Chapter 4 Cosmological models and their key parameters

The first three chapters of this book have introduced fundamental physical concepts including the cosmological principle and the theories of special and general relativity, and set out the metric for an expanding Universe and its key parameters.

In this chapter you will learn how those concepts can be used to build mathematical models that describe the Universe and its contents. Using these models it becomes possible to understand how the Universe has evolved in the past, and predict how it will continue to evolve in the future.

Objectives

Working through this chapter will enable you to:

- derive the two Friedmann equations and the fluid equation, and understand how they can be used to describe and model the expansion of the Universe
- use the Friedmann equations and the fluid equation to model the behaviour of some simple idealised universes
- predict how the densities and expansion rates of model universes containing different proportions of matter and radiation evolve with time
- understand where the cosmological constant appears in the Friedmann equations and the fluid equation, and explain how its existence affects the evolution of the Universe
- describe how the geometry and contents of the Universe influence its history and its ultimate fate.

4.1 The Friedmann equations

This section introduces two equations called the Friedmann equation and the acceleration equation, which are often collectively referred to as the Friedmann equations. Together with a third equation called the fluid equation, which we will also consider, they can be used to describe how the expansion, geometry and contents of the Universe depend on one another.

4.1.1 The Friedmann equation

The **Friedmann equation** is one of the most important equations in cosmology. It describes how the expansion rate of the Universe depends on its geometry, and the density of matter and energy within it.

A completely rigorous derivation of the Friedmann equation would entail solving Einstein's field equations and assuming that spacetime is described by the Robertson–Walker metric. However, such a derivation requires a much more mathematical foundation in general relativity than this module provides. For that reason, we will now derive the Friedmann equation using an approach that assumes that spacetime is Euclidean and uses arguments from Newtonian dynamics, and has a more intuitive physical description. We will still arrive at an expression that is equivalent to the general relativistic one we would have derived otherwise.

We start by considering the cube depicted in Figure 4.1, which shows part of a hypothetical universe. It is expanding *isotropically*, and is filled with a completely smooth and *homogeneous* fluid that has density $\rho(t)$ at time t. We will assume that the total fluid content of the universe is constant, so $\rho(t)$ decreases as the universe expands.

We can pick a point P in this hypothetical universe and imagine a spherical region with radius R, which is centred on P and is expanding along with the universe. A small test particle with mass m lies on the boundary of the sphere, and M_{sphere} is the total mass of material inside the spherical region.

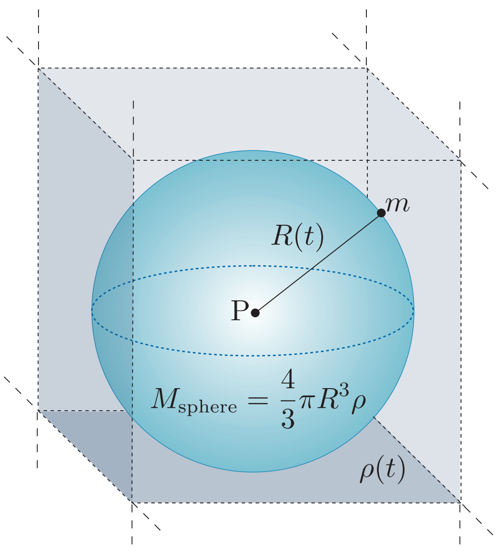

Figure 4.1 Part of a hypothetical universe filled with a homogeneous medium that has density $\rho(t)$ at time t. A spherical region with radius $R(t)$ is centred on a point P, with a small test particle of mass m on its boundary.

- Does it matter which point within the universe we choose to label as P?

- No. Our assumption of homogeneity means that any point in our hypothetical universe is completely identical to all others. Similarly, all other spheres with radius R centred on these arbitrary points are also completely equivalent.

Let's compute the total energy U of the test particle on the boundary of the spherical region. The particle will experience a gravitational force in the direction of P, given by

$$F = \frac{GM_{\text{sphere}}m}{R^2}$$

where $M_{\text{sphere}} = (4/3)\pi R^3 \rho(t)$ is the total mass of the material within the spherical region. This means that the gravitational potential energy of the particle is

$$E_{\mathrm{g}} = -\frac{GM_{\mathrm{sphere}}m}{R} = -\frac{4}{3}G\pi R^2\rho m$$

What about all the material in the universe that sits outside the sphere we've defined: does it also exert a gravitational force on the test particle? In fact, the answer is no. Newton was the first to show that a spherical *shell* of matter does not exert a net gravitational force on any object inside it. This result is true regardless of the object's location within the shell. To see that only the material inside the sphere exerts a gravitational force on our test particle, we could just divide all the remaining fluid in the universe (i.e. external to the sphere) into a series of concentric spherical shells, none of which would exert any force on our particle.

The only force acting on the particle points back towards P along the radial direction, so its kinetic energy must just be

Recall from the previous chapter that 'dot notation' can be used to indicate time-derivatives of quantities, so $\dot{x}$ means $\mathrm{d}x/\mathrm{d}t$ and $\ddot{x}$ means $\mathrm{d}^2x/\mathrm{d}t^2$.

$$E_{\mathrm{k}} = \frac{1}{2}m\dot{R}^2$$

where $\dot{R}$ denotes the time derivative of position along the radial direction. Energy conservation implies that the total energy in the system is constant, so

$$U = E_{\mathrm{k}} + E_{\mathrm{g}} = \frac{1}{2}m\dot{R}^2 - \frac{4}{3}G\pi R^2\rho m = \text{constant} \tag{4.1}$$

Our assumption of a homogeneous, isotropically expanding universe means that we can relate the physical distance $\mathbf{R}(t)$ between *any* two points at time t to the co-moving distance $\mathbf{x}$ between them (see Section 3.3.1) using

$$\mathbf{R}(t) = a(t)\mathbf{x}$$

where $a(t)$ is the scale factor of the universe at time t, and $\mathbf{x}$ is time-invariant by definition. We can now rewrite Equation 4.1 in terms of a and $x \equiv |\mathbf{x}|$:

$$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4}{3}G\pi a^2x^2\rho m \tag{4.2}$$

- If $\dot{a}$ is positive, is the universe expanding or contracting?
- If $\dot{a} > 0$ then the scale factor is increasing with time and the universe must be expanding.

Rearranging Equation 4.2 and defining a quantity $k = -2U/\left(mc^2x^2\right)$ yields our Newtonian expression for the Friedmann equation.

The Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{4.3}$$

Before comparing this expression with its equivalent general-relativistic form, we can observe some of its general properties. Note that changing the sign of $\dot{a}$ in Equation 4.3 does not change the value of the term on the left-hand side, which only depends on $\dot{a}^2$. Indeed, there is nothing in our derivation that formally requires $\dot{a} > 0$, which means that the Friedmann equation is equally valid when considering contracting *or* expanding universes.

The parameter k in Equation 4.3 is the spatial curvature parameter that appears in the definition of the Robertson–Walker metric. The value of k does not depend on spatial coordinates *or* time – it is an invariant property of the Universe. The following example builds on our derivation so far, to show that this must be the case.

Example 4.1

Assuming that the Universe is homogeneous and undergoing isotropic expansion, explain why the Friedmann equation implies that k must be spatially invariant.

Solution

Consider the three terms in the Friedmann equation. In a homogeneous universe undergoing isotropic expansion, a, $\dot{a}$ and ρ must be spatially invariant, so neither $(\dot{a}/a)^2$ nor $8\pi G\rho/3$ can depend on the value of x. Bearing in mind the spatial invariance of a, this means that the value of k must also be independent of x, and therefore spatially invariant, for the equality in Equation 4.3 to hold at all points in space.

The value of k has profound implications for the ultimate fate of universes that are described by the Friedmann equation. To see this, first note that the physical impossibility of negative densities means that the first term on the right-hand side of Equation 4.3 must always be positive.

Now, consider the case when $k < 0$, so both sides of Equation 4.3 are always positive. In an initially expanding universe, where $\dot{a} > 0$, the right-hand side of the equation must always be positive and so $\dot{a}$ cannot change sign. Any expanding universe that has a negative value for k will continue to expand forever.

When $k > 0$ in a universe that is initially expanding, the scale factor will increase and ρ will decrease until

The arrow symbol $\Longrightarrow$ is sometimes used as shorthand for 'implies'.

$$\frac{8\pi G\rho}{3} = \frac{kc^2}{a^2} \quad \Longrightarrow \quad a^2 = \frac{3kc^2}{8\pi G\rho}$$

When this happens, $\dot{a}$ will be zero and the universe will stop expanding. However, the second time derivative $\ddot{a}$ will be negative, which means the universe will begin to contract again. Universes with positive k will eventually recollapse to zero size at an epoch known colloquially as the 'big crunch'.

- How does a evolve in the limiting case when $k = 0$?
- When $k = 0$, the expansion continues forever but becomes slower and slower as $\dot{a}$ and ρ both tend towards zero.

As the following highlight box explains, cosmologists often use shorthand names to refer to universes with different, specific values of k.

Open, closed and flat universes

Hypothetical universes that have negative values of k are often described as **open universes**, while those with positive k are often referred to as **closed universes**. The term **flat universe** is used to refer to the special case when k equals zero.

The nomenclature comes from the geometric interpretation of k representing the shape of spacetime in the context of the Robertson–Walker metric, which you learned about in Section 3.3.2.

We will return to these different types of universe later in the chapter when we consider further models, but before we end this section on the Friedmann equation let's explore how our Newtonian expression (Equation 4.3) differs from the relativistic one that can be derived by solving the Einstein field equations. The relativistic expression is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{a^2} \tag{4.4}$$

The only difference is that the the density term ρ has been replaced by an energy density ϵ divided by c^2. General relativity states that the *energy* of a group of particles is the relevant quantity to consider when determining their gravitational influence, not their rest mass. For a single particle with rest mass m_0 and momentum p, the relativistic expression for its energy is

$$E = \sqrt{p^2c^2 + m_0^2c^4} \tag{4.5}$$

This means that for a collection of massive particles with $v \approx c$, their *motion* contributes a significant fraction of their overall energy density. Another consequence of general relativity is that even massless particles such as photons exert a gravitational influence, because their energies contribute to the energy density of any volume they occupy.

Exercise 4.1

For the remainder of this chapter we will write the energy density explicitly as ρc^2, rather than using the symbol ϵ that appeared in Equation 4.4.

With this in mind, and assuming a universe filled with non-relativistic particles that have velocities $v \ll c$, show that $\epsilon/c^2 \to \rho$, and therefore that the general-relativistic and Newtonian forms of the Friedmann equation are equivalent.

You should now have a good sense of the general properties of the Friedmann equation, and its capacity to predict how the geometry and density of the Universe affect its expansion. In the next section you will learn about another equation that describes the *inverse* relationship, and models how the densities of the Universe's contents depend on its expansion rate.

4.1.2 The fluid equation

In this section we will construct an equation called the **fluid equation** that describes how the energy densities, $\rho(t)c^2$, and pressures, P, of different fluids evolve in an expanding universe. Once we know how ρ changes with time, we will solve the Friedmann equation to predict the value of a at any time in the Universe's past or future.

Our derivation will consider the spherical region illustrated in Figure 4.1, and we will focus on the properties of the fluid it contains. Remember that the sphere is expanding isotropically along with the universe in which it is embedded, so its *co-moving* radius, $R' = R(t)/a(t)$, is constant. To simplify expressions in the rest of this section, we will assume that $R' = 1$.

We start with the first law of thermodynamics, which expresses how the heat Q flowing into and out of a fluid at pressure P is balanced by changes in its internal energy E and volume V.* For the fluid in our model sphere, we can write the change in heat with respect to time as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + P\frac{\mathrm{d}V}{\mathrm{d}t} = 0 \tag{4.6}$$

- Suggest one reason why we can assume that $\mathrm{d}Q/\mathrm{d}t = 0$ for the fluid in our expanding spherical region.

- The simplest way to see that this is true is to realise that isotropic expansion preserves the homogeneity of the fluid. Since there can be no temperature gradients in a homogeneous medium, no heat can flow into or out of the volume.

Alternatively, we could recognise that the only way that heat can be transferred into or out of the region is if some of the fluid filling the universe crosses its boundary. We have assumed that this fluid is completely homogeneous, which means that there are no overdensities or pressure gradients that might induce flows within it. In other words, the expanding fluid remains completely static in the *co-moving* coordinate frame.

We have also specified that the spherical region has a constant co-moving radius. This means that no fluid enters or leaves the region because its boundary is completely fixed in the same coordinate frame in which the fluid remains static. It follows from this observation that $\mathrm{d}Q/\mathrm{d}t$ must be equal to zero.

*It is important to recognise that the pressure P of a fluid may, in fact, depend on other quantities, such as the fluid's density or the ambient temperature, that vary with time. For conciseness, we will not explicitly indicate the time-dependence of $P(t)$ in the remainder of this section, but we will return to examine it in Section 4.2.1.

To derive the fluid equation, we need to express $\mathrm{d}E/\mathrm{d}t$ and $\mathrm{d}V/\mathrm{d}t$ in terms of a and ρ and their time derivatives. The volume of the spherical region in Figure 4.1 increases as the universe expands, and we can express the rate of this increase in terms of a:

$$\dot{V} = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4}{3}\pi R(t)^3\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4}{3}\pi a(t)^3\right] = 4\pi a^2\dot{a}$$

We can write the internal energy of the fluid as the product of its volume and energy density: $E = V(t)\rho(t)c^2$. Remember that we have specified $R(t)/a(t) = 1$, which means we can write:

$$\begin{aligned}\dot{E} = \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[V(t)\rho(t)c^2\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4}{3}\pi a^3\rho(t)c^2\right] \\ &= \frac{4}{3}\pi c^2\left(3a^2\dot{a}\rho + a^3\dot{\rho}\right) \qquad (4.7)\end{aligned}$$

Substituting these expressions for $\dot{V}$ and $\dot{E}$ in Equation 4.6, we find that

$$\mathrm{d}E + P\mathrm{d}V = \frac{4}{3}\pi c^2\left(3a^2\dot{a}\rho + a^3\dot{\rho}\right) + 4\pi a^2\dot{a}P = 0$$

All that remains is to rearrange and simplify this expression: we arrive at the fluid equation as you will see it written in many cosmology textbooks.

The fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \qquad (4.8)$$

The fluid equation shows us that $\dot{\rho}$ is governed by separate terms that depend on the current values for the density ρ and the pressure P of the fluid. The density term, which is proportional to $\dot{a}\rho/a$, just describes a dilution effect. If no new fluid is created as the universe expands, then conservation of energy means that the density must decrease. The interpretation of the pressure term is slightly more subtle. It arises because the fluid has lost internal energy by doing work as the volume it occupies expands.

- Explain why the pressure term in the fluid equation does *not* imply non-conservation of energy as the universe expands.

- Energy is still conserved, because although the *internal* energy of the fluid has decreased, this is exactly balanced by a corresponding increase in its gravitational potential energy, because the universe's expansion increases the distance between separate elements of the fluid.

This increase in gravitational potential might be easier to intuit by conceiving of the fluid as a finite number of massive particles distributed evenly throughout space. Isotropic expansion increases the distance between the particles, which increases the gravitational potential energy of the system.

4.1.3 The acceleration equation

In this section we will derive a third equation called the **acceleration equation** that explicitly describes the change in the rate at which a universe expands. The acceleration equation is not independent of the Friedmann and fluid equations but, as we will discover later in this chapter, having an equation that describes an accelerating scale factor is particularly useful for modelling our own Universe.

To derive the acceleration equation we will perform a series of operations to combine the Friedmann and fluid equations. We start by taking the time derivative of the Friedmann equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\dot{a}}{a}\right)^2\right] = 2\frac{\dot{a}}{a}\frac{a\ddot{a}-\dot{a}^2}{a^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}\right)\\ &= \frac{8\pi G}{3}\dot{\rho} + 2\frac{\dot{a}}{a}\frac{kc^2}{a^2}\end{aligned} \tag{4.9}$$

Next, we replace $\dot{\rho}$ in Equation 4.9 using the fluid equation. After simplifying and rearranging the result, we obtain:

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{P}{c^2}\right) + \frac{kc^2}{a^2}$$

The Friedmann equation can be used to replace $(\dot{a}/a)^2$ and a final reorganisation yields the acceleration equation.

The acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \tag{4.10}$$

Equation 4.10 tells us that increasing the density of the fluid in our model universe increases its gravitational influence and reduces the rate at which the scale factor a increases. A slightly less intuitive implication is that increasing the pressure of the fluid also *slows* the universe's expansion.

We are used to conceiving of positive pressure as a mechanism for transferring forces and inflating objects like balloons; however, forces are only transmitted when a pressure *gradient* is present. The balloon inflates because its internal pressure is higher than the ambient atmospheric pressure so there is a pressure gradient across its boundary. In a homogeneous universe there can be no pressure gradients and isotropic expansion preserves homogeneity. In the formalism of Einstein's field equations, the pressure is found to contribute to the overall **stress energy** of the Universe and, ultimately, it is stress energy that curves spacetime to produce gravitation.

4.2 Modelling the contents of the Universe

The two Friedmann equations and the fluid equation that you learned about in Section 4.1 are fundamental to the study of cosmology. Together, they allow us to model the histories and predict the futures of any universe that is described by the Robertson–Walker metric.

In this section you will learn how cosmologists use the Friedmann and fluid equations to model the evolution of the Universe. We will investigate how the scale factor behaves in homogeneous, isotropically expanding universes and how the contents of those universes affect the way they evolve.

4.2.1 The equation of state

The Friedmann equation and the fluid equation describe the co-evolution of a and ρ in a homogeneous, isotropically expanding universe. However, these equations can only be solved once we specify how the pressure term in the fluid equation evolves with time.

An equation that relates the pressure of a fluid to other quantities, such as its density or temperature, is called the fluid's **equation of state**. It is likely that you have met equations of state before. For example, the ideal gas law, which relates the pressure of a gas to its temperature and the volume it occupies, is also an equation of state.

The standard approach in cosmology is to model the contents of the Universe as a **perfect fluid**. Perfect fluids have particularly simple equations of state that depend *only* on their density and pressure.

Equation of state for a perfect fluid

$$P(\rho) = w\rho c^2 \tag{4.11}$$

The new parameter, w, in Equation 4.11 is the **equation of state parameter**. The evolution of universes containing different types of perfect fluid can be described by choosing different values of w to specify $P(\rho)$ before solving the Friedmann and fluid equations.

In subsequent sections we will examine the behaviour of model universes for which the curvature parameter $k = 0$. As noted in Section 4.1.1, such models are often called flat universes (you may also see them described as 'flat cosmologies'). This is not because the spacetime within them is completely flat everywhere: mass and energy still produce local spacetime curvature in flat universes. However, on very large scales, the spatial parts of their metrics are Euclidean.

We will also focus on universes that contain the same types of fluid that we know exist in the real Universe. You will discover that the densities of different fluids evolve at different rates as the model universes expand, which has interesting implications for the history of our own Universe.

4.2.2 Models for matter

Cosmologists use the term 'matter' to refer to any fluid comprising non-relativistic material that has negligible or zero pressure. For such fluids, the equation of state parameter $w_{\mathrm{m}} = 0$, where the subscript 'm' indicates that this property pertains to a matter-like fluid (or 'matter fluid'). The equation of state has a very simple form in this scenario.

Equation of state for matter

$$P_{\mathrm{m}} = 0 \tag{4.12}$$

You may be wondering whether there are any materials in the real Universe that behave like this idealised description of matter? The answer is yes! In fact, we know from astrophysical observations that around 50% of the baryonic matter in the Universe currently exists as a rarefied plasma of ions called the warm–hot intergalactic medium (or WHIM for short), which is distributed in the space between galaxies.

The WHIM contains some of the hottest gas in the Universe: its typical temperature is $T_{\mathrm{WHIM}} \approx 10^7\,\mathrm{K}$. Exercise 4.2 asks you to show that even this superheated plasma is effectively pressureless and contains only non-relativistic particles.

Exercise 4.2

The baryons in the WHIM are so diffuse that they almost never interact with one another, which means that they can be very accurately modelled as an ideal gas.

The ideal gas law can be written as

$$P = \frac{k_{\mathrm{B}}T}{\mu}\rho \tag{4.13}$$

where P is the pressure of the gas, T is the temperature, μ is the mean mass of the particles in the gas and k_{B} is the Boltzmann constant. The typical kinetic energy of non-relativistic particles with temperature T is simply $3k_{\mathrm{B}}T/2$.

Use this information to answer the following questions.

(a) A fluid can be considered to behave like matter if its equation of state parameter $w \ll 1$. Show that $w < 10^{-6}$ for a plasma containing only protons that have a temperature $T_{\mathrm{p}} \approx T_{\mathrm{WHIM}}$, therefore implying that the WHIM behaves like our cosmological model for a matter fluid.

(b) Assuming that the protons with $T_\mathrm{p} \approx T_\mathrm{WHIM}$ are non-relativistic, determine their typical squared velocity $\langle v_\mathrm{p}^2 \rangle$.

(c) Based on the result of part (b), was the assumption that the protons are non-relativistic appropriate?

A large fraction of the baryons in the Universe have now coalesced to form dense structures with non-negligible internal pressures. However, modelling the matter content of the Universe as having $w_\mathrm{m} = 0$ is still a good approximation. Furthermore, in Chapter 1 you learned that baryonic matter contributes less than one-fifth of the total matter content of the Universe. The remaining $\sim$80% is non-baryonic dark matter, which interacts so weakly with itself and the rest of the Universe that it can be very accurately described as being a pressureless fluid. This means that dark matter is also consistent with the description of matter that began this section.

Using Equation 4.12, and setting $P = 0$ in Equation 4.8, the fluid equation for matter can be written as

$$\frac{\dot{\rho}_\mathrm{m}}{\rho_\mathrm{m}} = -3\frac{\dot{a}}{a} \tag{4.14}$$

In the following example, you will see how this form of the fluid equation can be solved to predict how the density of matter evolves as the Universe expands, and how it depends on the value of the scale factor.

Example 4.2

Solve Equation 4.14 to show that $\rho_\mathrm{m} \propto a^{-3}$.

Solution

Begin by multiplying both sides of Equation 4.14 by an infinitesimal time interval $\mathrm{d}t$ to obtain

$$\frac{1}{\rho_\mathrm{m}}\,\mathrm{d}\rho_\mathrm{m} = -3\frac{1}{a}\,\mathrm{d}a$$

Integrating both sides yields

$$\begin{aligned} \int \frac{1}{\rho_\mathrm{m}}\,\mathrm{d}\rho_\mathrm{m} &= -3\int \frac{1}{a}\,\mathrm{d}a \\ \ln\rho_\mathrm{m} &= -3\ln(a) + C \end{aligned}$$

Now just apply the exponential function to both sides and simplify to find the required proportionality:

$$\begin{aligned} \exp\left(\ln\rho_\mathrm{m}\right) &= \exp\left[-3\ln(a) + C\right] \\ \implies \rho_\mathrm{m} &= \exp\left[-3\ln(a)\right]\exp\left(C\right) \\ &= a^{-3}\exp\left(C\right) \\ &\propto a^{-3} \end{aligned}$$

To fix the constant of proportionality, we need to specify some boundary conditions. We can use the convention that we described in Chapter 3 (Exercise 3.4) that sets $a = 1$ at the present time, when the age of the Universe is $t = t_0$. If we define $\rho_{\mathrm{m},0} = \rho_{\mathrm{m}}(t_0)$ to be the present density of matter in the Universe, then we can relate the scale factor and the density of matter as follows.

Scale factor dependence of matter density

$$\rho_{\mathrm{m}} = \frac{\rho_{\mathrm{m},0}}{a^3} \tag{4.15}$$

Now that we know how matter density depends on the scale factor, we can solve the Friedmann equation to determine how a evolves with time. Substituting for ρ_{m} in Equation 4.3 and assuming a flat universe with $k = 0$, we find:

$$\dot{a}^2 = \frac{8\pi G}{3a}\rho_{\mathrm{m},0} \tag{4.16}$$

It is important to remember that this form of the Friedmann equation only applies in universes that are both matter-dominated and spatially flat on large scales. In contrast, Equation 4.15 is true for the matter component of any universe. In the following exercise you will use Equation 4.16 to predict how the scale factor evolves with time in a universe that *is* flat and matter-dominated.

Exercise 4.3

Solve Equation 4.16 to show that in a flat, matter-dominated universe

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{4.17}$$

where $t_0 = (6\pi G\rho_{\mathrm{m},0})^{-1/2}$ is the time at the present day for a flat, matter-dominated universe.

We can use Equations 4.15 and 4.17 to predict how the density ρ_{m} evolves in a flat, matter-dominated universe.

Time dependence of matter density

$$\rho_{\mathrm{m}}(t) = \frac{\rho_{\mathrm{m},0}}{a^3} = \rho_{\mathrm{m},0}\frac{t_0^2}{t^2} \tag{4.18}$$

4.2.3 Models for radiation

When cosmologists talk about radiation-like fluids (or 'radiation fluids') in the context of model universes, they mean fluids consisting of relativistic particles, with $v \approx c$. Examples of real particles that are always relativistic include photons and neutrinos.

Photons are massless, and are therefore relativistic by definition: in a vacuum, their speed exactly equals c. Although most particle physicists now believe that neutrinos must have a non-zero mass,† this mass is so small that all neutrinos move at speeds very near c.

For relativistic radiation fluids, whose properties are indicated in this chapter with a subscript 'r', the equation of state parameter $w_\mathrm{r} = 1/3$, which is reflected in the resulting expression for their equation of state.

Equation of state for radiation

$$P_\mathrm{r} = \frac{1}{3}\rho_\mathrm{r} c^2 \tag{4.19}$$

As we did for the matter-dominated model in Section 4.2.2, we can substitute for P in Equation 4.8 to obtain the fluid equation for radiation:

$$\frac{\dot{\rho}_\mathrm{r}}{\rho_\mathrm{r}} = -4\frac{\dot{a}}{a}$$

Now, following exactly the same steps that we took for the universe that was matter-dominated, and introducing $\rho_{\mathrm{r},0} = \rho_\mathrm{r}(t = t_0)$ as the current density of radiation in this radiation-only universe, we find the following dependence.

Scale factor dependence of radiation density

$$\rho_\mathrm{r} = \frac{\rho_{\mathrm{r},0}}{a^4} \tag{4.20}$$

As before, we can write a simplified Friedmann equation that assumes $k = 0$ in this universe:

$$\dot{a}^2 = \frac{8\pi G}{3a^2}\rho_{\mathrm{r},0} \tag{4.21}$$

We know from our experience of solving Equation 4.16 that the solutions to equations like Equation 4.21 take the form of power laws with $a(t) \propto t^q$. To determine the value of the index q for a universe filled with radiation

† The belief that neutrinos have mass is related to the observation of neutrinos changing flavour or 'oscillating' as they propagate from their sources to the point at which they are detected. According to the Standard Model of particle physics, such oscillations would not be possible if neutrinos were truly massless.

we substitute this trial power law solution into Equation 4.21 and rearrange to find that

$$qt^{q-1} = \sqrt{\frac{8\pi G\rho_{\mathrm{r},0}}{3}}\,t^{-q} \tag{4.22}$$

For the equality in Equation 4.22 to hold, the powers of t must be equal on both sides of the equation, which means that $q = 1/2$, and our full solution for the radiation-only model becomes

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

Exercise 4.4

Derive an expression for the present age t_0 of a flat, radiation-only universe in terms of $\rho_{\mathrm{r},0}$. Your expression should be analogous to the one for t_0 in Exercise 4.3.

Substituting for a in Equation 4.20, we can now predict how the radiation density evolves with time.

Time dependence of radiation density

$$\rho_\mathrm{r}(t) = \frac{\rho_{\mathrm{r},0}}{a^4} = \rho_{\mathrm{r},0}\frac{t_0^2}{t^2} \tag{4.23}$$

Let's explore why the density of radiation (i.e. Equation 4.20) decreases faster than the density of matter (Equation 4.15) as the Universe expands.

Figure 4.2 shows two spheres with respective volumes V_m and V_r and constant co-moving radii. In other words, V_m and V_r are expanding along with the Universe. V_m contains a fixed number of massive particles and V_r contains a fixed number of photons. As the Universe expands, the volumes of the spheres increase by a factor a^3 and the *number* density of the particles and photons they contain decreases by the same factor.

The individual energies of the massive particles are not affected by the expansion, so their overall energy density $\rho_\mathrm{m}c^2 \propto a^{-3}$, as we saw in Example 4.2. The situation is different for the photons in V_r. The energy of an individual photon is proportional to its wavelength, λ, such that $E = hc/\lambda$, where h is the Planck constant. As the Universe expands, the wavelengths of the photons in V_r get stretched by a factor a, which decreases their energy by the same factor. So each individual photon has a times less energy after the Universe has expanded and, when combined with the decrease in photon number density, we find that the energy density is reduced by a factor of a^4, as shown in Equation 4.23.

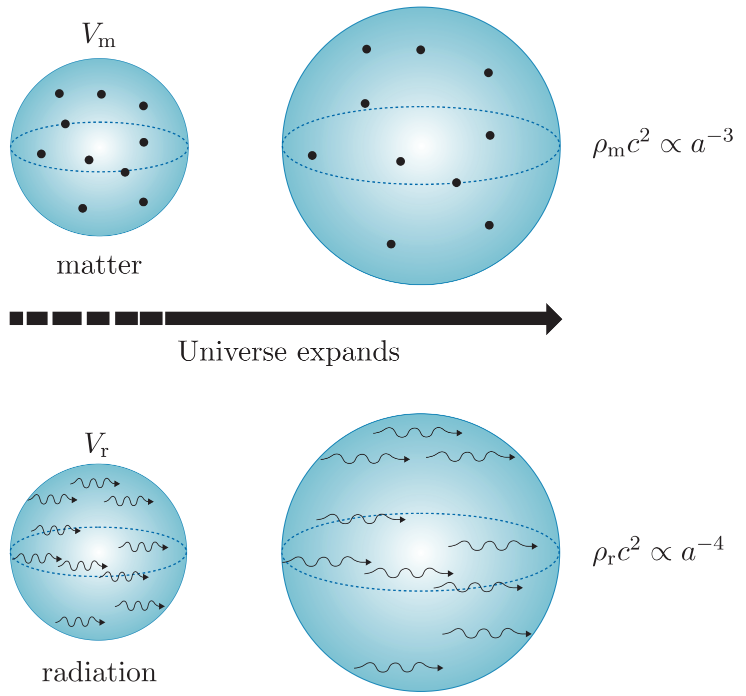

Figure 4.2 As the Universe expands the number densities of the photons in V_r and the massive particles in V_m both decrease by a factor $\propto a^{-3}$, but the photons lose energy corresponding to an additional factor of a as their wavelengths are stretched by the expansion.

4.2.4 Mixture models

The single-fluid universes that we investigated in the previous two sections are not very realistic. We know that the real Universe contains both matter and radiation, so in this section we will investigate the properties of more realistic 'mixed' universes that contain both relativistic and non-relativistic components.

To keep things simple in our discussions, we will assume that the two types of fluid in our mixed-model universes do not interact. However, in the *real* Universe, there are several ways that matter and radiation can interact and exchange energy. Two examples are given below.

- Matter is continuously being converted into radiation by nuclear fusion reactions in the cores of stars.
- In the space close to black hole event horizons, massive particles like protons and electrons can be accelerated to ultra-relativistic speeds. When the kinetic energy of these particles starts to dominate their rest mass energy, they start to behave like a radiation fluid.

However, stellar processes and extreme particle acceleration only affect fluids in very *localised* regions. Radiation will eventually be reabsorbed and converted into the kinetic energy of massive particles, and accelerated particles will eventually lose their kinetic energy and become non-relativistic again.

On the very large scales that all the models discussed in this chapter consider, relativistic and non-relativistic fluids can be considered as being separate. One *possible* mechanism for interaction between matter and radiation fluids on larger scales is that of matter–antimatter annihilation. When a massive particle encounters its antiparticle, the two are annihilated and their rest mass energy is converted entirely to radiation. In reality, the matter content of the Universe today is dominated by 'normal' matter, and antimatter is relatively scarce. This makes interactions between matter and antimatter rare enough that this mechanism for converting matter to radiation can normally be neglected on cosmological scales.

Our assumption that matter and radiation fluids do not interact means that separate fluid equations can be used to describe the evolution of their densities independently. The solutions to these independent fluid equations are identical to those we found for the single-component universes, and we would still find that $\rho_{\mathrm{m}} = \rho_{\mathrm{m},0}/a^3$ and $\rho_{\mathrm{r}} = \rho_{\mathrm{r},0}/a^4$.

Solving the Friedmann equation for multi-component universes is not as straightforward. Both fluids exert a gravitational influence, so the evolution of the scale factor depends on both of their densities at the same time. This means that we cannot construct independent Friedmann and fluid equations for matter and radiation in this scenario. Instead, we must use the total density, $\rho_{\mathrm{total}} = \rho_{\mathrm{m}} + \rho_{\mathrm{r}}$, to write:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{total}} - \frac{kc^2}{a^2} = \frac{8\pi G}{3}(\rho_{\mathrm{m}} + \rho_{\mathrm{r}}) - \frac{kc^2}{a^2} \tag{4.24}$$

We will not attempt to solve Equation 4.24 to find $a(t)$ and $\rho_{\mathrm{m,r}}(t)$ for arbitrary values of ρ_{m} and ρ_{r}. Even if we assume that $k = 0$, the solution is quite complicated when the values of ρ_{m} and ρ_{r} are comparable.

However, it is useful to consider solutions when the density of one component dominates over that of the other. In fact, we know that there have been times in the history of the Universe when its density has been dominated by radiation, and other times when it has been dominated by matter.

How would a spatially flat universe containing matter and radiation evolve in the radiation-dominated case? We can assume that a evolves as if there were no matter present, so $a(t) \propto t^{1/2}$, and the density of its radiation-fluid component $\rho_{\mathrm{r}}^{\mathrm{r}}$ is

The superscript 'r' indicates that this mixed universe is radiation-dominated, while the subscript 'r' implies we are describing a property of the radiation-fluid component.

$$\rho_{\mathrm{r}}^{\mathrm{r}}(t) \propto a^{-4} \propto t^{-2}$$

For the matter component, its density $\rho_{\mathrm{m}}^{\mathrm{r}}$ still evolves such that $\rho_{\mathrm{m}}^{\mathrm{r}}(a) \propto a^{-3}$, but because the evolution of a is dominated by ρ_r^r, the *time* evolution of $\rho_{\mathrm{m}}^{\mathrm{r}}$ is different from that in a matter-only universe:

$$\rho_{\mathrm{m}}^{\mathrm{r}}(t) \propto a^{-3} \propto t^{-3/2}$$

So in a radiation-dominated universe, $\rho_{\mathrm{m}}^{\mathrm{r}}$ decreases more slowly than $\rho_{\mathrm{r}}^{\mathrm{r}}$. This means that a flat, two-component universe that is initially radiation-dominated, with $\rho_{\mathrm{r}}^{\mathrm{r}} \gg \rho_{\mathrm{m}}^{\mathrm{r}}$, must eventually evolve to become matter-dominated. Conversely, a universe that also contains only matter and radiation but is currently matter-dominated must once have been dominated by radiation, regardless of the present-day densities $\rho_{\mathrm{m},0}$ and $\rho_{\mathrm{r},0}$.

When a two-component universe becomes matter-dominated, we can neglect the influence of the radiation component. Following the same argument as we did for the radiation-dominated case, we find that

Similar to the previous explanation, the superscript 'm' is used to indicate properties of fluids in a matter-dominated universe, while the subscript 'm' implies we are describing a property of the matter-fluid component.

$$a(t) \propto t^{2/3}$$
$$\rho_{\mathrm{m}}^{\mathrm{m}} \propto a^{-3} \propto t^{-2}$$
$$\rho_{\mathrm{r}}^{\mathrm{m}} \propto a^{-4} \propto t^{-8/3}$$

So in the matter-dominated case, $\rho_{\mathrm{r}}^{\mathrm{m}}$ decreases faster than $\rho_{\mathrm{m}}^{\mathrm{m}}$. This means that once this model universe becomes matter-dominated, it will stay that way forever.

Until the late 1990s, most cosmologists believed that a two-component model involving just matter and radiation provided a good description for the contents of the Universe. In Section 4.4 we will explain why this is no longer the case, and explore the implications of what is one of the most poorly understood properties of our Universe.

4.3 Density and curvature

In this section we will rewrite the Friedmann equation in a way that explicitly reflects the link between the spatial curvature k of the Universe and the densities of its various components.

4.3.1 The critical density

In Section 4.1.1 we demonstrated that the value of k is spatially and temporally invariant. Once we know k, the co-evolution of a and ρ is completely specified by the Friedmann equation, the fluid equation, and the equations of state for each of the Universe's component fluids. Conversely, if we *know* the value of a and how fast the Universe is expanding, we can ask what its density must be in order for k to have a particular value.

Cosmologists often make use of a quantity called the **critical density** (ρ_{c}), which is the density required for the Universe to be spatially flat if the Hubble parameter, $H(t) = \dot{a}/a$, has a particular value. In a flat Universe $k = 0$ and $\rho = \rho_{\mathrm{c}}$ by definition, so the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{c}}(t) \tag{4.25}$$

Making the substitution for $H(t)$, we can rearrange Equation 4.25 to obtain an expression for the critical density.

The critical density

$$\rho_c(t) = \frac{3H(t)^2}{8\pi G} \tag{4.26}$$

Note that $\rho_c(t)$ is a time-dependent quantity that varies in proportion to the Hubble parameter squared as the Universe expands or contracts. By replacing $H(t)$ with the Hubble constant H_0, we can also write down the present critical density of the Universe:

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \tag{4.27}$$

The following example should help to give you a physical intuition for the magnitude of the critical density today.

Example 4.3

Answer the following questions assuming that $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, and otherwise using values provided in the table of constants.

(a) What is the value of the present-day critical density, $\rho_{c,0}$, in $\mathrm{kg\,m^{-3}}$?

(b) How many protons per $\mathrm{m^3}$ does this value correspond to?

(c) If the mass of a typical galaxy is $10^{11}\,\mathrm{M_\odot}$ (where $\mathrm{M_\odot}$ is the mass of the Sun), how many galaxies per cubic megaparsec are required to make the Universe flat today?

Solution

(a) To evaluate $\rho_{c,0}$ using Equation 4.27, we need to convert the value we are given for the Hubble constant into SI units. One megaparsec equals $3.086 \times 10^{22}\,\mathrm{m}$, so:

$$H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = \frac{68 \times 1000\,\mathrm{m}}{3.086 \times 10^{22}\,\mathrm{m}}\,\mathrm{s^{-1}} = 2.2 \times 10^{-18}\,\mathrm{s^{-1}}$$

Now we can evaluate Equation 4.27:

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = \frac{3 \times (2.2 \times 10^{-18})^2}{8\pi \times 6.674 \times 10^{-11}}\,\mathrm{kg\,m^{-3}} = 8.7 \times 10^{-27}\,\mathrm{kg\,m^{-3}}$$

(b) The mass of a proton is $1.673 \times 10^{-27}\,\mathrm{kg}$, so the present critical density equates to approximately 5 protons per cubic metre.

(c) To answer the final part of the question we need to convert $\rho_{c,0}$ to solar masses per cubic megaparsec. We use the fact that $1\,\mathrm{M_\odot} = 1.99 \times 10^{30}\,\mathrm{kg}$ to deduce:

$$\begin{aligned}
\rho_{c,0} &= 8.7 \times 10^{-27}\,\mathrm{kg\,m^{-3}} \\
&= 8.7 \times 10^{-27} \times \frac{(3.086 \times 10^{22})^3}{1.99 \times 10^{30}}\,\mathrm{M_\odot\,Mpc^{-3}} \\
&= 1.3 \times 10^{11}\,\mathrm{M_\odot\,Mpc^{-3}}
\end{aligned}$$

So the present critical density corresponds to approximately one galaxy per cubic megaparsec. This value turns out to be quite a bit larger than the observed number density of galaxies in the Universe.

4.3.2 Density parameters

It is important to keep in mind that the critical density is not necessarily the true density of a universe. If a universe has *non-zero* curvature then its true density ρ will be different from its critical density ρ_c.

To quantify this difference, cosmologists define a dimensionless ratio called the **density parameter**.

Density parameter

$$\Omega(t) = \frac{\rho}{\rho_c} \tag{4.28}$$

- In general, $\Omega(t)$ is time-dependent and varies as a universe evolves, but under what circumstances would a universe have a time-invariant value of Ω?
- In flat universes, which are defined by having $k = 0$, Ω does not depend on time. Remember that k is time-invariant, so if $k = 0$ then ρ always equals ρ_c, and Ω is *always* exactly 1.

Using the definition of ρ_c in Equation 4.26, we can rewrite the Friedmann equation (Equation 4.3) in terms of the density parameter:

$$H(t)^2 = \frac{8\pi G}{3}\rho_c\Omega - \frac{kc^2}{a^2} = H(t)^2\Omega - \frac{kc^2}{a^2}$$

Rearranging this expression to isolate the density parameter, we obtain the Friedmann equation in terms of Ω:

$$\Omega - 1 = \frac{kc^2}{H(t)^2 a^2} \tag{4.29}$$

For universes with non-zero k, the value of $\Omega(t)$ varies as the universe evolves, but we can infer from Equation 4.29 that its *sign* never changes. If $k > 0$ then the density of the universe must always be larger than the critical density and, vice versa, if $k < 0$ then the density of the universe must always be less than ρ_c. Using the convention that the present-day scale factor $a(t_0) = 1$, Equation 4.29 can be expressed in terms of the current critical density and the Hubble constant as

$$\Omega_0 - 1 = \frac{\rho_0}{\rho_{c,0}} - 1 = \frac{kc^2}{H_0^2} \tag{4.30}$$

The following example shows how Equation 4.30 can be used to constrain the curvature of the real Universe at earlier times in its history.

Example 4.4

Current observational evidence indicates that $|\Omega_0 - 1| < 0.003$. Assuming that the Universe contains a mixture of matter and radiation, show that the value of Ω must have been even closer to 1 in the past.

Solution

We can rewrite Equation 4.29 as

$$\Omega - 1 = \frac{kc^2}{\dot{a}^2}$$

The value of k never changes, so to prove that $\Omega - 1$ was *smaller* in the past, we must show that $\dot{a}$ was *larger* in the past.

We showed in Sections 4.2.2 and 4.2.3 that $a \propto t^{2/3}$ for matter-dominated universes and $a \propto t^{1/2}$ for universes dominated by radiation. In both cases, $\dot{a}$ decreases with time and must have been larger in the past. These results from these hypothetical scenarios imply that if the real Universe contains only matter and radiation, then its rate of expansion should also be slowing down and would have been larger in the past.

Ω is commonly expressed as a sum of density parameters for each component of a universe's contents. For a universe containing matter and radiation, we would write:

$$\Omega = \frac{\rho}{\rho_\mathrm{c}} = \frac{\rho_\mathrm{m}}{\rho_\mathrm{c}} + \frac{\rho_\mathrm{r}}{\rho_\mathrm{c}} = \Omega_\mathrm{m} + \Omega_\mathrm{r} \tag{4.31}$$

To simplify the appearance of equations and calculations, cosmologists also define an effective density parameter for curvature:

$$\Omega_k = -\frac{kc^2}{a^2 H^2} \tag{4.32}$$

Ω_k contains no more *physical* information than k does, but using this convention we can rewrite the Friedmann equation very compactly as

$$\Omega + \Omega_k = 1 \tag{4.33}$$

4.4 The cosmological constant

4.4.1 Introducing Λ

When Einstein published his general theory of relativity in 1915, astronomical observations had only revealed a fraction of the Universe that is visible to modern astronomers. Based on the observational data that were available at the time, Einstein quite justifiably believed that the Universe was static, with $\dot{a} = 0$. It would be almost 15 years before astronomers established that the Universe was expanding by measuring the recession velocities of distant galaxies.

Einstein realised that if the Friedmann equation was correct (Equation 4.3 or, equivalently, Equation 4.4), and the Universe has non-zero energy density, then $\dot{a}$ could only be exactly zero at one instant in its history. Even if the Universe *was* momentarily static, the gravitational influence of its contents would make it start to collapse. We saw in Section 4.1.1 that the only possible solutions to Equation 4.3 when a universe is *not* empty describe a universe that expands forever, or one that expands initially and then recollapses.

To reconcile this apparent disagreement between his theory and the observations that were available in 1915, Einstein chose to modify his field equations. His modifications introduced a new repulsive term in the Friedmann equation that maintained $\dot{a} = 0$ by exactly balancing the gravitational attraction of the matter and radiation in the Universe. Einstein called this new term the **cosmological constant** and it is normally written in equations as Λ (the Greek upper-case letter lambda).

As the name suggests, the value of Λ is identical at all points in space and does not evolve with time. With this new term, the Friedmann equation can be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \tag{4.34}$$

The following exercise is designed to give you some practice using the Friedmann equation with a cosmological constant term.

Exercise 4.5

Einstein introduced a positive cosmological constant to enable the existence of a static, *non-empty* universe. Show that such a universe must have a positive curvature by considering whether each term in Equation 4.34 is positive or negative in this context.

Once astronomers had established that the Universe was indeed expanding, Einstein happily abandoned the cosmological constant. He had considered its somewhat arbitrary introduction to be an affront to the mathematical beauty of his theory and famously referred to this as his ‘greatest blunder’.

In 1930, Arthur Eddington published a paper showing that even with a cosmological constant, static universes are inherently unstable. Perturbing the matter or radiation density by a tiny amount would upset the precise balance between Λ and the gravitating components of such a universe. Once that happened, the universe would begin to expand or contract according to whether the initial density perturbation was negative or positive.

The resultant change in the scale factor would amplify the initial density perturbation and accelerate the expansion or contraction. Remember that Λ is constant by definition, and so the more the density perturbation grows, the less Λ is able to balance its gravitational influence. The inevitable result of any small density perturbation in Einstein's static universe is runaway expansion or collapse.

Despite its abandonment during the early twentieth century, the cosmological constant has enjoyed a renaissance in recent years. In the late 1990s, astronomers measured the brightness of distant supernovae and found that they appeared fainter than expected. In Chapter 5, you will learn that this has been interpreted as evidence that the expansion of the Universe is actually accelerating. Moreover, the observed acceleration seems to be exactly consistent with the expected effect of a real cosmological constant.

If the cosmological constant is real, then what generates it? At the time of writing, the physical origin of Λ and its true nature remain to be firmly established. Many cosmologists use the name 'dark energy' to refer to the unknown phenomenon that is responsible for the cosmological constant. You will learn more about dark energy and some popular theories for its physical origin later in this module.

4.4.2 Implications of the cosmological constant

A variety of observations strongly suggest that a cosmological constant term contributes to the current energy density of the real Universe. In this section we will explore how the introduction of Λ affects some of the equations that we derived earlier in this chapter.

First, to derive the acceleration equation for a universe with a cosmological constant, we can follow the steps in Section 4.1.3. Replacing Equation 4.3 with Equation 4.34 in that derivation gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3} \qquad (4.35)$$

We can now see explicitly that positive values of Λ increase the value of $\ddot{a}$. In other words, a positive cosmological constant acts like a repulsive force that increases the rate at which the Universe expands.

It is worth noting that there is no mathematical reason for Λ to be positive. Given our current lack of knowledge about the true origins of Λ, there is no theoretically motivated physical reason either! However, the effect of Λ on the behaviour of $\ddot{a}$ in the acceleration equation, coupled with the fact that the Universe appears to be accelerating in its expansion, strongly suggests that $\Lambda > 0$ in the real Universe.

A fluid description of the cosmological constant

Cosmologists often choose to model the cosmological constant as a third fluid component of the Universe, with its own *constant* energy density ($\epsilon_\Lambda = \rho_\Lambda c^2$) and pressure ($P_\Lambda$). If we adopt this approach, we can rewrite Equation 4.34 as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{a^2} \tag{4.36}$$

Comparing this expression with Equation 4.34, we see that

$$\frac{8\pi G}{3}\rho_\Lambda = \frac{\Lambda c^2}{3}$$

and so

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \tag{4.37}$$

We can also write down a separate fluid equation (see Equation 4.8) for Λ:

$$\dot{\rho_\Lambda} + 3\frac{\dot{a}}{a}\left(\rho_\Lambda + \frac{P_\Lambda}{c^2}\right) = 0 \tag{4.38}$$

For the cosmological constant, the equation of state parameter $w_\Lambda = -1$, which implies that it has a negative effective pressure. Remember that fluids with positive pressures exert an attractive gravitational effect. The fact that $P_\Lambda < 0$ is yet another way to see that Λ generates a repulsive force that counteracts gravity.

Using Equation 4.37 we can also express the density parameter, which you met in Section 4.3.2, for Λ:

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_\mathrm{c}} = \frac{\Lambda c^2}{8\pi G}\frac{8\pi G}{3H(t)^2} = \frac{\Lambda c^2}{3H(t)^2} \tag{4.39}$$

Using this expression together with Equation 4.32, the Friedmann equation can be rewritten as

$$\Omega = \Omega_\mathrm{m} + \Omega_\mathrm{r} + \Omega_\Lambda = 1 - \Omega_k \tag{4.40}$$

4.4.3 Models of the Universe with a cosmological constant

The introduction of a cosmological constant term adds a new degree of freedom to the Friedmann equation and permits many more possible histories and futures of the Universe. In this section we will explore some of these models and see how they differ from those for universes with $\Lambda = 0$.

To help our exploration, it will be useful to rewrite the Friedmann equation in terms of the density parameters that were introduced in Section 4.3.2. The following example illustrates how to derive the expression we need.

Example 4.5

Show that the Friedmann equation can be written as

$$\frac{H^2}{H_0^2} = \frac{\Omega_{\mathrm{m},0}}{a^3} + \frac{\Omega_{\mathrm{r},0}}{a^4} + \Omega_{\Lambda,0} - \frac{(\Omega_0 - 1)}{a^2} \tag{4.41}$$

Solution

Starting from the familiar Newtonian form of the Friedmann equation (Equation 4.3), we can use Equation 4.30 to replace k:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{H_0^2(\Omega_0 - 1)}{a^2}$$

Dividing this expression by H_0^2 and then using Equation 4.27 we find:

$$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\rho - \frac{\Omega_0 - 1}{a^2} = \frac{\rho}{\rho_{\mathrm{c},0}} - \frac{\Omega_0 - 1}{a^2}$$

Next, we express ρ as the sum of the densities of the Universe's components:

$$\frac{H^2}{H_0^2} = \frac{\rho_{\mathrm{m}} + \rho_{\mathrm{r}} + \rho_\Lambda}{\rho_{\mathrm{c},0}} - \frac{\Omega_0 - 1}{a^2}$$

By definition, $\rho_\Lambda = \rho_{\Lambda,0}$ is constant, and in Section 4.2 we showed that $\rho_{\mathrm{m}} = \rho_{\mathrm{m},0}a^{-3}$ and $\rho_{\mathrm{r}} = \rho_{\mathrm{r},0}a^{-4}$, so:

$$\begin{aligned}\frac{H^2}{H_0^2} &= \frac{\rho_{\mathrm{m},0}a^{-3} + \rho_{\mathrm{r},0}a^{-4} + \rho_{\Lambda,0}}{\rho_{\mathrm{c},0}} - \frac{\Omega_0 - 1}{a^2} \\ &= \frac{\Omega_{\mathrm{m},0}}{a^3} + \frac{\Omega_{\mathrm{r},0}}{a^4} + \Omega_{\Lambda,0} - \frac{\Omega_0 - 1}{a^2}\end{aligned}$$

The first three terms on the right-hand side of Equation 4.41 separately describe how the energy density of each component influences the evolution of a, while the final term encapsulates the influence of curvature. This form of the Friedmann equation is very helpful when exploring the current behaviour of model universes that include multiple components, and in particular those with a cosmological constant.

Figure 4.3 illustrates the possible behaviours of a simple model in which radiation contributes a very small proportion of the total energy density. If $\Omega_{\mathrm{r},0} \approx 0$, then Equation 4.41 becomes

$$\begin{aligned}\frac{H^2}{H_0^2} &= \frac{\Omega_{\mathrm{m},0}}{a^3} + \Omega_{\Lambda,0} - \frac{\Omega_0 - 1}{a^2} \\ &= \frac{\Omega_{\mathrm{m},0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}}{a^2} \qquad (4.42)\end{aligned}$$

This expression tells us that the behaviours of our simplified model depend on the values of $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ at different points in the $\Omega_{\mathrm{m},0}$–$\Omega_{\Lambda,0}$ plane. The curves in Figure 4.3 divide this plane based on different characteristics that a universe described by Equation 4.42 can exhibit. The remainder of this section considers these different characteristics in more detail.

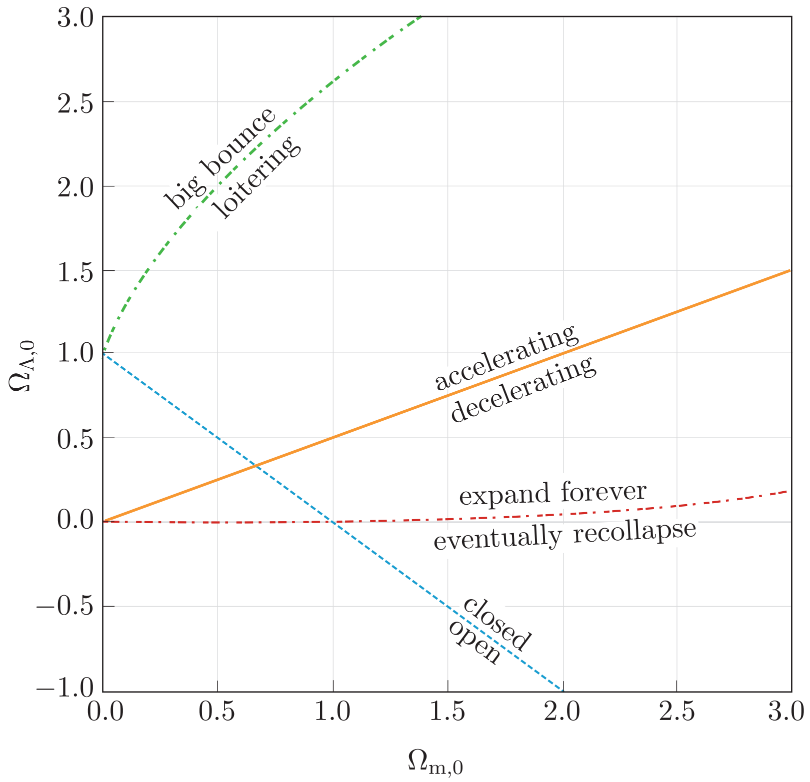

Figure 4.3 Possible characteristics of a model universe that contains matter and a cosmological constant, but in which the density of radiation is negligible.

Open, closed and flat universes

The easiest question to answer about a model universe is whether it is flat, open or closed. This depends solely on the sign of $\Omega_k = 1 - \Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}$. If $\Omega_{\mathrm{m},0} > 1 - \Omega_{\Lambda,0}$ then $\Omega_k < 0$, so $k > 0$ and the universe is closed; such universes lie above the blue dashed line in Figure 4.3. Conversely, if $\Omega_{\mathrm{m},0} < 1 - \Omega_{\Lambda,0}$ then the universe is open and would lie below the blue dashed line in the figure. Flat universes lie exactly on the blue dashed line, having $\Omega_{m,0} = 1 - \Omega_{\Lambda,0}$.

- Is a universe with positive Λ and total density $\rho > \rho_{\mathrm{c}}$ guaranteed to end by collapsing to zero size in a big crunch? (*Hint*: you may need to refer to Figure 4.3 to determine your answer.)

- No. If $\rho > \rho_{\mathrm{c}}$ then $k > 0$, $\Omega_k < 0$, and the universe must be closed, and lie above the blue dashed line on Figure 4.3. The plot shows us that there are a large number of universe types with positive Λ that are closed and that also lie above the red dot–dashed line. These universes will not collapse: rather they will expand forever.

Without a cosmological constant, the curvature k, the total density ρ and the ultimate fate of the Universe are all intimately related. If $\Lambda = 0$, then any universe that has $\rho > \rho_c$ has positive k and must end in a big crunch, while any universe that has $\rho \leq \rho_c$ has zero or negative k and will expand forever. However, the *inclusion* of a cosmological constant in our model for the Universe destroys this simple relationship, and we can now describe closed universes that expand forever as well as open universes that eventually recollapse.

Accelerating, decelerating and coasting universes

The next question we can ask is whether the expansion of the model universe we are investigating is currently accelerating or decelerating. To answer this question, we will derive a quantity called the **deceleration parameter**.

The deceleration parameter

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}(t_0)^2} \tag{4.43}$$

From Equation 4.43 we see that the value of q_0 is negative if $\ddot{a}$ is positive, and therefore the rate of expansion of the universe ($\dot{a}$) is currently increasing. Conversely, q_0 is positive if $\dot{a}$ is instead decreasing.

Our derivation of the deceleration parameter starts by writing down the first three terms of the Taylor expansion of the scale factor evaluated at the present time:

$$a(t) = a(t_0) + \dot{a}(t_0)\,(t - t_0) + \frac{1}{2}\ddot{a}(t_0)\,(t - t_0)^2 + \mathcal{O}(3) \tag{4.44}$$

The notation $\mathcal{O}(3)$ is a shorthand that is used to refer to any terms in the series that are proportional to powers of $(t - t_0)$ greater than 3. For values of t close to the present age of the Universe $(t - t_0) \ll 1$, and we expect that those terms represented by $\mathcal{O}(3)$ will be negligible. Next, we divide all terms by $a(t_0)$, which allows us to express the coefficients in terms of the Hubble constant, and simultaneously provides a way to define the deceleration parameter:

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{q_0}{2}H_0^2(t - t_0)^2 + \mathcal{O}(3) \tag{4.45}$$

Comparing the coefficients of $(t - t_0)^2$ in Equations 4.44 and 4.45, we find:

$$\frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)} = -\frac{q_0}{2}H_0^2$$

Rearranging this expression gives us the definition of the deceleration parameter q_0 in Equation 4.43.

Example 4.6

Show that if the energy density of a model universe is dominated by matter and a cosmological constant, then the deceleration parameter in that universe can be expressed as

$$q_0 = \frac{\Omega_{\mathrm{m},0}}{2} - \Omega_{\Lambda,0} \tag{4.46}$$

Solution

Start by writing Equation 4.35, assuming that $t = t_0$, $\rho = \rho_\mathrm{m}$ and $P = P_\mathrm{m} = 0$:

$$\frac{\ddot{a}(t_0)}{a(t_0)} = -\frac{4\pi G}{3}\left(\rho_{\mathrm{m},0} + \cancelto{0}{\frac{3P_\mathrm{m}}{c^2}}\right) + \frac{\Lambda c^2}{3} \tag{4.47}$$

Dividing by H_0^2 and using Equation 4.27 we obtain

$$\begin{aligned}\frac{1}{H_0^2}\frac{\ddot{a}(t_0)}{a(t_0)} &= -\frac{4\pi G}{3H_0^2}\rho_{\mathrm{m},0} + \frac{\Lambda c^2}{3H_0^2}\\ &= -\frac{1}{2}\frac{\rho_{\mathrm{m},0}}{\rho_{\mathrm{c},0}} + \frac{\Lambda c^2}{3H_0^2}\end{aligned}$$

To derive Equation 4.46 we need to recognise, using Equation 4.43, that the left-hand side of the previous expression is equal to $-q_0$. Then, using the definitions of Ω from Equation 4.28 (evaluated for the matter component at the present time) and Ω_Λ from Equation 4.39, we can write:

$$q_0 = -\frac{1}{H_0^2}\frac{\ddot{a}(t_0)}{a(t_0)} = \frac{\Omega_{\mathrm{m},0}}{2} - \Omega_{\Lambda,0}$$

From Equation 4.46 we can straightforwardly infer that when $\Omega_{\Lambda,0} > \Omega_{\mathrm{m},0}/2$ then $q_0 < 0$, and so $\ddot{a}$ is positive. Universes that fulfil this criterion lie above the solid orange line in Figure 4.3. Conversely, if $\Omega_{\Lambda,0} < \Omega_{\mathrm{m},0}/2$ for a universe then it would lie below the solid orange line in the figure, and its rate of expansion would be slowing down. Universes whose expansion is neither accelerating nor decelerating lie exactly on the solid orange line, and are often described as 'coasting' universes.

'Loitering' universes

By carefully fine-tuning the values of $\Omega_{\Lambda,0}$ and $\Omega_{\mathrm{m},0}$ it is possible for expanding universes with negative curvature to enter a state in which a hardly evolves at all. These solutions, where the density of matter is almost able to overcome a positive cosmological constant, are called 'loitering' universes. In Figure 4.3 they lie below, but very close to, the green dot–dashed line in the upper left-hand corner of the plot. An observer in such a universe might easily mistake it for a static universe unless they could make sufficiently precise measurements to detect the almost imperceptible expansion of spacetime.

Loitering universes eventually evolve away from their quasi-static state when their cosmological constant starts to dominate the gravitational effect of their matter content. Once that happens, a starts to grow exponentially and the universe will keep expanding forever.

'Big bounce' universes

Perhaps the strangest type of expanding universes that become possible when $\Lambda > 0$ are those in which there was no big bang, and in which a was never zero! In Figure 4.3 such universes lie above the green dot–dashed line. They all have positive curvature.

Let's consider the case of one that is initially Λ-dominated and contracting, with $H = \dot{a}/a < 0$. This model universe is positively curved, so Equation 4.40 tells us that the third term in Equation 4.42, which can be expressed as Ω_{k}/a^2, must be negative. If $\Omega_{\Lambda,0}$ and $\Omega_{\mathrm{m},0}$ are both positive and have appropriate values, it is possible for this third term in Equation 4.42 to become sufficiently dominant in time to halt the contraction (reducing H to zero) and allow an expansion phase with $\dot{a} > 0$ to start. Universes like this are sometimes referred to as 'big bounce' universes, in reference to the rebounding evolution of their scale factors.

4.5 Summary of Chapter 4

- The **Friedmann equation** describes how the rate of the Universe's expansion evolves with time, and how that evolution depends on the Universe's contents:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \qquad \text{(Eqn 4.34)}$$

- The **fluid equation** describes how the densities of the different components of the Universe evolve as the Universe itself expands or contracts:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \qquad \text{(Eqn 4.8)}$$

- The **acceleration equation** can be derived from the Friedmann and fluid equations, and can be used to determine whether the expansion of the Universe is speeding up or slowing down:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3} \qquad \text{(Eqn 4.35)}$$

- The contents of the universe can be modelled as non-interacting **perfect fluids** with different **equations of state**.
- In matter-dominated universes the energy density decreases over time as the universe expands, and the scale factor increases such that

$$\rho_{\mathrm{m}}(t) = \frac{\rho_{\mathrm{m},0}}{a^3} = \rho_{\mathrm{m},0}\frac{t_0^2}{t^2} \qquad \text{(Eqn 4.18)}$$

- In radiation-dominated universes the energy density decreases over time as the universe expands and the scale factor increases such that

$$\rho_{\mathrm{r}}(t) = \frac{\rho_{\mathrm{r},0}}{a^4} = \rho_{\mathrm{r},0}\frac{t_0^2}{t^2} \qquad \text{(Eqn 4.23)}$$

- The geometric curvature of the Universe is time-invariant. If the curvature is known then the Friedmann equation, the fluid equation and the equations of state for all components of a model universe can be used to predict how that universe's density and expansion rate will evolve throughout its lifetime.
- The **critical density** ρ_{c} refers to the total density a universe must have to be spatially flat, with its curvature parameter k exactly equal to zero. In general, ρ_{c} varies as the universe expands or contracts.
- For each component of the Universe a corresponding **density parameter** can be defined. This time-varying quantity is equal to the ratio of the component's density to the critical density of the Universe. It is also possible to define an effective density parameter corresponding to the effect of the spatial curvature, k.
- There is observational evidence that the real Universe contains a **cosmological constant** component, Λ, that counteracts the gravitational influence of the other components and can be modelled as a perfect fluid with a constant, negative pressure.
- The cosmological constant greatly increases the number of possible evolutionary histories that a model universe can follow. The exact path followed depends in a detailed way on the relative densities of the universe's different components, including the cosmological constant, and the universe's geometry.