}$ directly using low-redshift supernovae so the value of H_0 was not required. Similarly, using Equations 5.9 and 5.12 allows the composite quantity $H_0 d_\mathrm{L}$ to be predicted directly given values of the density parameters. In Section 5.3.1 you saw that measuring H_0 is complicated, and requires the true luminosities of standard candles to be known. The ability to measure other cosmological parameters independently is therefore a big advantage!

Figure 5.11 shows the Hubble diagram for 60 Type Ia supernovae that were analysed by the Supernova Cosmology Project (SCP) team. The supernova magnitudes you used in Example 5.5 are actually a subset of these data. The curves in Figure 5.11 show predictions for how the corrected apparent peak magnitudes of Type Ia supernovae should evolve as a function of redshift for different cosmological models. It might not be obvious under visual inspection, but a careful statistical analysis by the SCP team showed that the high-redshift data points ($z \gtrsim 0.2$) are *inconsistent* with a universe in which Ω_Λ is zero. This was the historic discovery for which the SCP team jointly won the 2011 Nobel Prize in Physics.

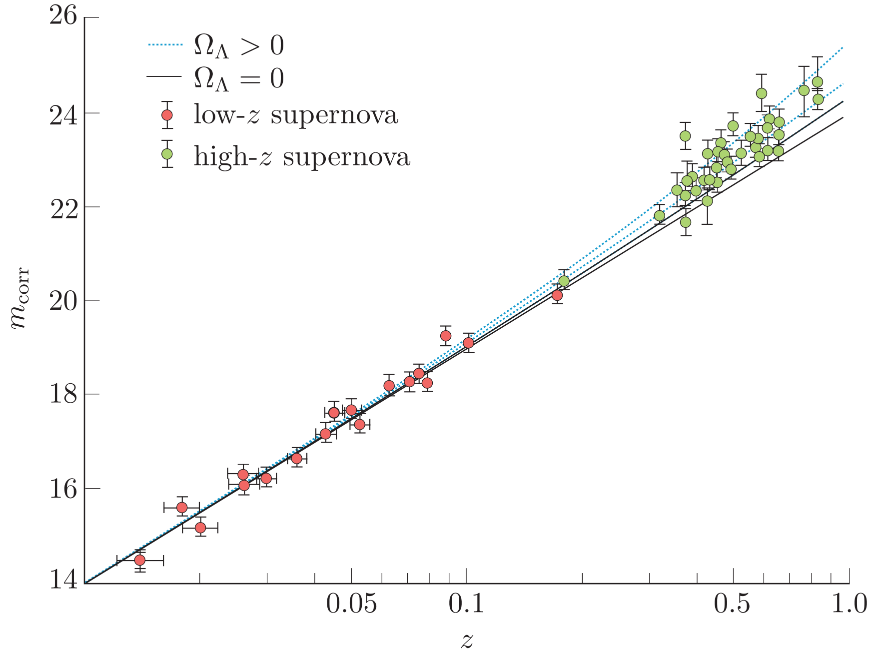
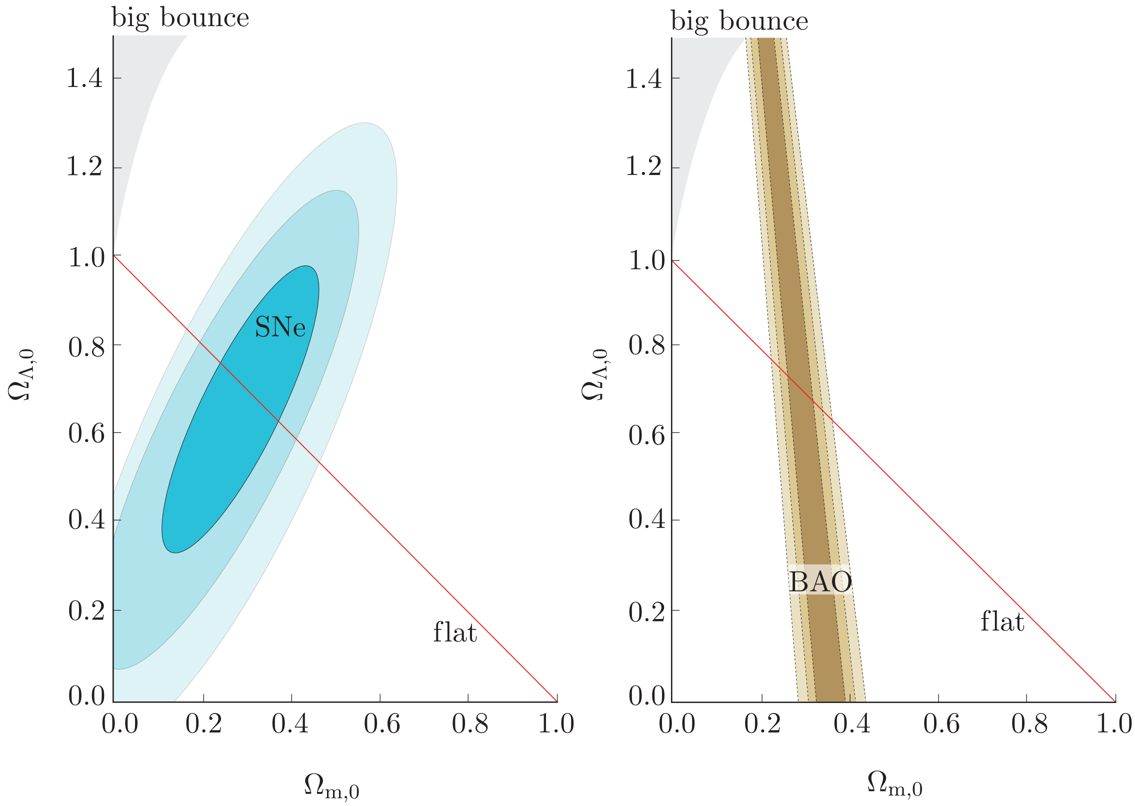


Figure 5.11 A Hubble diagram showing the corrected peak magnitudes of 18 low-redshift supernovae (red points) and 42 high-redshift supernovae (green points) versus their redshifts (after Perlmutter *et al.*, 1999) The different curves show predictions for different model universes.

Our constraints on the cosmological parameters are constantly improving as new observational data are collected. The shaded contours in Figure 5.12a illustrate the ranges of $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ that are consistent at different significance levels with a recent analysis of 580 Type Ia supernovae. The supernova-derived constraints *alone* show very strong evidence that $\Omega_{\Lambda,0} > 0$.

In Chapter 11 you will learn about another observable phenomenon called baryon acoustic oscillations (BAOs) that allows the spatial distribution of galaxies in the Universe to be used as a standard rod. The contours in Figure 5.12b show the constraints on $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ that BAOs can provide. BAOs and Type Ia supernovae exclude different values of the density parameters; by combining these complementary constraints, cosmologists can completely rule out a $\Omega_{\Lambda,0} = 0$ universe.

In the next chapter you will learn how the properties of the cosmic microwave background can further restrict the range of plausible models for the real Universe.



(a) (b)

Figure 5.12 Constraints on $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ derived using observations of (a) Type Ia supernovae, and (b) baryon acoustic oscillations. The shaded contour levels indicate 1σ (darkest), 2σ and 3σ (lightest) uncertainty regions.

5.4 Summary of Chapter 5

- The **proper distance** $d_{\mathrm{p}}(t)$ between two points measures the length of a spatial geodesic connecting them at a particular cosmic time, t:

$$d_{\mathrm{p}}(t) = a(t)\int_0^R \frac{\mathrm{d}r}{\sqrt{1-kr^2}} \qquad \text{(Eqn 5.1)}$$

  Assuming $a(t_0) = 1$, the *present* proper distance between two points is therefore equal to the co-moving distance d_{c} between them:

$$d_{\mathrm{p}}(t_0) = a(t_0)d_{\mathrm{c}} = d_{\mathrm{c}} \qquad \text{(Eqn 5.3)}$$

- The **luminosity distance** d_{L} relates the observed flux F of a distant object to its intrinsic luminosity L:

$$d_{\mathrm{L}} = \sqrt{\frac{L}{4\pi F}} \qquad \text{(Eqn 5.11)}$$

  If $a(t)$ is *known* then the measured luminosity distance to a distant object can be used to calculated its intrinsic luminosity.

- The **angular diameter distance** d_{A} relates the observed angular size θ of a distant object to its physical size l.

$$d_{\mathrm{A}} = \frac{l}{\sin\theta} \approx \frac{l}{\theta} \qquad \text{(Eqn 5.13)}$$

  If $a(t)$ is *known* then the measured angular diameter distance to a distant object can be used to calculated its physical size.

- In a spatially flat universe, the present proper distance to a distant object, its luminosity distance and its angular diameter distance can be related in terms of its redshift z:

$$d_{\mathrm{p}}(t_0) = \frac{d_{\mathrm{L}}}{1+z} = (1+z)d_{\mathrm{A}} \qquad (5.38)$$

- Estimates of the proper distance to objects that have a wide range of redshifts can be used to measure $a(t)$, and thereby constrain cosmological parameters like $\Omega_{\mathrm{m},0}$, $\Omega_{\Lambda,0}$ and H_0.
- **Stellar parallax** can be used to directly measure the proper distance to objects within the Milky Way and its satellite galaxies.
- Objects with *known* intrinsic luminosity are called **standard candles**. Measuring the *observed* brightness of standard candles allows their luminosity distances to be determined. These luminosity distances can be used to calculate the corresponding proper distances if the standard candles' redshifts are known.
- **Classical Cepheid** (CC) stars can be used as standard candles because their mean **absolute magnitude** $\langle M\rangle$ can be inferred from measurements of their pulsation periods P using a **period–luminosity relation**:

$$\langle M\rangle = A\left[\log_{10}(P) - 1\right] - B \qquad \text{(Eqn 5.24)}$$

To calibrate the relation, the coefficients A and B must be empirically determined by observing nearby CCs whose distances can be directly measured using stellar parallax. Periods have been measured for CCs at distances up to 33 Mpc, where the **peculiar velocities** of their host galaxies are comparable to the recession velocities that result from the expansion of the Universe.

- All **Type Ia supernovae** explode with similar, but not identical, peak luminosity. Even though their luminosities are not identical, they can still be used as standard candles because the *shapes* of their **light curves** can be used to reliably 'correct' their peak apparent magnitudes, and make them reflect the apparent peak magnitude of an idealised 'standard' Type Ia supernova.
- Type Ia supernovae have been observed and used as standard candles out to redshifts $z \approx 2$, where their velocities are dominated by the Hubble flow.
- Cosmologists use a technique known as the **cosmological distance ladder**, which is 'anchored' using measured stellar parallaxes, to calibrate the CC period–luminosity relation and the relation between a Type Ia supernova's light curve shape and its intrinsic luminosity.
- The corrected peak *apparent* magnitudes of Type Ia supernova populations at low and high redshift can be used in combination to directly measure H_0 if the *absolute* magnitude of the idealised 'standard' supernova is known.
- In addition, the corrected peak *apparent* magnitudes of such supernova populations can be used in combination to measure $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$, even if H_0 is not known.