How To: Asymptotics

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Contents

Disclaimer

The following guide is **not** part of the official lecture material and is subject to ongoing revision.

Sections that are marked with (*) are not lecture relevant. Examples are marked with $\star$, which indicate the difficulty level (more stars is more difficult). Try solving the examples before reading hints or the solution.

1 Introduction

Imagine you have two algorithms A and B. How do you measure, which one is *better*?

One way to measure this is by comparing the runtime of each algorithm, i.e. the time it takes for the algorithm to do, what it's supposed to do. In this lecture, when we talk about *runtime* we really talk about *the number of elementary operations that are executed until the algorithm terminates* (instead of e.g. seconds).

Since the input size for algorithms of real world problems is usually very large, it only makes sense to look at the runtime using very large input sizes. Here is where limits come into play. Instead of looking at runtimes of algorithms using a fixed input size, we want to observe how runtimes behave as the input size *grows*.

When we asked ourselves which algorithm is better, we were essentially looking for a way of *categorising* algorithms according to their runtime. This is exactly what O-notation allows us to do.

> *In computer science, O-notation is used to classify algorithms according to how their runtime grows as the input size grows.*

2 O-notation

2.1 Set Definitions

Here is the definition of $\mathcal{O}(f)$ taken from the script.

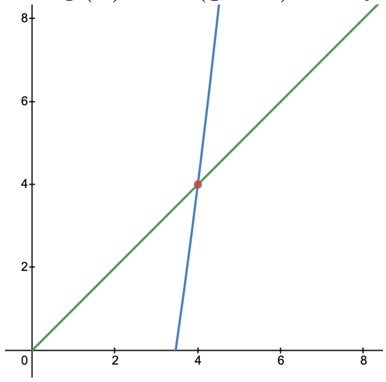
$$\mathcal{O}(f) := \left\{g\colon \mathbb{N} \to \mathbb{R}^+ \mid \text{there exists } c > 0, n_0 \in \mathbb{N} \text{ such that } g(n) \leq cf(n) \text{ for all } n \geq n_0\right\}$$

So what does that mean? I will translate it from mathematical notation to words:

> $\mathcal{O}(f)$:= "in this set are all functions g, that fulfil a certain property. Namely, these functions must be smaller than the function f multiplied by some positive constant c. However, g can be bigger than f multiplied by c for some time as long as there exists some point n_0, after which g will be smaller than f multiplied by c *forever*."

Consider figure 1. If I were to ask you, is the function g in $\mathcal{O}(f)$, you'd see that, yes, after the functions meet at the red dot (we find $n_0 = 4$), the function g will be smaller than f multiplied by some constant forever. Thus $g \in \mathcal{O}(f)$.

Figure 1: The functions $g(n) = n$ (green) and $f(n) = n^2 - 12$ (blue).



We say: *"g is bounded above by f asymptotically"*.

Now what about Ω?

$\Omega(f) \coloneqq \left\{g \colon \mathbb{N} \to \mathbb{R}^+ \mid \text{es gibt } c > 0, n_0 \in \mathbb{N} \text{ so, dass } g(n) \geq cf(n) \text{ für alle } n \geq n_0\right\}$

Notice how the definitions of $\mathcal{O}$ and Ω only differ at the bigger/smaller sign. Coming back to our image from earlier, if the green line is the function $g(n) = n$ and the blue line is the function $f(n) = n^2 - 12$, can we say $g \in \Omega(f)$? No, because now we want the function g to be *bigger* than f multiplied by some constant c. However, what we can say is that $f \in \Omega(g)$, by the same logic as earlier.

We say: *"f is bounded below by f asymptotically"*.

If we were to say, that a function f is bounded both above and below by a function g, meaning $g \in \mathcal{O}(f)$ and $g \in \Omega(f)$, then we say that f and g are equal asymptotically. The set of functions that are asymptotically equal to a function f is what we call $\Theta(f)$.

$\Theta(f) \coloneqq \{g \colon \mathbb{N} \to \mathbb{R}^+ \mid \text{es gibt } c_1, c_2 > 0, n_0 \in \mathbb{N} \text{ so, dass } c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ für alle } n \geq n_0\}$

The way of figuring out what function belongs to which set will be covered in the later sections.[1]

[1] You will not have to draw functions.

2.2 Limit Definitons

Theorem 1. Seien $f, g \colon \mathbb{N} \to \mathbb{R}^+$ zwei Funktionen. Dann gilt:

1. Ist $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, dann ist $f \in \mathcal{O}(g)$, und $\mathcal{O}(f) \subsetneq \mathcal{O}(g)$.
2. Ist $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C > 0$ (wobei C konstant ist), dann ist $f \in \Theta(g)$.
3. Ist $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, dann ist $g \in \mathcal{O}(f)$, und $\mathcal{O}(g) \subsetneq \mathcal{O}(f)$.

Note that in the lecture we generally use $\leq$ instead of $\in$.[2]
Whenever we see an exercise like $n \in \mathcal{O}(n^2)$, we simply use the definition and calculate some limit.

Example 1. ($\star$) Is it true that $n \in \mathcal{O}(n^2)$?

Solution. We just use apply Theorem 1 and calculate the resulting limit.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

As we can clearly see, the functions f, g satisfy point 1. of Theorem 1, so we can deduce that $n \in \mathcal{O}(n^2)$ is correct.

(*) But what is a limit anyways?

Informally, when we talk about a limit, we talk about the value a function approaches as the input approaches some other value.
For example if f is a function, n is the input and n approaches infinity (we write $n \to \infty$), for the limit L of f as the input n approaches infinity we write:

$$\lim_{n\to\infty} f(n) = L$$

Note that the input n could approach many different kind of values (e.g. $3, -10, -\infty$) but in the lecture it approaches ∞ most of the time. This is because, when we talk about the asymptotic behaviour of an algorithm, we are interested in seeing what happens as the input size grows, i.e. it gets bigger and bigger (approaches

[2] This is because we are comparing functions, so we are just trying to say "some function f grows slower/faster than some other function g", which makes sense intuitively, but not mathematically (since we define $\mathcal{O}(f)$ to be a set of functions with certain properties).

positive infinity).

But what happens *behind the scenes*? Intuitively, we can think of the limit of a function in the following way:
We'll consider the limit of a function f as its input n approaches ∞. Consider

Figure 2: This image was taken from Wikipedia.

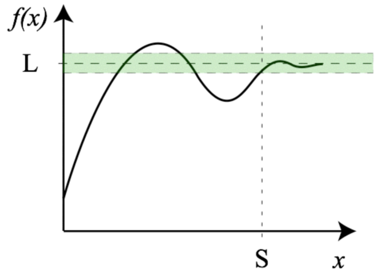


figure 2. Note that on the x axis we have the input size n. So the further right we go, the larger n gets.
At some point, the functions value will be effectively indistinguishable from the limit L, or rather *very very close*. In the graphic this is modeled by the little green tunnel. Notice how at some point S our function f doesn't leave the tunnel anymore. Depending on the thickness of this green tunnel, the point S can be a different point on the x axis, the important thing is just that it exists somewhere.

Coming back to the equation from earlier:

$$\lim_{n \to \infty} f(n) = L.$$

What we could also say is: No matter how thick or thin our green tunnel is, there will always be some point on the x axis, such that from this point onwards, the function f doesn't leave the tunnel anymore (forever).

As I have said before, this is a **very informal** way of describing a limit.

Again, none of this is relevant for the lecture, it should just give you some intuition.

2.3 Calculation Rules

The following theorem was taken from exercise sheet 2 from fall semester 2022 (you can find it here). I couldn't find it in the script.

Theorem 2. Let $f, g, h : \mathbb{N} \to \mathbb{R}^+$. If $f \leq \mathcal{O}(h)$ and $g \leq \mathcal{O}(h)$, then

1. for every constant $c \geq 0, c \cdot f \leq \mathcal{O}(h)$.
2. $f + g \leq \mathcal{O}(h)$.

Both 1. and 2. can be proven using the set definitions introduced earlier.

3 Working with Limits

Since we use limits to categorise functions according to their asymptotic behaviour, being able to solve asymptotics exercises mainly comes down to being able to calculate limits. For this reason, I will introduce rules and tricks that will help with just that.

3.1 Limit Calculation Rules

Limit Calculation Rules: Let f, g be two functions. If the limits $\lim_{n\to\infty} f(n)$ and $\lim_{n\to\infty} g(n)$ exist[a], then:

1. $\lim_{n\to\infty}(f(n) \pm g(n)) = \lim_{n\to\infty} f(n) \pm \lim_{n\to\infty} g(n)$.
2. $\lim_{n\to\infty}(c_1 \cdot f(n) + c_2) = c_1 \cdot (\lim_{n\to\infty} f(n)) + c_2$, for some constants c_1, c_2
3. $\lim_{n\to\infty} f(n) \cdot g(n) = \lim_{n\to\infty} f(n) \cdot \lim_{n\to\infty} g(n)$.
4. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\lim_{n\to\infty} f(n)}{\lim_{n\to\infty} g(n)}$, if $\lim_{n\to\infty} g(n) \neq 0$.

[a] There are in fact limits that don't exist, e.g. $\lim_{n\to\infty}(-1)^n$ or $\lim_{n\to\infty}\sin(n)$. However, this is not relevant for this lecture (I never encountered an asymptotics problem where some limit didn't exist). Still, it's good to know.

Let's go over an example.

Example 2. ($\star$) Determine the following limit: $\lim_{n\to\infty} \frac{2023^2+n}{n^2}$.

Solution.

$$\lim_{n\to\infty} \frac{2023^2 - n}{n^2} = \lim_{n\to\infty} \frac{2023^2}{n^2} - \frac{n}{n^2} = \lim_{n\to\infty} \frac{2023^2}{n^2} - \frac{1}{n}.$$

Now we apply rule 1. to obtain:

$$\lim_{n\to\infty} \frac{2023^2}{n^2} - \frac{1}{n} = \lim_{n\to\infty} \frac{2023^2}{n^2} - \lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{2023^2}{n^2} - 0.$$

To determine the first limit, we use rule 2. with $c_1 = 2023^2, c_2 = 0$:

$$\lim_{n\to\infty} \frac{2023^2}{n^2} - 0 = 2023^2 \cdot \lim_{n\to\infty} \frac{1}{n^2} - 0.$$

Now we can use rule 3. for $f(n) = g(n) = \frac{1}{n}$:

$$2023^2 \cdot \lim_{n\to\infty} \frac{1}{n^2} - 0 = 2023^2 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} - 0 = 2023^2 \cdot 0 \cdot 0 - 0 = 0.$$

Thus

$$\lim_{n\to\infty} \frac{2023^2 + n}{n^2} = 0.$$

Note that you don't have to be that rigorous, I just wanted to show as many rules as possible. This solution would have been totally fine as well:

Solution.

$$\lim_{n\to\infty} \frac{2023^2 + n}{n^2} = 2023^2 \cdot \lim_{n\to\infty} \frac{1}{n^2} - \lim_{n\to\infty} \frac{1}{n} = 0.$$

3.2 Exponent-Limit Trick

Exponent-Limit Trick: Let f, g be two functions. If f is continuous[a] then:

$$\lim_{n\to\infty} f(n)^{g(n)} = \lim_{n\to\infty} f(n)^{\lim_{n\to\infty} g(n)}$$

[a] A very simple intuition for continuous functions is the following: *A function is considered continuous if you can draw them without lifting your pen (a continuous pen stroke, without any jumps).* You will learn about continuous functions and why it matters in Analysis 1, in the second semester. For this lecture, it doesn't really matter.

Example 3. ($\star$) Determine the following limit: $\lim_{n\to\infty} e^{\frac{2023}{n^3}}$.

Solution. The function e^n is continuous, thus we have:

$$\lim_{n\to\infty} e^{\frac{2023}{n^3}} = \lim_{n\to\infty} e^{\lim_{n\to\infty} \frac{2023}{n^3}}.$$

We determine the limit in the exponent by applying the rules from earlier:

$$\lim_{n\to\infty} \frac{2023}{n^3} = 2023 \cdot 0 = 0.$$

Coming back to the original problem, we have:

$$\lim_{n\to\infty} e^{\lim_{n\to\infty} \frac{2023}{n^3}} = \lim_{n\to\infty} e^0 = \lim_{n\to\infty} 1 = 1.$$

Thus

$$\lim_{n\to\infty} e^{\frac{2023}{n^3}} = 1.$$

Common continuous functions that you will encounter in AnD are: e^n and $\log n$.

3.3 L'Hôpital's Rule

Theorem 3 (Bernoulli-de L'Hôpital). Let f, g be two differentiable functions, such that $g' \neq 0$. If either $\lim_{n\to\infty} \frac{f(n)}{g(n)} =$ "$\frac{0}{0}$" or $\lim_{n\to\infty} \frac{f(n)}{g(n)} =$ "$\frac{\infty}{\infty}$" then:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

My guess is that most students will be familiar with derivatives from school (if not I would recommend looking at basic derivation rules).

Just like with continuous functions, I haven't encountered functions that weren't differentiable in AnD a single time. The most important part is that the limit is of the form "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$" then you should be good to go.
Common differentiable functions that you will encounter in AnD are: $e^n, \log n$ and all polynoms (e.g. $n^2 + n + 1$).

Example 4. ($\star$) Determine the following limit: $\lim_{n\to\infty} \frac{e^n}{n}$

Solution. The limit is of the form "$\frac{\infty}{\infty}$", as both n and e^n get bigger and bigger. We apply L'Hôpital's rule for $f(n) = e^n$ and $g(n) = n$, where $f'(n) = e^n$ and $g'(n) = 1$:

$$\lim_{n\to\infty} \frac{e^n}{n} = \lim_{n\to\infty} \frac{e^n}{1} = \lim_{n\to\infty} e^n = \infty.$$

Example 5. ($\star$) Determine the following limit: $\lim_{n\to\infty} \frac{e^n}{n^3}$

Hint: We can also apply L'Hôpital multiple times.

Solution. The limit is of the form "$\frac{\infty}{\infty}$", as both n^3 and e^n get bigger and bigger. We apply L'Hôpital's rule three times for $f(n) = e^n$ and $g(n) = n^3$, where $f'(n) = f''(n) = f'''(n) = e^n$ and $g'(n) = 3n^2, g''(n) = 6n, g'''(n) = 6$:

$$\lim_{n\to\infty} \frac{e^n}{n^3} = \lim_{n\to\infty} \frac{e^n}{3n^2} = \lim_{n\to\infty} \frac{e^n}{6n} = \lim_{n\to\infty} \frac{e^n}{6} = 6 \cdot \lim_{n\to\infty} e^n = \infty.$$

3.4 Logarithm and Exponent Rules

Another thing that is very helpful many topics in AnD are various logarithm and exponentiation rules. Here are a few that I consider the most important one's for AnD. For a complete list please refer to the internet.

Logarithm Rules:

1. $\log_a x \cdot y = \log_a x + \log_a y$.
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$.
3. $\log_a x^n = n \cdot \log_a x$.
4. $\log_a b = \frac{\log_c b}{\log_c a}$.
5. $\log_a b = \frac{1}{\log_b a}$.

Exponent Rsles:

1. $x^m \cdot x^n = x^{m+n}$.
2. $\frac{x^m}{x^n} = x^{m-n}$.

3. $(x^m)^n = x^{m \cdot n}$.

4. $(x \cdot y)^n = x^n \cdot y^n$ and $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.

5. $x^0 = 1$.

6. $x^{-n} = \frac{1}{x^n}$.

7. $x^{\frac{m}{n}} = \sqrt[n]{x^m}$.

Example 6. ($\star$) Is it true that $\ln n^{42} \leq \Theta(\ln n^{2023})$?

Solution. Consider the following limit $\lim_{n\to\infty} \frac{\ln n^{42}}{\ln n^{2023}}$. We use logarithm rule 3. on both the nominator and denominator to get:

$$\lim_{n\to\infty} \frac{\ln n^{42}}{\ln n^{2023}} = \lim_{n\to\infty} \frac{42 \cdot \ln n}{2023 \cdot \ln n} = \lim_{n\to\infty} \frac{42}{2023} = \frac{42}{2023}.$$

Since $\frac{42}{2023}$ is just some constant we see that statement 2. from Theorem 1 holds, thus we can conclude:

$$\ln n^{42} \leq \Theta(\ln n^{2023}).$$

3.5 $e^{\ln n}$ - Trick

Now that we know common logarithm and exponentiation rules, I want to introduce another trick that can be useful at times:

$e^{\ln n}$ **- Trick:** Let f f We have: $f(n) = e^{\ln f(n)}$ for all n, where $\ln n$ denotes $\log_e n$, the so-called natural logarithm.

Example 7. ($\star\star$) Determine the following limit: $\lim_{n\to\infty} \sqrt[n]{n}$

Hint 1: The solution requires multiple rules and tricks that we have learned.
Hint 2: Can you rewrite $\sqrt[n]{n}$ and apply some trick involving exponents?

Solution. We rewrite $\sqrt[n]{n}$ to $n^{\frac{1}{n}}$ and apply the $e^{\ln n}$ - trick.

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{1}{n}\cdot\ln n}.$$

We know that e^n is continuous, thus we get:

$$\lim_{n\to\infty} e^{\frac{1}{n}\cdot\ln n} = \lim_{n\to\infty} e^{\lim_{n\to\infty} \frac{1}{n}\cdot\ln n}.$$

Now we look at the limit in the exponent separately. $\lim_{n\to\infty} \frac{\ln n}{n}$ is of the form "$\frac{\infty}{\infty}$", we use L'Hôpital:

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Putting everything together we get:

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\lim_{n\to\infty} \frac{1}{n}\cdot\ln n} = \lim_{n\to\infty} e^0 = \lim_{n\to\infty} 1 = 1.$$

3.6 Estimation Tricks for $\sum$ and $\prod$

In some situations, using estimations instead of the actual terms can be very helpful.
Consider the following example:

Example 8. ($\star$) Is it true that $\sum_{i=1}^{n} i \leq \mathcal{O}(n^2)$?

Solution. If you already know about Gauss' summation formula this problem becomes $\frac{n(n+1)}{2} \overset{!}{\leq} \mathcal{O}(n^2)$ which is obviously true.

But how about if we don't? We can try to approximate the sum. Since the i is always smaller or equal to n, we just substitute i with the largest value n.

$$\sum_{i=1}^{n} i \leq \sum_{i=1}^{n} n = \underbrace{n + \cdots + n}_{n\text{-times}} = n^2 \leq \mathcal{O}(n).$$

The direction of the smaller equal sign is very important here! When we want to show that something is in $\mathcal{O}(f)$ using approximations, we have to use approximations that are *larger*.

Example 9. ($\star\star$) Is it true that $\ln n! \leq \mathcal{O}(n \ln n)$?[a]

[a] $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.

Hint: Can you use logarithm rules?

Solution. Notice that:

$$\ln n! = \ln(1 \cdot 2 \cdots (n-1) \cdot n).$$

We use logarithm rules to get:

$$\ln(1 \cdot 2 \cdots (n-1) \cdot n) = \sum_{i=1}^{n} \ln i.$$

We approximate the sum:

$$\sum_{i=1}^{n} \ln i \leq \sum_{i=1}^{n} \ln n \leq \mathcal{O}(n \ln n).$$

Example 10. ($\star\star\star$) Is it true that $n! \leq \mathcal{O}(n^{n/2})$?[a]

[a] This exercise was taken from exercise sheet 4 of fall semester 2022, you can find it here. On the sheet it was noted: "Note that the last claim is challenge. It was one of the hardest tasks of the exam. If you want a 6 grade, you should be able to solve such exercises."

Solution. TODO.

4 Miscellaneous

4.1 Master Theorem

TODO.

4.2 Why the logarithm base doesn't matter for O-notation

The base of the logarithm doesn't have to be written in O-notation, i.e. $\mathcal{O}(\log_a n)$ becomes $\mathcal{O}(\log n)$.

Can you see why? You should be able to prove this using everything we have learned so far.

Solution. Let $a, b > 1$ be some arbitrary bases for the logarithms. Using logarithm rules we get:

$$\log_a n = \frac{\log_b n}{\log_b a}.$$

thus we have:

$$\log_a n \cdot \log_b a = \log_b n.$$

Since a, b were chosen arbitrarily, we just showed that we can write a logarithm of any base, as a logarithm with some other base, multiplied a *constant* factor which doesn't change anything about the asymptotics as stated in theorem 2.

4.3 Stirling Approximation

Something that can be useful in some cases is the so-called Stirling approximation $n! = 1 \cdot 2 \cdots n \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

TODO