Chapter 3 The geometry of the Universe

The previous chapter demonstrated that any physical theory of the Universe must account for the interlinked nature of space and time, and introduced some geometric methods for describing spacetime. This chapter puts those ideas together with a crucial insight about the nature of gravity, which led to the development of the general theory of relativity.

In this chapter you will explore the ideas underpinning general relativity as well as some key evidence in support of the theory, before moving on to explore two of the most important metrics for understanding and observing the distant Universe: the Schwarzschild metric, which describes spacetime surrounding a massive object, and the Robertson–Walker metric, which describes the geometry of our expanding Universe.

Objectives

Working through this chapter will enable you to:

- explain the equivalence principle, and describe its significance for our understanding of gravity
- describe Einstein's field equations and understand and explain the meaning of the terms included in them
- summarise key evidence in support of the theory of general relativity
- describe the behaviour of spacetime in the vicinity of a massive object such as a planet or a black hole, and apply the Schwarzschild metric to solve problems in this situation
- understand and explain the form of the Robertson–Walker metric that describes an expanding Universe
- define and manipulate three key cosmological parameters linked to this metric: the scale factor, curvature parameter and Hubble parameter.

3.1 Gravity as geometry

3.1.1 Free fall and the equivalence principle

The extension of the theory of special relativity to encompass the behaviour of gravity began with Einstein's realisation that for an observer falling freely from a great height, accelerating downwards as a result of a gravitational field, physics behaves as though gravity is 'switched off'.

Figure 3.1 Effect of dropping an object in a freely falling lift.

Figure 3.1 shows an observer in a lift falling down an airless lift shaft with no friction, who has let go of an object while falling freely. The downward accelerations of the lift, the observer and the object will all be the same, because a principle called the **universality of free fall** guarantees that acceleration under gravity is independent of an object's mass or composition. This means that the released object will appear to remain stationary to the observer; it will not fall to the floor of the lift, because the lift and the object are accelerating by the same amount. If the observer exerts a force on the object (e.g. pushing it sideways) they will see it behave according to Newton's laws of motion (e.g. being displaced horizontally in the direction of the applied force), so the freely falling observer's frame is acting like an inertial frame of reference.

- If the released object in the freely falling lift appears to 'float' rather than drop to the floor, should the same apply to the person?

- If the person was standing on the floor at the time the lift began freely falling, then they would remain in the same place relative to the top and bottom of the lift, as shown in Figure 3.1; similarly, if they raised their feet off the ground they would not drop back to the bottom of the lift.

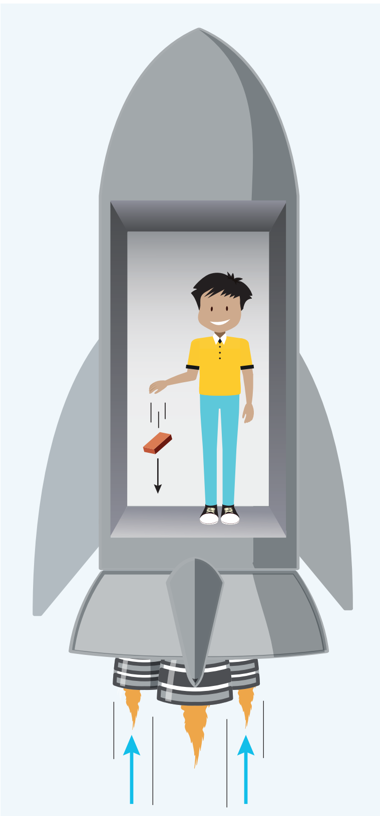


Figure 3.2 Effect of dropping an object in a rocket undergoing uniform acceleration.

In contrast to the situation shown in Figure 3.1, where the dropped object appeared *not* to fall from the perspective of the person in the lift, we can consider an observer located in a region where there is no gravitational field. Figure 3.2 shows what would happen if someone dropped an object while in a rocket that is undergoing uniform acceleration and is not subject to gravitational forces. If the uniform acceleration, a, has the same magnitude as g, the gravitational acceleration on the surface of the Earth, then the observer will see behaviour that appears identical to gravity on Earth – namely, the object will appear to fall. The relative motion between the dropped object and the floor of the rocket will be equivalent to that of a released object that falls to the ground on Earth, because their relative accelerations are the same in the two situations.

- How are these two situations equivalent if the rocket in Figure 3.2 is accelerating in an *upward* direction relative to the person inside it, while a person on the surface of the Earth feels a *downward* gravitational pull?

- A person of mass m on the surface of the Earth exerts a force on the Earth's surface of $F = -mg$, and so by Newton's third law the Earth's surface exerts an opposing force of the same magnitude, mg. In Figure 3.2, the floor of the accelerating rocket exerts an upward force of $F = ma$ on the person, so that if the magnitudes of a and g are the same, then the effect is equivalent.

This agreement between the physics in regions of uniform acceleration and in regions under gravity is known as the **equivalence principle**. This principle has a 'weak' and a 'strong' form. The weak equivalence principle refers specifically to the *motion* of objects and can be stated as follows.

The weak equivalence principle

Within a localised region of spacetime near to a concentration of mass, the motion of objects caused by gravitational effects alone cannot be distinguished by any experiment from the motion of objects within a region of appropriate uniform acceleration.

It is important to note the reference to a *localised* region of space. Unlike the theory of special relativity, when thinking about general relativity we need to consider **local inertial frames**: the location of an observer matters, as well as their relative motion.

To see why this is the case, consider an observer in a lab on the surface of the Earth, who drops two objects separated by a certain horizontal distance. If the objects are falling towards the centre of the Earth under gravity, then the directions of their acceleration vectors will be very slightly different, as represented in Figure 3.3a. By contrast, for the equivalent situation we considered earlier – namely, an upwardly accelerating rocket that is *not* subject to a gravitational field – the directions in which the two objects fall are exactly the same, as shown in Figure 3.3b.

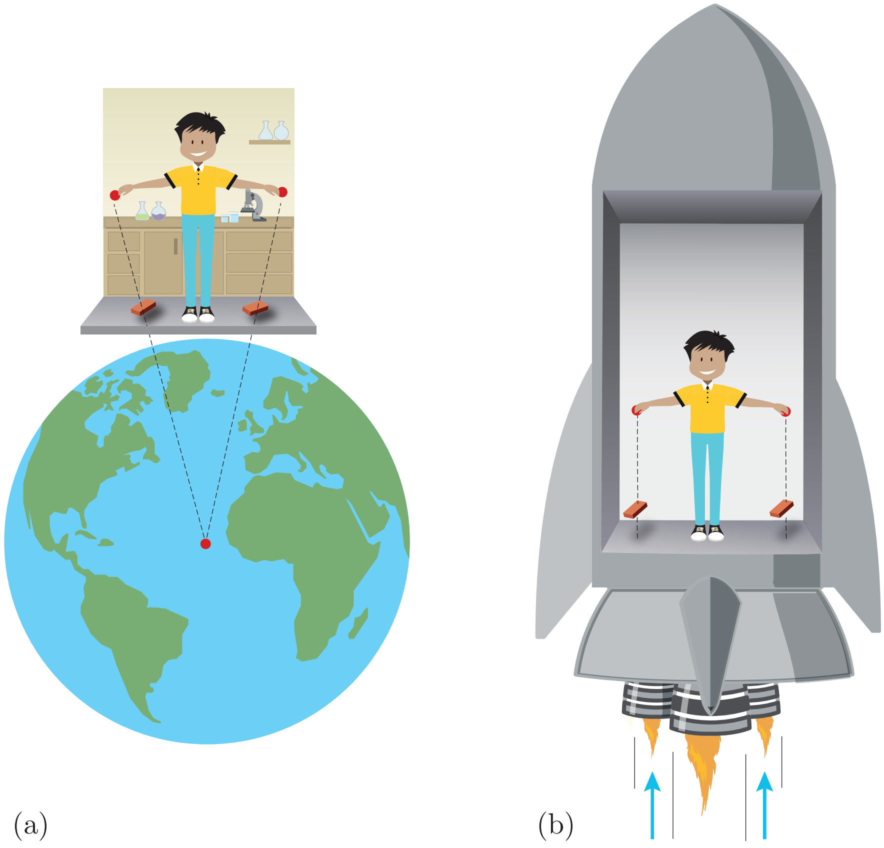


Figure 3.3 Dashed lines showing the motion of objects falling (a) under gravity on Earth and (b) in a rocket with uniform acceleration that is unaffected by gravitational forces. Differences in the motions of the objects become noticeable if the reference frames applied are insufficiently local.

Although the difference in the situation of Figure 3.3a would be undetectable in a lab-sized region, the slight deviation in the direction of acceleration from the exactly vertical would become measurable with an experiment if the reference frame, and the distance between the dropped objects, spanned a significant angular extent relative to the Earth's centre. It is therefore *crucial* to always define reference frames that cover a region and time interval that are sufficiently small as to be inertial within the accuracy needed to draw useful conclusions.

The weak equivalence principle has been very well tested experimentally, and is known to be accurate to an uncertainty level of less than 1 part in 10^{11}. The 'strong' form of the equivalence principle states that not just motion, but *any* physical behaviour of objects cannot be experimentally distinguished. This is less definitively proven, but any deviations must again be small. Much of the behaviour we will consider in the rest of the chapter is a consequence of the weak equivalence principle.

3.1.2 Einstein's field equations

The equivalence principle led Einstein to the idea that a *metric theory of gravity* is possible. Under this model, whose postulates are summarised below, the motion of test particles that are subject to gravity is determined by the form of the metric in a particular location, instead of being caused by gravitational forces. More specifically, in general relativity, test particles (including light) will follow geodesics.

- What is a geodesic, and why might a test particle follow such a path?

- Geodesics are the shortest routes between two points, and so are likely to be the path requiring the least energy. Geodesic paths will differ in flat and curved geometries, as discussed in the previous chapter.

Postulates of a metric theory of gravity

- The geometric properties of spacetime are described by a metric that encapsulates the influence of gravity on spacetime.
- The world line of a test particle subject only to gravitational influence (i.e. not subject to electromagnetic or nuclear forces) is a geodesic of this metric.
- The world line of a light ray or other electromagnetic signal travelling in a vacuum is a null geodesic of this metric ($ds^2 = 0$).

Geodesics in Minkowski (flat) spacetime are straight lines, which is consistent with the way Newton's first law describes the motion of bodies that are not subject to forces. The influence of gravity must change the form of the geodesics, and so cause spacetime to be curved. The general theory of relativity sets out Einstein's theory of how the presence of matter determines the metric of four-dimensional spacetime.

More specifically, this theory is encapsulated in **Einstein's field equations**, which describe in mathematical form the relationship between geometric curvature and the distribution of mass and energy. As noted in the previous chapter, it is beyond the scope of this module to teach the full mathematics of Riemannian geometry necessary to derive the field equations. Therefore, we can only introduce them and describe their meaning in a qualitative way. This will be sufficient to allow us to use some important solutions to the field equations in order to study cosmology.

Einstein's field equations can be set out as in Equation 3.1, where G is the gravitational constant and the other terms are summarised in Table 3.1.

Einstein's field equations of general relativity

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.1}$$

The tensor quantities in Equation 3.1 have two indices, which cycle through each dimension of spacetime. This means that Equation 3.1 describes $2^4 = 16$ equations, which need to be solved jointly to find the metric for a particular energy–momentum distribution. In practice, only 10 of the equations are independent, so there are usually 10 equations to solve, not 16.

Table 3.1 The meaning of terms in Einstein's field equations

Quantity	Definition
$R_{\mu\nu}$	The Ricci curvature – this is a simplified form of the Riemann curvature tensor.
$g_{\mu\nu}$	The metric tensor.
R	The curvature scalar – a further simplification of the Ricci curvature.
$T_{\mu\nu}$	The **energy–momentum tensor** (or stress-energy tensor) – a set of terms describing the distribution and flow of energy and momentum caused by the presence of matter and radiation.

In other descriptions of general relativity you may sometimes see the left-hand side of Equation 3.1 written in simplified form as a single tensor $G_{\mu\nu}$, known as the **Einstein tensor**.

Exercise 3.1

In Chapter 1 you were introduced to a famous description of the theory of general relativity by John Wheeler: 'Space tells matter how to move; matter tells space how to curve' (Misner, Thorne and Wheeler, 1973, p. 5). Write a short explanation of which elements of the theory of general relativity as introduced in this chapter relate to each half of this description.

The field equations are a very concise way to encapsulate a complete description of how gravity behaves anywhere in the Universe, for any possible distribution of matter and energy. Solving the field equations is not an easy task, and you will not be asked to do it in the module. For situations other than the simplest distributions of mass it is usually necessary to use computational methods, rather than doing the algebra by hand.

Later in this chapter you will investigate two of the most important solutions of the field equations: the metric that describes spacetime near a single massive object (e.g. a planet or a star) and the metric that describes the expansion of the Universe as a whole. However, we will first briefly discuss how the theory of general relativity can be tested, and what evidence exists to convince us we can rely on it for studying cosmology.

3.1.3 Evidence for general relativity

A wide range of experiments have been conducted to test the theory of general relativity. To date, the theory has passed all of these tests successfully, with no evidence of discrepancies.

A famous early triumph for the theory was its ability to predict the **precession** of the perihelion of Mercury. Midway through the nineteenth century, it was already known that the orbit of the planet Mercury exhibited some unexplained behaviour: the planet's perihelion – the point of closest approach to the Sun – had been measured to shift position with each successive orbit, as illustrated in Figure 3.4a. This effect was not explained by Newtonian mechanics, but Einstein demonstrated that it is expected in general relativity, with the predicted angular shift being in very good agreement with what was observed in practice.

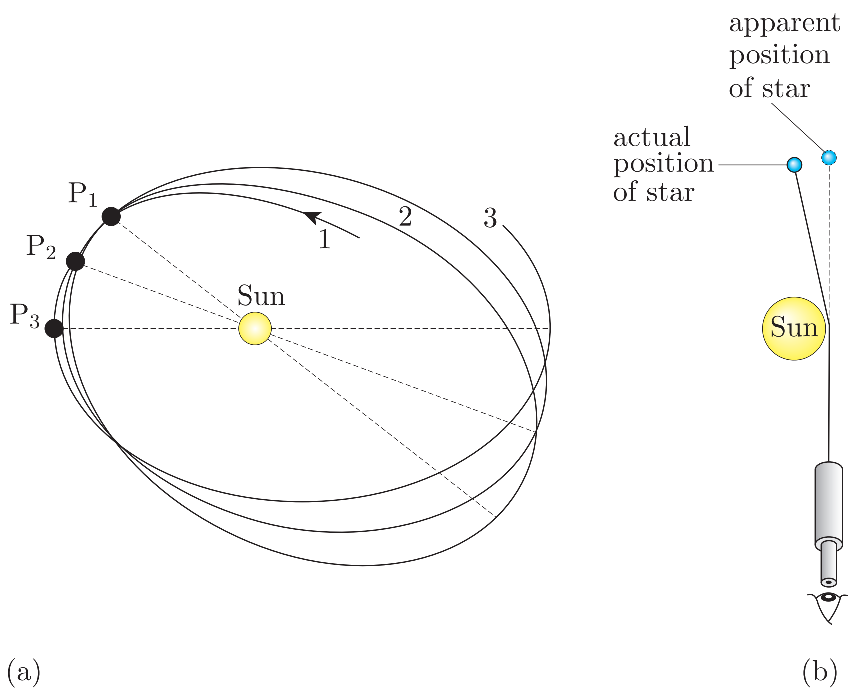


Figure 3.4 Two predictions of general relativity: (a) the changing orbital shape of Mercury with time, with the perihelion location P advancing with each successive orbit (labelled 1, 2, 3, . . .); (b) the deflection of a star's light when the Sun passes near to the star's light path to Earth.

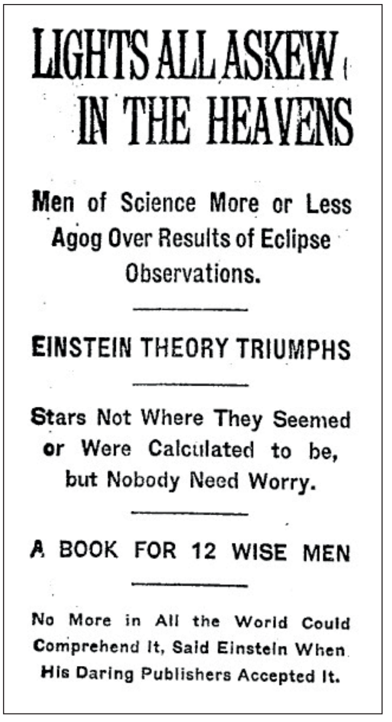
General relativity also predicts a comparatively large deflection in the path of light travelling from a distant star and passing near to the Sun, as illustrated in Figure 3.4b. Testing this prediction requires making measurements of the positions of stars and recording how these change when the Sun passes very close to their projected location on the sky. Measuring the resulting deflections is tricky because of the brightness of the Sun, but is possible during a total eclipse.

Arthur Eddington carried out a series of such measurements in the early twentieth century, which helped to promote Einstein's theory and made international headlines (Figure 3.5). More recently, the predictions of light bending by the Sun have been verified to an accuracy of $< 0.04\%$ using observations of quasars via radio interferometry, a method of measuring radio waves to very high spatial precision.

Another famous experiment measured an effect known as gravitational redshift, which is a consequence of relativistic time dilation due to gravity, and is discussed further in Section 3.2.3. The **Pound–Rebka experiment** of 1960 confirmed that the Earth's gravity causes a detectable redshifting of light, and calculated the magnitude of this to be less than 1% with the general relativity prediction. The **Shapiro delay** is another important general relativistic effect, in which radar signals are delayed when they pass close to a massive object.

Modern astronomical observations also provide extensive evidence for general relativity. This includes the gravitational lensing of distant galaxies and quasars (the distortion of their appearance in images due to the path their light has taken to reach us) and the behaviour of neutron stars and black holes, which will be discussed further in the next section.

One of the most exciting recent discoveries in astrophysics was the direct detection in 2015 of **gravitational waves** with an instrument on Earth. General relativity predicts that moving sources of gravitation will emit waves that propagate as time-varying distortions in spacetime. Detections of such gravitational waves were made by the Laser Interferometer Gravitational-wave Observatory (LIGO) and the Virgo interferometer (located in the US and Italy, respectively), and led to the Nobel Prize in Physics being awarded to Thorne, Weiss and Barish in 2017. Gravitational waves had actually been detected more indirectly in the 1970s by radio monitoring of a binary pulsar system, in which two neutron stars are gradually spiralling closer together, with this shrinking orbit caused by the loss of energy via gravitational waves. This earlier discovery also received a Nobel Prize in Physics, which was awarded to Hulse and Taylor in 1993.


LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

Figure 3.5 A *New York Times* newspaper headline describing one of Eddington's solar eclipse expeditions to test general relativity using the light-deflection method.

Figure 3.6 shows results from the two Nobel Prize-winning studies mentioned above. Panel (a) shows the first *direct* detection of gravitational waves. The data are plotted as a quantity called strain, which measures the relative distortion of spacetime in different directions caused by the passage of gravitational waves from distant events, such as mergers of black holes and neutron stars. The experimental data are presented alongside models that are used to infer the properties of the merging objects.

Panel (b) shows observations of the decay of the orbit of the binary pulsar system identified by Hulse and Taylor over 17 years. Red dots plot the shift in the time of periastron (closest approach) compared with the orbital change predicted by general relativity (solid black curve beneath the dots). The lower panel highlights the close alignment between the data and the predicted model.

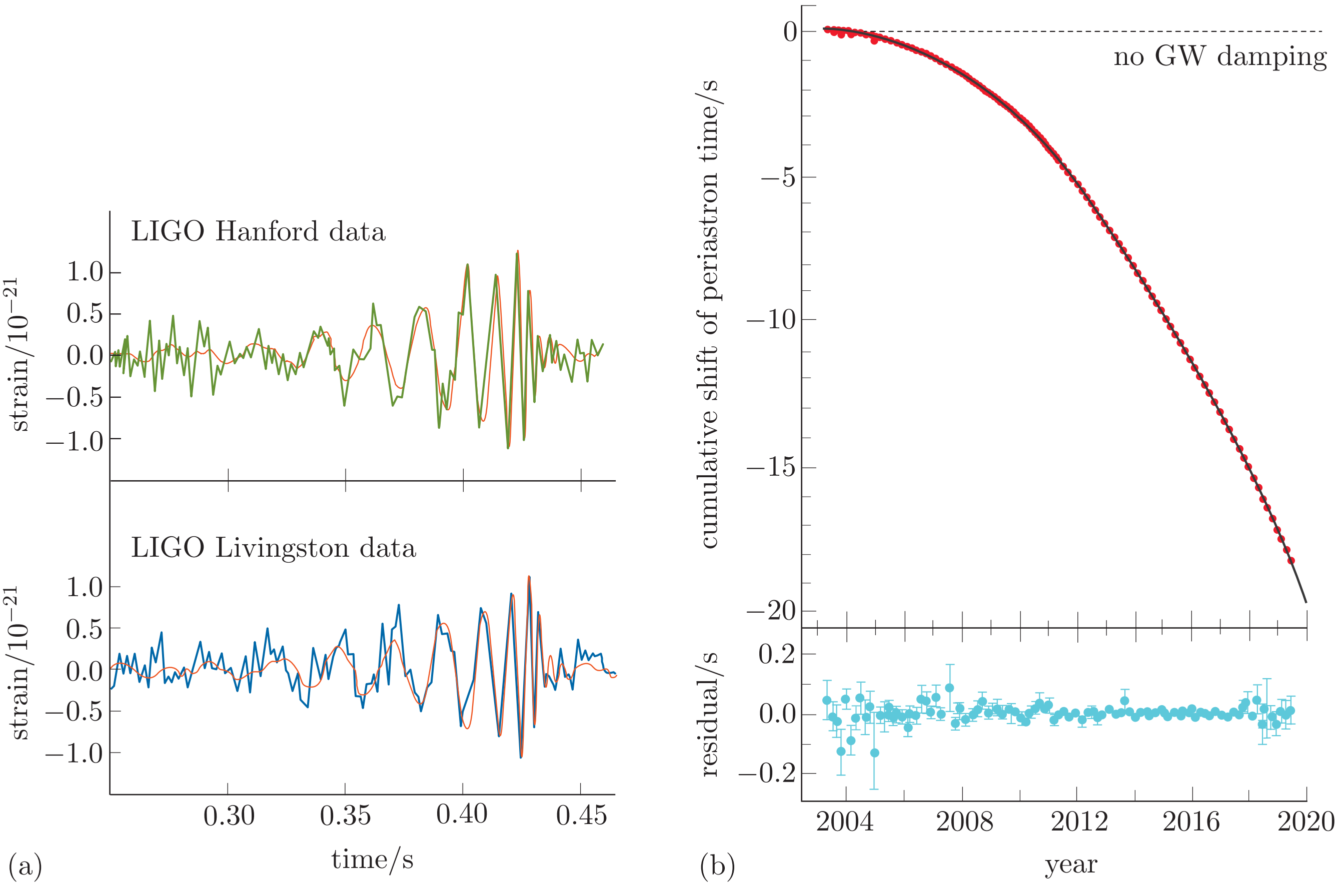


Figure 3.6 (a) The first gravitational wave signals, measured independently at the LIGO Hanford and Livingston detectors, compared with a black-hole merger model (red–orange lines). (b) Radio monitoring of the orbital decay of the Hulse–Taylor binary pulsar system (red dots) plotted against the prediction of general relativity (black curve). The horizontal dashed line shows the expected behaviour if no gravitational waves (GW) were transporting energy away; the lower panel highlights the close alignment between the data and the predicted model; the lower plot shows the residual difference between the model and the data.

3.2 Spacetime near planets and black holes

The first full solution to Einstein's field equations was derived by Karl Schwarzschild in 1915, and describes the behaviour of spacetime near a single concentration of mass. In other words, it describes the gravitational effects we see in everyday life, caused by the Earth's gravity, as well as the effects of gravity close to other planets.

One of the solution's most interesting applications is to the spacetime around very dense objects: the **Schwarzschild metric** describes the existence of the objects now known as black holes. In this section you will examine the Schwarzschild metric and some of its peculiar consequences in the vicinity of very dense objects.

3.2.1 The Schwarzschild metric and its properties

The Schwarzschild metric is a 'vacuum solution' to the Einstein field equations. This means that it describes the geometry of regions that don't themselves contain significant matter or energy (i.e. where the energy–momentum tensor is equal to zero). The geometry is instead determined by the presence of an external, nearby mass.

We can consider a region of spacetime within the gravitational influence of some sort of mass concentration, which could be a planet, a star or some other type of object, with total mass M. The influence of such a mass is spherically symmetric, and so the spatial part of the metric is usually written using spherical coordinates, r, θ and ϕ, whose origin is the centre of the dominating mass.

The Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2\,\mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2 \qquad (3.2)$$

- What are the units of the metric coefficients for the ct and r coordinates?

- The term in brackets multiplying each coordinate is 1 minus a ratio of terms involving the central mass M and r. The numerator of this ratio has units of $G \times M$, which are $\mathrm{m^3\,s^{-2}}$, and the denominator has units of $c^2 \times r$, which are also $\mathrm{m^3\,s^{-2}}$. Hence the coefficients are dimensionless.

The $2GM/c^2r$ term in the ct and r coefficients can also be written as a dimensionless ratio of distances, R_S/r, where R_S corresponds to a radius known as the **Schwarzschild radius**:

$$R_\mathrm{S} = \frac{2GM}{c^2} \qquad (3.3)$$

The significance of this radius is that when the r coordinate is equal to R_S, the $c^2\,\mathrm{d}t^2$ term in the metric vanishes and the $\mathrm{d}r^2$ term tends to infinity, hence the spacetime interval $\mathrm{d}s$ itself becomes infinite. In other words, something peculiar happens to the behaviour of spacetime at the Schwarzschild radius. You will explore behaviour at the Schwarzschild radius in the next section, but in the following exercise you will first investigate the size of this radius for different astronomical objects.

Exercise 3.2

Calculate the Schwarzschild radii for the Earth ($M_\oplus = 5.97 \times 10^{24}$ kg), the Sun ($M_\odot = 1.99 \times 10^{30}$ kg) and a neutron star ($M_{\rm NS} = 2\,{\rm M}_\odot$). Comment on how these values compare to the actual radii of these bodies (which are 6400 km, 7.0×10^8 m and 15 km, respectively).

The previous exercise points to the importance of the *density* of the massive object for the behaviour of spacetime in its vicinity: at high densities the Schwarzschild radius starts to approach the size of the object itself. In fact, we have ample evidence for the existence of astronomical objects that are sufficiently dense for their Schwarzschild radius to be larger than their physical size. Such objects are known as black holes.

Before we discuss the properties of black holes and what actually happens at the Schwarzschild radius in more detail, the following example considers how the metric behaves at the opposite extreme, namely, at locations a long way from the massive object.

Example 3.1

The limit in which $r \gg R_{\rm S}$ corresponds to spacetime at a very large distance from the dominating mass M. Write a simplified expression for the Schwarzschild metric in this limit, and compare this to our Newtonian understanding of the effect of gravity at large distances from a large mass.

Solution

We first combine Equations 3.2 and 3.3 to write the Schwarzschild metric in terms of $R_{\rm S}$:

$$\mathrm{d}s^2 = \left(1 - \frac{R_{\rm S}}{r}\right) c^2\,\mathrm{d}t^2 - \left(1 - \frac{R_{\rm S}}{r}\right)^{-1} \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2$$

In the limit where $r \gg R_{\rm S}$, we know that $R_{\rm S}/r$ tends to $1/\infty$, i.e. 0. This means that the two terms in brackets both simplify to 1. The metric therefore becomes:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2$$

Ignoring the ct coordinate for a moment, you should recognise the last three terms in the metric: this is just -1 times the metric of a flat three-dimensional space, as written in spherical coordinates. With the ct term included, it is the Minkowski spacetime metric (see Section 2.2.3), written in spherical coordinates.

The example has therefore shown that spacetime at very large distances from a mass concentration has a flat geometry. This is consistent with our Newtonian expectation that the effect of gravity will be very much reduced at large distances from a particular object.

3.2.2 Black holes and the event horizon

As mentioned briefly in the preceding section, **black holes** are objects whose density is sufficiently high for their Schwarzschild radius to be located outside the mass concentration itself. This means that there is a region within R_S into which matter can fall and be trapped, without possibility of escape. Such a radius is also known as an **event horizon**.

If an object of mass m is launched from the surface of a spherical body of mass M and radius R then, in order to escape from the gravitational influence of the body, the kinetic energy of the object must exceed its gravitational potential energy. This leads to the definition of an **escape speed**, v_{esc}, that is high enough to overcome gravity:

$$\frac{1}{2}mv_{\text{esc}}^2 = \frac{GMm}{R}, \quad \text{and so} \tag{3.4}$$

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \tag{3.5}$$

If we consider the escape speed for an object with radius $R < R_S$, as is the case for a black hole, then we find that $v_{\text{esc}} > c$. In other words, for the densest objects there is a radius from within which matter and light (if similarly affected by gravity) cannot escape, unless travelling faster than the speed of light. This radius corresponds to what we now refer to as the event horizon for a non-spinning, spherical body.

The idea of 'dark stars', so dense that neither light nor matter can escape from them, dates back to the eighteenth century. The concept was revived in the twentieth century as a result of the Schwarzschild solution, together with advances in stellar astrophysics and the observational discovery in 1967 of highly dense neutron stars (pulsars) by Jocelyn Bell and Anthony Hewish. Evidence has since accumulated for the existence of black holes throughout the Universe. Table 3.2 summarises some of the many forms of evidence for astronomical black holes, a number of which are depicted in Figures 3.7 and 3.8.

Table 3.2 A summary of evidence for the existence of black holes.

Observations	**Evidence**
Star orbits at the centre of the Milky Way	Detailed monitoring of the motion of stars at the centre of the Milky Way over several decades demonstrates the existence of a concentration of invisible mass of $\sim 4 \times 10^6\ \text{M}_\odot$ in a central, Solar System-sized region. (Figure 3.7a)
Gas orbits in galaxy centres	Doppler shift measurements of gas close to the centres of nearby galaxies show very high speeds, which require the presence of a central black hole.
X-ray observations of active galaxies	The extreme luminosities of active galaxies and quasars can only be explained by large amounts of matter falling in towards a black hole.
Imaging black-hole 'shadows'	The Event Horizon Telescope has imaged the shadow of a black hole in the nearby galaxy M87: a dark central region surrounded by the light of infalling and outflowing matter. (Figure 3.7b)
Radio jets	The existence of powerful radio jets travelling at close to the speed of light from galaxy centres can only be powered by matter falling onto a central supermassive black hole. (Figure 3.7c)
Gravitational waves	Recent detections of gravitational waves provide evidence for many mergers of black holes produced by stellar evolution. (Figure 3.8)

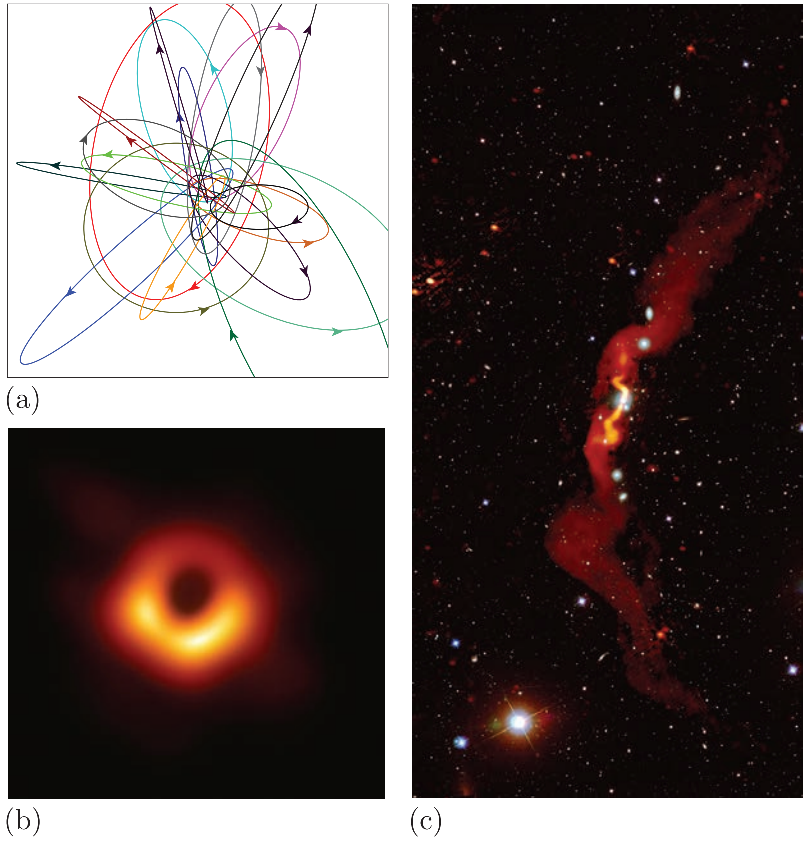


Figure 3.7 Observational evidence for black holes: (a) the orbits of stars around the centre of the Milky Way; (b) the Event Horizon Telescope image of a black-hole shadow; (c) powerful radio-emitting jets, travelling at close to the speed of light, powered by matter falling in towards a supermassive black hole in the central galaxy.

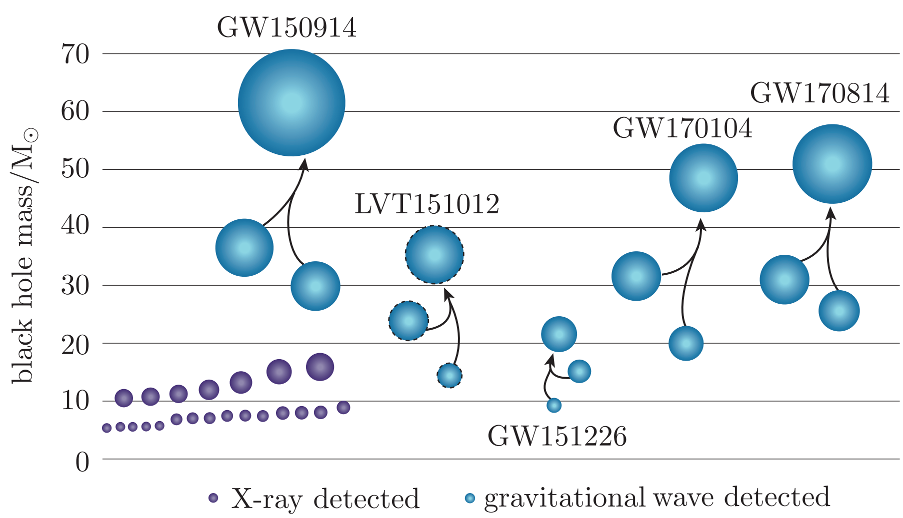


Figure 3.8 A chart of black-hole mergers (blue circles) discovered via gravitational waves, together with measured masses of some X-ray-detected black holes (purple circles). LVT151012 is a 'candidate event' that was too weak to be conclusively claimed as a detection.

3.2.3 Behaviour of spacetime near black holes

It is possible for individual particles (or large objects, e.g. an entire star or planet) to cross over an event horizon from the outside, but not to escape it from the inside. A body released from rest at a large distance from a non-rotating black hole requires only a finite proper time to reach the gravitational **singularity** at its centre (see Figure 3.9). At this location the metric tends to infinity, irrespective of the coordinate system used to describe it.

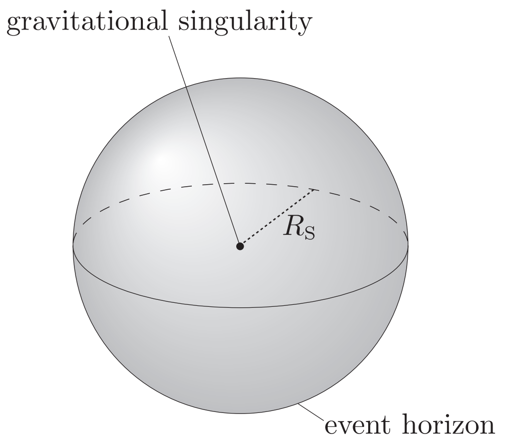


Figure 3.9 Simplified geometry of a black hole

What happens to mass that arrives at the singularity is the subject of theoretical speculation but, unlike behaviour near the event horizon, modern science provides no way to study it. However, it can be shown that the time taken for a body following a direct radial path (i.e. with constant θ and ϕ coordinates) to travel from the event horizon to the central singularity is

$$d\tau_{\text{sing}} = \frac{2}{3}\frac{R_S}{c} \tag{3.6}$$

The next exercise allows you to explore the timescales involved.

Exercise 3.3

(a) What is the proper time interval required for a falling body to travel from the Schwarzschild radius to the singularity of a black hole with three times the mass of the Sun?

(b) What is the corresponding proper time if instead we consider a supermassive black hole of mass $10^9\,M_\odot$?

The previous exercise illustrated that matter can very quickly fall from outside a black hole to its very centre. An observer travelling with the infalling matter will not notice anything unusual happen when crossing the event horizon. However, the situation looks very different to a distant observer watching matter approach a black hole.

We saw in Example 3.1 that for a distant observer (i.e. $r \gg R_S$) the metric is locally that of special relativity (the flat Minkowski metric). The next example explores what will be measured from this distant observer's perspective.

Example 3.2

A distant observer is located on the same radial line as an object falling into a black hole.

(a) Find an expression for the time interval $t_2 - t_1$ between the emission of a photon by the falling object near to the black hole at coordinate r_1 and the photon's arrival at the distant observer's location r_2. You may assume that the relevant light signals are also travelling only in the radial direction, so that $d\theta$ and $d\phi$ can be neglected.

(b) Comment on how the time taken for light signals from near the event horizon to reach r_2 compares to the r coordinate separation divided by the speed of light, and explain what happens to the interval $t_2 - t_1$ when the location r_1 of the signal being emitted tends to R_S.

Solution

(a) The world line of a photon must be a null geodesic (see Section 2.3.3), and so the spacetime separation ds between the two events must be zero. Therefore, the Schwarzschild metric becomes

$$ds^2 = (1 - R_S/r)\, c^2\, dt^2 - \frac{dr^2}{1 - R_S/r} = 0$$

Rearranging this expression and taking square roots gives

$$c\, dt = \frac{1}{1 - R_S/r}\, dr$$

We can integrate each side for a photon travelling from r_1 to r_2:

$$c \int_{t_1}^{t_2} dt = \int_{r_1}^{r_2} \frac{1}{1 - R_S/r}\, dr$$

Dividing by c and evaluating the (simpler) left-hand integral gives:

$$t_2 - t_1 = \frac{1}{c} \int_{r_1}^{r_2} \frac{1}{1 - R_S/r}\, dr$$

The remaining integral can be solved by multiplying the top and bottom of the fraction by r and using the substitution $u = r - R_S$, so that:

$$t_2 - t_1 = \frac{1}{c} \int_{r_1 - R_S}^{r_2 - R_S} \frac{u + R_S}{u}\, du = \frac{1}{c} \int_{r_1 - R_S}^{r_2 - R_S} \left(1 + \frac{R_S}{u}\right) du$$

and so

$$t_2 - t_1 = \frac{r_2 - r_1}{c} + \frac{R_S}{c}\ln\left(\frac{r_2 - R_S}{r_1 - R_S}\right) \tag{3.7}$$

(b) The logarithm term in Equation 3.7 means that the time interval must be larger than the radial separation divided by the speed of light. In the limit $r_1 \to R_S$, the denominator of the log term in Equation 3.7 tends to zero, so that the fraction inside the logarithm tends to infinity. Hence the time interval for a light signal to reach a distant observer from very close to the event horizon is infinite.

The calculation above reinforces one of the peculiar properties of black holes: that matter can fall into a black hole quite quickly, without anything unusual happening to it when passing the event horizon, but, to a distant observer, the matter is seen to get closer and closer to the event horizon, but not observed to cross it.

Figure 3.10a illustrates the path of infalling matter as viewed by an observer travelling with it (the proper time measured along the world line of the infalling matter), with panel (b) comparing this with what is seen by a distant observer (the coordinate time at which light signals from the infalling material at a particular radius will be received).

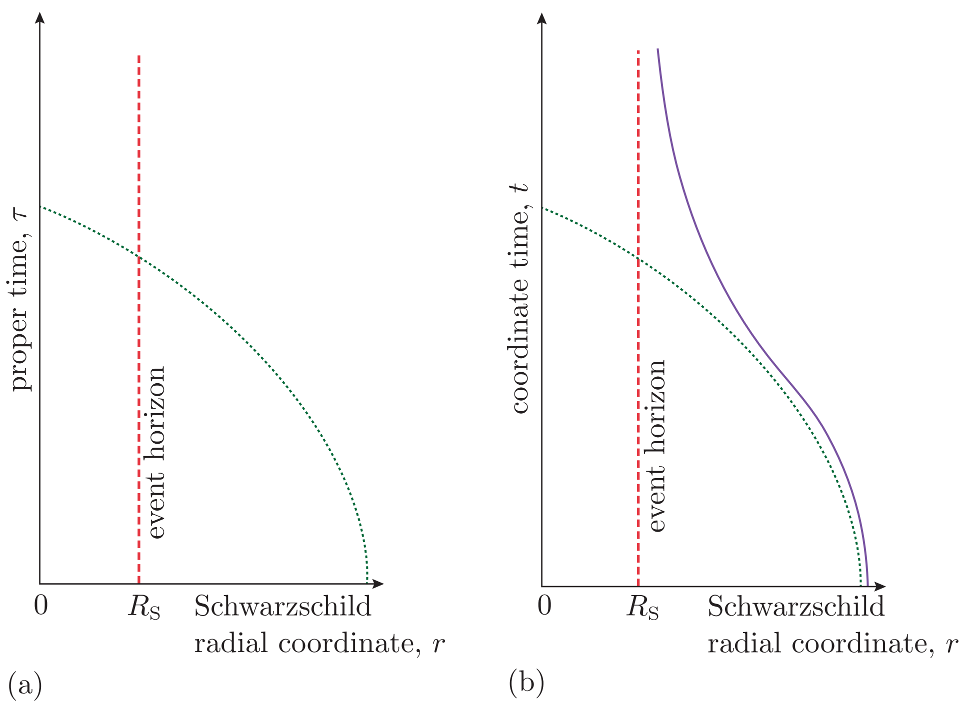
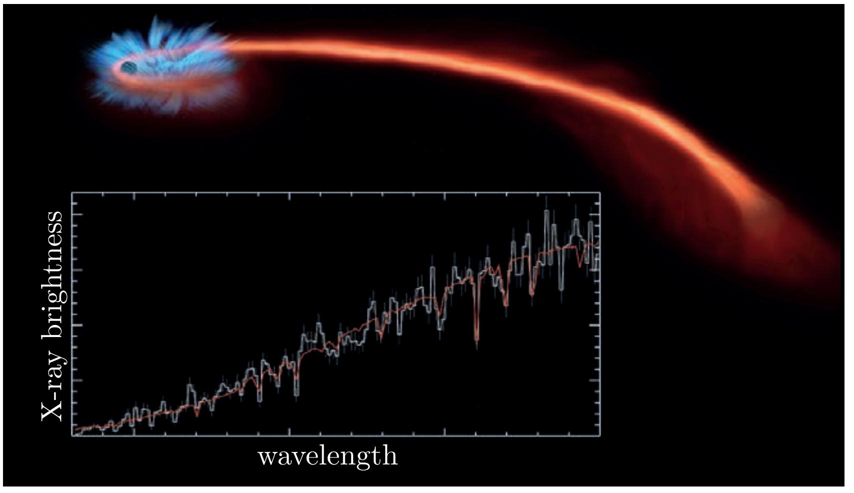


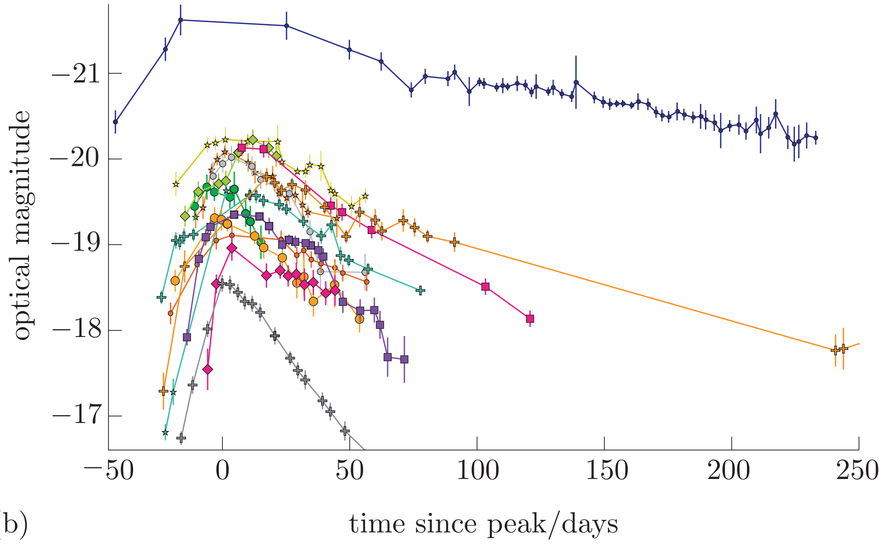
Figure 3.10 The path of an object falling radially into a black hole in Schwarzschild coordinates, as observed by: (a) an observer falling with the object (green dotted line indicates the proper time, τ, along the world line); (b) a distant observer at a location where the effects of the central mass are negligible (blue solid line indicating the coordinate time at which signals are received).

Of course, matter falling into a black hole will not always take a radial path, and it *will* also be affected by the strong gravitational field of the object. Tidal forces, that is, strong differences in the gravitational field strength at different nearby locations, can have a very dramatic effect on an infalling body, depending on the black hole's mass.

Figure 3.11 shows an artist's impression of such tidal disruption events (TDEs), as well as some observational evidence for them. The plots in panels (a) and (b) show X-ray and optical flares respectively, which are thought to occur as stars fall into black holes. Such observations can allow astronomers to determine the properties of the star being disrupted.



(a)



(b)

Figure 3.11 Observational evidence of stars disrupted by tidal forces in the vicinity of black holes: (a) an artist's impression of a TDE, with an X-ray spectrum showing peaks in brightness as more material falls into the black hole; (b) optical light curves from a sample of TDEs, showing how they brighten and then fade with time (after van Velzen *et al.*, 2021).

The discussion above has demonstrated that gravity can cause time dilation (disagreement about the time interval between events), just as time dilation can be caused by motion at relativistic speeds. Another consequence of gravitational time dilation is a shifting in the frequency of light signals, because the frequency of light is the inverse of the time interval between successive wavefronts.

The decrease in frequency (increase in wavelength) of an electromagnetic signal emitted close to a massive object when observed at a distance that is less influenced by the object is termed **gravitational redshift**, and is given by Equation 3.8.

Gravitational redshift

For an electromagnetic signal emitted at a distance r_{em} from an object of mass M, the observed frequency at large distance, ν_∞, is related to the emitted frequency, ν_{em}, by

$$\nu_\infty = \nu_{\mathrm{em}} \left(1 - \frac{2GM}{c^2 r_{\mathrm{em}}}\right)^{1/2} \tag{3.8}$$

As noted in Section 3.1.3, gravitational redshift does not require a black hole – measurements of the Earth's gravitational redshift are one of the ways general relativity has been experimentally verified – but this effect is a crucial one for interpreting astronomical observations of black holes.

To close, we have only been able to provide a brief introduction to the spacetime of black holes in this chapter. An important complication not discussed here is that the Schwarzschild metric applies to *non-rotating* black holes. In reality, we expect that the majority of black holes are spinning (indeed, it is possible to make measurements of their spin in some cases), which alters the metric. The metric for a spinning black hole is the **Kerr metric**, which has both inner and outer event horizons, and some further interesting behaviour near to those regions. We will return to the topic of black holes in the second part of the module.

3.3 The geometry of the expanding Universe

The previous section introduced one important example of a curved spacetime metric, in which a single concentration of mass influences the geodesics for matter and light and how they are seen by different observers. In this section we consider how to construct a metric appropriate for describing the geometry and time evolution of the entire Universe. This is a rather ambitious undertaking, but the cosmological principle – which you met in Chapter 1 – helps make it possible to construct a metric straightforward enough to use to build models of the Universe that can be directly tested against astronomical observations in many different ways.

3.3.1 Cosmic time and co-moving coordinates

The starting point for constructing a metric for the entire Universe and its expansion must be the behaviour we infer from astronomical observations made from our cosmic location on Earth. However, the previous section has shown that observers may differ in the conclusions they draw about events, either because they are in relative motion or because they are at different locations relative to concentrations of mass.

We cannot travel to distances far enough away on cosmic scales to get a different perspective, so it's only possible to proceed by making some big assumptions about the nature of the Universe and then finding ways to test whether all of the available evidence supports them. The first necessary assumption is the cosmological principle: on large scales, the Universe is homogeneous and isotropic.

- What evidence do we have that the cosmological principle is correct?
- Two key observations are that maps of the large-scale structure of the Universe (clusters, superclusters and voids between them) appear homogeneous and isotropic on 100 Mpc scales, and the cosmic microwave background (corrected for distortion due to local motions) is uniform in all directions, with fluctuations of less than 1 part in 10 000 (see Chapter 1).

The cosmological principle makes no mention of time, but we know from observations that the large-scale structure of the Universe is *not* constant in time – it is expanding. A second crucial assumption, known as **Weyl's postulate** after its originator Hermann Weyl, greatly simplifies the way time is incorporated into cosmological models. It proposes a set of hypothetical observers whose motion in spacetime matches the overall expansion of the Universe, sometimes known as the **Hubble flow**. These **fundamental observers** will perceive the Universe as obeying the cosmological principle, and will agree about the nature of the Universe's expansion.

Weyl's postulate

In cosmic spacetime there exists a set of fundamental observers, whose world lines are time-like geodesics that do not ever meet, except possibly at an initial singularity in the past and/or a final singularity in the future.

- Where might such fundamental observers be located?
- A hypothetical fundamental observer could be moving with their local supercluster or their home galaxy. In the latter case they would need to correct for any local motions within their wider environment, such as a galaxy orbit within a cluster or supercluster.

Figure 3.12 illustrates how the existence of fundamental observers (which could, for example, represent the locations of particular galaxies moving with the Hubble flow) allows a universal definition of **cosmic time** to be agreed. Hypersurfaces are shown to represent the spatial dimensions of the Universe at a particular time. The figure uses a two-dimensional surface to represent the spatial dimensions of the Universe, but in reality expansion is taking place in three dimensions. This is why the surfaces are labelled as 'hypersurfaces', indicating they have more than two spatial dimensions.

The spatial separations between the fundamental observers, as measured on the hypersurfaces, increase with time. But the cosmological principle requires that time elapses at the same rate along the world lines of all fundamental observers, so that all such observers can agree on the rate at which the spatial dimensions of the Universe expand.

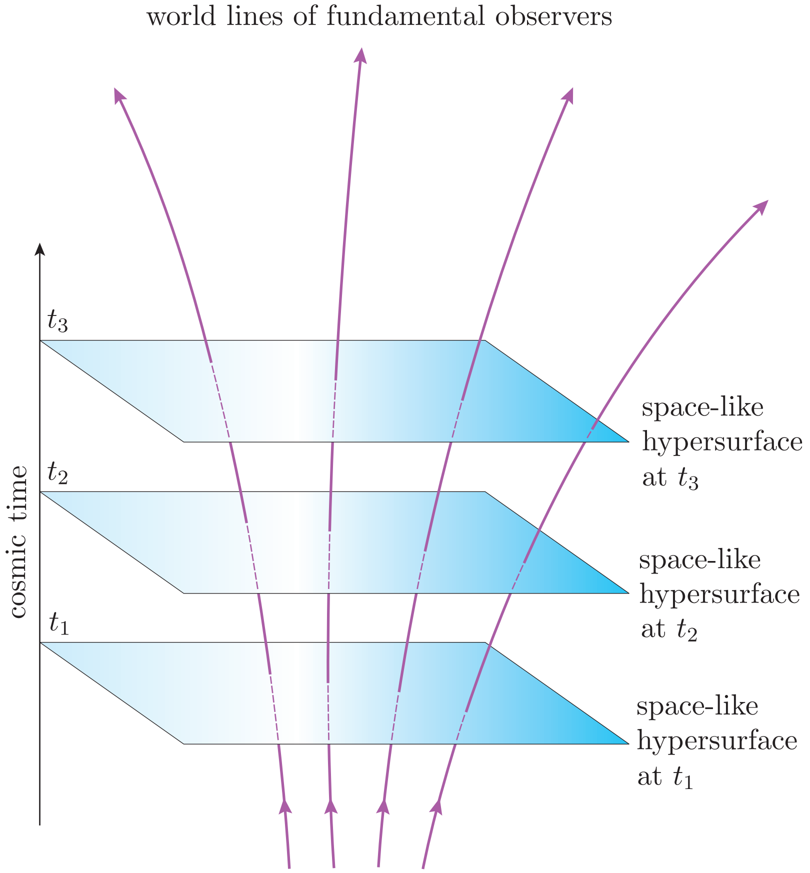


Figure 3.12 The world lines in cosmic spacetime of fundamental observers who see the Universe as homogeneous and isotropic. The vertical axis represents a universal cosmic time that the fundamental observers can agree on, while the 2D surfaces represent the 3D spatial geometry that expands homogeneously and isotropically with time.

Having defined a cosmic time in relation to the Hubble flow, it is now possible to determine a useful set of spatial coordinates that describe the hypersurfaces of Figure 3.12. We can specify a grid of coordinates such that the world line of a particular fundamental observer is assigned the same coordinate values at all times, that is, for every instance of the space-like hypersurface in Figure 3.12 (corresponding to a particular cosmic time). These values are known as **co-moving coordinates**, and although the grid on which they're based expands with cosmic time, the coordinate distances between locations stay fixed, as illustrated in Figure 3.13.

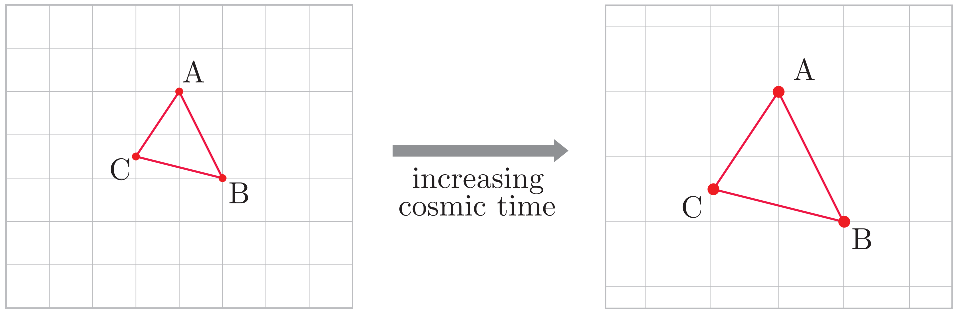


Figure 3.13 A grid of co-moving coordinates will track the expanding Hubble flow. Points A, B and C move with the flow and so have coordinate values that remain constant with cosmic time.

- The separation distances between fundamental observers are now twice as large as they were at a particular time in the past, $t_{1/2}$. How does the current co-moving distance between the Milky Way and a distant galaxy – NGC 1262, for example – compare to their co-moving distance at time $t_{1/2}$?

- The co-moving distance between the Milky Way and NGC 1262 will be unchanged (except from any correction for local motions that are not part of the Hubble flow).

The co-moving coordinate distance at a particular time is not a useful measurement for all possible purposes. For example, it won't, on its own, tell us how long it would take for light or a spacecraft to travel between two galaxies. But co-moving coordinates provide a straightforward way to simplify the metric for cosmic spacetime.

The final ingredient we need for a cosmological metric is the **scale factor**, $a(t)$, where t is the cosmic time, which describes the relationship between co-moving coordinate distances and the true physical distance between objects at a particular cosmic time. In other words, the scale factor captures the rate at which the coordinate grid in Figure 3.13 expands.

- Which parameter that describes the Universe's expansion and that you met earlier in the module must be closely related to the scale factor $a(t)$?

- The Hubble constant, introduced in Chapter 1, describes the rate of expansion of the Universe at the present time, and so must be related to the scale factor. (The precise relationship will be discussed in a later section.)

Together, the concepts of a universal cosmic time, co-moving coordinates, and a scale factor to describe cosmic expansion allow a metric to be constructed to describe the large-scale geometry of the Universe. The next section introduces that metric.

3.3.2 The Robertson–Walker metric

To get a feel for the form that a cosmological metric must take, we will initially make the simplifying assumption (which is not necessarily correct) that the large-scale spatial geometry of the Universe is flat. That assumption leads to the following metric for an expanding spacetime:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right) \tag{3.9}$$

where t is cosmic time, $a(t)$ is the scale factor, and r, θ and ϕ are co-moving coordinates for a three-dimensional spherical geometry centred on a fundamental observer.

- How does the metric in Equation 3.9 differ from the Minkowski metric of the previous chapter, and why?

- The spatial part of the metric shown in Equation 3.9 is expressed in spherical coordinates, but the part inside the brackets corresponds to flat space (see Section 2.3.1). This spatial part is then multiplied by a scaling factor a, which depends on time. This means that although the geometry changes with time, at any particular cosmic time, t, it is equivalent to the Minkowski metric.

The question of whether the Universe is spatially flat (in which case Equation 3.9 correctly describes its geometry) or curved (so that a more complex metric is needed) is one of the most important in cosmology. You will see later that this question has significant implications for the eventual fate of the Universe.

- What physical properties of the Universe would you expect to determine whether or not it has spatial curvature?

- Einstein's field equations relate curvature to the overall mass and energy content of the Universe. Therefore, the amount of matter and energy in the Universe should determine its curvature.

You will notice that, in contrast to the Schwarzschild metric (Equation 3.2), there is no obvious mass term in Equation 3.9, and so it is not immediately clear that it can be a solution to Einstein's field equations. The explanation for this is that the scale-factor term, $a(t)$, depends on the mass and energy content of the Universe, which influence how the scale factor changes with time. We will explore how a is related to mass and energy in the next chapter, but it is sufficient to note that this relationship leads us to cosmological models that we can test directly with observations.

The best observational evidence we have at the moment indicates that the Universe *is* spatially flat, as you will see later. But we need to incorporate the possibility of curvature into the metric to be able to explore the range of possible cosmological models and understand how modern observations are constraining them.

The full cosmological metric, including curvature of space, is known as the **Robertson–Walker metric**. It is most commonly written as follows, where k is a *spatial* **curvature parameter**:

The Robertson–Walker metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right) \tag{3.10}$$

For a flat 3D geometry, $k = 0$, but it can also take positive or negative values,* indicating positively and negatively curved geometries, respectively (as introduced in Section 2.3.1). For the $k = 0$ case, Equation 3.10 simplifies to Equation 3.9.

It is also important to note that because the 3D spatial geometry (the part of the Robertson–Walker metric inside the brackets) is time-independent (as illustrated in the 2D schematic of Figure 3.12), k is also constant with time. It does not describe the curvature of four-dimensional spacetime, but only the curvature of the co-moving grid that scales uniformly with $a(t)$.

- What are the units for the quantities $a(t)$ and k?
- Each term in the metric must have the same units as $\mathrm{d}s^2$, which has units of distance squared (e.g. m^2). This requires the scale factor a to be dimensionless, and k to have units of the inverse of distance squared (e.g. m^{-2}).

3.3.3 The Hubble parameter and the scale factor

We conclude this chapter by making some initial connections between the rather abstract topic of metrics and the quantities that can be directly measured using astronomical observations.

A crucial observation is that the Universe is expanding at the present time; you saw in Chapter 1 that this expansion is measured using a quantity known as the Hubble constant, H_0, which corresponds to the rate of expansion over a given length scale. H_0 relates to the cosmic scale factor a according to

$$H_0 = \frac{1}{a(t_0)}\frac{\mathrm{d}a}{\mathrm{d}t}(t_0) = \frac{\dot{a}(t_0)}{a(t_0)} \tag{3.11}$$

where t_0 is the present time. We have also introduced the standard notation $\dot{a} = \mathrm{d}a/\mathrm{d}t$; throughout the module, a dotted quantity corresponds to the time derivative of that quantity.

The next exercise involves considering quantitatively how the scale factor has changed with time over the recent history of the Universe.

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*Some cosmology textbooks use a scaled coordinate system in which k is normalised to take only one of the three values $+1$, 0 or -1, rather than allowing any value for k as we do here. This requires a slightly different definition for the scale factor. Either convention makes certain cosmological calculations simpler, but for the purposes of this book k is not normalised and can take any value.

Exercise 3.4

Assuming (for the purpose of this exercise) that $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, and has not changed significantly over the past billion years, calculate the factor by which the scale factor of the Universe has changed:

(a) over the last 1000 years

(b) over the last 100 million years.

(*Hint*: assume the scale factor at the present day is $a(t_0) = 1$.)

Measuring H_0 can therefore give us information about how a is changing. But the assumption made in Exercise 3.4 that H_0 is the same for all cosmic epochs is not correct: in some cosmological models it can change significantly over the lifetime of the Universe. A more general **Hubble parameter** is defined as

$$H(t) = \frac{\dot{a}}{a} \tag{3.12}$$

where H now represents the value of the Hubble constant at some particular time t, which could be in the past or the future.

The scale factor is also related to another important observational quantity linked to cosmic time: the cosmological redshift. The redshift of a distant light signal (z) is related to the emitted and observed wavelengths (λ) or frequencies (ν) of the signal according to

$$1 + z = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{em}}} = \frac{\nu_{\mathrm{em}}}{\nu_{\mathrm{obs}}} \tag{3.13}$$

We can derive a relationship between a and z by considering the behaviour of a light signal from a distant galaxy, located at a fixed radial coordinate, r_{gal}. The light signal is emitted at time t_{em} and observed at time t_{obs}.

To simplify the algebra, we will start by using the Robertson–Walker metric for a flat geometry ($k = 0$) only. Recalling that light follows null geodesics, for which $\mathrm{d}s^2 = 0$, the metric can be greatly simplified in this situation. The light is also travelling on a purely radial path, so the $\mathrm{d}\theta$ and $\mathrm{d}\phi$ terms are zero because we can define the distant galaxy to have the same θ and ϕ coordinates as the observer. Equation 3.10 therefore becomes

$$0 = c^2\,\mathrm{d}t^2 - a^2(t)\,\mathrm{d}r^2$$

We can obtain an expression for the radial coordinate of the galaxy, r_{gal}, by first rearranging and then taking square roots to obtain

$$\frac{c\,\mathrm{d}t}{a(t)} = \mathrm{d}r$$

and we can then integrate to get

$$c\int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)} = \int_0^{r_{\mathrm{gal}}} \mathrm{d}r = r_{\mathrm{gal}}$$

Here, the integration limits for the left-hand integral correspond to the time range between the light being emitted and observed, and those for the right-hand integral correspond to the change in coordinate distance for the same spacetime interval under consideration.

How can we use this to find a relationship with redshift? Because z is related to wavelength, which can be thought of as the interval between the peaks of light signals, we can start by considering a second signal, corresponding to the next wavefront of the light from the galaxy. This signal is emitted at time $t_{\mathrm{em}} + \Delta t_{\mathrm{em}}$ and observed at a time $t_{\mathrm{obs}} + \Delta t_{\mathrm{obs}}$.

The situation here is the same as for the first signal, except that the start and end times for the left-hand integral are modified:

$$c \int_{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)} = \int_0^{r_{\mathrm{gal}}} \mathrm{d}r = r_{\mathrm{gal}}$$

Because the coordinate distance travelled by the light, r_{gal}, will be the same in both cases, the two expressions can be equated:

$$c \int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)} = c \int_{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)}$$

We don't actually know how a depends on time, and so we need to find a way to take it out of the integral. Although the scale factor may change significantly between t_{em} and t_{obs} – that is, $a(t_{\mathrm{em}}) \neq a(t_{\mathrm{obs}})$ – it cannot change significantly within the timescale between the emission of the two wavefronts or that of their receipt (e.g. see the solution to Exercise 3.4). Therefore, we can take $a(t_{\mathrm{em}}) = a(t_{\mathrm{em}} + \Delta t_{\mathrm{em}})$ and $a(t_{\mathrm{obs}}) = a(t_{\mathrm{obs}} + \Delta t_{\mathrm{obs}})$.

In order to make use of these simplifications we need to take the (not at all obvious) step of subtracting the integral

$$c \int_{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)}$$

from both sides, which effectively changes the integration ranges to give

$$\int_{t_{\mathrm{em}}}^{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}} \frac{\mathrm{d}t}{a(t)} = \int_{t_{\mathrm{obs}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)}$$

where we have also cancelled out factors of c. Now both integrals only cover a very small time interval (e.g. for typical light frequencies the interval between wavelengths is $\sim 10^{-15}\,\mathrm{s}$) compared to the scale on which the Universe expands, and so now we can take a to be constant for each integral, giving

$$\frac{1}{a(t_{\mathrm{em}})} \int_{t_{\mathrm{em}}}^{t_{\mathrm{em}}+\Delta t_{\mathrm{em}}} \mathrm{d}t = \frac{1}{a(t_{\mathrm{obs}})} \int_{t_{\mathrm{obs}}}^{t_{\mathrm{obs}}+\Delta t_{\mathrm{obs}}} \mathrm{d}t$$

Integrating now gives us

$$\frac{\Delta t_{\mathrm{em}}}{a(t_{\mathrm{em}})} = \frac{\Delta t_{\mathrm{obs}}}{a(t_{\mathrm{obs}})} \tag{3.14}$$

In the following exercise you will use this expression to finally link a and redshift.

Exercise 3.5

Noting that the frequency of light is the inverse of the time interval between successive wavefronts (i.e. $\Delta t = 1/\nu$), use Equation 3.14 to find an expression relating scale factor and redshift, i.e. a and z.

For simplicity, the discussion and exercise above considered only the case of $k = 0$; however, the same derivation can be carried out for the full Robertson–Walker metric. The only difference is that the term in r_{gal} is more complicated, but it still corresponds to the same distance measure for both wavefronts. Hence the same relationship between a and z holds for all geometries.

Scale factor and redshift

$$a \propto \frac{1}{1+z} \tag{3.15}$$

The redshift of a distant galaxy therefore tells us how much smaller the scale factor of the Universe was at the time the light we measure was emitted.

Conventionally, the present-day value of the scale factor, $a(t_0)$, is set to 1, and so Equation 3.15 becomes an equality.

The final exercise in this chapter allows you to further explore the relationship between redshift and the scale factor.

Exercise 3.6

Two distant galaxies are measured to have redshifts of $z = 10.1$ and $z = 3.6$. Calculate by what factor the distance between two locations in the Universe has increased in size since the time when the light we observe from each galaxy was emitted.

3.4 Summary of Chapter 3

- The (weak) **equivalence principle** states that within a local region of spacetime, close to a concentration of mass, the motion of an object cannot be distinguished by any experiment from how it would behave in a region of (appropriately chosen) uniform acceleration.
- The theory of general relativity is a metric theory of gravity, encapsulated in **Einstein's field equations** (Equation 3.1). These equations relate a function of the curvature tensor to the **energy–momentum tensor**, such that 'matter tells space[time] how to curve' and 'space[time] tells matter how to move' (Misner, Thorne and Wheeler, 1973, p. 5).
- The predictions of general relativity are well tested, and supported by a range of evidence including measurements of the **precession** of the perihelion of Mercury, deflection of light passing near to the Sun, and the measurements of **gravitational redshift** of the **Pound–Rebka experiment**. **Shapiro delay** and both indirect and direct detection of **gravitational waves** also support the predictions of the theory.
- Einstein's field equations are effectively a set of 16 separate equations that can, in principle, be solved for a given energy–momentum tensor (mass and energy distribution) to determine the metric.
- The **Schwarzschild metric** is a solution to the Einstein field equations that applies to regions in the vicinity of a localised mass, and so describes the behaviour of spacetime near astronomical objects such as planets, stars and **black holes**:

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2\,\mathrm{d}t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2 \quad \text{(Eqn 3.2)}$$

- In the vicinity of a black hole the Schwarzschild metric predicts the presence of an **event horizon**, defined in the case of a non-rotating black hole by the **Schwarzschild radius**, from within which no light signals nor matter can exit. Distant observers measure time intervals for events close to the event horizon as tending towards becoming infinite, and measure light signals emitted near the central mass to be gravitationally redshifted.
- A metric to describe the geometry and expansion of the Universe can be constructed by defining a set of **fundamental observers** whose locations trace the **Hubble flow** and define a set of fixed **co-moving coordinates**. All such observers agree on a definition of **cosmic time**.
- The resulting metric is known as the **Robertson–Walker metric** and describes the spacetime of a homogeneous and isotropic expanding Universe:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2\right) \quad \text{(Eqn 3.10)}$$

- The **scale factor**, a, describes the rate at which any region of the Universe expands (or contracts) with time and is related to cosmic redshift, z, via

$$a \propto \frac{1}{1+z} \qquad \text{(Eqn 3.15)}$$

- The **Hubble parameter**, $H(t)$, describes the rate at which the scale factor evolves with time over a given length scale, and is defined as

$$H(t) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\dot{a}}{a} \qquad \text{(Eqn 3.12)}$$

  The value of the Hubble parameter in the present epoch is known as the Hubble constant, H_0.

- The spatial curvature of the Universe is parameterised by the (time-independent) **curvature parameter**, k, which is zero for a flat geometry, but can also take positive or negative values, indicating curved geometries.