Solving DSGE Models with Incomplete Markets by Perturbation

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Abstract

This paper proposes to solve DSGE models with incomplete markets by first approximating the local dynamics of a tractable auxiliary model, and then applying regular perturbation to some of the parameters to reach the model of interest. The method is easy to implement with available solution packages, and can approximate models around a large subset of the state-space, including the stochastic steady state. The lead application extends the two-period, multi-asset model of Coeurdacier and Gourinchas (2016, JME) to an infinite horizon setup. The calibrated model with bonds and equities delivers a large level of equity home bias, and a natural link between trade and financial openness with external asset positions comparable to the data. However, the model generates excessive risk-sharing, and counterfactual co-movements of gross capital flows.

Keywords: Equity home bias, incomplete markets, international capital flows, open economy DSGE models, perturbation, portfolio allocations, solution methods.

JEL classification: E32, E44, F41

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1 Introduction

Perturbation methods are a widely popular tool for solving Dynamic Stochastic General Equilibrium (DSGE) models. They tend to be fast, locally accurate as one increases the order of approximation, and can handle many state variables without suffering from the curse of dimensionality. By far, their most common application consists of building Taylor series to approximate policy functions of the model around its deterministic steady state using implicit-function theorems (FernÃ¡ndez-Villaverde, Rubio-RamÃrez, and Schorfheide, 2016).

However, standard regular perturbation falls short in dealing with many incomplete markets models of interest because the properties of these models in a stochastic setting can differ radically from their deterministic counterparts. This is precisely what happens in DSGE models with portfolio choice: in a deterministic arbitrage-free equilibrium, all expected asset returns must be identical, leading to arbitrary portfolio allocations that make the analysis of these models intractable with the conventional perturbation approach (Devereux and Sutherland, 2011; Tille and van Wincoop, 2010). Another classic example is the small open economy with a stochastic endowment. While this model is dependent on initial conditions and displays non-stationary dynamics in a deterministic setup (Schmitt-GrohÃ© and Uribe, 2003), a precautionary-savings motive induces local stability and stationarity in a stochastic context (Chamberlain and Wilson, 2000).[1]

The first contribution of this paper is to propose a small generalization of the standard perturbation procedure to bypass these technical difficulties. The method involves using two perturbation parameters, ε and σ, to link a family of models. While σ is the conventional parameter scaling random innovations, functions of the new parameter (e.g., $1-\varepsilon$) enter directly into the equilibrium conditions, scaling auxiliary modifications that resolve the indeterminacy problem in the absence of risk. Consider a setup where the model of interest $(\varepsilon, \sigma) = (1, 1)$ is well-behaved (e.g., a small open economy model with a stochastic endowment), but intractable with $(\varepsilon, \sigma) = (1, 0)$ (the deterministic version of this model). However, the case $(\varepsilon, \sigma) = (0, 0)$ activates portfolio adjustment costs that make the model tractable. Then, the same algorithm used in standard perturbation can be applied to obtain approximations to the policy functions of the model of interest around the deterministic steady state implied by $(\varepsilon, \sigma) = (0, 0)$. As such, the joint perturbation of the pair (ε, σ) allows us to effectively reach the model of interest starting from the nearby auxiliary one.[2]

In theory, one could obtain the same result with a single-parameter perturbation model where σ interacts with both the random innovations and the auxiliary modifications, since the two procedures are mathematically equivalent. The advantage of a two-parameter approach is eminently practical: popular solution toolboxes (e.g., Dynare) only allow for the classical

[1] This issue is pervasive in open-economy macroeconomics. For example, Corsetti, Dedola and Leduc (2023) analyze a DSGE one-bond economy that delivers non-stationary wealth dynamics because, for simplicity, they only consider the perfect-foresight case.

[2] The technique works because we can always set the parameter controlling the auxiliary modification to a small value, thereby narrowing the gap between the models. Following the same logic as in standard perturbation, we can also scale features of the model of interest by ε, so that they are deactivated in the tractable auxiliary deterministic model.

perturbation scheme. But thanks to the variable-parameter duality of perturbation objects, we can treat the new parameter ε as an exogenous state that is constant over time.[3] Thus, at the very low cost of augmenting the auxiliary model with an additional exogenous state, we can compute approximations to the model of interest using available toolboxes.

Furthermore, we can use two-parameter perturbation to compute local solutions around a large subset of points in the state-space while still relying on the implicit function theorems that ensure the convergence of the Taylor series. This result follows because we can introduce auxiliary parameters that affect the deterministic steady state of the auxiliary model, allowing it to differ from the one implied by the model of interest (which may not even exist). Perturbation of ε then removes the effect of these parameters in the model dynamics. Exploiting this property, the paper develops a simple fixed-point algorithm that, akin to standard calibration routines of DSGE models, approximates the stochastic steady state of the model and builds the perturbation solution around it.[4] A proposition ensures that, under standard regularity conditions, the output of the algorithm converges to the true stochastic steady state of the model as the order of the Taylor series goes to infinity.

In the applications, the algorithm delivers the same zero-order portfolios obtained with bifurcation methods (Judd, 1998; Devereux and Sutherland, 2011), and a stochastic steady state of DSGE models identical to the one obtained with global methods. The approximations also follow the same pattern emphasized by Devereux and Sutherland (2010) and Tille and van Wincoop (2010): a second-order solution might suffice to approximate the stochastic steady state, but we require at least a third-order solution to capture well portfolio dynamics.

The second contribution of the paper consists of applying two-parameter perturbation to analyze hedging motives behind equity home bias in a two-country DSGE model with bonds and equities. To achieve this, I extend the two-period model by Coeurdacier and Gourinchas (2016) to an infinite-horizon setting, thus allowing for rigorous quantitative analysis. The baseline model features an endowment economy with two symmetric countries, two goods, up to four assets, and various sources of risk (income, redistributive, and preference shocks) that make financial markets incomplete. With two-parameter perturbation, what the literature considered a very challenging task now becomes standard DSGE work. In particular, introducing portfolio adjustment costs as auxiliary devices suffices to obtain a third-order solution to the model of interest around the stochastic steady state that is highly accurate (average log Euler errors below -7) and very fast to compute.[5]

The calibrated model generates results for long-run gross asset positions that are consistent with various stylized facts. First, it confirms that bonds matter: incorporating them into the menu of available assets is key to reproducing the observed large level of equity home bias. Second, the model yields large, leveraged gross debt positions comparable to the data, with countries holding long positions in the home currency and short positions in the foreign one

[3] Levintal (2017) exploits this duality to build an efficient solution algorithm for the standard perturbation model.

[4] Coeurdacier, Rey and Winant (2011) call this point the risky steady state.

[5] See https://github.com/ghausmann/two_parameter_perturbation.git for a repository with all the replication files to solve the main application and examples of this paper, using both Dynare and the solution package by Levintal (2017).

(Lane and Shambaugh, 2010; Maggiori, Neiman and Schreger, 2020). Third, it predicts a strong positive relationship between trade and financial openness (Collard et al., 2007; Heathcote and Perri, 2013) that is mostly driven by a large increase in the external debt position of countries (Khalil, 2019). The mechanism generating this positive link is novel: holding leveraged debt positions delivers risk-sharing transfers because the returns of Home and Foreign bonds respond differently to the same income shocks. But as deeper trade integration synchronizes returns, countries find it optimal to increase their gross debt positions just to maintain the same degree of risk-sharing.

A key strength of two-parameter perturbation is that the quantitative analysis naturally yields insights into the dynamics of equilibrium portfolios and international capital flows. As in Sauzet (2022a, 2022b), the allocation of time-varying wealth across countries emerges as a key driver of portfolio reallocation—an aspect largely ignored by previous literature. Since relative wealth is a slow moving object, this dependence generates strong portfolio inertia despite the absence of financial frictions in the model of interest.[6]

The model, however, struggles to match key second moments from the data. First, it produces counterfactual co-movements of gross capital flows: stochastic simulations and impulse responses reveal an almost perfect negative correlation between capital inflows and outflows, implying that gross capital flows are much less volatile than net capital flows. As documented by Broner et al. (2013) and Davis and van Wincoop (2018), both predictions are at odds with international time-series data. Second, the simulations fail to capture the large volatility and persistence of the real exchange rate and cannot fully explain its co-movement with relative consumption, despite a considerable quantitative improvement compared to the complete-markets models popular in the literature (Backus and Smith, 1993). Another symptom of excessive risk-sharing is the model's inability to account for the consumption correlation puzzle (Backus, Kehoe, and Kydland, 1992), as the cross-correlation between Home and Foreign consumption is too strong, both compared to the data and relative to the cross-correlation of GDP.[7]

2 Related literature

Solution methods. First and foremost, the paper contributes to the existing body of literature on solution methods for portfolio-choice models that rely on local approximations.[8] Within this literature, it aligns closely with the methodological breakthroughs of Devereux and Sutherland (2010, 2011) and Tille and van Wincoop (2010), who build approximations based on the so-called zero-order portfolios (those held by investors in a symmetric deterministic steady state

[6]See Bacchetta, Davenport and van Wincoop (2022), and Bacchetta, van Wincoop and Young (2023) for recent contributions introducing portfolio frictions in DSGE models to generate this inertia.

[7]Since the model does not include features such as recursive preferences, stochastic volatility, or disaster risk, it also fails to reproduce many stylized asset-pricing facts (see e.g. Andreasen, 2012). The good news is that two-parameter perturbation can easily handle all these extensions.

[8]See Bacchetta, van Wincoop and Young (2023), Cao, Luo and Nie (2023), Sauzet (2022b), and Stepanchuk and Tsyrennikov (2015) for examples of global solution methods. Compared to these algorithms, the usual trade-offs between global and local methods apply (higher global accuracy and ability to handle strong non-linearities, versus acute curse of dimensionality and sizable entry costs).

as risk goes to zero). Alternatives to this approach include contributions based on bifurcation theorems (Judd, 1998; Judd and Guu, 2001; Winant, 2014) or continuous-time approximations (Campbell and Viceira, 1999; Evans and Hnatkovska, 2012; Bacchetta, Davenport and van Wincoop, 2022). To this literature, I bring a solution method fully based on regular perturbation, which benefits from all the theoretical results and efficient, user-friendly algorithms available for this mathematical tool.[9] In addition, approximations based on zero-order portfolios suffer from two main limitations. First, they can become highly inaccurate for models with asymmetric countries (Rabitsch, Stepanchuk and Tsyrennikov, 2015). Second, being based on a second-order approximation to the portfolio Euler equations, they cannot account for common features in DSGE macro-finance models involving higher-order moments of the shocks, such as non-symmetric distributions or stochastic volatility. By building approximations centered at the stochastic steady state, two-parameter perturbation provides a satisfactory solution to these shortcomings.

Second, the paper contributes to a literature proposing risk-corrected approximations of DSGE models around the stochastic steady state. Examples include Collard and Juillard (2001), Coeurdacier, Rey and Winant (2011), Juillard (2011), de Groot (2013), Hausmann-Guil (2022), and Andreasen et al. (2024). Essentially, what these papers propose are different versions of Taylor-projection (Levintal, 2018), where the unknown policy rules are replaced with polynomials, and —as in standard perturbation— the derivatives of the equilibrium conditions with respect to the states are set to zero, which provides enough equations to solve for the coefficients of the polynomials. Since this is done while keeping risk strictly positive (achieved in these papers through a second-order approximation of expectations), the coefficients already capture risk effects even at first order.

Nevertheless, these methods suffer from a scalability problem that limits their applicability for the models I consider in this paper. Unlike with standard perturbation, the system of equations that pins down the coefficients is always nonlinear, thus requiring Newton methods to solve it. Except for very small-scale models, this quickly becomes an issue because Newton methods are very sensitive to the initial guess and may fail to converge. Worse still, the resulting system of equations is such that coefficients from lower and higher-order terms appear in every equation, meaning that one must solve for all coefficients simultaneously. Since their number grows exponentially with the order of the approximation, even a second-order solution of a medium-scale model involves solving a nonlinear system with hundreds or thousands of unknowns. While applications like FernÃ¡ndez-Villaverde and Levintal (2018) and Andreasen et al. (2024) circumvent this by using the solution from standard perturbation as the initial guess, this approach is not feasible when dealing with portfolio-choice models or small open economy models without closing devices.[10]

[9] See Jin and Judd (2002), Schmitt-GrohÃ© and Uribe (2004), Andreasen (2012), and Levintal (2017), among others.

[10] Another concern with the linear solutions obtained with these methods is accuracy. Both den Haan, Kobielarz and Rendahl (2015) and de Groot, Durdu and Mendoza (2023) conduct accuracy tests of the solution proposed by Coeurdacier, Rey and Winant (2011) and find rather disappointing results. While more refined approaches like Andreasen et al. (2024) can fix this issue, they require the use of higher-order polynomials, which again conflicts

Third, the perturbation technique employed to solve incomplete-market models relate this paper to Mertens and Judd (2018). To my knowledge, this is the only other paper in the DSGE literature that introduces auxiliary modifications in the equilibrium conditions to address an indeterminacy problem. Specifically, they use a one-parameter perturbation approach to solve a Heterogeneous-agent model a lÃ Krusell-Smith where a factor $(1-\sigma)$ scales a penalty function that ensures all households hold the same amount of capital in the auxiliary deterministic model. My contribution, however, differs from theirs in several key dimensions. First, the focus of Mertens and Judd (2018) is on a completely different topic (exploiting symmetry to handle models with a large number of heterogeneous agents), and they do not explore how the technique could be applied to the models I study here.[11] Second, the idea of introducing modifications and/or auxiliary parameters to solve the model around a point other than the deterministic steady state, and the resulting stochastic steady-state algorithm—critical for solving portfolio-choice models—is entirely new. Third, the introduction of a second perturbation parameter can be a game changer from a practical standpoint, as it allows practitioners to implement the method with user-friendly DSGE software like Dynare.

Applications. First, the main applications of this paper contribute to the international finance literature that builds general equilibrium models to account for the international diversification puzzle and, more generally, for the determinants of gross external positions. Seminal contributions include Baxter and Jermann (1997), Obstfeld and Rogoff (2000), Engel and Matsumoto (2009), Heathcote and Perry (2013), and Coeurdacier and Gourinchas (2016).[12] Apart from the obvious connection to the latter, the present work strongly relates to Heathcote and Perry (2013) and Khalil (2019) in generating a strong link between trade and financial integration, and to Coeurdacier, Kollmann and Martin (2010) and Sauzet (2022a) in delivering quantitative predictions for the dynamics of capital flows. To my knowledge, this is the first study delivering a solution for a model with two (potentially asymmetric) countries, two goods, four assets, and incomplete markets with implications for gross capital flows.

The paper also contributes to the vast literature that, following Schmitt-GrohÃ© and Uribe (2003), closes small open-economy models by introducing technical modifications solely for the purpose of generating stable dynamics around a well-defined deterministic steady state. The problem with this approach is twofold. First, the *ad hoc* modifications prevent researchers from studying the determinants of long-run imbalances in small open economies, as the steady state of net foreign assets becomes a direct function of the modification involved. Second, these extra assumptions are not innocuous for the numerical results (Seoane, 2015; de Groot, Durdu and Mendoza, 2023). With two-parameter perturbation, both problems are gone: now we can view these modifications as useful auxiliary devices to be shut down at the model of interest.[13]

with the scalability problem.

[11] In their Section 6 (on extending their methods), they speculate about using penalty functions to deal with portfolio-choice problems but, surprisingly, do not consider scaling the penalty parameter with a factor $(1-\sigma)$.

[12] See Coeurdacier and Rey (2013) for a thorough review of the literature on the equity home bias.

[13] To illustrate this, I use both Uzawa preferences and portfolio adjustment costs as auxiliary devices to solve a small open-economy model with interest rate shocks calibrated to match aggregate Mexican data. Regardless of

The ability of two-parameter perturbation to handle DSGE models with *ex-ante* agent heterogeneity across various settings connects this paper to other strands of literature beyond international economics. First, the method can be applied to infinite-horizon models in finance that, following the seminal contribution by Dumas (1989), combine agent heterogeneity and market incompleteness to account for a variety of asset pricing and business cycle phenomena.[14] Second, two-parameter perturbation can solve DSGE models with patient and impatient agents (or other types of *ex-ante* heterogeneity) without relying on *ad hoc* borrowing constraints that are always binding for the impatient agent.[15] Third, the method also applies to DSGE models where financial intermediaries must make portfolio decisions, as in the contributions of Gertler, Kiyotaki, and Queralto (2011) and Itskhoki and Mukhin (2021).

Finally, by solving models around a point much closer to where the model dynamics actually take place, two-parameter perturbation can handle macro-finance models where the combination of strong volatility and high levels of risk aversion makes the analysis of these models challenging with standard local methods. Recent examples along these lines include the applications by FernÃ¡ndez-Villaverde and Levintal (2018), and Pohl, Schmedders, and Wilms (2018).

3 Two-parameter perturbation: a simple example

To introduce two-parameter perturbation in an accessible way, I start by solving the same example used by Devereux and Sutherland (2011, hereafter DS) to explain their zero-order portfolio method. The example consists of an infinite-horizon model with two countries, stochastic endowments of a single consumption good, and two internationally traded assets. Agents from both countries must therefore make non-trivial portfolio choices. Compared to the original exposition, here I use a different notation (the same as in their appendix C) that uses holdings of international assets as state variables instead of their real expenditures, as it simplifies a bit the solution procedure. I also slightly extend the original model by allowing for persistent income shocks, and introducing a global shock to the capital income share that helps to illustrate a point about portfolio dynamics.

3.1 Model description

Time is discrete and infinite. There are two symmetric countries, Home (H) and Foreign (F), each populated by a representative agent. When necessary, I distinguish Foreign's variables with an asterisk.

the modification employed, I find that two-parameter perturbation delivers well-defined ergodic distributions of net foreign assets, with average log Euler errors below -5 using a third-order approximation, and well below -6 using a fifth-order one. Since the model is essentially the same as the one studied by de Groot, Durdu and Mendoza (2023), this indicates that two-parameter perturbation resolves all the concerns raised by these authors regarding previous local methods.

[14]Recent examples in this field include Chabakauri (2013), Brunnermeier and Sannikov (2014), GÃ¢rleanu and Panageas (2015), and Drechsler, Savov, and Schnabl (2018).

[15]See, among others, Iacoviello (2005), Monacelli (2009), and Eggertsson and Krugman (2012).

Preferences. Home agents maximize the following utility function:

$$\mathbb{E}_t\left[\sum_{t=0}^{\infty}\theta_t\frac{C_t^{1-\gamma}-1}{1-\gamma}\right],$$

where C_t is consumption, and the Uzawa discount factor θ_t evolves over time according to

$$\begin{aligned}\theta_0 &= 1,\\ \theta_{t+1} &= \theta_t\beta\left(\widetilde{C}_t\right)^{-\kappa},\end{aligned}$$

where $\widetilde{C}_t$ denotes Home average consumption, and $\beta \in (0,1)$ and $\kappa \geq 0$ are parameters. When $\kappa = 0$, we have standard exogenous discounting. In this case, the DS method cannot be directly applied because the deterministic steady state of the non-portfolio variables (e.g., wealth) is not uniquely defined. However, the model still admits a unique solution because a standard precautionary-saving motive induces equilibrium stability. As shown below, Two-parameter perturbation will take care of this case too.

Endowments. At each period, Home agents receive an endowment Y_t of a single perishable good that can be decomposed as the weighted sum of two components, capital income $e^{y_{K,t}}$ and labor income $e^{y_{L,t}}$, as follows:

$$Y_t = \delta_t e^{y_{K,t}} + (1-\delta_t)e^{y_{L,t}}. \tag{3.1}$$

Here, the share of capital income in total output $\delta_t \in (0,1)$ is common to both countries and has law of motion:

$$\left(\delta_{t+1}-\overline{\delta}\right) = \rho_\delta\left(\delta_t-\overline{\delta}\right), \tag{3.2}$$

where $\overline{\delta} \in (0,1)$ is the weighting parameter. Absent unexpected shocks, δ_t is expected to converge to $\overline{\delta}$ in the long-run. In turn, the logs of the income components follow simple AR(1) processes:

$$y_{K,t+1} = \rho_y y_{K,t} + \mu_K u_{K,t+1}, \tag{3.3}$$

$$y_{L,t+1} = \rho_y y_{L,t} + \mu_L u_{L,t+1}, \tag{3.4}$$

where $u_{K,t+1}$ and $u_{L,t+1}$ are i.i.d. standard normal innovations, so that the conditional standard deviations of $y_{K,t+1}$ and $y_{L,t+1}$ are $\mu_K > 0$ and $\mu_L > 0$, respectively.[16] The reciprocal equations in Foreign are:

$$Y_t^* = \delta_t e^{y_{K,t}^*} + (1-\delta_t)e^{y_{L,t}^*}, \tag{3.5}$$

[16] In the original DS model $\rho_y = 0$. Since I will solve for the model dynamics and perform accuracy tests, it is convenient to be a bit more general here.

and

$$
\begin{aligned}
y^*_{K,t+1} &= \rho_y y^*_{K,t} + \mu_K u^*_{K,t+1}, && (3.6)\\
y^*_{L,t+1} &= \rho_y y^*_{L,t} + \mu_L u^*_{L,t+1}. && (3.7)
\end{aligned}
$$

Within each period, the covariance matrix of the vector $\mathbf{u_t} = [u_{K,t+1}, u_{L,t+1}, u^*_{K,t+1}, u^*_{L,t+1}]$ is:

$$
\mathrm{Cov}[\mathbf{u_t}] = \begin{bmatrix} 1 & \mu_{KL} & 0 & 0 \\ \mu_{KL} & 1 & 0 & 0 \\ 0 & 0 & 1 & \mu_{KL} \\ 0 & 0 & \mu_{KL} & 1 \end{bmatrix},
$$

where μ_{KL} is the correlation parameter between capital and labor shocks.

Asset trading, budget constraints, and market-clearing. At each period, agents can trade one-period equity claims on the capital income from the two countries. Let $a_{H,t}$ and $a_{F,t}$ denote the start-of-period holdings of home and foreign assets, and $z_{H,t}$ and $z_{F,t}$ their respective end-of-period prices. The budget constraint faced by Home agents is:

$$
C_t + z_{H,t} a_{H,t+1} + z_{F,t} a_{F,t+1} = Y_t + a_{H,t} \delta_t e^{y_{K,t}} + a_{F,t} \delta_t e^{y^*_{K,t}}, \qquad (3.8)
$$

and the corresponding one for Foreign agents is:

$$
C^*_t + z_{H,t} a^*_{H,t+1} + z_{F,t} a^*_{F,t+1} = Y^*_t + a^*_{H,t} \delta_t e^{y_{K,t}} + a^*_{F,t} \delta_t e^{y^*_{K,t}}, \qquad (3.9)
$$

The market-clearing conditions are as follows. The good's market clears:

$$
C_t + C^*_t = Y_t + Y^*_t,
$$

and the two assets are in zero net supply:

$$
\begin{aligned}
a_{H,t} + a^*_{H,t} &= 0, && (3.10)\\
a_{F,t} + a^*_{F,t} &= 0. && (3.11)
\end{aligned}
$$

Equilibrium conditions. The first-order conditions of inter-temporal optimization for Home agents consist of two Euler equations (one for each asset):

$$
\begin{aligned}
\frac{z_{H,t}}{(C_t)^\gamma} &= \beta (C_t)^{-\kappa} \mathbb{E}_t \left[\frac{\delta_{t+1} e^{y_{K,t+1}}}{(C_{t+1})^\gamma} \right], && (3.12)\\
\frac{z_{F,t}}{(C_t)^\gamma} &= \beta (C_t)^{-\kappa} \mathbb{E}_t \left[\frac{\delta_{t+1} e^{y^*_{K,t+1}}}{(C_{t+1})^\gamma} \right], && (3.13)
\end{aligned}
$$

where I have already imposed that, in equilibrium, $\widetilde{C}_t = C_t$. Similarly, the two Euler equations for Foreign agents are:

$$\frac{z_{H,t}}{(C_t^*)^\gamma} = \beta (C_t^*)^{-\kappa} \mathbb{E}_t \left[\frac{\delta_{t+1} e^{y_{K,t+1}}}{\left(C_{t+1}^*\right)^\gamma} \right], \tag{3.14}$$

$$\frac{z_{F,t}}{(C_t^*)^\gamma} = \beta (C_t^*)^{-\kappa} \mathbb{E}_t \left[\frac{\delta_{t+1} e^{y_{K,t+1}^*}}{\left(C_{t+1}^*\right)^\gamma} \right]. \tag{3.15}$$

We can exploit equations (3.10) and (3.11) to write the equilibrium conditions in terms of 2 endogenous states only (Home's holdings of assets $a_{H,t}$ and $a_{F,t}$), by substituting $a_{H,t}^* = -a_{H,t}$ and $a_{F,t}^* = -a_{F,t}$ into (3.9). The 5 exogenous states are capital log-income components $y_{K,t}$ and $y_{K,t}^*$, labor log-income components $y_{L,t}$ and $y_{L,t}^*$, and the share δ_t. The 6 control variables are Home and Foreign consumptions C_t and C_t^*, endowments Y_t and Y_t^*, and asset prices $z_{H,t}$ and $z_{F,t}$. The 13 equilibrium conditions that these variables must satisfy are the income processes (3.3), (3.4), (3.6) and (3.7), law of motion (3.2), endowment equations (3.1) and (3.5), budget constraints (3.8) and (3.9), and the 4 Euler equations (3.12)-(3.15).

3.2 Failure of standard perturbation

The model of interest just described cannot be solved by standard perturbation because in the deterministic version of the model optimal portfolios are not uniquely defined. To see why, we can follow DS and combine equations (3.12)-(3.13) and (3.14)-(3.15) to derive the Portfolio Euler equations for Home and Foreign agents:

$$\mathbb{E}_t \left[\left(R_{H,t+1} - R_{F,t+1}\right) \frac{1}{\left(C_{t+1}\right)^\gamma} \right] = 0, \tag{3.16}$$

$$\mathbb{E}_t \left[\left(R_{H,t+1} - R_{F,t+1}\right) \frac{1}{\left(C_{t+1}^*\right)^\gamma} \right] = 0, \tag{3.17}$$

where $R_{H,t+1} = \delta_{t+1} \frac{e^{y_{K,t+1}}}{z_{H,t}}$ and $R_{F,t+1} = \delta_{t+1} \frac{e^{y_{K,t+1}^*}}{z_{F,t}}$ are the future gross returns of home and foreign assets.

An alternative (but equivalent) representation of the system of equilibrium conditions consists of replacing equations (3.13) and (3.15) by (3.16) and (3.17). In the deterministic economy the expectation operator goes away, and both equations convey the same information: in equilibrium, the two assets deliver the same expected returns ($R_{H,t+1} = R_{F,t+1}$). Since one of the equations is redundant, we lack enough independent equations to characterize a unique solution to the model, which prevents the use of standard perturbation.[17] Intuitively, in the model of interest the two assets only differ in their risk characteristics; absent this risk, the assets become perfect substitutes, thus making the portfolio decision trivial.

Regarding the deterministic steady state (DSS), endogenous discounting ($\kappa > 0$) allows us

[17] Another way to see this is to work directly with the linearized Euler equations (3.12)-(3.15), and verify that one of them can be written as a linear combination of the others (e.g., (3.12)-(3.13)+(3.15)=(3.14)).

to pin down symmetric consumptions $C_d = C_d^* = 1$ (where the subscript d denotes the DSS), asset prices $z_{H,d} = z_{F,d} = \beta\bar{\delta}$, and a zero net foreign asset position $a_{H,d} + a_{F,d} = 0$, but not the gross asset positions $a_{H,d}$ and $a_{F,d}$. If instead we use exogenous discounting ($\kappa = 0$) the problem worsens because, in a deterministic World, $a_{H,d} + a_{F,d} = 0$ no longer needs to hold. This means that even if one imposes a trivial portfolio allocation (e.g., $a_{F,t+1} = 0$ at all times), and drops the 2 portfolio Euler equations (3.16) and (3.17), the DSS of the resulting model is still indeterminate, and the first-order solution has a unit root (Schmitt-GrohÃ© and Uribe, 2003).

3.3 Auxiliary perturbation model

To make progress, consider the following auxiliary model in terms of two dimensionless perturbation parameters, ε and σ. As in standard perturbation, the parameter σ scales future innovations to the log-income processes (3.3), (3.4), (3.6) and (3.7), so that when $\sigma = 0$ we obtain the deterministic version of the model, and when $\sigma = 1$ we recover the stochastic model that we care about.

The core idea of this paper is to add auxiliary modifications to the equilibrium conditions of the original model to resolve the indeterminacy problem in the deterministic economy, and then use perturbation to remove their effect on the approximate solution. We can accomplish this task by interacting the modifications with a differentiable function $f(\varepsilon)$ that is decreasing in the interval $[0,1]$, and satisfies $f(0) = 1$ and $f(1) = 0$. Hence, the modifications are fully active at the approximation point $\varepsilon = 0$, and the model of interest is recovered by evaluating the approximate solution at $\varepsilon = 1$.

In the current example, a simple modification that does the job is the following. The new equations replacing the Home Euler equations (3.12) and (3.13) are:

$$\frac{z_{H,t}}{(C_t)^\gamma}\left[1 + \left(1-\varepsilon^2\right)\psi\left(a_{H,t+1} - \bar{a}\right)\right] = \beta\left(C_t\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}e^{y_{K,t+1}}}{\left(C_{t+1}\right)^\gamma}\right], \tag{3.18}$$

$$\frac{z_{F,t}}{(C_t)^\gamma}\left[1 + \left(1-\varepsilon^2\right)\psi\left(a_{F,t+1} + \bar{a}\right)\right] = \beta\left(C_t\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}e^{y^*_{K,t+1}}}{\left(C_{t+1}\right)^\gamma}\right], \tag{3.19}$$

where $\psi > 0$ and $\bar{a}$ are two auxiliary parameters. The proposed function $f(\varepsilon) = \left(1-\varepsilon^2\right)$ ensures that perturbation corrections to ε in the Taylor series of the policy functions behave in the same way as corrections to σ (no corrections at first order, only a constant term at second order, etc.). This property is not required, but as will become clear later, it helps speed up the stochastic steady-state algorithm without compromising accuracy. Similarly, the new equations replacing the Foreign Euler equations (3.12) and (3.13) are:

$$\frac{z_{H,t}}{(C^*_t)^\gamma}\left[1 - \left(1-\varepsilon^2\right)\psi\left(a_{H,t+1} - \bar{a}\right)\right] = \beta\left(C^*_t\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}e^{y_{K,t+1}}}{\left(C^*_{t+1}\right)^\gamma}\right], \tag{3.20}$$

$$\frac{z_{F,t}}{(C^*_t)^\gamma}\left[1 - \left(1-\varepsilon^2\right)\psi\left(a_{F,t+1} + \bar{a}\right)\right] = \beta\left(C^*_t\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}e^{y^*_{K,t+1}}}{\left(C^*_{t+1}\right)^\gamma}\right]. \tag{3.21}$$

When $\varepsilon = 0$, these modifications make sure that the four equations are independent of each other, and that they deliver a stable DSS for all the endogenous variables. In particular, in the symmetric equilibrium we have $a_{H,d} = \bar{a}$ and $a_{F,d} = -\bar{a}$. Note that this is true even with exogenous discounting ($\kappa = 0$), so the auxiliary model covers this case too. When $\varepsilon = 1$, the modifications receive zero weight and we recover the Euler equations from the model of interest (3.12)-(3.15). Putting it all together, the auxiliary perturbation model is effectively nesting a family of models such that $(\varepsilon, \sigma) = (0, 0)$ is a tractable deterministic model around which we can build our approximation, and $(\varepsilon, \sigma) = (1, 1)$ corresponds to the model of interest that we aim to solve.

Interpretation of the auxiliary modifications. The reader familiar with the open economy macro literature might recognize these modifications as those that would have arisen if the original model had incorporated Portfolio Adjustment Costs (PAC), a well-known device to induce stationarity in small open economies.[18] In this model, the budget constraints of the two countries would include quadratic costs of holding stocks in a quantity different from some long-run exogenous levels $\bar{a}$ and $-\bar{a}$, to be paid to a national agency that simply returns them to the households as lump-sum transfers T_t and T_t^*. Home's budget constraint would be:

$$C_t + z_{H,t}a_{H,t+1} + z_{F,t}a_{F,t+1} + \left(1 - \varepsilon^2\right) \frac{\psi}{2} \left[z_{H,t} \left(a_{H,t+1} - \bar{a}\right)^2 + z_{F,t} \left(a_{F,t+1} + \bar{a}\right)^2\right] = Y_t + a_{H,t}\delta_t e^{y_{K,t}} + a_{F,t}\delta_t e^{y^*_{K,t}} + T_t.$$

The equilibrium conditions of such a model are identical to the auxiliary model used here. Intuitively, PAC generates stable asset dynamics because the effective rate of return of each asset decreases when next-period holdings are above their long-run level, and increases when assets holdings are below their long-run level. Again, this stabilization mechanism is independent from but complementary to the Uzawa discount factor, so it operates even if $\kappa = 0$.

While it can be helpful for intuition purposes to motivate the modifications as coming from the original agent's problem, this is by no means necessary. From a mathematical point of view, all that is required from the auxiliary equilibrium conditions is that they form a consistent set of equations amenable to perturbation with a well-defined DSS. From the Economics point of view, only the equations evaluated at $(\varepsilon, \sigma) = (1, 1)$ need to be interpreted as first-order conditions encoding optimal behavior. This is a convenient feature, because it allows researchers to use as auxiliary modifications the various *ad hoc* devices that previous literature routinely employed to close open-economy models without having to justify them with proper microfoundations. As long as you interact them with $f(\varepsilon)$, perturbation will remove their effect from the equilibrium conditions.

[18] See for example Schmitt-GrohÃ© and Uribe (2003).

3.4 Implementation

The reader might wonder why the need for two perturbation parameters. Couldn't one accomplish the same result with a single perturbation parameter (say, σ) controlling both future innovations and auxiliary modifications? The theoretical answer is yes: the two procedures are mathematically equivalent, as perturbation yields the exact same approximations once $\varepsilon = \sigma$ is imposed in the approximated decision rules.[19] However, current algorithms implemented by popular DSGE software (e.g., Dynare) restrict σ to scale innovations only, which optimizes performance in standard applications since one can tell in advance when certain coefficients will be zero.[20] A two-parameter approach overcomes this limitation because, as shown in section 5.2, we can exploit the variable-parameter duality of perturbation objects by treating the perturbation parameter ε as an exogenous state variable that is constant over time.

To exploit this duality, simply rewrite ε_t instead of ε in equations (3.18)-(3.21), and add to the set of equilibrium conditions the following law of motion:

$$\varepsilon_{t+1} = \varepsilon_t \tag{3.22}$$

From the theoretical side, having a unit root is valid because ε_t is a purely exogenous state that only depends on itself. From the practical side, popular solution packages such as Dynare and the MATLAB function gx_hx.m by Schmitt-GrohÃ© and Uribe (2004) accept the equation $\varepsilon_{t+1} = \rho_\varepsilon \varepsilon_t$ with $\rho = 1$ without problems (the latter once we change the default threshold of unit eigenvalues).[21] Alternatively, we can set a value $\rho_\varepsilon < 1$ before executing the solution algorithm, and impose $\rho_\varepsilon = 1$ in the solution matrices right after executing the solver. In practice, this procedure delivers numerical results almost identical to the unit root case if ρ_ε is sufficiently close to one.

At this point, we already have a set of variables and equilibrium conditions that match the canonical form (FernÃ¡ndez-Villaverde, Rubio-RamÃrez, and Schorfheide, 2016), so we can solve the auxiliary perturbation model with any standard DSGE software. Specifically, we have 2 endogenous states ($a_{H,t}$ and $a_{F,t}$), 6 exogenous states (the 4 log-income components $y_{K,t}$, $y^*_{K,t}$, $y_{L,t}$ and $y^*_{L,t}$, the share δ_t, and the perturbation variable ε_t), and the same 6 control variables as before (consumptions C_t and C^*_t, endowments Y_t and Y^*_t, and prices $z_{H,t}$ and $z_{F,t}$). The 14 equilibrium conditions that these variables must satisfy are the log-income processes (3.3), (3.4), (3.6) and (3.7) (all with future innovations scaled by σ), law of motion (3.2), endowment equations (3.1) and (3.5), budget constraints (3.8) and (3.9), the 4 auxiliary Euler equations (3.18)-(3.21), and the law of motion (3.22). In turn, it is easy to verify that the DSS of the endogenous variables in the auxiliary deterministic model is given by symmetric consumptions

[19] To see this, note that the classical procedure for deriving the multivariate Taylor expansion (by expanding over a uni-variate auxiliary function $g(t)$ and using the chain rule) delivers the same derivatives regardless of whether we start with $\varepsilon(t) = t\varepsilon$ and impose $\varepsilon = \sigma$ afterward, or impose $\varepsilon(t) = t\sigma$ from the start. As a practical illustration, the online repository provides codes to solve the SOE model of Appendix D.3 both ways, and they both deliver the same results reported in the appendix.

[20] See among others Schmitt-GrohÃ© and Uribe (2004), Andreasen (2012), and Levintal (2017).

[21] In the online repository, the function gx_hx.m already has a modified default threshold for eigenvalues that is larger than (but extremely close to) one. I thank Oren Levintal for this suggestion.

$C_d = C_d^* = 1$, asset prices $z_{H,d} = z_{F,d} = \beta\overline{\delta}$, and gross asset positions $a_{H,d} = \bar{a}$ and $a_{F,d} = -\bar{a}$.

The perturbation algorithm implemented by the DSGE software computes the derivatives of the decision rules (or policy functions) with respect to the states, and uses this input to approximate the rules by a Taylor series around the DSS. Once these tasks are completed, we must evaluate the resulting functions at $(\varepsilon, \sigma) = (1, 1)$ to obtain a valid approximate solution to the model of interest, which can be used to generate standard outputs from DSGE models (e.g., moments from simulations, impulse responses). If the solution package sets $\sigma = 1$ by default (as is often the case), we only need to ensure that $\varepsilon_t = 1$ is imposed in the vector of state variables.

3.5 Calibration

While the DS model is too stylized to allow for serious quantitative work, I still provide numerical results to illustrate some key properties of the solution method. The chosen parameterization borrows mostly from Table 2, which reports the calibration of the infinite-horizon version of the Coeurdacier and Gourinchas (2016) model. Specifically, I set the same values for the parameters of the utility function ($\beta = 0.96$, $\gamma = 2$, $\kappa = 0.007$), and for the income processes ($\rho_y = 0.51$ and $\mu_K = \mu_L = 0.018$). For the remaining parameters in the model of interest, I follow Coeurdacier and Gourinchas (2016, Table 1) and set $\overline{\delta} = 0.167$ to match the OECD non-naive average share of financial income in output, and choose a sizable negative correlation between capital and labor shocks of $\mu_{KL} = -0.5$ to generate home bias in international assets.

Regarding the auxiliary parameters, I implement the algorithm described in the next subsection to find the value $\bar{a} = 1.464$ such that this approximation point coincides with the stochastic steady state of the model of interest. When analyzing the properties of the stochastic steady state, I set $\rho_y = 0$ to match the original specification and theoretical formulas of DS, which delivers $\bar{a} = 0.747$ instead. Finally, I fix a very small value of the auxiliary PAC parameter $\psi = 5 \times 10^{-6}$ to ensure convergence to the model of interest (since ψ is the parameter that is effectively being perturbed).[22]

3.6 The stochastic steady state

The auxiliary perturbation model allows the researcher to build approximate solutions around a large subset of points in the endogenous state-space, because $\bar{a}$ is a free auxiliary parameter with no role in the model of interest. Here, I exploit this property to introduce a simple algorithm that finds the value of $\bar{a}$ such that the pair $a_{H,s} = \bar{a}$ and $a_{F,s} = -\bar{a}$ correspond to the stochastic steady state (SSS) of the endogenous states in the model of interest.[23] Following Coeurdacier, Rey and Winant (2011), this is the point where agents choose to stay at a given date if the realization of shocks is zero at this date (here $y_{K,t} = y_{K,t}^* = 0$ and $y_{L,t} = y_{L,t}^* = 0$), but if they expect future risk. As shown below, the SSS portfolios obtained with the algorithm are

[22] See section 4.2 for a discussion of how to set this type of parameters.

[23] See section 5.3 for a formal description of the algorithm general enough to cover many different settings.

quite related to the zero-order portfolios advocated by DS and Tille and van Wincoop (2010), to the point that, depending on the application, the two concepts may deliver identical answers.[24]

3.6.1 Deriving the stochastic steady state condition

The key idea behind the algorithm is to impose enough conditions on the approximate decision rules such that the DSS of the endogenous states in the auxiliary deterministic model coincides with the SSS implied by the model of interest: $a_{H,d} = a_{H,s}$ and $a_{F,d} = a_{F,s}$. In this example, this task is simplified because symmetry between countries delivers a zero net foreign asset position at the steady state (both deterministic and stochastic), which implies $a_{F,d} = -a_{H,d}$ and $a_{F,s} = -a_{H,s}$. Thus, we can forget about foreign assets and focus exclusively on holdings of the home asset. Since $a_{H,d} = \bar{a}$ in the auxiliary economy, our job reduces to find the value of $\bar{a}$ that satisfies $a_{H,s} = \bar{a}$.

To capture risk effects, inherent to the concept of the SSS, we have to work with at least a second-order decision rule for $a_{H,t+1}$. Treating both ε and σ as parameters, this function can be expressed in a compact way as follows:

$$a_{H,t+1} = \bar{a} + h\left(\widehat{\mathbf{x}}_{\mathbf{t}}\right) + \frac{1}{2}\left(h_{\varepsilon\varepsilon}\varepsilon^2 + h_{\sigma\sigma}\sigma^2\right), \tag{3.23}$$

where h denotes a second-order polynomial that depends on the vector $\widehat{\mathbf{x}}_{\mathbf{t}}$ (DSS deviations of the states), and $h_{\varepsilon\varepsilon}$ and $h_{\sigma\sigma}$ are the only two derivatives with respect to ε and σ that can be non-zero at this order of approximation. In line with standard perturbation, $h_{\sigma\sigma}$ captures the correction for risk that breaks certainty equivalence, and the new coefficient $h_{\varepsilon\varepsilon}$ captures the correction that removes the effect of the auxiliary modifications. Also, note that the polynomial satisfies $h(0) = 0$.

Section 5.3 defines the SSS of the endogenous states as the fixed-point of the policy functions when these are evaluated at the model of interest $(\varepsilon, \sigma) = (1, 1)$. Since we are looking for a SSS of the state variables that coincides with their DSS in the auxiliary economy, substitute $\widehat{\mathbf{x}}_{\mathbf{t}} = 0$ (so that $a_{H,t} = \bar{a}$) and $(\varepsilon, \sigma) = (1, 1)$ in (3.23) to obtain:

$$a_{H,t+1} = \bar{a} + \frac{1}{2}\left(h_{\varepsilon\varepsilon} + h_{\sigma\sigma}\right). \tag{3.24}$$

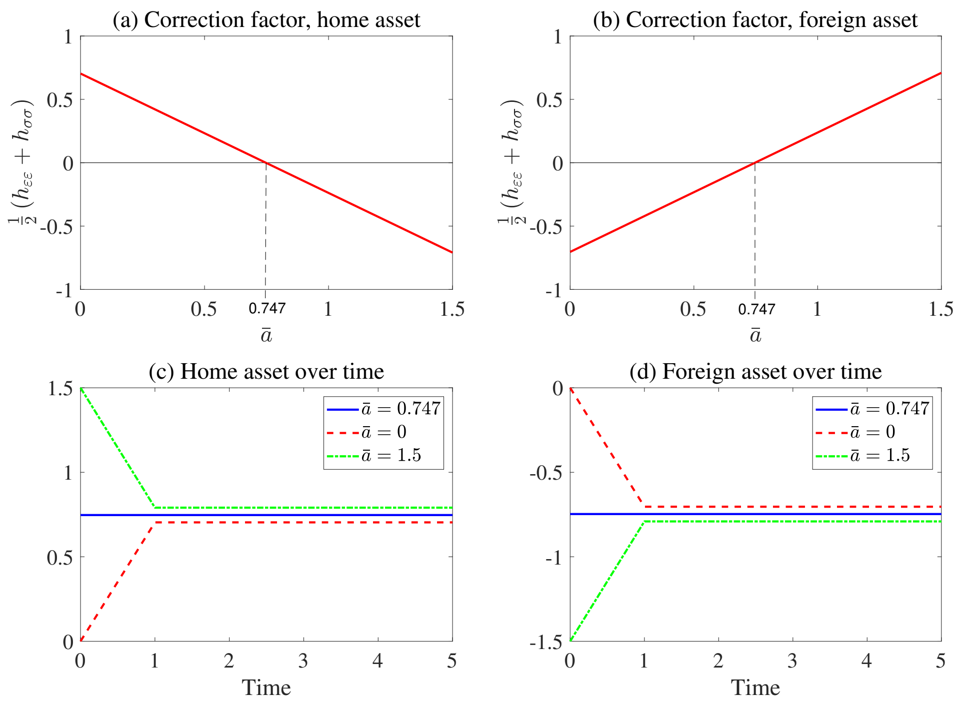
The only way for this equation to return $a_{H,t+1} = \bar{a}$ is that the sum of the two perturbation coefficients equals zero. The SSS condition that we are looking for is therefore:

$$h_{\varepsilon\varepsilon} + h_{\sigma\sigma} = 0. \tag{3.25}$$

Since both coefficients are functions of the parameters of the auxiliary model, the SSS condition pins down the value for $\bar{a}$ such that $a_{H,d} = a_{H,s} = \bar{a}$, which, as explained in section 5.3, can be found numerically with a standard nonlinear solver,

[24] See Devereux and Sutherland (2009) and Rabitsch, Stepanchuk, and Tsyrennikov (2015) for earlier attempts to link the two solution concepts, and especially the latter for a discussion of the difficulties encountered by previous methodologies.

Figure 1: The stochastic steady state condition in the DS model



3.6.2 Graphical illustration

Figure 1 provides an illustration of the SSS condition. In Panel (a), I solve the auxiliary perturbation model for different values of the location parameter $\bar{a}$, ranging from $\bar{a} = 0$ to $\bar{a} = 1.5$, and plot the correction factor $\frac{1}{2}(h_{\varepsilon\varepsilon} + h_{\sigma\sigma})$ from equation (3.24). The results (blue solid line) show a linear negative relationship between the choice of $\bar{a}$ and the correction factor. Since ψ is small, the coefficient $h_{\varepsilon\varepsilon}$ is also small, meaning that this relationship is driven by changes in the risk-correction coefficient $h_{\sigma\sigma}$. At $\bar{a} = 0.747$, the correction factor becomes zero, indicating that the SSS condition (3.25) is satisfied at this approximation point. Panel (b) plots the correction factor from the policy function of the foreign asset against the same range of values of $\bar{a}$. The behavior in this case is the mirror image of Panel (a): a positive linear relationship that also becomes zero at $\bar{a} = 0.747$.

Panels (c) and (d) plot sequences of $a_{H,t}$ and $a_{F,t}$, generated by a second-order solution to the model of interest, for three choices of $\bar{a}$ (blue solid line for $\bar{a} = 0.747$, red dashed line for $\bar{a} = 0$, and green dotted line for $\bar{a} = 1.5$), all starting from their respective DSS. The results confirm that only the approximation point satisfying the SSS condition is also a SSS of the associated policy functions for the endogenous states. In the other two cases, correction factors different from zero imply a SSS for $a_{H,t}$ and $a_{F,t}$ different from the DSS.[25]

The economic intuition behind these results is as follows: since the share of labor income over the total endowment $1 - \bar{\delta}$ is large, agents from both countries have incentives to build

[25] Convergence to the SSS only takes one period because of the strong symmetries imposed in the auxiliary model.

gross asset positions that provide insurance against labor income shocks. As these shocks are negatively correlated with capital income shocks, the return differential $R_{H,t} - R_{F,t}$ is negatively correlated with labor income shocks from Home and positively correlated with labor income shocks from Foreign. It follows that, for each country, a portfolio biased toward their own equity is a good hedge against labor income risk, as it tends to deliver positive transfers when their labor income is relatively low. However, this bias cannot be too large, or the resulting transfers could undermine their risk-sharing purpose.

Armed with this intuition, the economic interpretation of Figure 1 is straightforward. When the initial Home bias is too low (e.g., $\bar{a} = 0$), the correction factors allow Home agents to choose a portfolio for the following periods that is more biased toward the home asset. As the initial Home bias increases ($\bar{a}$ goes up), smaller corrections are needed. In contrast, if the initial Home bias is too large (e.g., $\bar{a} = 1.5$), the correction factors ensure that agents choose a less biased portfolio for the next periods. At $\bar{a} = 0.747$, agents are satisfied with the initial level of Home bias and choose to maintain it, thereby fulfilling the SSS definition of Coeurdacier, Rey, and Winant (2011).

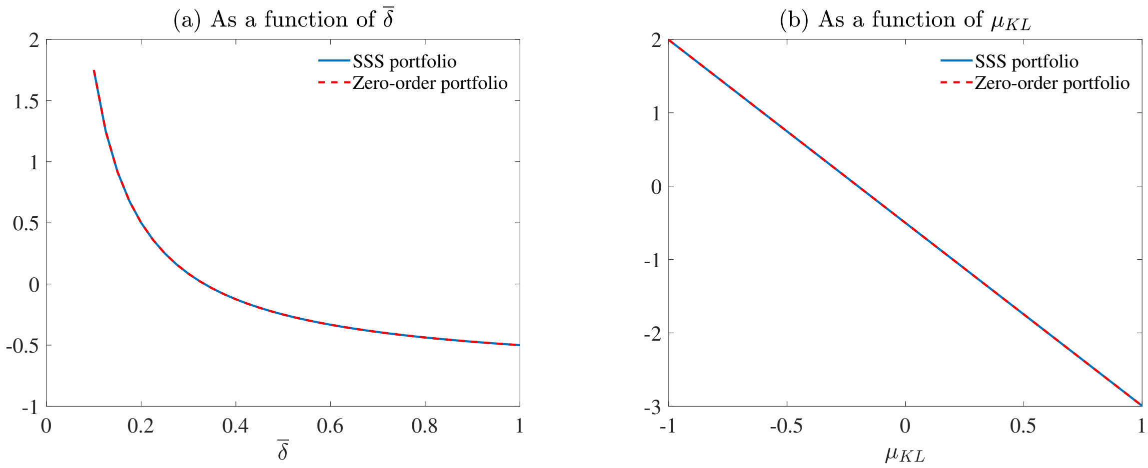
Comparison to Devereux and Sutherland (2011). In their paper, DS derive a formula for the zero-order component of real holdings of home assets $w_{H,0} = z_{H,0} a_{H,0}$ in terms of variances and covariances of the innovations. Since they use the DSS for all non-portfolio variables, $z_{H,0} = \beta\overline{\delta}$, and their formula can be used to obtain an expression for the zero-order component of nominal holdings. Using my notation:

$$a_{H,0} = -\frac{1}{2}\left(1 + \left(\frac{1-\overline{\delta}}{\overline{\delta}}\right)\mu_{KL}\frac{\mu_L}{\mu_K}\right), \tag{3.26}$$

and, by symmetry, $a_{F,0} = -a_{H,0}$. The formula captures the hedging motives highlighted above. In particular, a combination of a low $\overline{\delta}$ (high labor income share) and a sizable negative correlation μ_{KL} delivers a portfolio biased toward the home asset. Indeed, evaluating this expression at the chosen calibration gives $a_{H,0} = 0.747$. Thus, the zero-order portfolio derived by DS coincides with the SSS pinned down by equation (3.25).

In Figure 2, I explore this relationship further by performing comparative statics. Panel (a) plots the SSS (blue solid line) and the zero-order component (red dashed line) of $a_{H,t}$ for different values of the capital income share ranging from $\overline{\delta} = 0.1$ to $\overline{\delta} = 1$. The upper bound is of particular interest, as it corresponds to a scenario with complete markets where the formula (3.26) provides the exact, time-invariant solution for $a_{H,t}$ (see DS). Panel (b) reports a similar experiment, allowing the correlation between capital and labor income shocks to vary between $\mu_{KL} = -1$ and $\mu_{KL} = 1$. The results from both panels confirm that, in this model, the SSS of $a_{H,t}$ computed by solving condition (3.25) is identical to the zero-order component of DS, with numerical differences between the two approximation points being on the order of 10^{-12} or less.

Figure 2: Comparative statics of the DS model



3.6.3 What approximation order?

Since this example employs a second-order solution, the algorithm can only deliver, at best, an approximation to the true SSS portfolio. Section 5.3 generalizes the SSS condition to higher-order solutions for a broad class of models, and Appendix C shows that, under standard regularity conditions, the solution from the algorithm converges to the true SSS as the approximation order increases. In this example, the regularity conditions hold, and regardless of the function $f(\varepsilon)$ used to scale the PAC-like modifications, SSS portfolios from higher-order solutions are nearly identical to those reported here. In particular, moving to a third-order solution adds nothing in terms of risk because, with symmetric innovations, the coefficient of σ^3 in the Taylor series is zero (Andreasen, 2012). This justifies using $f(\varepsilon) = 1 - \varepsilon^2$ to scale modifications: since the correction to ε^3 is also zero with this functional form, we can obtain the same SSS with the much faster second-order solution and use it as the approximation point for a third-order solution to accurately study portfolio dynamics, as explained below.

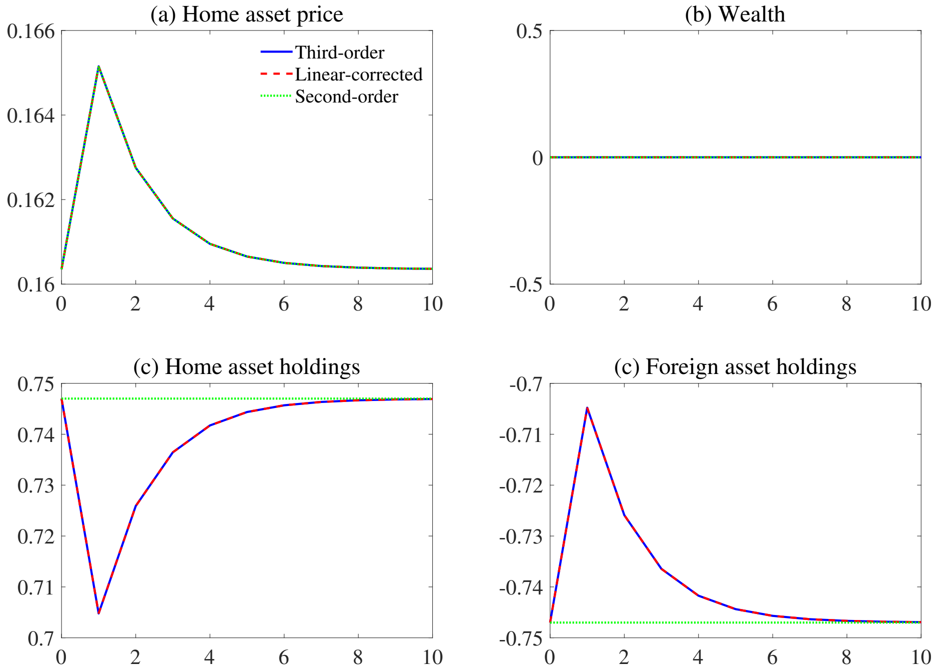
Sometimes, however, resorting to higher-order solutions may be unavoidable if we suspect that higher-order moments affect portfolios in non-trivial ways. The good news is that, since only powers of σ and ε enter in the SSS condition, we can use existing analytical results (Levintal, 2017) to choose the appropriate approximation order needed to capture the moments we care about.[26]

3.7 Portfolio dynamics

Recall that, in the model of interest, assets are only distinguishable in terms of their risk characteristics. This implies that only Taylor series of order 3 or higher are valid approximations for studying portfolio dynamics. A first-order approximation does not include any perturbation term with a coefficient different from zero and is therefore identical to that of the auxiliary

[26]If this order needs to be third or higher, I strongly recommend implementing the SSS algorithm with the perturbation package of Levintal (2017), using as a initial guess for the nonlinear solver the SSS obtained with the fast second-order solution.

Figure 3: Impulse responses to a redistributive shock in the DS model



deterministic model $(\varepsilon, \sigma) = (0, 0)$. As already shown, a second-order solution is very helpful for computing SSS portfolios, but the resulting Taylor series are still identical to that of the auxiliary deterministic model. It is only with at a third-order approximation that cubic terms with non-zero coefficients, such as $\sigma^2 (a_{H,t} - \bar{a})$ or $\varepsilon^2 y_{K,t}$, enter as new arguments in the Taylor series, which, if evaluated at $(\varepsilon, \sigma) = (1, 1)$, correct for the first-order effects of the state variables on the equilibrium ones.[27]

However, working with higher-order solutions comes with the technical difficulty of dealing with the spurious explosive paths that these approximations sometimes generate. One way to address this problem is to resort to a pruning scheme (Andreasen, FernÃ¡ndez-Villaverde, and Rubio-RamÃrez, 2018). Alternatively, nothing prevents us from computing numerical jacobians of higher-order policy functions (treating (ε, σ) as parameters), and using them to build linear-corrected rules centered at the approximation point, which might help in applications involving the estimation of the DSGE model. Appendix B provides a formal discussion of these two approaches.

Example: impulse responses for different approximations. Figure 3 reports impulse responses of selected variables following a surprise worldwide shock to the share of capital in-

[27] In this sense, two-parameter perturbation mimics the patterns emphasized by Devereux and Sutherland (2010) and Tille and van Wincoop (2010). One difference between their approaches and mine is that, while their method does not deliver portfolio dynamics at all unless one works with a third-order approximation to the model equations, the auxiliary perturbation model delivers portfolio dynamics even at first-order, though likely inaccurate ones since they lack correction terms.

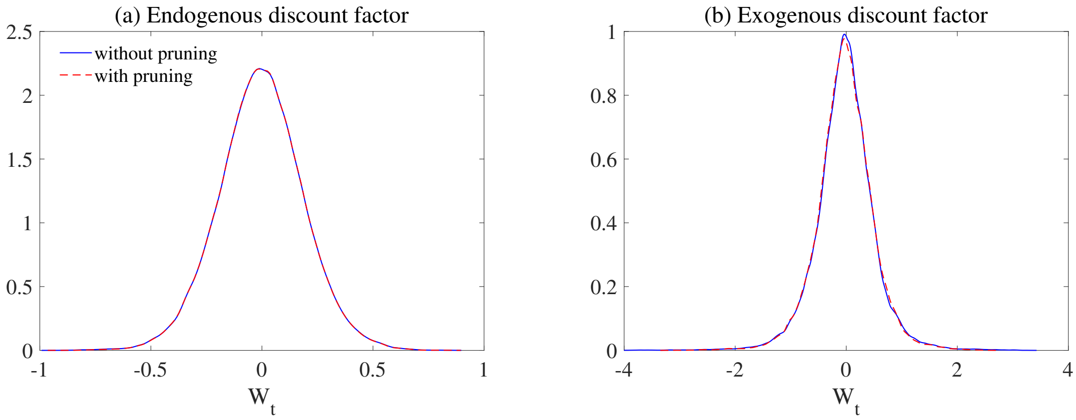
come by one percent point at $t = 1$, starting from the SSS, such that $\delta_1 = 0.177$, and afterwards δ_t follows its law of motion (with persistence $\rho_\delta = 0.5$) and converges to the long-run value $\overline{\delta} = 0.167$. All the variables are expressed in their natural units and refer to Home measures (except the asset price, which is the same in both countries). In all panels, a blue solid line indicates a third-order solution, a green dotted line indicates a second-order solution, and a red dashed line indicates a linear-corrected solution.

Consider first the top row of Figure 3. A positive and persistent shock to δ_t temporarily increases the expected cash flows of the two assets. It is therefore not surprising that in Panel (a) $z_{H,t}$ increases on impact, and then gradually returns to its long-run level. The response of the foreign asset is identical and thus omitted. Since initial capital and labor incomes are equal in both countries, this shock does not cause any income differences between them. Consequently, in Panel (b), end-of-period domestic wealth $W_t = z_{H,t}a_{H,t+1} + z_{F,t}a_{F,t+1}$ (also the net foreign asset position) remains unchanged and equal to its SSS level of zero. Note that, in these two panels, all three solutions deliver identical results.

However, the bottom row of Figure 3 tells a different story. According to a second-order solution, a shock to the capital income share should have no effect whatsoever on the gross asset position of Home agents, so in Panels (c) and (d), holdings of home and foreign assets simply stay at their SSS levels. In contrast, a third-order solution predicts international retrenchment. On impact, a positive and persistent shock to δ_t leads to a drop in $a_{H,t}$, and a simultaneous change in $a_{F,t}$ in the opposite direction but with identical magnitude, therefore preserving the net asset position at zero. As δ_t returns to its long-run level, so too do $a_{H,t}$ and $a_{F,t}$. In turn, the linear-corrected rules produce the same effects as the third-order solution, with identical impulse responses as shown in the figure.

The intuition behind the strong disagreement between solutions is as follows. A second-order solution, lacking any correction terms with non-zero coefficients proportional to the states, is dynamically equivalent to a deterministic PAC-like model where assets only differ in how far away they are from their respective location points. Since an increase in δ_t is irrelevant in this respect, agents in that model simply keep their portfolios at their DSS levels. However, the increase in δ_t changes the portfolio allocation problem in the model of interest because it rebalances the weights of capital and labor income risk in total output. Specifically, a positive and persistent shock to δ_t leads to future output that is relatively more correlated with capital income and less correlated with labor income. This causes initial portfolios to generate future transfers that are too correlated with total income, diminishing international risk-sharing. As a result, Home and Foreign agents trade with each other to rebalance their portfolios, making them less biased toward their domestic asset. Since these effects are captured in the third-order policy functions by correction terms that are first-order with respect to the states (e.g., $\frac{1}{2}\sigma^2\left(\delta_t - \overline{\delta}\right)$) , they become naturally incorporated in the linear-corrected solution, accounting for the identical impulse responses in panels (c) and (d).

Figure 4: Ergodic wealth distribution of the DS model



3.8 Stochastic simulations, accuracy, and computing times

As a last exercise, I generate stochastic simulations of the approximate solutions to explore some of their properties and conduct global accuracy tests. The benchmark case is a third-order solution to the model of interest around the SSS, which in turn is computed with a second-order solution as discussed in section 3.6.

Ergodicity. To look at the long-run properties of the third-order solution, I simulate the economy for $1,000,000$ periods (with a burn-in of $1,000$ periods), feeding policy functions evaluated at $(\varepsilon, \sigma) = (1, 1)$ with pseudo-random innovations. Then, I use the resulting time series for domestic net foreign assets $W_t = z_{H,t}a_{H,t+1} + z_{F,t}a_{F,t+1}$ to compute kernel distributions (using the Epanechnikov method) that I plot in Figure 4, Panel (a). The blue solid line corresponds to a basic simulation, and the red dashed line corresponds to a simulation obtained using a pruning scheme similar to Andreasen, FernÃ¡ndez-Villaverde, and Rubio-RamÃrez (2018). The first result is that the two simulations deliver well-behaved ergodic distributions of net foreign assets centered around zero. The second result is that the two distributions are very close to each other, indicating near-identical long-run dynamics.

In Panel (b), I repeat the experiment after setting the Uzawa parameter to zero, which makes the discount factor exogenous. Despite the absence of stationarity-inducing devices in the model of interest, the two simulations still deliver well-behaved ergodic distributions of net foreign assets centered around zero. This result is due to the same precautionary-saving mechanism that induces a stationary wealth distribution in Heterogeneous-agent models (see, e.g., Aiyagari (1994)). Specifically, expected returns are lower than the inverse of the discount factor, which, combined with a standard precautionary-saving effect, delivers policy functions locally stable around the SSS.[28] Stationarity is achieved because, while evaluating the policy functions at $\varepsilon = 1$ removes the auxiliary PAC-like device (by itself sufficient to produce stable dynamics), the simultaneous evaluation at $\sigma = 1$ introduces corrections to the arguments $a_{H,t}$ and $a_{F,t}$ that capture precautionary-saving behavior. Since the policy functions are almost linear with re-

[28]Specifically, the SSS returns satisfy $R_{H,s} = R_{F,s} < 1/\beta$.

Table 1: Euler equation errors of the DS model

Approximation order	Consumption equation			Portfolio equation		
	Mean	Median	Max.	Mean	Median	Max.
Second	-5.43	-5.33	-4.31	-5.54	-5.47	-3.94
Third	-6.04	-5.95	-4.43	-7.31	-7.27	-5.83
Fourth	-6.68	-6.59	-5.09	-7.91	-7.79	-6.98
Fifth	-7.17	-7.11	-5.61	-8.07	-7.97	-7.30

Notes. The table reports the mean, median and maximum values of the Euler errors (in log10 scale) across the ergodic set of each approximation.

spect to $a_{H,t}$ and $a_{F,t}$ (higher-order corrections to these arguments are very small) local stability translates into global stability, which explains the ergodicity of the distributions.

Euler equation errors. To measure the accuracy of the perturbation solution, I follow Aruoba et al. (2006) by computing unit-free errors of the consumption Euler equation (3.12) and the portfolio Euler equation (3.16), obtained from simulated time series of $10,000$ periods (with a burn-in of $1,000$ periods) for second, third, fourth, and fifth order solutions around the same SSS $a_{H,s} = \bar{a} = 1.464$. Table 1 reports the mean, median and maximum of Euler errors (in log10 scale) from each simulation and equation.[29] The first result is the large improvement in accuracy when moving from a second-order to a third order solution, particularly marked for the Portfolio Euler equation: on average, the third-order solution delivers Euler errors 59 times smaller than those from a second-order solution. The second result is that Two-parameter perturbation delivers solutions to the DS model that are globally very accurate, converging to the true solution as the order of approximation increases.[30]

Computing times. I compute all the numerical results of this paper with MATLAB R2021a, installed in a laptop with a 2.10GHz AMD Ryzen 5 5500U processor with six cores. Using the perturbation package by Levintal (2017) to perform calculations, it takes 0.04 seconds to compute the SSS starting from the agnostic initial guess $a_H = 0$, and 0.02 seconds to compute the third-order solution. Using Dynare instead, it takes 0.04 seconds to compute the SSS, and 1.77 seconds to compute the third-order solution.

[29] The log10 scale allows for an intuitive interpretation: a value of -3 means $1 mistake for each $1,000, a value of -4 a $1 mistake for each $10,000, and so on. To approximate expectations, I discretize the innovations with monomials following Judd, Maliar and Maliar (2011).

[30] To put this result in context, Aruoba et al. (2006, Table 5) find that a standard fifth-order approximation of the stochastic growth model delivers a mean error of -5.43, and a max. error of -3.33.

3.9 Original notation

This section has followed the notation from appendix C of DS to solve the example. In their main text, however, DS write down their model in terms of real holdings of the home asset α_t and Home wealth W_t. Using my notation, these variables can be defined as follows:

$$\alpha_t = z_{H,t-1} a_{H,t}, \tag{3.27}$$

$$W_t = z_{H,t-1} a_{H,t} + z_{F,t-1} a_{F,t}. \tag{3.28}$$

Appendix A of this paper shows how, through a change of variables, the equilibrium conditions of the auxiliary model can be recast in terms of α_t, W_t, and pre-determined prices $z_{H,t-1}$ and $z_{H,t-1}$ as the endogenous states, so that it can be solved by two-parameter perturbation.

4 Scope and choice of auxiliary models

The theory developed in this paper imposes very mild restrictions on how the new perturbation parameter ε is introduced into the equilibrium conditions or on the type of auxiliary modifications its controls. This section discusses various ways of exploiting this flexibility to widen the scope of two-parameter perturbation and to improve both its accuracy and efficiency.

4.1 Scope and examples

Portfolio choice. The focus of this paper is on solving DSGE models with portfolio-choice by introducing modifications in the auxiliary model that are shut down as we approach the model of interest. This technique allows us to build an approximate solution to the DSGE economy by starting from a nearby model whose solution we have access to, *even when the deterministic version of the original economy is intractable with standard perturbation*. In particular, the DS example demonstrates that an auxiliary model with easy-to-use PAC-like devices, scaled by a decreasing function $f(\varepsilon)$, does a good job of resolving the indeterminacy issue inherent to deterministic settings, which we can later remove through perturbation. This is also the technique used to solve the more complex quantitative version of the Coeurdacier and Gourinchas (2016) model of section 6.

While the two main applications consist of open economy models, there is nothing in the theory that prevents the use of the method in other contexts. Since section 5 generalizes two-parameter perturbation as an extension of the standard perturbation model, we can use it right away to solve any DSGE application that can be cast into the canonical framework (Schmitt-GrohÃ© and Uribe, 2004; FernÃ¡ndez-Villaverde, Rubio-RamÃrez, and Schorfheide, 2016). The method works well even with finite-period models that have been popular so far in the finance literature, such as the original Coeurdacier and Gourinchas (2016) model. To show this, Appendix D.1 solves the single-agent example used by Judd (1998, chapter 15) to introduce bifurcation in economics.

Deactivating devices from the model of interest. We can also use two-parameter perturbation to handle devices in the model of interest that may cause trouble in a deterministic setting (like a parameter that makes agents heterogeneous *ex-ante*). In such cases, the technique consists of scaling the device by another increasing function (call it $f_2(\varepsilon)$), satisfying $f_2(0) = 0$ and $f_2(1) = 1$ (e.g, $f_2(\varepsilon) = \varepsilon^2$). Then, evaluating the equilibrium conditions at $\varepsilon = 0$ deactivates the device, yielding a tractable deterministic model that we can solve. The effect of the device on the equilibrium dynamics can be recovered later through perturbation.[31] Since this approach is compatible with the introduction of auxiliary devices, we can, even within the same Euler equation, activate some devices and deactivate others simultaneously. To illustrate this, Appendix D.4 solves a one-bond economy with a patient and an impatient country, where the asymmetry between discount factors makes it impossible to solve the model with standard perturbation because a DSS does not even exist.[32]

Approximation point and the stochastic steady state. Another nice implication of a two-parameter approach is that it allows us to approximate the policy functions around a point different from the DSS of the endogenous state variables implied by the model of interest (which may not even exist), *while still relying on the theorems ensuring convergence of the Taylor series*.[33] In general, we can achieve this by properly interacting ε with auxiliary parameters affecting the DSS. Since these parameters do not belong to the model of interest, we can set their values to target a DSS for the endogenous states that matches some point in the state-space. Then, we can use perturbation to remove any effect they may have on the equilibrium dynamics, so that all is left are policy functions centered around the desired point.

As shown in section 3.6, a straightforward application of this technique consists of imposing sufficient conditions on the correction terms of the Taylor series to find an approximation to the SSS of the endogenous states that can be used as the expansion point of the Taylor series. When working with portfolio-choice models, this is very useful as it allows us to compute SSS portfolios that we can compare with the data. Section 5.3 demonstrates how this SSS algorithm can be applied to more general settings and integrated into calibration or estimation exercises, *even if the DSS implied by the model of interest is ill-defined*. The algorithm is also theorem-proof: Appendix C shows that as the order of the Taylor series increases, the outcome of the algorithm converges to the SSS of the true policy functions.

To illustrate these capabilities, Appendix D.3 considers the open economy version of the classical income fluctuation problem studied by Coeurdacier, Rey, and Winant (2011). In this model, the well-known parameter restriction $R\beta < 1$ that induces stationarity in the stochastic economy also rules out a DSS, making standard perturbation inapplicable. By using two different auxiliary models (Uzawa preferences and PAC), two-parameter perturbation is able to

[31] This is essentially what standard perturbation does: by scaling innovations with σ and evaluating the model at $\sigma = 0$, it deactivates uncertainty so that we can work with a tractable deterministic economy. My point is that we can do the same with other features of the model.

[32] See Iacoviello (2005), Monacelli (2009), and Eggertsson and Krugman (2012) for examples of models with patient and impatient agents. Since they all solve their models by standard perturbation, they are forced to introduce an *ad hoc* borrowing constraint that is always binding for the impatient agent.

[33] See in particular Theorem 6 in Jin and Judd (2002).

match the same SSS obtained with the endogenous grid method. In a more complex setting with interest rate shocks calibrated to match aggregate Mexican data, the method also delivers well-defined ergodic distributions of net foreign assets, with Euler errors below -5 using a third-order approximation, and well below -6 using a fifth-order one.

Finally, we can use two-parameter perturbation to approximate standard DSGE models around the SSS instead of the DSS. This methodological change can be useful for studying macrofinance models where the combination of strong volatility and high levels of risk aversion makes the analysis of these models challenging with standard perturbation (FernÃ¡ndez-Villaverde and Levintal, 2018). To illustrate this point, Appendix D.2 compares the approximate solution of the neoclassical growth model around the SSS with the standard perturbation solution around the DSS. The results show that two-parameter perturbation is about five times more accurate in scenarios with high levels of volatility and risk aversion, thanks to approximations built around a point much closer to where the model dynamics actually take place.

4.2 Choosing the "right" auxiliary model

An implication of the flexibility allowed by the theory is that, in most cases, there will be many (possibly infinite) candidate auxiliary models that can, a priori, perform the same job. Therefore, it is up to the practitioner to find the auxiliary model that best balances accuracy and efficiency to suit their specific needs. In this sense, choosing the "right" model can be (as Thomas Sargent would put it) a bit of an art. This subsection aims to provide some guidance based on my experience using the method.[34]

Before proceeding, it is important to keep in mind the following limitation. By construction, perturbation methods are only guaranteed to work well when the parametric departure from the auxiliary model is small, and the resulting approximations are valid around the point of approximation. Thus, the advice that follows (specially when it comes to accuracy) can make a difference in applications where strong volatility and rapidly changing dynamics push local methods to their limits.

Comparing accuracy. This paper uses (unconditional) average Euler equation errors over a ergodic set (obtained by simulation) as the primary metric to assess the global accuracy of the approximate solutions, and we can use them to compare and discriminate between different auxiliary models. Thanks to the use of monomials to approximate expectations (Judd, Maliar and Maliar, 2011), we can compute Euler errors conditional on a given point in the state-space very efficiently. Just to give a number, a single evaluation of a third-order solution to the model of section 6 (the most involved application, with 10 state variables and 6 innovations) takes less than 0.003 seconds. However, if obtaining a well-behaved ergodic distribution requires large simulations (e.g., near unit-root times series), computing average Euler errors can still be quite time-consuming.

[34]This guidance is fully compatible with well-known practices used to solve DSGE models by standard perturbation, such as imposing a change of variables to ensure that states and controls are bound within their natural domain.

To overcome this issue, I propose two possible proxy metrics that are much faster to compute. One is simply to employ average errors from a small simulation of the economy (say, $T = 100$) using the same innovations. The other proxy is to calculate average errors conditional on a given point in the state-space, such as the SSS (that is, the expected value of the Euler errors tomorrow if today we are at the SSS). In practice, I found that these alternative measures of accuracy correlate well with the unconditional average errors (by ranking different auxiliary models in the same way), because what typically matters the most for the accuracy of portfolio choice dynamics are the perturbation corrections to the exogenous states in the policy rules.[35]

Functional form of auxiliary modifications. In models with portfolio choice, the minimum requirement for an auxiliary modification is clear: it should eliminate the indeterminacy that arises in a deterministic setting. However, a more subtle point is that the specific functional form of the modification can help to improve accuracy. In particular, working with auxiliary models whose functional form affects equilibrium variables in a way similar to the model of interest is useful because, by narrowing the gap between the two models, it helps perturbation converge to the model of interest. This is partly why PAC-like modifications perform so well: the stable dynamics of asset holdings they generate resemble those implied by precautionary saving, a feature that is only active at the model of interest.[36]

Regarding efficiency, I recommend using modifications that result in a DSS that can be solved analytically whenever possible. By eliminating the bottleneck of resorting to a nonlinear solver, this speeds up the SSS algorithm and makes it easier to integrate two-parameter perturbation into complex calibration and estimation routines. Again, PAC-like modifications excel in this task, as their location parameters immediately pin down the DSS of asset holdings.

Auxiliary parameters. It is useful to distinguish between two types of auxiliary parameters: those that affect the DSS of the endogenous states, and those that do not. As already explained, the first type is useful because they allow us to approximate the solution around a large subset of the state-space. But how large is this subset depends on how the parameters are introduced. Specifically, parameters directly scaled by a perturbation function $f(\varepsilon)$ must remain small for perturbation to yield accurate results. In these cases, I recommend perturbing parameters where small changes lead to relatively large shifts in the DSS (e.g., the discount factor in the Growth model of Appendix D.2). A related piece of advice is, where feasible, to rely more on location parameters that are not directly perturbed, such as $\bar{a}$ in the DS example.

The second type of parameters typically control the auxiliary modifications (such as ψ in the auxiliary DS model), so perturbing them is the direct way to remove the dynamic effects of these modifications in the approximate solution. Thus, setting their values becomes purely

[35]If the focus of the application is on unexpected shocks (or shocks with a low probability of occurrence), my suggestion is to compute impulse responses and compare the Euler errors generated along the transition path.

[36]An easy way to test this is to flip the signs of the PAC-like modifications in equations (3.18)-(3.21). Provided that the Uzawa elasticity κ is large enough as in the calibration, the modifications still manage to eliminate the indeterminacy and the SSS algorithm still delivers the same SSS. However, compared to the numbers reported in Table 1, average portfolio Euler errors significantly worsen: from -7.31 to -6.55 using a third-order solution, and from -7.91 to -6.53 using a fourth-order one.

a matter of improving accuracy, much like choosing the number of grid points in a global solution method. One might think the obvious choice is to make them as small as possible, but this is not necessarily the case if the functional form of the modifications helps to narrow the gap between models, or if extreme low values affect the precision of higher-order derivatives. Since we can test for these effects directly by comparing average Euler errors (or the proxy metrics discussed), my recommendation is to experiment with different values for these type of parameters and check whether this leads to substantial gains in accuracy.[37]

The perturbation function. The final choice to make is the functional form of the scaling function $f(\varepsilon)$ (and $f_2(\varepsilon)$, if we aim to deactivate devices from the model of interest). The minimum requirements imposed by the theory are clear: (i) both functions must be differentiable; (ii) $f(\varepsilon)$ must be decreasing on the interval $[0, 1]$ and satisfy $f(0) = 1$ and $f(1) = 0$; and (iii) $f_2(\varepsilon)$ must be increasing on the interval $[0, 1]$ and satisfy $f_2(0) = 0$ and $f_2(1) = 1$. Following these requirements, almost all the applications in this paper use the simple functional forms $f(\varepsilon) = 1 - \varepsilon^2$ and $f_2(\varepsilon) = \varepsilon^2$, because they ensure that perturbation corrections to ε behave the in same way as corrections to σ (no corrections at first order, only a constant term at second order, etc.). If innovations are symmetric, this improves efficiency because, when implementing the SSS algorithm, the much faster second-order solution yields the same SSS that a third-order one would. Moreover, this choice does not conflict with accuracy, as I found that different functional forms deliver the same SSS and nearly identical dynamics. The only exception is the Growth model, where the choice of $f(\varepsilon) = 1 - \varepsilon$ delivers much more accurate results in an extreme calibration that pushes perturbation to its limits.

5 The two-parameter perturbation model

This section has three main goals. First, it generalizes the two-parameter perturbation model as a straightforward extension of the canonical one-parameter perturbation model, where the new perturbation parameter ε enters directly into the equilibrium conditions. Second, it shows that this new model can be solved using standard DSGE software by treating ε_t as an additional exogenous state that is constant over time. Third, it provides a formal description of the stochastic steady-state algorithm introduced in section 3.6 and explains how to implement it numerically within this general framework.

5.1 The setup

The structure and notation of the model follows closely the expositions by Schmitt-GrohÃ© and Uribe (2004), and FernÃ¡ndez-Villaverde, Rubio-RamÃrez, and Schorfheide (2016). Con-

[37]One can even set an optimization routine to find the values of these auxiliary parameters that minimize the accuracy metric, as in FernÃ¡ndez-Villaverde and Rubio-RamÃrez (2006). To illustrate this, the online repository provides an example code that uses the package by Levintal (2017) to find the optimal ψ in the auxiliary DS model. However, unless accuracy is the absolute priority, this approach seems like overkill. As FernÃ¡ndez-Villaverde, Rubio-RamÃrez and Schorfheide (2016) put it, when finding parameters that improve accuracy, optimality is desirable but not essential, so it can be traded off against computational costs.

sider an auxiliary perturbation model with a set of equilibrium conditions of the form

$$\mathbb{E}_t \mathbf{f}\left(\mathbf{y_{t+1}}, \mathbf{y_t}, \mathbf{x_{t+1}}, \mathbf{x_t}, \varepsilon\right) = 0. \tag{5.1}$$

Here, $\mathbb{E}_t$ denotes conditional expectations at time t , $\mathbf{y_t}$ is a vector of n_y control variables, $\mathbf{x_t}$ is a vector of n_x state variables, and $\varepsilon \geq 0$ is the new perturbation parameter. The function $\mathbf{f}$ maps $\mathbb{R}^{2(n_y+n_x)+1}$ into $\mathbb{R}^{(n_y+n_x)}$. In turn, the state vector can be partitioned as $\mathbf{x_t} = [\mathbf{x_{1,t}}; \mathbf{x_{2,t}}]$, where $\mathbf{x_{1,t}}$ consists of the n_{x1} endogenous states, and $\mathbf{x_{2,t}}$ of the n_{x2} exogenous states.[38] The latter follows a stochastic process of the form

$$\mathbf{x_{2,t+1}} = \mathbf{C}(\mathbf{x_{2,t}}) + \sigma \eta_u \mathbf{u}_{t+1},$$

where $\mathbf{C} : \mathbb{R}^{n_{x2}} \to \mathbb{R}^{n_{x2}}$ is a differentiable function that generates stationary dynamics, $\mathbf{u}_t$ is a vector of n_u zero-mean innovations, and η_u is a known matrix with dimensions $n_{x2} \times n_u$ scaled by the standard perturbation parameter $\sigma \geq 0$.

The solution to the model consists of a set of policy functions $\mathbf{g}$ and $\mathbf{h}$ for the control and state variables:

$$\begin{aligned} \mathbf{y_t} &= \mathbf{g}(\mathbf{x_t}, \varepsilon, \sigma), \\ \mathbf{x_{t+1}} &= \mathbf{h}(\mathbf{x_t}, \varepsilon, \sigma) + \sigma \eta \mathbf{u}_{t+1}, \end{aligned} \tag{5.2}$$

where $\mathbf{g}$ maps $\mathbb{R}^{(n_x+2)}$ into $\mathbb{R}^{n_y}$, $\mathbf{h}$ maps $\mathbb{R}^{(n_x+2)}$ into $\mathbb{R}^{n_x}$, and $\eta = [0_{n_{x1} \times n_u}; \eta_u]$ is a $n_x \times n_u$ matrix. Note that, under this formulation, the function $\mathbf{C}$ is the lower block of $\mathbf{h}$, and the only one updated by innovations.

The deterministic steady state (DSS) of the model consists of vectors $(\mathbf{x_d}, \mathbf{y_d})$ that satisfy

$$\mathbf{f}\left(\mathbf{y_d}, \mathbf{y_d}, \mathbf{x_d}, \mathbf{x_d}, 0\right) = 0. \tag{5.3}$$

Assume that the DSS exists and that it is unique. Note that, by construction, we have that

$$\begin{aligned} \mathbf{x_d} &= \mathbf{h}(\mathbf{x_d}, 0, 0), \\ \mathbf{y_d} &= \mathbf{g}(\mathbf{x_d}, 0, 0). \end{aligned} \tag{5.4}$$

That is, the vector $\mathbf{x_d}$ is the fixed-point of $\mathbf{h}$ conditional on $(\varepsilon, \sigma) = (0, 0)$, and it can be partitioned as $\mathbf{x_d} = [\mathbf{x_{1,d}}; \mathbf{x_{2,d}}]$.

The goal of perturbation is to approximate the functions $\mathbf{g}$ and $\mathbf{h}$ with Taylor series around $\mathbf{x_t} = \mathbf{x_d}$. To find their partial derivatives, plug-in the unknown policy functions on $\mathbf{f}$ and define the new operator $\mathbf{F} : \mathbb{R}^{n_x+2} \to \mathbb{R}^{n_x+n_y}$ as follows:

$$\mathbf{F}\left(\mathbf{x_t}, \varepsilon, \sigma\right) \equiv \mathbb{E}_t \mathbf{f}\left(\mathbf{g}\left(\mathbf{h} + \sigma \eta \mathbf{u}_{t+1}\right), \mathbf{g}, \mathbf{h} + \sigma \eta \mathbf{u}_{t+1}, \mathbf{x_t}, \varepsilon\right) = 0.$$

[38] Throughout the text I will also refer to the exogenous states as shocks.

Since $\mathbf{F}(\mathbf{x_t}, \varepsilon, \sigma) = 0$ must hold for all values of $(\mathbf{x_t}, \varepsilon, \sigma)$, any of its partial derivatives evaluated at $(\mathbf{x_d}, 0, 0)$ must also equal zero:

$$\mathbf{F}_{x_i^j \varepsilon^l \sigma^m}(\mathbf{x_d}, 0, 0) = 0,$$

where $\mathbf{F}_{x_i^j \varepsilon^l \sigma^m}$ is the partial derivative of $\mathbf{F}$ with respect to the i-th component of $\mathbf{x_t}$ taken j times, with respect to ε taken l times, and with respect to σ taken m times. It follows that, by taking the proper number of derivatives, we end up with a system of equations to solve for the coefficients of the Taylor series of $\mathbf{g}$ and $\mathbf{h}$ up to a given order k.

Once the partial derivatives are computed, we can use them to construct the Taylor series $\mathbf{h}^k$ and $\mathbf{g}^k$, given by:

$$\mathbf{h}^k(\mathbf{x_t}, \varepsilon, \sigma) = \mathbf{x_d} + \sum_{i=1}^{k} \frac{1}{i!} H_i \begin{pmatrix} \mathbf{x_t} - \mathbf{x_d} \\ \varepsilon \\ \sigma \end{pmatrix}^{\otimes i}, \tag{5.5}$$

$$\mathbf{g}^k(\mathbf{x_t}, \varepsilon, \sigma) = \mathbf{y_d} + \sum_{i=1}^{k} \frac{1}{i!} G_i \begin{pmatrix} \mathbf{x_t} - \mathbf{x_d} \\ \varepsilon \\ \sigma \end{pmatrix}^{\otimes i}, \tag{5.6}$$

where each H_i and G_i are matrices of order i derivatives with dimensions $n_x \times (n_x + 2)^i$ and $n_y \times (n_x + 2)^i$, and the symbol $^{\otimes i}$ denotes a "Kronecker power", that is $z^{\otimes i} = z \otimes ... \otimes z$ i times.

Finally, it will prove convenient to work with the upper block of $\mathbf{h}^k$ directly, denoted by $\mathbf{h}_1^k$, and given by:

$$\mathbf{h}_1^k(\mathbf{x_t}, \varepsilon, \sigma) = \mathbf{x_{1,d}} + \sum_{i=1}^{k} \frac{1}{i!} H_{1,i} \begin{pmatrix} \mathbf{x_t} - \mathbf{x_d} \\ \varepsilon \\ \sigma \end{pmatrix}^{\otimes i}, \tag{5.7}$$

where each matrix $H_{1,i}$ with dimension $n_{x1} \times (n_x + 2)^i$ is the upper block of H_i.

5.2 Implementation with standard perturbation algorithms

The two-parameter perturbation model can be implemented using any algorithm designed to solve the standard perturbation model. To show this, I follow Levintal (2017) and exploit the variable-parameter duality of perturbation objects to rewrite the model with a new notation that treats the perturbation parameter ε as a state variable.

As discussed in section 3.4, the starting point is to denote ε_t as an exogenous state variable that is constant over time: $\varepsilon_{t+1} = \varepsilon_t$. Next, define the new vectors:

$$\widehat{\mathbf{f}} = \begin{pmatrix} \mathbf{f} \\ -\varepsilon_{t+1} + \varepsilon_t \end{pmatrix}, \ \widehat{\mathbf{x}}_\mathbf{t} = \begin{pmatrix} \mathbf{x_t} \\ \varepsilon_t \end{pmatrix}, \ \widehat{\eta} = \begin{pmatrix} \eta \\ 0_{1 \times n_u} \end{pmatrix}.$$

Using this notation, the new set of equilibrium conditions is:

$$\mathbb{E}_t \widehat{\mathbf{f}}(\mathbf{y_{t+1}}, \mathbf{y_t}, \widehat{\mathbf{x}}_\mathbf{t+1}, \widehat{\mathbf{x}}_\mathbf{t}) = 0, \tag{5.8}$$

and we can restate the solution to the model as:

$$\mathbf{y_t} = \widehat{\mathbf{g}}(\mathbf{x_t}, \sigma), \tag{5.9}$$
$$\widehat{\mathbf{x}}_{\mathbf{t+1}} = \widehat{\mathbf{h}}(\widehat{\mathbf{x}}_{\mathbf{t}}, \sigma) + \sigma \widehat{\eta} \mathbf{u}_{t+1},$$

where $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{h}}$ are the policy functions in the new notation. Finally, let $\widehat{\mathbf{x}}_{\mathbf{d}} = [\mathbf{x_d}; 0]$. The new DSS consists of vectors $(\widehat{\mathbf{x}}_{\mathbf{d}}, \mathbf{y_d})$ that satisfy

$$\widehat{\mathbf{f}}\,(\mathbf{y_d}, \mathbf{y_d}, \widehat{\mathbf{x}}_{\mathbf{d}}, \widehat{\mathbf{x}}_{\mathbf{d}}) = 0. \tag{5.10}$$

It follows that any algorithm capable of solving a model with the form of (5.8)-(5.10) will also solve the two-parameter perturbation model.

5.3 Approximating the stochastic steady state

This subsection describes the algorithm to approximate the stochastic steady state (SSS) of the model of interest $(\varepsilon, \sigma) = (1, 1)$ such that the perturbation is centered at this point. The procedure is akin to standard calibration routines, and it builds on imposing enough conditions such that the DSS of the endogenous state variables in the auxiliary model coincides with their SSS in the model of interest. If the auxiliary model incorporates the proper modifications, the algorithm will approximate the true SSS even if the DSS implied by the model of interest (the $(\varepsilon, \sigma) = (1, 0)$ case) is ill-defined.

To start, define the SSS of the model of interest as the vectors $\mathbf{x_t} = \mathbf{x_s}$ and $\mathbf{y_t} = \mathbf{y_s}$ satisfying

$$\mathbf{x_s} = \mathbf{h}(\mathbf{x_s}, 1, 1), \tag{5.11}$$
$$\mathbf{y_s} = \mathbf{g}(\mathbf{x_s}, 1, 1). \tag{5.12}$$

In a way analogous to Eq. (5.4), $\mathbf{x_s}$ is the fixed point of $\mathbf{h}$ conditional to $(\varepsilon, \sigma) = (1, 1)$, assumed to be unique. As with the DSS, $\mathbf{x_s}$ can be partitioned as $\mathbf{x_s} = [\mathbf{x_{1,s}}; \mathbf{x_{2,s}}]$, where $\mathbf{x_{1,s}}$ is the SSS of the endogenous states, and $\mathbf{x_{2,s}}$ the SSS of the exogenous states. Note that, while $\mathbf{x_{2,s}} = \mathbf{x_{2,d}}$ by construction (and hence already known), $\mathbf{x_{1,s}}$ does not need to coincide with $\mathbf{x_{1,d}}$ (due to, for example, a precautionary savings effect). In general, finding the true $\mathbf{x_{1,s}}$ requires previous knowledge of $\mathbf{h}$, a function we do not have access to.

5.3.1 The stochastic steady state condition

On what follows, think of $\mathbf{x_s}$ as the fixed point of the approximate rule $\mathbf{h}^k\,(\mathbf{x_t}, 1, 1)$, with $k \geq 2$ to ensure that it captures risky behavior. Assume that a $n_{x1} \times 1$ vector $\overline{\mathbf{x}}$ completely pins down the DSS of the endogenous states: $\mathbf{x_{1,d}} = \overline{\mathbf{x}}$. In most instances, we can achieve this by introducing up to $n_{x1} \times 1$ auxiliary parameters $\overline{\boldsymbol{\psi}}$ that do not show up in the $(\varepsilon, \sigma) = (1, 1)$ model but affect the DSS of the $(\varepsilon, \sigma) = (0, 0)$ model.[39] Since we are looking for an approximation centered

[39]Sometimes we will require less than n_{x1} auxiliary parameters if the SSS of some endogenous states is a function of the others. The DS model is a clear example of this.

at the SSS, it must be the case that $\mathbf{x_{1,s}} = \mathbf{x_{1,d}} = \overline{\mathbf{x}}$. It follows that the DSS is $\mathbf{x_d} = [\overline{\mathbf{x}}; \mathbf{x_{2,d}}]$, where the exogenous lower block $\mathbf{x_{2,d}}$ is already known and completely independent of $\overline{\mathbf{x}}$. As this makes $\mathbf{x_d}$ the fixed point of $\mathbf{h}^k(\mathbf{x_t}, 1, 1)$, evaluating the approximate law of motion of the endogenous states $\mathbf{x_{1,t+1}} = \mathbf{h}_1^k(\mathbf{x_t}, \varepsilon, \sigma)$ at $(\mathbf{x_d}, 1, 1)$ must return $\mathbf{h}_1^k = \overline{\mathbf{x}}$. Then, by using (5.7) we obtain the following SSS condition:

$$\sum_{i=1}^{k} \frac{1}{i!} H_{1,i} \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i} = 0, \tag{5.13}$$

Since each $H_{1,i}$ has dimension $n_{x1} \times (n_x + 2)^i$ and all its derivatives are function of $\overline{\mathbf{x}}$, the last expression constitutes a system of n_{x1} equations to solve for each of the components of $\overline{\mathbf{x}}$. Intuitively, if $\mathbf{x_d}$, by construction a fixed-point of $\mathbf{h}^k(\mathbf{x_t}, 0, 0)$, is also a fixed-point of $\mathbf{h}^k(\mathbf{x_t}, 1, 1)$, it must be the case that the sum of all coefficients proportional to powers of ε and/or σ in the Taylor series equals zero for each of its rows. Appendix C shows that, under standard regularity assumptions, the solution to (5.13) converges to the true $\mathbf{x_{1,s}}$ as k goes to infinity.

Once a candidate solution for $\mathbf{x_{1,s}}$ has been found, computing the SSS of the control variables just requires evaluating (5.6) at $(\mathbf{x_s}, 1, 1)$. That is:

$$\mathbf{y_s} = \mathbf{g}^k(\mathbf{x_s}, 1, 1). \tag{5.14}$$

5.3.2 Numerical implementation

The system of equations (5.13) can be solved numerically with a standard nonlinear solver. All that is needed is to code a residual function that takes the vector $\overline{\mathbf{x}}$ as an input and returns $\mathbf{h}_1^k([\overline{\mathbf{x}}; \mathbf{x_{2,d}}], 1, 1) - \overline{\mathbf{x}}$ as an output. The nonlinear solver will call this function and search for a candidate $\overline{\mathbf{x}}$ that makes it zero. Here I describe in detail the pseudo-code for this residual function.

I assume that the user will implement the algorithm with an available DSGE solution toolbox. For this reason, it is convenient to treat ε_t as an exogenous state and follow the notation of Section 5.2. Therefore, the k-order Taylor series of $\widehat{\mathbf{h}}$ is:

$$\widehat{\mathbf{h}}^k(\widehat{\mathbf{x}}_\mathbf{t}, \sigma) = \mathbf{x_d} + \sum_{i=1}^{k} \frac{1}{i!} H_i \begin{pmatrix} \widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d} \\ \sigma \end{pmatrix}^{\otimes i}$$

where $(\widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d})$ are deviations of $\widehat{\mathbf{x}}_\mathbf{t} = [\mathbf{x_t}; \varepsilon_t]$ from the extended DSS $\widehat{\mathbf{x}}_\mathbf{d} = [\mathbf{x_d}; 0]$.

For any given $\overline{\mathbf{x}}$, the residual function performs three tasks. First, it recalculates the new DSS for the control variables $\mathbf{y_d}$ and the values of the auxiliary parameters $\overline{\boldsymbol{\psi}}$ consistent with $\mathbf{x_{1,d}} = \overline{\mathbf{x}}$. Second, it calls an external algorithm (the DSGE toolbox) to calculate the matrices of derivatives $H_1, ..., H_k$. Third, it constructs the Taylor series $\widehat{\mathbf{h}}^k$, and evaluates it at the model of interest (that is, $\widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d} = [0_{n_x \times 1}; 1]$ and $\sigma = 1$). The function then returns the output $y = \widehat{\mathbf{h}}_1^k - \overline{\mathbf{x}}$ (where $\widehat{\mathbf{h}}_1^k$ is the upper block of $\widehat{\mathbf{h}}^k$). Let **S** be all the structure arrays required by the external algorithm.[40] The pseudocode summarizing these steps is:

Algorithm 1 Residual SSS function

function y = eval_sss($\overline{\mathbf{x}}$;S,k)

STEP 1: Given $\overline{\mathbf{x}}$, recalculate $\mathbf{y_d}$ and $\overline{\boldsymbol{\psi}}$, modifying S when necessary.
STEP 2: Call an external algorithm to compute the matrices $H_1, ..., H_k$.
STEP 3: Construct $\widehat{\mathbf{h}}^k$ and evaluate it at $\widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d} = [0_{n_x \times 1}; 1]$ and $\sigma = 1$.

y = $\widehat{\mathbf{h}}^k_1 - \overline{\mathbf{x}}$.

Steps 2 and 3 can be automated conditional on the external algorithm. To help with this, the Github repository includes MATLAB functions that construct and evaluate $\widehat{\mathbf{h}}^k$ and $\widehat{\mathbf{g}}^k$ for Dynare and the perturbation algorithm by Levintal (2017).

Calibration mode. We can easily adapt the previous residual function to solve for values of a $n_{x1} \times 1$ vector $\bar{\theta}$ of parameters of the model of interest that target the desired $\mathbf{x_{1,s}} = \overline{\mathbf{x}}$. In this case, the main argument of the residual function is $\bar{\theta}$, and the vector $\overline{\mathbf{x}}$ is kept fixed. This way, the nonlinear solver will search for a candidate $\bar{\theta}$ that effectively zeroes the left hand-side of (5.13). Appendix D.3.4 uses this algorithm to calibrate the exogenous discount factor targeting the observed long-run level of net foreign assets of a small open economy. We can also implement a hybrid approach to solve for a subset of $\bar{\theta}$ and $\overline{\mathbf{x}}$ while keeping the other subset fixed. For example, the baseline application section 6 calibrates the standard deviation of preference shocks consistent with the observed level of equity home bias, and solves for the long-run level of bond holdings. Finally, we can integrate the SSS algorithm into a more complex one targeting a variety of observables. It all comes down to ensuring that $\overline{\mathbf{x}}$ zeroes the left hand-side of (5.13).

6 Equity home bias in a multi-asset DSGE model

The model is the infinite-period version of Coeurdacier and Gourinchas (2016) (from hereon CG). Specifically, I consider a version of the model with endowment, redistributive and preference shocks (appendix A.4 of their paper) that makes financial markets incomplete. As shown in the results, an incomplete markets setup with bonds and equities is essential to deliver external asset positions consistent with the data.

[40] If the DSGE toolbox is Dynare, **S** includes the arrays M_, options_, and oo_. The external Dynare function resol.m takes these as inputs to compute the matrices $H_1, ..., H_k$.

6.1 Model description

Consider a World with two symmetric countries, Home (H) and Foreign (F). Below I describe the model of interest from Home's perspective. When necessary, I distinguish Foreign's variables with an asterisk.

Preferences. A representative household at Home maximizes the following utility function:

$$\mathbb{E}_t \left[\sum_{t=0}^{\infty} \theta_t \frac{C_t^{1-\gamma} - 1}{1-\gamma} \right].$$

The risk-aversion parameter γ is the same for both countries in the baseline scenario, but I will also explore the consequences of relaxing this assumption (which breaks perfect symmetry across countries). The consumption index C_t is a constant-elasticity aggregator over consumption of Home and Foreign goods $c_{H,t}$ and $c_{F,t}$:

$$C_t = \left[\alpha^{\frac{1}{\phi}} \left(q_t c_{H,t} \right)^{(\phi-1)/\phi} + (1-\alpha)^{\frac{1}{\phi}} \left(q_t^* c_{F,t} \right)^{(\phi-1)/\phi} \right]^{\phi/(\phi-1)},$$

where $\phi > 0$ is the elasticity of substitution between the two goods, and the weight $\alpha \in (\frac{1}{2}, 1)$ captures Home bias towards $c_{H,t}$. In turn, q_t and q_t^* are worldwide shocks to the preference (or quality) for Home and Foreign goods.[41]

The Uzawa discount factor θ_t evolves over time according to

$$\begin{aligned} \theta_0 &= 1, \\ \theta_{t+1} &= \theta_t \beta \left(\frac{\widetilde{C}_t}{\overline{C}} \right)^{-\kappa}. \end{aligned}$$

where $\widetilde{C}_t$ denotes Home average consumption, and $\overline{C} > 0$, $\beta \in (0,1)$ and $\kappa \geq 0$ are parameters. As in the DS model, the special case $\kappa = 0$ makes the discount factor fully exogenous and equal to β.

Let the Home good be the numeraire, and denote the relative price of Foreign by $p_{F,t}$. Then, the implied consumer price indices for Home and Foreign are:

$$P_t = \left[\alpha \left(\frac{1}{q_t} \right)^{1-\phi} + (1-\alpha) \left(\frac{p_{F,t}}{q_t^*} \right)^{1-\phi} \right]^{1/(1-\phi)}, \tag{6.1}$$

$$P_t^* = \left[(1-\alpha) \left(\frac{1}{q_t} \right)^{1-\phi} + \alpha \left(\frac{p_{F,t}}{q_t^*} \right)^{1-\phi} \right]^{1/(1-\phi)}. \tag{6.2}$$

In turn, optimal intra-temporal allocation yields the following relative demands for the two

[41] See Pavlova and Rigobon (2010) and Khalil (2019) for examples of DSGE portfolio-choice models with a similar type of shock.

goods:

$$
\begin{aligned}
c_{H,t} &= \alpha P_t^{\phi} (q_t)^{\phi-1} C_t, \\
c_{F,t} &= (1-\alpha) \left(\frac{P_t}{p_{F,t}}\right)^{\phi} (q_t^*)^{\phi-1} C_t,
\end{aligned} \tag{6.3}
$$

for Home, and

$$
\begin{aligned}
c_{H,t}^* &= (1-\alpha) (P_t^*)^{\phi} (q_t)^{\phi-1} C_t^*, \\
c_{F,t}^* &= \alpha \left(\frac{P_t^*}{p_{F,t}}\right)^{\phi} (q_t^*)^{\phi-1} C_t^*,
\end{aligned} \tag{6.4}
$$

for Foreign.

Asset trading and budget constraint. In each country there is a Lucas tree whose supply is normalized to unity. At each period, the Home tree delivers a endowment Y_t of the Home good. Out of this endowment, a Home share $\delta_t \in (0,1)$ is distributed to shareholders as dividends, while the remaining fraction $(1-\delta_t)$ is distributed to Home households as non-financial income. Similarly, the Foreign tree delivers an endowment of Foreign goods Y_t^*, so that shareholders receive a financial income of $\delta_t^* p_{F,t} Y_t^*$ as dividends (with $\delta_t^* \in (0,1)$), while Foreign households retaining a non-financial income of $(1-\delta_t^*)\, p_{F,t} Y_t^*$. Let $z_{H,t}^S$ and $z_{F,t}^S$ denote the prices of Home and Foreign equity, and $S_{H,t}$ and $S_{F,t}$ the start-of-period holdings of Home and Foreign stocks.

Households can also trade Home and Foreign one-period bonds in zero net supply. Let $z_{H,t}^B$ and $z_{F,t}^B$ denote the prices of Home and Foreign bonds, and $B_{H,t}$ and $B_{F,t}$ the start-of-period holdings of Home and Foreign bonds. Following CG, bonds of each country pays one unit of their respective consumption index *not* adjusted for preference shocks.[42] That is, the cash-flows of one unit of the Home and Foreign bond purchased at $t-1$ are:

$$
\widetilde{P}_t = \left[\alpha + (1-\alpha) (p_{F,t})^{1-\phi}\right]^{1/(1-\phi)}, \tag{6.5}
$$

$$
\widetilde{P}_t^* = \left[(1-\alpha) + \alpha (p_{F,t})^{1-\phi}\right]^{1/(1-\phi)}. \tag{6.6}
$$

Putting it all together, Home's representative household faces a budget constraint given by

$$
\begin{aligned}
& P_t C_t + z_{H,t}^S S_{H,t+1} + z_{F,t}^S S_{F,t+1} + z_{H,t}^B B_{H,t+1} + z_{F,t}^B B_{F,t+1} \\
&= (1-\delta_t) Y_t + S_{H,t} \left(\delta_t Y_t + z_{H,t}^S\right) + S_{F,t} \left(\delta_t^* p_{F,t} Y_t^* + z_{F,t}^S\right) + \widetilde{P}_t B_{H,t} + \widetilde{P}_t^* B_{F,t}.
\end{aligned} \tag{6.7}
$$

[42] CG introduce this assumption in their two-period model to generate a wedge between relative bond returns and changes in the real exchange rate that makes the inclusion of preference shocks meaningful. Since I have verified that the result holds in this infinite-horizon setup, I keep it as they have it.

Similarly, the budget constraint faced by Foreign's households is:

$$\begin{aligned}&P_t^* C_t^* + z_{H,t}^S S_{H,t+1}^* + z_{F,t}^S S_{F,t+1}^* + z_{H,t}^B B_{H,t+1}^* + z_{F,t}^B B_{F,t+1}^* \\ = &\left(1-\delta_t^*\right) p_{F,t} Y_t^* + S_{H,t}^* \left(\delta_t Y_t + z_{H,t}^S\right) + S_{F,t}^* \left(\delta_t^* p_{F,t} Y_t^* + z_{F,t}^S\right) + \widetilde{P}_t B_{H,t}^* + \widetilde{P}_t^* B_{F,t}^*. \end{aligned} \tag{6.8}$$

Stochastic processes. There are six independent sources of risk that makes financial markets incomplete. Home and Foreign income follow log-normally distributed AR(1) processes:

$$\ln Y_{t+1} = \rho_y \ln Y_t + \eta_y u_{y,t+1} \tag{6.9}$$

$$\ln Y_{t+1}^* = \rho_y \ln Y_t^* + \eta_y u_{y,t+1}^* \tag{6.10}$$

where $\left(u_{t+1}, u_{t+1}^*\right)$ are standard normal innovations. Redistributive shocks $\widetilde{\delta}_t = \ln\left(\delta_t/(1-\delta_t)\right)$ and $\widetilde{\delta}_t^* = \ln\left(\delta_t^*/(1-\delta_t^*)\right)$ follow AR(1) processes of the form:

$$\widetilde{\delta}_{t+1} = \left(1-\rho_\delta\right)\widetilde{\delta} + \rho_\delta \widetilde{\delta}_t + \eta_\delta u_{\delta,t+1} \tag{6.11}$$

$$\widetilde{\delta}_{t+1}^* = \left(1-\rho_\delta\right)\widetilde{\delta} + \rho_\delta \widetilde{\delta}_t^* + \eta_\delta u_{\delta,t+1}^* \tag{6.12}$$

These specifications ensure that the shares δ_t and δ_t^* are bounded within their natural domain. In particular, $\widetilde{\delta} = \ln\left(\overline{\delta}/(1-\overline{\delta})\right)$ ensures that the DSS of both δ_t and δ_t^* equals $\overline{\delta} \in (0,1)$. Finally, preference shocks q_t and q_t^* follow log-normally distributed AR(1) processes:

$$\ln q_{t+1} = \rho_q \ln q_t + \eta_q u_{q,t+1} \tag{6.13}$$

$$\ln q_{t+1}^* = \rho_q \ln q_t^* + \eta_q u_{q,t+1}^* \tag{6.14}$$

I assume that all innovations are serially uncorrelated, but will allow for intra-period correlation to match observed long-term correlations of output across countries, and between output and financial shares within countries.

Market-clearing conditions. At each period, equilibrium in the two-good markets requires

$$\begin{aligned} c_{H,t} + c_{H,t}^* &= Y_t, \\ c_{F,t} + c_{F,t}^* &= Y_t^*. \end{aligned} \tag{6.15}$$

Similarly, equilibrium in equity markets requires

$$S_{H,t} + S_{H,t}^* = 1, \tag{6.16}$$

$$S_{F,t} + S_{F,t}^* = 1, \tag{6.17}$$

and equilibrium in bond markets requires

$$B_{H,t} + B_{H,t}^* = 0, \tag{6.18}$$

$$B_{F,t} + B_{F,t}^* = 0. \tag{6.19}$$

6.2 Equilibrium conditions and solution

The model of interest just described cannot be solved by standard perturbation due to the same indeterminacy reasons highlighted in section 3.2. Appendix E.1 follows the same steps of section 3.3 by considering instead an auxiliary model where σ scales future innovations, and a function $f(\varepsilon) = (1 - \varepsilon^2)$ scales PAC-like modifications in the Euler equations encoding optimal behavior. As in the DS model, we can exploit the asset market-clearing conditions (6.16)-(6.19) to reduce the endogenous state-space in terms of Home states only. This leads to a set of 21 equilibrium conditions in terms of 10 state variables and 11 control variables. The 4 endogenous states are bond holdings $B_{H,t}$ and $B_{F,t}$, and stock holdings $S_{H,t}$ and $S_{F,t}$. The 6 exogenous states are Home and Foreign income Y_t and Y_t^*, Home and Foreign shares δ_t and δ_t^*, and preference shocks q_t and q_t^*. In turn, the 11 control variables are: consumptions C_t and C_t^*, equity prices $z_{H,t}^S$ and $z_{F,t}^S$, bond prices $z_{H,t}^B$ and $z_{F,t}^B$, price indices P_t, P_t^*, $\widetilde{P}_t$ and $\widetilde{P}_t^*$, and terms of trade $p_{F,t}$.

Appendix E.2 shows that the location parameters $\overline{a}$ and $\overline{b}$ from the PAC-like modifications pin down the DSS of the endogenous states in the deterministic auxiliary model: $B_{H,d} = \overline{b}$, $B_{F,d} = -\overline{b}$, $S_{H,d} = \overline{a}$, and $S_{F,d} = 1 - \overline{a}$. As in the DS example, I use the algorithm of section 5.3 (fed with second-order approximations) to solve for the pair $\left(\overline{a}, \overline{b}\right)$ that approximate the SSS of the endogenous states in the model of interest, and build a third-order solution around this point to study the equilibrium dynamics.

Once we have a solution for the equilibrium variables, we can use it to generate an additional set of variables of interest for which we have data (e.g., the real exchange rate, current accounts). Appendix E.3 lists these additional observed variables and provides their definitions in terms of the equilibrium ones.

6.3 Calibration

Table 2 reports the baseline calibration of the two-country model, chosen to match key moments of annual data for advanced economies (mostly the United States). The discount factor $\beta = 0.96$ delivers a DSS net return of 4% for all assets.[43] The risk aversion parameter $\gamma = 2$ is large enough to ensure that households want to increase their income when their consumption goods are more expensive (Coeurdacier and Rey, 2013). The home bias in consumption $\alpha = 0.85$ is set to match the average U.S. import ratio, and the trade elasticity is set to $\phi = 2$, which falls within the conservative range of empirical estimates using sectoral trade data (Imbs and Mejean (2015), Boehm et al. (2023)). In the results section I explore how changes to these two parameters affects the equilibrium outcomes.

Regarding the stochastic processes (6.9)-(6.12), the parameters are set to match estimates from AR(1) regressions using U.S. data. Specifically, I collect annual data from the FRED database for U.S. real GDP and net corporate dividends (as a share of GDP) for the time period 1947-2022. The average dividend share of 3.6% pins down the DSS share $\overline{\delta} = 0.036$. Then, I

[43]Since this model lacks features such as time-varying volatility or disaster risk, the SSS returns will be close to 4% for all assets too.

Table 2: Calibration of the two-country DSGE model

Parameter		Value
	Model parameters	
β	Discount factor	0.96
γ	Risk aversion	2
α	Share of domestic consumption	0.85
ϕ	Trade elasticity	2
κ	Uzawa parameter	0.007
	Stochastic processes	
$\overline{\delta}$	DSS dividend share	0.036
ρ_y	Persistence income shocks	0.51
ρ_δ	Persistence redistributive shocks	0.42
ρ_q	Persistence preference shocks	0.46
η_y	Std. income shocks	0.018
η_δ	Std. redistributive shocks	0.059
η_q	Std. preference shocks	0.007
ρ_{yy^*}	$\text{Corr}\left(u_{y,t+1}, u^*_{y,t+1}\right)$	0.68
$\rho_{y\delta}$	$\text{Corr}\left(u_{y,t+1}, u_{\delta,t+1}\right)$	0.12

apply the Hodrick-Prescott (HP) filter to detrend the variables $\ln Y_t$ and $\widetilde{\delta}_t$, and run AR(1) regressions using their cyclical components. The procedure yields auto-correlations of $\rho_y = 0.51$ and $\rho_\delta = 0.42$, and conditional standard deviations of $\eta_y = 0.018$ and $\eta_\delta = 0.059$. Moreover, the correlation between the two time-series pins down the correlation of income and redistributive shocks to $\rho_{y\delta} = 0.12$.

The remaining parameters are calibrated as follows. The correlation between Home and Foreign income shocks $\rho_{yy^*} = 0.68$ helps to match the observed cross-country correlation of GDP between the U.S. and a set of OECD countries reported by Corsetti, Dedola and Leduc (2008, Table 3). Since preference shocks are unobserved, I set the auto-correlation $\rho_q = 0.42$ such that $\rho_\delta < \rho_q < \rho_y$ and, as explained in Appendix E.4, calibrate a standard deviation $\eta_q = 0.0067$ such that the model with bonds and equities matches a SSS of equity home bias equal to 0.66 (the index value for the U.S. in 2008 reported by Coeurdacier and Rey (2013, Table 1)).[44] Finally, the Uzawa parameter $\kappa = 0.007$ helps to match the auto-correlation of net exports over GDP ratio for the U.S., equal to 0.62 using (HP detrended) FRED data.

[44] A value of 1 for this index indicates full bias, and a value of 0 perfect diversification. See Appendix E.3 for the precise construction of this index as well as for other observed variables.

6.4 Results

I study the quantitative implications of the CG economy by generating standard outputs from DSGE models, namely comparative static exercises, impulse responses to shocks, and stochastic simulations. To economize on space, here I report results from the comparative statics and simulations, and leave the analysis of impulse responses to Appendix E.4.

6.4.1 Comparative statics at the stochastic steady state

A natural place to start the analysis is by looking at the long-run implications of the CG model, which I do by performing several comparative static exercises on the SSS of the Home economy in the model of interest. The first experiment, replicating the one conducted by CG with their two-period economy, compares the full model with bonds and equities against a simpler version with equities only. It explores how changes in the standard deviation of shocks affect the SSS of asset holdings, and confirms that, as CG put it, bonds matter, in that they are essential to generate a level of equity home bias consistent with the data. The second experiment investigates the implications of introducing a parameter asymmetry, wherein domestic households become less risk-averse than foreigners, on the external asset position of the countries. Appendix E.4 provides the details and results of these two experiments.

The third experiment, reported here, analyzes the trade and financial linkages generated by the model. In the steady state, trade openness (exports plus imports over GDP) is essentially exogenous, as it is given by $2(1-\alpha)$, where $\alpha > 0.5$ controls home bias in goods. To explore the effects of trade openness on external asset positions, I vary $1-\alpha$ between 0.1 and 0.45, compute the associated steady states of bonds and equities, and use them to construct measures of external equity and debt assets as indicated in Appendix E.3. Figure 5 shows the results, where the dashed green line is equity openness (external equity assets and liabilities over GDP), the red dotted line is debt openness (external debt assets and liabilities over GDP), and the blue solid line is financial openness (the sum of equity and debt openness), a common measure of financial integration in the empirical literature.[45]

The results in Figure 5 are consistent with the data. First, the model predicts a strong positive relationship between trade and financial openness, as documented by Heathcote and Perry (2013).[46] Furthermore, this positive relationship is almost entirely driven by a large increase in the external debt position of countries, whereas the change in the external equity position is much more modest and can even decrease for sufficiently large levels of trade openness (Khalil, 2019). Since these patterns hold empirically both across countries and over time, the findings of Figure 5 help to rationalize the slow reduction in equity home bias despite decades of intense trade globalization and large increases in external gross positions (Coeurdacier and Rey, 2013; Gourinchas and Rey, 2014). Last but not least, the magnitudes of external asset positions in

[45] Since countries are symmetric, equity and debt liabilities equal their asset counterparts, and total external assets equals external liabilities.

[46] Indeed, the fast rate of increase of financial openness in this model does a better job that theirs in rationalizing Figure 2 of their paper, where they plot a sample of advanced economies in the trade-financial space. See also IMF (2002).

Figure 5: Trade and financial linkages in the CG model

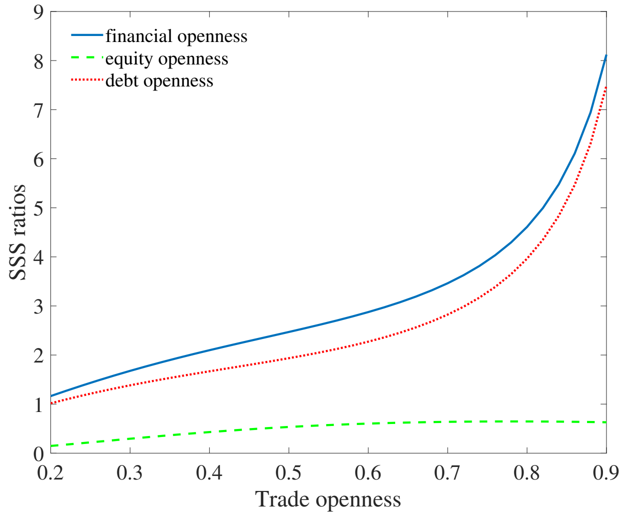


Figure 5 are comparable to their empirical counterparts for advanced economies in the Lane and Milesi-Ferretti (2018) dataset: just as predicted by the model, the debt openness ratio for the more trade-integrated countries in 2008 (such as Belgium, the Netherlands, and the U.K.), is slightly above 5, and the equity openness ratio is always well below 1.

What is the intuition behind these predictions? As in the DS model, countries choose to hold a leveraged bond position that is long in Home and short in Foreign. This portfolio provides stabilizing transfers because the cash-flows of Home and Foreign bonds respond differently to the same income shocks. But since the degree of this asymmetry is a direct function of home bias in goods, lower home bias (higher trade openness) implies smaller transfers other things equal. To compensate for this effect, more trade integrated countries must leverage up their bond positions just to maintain the same transfer size. As α approaches 0.5, the asymmetry between cash-flows almost entirely vanishes, and a extremely large gross debt position is needed for the transfer scheme to work at all.

As for equities, the equilibrium portfolio is determined by two opposing forces: redistributive shocks incentivize a less diversified portfolio (smaller foreign equity holdings), while preference shocks encourage a more diversified one. In this model, the relative strength of these two forces changes with α in a non-monotonic way. Specifically, the net incentive towards diversification is strong for low levels of trade openness, but it weakens as trade openness increases.[47] As such, the pattern of equity openness in Figure 5 is the logical outcome of these changing incentives.

[47]Consistent with this argument, the size of impulse responses to the return differential of stocks changes with α in a non-monotonic way.

6.4.2 Stochastic Simulations

Table 3 reports moments from stochastic simulations of the model. The "Baseline" column corresponds to a simulation using the calibration specified in Table 2. Relative to this scenario, the "High ϕ" column doubles the trade elasticity ($\phi = 4$), the "Low α" column considers a World with lower Home bias in goods ($\alpha = 0.75$), and the "Different γ's" column introduces a parameter asymmetry between countries by making domestic households less risk-averse than foreigners: $\gamma = 1.75$ for Home, and $\gamma = 2.25$ for Foreign (the moments reported are from the Home economy). For reference, the "Data" column reports the empirical counterparts for the United States. In all cases, I simulate the model $100,000$ periods (with a burn-in of $10,000$ additional periods) feeding the decision rules with pseudo-random innovations.

Despite the absence of a pruning scheme, these simulations produce well-behaved unimodal distributions for all model variables. In the first three columns, I calibrate the volatility of preference shocks to match a SSS with an index of equity home bias equal to 0.66 as in the data. Thus, the fact that the averages of this variable equal 0.67 shows that, at least in this model, the SSS is very close to the ergodic mean. It is then no surprise that the model performs reasonably well in delivering average external asset positions (all measured as ratios to GDP) comparable to those from the U.S., with external equity assets of 0.14 in the model versus 0.19 in the data, and total external assets of 0.84 in the model versus 1.04 in the data.[48]

The model is less successful in matching empirical second moments. On the positive side, the model closely matches the volatility of the trade balance (std. of 0.55%), and it correctly predicts that consumption is less volatile than GDP. But compared to the data, the std. of the real exchange rate (RER) of 0.54% is about seven times lower, the std. of the current account (0.30%) is about two times lower, and the std. of the change in net foreign assets (NFA) of 0.53% is about five times lower.

More troubling are the results for gross capital flows. Broner et al. (2013) and Davis and van Wincoop (2018) show that, empirically, gross capital flows are much more volatile than net capital flows (the negative of the current account). In Table 3, the opposite is true. To understand this result, note that in all figures from Appendix E.4 depicting impulse responses, capital inflows (CIF) and capital outflows (COD) are the opposite mirror of each other: if CIF rises, COD falls by almost the same magnitude. It follows that CIF and COD are almost perfectly negatively correlated in the model (whereas in the data the correlation is 0.78), and thus they cancel each other as a sum, which is the definition of gross capital flows.[49]

The model closely matches the empirical serial correlations of the trade balance (0.63) and of the change in NFA (-0.01), but it underestimates the one of the current account (0.46 in the model versus 0.67 in the data). In turn, the equity home bias follows an extremely persistent process with a serial correlation of 0.99, which is consistent with the impulse responses of

[48]Since in the first three columns countries are perfectly symmetric, there are no long-run global imbalances: on average, total external liabilities match total external assets.

[49]See Tille and van Wincoop (2010) for a model with a positive correlation between CIF and COD, driven by time-varying second moments of the return differentials. While in the current model these moments are also time-varying, their impact on portfolio reallocation is almost nil.

Table 3: Simulated Moments of the CG model

	Baseline	High ϕ	Low α	Different γ's	Data
Averages:					
Equity home bias	0.67	0.67	0.67	0.49	0.66
External equity assets	0.14	0.15	0.15	0.30	0.19
Total external assets	0.84	1.97	1.43	0.96	1.04
Total external liabilities	0.84	1.97	1.44	0.70	1.38
Euler errors (in log10)	-7.49	-7.41	-7.45	-7.40	
Standard deviations (relative to GDP):					
Consumption	0.98	0.93	0.95	1.07	0.94
RER	0.54	0.26	0.27	0.55	3.90
Absolute standard deviations:					
Trade balance	0.55	0.66	0.55	0.56	0.56
Current account	0.30	0.29	0.30	0.31	0.66
Δ(Net foreign assets)	0.53	0.63	0.53	1.12	2.36
Gross capital flows	0.02	0.02	0.01	0.02	
Equity home bias	0.05	0.05	0.05	0.03	
Serial correlations:					
Trade balance	0.63	0.58	0.61	0.62	0.62
Current account	0.46	0.46	0.46	0.44	0.67
Δ(Net foreign assets)	-0.01	-0.09	-0.02	-0.19	0.03
Equity home bias	0.99	0.99	0.99	0.99	
Cross-Correlations:					
RER and relative consumption	-0.34	-0.17	-0.27	-0.25	0.71
Home and Foreign consumption	0.89	0.94	0.93	0.88	0.60
Capital inflows and outflows	-0.99	-0.99	-0.99	-0.99	0.78
Current account and GDP	0.06	0.10	0.10	0.01	-0.52
Δ(Net foreign assets) and GDP	-0.11	-0.12	-0.09	0.39	-0.27
Conditional hedge ratios:					
Relative bond returns	0.81	0.82	0.76	0.82	0.94
Relative equity returns	-0.18	-0.07	-0.09	-0.17	-0.01

Notes. The table reports simulated moments of the two-country DSGE model, using a third-order approximation to the policy rules around the SSS. Each simulation contains $100,000$ observations to generate an ergodic set. RER is the real exchange rate. The column "Baseline" reports results using the baseline calibration. The column "High ϕ" doubles the elasticity of substitution between Home and Foreign goods. The column "Low α" sets $\alpha = 0.75$ to match the World's import share. The column "Different γ's" makes Home and Foreign heterogeneous by setting $\gamma_H = 1.75$ and $\gamma_F = 2.25$, and reports results from Home's perspective. The sources and periods of the data are as follows: Coeurdacier and Rey (2013) for the 2008 U.S. home bias; the dataset by Lane and Milesi-Ferretti (2018) for 2008 U.S. external positions; Corsetti, Dedola and Leduc (2008) for U.S. relative standard deviations, and cross-correlations of RER and consumption; Coeurdacier, Kollman and Martin (2010) for data on U.S. net foreign assets, Broner et al. (2013) for the correlation between inflows and outflows for advanced economies; Coeurdacier and Gourinchas (2016) for U.S. hedge ratios, and the author's calculations using U.S. data from the FRED for the remaining variables.

Appendix E.4 that highlight time-varying wealth as a key driver of portfolio reallocation.

The model also struggles reproducing international co-movements. The simulations deliver a negative correlation between the RER and relative consumption (-0.34 in the baseline scenario), while in the data this correlation is strongly positive, at least for the U.S. (0.71). Still, this result is an improvement relative to complete-markets models where full risk-sharing leads to perfectly negative correlations (Backus and Smith, 1993).[50] Moreover, if one uses instead a measure of RER not adjusted for preference shocks, the correlation raises significantly (-0.09). As an additional symptom of excessive risk-sharing, Home and Foreign consumptions are too synchronized, both relative to the empirical counterpart (correlation of 0.89 or above in the model, versus 0.60 in the data), and relative to the cross-correlation between Home and Foreign GDP (0.68), leading to what is commonly referred to as the Consumption Correlation Puzzle (Backus, Kehoe, and Kydland, 1992).

Another problem, shared with many open-economy models, is that the simulations fail to reproduce the empirical negative co-movement between the current account and GDP (0.06 of above in the model, versus -0.52 in the data). On a positive note, the model correctly predicts a small and negative correlation between NFA and GDP (-0.11 in the model, -0.27 in the data).

The model is relatively successful in reproducing observed conditional hedge ratios between changes in the RER and relative asset returns. Similar to CG, these ratios correspond to estimates from OLS regressions, where the dependent variable is the change in the RER, and the regressors are (the logs of) the return differentials of equities and bonds. The model simulations successfully reproduce a large hedge ratio for relative bond returns (0.81 in the model, compared to 0.94 in the empirical application of CG), and a small, negative hedge ratio for relative equity returns (-0.18 in the model, -0.01 in the empirical application of CG).[51] Again, using a measure of RER not adjusted for preference shocks improves the results substantially, yielding hedge ratios of 1 for bond returns, and 0 for equity returns. Overall, this outcome simply confirms a point already made in CG: that bonds provide a much effective hedge against income shocks compared to equities.

Finally, Table 3 reveals that changes to the trade elasticity or the home bias in goods make little difference in shaping the ergodic distributions of the simulations, as all moments remain quite similar. However, the picture changes when domestic households are less risk-averse than foreigners. On the one hand, average total external assets (0.96) are larger than average total external liabilities (0.70), indicating a long-run global imbalance where Home, on average, serves as a net creditor. On the other hand, domestic consumption becomes more volatile than GDP (and foreign consumption, not reported, becomes substantially less volatile). That is, the price paid by Home for a larger average consumption is an increase in its volatility.[52]

[50]The impulse responses from Appendix E.4 show that preference shocks are the main driver behind this departure from perfect risk-sharing, since they trigger positive transfers from Foreign when Home is doing fine. Indeed, the correlation between RER and relative consumption converges to -1 as I shut down preference risk.

[51]See van Wincoop and Warnock (2010) for a similar result concerning relative equity returns.

[52]While these results are at odds with U.S. data (Gourinchas, Rey, and Govillot, 2017), they are consistent with recent empirical findings by Zhang (2023), who shows that creditor countries tend to exhibit more diversified international portfolios.

6.4.3 Accuracy and computing times.

To test the accuracy of the third-order solution, I proceed as in the DS example by feeding the wealth Euler equation (see Appendix E.3) with the policy rules to compute average unit-free Euler errors (in log10 scale) of the simulated time-series, discretizing the innovations with monomials to approximate expectations. The last row of "Averages" in Table 3 shows that the simulations are, indeed, highly accurate. In all scenarios, the mean errors are well below -7, and the max. errors (not reported) are around -6.

Regarding computing times, in the baseline calibration, it takes 0.06 seconds to compute the SSS starting from the agnostic guess $S_H = B_H = 0$, 0.06 seconds to compute the SSS in calibration mode (i.e., fixing $S_H = 0.83$ to target the observed equity home bias and solving jointly for the std. of preference shocks and B_H), and 0.02 seconds to compute the third-order solution. Using Dynare instead, it takes 0.07 seconds to compute the SSS (0.08 seconds in calibration mode), and 1.85 seconds to compute the third-order solution.

7 Conclusion

This paper has proposed a small generalization to the standard perturbation approach by introducing a new perturbation parameter that scales auxiliary modifications in the equilibrium conditions of the DSGE model. The resulting two-parameter perturbation model is well suited for solving DSGE models with incomplete markets, as it allows to solve for the dynamics of a nearby auxiliary model, and then use perturbation to reach the model of interest. Exploiting the ability of two-parameter perturbation to approximate the model of interest around a point other than its implied deterministic steady state, the paper has developed a simple algorithm that, backed by theoretical results, approximates the DSGE economy around the stochastic steady state. Since the method is fully compatible with popular solution toolboxes, researchers can now solve DSGE models with non-trivial wealth distributions, portfolio choice, and *ex-ante* heterogeneity just like any other standard model.

As a main application, the paper has extended the two-period, multi-asset model of Coeurdacier and Gourinchas (2016) to an infinite-horizon setting, and has performed a rigorous quantitative analysis that includes the study of portfolio dynamics. Among other results, the paper has successfully demonstrated that an incomplete-market structure with bonds and equities is essential to generate long-run external gross positions comparable to the data, and has uncovered a new natural link between trade and financial integration consistent with the main patterns of globalization during the last decades.

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Online Appendix to "Solving DSGE Models with Incomplete Markets by Perturbation"

A The Devereux-Sutherland model with the original notation

Combining equations (3.27) and (3.28) with the market-clearing conditions (3.10) and (3.11), it is easy to verify that in equilibrium their counterparts in Foreign satisfy $\alpha_t^* = -\alpha_t$, and $W_t^* = -W_t$. In addition, we can define

$$R_{H,t} = \delta_t \frac{e^{y_{K,t}}}{z_{H,t-1}}, \tag{A.1}$$

$$R_{F,t} = \delta_t \frac{e^{y_{K,t}^*}}{z_{F,t-1}}, \tag{A.2}$$

as the current gross returns of home and foreign assets. Note that, because asset prices $z_{H,t-1}$ and $z_{F,t-1}$ are pre-determined in the definitions, they become state variables under the DS notation.

With this change of variables, the new equilibrium budget constraints for Home and Foreign replacing (3.8) and (3.9) are:

$$C_t + W_{t+1} = Y_t + \left(R_{H,t} - R_{F,t}\right)\alpha_t + R_{F,t}W_t, \tag{A.3}$$

$$C_t^* - W_{t+1} = Y_t^* - \left(R_{H,t} - R_{F,t}\right)\alpha_t - R_{F,t}W_t, \tag{A.4}$$

which are the same as equations (3) and (5) in DS.[53] In turn, the Euler equations written in terms of future returns are:

$$\frac{1}{(C_t)^{\gamma}} = \beta\left(C_t\right)^{-\kappa}\mathbb{E}_t\left[\frac{R_{H,t+1}}{(C_{t+1})^{\gamma}}\right], \tag{A.5}$$

$$\frac{1}{(C_t)^{\gamma}} = \beta\left(C_t\right)^{-\kappa}\mathbb{E}_t\left[\frac{R_{F,t+1}}{(C_{t+1})^{\gamma}}\right], \tag{A.6}$$

for Home, and

$$\frac{1}{(C_t^*)^{\gamma}} = \beta\left(C_t^*\right)^{-\kappa}\mathbb{E}_t\left[\frac{R_{H,t+1}}{(C_{t+1}^*)^{\gamma}}\right], \tag{A.7}$$

$$\frac{1}{(C_t^*)^{\gamma}} = \beta\left(C_t^*\right)^{-\kappa}\mathbb{E}_t\left[\frac{R_{F,t+1}}{(C_{t+1}^*)^{\gamma}}\right], \tag{A.8}$$

for Foreign.

[53] Note that the definitions (3.27) and (3.28) imply that α_t and W_t appear as pre-determined in the budget constraints. Since this is also the case for asset prices in the definitions (A.1) and (A.2), we have a total of 4 endogenous state variables that, with regular perturbation, need to be treated as such. This result contrasts sharply with DS, who postulate that policy functions (α_t in particular) only depend on W_t. While this might come true *ex-post*, we cannot rule out dependence on the other states *ex-ante* (at least not when working with regular perturbation).

Equilibrium with the DS notation. The 4 endogenous state variables are real holdings α_t, wealth W_t, and asset prices $z_{H,t-1}$ and $z_{F,t-1}$. As in section 3.1, the 5 exogenous states are capital log-income components $y_{K,t}$ and $y^*_{K,t}$, labor log-income components $y_{L,t}$ and $y^*_{L,t}$, and the share δ_t. The 6 control variables are Home and Foreign consumptions C_t and C^*_t, endowments Y_t and Y^*_t, and returns $R_{H,t}$ and $R_{F,t}$. The 15 equilibrium conditions that these variables must satisfy are the income processes (3.3), (3.4), (3.6) and (3.7), law of motion (3.2), endowment equations (3.1) and (3.5), the definitions of returns (A.1) and (A.2), the new budget constraints (A.3) and (A.4), and the new Euler equations (A.5)-(A.8).

Auxiliary model. The steps to solve the model via two-parameter perturbation are very similar to those of section 3.3. The small twist is that, since now we also have asset prices as endogenous states, consistency requires that we must solve for their SSS values simultaneously with the SSS of α_t and W_t. As shown below, we can accomplish this by introducing in the auxiliary model an additional auxiliary parameter.

Same as in section 3.3, equations (**??**)-(**??**) with the parameter σ scaling innovations replace the log-income processes (3.3), (3.4), (3.6) and (3.7). As for the Euler equations, the new equilibrium conditions replacing (A.5)-(A.8) include, in addition to the PAC-like modifications, a new auxiliary parameter ψ_2 that makes the exogenous component of the discount factor equal to $\beta + \psi_2$ in the deterministic auxiliary model. Specifically, we have:

$$\frac{1}{(C_t)^\gamma}\left[1 + \left(1-\varepsilon^2\right)\psi_1\left(\frac{\alpha_{t+1}}{z_{H,t}} - \bar{a}\right)\right] = \left[\beta + \left(1-\varepsilon^2\right)\psi_2\right](C_t)^{-\kappa}\,\mathbb{E}_t\left[\frac{R_{H,t+1}}{(C_{t+1})^\gamma}\right], \tag{A.9}$$

$$\frac{1}{(C_t)^\gamma}\left[1 + \left(1-\varepsilon^2\right)\psi_1\left(\frac{W_t - \alpha_{t+1}}{z_{F,t}} + \bar{a}\right)\right] = \left[\beta + \left(1-\varepsilon^2\right)\psi_2\right](C_t)^{-\kappa}\,\mathbb{E}_t\left[\frac{R_{F,t+1}}{(C_{t+1})^\gamma}\right], \tag{A.10}$$

for Home, and

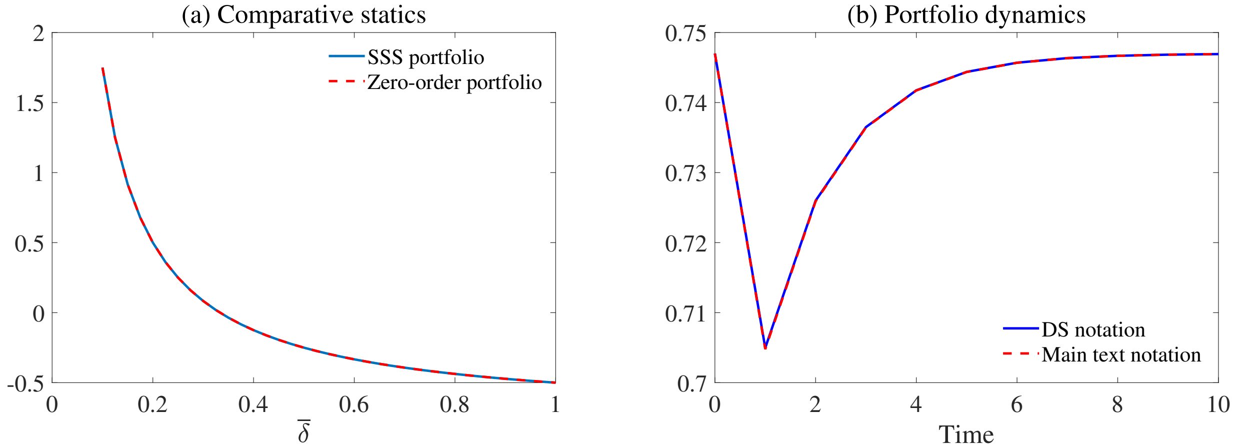
$$\frac{1}{(C^*_t)^\gamma}\left[1 - \left(1-\varepsilon^2\right)\psi\left(\frac{\alpha_{t+1}}{z_{H,t}} - \bar{a}\right)\right] = \left[\beta + \left(1-\varepsilon^2\right)\psi_2\right](C^*_t)^{-\kappa}\,\mathbb{E}_t\left[\frac{R_{H,t+1}}{(C_{t+1})^\gamma}\right], \tag{A.11}$$

$$\frac{1}{(C^*_t)^\gamma}\left[1 - \left(1-\varepsilon^2\right)\psi\left(\frac{W_t - \alpha_{t+1}}{z_{F,t}} + \bar{a}\right)\right] = \left[\beta + \left(1-\varepsilon^2\right)\psi_2\right](C^*_t)^{-\kappa}\,\mathbb{E}_t\left[\frac{R_{F,t+1}}{(C^*_{t+1})^\gamma}\right], \tag{A.12}$$

for Foreign. When $\varepsilon = 0$, the modifications make sure that these equations are independent of each other, and that the DSS of the state variables is a direct function of the auxiliary parameters $\bar{a}$ and ψ_2. In particular, we have $z_{H,d} = z_{F,d} = (\beta + \psi_2)\,\overline{\delta}$, $\alpha_d = (\beta + \psi_2)\,\overline{\delta}\bar{a}$, and, by symmetry, $W_d = 0$.

The additional degree of freedom obtained by introducing ψ_2 gives us full control of the symmetric DSS of asset prices and real holdings, allowing us to solve the model around a large subset of points in the state-space. It follows that we can use the SSS algorithm of section 5.3 to solve for the SSS of the state variables. Since symmetry implies $W_s = 0$ and $z_{H,s} = z_{F,s}$, the auxiliary parameters $\bar{a}$ and ψ_2 are sufficient to pin down $z_{H,s}$ and α_s simultaneously. Once the SSS of the states is found, it is straightforward to compute a third-order solution around this

Figure 6: Solving the DS model with different notations



point.

Figure 6 compares the solution to the DS model obtained with the original notation against the solution from the main text. Panel (a) replicates the same comparative statics exercise of figure 2, panel (a), and panel (b) replicates the same impulse response experiment of figure 3, panel (c). In both panels the variable under study is nominal asset holdings and the solid blue line represents the the solution with the original DS notation. In turn, a dashed red line indicates the zero-order portfolio obtained with the formula 3.26 in panel (a), and the same impulse response from Figure 3 in panel (b). Since in both panels the two lines are identical, I conclude that, at least in this model, the change of variables does not affect the equilibrium outcomes.

B Linear solutions and pruning

This appendix discusses two alternatives to the plain higher-order solutions obtained with two-parameter perturbation. Since it uses the notation and results from section 5 extensively, I recommend the reader to review that section before proceeding.

B.1 Linear solutions

The goal is to derive a linear solution valid for the model of interest. As explained in section 3.7, the first-order solution obtained with the auxiliary model does not work because it exhibits certainty-equivalence and, depending on how ε is introduced, may fail to correct for the auxiliary modifications. The same goes with a second-order solution if we use the SSS algorithm. Instead, what we can do is compute the Jacobian of a Taylor series (of order 3 or higher) evaluated at the model of interest ($\varepsilon = \sigma = 1$) and use it to construct a linear, risk-corrected solution around the SSS. The outcome of this approach is conceptually very similar to the linear solution proposed by Coeurdacier, Rey and Winant (2011).[54]

To formalize this idea, let $\widetilde{\mathbf{h}}^k(\mathbf{x_t}) = \mathbf{h}^k(\mathbf{x_t}, 1, 1)$ be the order $k \geq 3$ decision rule for the endogenous states, evaluated at $\varepsilon = \sigma = 1$. Note that $\widetilde{\mathbf{h}}^k(\mathbf{x_t})$ is a polynomial function that maps $\mathbb{R}^{n_x}$ into $\mathbb{R}^{n_x}$. Moreover, suppose that we have implemented the SSS algorithm so that $\mathbf{x_d} = \mathbf{x_s}$ holds. Thus, the Jacobian of $\widetilde{\mathbf{h}}^k(\mathbf{x_t})$ about $\mathbf{x_t} = \mathbf{x_s}$, denoted by $\mathbf{J_h^k}$, is a $n_x \times n_x$ matrix of coefficients that includes terms correcting for risk and the auxiliary modifications. We can then use $\mathbf{J_h^k}$ to build the following linear law of motion for the states:

$$\mathbf{x_{t+1}} = \mathbf{x_s} + \mathbf{J_h^k}\left(\mathbf{x_t} - \mathbf{x_s}\right).$$

We can apply the same idea to obtain a linear law of motion for the control variables $\mathbf{y_t}$. Let $\widetilde{\mathbf{g}}^k(\mathbf{x_t}) = \mathbf{g}^k(\mathbf{x_t}, 1, 1)$ be the order $k \geq 3$ decision rule for $\mathbf{y_t}$, evaluated at $\varepsilon = \sigma = 1$. Since this polynomial function maps $\mathbb{R}^{n_x}$ into $\mathbb{R}^{n_y}$, its Jacobian $\mathbf{J_g^k}$ about $\mathbf{x_t} = \mathbf{x_s}$ is a $n_y \times n_x$ matrix of coefficients that, as before, includes all the corrections terms. Moreover, we can use equation (5.14) to compute the SSS $\mathbf{y_s}$. The proposed linear rule for the controls is thus:

$$\mathbf{y_t} = \mathbf{y_s} + \mathbf{J_g^k}\left(\mathbf{x_t} - \mathbf{x_s}\right).$$

The Github repository includes MATLAB functions that construct, evaluate and simulate these linear rules for Dynare and the perturbation algorithm by Levintal (2017).

B.2 Pruning

It is well known that high-order perturbation solutions may sometimes generate spurious explosive simulations. To resolve this issue for orders $k \geq 3$, we can implement the pruning method by Andreasen, FernÃ¡ndez-Villaverde, and Rubio-RamÃrez (2018). However, their approach treats σ as a state variable, and therefore it only affects second and higher-order effects. This can be problematic if we use two-parameter perturbation to solve portfolio-choice models, because it imposes that the first-order effects —the main driver of dynamics in a pruned state-space system— are those from the auxiliary model. [55]

My proposed solution is to slightly modify their pruning method by treating ε and σ (or at the very least just σ) as parameters. To do so, impose $\varepsilon = \sigma = 1$, and rewrite the Taylor series (5.5) and (5.6) as follows:

$$\begin{aligned} \widetilde{\mathbf{h}}^k(\mathbf{x_t}) &= \widehat{H}_0 + \sum_{i=1}^{k} \frac{1}{i!}\widehat{H}_i \left(\mathbf{x_t} - \mathbf{x_d} \right)^{\otimes i} \\ \widetilde{\mathbf{g}}^k(\mathbf{x_t}) &= \widehat{G}_0 + \sum_{i=1}^{k} \frac{1}{i!}\widehat{G}_i \left(\mathbf{x_t} - \mathbf{x_d} \right)^{\otimes i} \end{aligned}$$

[54] Under the assumptions of Appendix C, it is easy to show that the proposed solution converges to the first-order expansion of the true policy functions around their SSS.

[55] The problem also arises in SOE models with modifications controlled by parameters set to very small values (e.g. FernÃ¡ndez-Villaverde et al., 2011), as these lead to nearly unit-root first-order dynamics that only get corrected by third-order terms.

where $\widehat{H}_0$ and $\widehat{G}_0$ are vectors of dimension $n_x \times 1$ and $n_y \times 1$, and each H_i and G_i are matrices with dimensions $n_x \times (n_x)^i$ and $n_y \times (n_x)^i$.[56] From here, one can follow Andreasen, FernÃ¡ndez-Villaverde, and Rubio-RamÃrez (2018) and construct the laws of motions for each n-order effect using the rewritten Taylor series. The Github repository includes MATLAB functions that implement the modified pruning method (third-order only) for users of Dynare and the algorithm by Levintal (2017).[57]

C Convergence to the true stochastic steady state

Let $\overline{\mathbf{x}}^k$ be the vector that solves the system (5.13) for a given order k. The following assumptions are sufficient to prove that $\overline{\mathbf{x}}^k$ converges to $\mathbf{x_{1,s}}$ as k goes to infinity:

Assumption 1. *There exists a vector $\overline{\boldsymbol{\psi}}$ of values of the auxiliary parameters $\boldsymbol{\psi}$ such that $\mathbf{x_s}$ satisfies the DSS condition (5.3).*

Assumption 2. *The function $\mathbf{h}(\mathbf{x}, \varepsilon, \sigma)$ is analytic in an open set around $(\mathbf{x_s}, 0, 0)$ that includes at least all points $(\mathbf{x_s}, \varepsilon, \sigma)$ satisfying $\varepsilon = \sigma$ up to $(\mathbf{x_s}, 1, 1)$.*

The first assumption ensures that the true SSS can be the DSS of the auxiliary model. The second assumption ensures that the Taylor series of $\mathbf{h}$ about $(\mathbf{x_s}, 0, 0)$ converges to $\mathbf{h}(\mathbf{x}, 1, 1)$ when evaluated at the true SSS.[58] In practice, this will be satisfied provided that both $\overline{\psi}$ and the matrix η scaling innovations are small enough in magnitude.

Under these regularity assumptions, the following proposition holds:

PROPOSITION. *If Assumptions 1 and 2 hold, $\overline{\mathbf{x}}^k$ converges to $\mathbf{x_{1,s}}$ as $k \to \infty$.*

PROOF. Let $\mathbf{h^T}_1(\mathbf{x})$ be the Taylor series of $\mathbf{h}_1$ about $(\mathbf{x}, 0, 0)$ evaluated at $(\mathbf{x}, 1, 1)$, given by:

$$\mathbf{h^T}_1(\mathbf{x}) = H_{1,0} + \sum_{i=1}^{k} \frac{1}{i!} H_{1,i} \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i} + \zeta^k,$$

where $H_{1,0} = \mathbf{h}_1(\mathbf{x}, 0, 0)$ is a $n_{x1} \times 1$ vector, and

$$\zeta^k = \sum_{i=k+1}^{\infty} \frac{1}{i!} H_{1,i} \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i}$$

[56]Note that, if we use the SSS algorithm, $\widehat{H}_0 = \mathbf{x_s} = \mathbf{x_d}$ and $\widehat{G}_0 = \mathbf{y_s}$.

[57]For Dynare, I implement a very mild modification to the original Dynare function simul_.m that only treats σ as a parameter.

[58]Note that Assumption 2 is equivalent to assuming analyticity within an open ball around $\sigma = 0$ that includes $\sigma = 1$ in a one-parameter approach. Importantly, the open set does not need to include the point $(\mathbf{x_s}, 1, 0)$, which depending on the application (e.g., the one-bond economy of Appendix D.4) might be problematic.

is the error term of the k-order Taylor series evaluated at $(\mathbf{x}, 1, 1)$. Fix $\boldsymbol{\psi} = \overline{\boldsymbol{\psi}}$ and evaluate $\mathbf{h^T}_1$ at $\mathbf{x} = \mathbf{x_s}$. Since Assumption 1 holds, $\mathbf{x_s}$ is the fixed-point of $\mathbf{h}(\mathbf{x}, 0, 0)$, and therefore $H_0 = \mathbf{x_{1,s}}$. Since Assumption 2 holds, convergence of the Taylor series gives $\mathbf{h^T}_1 (\mathbf{x_s}) = \mathbf{h}_1 (\mathbf{x_s}, 1, 1)$, and by definition (5.11) we have $\mathbf{x_{1,s}} = \mathbf{h_1} (\mathbf{x_s}, \mathbf{1}, \mathbf{1})$. Combining these two equalities gives $\mathbf{h^T}_1 (\mathbf{x_s}) = \mathbf{x_{1,s}}$, which is equivalent to

$$\sum_{i=1}^{k} \frac{1}{i!} H_{1,i} \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i} + \zeta^k = 0. \tag{C.1}$$

Note that each matrix $H_{1,i}$ is a continuous function of $\mathbf{x_{1,s}}$. If $\mathbf{h^T}_1 (\mathbf{x_s})$ is an exact k-order Taylor series, then $\zeta^k = 0$ and the system (C.1) is identical to (5.13) in the main text, which automatically gives $\overline{\mathbf{x}}^k = \mathbf{x_{1,s}}$. Otherwise Assumption 2 implies that $\lim_{k\to\infty} \zeta^k = 0$ holds, and the systems (C.1) and (5.13) become identical as $k \to \infty$, leading to $\overline{\mathbf{x}}^k \to \mathbf{x_{1,s}}$. ■

D Examples

D.1 Portfolio choices for small risks

Same example used by Judd (1998, Chapter 15) to introduce bifurcation. It consists of a one-period model whose approximation can be worked out using pen and paper, and it examines the portfolio choice problem of an individual investor with access to a safe and a risky asset. The auxiliary model introduces portfolio adjustment costs (PAC) to pin down the proportion of wealth allocated in the risky asset. Remarkably, two-parameter perturbation yields the same approximation formulas obtained by Judd.

An investor has one unit of wealth to invest in two assets. The safe asset yields 1 dollar per dollar invested, and the risky asset yields Z dollars per dollar invested, where Z is stochastic and given by

$$Z = 1 + \pi + z \tag{D.1}$$

Here $\pi \geq 0$ is the premium paid by the risky asset, and z a random variable with moments $\mathbb{E}\{z\} = 0$, $\mathbb{E}\{z^2\} = \mu_2 > 0$, and $\mathbb{E}\{z^3\} = \mu_3$. Let ω be the proportion of wealth invested in the risky asset, and $Y = (1-\omega) + \omega Z$ the investor's final wealth. The investor has CRRA preferences, and her optimization problem is

$$\max_{\omega} U = \mathbb{E}\left\{\frac{Y^{1-\gamma} - 1}{1-\gamma}\right\},$$

with first-order condition

$$0 = \mathbb{E}\left\{\frac{\pi + z}{Y^{\gamma}}\right\}. \tag{D.2}$$

The goal is to approximate the proportion ω^* that solves (D.2). Regular perturbation around the deterministic case $z = 0$ does not work because a unique finite solution for ω^* does not exist: if $\pi > 0$, ω^* should be infinity, and if $\pi = 0$ we find that (D.2) is satisfied for all ω. To deal with this issue, Judd (1998) considers an auxiliary model with $Z(\sigma) = 1 + \sigma^2\pi + \sigma z$ depending on the perturbation parameter σ, and then applies a bifurcation theorem to approximate ω^*.

Instead, consider the following auxiliary model in terms of perturbation parameters ε and σ. We have

$$Z(\varepsilon, \sigma) = 1 + \varepsilon^2\pi + \sigma z.$$

The investor faces quadratic costs of holding a proportion ω different from some given value $\overline{\omega}$, as follows:

$$Y(\omega, \varepsilon, \sigma) = (1-\omega) + \omega Z(\varepsilon, \sigma) - \frac{\psi}{2}\left(1-\varepsilon^2\right)(\omega - \overline{\omega})^2,$$

where $\psi > 0$ is an auxiliary parameter. Note that if $(\varepsilon, \sigma) = (1, 1)$ we recover the model of interest.[59] The first-order condition of the auxiliary model is

$$0 = \mathbb{E}\left\{\frac{\varepsilon^2\pi + \sigma z - \psi\left(1-\varepsilon^2\right)(\omega - \overline{\omega})}{\left[Y(\omega, \varepsilon, \sigma)\right]^\gamma}\right\} = G(\omega, \varepsilon, \sigma). \tag{D.3}$$

Now we can use regular perturbation. First, at the deterministic case $(\varepsilon, \sigma) = (0, 0)$ we find a unique and finite solution $\omega^* = \overline{\omega}$. Second, the Implicit function theorem applies and there exists a well-defined function $\omega^* = \omega(\varepsilon, \sigma)$ around $(\varepsilon, \sigma) = (0, 0)$ that satisfies (D.3). Thus, we can write:

$$G(\omega(\varepsilon, \sigma), \varepsilon, \sigma) = 0. \tag{D.4}$$

The second-order approximation to ω^* is:

$$\omega^* = \overline{\omega} + \omega_\varepsilon \varepsilon + \omega_\sigma \sigma + \frac{1}{2}\left(\omega_{\varepsilon\varepsilon}\varepsilon^2 + \omega_{\sigma\sigma}\sigma^2 + 2\omega_{\varepsilon\sigma}\varepsilon\sigma\right). \tag{D.5}$$

We can obtain the derivatives of $\omega(\varepsilon, \sigma)$ by taking partial derivatives of G with respect to σ and ε and equating them to zero. The procedure yields $\omega_\varepsilon = \omega_\sigma = \omega_{\varepsilon\sigma} = 0$, and

$$\begin{aligned}
\omega_{\varepsilon\varepsilon} &= 2\frac{\pi}{\psi}, \\
\omega_{\sigma\sigma} &= -2\left(\frac{\gamma\mu_2}{\psi}\right)\overline{\omega}.
\end{aligned}$$

Plugging these results into (D.5) and evaluating at the model of interest $(\varepsilon, \sigma) = (1, 1)$ gives

$$\psi(\omega^* - \overline{\omega}) = \pi - \gamma\mu_2\overline{\omega}.$$

Then, imposing $\omega^* = \overline{\omega}$ (a condition analogous to the stochastic steady state of infinite-horizon models) delivers a solution that depends on second moments:

$$\omega^* = \frac{\pi}{\gamma\mu_2},$$

[59] Following Judd (1998), I use second powers of ε to ensure that, whenever $\sigma = \varepsilon$, risk premia is proportional to variance.

which is the same formula obtained by Judd (1998). Intuitively, if the approximation point $\overline{\omega}$ is already the optimal proportion of wealth, the sum of the perturbation components $\omega_{\varepsilon\varepsilon}$ and $\omega_{\sigma\sigma}$ must equal zero, which pins down ω^*.

We can proceed further and use a third-order approximation to obtain a solution that depends on both μ_2 and μ_3:

$$\omega^* = \overline{\omega} + \frac{1}{2}\left(\omega_{\varepsilon\varepsilon}\varepsilon^2 + \omega_{\sigma\sigma}\sigma^2\right) + \frac{1}{6}\left(\omega_{\varepsilon\varepsilon\varepsilon}\varepsilon^3 + 3\omega_{\varepsilon\varepsilon\sigma}\varepsilon^2\sigma + 3\omega_{\varepsilon\sigma\sigma}\varepsilon\sigma^2 + \omega_{\sigma\sigma\sigma}\sigma^3\right). \tag{D.6}$$

The new derivatives are $\omega_{\varepsilon\varepsilon\varepsilon} = \omega_{\varepsilon\varepsilon\sigma} = \omega_{\sigma\sigma\varepsilon} = 0$, and

$$\omega_{\sigma\sigma\sigma} = \left(\frac{\gamma\left(\gamma+1\right)\mu_3}{\psi}\right) 3\left(\overline{\omega}\right)^2.$$

Plugging the results for $\omega_{\sigma\sigma}$, $\omega_{\varepsilon\varepsilon}$ and $\omega_{\sigma\sigma\sigma}$ into (D.6), evaluating at $(\varepsilon,\sigma) = (1,1)$, and imposing $\omega^* = \overline{\omega}$ gives

$$0 = \pi - \gamma\mu_2\omega^* + \left(\frac{\gamma\left(\gamma+1\right)}{2}\right)\mu_3\left(\omega^*\right)^2.$$

While formally a quadratic equation in ω^* , we can use implicit differentiation to obtain the following expression for ω^* around $\mu_3 = 0$:

$$\omega^* = \frac{\pi}{\gamma\mu_2} + \frac{(\gamma+1)}{2}\left(\frac{\mu_3}{\mu_2}\right)\left(\frac{\pi}{\gamma\mu_2}\right)^2.$$

Again, the second term proportional to μ_3 is the same one obtained by Judd (1998).

D.2 Neoclassical growth model

The equilibrium equations of the model of interest are standard:

$$y_t = c_t + i_t, \tag{D.7}$$

$$y_t = e^{z_t}k_t^{\alpha}, \tag{D.8}$$

$$k_{t+1} = (1-\delta)\,k_t + i_t, \tag{D.9}$$

$$\frac{1}{c_t^{\gamma}} = \beta\mathbb{E}_t\left[\left(\alpha\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right)\frac{1}{c_{t+1}^{\gamma}}\right], \tag{D.10}$$

$$z_{t+1} = \rho z_t + \eta u_{t+1}, \tag{D.11}$$

where y_t is output, c_t consumption, i_t investment, z_t the log of TFP, k_t current capital, and u_{t+1} the iid standard normal innovation.

The auxiliary model includes the perturbation parameters ε and σ, and introduces an auxiliary parameter ψ such that the effective discount factor is $\beta\left[1+\psi\left(1-\varepsilon\right)\right]^{-1}$. This way we allow for a small change to the discount factor such that the DSS of the auxiliary model is different from the one implied by the model of interest.[60] Perturbation around $(\varepsilon,\sigma) = (0,0)$ then

[60] Another natural candidate for perturbation is δ. Perturbing α is also feasible but less efficient because one

corrects for this deviation and risk simultaneously. The new equilibrium conditions replacing (D.10) and (D.11) are:

$$\frac{1}{c_t^{\gamma}} = \beta \left[1+\psi\left(1-\varepsilon\right)\right]^{-1} \mathbb{E}_t \left[\left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) \frac{1}{c_{t+1}^{\gamma}} \right],$$

$$z_{t+1} = \rho z_t + \sigma \eta u_{t+1}.$$

As always, we recover the model of interest by setting $(\varepsilon, \sigma) = (1,1)$.

At $(\varepsilon, \sigma) = (0,0)$, the deterministic steady state (DSS) of capital in the auxiliary model is:

$$k_d = \left[\frac{\alpha}{\left(\left[1+\psi\right]/\beta - (1-\delta)\right)} \right]^{1/(1-\alpha)}.$$

It follows that by setting

$$\psi\left(\bar{k}\right) = \beta \left[\frac{\alpha}{\left(\bar{k}\right)^{1-\alpha}} + (1-\delta) \right] - 1$$

we can choose a wide range of values for $\bar{k}$ such that $k^d = \bar{k}$. Since $\bar{k}$ uniquely pins down k_d, we can use the algorithm of section 5.3 to build a local solution around the stochastic steady state (SSS) of the model of interest such that $k_s = k_d = \bar{k}$.

D.2.1 Numerical performance

The baseline calibration of the model of interest is standard: $\beta = 0.99$, $\gamma = 2$, $\alpha = 0.35$, $\delta = 0.025$, $\rho = 0.95$, and $\eta = 0.01$. In addition, I consider an extreme calibration combining high levels of risk and risk aversion: $\eta = 0.025$ and $\gamma = 20$. I compute third-order approximations to the policy rule for capital using both standard perturbation and two-parameter perturbation. To evaluate accuracy, I compare their performance against a global solution computed with the endogenous grid method (Carroll, 2006) using a grid of 400 grid-points for capital and 15 grid-points for productivity, discretizing the AR(1) process for z_t with the Rouwenhorst method.

First, I perform a comparative statics exercise by looking at the implied SSS of the Growth model across methods and calibrations. Specifically, for each method I solve the model for different values of risk (with the conditional standard deviation of productivity shocks η ranging from 0.0025 to 0.03), and then calculate the implied SSS of each policy rule.[61] Figure 7 reports the results. In Panel (a) the risk-aversion parameter is set to $\gamma = 2$, and the key finding is that the three solutions deliver virtually the same SSS, which increases with η due to the precautionary-saving motive.

In contrast, in panel (b) I set a high value of $\gamma = 20$, and find that the SSS of standard perturbation (green dashed line) deviates from the global solution (blue solid line) for enough

cannot solve analytically for ψ as a function of the target DSS of capital .

[61] For standard perturbation, I find the SSS by iterating the state vector (k_t, z_t) over time until convergence using the third-order policy rule. For the global solution, the SSS is the point where the policy rule of capital (conditional on $z = 0$) crosses the 45° line.

Figure 7: Comparative statics of the Growth model across methods

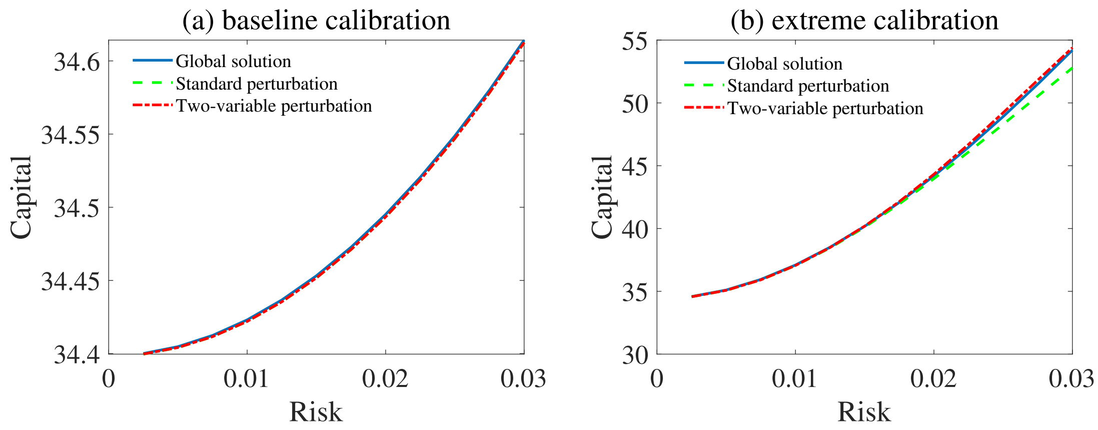


Table 4: Accuracy check for the Growth model

Approximation	Baseline		Extreme	
	Mean	Max	Mean	Max
Standard perturbation	0.016	0.017	2.13	8.02
Two-parameter perturbation	0.016	0.017	0.40	1.32

Notes. The table reports key statistics of the time-series for absolute percent errors of two local approximations (relative to the global solution) under two different calibrations.

large risk, whereas the one computed with two-parameter perturbation (red dash-dotted line) manages to follow the global solution closely. This happens because the scenario with strong uncertainty and large risk-aversion boosts the precautionary-saving effect, leading to a true SSS far away from the deterministic one. Since a large value of $\gamma = 20$ greatly increases the non-linearity of the policy rule, the region of the state-space where the true SSS belongs falls beyond the radius of convergence of a local approximation around the DSS, thus worsening the accuracy.

Next, I fix $\gamma = 2$ and $\eta = 0.01$ in the baseline calibration, $\gamma = 20$ and $\eta = 0.025$ in the extreme calibration, and run a stochastic simulation of the model with a length of $10,000$ periods (burn-in of $1,000$ periods) for the three solution methods. To make a fair comparison, in all three cases productivity z follows the same Markov chain as in the global solution. Hence different time-paths of capital across models are due to differences in the policy rules only. Then, for the two local approximations I compute the (absolute) percent errors of the time-series of capital with respect to the time-series of the global solution, which I treat as the "true" solution.

Table 4 reports mean and maximum values for each simulation. Under the baseline calibration, these statistic are very small and virtually identical across local approximations. However, under the extreme calibration the simulation of two-parameter perturbation clearly outperforms standard perturbation: its mean absolute error of 0.4% is about five times smaller, and its maximum of 1.32% is about six times smaller. The intuition behind this result is the same as in the comparative statics exercise: under the extreme calibration, the dynamics of capital gravitate around a true SSS that is far above the DSS. By changing the point of approximation, the two-parameter perturbation corrects for this effect, leading to more accurate time-series.

D.3 Small open economy model

A representative agent in a small open economy (SOE) seeks to maximize a standard CRRA utility function:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma} \right].$$

At each period the agent receives an endowment income Y_t and can save or borrow net foreign assets (NFA, here bonds) b_t with a world interest rate R_t. The budget constraint is:

$$c_t = Y_t + b_t - \frac{b_{t+1}}{R_t}. \tag{D.12}$$

The exogenous states $y_t = \ln Y_t$ and $z_t = \ln\left(\frac{R_t}{\overline{R}}\right)$ (with $\overline{R}$ as the DSS level of R_t) follow AR(1) processes:

$$y_{t+1} = \rho_y y_t + \sigma \eta_y u_{y,t+1}, \tag{D.13}$$

$$z_{t+1} = \rho_r z_t + \sigma \eta_r u_{r,t+1}, \tag{D.14}$$

where and $u_{y,t+1}$ and $u_{r,t+1}$ are serially uncorrelated standard normal disturbances, already scaled by the perturbation parameter σ.

Equilibrium conditions. The first-order conditions of this problem, together with a no-Ponzi-game condition, lead to the following Euler Equation for bonds:

$$c_t^{-\gamma} = R_t \beta \mathbb{E}_t \left[c_{t+1}^{-\gamma} \right]. \tag{D.15}$$

The state variables are b_t, y_t, and z_t, and the control variable is c_t. The equilibrium conditions of the model of interest consists of equations (D.12)-(D.15).

D.3.1 Problems with standard perturbation

If $\beta = 1/\overline{R}$, implementing standard perturbation is problematic because the DSS of bond holdings is indeterminate, and its first-order solution exhibits a unit root. If we have instead $\beta < 1/\overline{R}$ (or $\beta > 1/\overline{R}$) standard perturbation is inapplicable because the deterministic version of the model does not have a DSS to begin with. Note, however, that if $\beta < 1/\overline{R}$ the true solution to the stochastic version of the model generates stationary dynamics because the precautionary savings force compensates for the household's relative impatience.[62] In particular,

[62] See Aiyagari (1994), and Chamberlain and Wilson (2000).

the economy has a well-defined SSS, so we can solve the model with two-parameter perturbation.

D.3.2 Auxiliary models

I consider two well-known modifications of the standard SOE model that have no other purpose than to induce stationarity of the equilibrium dynamics (Schmitt-GrohÃ© and Uribe, 2003). As emphasized in section 3.3, motivating these from first principles is not required. I do it here because it might help the reader to build intuition, and because I want to make the point that previous *ad hoc* devices used by the literature can now be reinterpreted as excellent auxiliary models.

(i) Uzawa preferences. The new lifetime utility function is:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \theta_t \frac{c_t^{1-\gamma} - 1}{1-\gamma} \right].$$

The discount factor θ_t depends on the perturbation parameter ε, and evolves over time according to

$$\begin{aligned} \theta_0 &= 1, \\ \theta_{t+1} &= \theta_t \overline{\beta} \left(1 + \psi_2 \varepsilon^2\right) \left(\kappa \widetilde{c}_t\right)^{-\psi_1 (1-\varepsilon^2)}, \end{aligned}$$

where $\widetilde{c}_t$ denotes average consumption, κ is a calibration parameter discussed below, $\overline{\beta} = 1/\overline{R}$, and $\psi_1 > 0$ and $\psi_2 < 0$ are two auxiliary parameters.

The new Euler equation replacing (D.15) is:

$$c_t^{-\gamma} = R_t \overline{\beta} \left(1 + \psi_2 \varepsilon^2\right) \left(\kappa c_t\right)^{-\psi_1 (1-\varepsilon^2)} \mathbb{E}_t \left[c_{t+1}^{-\gamma}\right] \tag{D.16}$$

where I have already imposed $\widetilde{c}_t = c_t$. Thus, the equilibrium conditions of the new model are given by equations (D.12), (D.13), (D.14), and (D.16). The deterministic auxiliary model corresponds to the case $(\varepsilon, \sigma) = (0, 0)$, and the model of interest is $(\varepsilon, \sigma) = (1, 1)$.

Note how this auxiliary model exploits the dual capability of two-parameter perturbation to activate and deactivate devices at will. On the one hand, a decreasing function $f_1(\varepsilon) = (1 - \varepsilon^2)$ scales the Uzawa elasticity parameter ψ_1, so that when $\varepsilon = 0$ the discount factor depends on current consumption, and when $\varepsilon = 1$ it becomes fully exogenous. On the other hand, scaling ψ_2 by the increasing function $f_2(\varepsilon) = \varepsilon^2$ implies that the exogenous component of the discount factor equals $\beta = \overline{\beta}$ when $\varepsilon = 0$, and $\beta = \overline{\beta}(1 + \psi_2) < 1/\overline{R}$ when $\varepsilon = 1$. Together, the two modifications ensure that the case $(\varepsilon, \sigma) = (0, 0)$ corresponds to a tractable deterministic model with a well-defined DSS.[63] In particular, imposing stationarity to (D.16) gives $c_d = 1/\kappa$, which

[63] In this example perturbation of ψ_2 is not strictly necessary, but it (slightly) helps improve accuracy because the resulting DSS does not depend on ψ_1. Since ψ_1 controls an auxiliary device, it must be set to a very small value for two-parameter perturbation to function properly, which can lead to small inaccuracies if it affects the DSS. The

we can substitute into (D.12) to pin down $b_d = (1/\kappa - 1)/(1 - 1/\overline{R})$. Since this expression is invertible in κ, we can set $\kappa = \left[1 + \overline{b}\left(1 - 1/\overline{R}\right)\right]^{-1}$, and target a wide range of DSS values for external assets such that $b_d = \overline{b}$.[64]

(ii) Portfolio adjustment costs (PAC). The new budget constraint takes into account that agents face convex costs of holding external assets in a quantity different from some long-run level $\overline{b}$:

$$c_t = y_t + b_t - \frac{b_{t+1}}{R_t} - \frac{\psi_1}{2}\left(1 - \varepsilon^2\right)\left(b_{t+1} - \overline{b}\right)^2 \tag{D.17}$$

where $\psi_1 > 0$ is an auxiliary parameter. The exogenous discount factor is $\beta = \overline{\beta}\left(1 + \psi_2 \varepsilon^2\right)$, with $\psi_2 < 0$ and $\overline{\beta} = 1/\overline{R}$. It follows that the new Euler equation replacing (D.15) is:

$$\left(\frac{1}{R_t} + \psi_1\left(1 - \varepsilon^2\right)\left(b_{t+1} - \overline{b}\right)\right) c_t^{-\gamma} = \overline{\beta}\left(1 + \psi_2 \varepsilon^2\right) \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] \tag{D.18}$$

The equilibrium conditions of the new model are given by equations (D.13), (D.14), (D.17), and (D.18). As in the Uzawa model, the modifications ensure that $(\varepsilon, \sigma) = (0, 0)$ is a tractable auxiliary model with a well-defined DSS of $b_d = \overline{b}$, and $(\varepsilon, \sigma) = (1, 1)$ is the model of interest we care about.

D.3.3 Comparative statics

Consider the special case where there are only income shocks ($\eta_r = 0$), so that $R_t = \overline{R}$ at all times. This allows me to compare the performance of the approximations with a global solution to the income fluctuation problem based on the endogenous grid method. The baseline calibration is $\overline{R} = 1.04$, $\gamma = 4$, $\rho_y = 0.85$, and $\eta_y = 0.02$. The auxiliary parameters are set to $\psi_1 = 10^{-5}$ and $\psi_2 = -10^{-4}$. These low values helps perturbation to reach the model of interest starting from the auxiliary ones.

To implement the global algorithm, I introduce the borrowing constraint $b_{t+1} \geq \underline{b}$ (as opposed to a no-Ponzi-game condition), and set a value of $\underline{b}$ low enough such that its effects on the dynamics are negligible. To maximize accuracy, I use a dense grid of 1,000 nodes for b_t and 25 nodes for income Y, discretizing its AR(1) process with the Rouwenhorst method. The solution delivers a SSS of $b_s = -2.0582$, which corresponds to the point where the policy rule of b_{t+1} (conditional on $Y = 1$) crosses the 45° line.

When I approximate the SSS with the algorithm of section 5.3 I obtain the following values: $b_s = -2.0510$ and $b_s = -2.0583$ using second and fourth-order approximations to the auxiliary PAC model, and $b_s = -2.0855$ and $b_s = -2.0584$ using second and fourth-order approximations to the auxiliary Uzawa model. Thus, regardless of the auxiliary model used, a second-order perturbation already provides a good approximation to the true SSS, which becomes a near-perfect fit as the order of perturbation increases.

same applies to the alternative PAC model.

[64] The restriction is that DSS consumption must be positive: $\overline{b} > -\left(1 - 1/\overline{R}\right)^{-1}$.

Figure 8: Comparative statics of the SOE model across methods

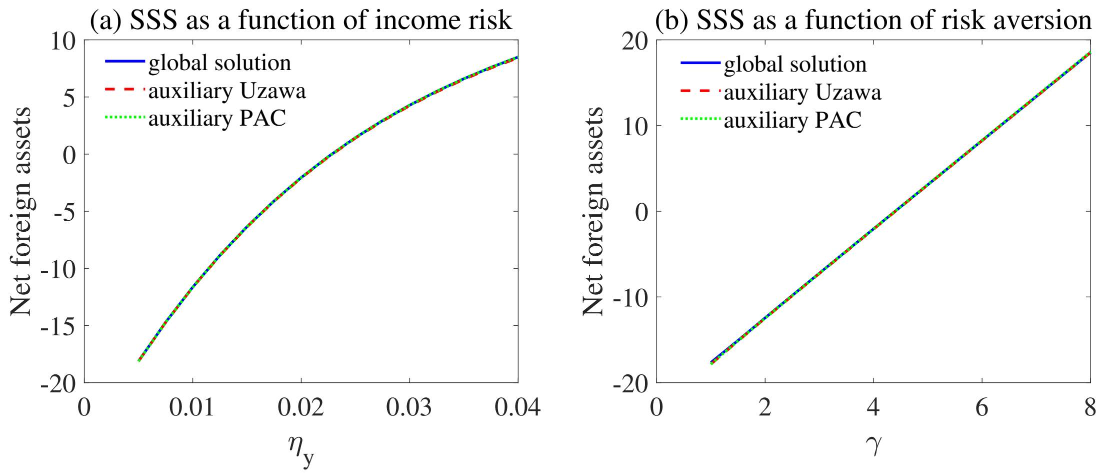


Table 5: Calibration of the SOE model

Parameter		Value
$\overline{R}$	DSS gross interest rate	1.0144
β	Discount factor	0.9851
γ	Risk aversion	2
ρ_y	Auto-correlation of income	0.749
ρ_r	Auto-correlation of interest rate	0.572
η_y	Std. of income shocks	0.0180
η_r	Std. of interest rate shocks	0.0161
$\rho_{y,r}$	Correlation of innovations	-0.62

Next, I perform a comparative statics exercise by computing the SSS for different values of income risk (η_y ranging from 0.005 to 0.04), and risk-aversion (γ ranging from 1 to 8). Figure 8 reports the results. Panel (a) plots the SSS values as a function of income risk, and Panel (b) plots the SSS values as a function of risk-aversion. In both panels the blue solid line corresponds to the global solution, the green dotted line to the second-order perturbation using the auxiliary PAC model, and the red dashed line to the second-order perturbation using the auxiliary Uzawa model. Since the three lines are virtually identical in both panels, the main result here is that two-parameter perturbation does an excellent job in approximating the true SSS of the model, regardless of the auxiliary model involved.

D.3.4 Simulations with a realistic calibration

Table 5 reports the calibration used to simulate the SOE economy. Here I follow Mendoza (2010) and set values to the structural parameters that help to match key moments from Mexican data for the period 1993:I-2005:II. The only parameters different from this study are $\overline{R}$ (set to match

Figure 9: Ergodic distribution of the SOE model

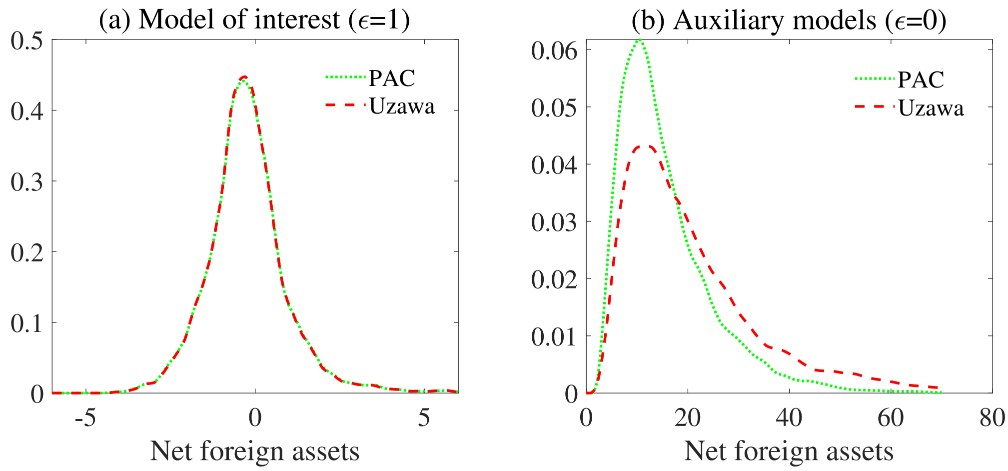


Table 6: Euler errors of the SOE model

Approximation	Auxiliary Uzawa		Auxiliary PAC	
	Mean	Max	Mean	Max
Third order	-5.19	-3.42	-5.28	-3.50
Fifth order	-6.92	-4.32	-6.82	-4.25

Notes. The table reports the mean and maximum values of the Euler errors (in log10 scale) across the ergodic set of each approximation.

the average (annual equivalent) of 5.9 percent from Uribe and Yue (2006)), and β, which I calibrate to deliver a SSS ratio of net foreign assets to GDP of -44 percent (Mexico's average ratio over the period 1985–2004 in the Lane and Milesi-Ferretti (2018) database).

Using a third-order perturbation to the model of interest for each auxiliary model, I simulate the economy for $100,000$ periods (burn-in of $1,000$ periods), feeding the decision rules with the same pseudo-random innovations. The resulting time-series for NFA allows me to compute kernel distributions (using the Epanechnikov method) that I plot in Figure 9, Panel (a). The first result is that the approximations deliver well-defined ergodic distributions of NFA (dotted green line for PAC, and red dashed line for Uzawa), despite the use of a third-order approximation without pruning the state-space. The second result is that the two distributions are extremely close to each other, indicating near-identical long-run dynamics. To show that this is not an artifact implied by similar underlying auxiliary models, Panel (b) repeats the same exercise evaluating the approximations without correcting for the stationary-inducing modifications (that is, $(\varepsilon, \sigma) = (0, 1)$). Now the two kernel distributions are widely different, both from each other and from the ones implied by the model of interest.

Finally, I compute the mean and maximum of unit-free Euler errors (in log10 scale) of the simulated time-series. Table 6 reports the results for third and fifth-order approximations to the policy rules. The overall message is that, regardless of the auxiliary model used, the two-variable perturbation method delivers approximations with good global properties, with mean Euler errors below -5 using a third-order approximation, and well below -6 using a fifth-order one.

D.4 One-bond economy with ex-ante heterogeneous countries

The model is a straightforward extension of the previous SOE economy to a general equilibrium setting with two countries, Home (H) and Foreign (F), that makes the interest rate endogenous. A non-trivial feature is that Home agents are assumed to be more patient than Foreign ones, which leads to long-run imbalances in equilibrium.

Home agents maximize:

$$\mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta_H \frac{c_t^{1-\gamma} - 1}{1-\gamma} \right],$$

where the exogenous discount factor β_H captures the degree of patience for Home agents. The utility function of Foreign only differs from Home's in that β_F can be lower than β_H. Specifically, we have $\beta_H = \beta + \delta$ and $\beta_F = \beta - \delta$, where β is the discount factor under perfect symmetry, and $\delta \geq 0$ represents the gap that makes countries heterogeneous *ex-ante*.

At each period Home agents receive an endowment income Y_t and can save or borrow net foreign assets (NFA, here bonds) b_t with a world interest rate R_t. Their budget constraint is:

$$c_t = Y_t + b_t - \frac{b_{t+1}}{R_t}, \tag{D.19}$$

and the corresponding budget constraint faced by Foreign agents is:

$$c_t^* = Y_t^* + b_t^* - \frac{b_{t+1}^*}{R_t}. \tag{D.20}$$

The exogenous states $y_t = \ln Y_t$ and $y_t^* = \ln (Y_t^*)$ follow AR(1) processes:

$$y_{t+1} = \rho_y y_t + \sigma \eta_y u_{t+1}, \tag{D.21}$$

$$y_{t+1}^* = \rho_y y_t^* + \sigma \eta_y u_{t+1}^*, \tag{D.22}$$

where I have already scaled innovations by the perturbation parameter σ. The model of interest is the case $\sigma = 1$.

The market-clearing conditions are as follows. At each period, good's market clears:

$$c_t + c_t^* = Y_t + Y_t^*,$$

and the bond's market is always in zero net supply:

$$b_t + b_t^* = 0. \tag{D.23}$$

Equilibrium conditions. The first-order conditions of inter-temporal optimization consists of two Euler equations (one for each country):

$$(c_t)^{-\gamma} = R_t\,(\beta+\delta)\,\mathbb{E}_t\left[(c_{t+1})^{-\gamma}\right], \tag{D.24}$$

$$(c_t^*)^{-\gamma} = R_t\,(\beta-\delta)\,\mathbb{E}_t\left[(c_{t+1}^*)^{-\gamma}\right]. \tag{D.25}$$

We can use equation (D.23) to define an equilibrium with Home bonds b_t as the only endogenous state by replacing $b_t^* = -b_t$ in (D.20). The 2 exogenous states are the log-incomes y_t and y_t^*. The 3 control variables are the interest rate R_t, and consumptions c_t and c_t^*. The 6 equilibrium conditions that these variables need to satisfy are the budget constraints (D.19) and (D.20), stochastic processes (D.21) and (D.22), and Euler equations (D.24) and (D.25).

D.4.1 Problems with standard perturbation

The situation parallels the case of the SOE. With symmetric countries ($\delta = 0$), implementing standard perturbation is problematic because the DSS of bond holdings is indeterminate, and its first-order solution exhibits a unit root. When $\delta > 0$, standard perturbation is inapplicable because the model does not have a DSS to begin with. One way to go around this technical issue is to assume a borrowing constraint on bond holdings such as $b_{t+1} \geq -\mu$ (as opposed to relying on an implicit no-Ponzi-game condition), and impose enough conditions on the model parameters such that the constraint is always binding for the impatient agent (here $b_t^* = -\mu$ at all times). By doing so, one can drop the Euler equation of the impatient agent, pin down the DSS from the other equilibrium conditions, and build an approximate solution around it.[65]

A drawback of this approach is that NFA and capital flows in the resulting model are completely determined by the (typically *ad hoc*) properties of the constraint. For example, the one I suggested shuts down capital flows as the NFA of both countries is always constant. More importantly though, the whole procedure misses the fact that the true solution to the model *does* deliver stationary dynamics with capital flows even without a borrowing constraint because, in a stochastic world, precautionary savings serves as a stabilizing force that prevents agents from following explosive consumption paths.

D.4.2 Auxiliary model

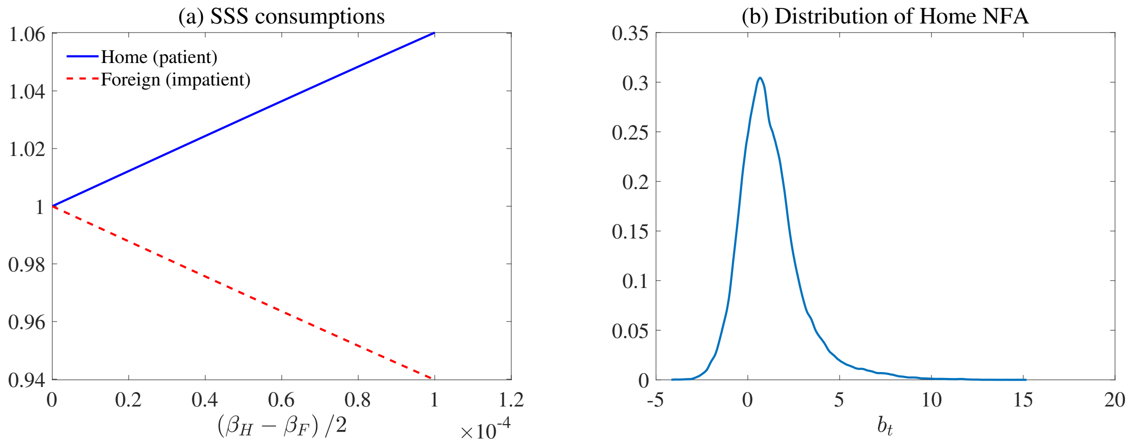
An easy and effective way of solving the model with two-parameter perturbation consists of modifying the Euler equations (D.24) and (D.25) as follows:

$$\left[1+\psi\left(1-\varepsilon^2\right)\left(b_{t+1}-\bar{b}\right)\right](c_t)^{-\gamma} = R_t\left(\beta+\varepsilon^2\delta\right)\mathbb{E}_t\left[(c_{t+1})^{-\gamma}\right], \tag{D.26}$$

$$\left[1-\psi\left(1-\varepsilon^2\right)\left(b_{t+1}-\bar{b}\right)\right](c_t^*)^{-\gamma} = R_t\left(\beta-\varepsilon^2\delta\right)\mathbb{E}_t\left[(c_{t+1}^*)^{-\gamma}\right]. \tag{D.27}$$

[65] Examples in the literature following this approach include Iacoviello (2005), Monacelli (2009), and Eggertsson and Krugman (2012).

Figure 10: Quantitative experiments with the one-bond, two-country model



Note how this auxiliary model fully exploits the dual capability of two-parameter perturbation to activate and deactivate devices at will. As in other applications in this paper, a decreasing function $f_1(\varepsilon) = \left(1-\varepsilon^2\right)$ scales the PAC-like modifications that pin down the DSS of bond holdings and induce stationarity in the first-order solution. At the same time, an increasing function $f_2(\varepsilon) = \varepsilon^2$ scales δ, which captures the asymmetry of the discount factor between countries. Together, these two modifications ensure that, at the approximation point $(\varepsilon, \sigma) = (0, 0)$, we have a tractable deterministic model with an easy-to-solve DSS: $b_d = \bar{b}$, $R_d = 1/\beta$, $c_d = 1 + (1-\beta)\bar{b}$, and $c_d^* = 1 - (1-\beta)\bar{b}$. As always, we recover the model of interest by evaluating (D.26) and (D.27) at $(\varepsilon, \sigma) = (1, 1)$.

D.4.3 Results

The calibration of the model uses the same parameter values as in Table 5, which reports the calibration for the SOE. Hence, the only new parameter here is δ. As a first experiment, I conduct comparative statics by varying the values of this parameter (ranging from $\delta = 0$ to $\delta = 10^{-4}$), and use the SSS algorithm to find the associated SSS of Home's NFA. This, in turn, allows me to compute the SSS consumption for the two countries. Figure 10, panel (a) shows the results, where the solid blue line represents the Home (patient) country, and the dashed red line represents the Foreign (impatient) country. The intuitive result is that, as the gap between discount factors widens, Home's NFA at the SSS increases. In the absence of shocks, this makes Home a net importer and Foreign a net exporter, enabling Home residents to enjoy a higher consumption stream at Foreign's expense. Moreover, in all cases the SSS return satisfies $\beta_H R_s < 1$, which, combined with the precautionary-savings motive, rules out explosive consumption paths in equilibrium.

As a second experiment, I use the SSS algorithm in calibration mode to find the value $\delta = 2.37 \times 10^{-5}$ that is consistent with a SSS of NFA of 1 (and hence equal to SSS output). I then compute a third-order solution around this point and run a large stochastic simulation of one million periods (with a burn-in of the first thousand). The resulting time series of NFA allows me to compute a kernel distribution (using the Epanechnikov method), which I plot in Figure 10, panel (b). Consistent with the positive gap between discount factors, the approximate solution delivers a well-behaved ergodic distribution for NFA that is biased towards positive values. This indicates that, on average, there is a global imbalance with Home as a net importer and Foreign as a net exporter.[66]

Finally, I use the first $100,000$ periods of the time series to compute Euler equation errors for both countries. The summary statistics are as follows: average Euler errors of -6.30 for Home and -6.19 for Foreign, and maximum errors of -4.66 for Home and -4.34 for Foreign. Therefore, as with the other applications of this paper, two-parameter perturbation produces local approximations that turn out to be globally very accurate.

E Additional material for the multi-asset DSGE model

E.1 Equilibrium conditions of the auxiliary model

Euler equations. As in the DS example, a simple way of solving the model consists of introducing PAC-like modifications in the Euler equations of the auxiliary model. Evaluating these conditions at $(\varepsilon, \sigma) = (0, 0)$ gives a tractable deterministic model, and when $(\varepsilon, \sigma) = (1, 1)$ they become the optimality conditions of the model of interest. The Euler equations of the Home agent for Home and Foreign stocks are:

$$\beta \left(\frac{C_t}{\overline{\overline{C}}} \right)^{-\kappa} \mathbb{E}_t \left[\frac{\delta_{t+1} Y_{t+1} + z^S_{H,t+1}}{P_{t+1} C^{\gamma}_{t+1}} \right] = \frac{z^S_{H,t}}{P_t C^{\gamma}_t} \left[1 + \psi_1 \left(1 - \varepsilon^2 \right) \left(S_{H,t+1} - \overline{a} \right) \right], \tag{E.1}$$

$$\beta \left(\frac{C_t}{\overline{\overline{C}}} \right)^{-\kappa} \mathbb{E}_t \left[\frac{\delta^*_{t+1} p_{F,t+1} Y^*_{t+1} + z^S_{F,t+1}}{P_{t+1} C^{\gamma}_{t+1}} \right] = \frac{z^S_{F,t}}{P_t C^{\gamma}_t} \left[1 + \psi_1 \left(1 - \varepsilon^2 \right) \left(S_{F,t+1} - \overline{a}_F \right) \right]. \tag{E.2}$$

where I have already imposed $\widetilde{C}_t = C_t$ in the Uzawa discount factor, and ψ_1, $\overline{a}$ and $\overline{a}_F$ are three auxiliary parameters. Specifically, $\psi_1 > 0$ controls PAC-like modifications for equities, $\overline{a}$ is the location parameter for $S_{H,t+1}$, and $\overline{a}_F$ the location parameter for $S_{F,t+1}$. As shown in Appendix E.2, the parameter restriction $\overline{a}_F = 1 - \overline{a}$ holds in the baseline symmetric model. Nevertheless, having $\overline{a}_F$ as a potentially different parameter is useful when introducing asymmetries in structural parameters (e.g. different γ's) that lead to long-run imbalances. As for the choice of $f(\varepsilon) = 1 - \varepsilon^2$, squaring ε ensures that the SSS of a second-order solution is identical to the SSS of a third-order one. This leads to significant gains in computing times since we can feed the SSS algorithm with the much faster second-order solution.

[66]The average of NFA is 1.26, slightly above the SSS. Therefore, a more serious quantitative exercise would require calibrating δ such that the average of NFA equals the one observed in the data. As explained in section 5.3.2, the SSS algorithm can be easily extended in this direction. Another issue is that the model can produce unrealistically large values for NFA. While I could have easily fixed that by introducing an endogenous component into the discount factor (as in the DS and CG models), I refrained from doing so because I wanted to keep the model as a straightforward extension to the SOE.

The Euler equations of the Home agent for Home and Foreign bonds are:

$$\beta\left(\frac{C_t}{\overline{C}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}_{t+1}}{P_{t+1}C_{t+1}^{\gamma}}\right]=\frac{z_{H,t}^{B}}{P_tC_t^{\gamma}}\left[1+\psi_2\left(1-\varepsilon^2\right)\left(B_{H,t+1}-\overline{b}\right)\right],\tag{E.3}$$

$$\beta\left(\frac{C_t}{\overline{C}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}_{t+1}^{*}}{P_{t+1}C_{t+1}^{\gamma}}\right]=\frac{z_{F,t}^{B}}{P_tC_t^{\gamma}}\left[1+\psi_2\left(1-\varepsilon^2\right)\left(B_{F,t+1}-\overline{b}_F\right)\right].\tag{E.4}$$

where ψ_2, $\overline{b}$ and $\overline{b}_F$ are three auxiliary parameters. Similar to ψ_1, the parameter $\psi_2>0$ controls PAC-like modifications for bonds. Allowing for ψ_1 and ψ_2 to take different values is not essential, but the extra degree of freedom helps to increase accuracy. In turn, $\overline{b}$ and $\overline{b}_F$ are location parameters for $B_{H,t+1}$ and $B_{F,t+1}$ that, in the baseline model, satisfy the restriction $\overline{b}_F=-\overline{b}$.

In turn, the Euler equations of the Foreign agent for Home and Foreign stocks are:

$$\beta\left(\frac{C_t^{*}}{\overline{C}^{*}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}Y_{t+1}+z_{H,t+1}^{S}}{P_{t+1}^{*}\left(C_{t+1}^{*}\right)^{\gamma}}\right]=\frac{z_{H,t}^{S}}{P_t^{*}\left(C_t^{*}\right)^{\gamma}}\left[1-\psi_1\left(1-\varepsilon^2\right)\left(S_{H,t+1}-\overline{a}\right)\right],\tag{E.5}$$

$$\beta\left(\frac{C_t^{*}}{\overline{C}^{*}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}^{*}p_{F,t+1}Y_{t+1}^{*}+z_{F,t+1}^{S}}{P_{t+1}^{*}\left(C_{t+1}^{*}\right)^{\gamma}}\right]=\frac{z_{F,t}^{S}}{P_t^{*}\left(C_t^{*}\right)^{\gamma}}\left[1-\psi_1\left(1-\varepsilon^2\right)\left(S_{F,t+1}-\overline{a}_F\right)\right],\tag{E.6}$$

where I have used the market conditions (6.16) and (6.17) to impose $S_{H,t+1}^{*}=1-S_{H,t+1}$, and $S_{F,t+1}^{*}=1-S_{F,t+1}$. Also, $\overline{C}^{*}$ is the DSS of C_t^{*} in the auxiliary model. The Euler equations of the Foreign agent for Home and Foreign bonds are:

$$\beta\left(\frac{C_t^{*}}{\overline{C}^{*}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}_{t+1}}{P_{t+1}^{*}\left(C_{t+1}^{*}\right)^{\gamma}}\right]=\frac{z_{H,t}^{B}}{P_t^{*}\left(C_t^{*}\right)^{\gamma}}\left[1-\psi_2\left(1-\varepsilon^2\right)\left(B_{H,t+1}-\overline{b}\right)\right],\tag{E.7}$$

$$\beta\left(\frac{C_t^{*}}{\overline{C}^{*}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}_{t+1}^{*}}{P_{t+1}^{*}\left(C_{t+1}^{*}\right)^{\gamma}}\right]=\frac{z_{F,t}^{B}}{P_t^{*}\left(C_t^{*}\right)^{\gamma}}\left[1-\psi_2\left(1-\varepsilon^2\right)\left(B_{F,t+1}-\overline{b}_F\right)\right],\tag{E.8}$$

where I have used the market-clearing conditions (6.18) and (6.19) to impose $B_{H,t}^{*}=-B_{H,t}$, and $B_{F,t}^{*}=-B_{F,t}$.

Other conditions. The equilibrium budget constraints are (6.7) for Home, and

$$\begin{aligned}&P_t^{*}C_t^{*}+z_{H,t}^{S}\left(1-S_{H,t+1}\right)+z_{F,t}^{S}\left(1-S_{F,t+1}\right)-z_{H,t}^{B}B_{H,t+1}-z_{F,t}^{B}B_{F,t+1}\\&=\left(1-\delta_t^{*}\right)p_{F,t}Y_t^{*}+\left(1-S_{H,t}\right)\left(\delta_tY_t+z_{H,t}^{S}\right)+\left(1-S_{F,t}\right)\left(\delta_t^{*}p_{F,t}Y_t^{*}+z_{F,t}^{S}\right)-\widetilde{P}_tB_{H,t}^{*}-\widetilde{P}_t^{*}B_{F,t}^{*}.\end{aligned}\tag{E.9}$$

for Foreign, where I have already imposed the market-clearing conditions. Substituting the relative demands (6.3) and (6.4) into the market-clearing condition (6.15) gives

$$\alpha P_t^{\phi}\left(q_t\right)^{\phi-1}C_t+\left(1-\alpha\right)\left(P_t^{*}\right)^{\phi}\left(q_t\right)^{\phi-1}C_t^{*}=Y_t.\tag{E.10}$$

As for the stochastic processes, one only has to scale future innovations by the parameter σ in equations (6.9)-(6.14) to obtain their auxiliary counterparts, so in the interest of space I skip reproducing them.

Bonds and equities model. The 21 equilibrium conditions of the full model are: Euler equations (E.1)-(E.8), budget constraints (6.7) and (E.9), price-index equations (6.1), (6.2), (6.5) and (6.6), market-clearing condition (E.10), and AR(1) processes (6.9)-(6.14) with innovations scaled by σ.

Equities-only model. In the quantitative experiments, I also consider an equities-only version of the model where $B_{H,t} = B_{F,t} = 0$ at all times. In this case, the 15 equilibrium conditions are: Euler equations (E.1), (E.2), (E.5) and (E.6), budget constraints (6.7) and (E.9), price-index equations (6.1) and (6.2), market-clearing condition (E.10), and AR(1) processes (6.9)-(6.14) with innovations scaled by σ.

Implementation. As explained in section 3.4, I treat ε as a state variable constant over time so that I can solve the model with available DSGE software (see Appendix 5.2). This simply requires adding ε_t in the set of exogenous state variables, and augmenting the set of equilibrium conditions with $\varepsilon_{t+1} = \varepsilon_t$. Also, when writing the codes with the equilibrium conditions (e.g., Dynare MOD files), I typically apply a change of variables to approximate logs of the original equilibrium variables (e.g., $c_t = \ln\left(\frac{C_t}{\overline{C}}\right)$ instead of C_t). This is a common practice in the DSGE literature that helps to improve accuracy and ensures that variables are bound within their natural domain.

Finally, I set $\psi_1 = 0.0001$ and $\psi_2 = 0.1$, which delivers the excellent average Euler errors reported in Table 3. To be clear, I do not came up with these numbers by using a minimization routine (which I could, but not worth it; see the discussion in section 4), but by doing a quick manual search instead. The effect of small changes to $\psi_1 = 0.0001$ and $\psi_2 = 0.1$ on the approximations is essentially nil, and the simulations from a previous version of the paper (where I had $\psi_1 = \psi_2 = 0.001$) yielded results very similar to those reported in Table 3, except that average Euler errors were around -6, and max. errors below -4.

E.2 Deterministic steady state

The first step is to pin down the DSS of asset holdings. Let $\overline{C} = C$ and $\overline{C}^* = C^*$ be the DSS of Home and Foreign consumption. Evaluating at $(\varepsilon, \sigma) = (0, 0)$ and imposing stationarity, Euler equations (E.1) and (E.5) become:

$$\beta \left[\frac{\overline{\delta} + z_H^S}{z_H^S} \right] = 1 + \psi \left(S_H - \overline{a} \right),$$

$$\beta \left[\frac{\overline{\delta} + z_H^S}{z_H^S} \right] = 1 - \psi \left(S_H - \overline{a} \right).$$

Equating the left hand-sides of these two equations delivers $S_H = \bar{a}$, which implies $z_H^S = \frac{\beta}{1-\beta}\bar{\delta}$. An identical procedure with the pair (E.2) and (E.6) gives $S_F = \bar{a}_F$, and $z_F^S = \frac{\beta}{1-\beta} p_F \bar{\delta}$. In turn, combining the pairs (E.3)-(E.7) and (E.4)-(E.8) yields DSS bond holdings $B_H = \bar{b}$ and $B_F = \bar{b}_F$, and DSS bond prices $z_H^B = \beta P$ and $z_F^B = \beta P^*$.[67] Thus, it follows that the vector $\left(\bar{a}, \bar{a}_F, \bar{b}, \bar{b}_F\right)$ uniquely pins down the DSS of the endogenous states.

Next, impose stationarity in the budget constraints (6.7) and (E.9), and use the previous results to obtain:

$$C = \left(\frac{1}{P}\right) \left[(1-\delta) + \delta\left(a + p_F a_F\right) + (1-\beta)\left(Pb + P^* b_F\right)\right] \tag{E.11}$$

$$C^* = \left(\frac{1}{P^*}\right) \left[(1-\delta)\, p_F + \delta\left((1-a) + p_F\left(1-a_F\right)\right) - (1-\beta)\left(Pb + P^* b_F\right)\right] \tag{E.12}$$

where I have omitted hat symbols to ease notation. Likewise, the stationary versions of the price-index equations (6.1) and (6.2) are:

$$P = \left[\alpha + (1-\alpha)\, p_F^{1-\phi}\right]^{1/(1-\phi)}, \tag{E.13}$$

$$P^* = \left[(1-\alpha) + \alpha p_F^{1-\phi}\right]^{1/(1-\phi)}, \tag{E.14}$$

and the stationary version of the market-clearing equation (E.10) is:

$$\alpha P^{\phi} C + (1-\alpha)\left(P_t^*\right)^{\phi} C_t^* = 1. \tag{E.15}$$

Thus, we can substitute (E.13) and (E.14) into (E.11) and (E.12), and in turn these into (E.15) to obtain an equation with p_F as the only unknown. Once this variable is solved, we can immediately compute the remaining DSS values of price indices (P, P^*), consumptions (C, C^*), and asset prices z_F^S, z_H^B, and z_F^B. Finally, the calculations for the equities-only version of the model are identical after setting $b = b_F = 0$.

Symmetrical case. The previous solution is general enough to allow for approximations to the policy rules around arbitrary values of $\left(\bar{a}, \bar{a}_F, \bar{b}, \bar{b}_F\right)$. But if one is working with perfectly symmetric countries, exploiting this symmetry simplifies calculations. In particular, perfect symmetry imposes a strong parameter restriction: $\bar{a}_F = 1 - \bar{a}$, and $\bar{b}_F = -\bar{b}$. To see this, note that under perfect symmetry it must be the case that $p_F = 1$, and $P = P^* = 1$. Plugging these results into (E.11) and (E.12), and substituting the resulting expressions into (E.15) gives

$$(1-\delta) + 2(1-\alpha)\,\delta + (2\alpha - 1)\left(\delta A + (1-\beta)\, B\right) = 1.$$

where $A = a + a_F$, and $B = b + b_F$. For this equation to hold good for arbitrary values of δ and α, we require $A = 1$ and $B = 0$, which delivers the result. Moreover, since one obtains an

[67] Note that the steady state values of P_t and its non-adjusted counterpart $\widetilde{P}_t$ are identical: $P^d = \widetilde{P}^d = P$. Similarly, $\left(P^*\right)^d = \left(\widetilde{P}^*\right)^d = P^*$.

identical expression by imposing stationarity in the model of interest $(\varepsilon, \sigma) = (1, 1)$, the same restrictions apply to the SSS of asset holdings. The consequence is that, when solving for the SSS with the algorithm of section 5.3, (i) we just have to solve for $\overline{a}$ and $\overline{b}$, and (ii) recalculating the DSS starting from arbitrary values of $\overline{a}$ and $\overline{b}$ does not require any intermediate numerical procedure.

E.3 Additional variables of interest

Domestic wealth, portfolio shares, and asset returns. End-of-period domestic wealth W_t is the sum of all asset gross purchases:

$$W_t = z^S_{H,t} S_{H,t+1} + z^S_{F,t} S_{F,t+1} + z^B_{H,t} B_{H,t+1} + z^B_{F,t} B_{F,t+1}. \tag{E.16}$$

The portfolio shares are: $w^S_{H,t} = z^S_{H,t} S_{H,t+1}/W_t$ for Home equity, $w^S_{F,t} = z^S_{F,t} S_{F,t+1}/W_t$ for Foreign equity, $w^B_{H,t} = z^B_{H,t} B_{H,t+1}/W_t$ for Home bonds, and $w^B_{F,t} = z^B_{F,t} B_{F,t+1}/W_t$ for Foreign bonds.[68] The returns of each asset are: $R^S_{H,t} = \left(\delta_t Y_t + z^S_{H,t}\right)/z^S_{H,t-1}$ for Home equity, $R^S_{F,t} = \left(\delta^*_t p_{F,t} Y^*_t + z^S_{F,t}\right)/z^S_{F,t-1}$ for Foreign equity, $R^B_{H,t} = \widetilde{P}_t / z^B_{H,t-1}$ for Home bonds, and $R^B_{F,t} = \widetilde{P}^*_t / z^B_{F,t-1}$ for Foreign bonds.

Using the previous definitions, the endogenous return to wealth (specific to each country) is the weighted sum of asset returns, where the weights are the portfolio shares chosen the previous period. For Home, we have:

$$R^W_{H,t} = w^S_{H,t-1} R^S_{H,t} + w^S_{F,t-1} R^S_{F,t} + w^B_{H,t-1} R^B_{H,t} + w^B_{F,t-1} R^B_{F,t}. \tag{E.17}$$

Combining (E.16) and (E.17), Home's budget constraint (6.7) can be rewritten as a standard wealth accumulation equation:

$$W_t = W_{t-1} R^W_{H,t} + (1 - \delta_t) Y_t - P_t C_t.$$

Its associated wealth Euler equation in the model of interest is:

$$\beta \left(\frac{C_t}{\overline{C}}\right)^{-\kappa} \mathbb{E}_t \left[\left(\frac{P_t}{P_{t+1}}\right) \frac{R^W_{H,t+1}}{C^\gamma_{t+1}}\right] = \frac{1}{C^\gamma_t}, \tag{E.18}$$

which I use to compute the Euler equation errors shown in Table 3.[69]

External assets positions and Equity home bias. Following the conventions of Lane and Milesi-Ferretti (2018), Home's external equity assets are its gross purchases of Foreign equity $z^S_{F,t} S_{F,t+1}$, and Home's external equity liabilities are Foreign's gross purchases of Home equity $z^S_{H,t} S^*_{H,t+1}$. In equilibrium, each country is long in the bond denominated in its own price index,

[68] Note that, by construction, $w^S_{H,t} + w^S_{F,t} + w^B_{H,t} + w^B_{F,t} = 1$.

[69] This equation can also be obtained as the weighted sum of the Euler equations (E.1)-(E.8) evaluated at $\varepsilon = 1$.

and short in the other. From Home's perspective, this means that $z^B_{H,t} B_{H,t+1}$ are its external debt assets, and $z^B_{F,t} B^*_{F,t+1}$ its external debt liabilities. Thus, total external assets TEA_t and liabilities TEL_t are:

$$\begin{aligned}\text{TEA}_t &= z^S_{F,t} S_{F,t+1} + z^B_{H,t} B_{H,t+1},\\ \text{TEL}_t &= z^S_{H,t} S^*_{H,t+1} + z^B_{F,t} B^*_{F,t+1}.\end{aligned}$$

In turn, net foreign assets NFA_t is the difference between the two. Combining (E.16) with (6.17) and (6.19), it can be shown that $\text{NFA}_t = W_t - z^S_{H,t}$.

The share of foreign equities in Home's equity holdings is $z^S_{F,t} S_{F,t+1} / \left(z^S_{H,t} S_{H,t+1} + z^S_{F,t} S_{F,t+1} \right)$, and the share of foreign equities in the World market portfolio is $z^S_{F,t} / \left(z^S_{H,t} + z^S_{F,t} \right)$. Following Coeurdacier and Rey (2013), the measure of equity home bias is:

$$\text{EHB}_t = 1 - \frac{\text{Share of foreign equities in Home's equity holdings}}{\text{Share of foreign equities in the World portfolio}}$$

Provided that countries are symmetric, it is easy to verify that the SSS of this equation is:

$$\text{EHB} = 2S_H - 1 = 1 - 2S_F. \tag{E.19}$$

Real exchange rate, trade balance, and current account. The real exchange rate is the ratio between Home and Foreign price indices: $\text{RER}_t = P_t / P^*_t$. In the stochastic simulations, I also consider an alternative measure not adjusted for preference shocks ($\text{RER}_t = \widetilde{P}_t / \widetilde{P}^*_t$), which might be relevant given the difficulty of quantifying these shocks in the data. Home's trade balance (or net exports) is the difference between Home's output and consumption expenditures:

$$\text{TB}_t = Y_t - P_t C_t. \tag{E.20}$$

The current account is defined as the sum of portfolio reallocations, excluding valuation effects:

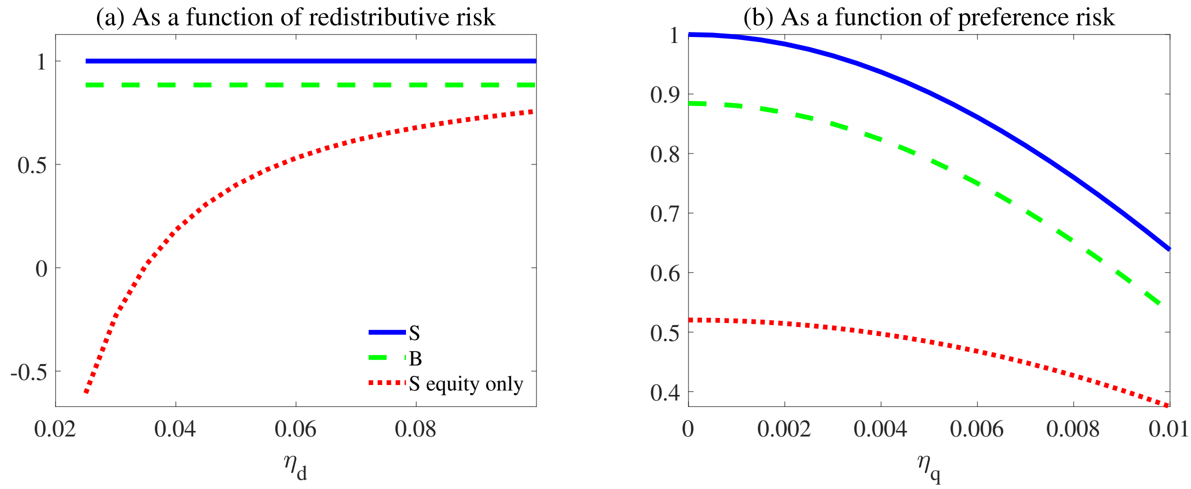
$$\text{CA}_t = z^S_{H,t} \Delta S_{H,t+1} + z^S_{F,t} \Delta S_{F,t+1} + z^B_{H,t} \Delta B_{H,t+1} + z^B_{F,t} \Delta B_{F,t+1},$$

where the operator Δ denotes first differences: $\Delta X_{t+1} = X_{t+1} - X_t$.

Capital inflows and outflows. Following Broner et al. (2013), end-of-period capital inflows by Foreign CIF_t is equal to the net purchases of domestic assets by Foreign, and capital outflows by domestic (Home) agents COD_t is equal to the net purchases of foreign assets by domestic agents. Given that in equilibrium each country is long in the bond denominated in its own price index, we can write:

$$\begin{aligned}\text{CIF}_t &= z^S_{H,t} \Delta S^*_{H,t+1} + z^B_{F,t} \Delta B^*_{F,t+1},\\ \text{COD}_t &= z^S_{F,t} \Delta S_{F,t+1} + z^B_{H,t} \Delta B_{H,t+1},\end{aligned}$$

Figure 11: SSS Portfolios in the multi-asset model



where, consistent with the dataset of Broner et al. (2013), the measures exclude valuation effects. Then, total gross flows is the sum $\text{CIF}_t + \text{COD}_t$, and net capital flows are equal to the difference $\text{CIF}_t - \text{COD}_t$. It is easy to verify that, in equilibrium, net capital flows are the negative of the current account.

E.4 Additional results

E.4.1 Comparative statics

Bonds still matter. This numerical exercise compares the SSS properties of the model with bonds and equities against the equities-only model by performing the same comparative statics studied by CG in their online appendix. The first experiment shuts down preference risk by setting a very small value of $\eta_q = 0.0001$. I then let η_δ vary between 0.025 and 0.1, compute the associated SSS asset values for both models, and plot the results in Figure 11, panel (a). The second experiment sets redistributive risk at its calibrated value $\eta_\delta = 0.059$. I then let η_q vary between 0 and 0.01, compute the associated SSS asset values, and report them in Figure 11, panel (b). In both panels, the red dotted line is the value of S_H in the equities-only model, and the blue solid and green dashed lines correspond to S_H and B_H in the model with bonds and equities.

The qualitative findings of Figure 11 are identical to those reported by CG in Figure 1 of their online appendix (top and bottom panels with trade elasticity above unity). In that sense, their key theoretical results survive in a quantitative model. The intuition behind these findings is therefore the same as in their two-period model. Consider first the case with only equities. In panel (a), sufficiently small values of η_δ deliver a worse-than-you-think-puzzle scenario (Baxter and Jermann, 1997): holding a portfolio strongly biased towards Foreign ($S_H < 0.5$) is optimal because Home equity returns $R^S_{H,t}$ are strongly correlated with Home's non-financial income $(1 - \delta_t) Y_t$. However, this correlation decreases as η_δ increases because a positive redistributive shock increases Home's dividends $d_t Y_t$ at the expense of Home's non-financial income. For enough large redistributive shocks the equities-only model can generate a significant Home bias ($S_H > 0.5$), just as CG found in their two-period model.

Therefore, whether an incomplete-markets structure with only equities can account for the International diversification puzzle is a quantitative question that only this model can address. Panels (a) and (b) together provide a clear negative answer: the calibrated value $\eta_\delta = 0.059$ delivers a S_H barely above 0.5 (zero Home bias), and increasing preference risk only pushes the model into worse-than-you-think-puzzle territory. Since S_H should be above 0.80 to match the high levels of equity home bias observed in the data (0.66 for the U.S. and 0.62 for the Eurozone in 2008), the equities-only model is strongly rejected by the data.[70]

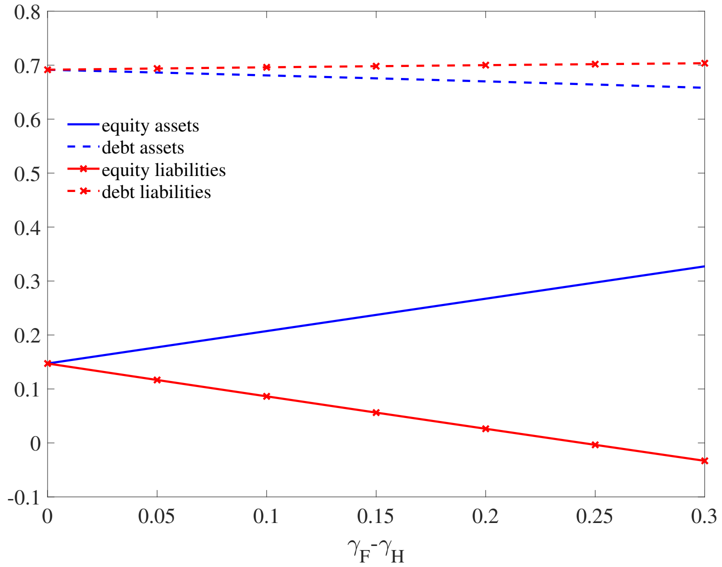
Next, consider the model with bonds and equities. Since panel (a) effectively shuts down preference risk, financial markets are complete, leading to the perfect risk-sharing scenario that CG cover in the main text of their paper. In this case, a large external debt position that is long in the Home bond and short in the Foreign one provides an optimal hedge against relative income shocks. Intuitively, shocks that increase Home's relative income lead to a RER depreciation, which lowers the bond return differential $\ln\left(R^B_{H,t}/R^B_{F,t}\right)$ and generates a transfer from Home to Foreign (see also the IRFs of Figure 13). As CG emphasize, all what is left for equities is to provide a hedge against redistributive risk, which residents in each country can achieve by holding all their domestic equity. The result is therefore full equity home bias ($S_H = 1$).

While this extreme case is also rejected by the data, panel (b) shows that as η_q increases both B_H and S_H decrease. Intuitively, a positive shock to the preference for Home goods leads to a (welfare based) RER depreciation and to a simultaneous positive bond return differential (see also the IRFs of Figure 15). Thus, a bond portfolio long in Home and short in Foreign is a bad hedge against this shock because it generates a positive transfer from Foreign to Home just when Home's goods are relatively cheaper. This gives countries an incentive to reduce their gross external debt positions and partially replace them with foreign equity. In particular, calibrating the standard deviation of (unobserved) preference shocks to $\eta_q = 0.0067$ delivers $S_H = 0.83$, which matches the U.S. equity home bias of 0.66 (see equation (E.19)), and $B_H = 0.721$. Since the latter implies a external gross debt position comparable to the one observed for the U.S. in 2008 (see Table 3), I conclude that, from the point of view of SSS portfolios, the incomplete-markets model with bonds and equities is consistent with the data.

Long-run global imbalances. In this experiment I explore the implications of introducing a parameter asymmetry between countries, wherein domestic households become less risk-averse than foreigners. The goal of this exercise is twofold. Firstly, Rabitsch, Stepanchuk, and Tsyrennikov (2015) document that the solution method proposed by Devereux and Sutherland (2010) struggles to generate accurate portfolio dynamics in asymmetric settings induced by risk. This issue arises because the approximation point of the solution method is the sym-

[70]The numbers come from Coeurdacier and Rey (2013). Note that even a massive value $\eta_\delta = 0.1$ that nearly doubles redistributive risk is unable to match the observed level of home bias. More generally, the equities-only model cannot generate a large equity home bias unless one imposes strongly counterfactual calibrations.

Figure 12: External wealth with asymmetric countries



metric DSS. In contrast, the SSS naturally incorporates any asymmetry related to risk, making two-parameter perturbation a more suitable tool for studying portfolio-choice models with long-run global imbalances.[71] Secondly, recent papers, such as Gourinchas, Rey, and Govillot (2017), Sauzet (2022a), and Stepanchuk and Tsyrennikov (2015) have considered this type of asymmetry to rationalize the international role of the United States as a global banker. It is then natural to examine the predictions of this model and compare them with the data.

To introduce the asymmetry, the new risk-aversion parameters are $\gamma_H = \overline{\gamma} - \tau$ for Home, and $\gamma_F = \overline{\gamma} + \tau$ for Foreign, where $\overline{\gamma}$ is the level of risk aversion in the baseline model, and $\tau \geq 0$ represents the gap between countries' risk aversion. I then vary τ between 0 and 0.15, compute the associated steady states, and use them to construct measures of external assets and liabilities. Figure 12 presents the results. Here, red lines with markers represent external liabilities, and blue lines without them correspond to external assets. Similarly, solid lines indicate equity positions, and dashed lines indicate debt positions.

The success of the model in replicating the long-run patterns of the international financial system is mixed. On the one hand, the model correctly predicts that the country with lower risk aversion (akin to the U.S. in the data) significantly increases its exposure to foreign equity assets. On the other hand, the model predicts that the same country should be a net creditor with a long-run positive NFA position, while the data shows the opposite. Indeed, the trade-off generated by the model, as highlighted in Table 3 ("Different γ's" column), is that Home enjoys a larger long-run consumption in exchange for an increase in its volatility relative to Foreign.[72]

[71] Indeed, Table 3 shows that the stochastic simulation with the parameter asymmetry is as accurate as the baseline scenario.

[72] The positive NFA position might follow because the risk premia generated by the model is too low. Thus, a promising avenue for future research is to incorporate disaster risk, as in Gourinchas, Rey, and Govillot (2017).

E.4.2 Impulse responses

Income shock. Figure 13 reports impulse responses of selected variables following a surprise shock to Home output Y_t of one standard deviation at $t = 1$, starting from the SSS. All variables are expressed in their natural units and, unless otherwise specified, refer to Home measures.

The economic effects can be summarized as follows. The shock to domestic income is persistent but mean-reverting, and it dies out after 10 years. Home goods are relatively more abundant during this period, which worsens Home's terms of trade (larger $p_{F,t}$) and causes a RER depreciation that mirrors the income shock. Due to a substantial trade elasticity, domestic non-financial income increases compared to Foreign.

The Return bond differential is negative on impact but zero afterwards, resulting in a one-time transfer from Home to Foreign (since the SSS bond position is long in Home and short in Foreign). However, this transfer does not fully offset the relative increase of Home's non-financial income. Exploiting this situation, domestic households accumulate both domestic and foreign assets, leading to Home's current account surpluses driven by a simultaneous decline of capital inflows and a surge of capital outflows.

Net Foreign Assets (NFA) decline on impact due to valuation effects, as the RER depreciation implies cheaper Home bonds compared to Foreign ones. Consequently, the wealth ratio falls on impact too. However, as the RER reverts to unity, both variables converge to medium-run values above their SSS. This allows domestic households to enjoy a consumption stream larger than foreigners in both the short and medium-run, as reflected in the impulse response of the Home-to-Foreign consumption ratio that outlasts the income shock.

The last row of Figure 13 shows the effects of the income shock on portfolio reallocation. Equity home bias gradually decreases until convergence to a medium-run level below its SSS.[73] We observe a similar pattern with the share of equity over total external assets: following the initial jump, there is convergence to a medium-run level above the SSS. What explains this extremely persistent portfolio reallocation? The answer is that, as Home becomes richer than Foreign, it also becomes less risk averse relative to Foreign. As a result, Home is inclined to embrace a riskier external position.[74] The increased (ex-ante) correlation between Home's endogenous wealth returns $R^W_{H,t}$ and non-financial income confirms the willingness of domestic households to bear a riskier portfolio than foreigners, in both the short and medium-run.

Redistributive shock. Figure 14 reports impulse responses of selected variables following a surprise shock to the Home dividend share of one standard deviation at $t = 1$, starting from the SSS. Since this shock implies an income transfer from domestic households to shareholders of Home's equity (via a positive return stock differential), and domestic households are the main owners of the asset, the resulting effect is a small transfer from domestic to foreign households.

All the impulse responses can therefore be interpreted under this light. The RER slightly depreciates (due to Foreign bias towards their goods), which leads to a (one-time) very small neg-

[73] In the long-run there is full convergence back to the initial SSS.

[74] See Sauzet (2022a) for a similar result linking relative wealth to portfolio reallocations.

ative return bond differential. To smooth out the income transfer over time, foreign households accumulate both Home and Foreign assets, as reflected by domestic current account deficits driven by an influx of capital inflows, and a decline of capital outflows. Consequently, there is a very persistent (albeit small) decrease of Home's NFA and consumption, and a gradual decline of relative wealth converging to a medium-run level slightly below the steady state. Consistent with these patterns, Home opts for a relatively safer portfolio, as illustrated by the increase in equity home bias, the decline of the share of equity over total external assets, and the reduced correlation between $R^W_{H,t}$ and non-financial income.

Preference shock. Figure 15 reports impulse responses of selected variables following a surprise shock to the preference for the Home good of one standard deviation at $t = 1$, starting from the SSS. This shock implies a welfare-based RER depreciation coupled with a simultaneous improvement of Home's terms of trade (lower $p_{F,t}$). Since the return differentials of bonds and stocks are positive on impact, domestic households receive an income transfer just when their non-financial income is relatively larger, and their consumption expenditures are relatively cheaper. To smooth out these positive effects over time, domestic households accumulate both Home and Foreign assets, as reflected by domestic current account surpluses driven by a decline of capital inflows, and a rise of capital outflows. As a result, there is a very persistent increase of Home's NFA and consumption, and a gradual rise of relative wealth converging to a medium-run level slightly above the steady-state. Consistent with these patterns, Home chooses a relatively riskier portfolio, as shown by the decrease in equity home bias, the increase of the share of equity over total external assets, and the larger correlation between $R^W_{H,t}$ and non-financial income.

Figure 13: Impulse responses to a Home income shock

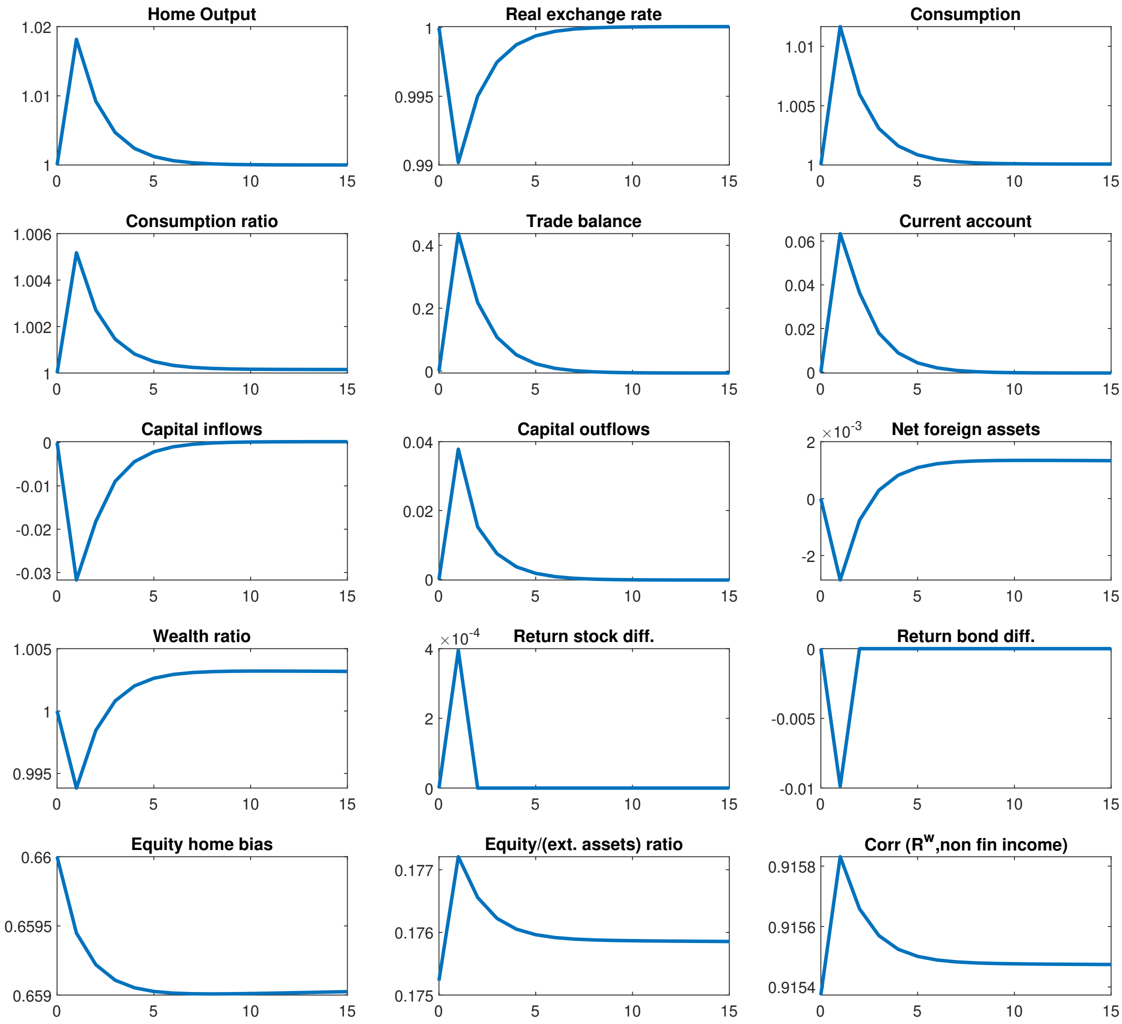


Figure 14: Impulse responses to a Home redistributive shock

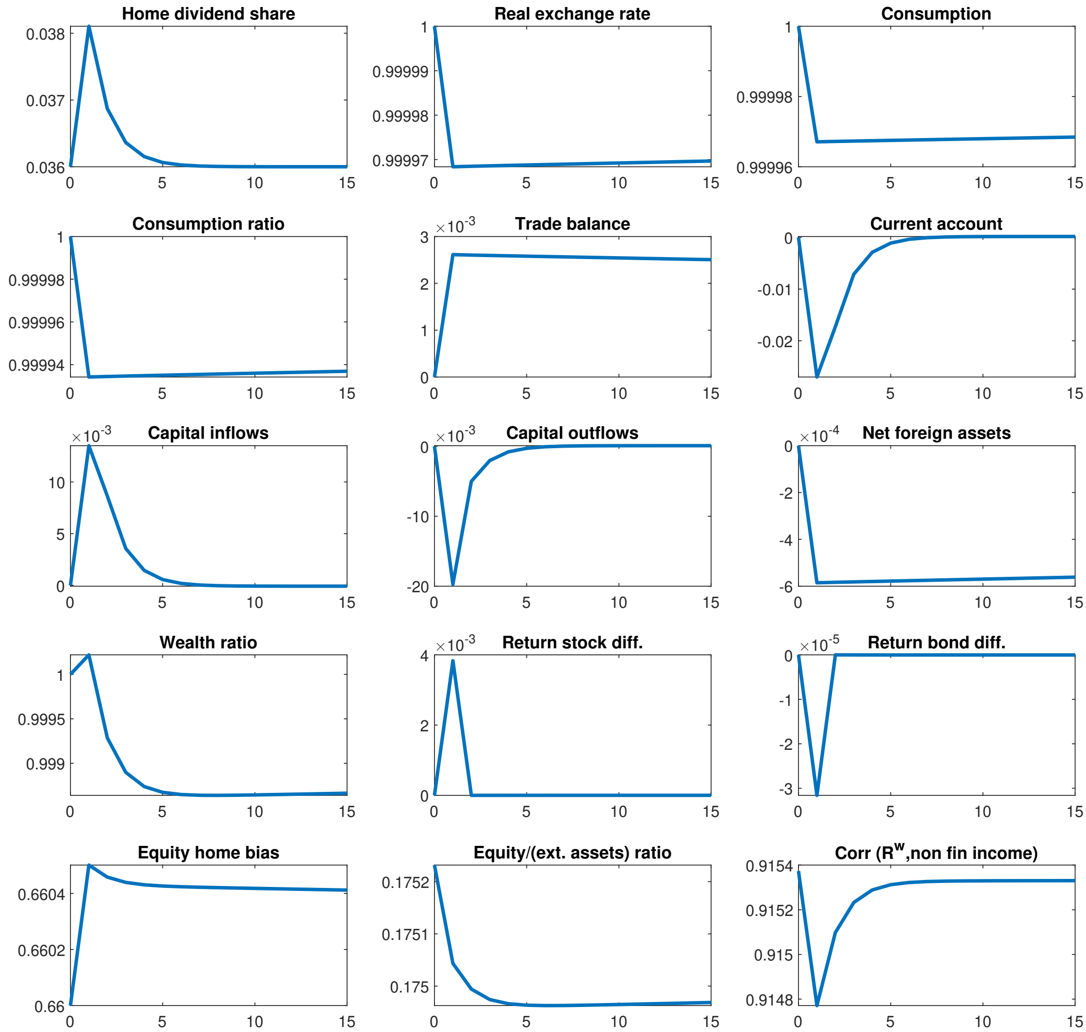


Figure 15: Impulse responses to a Home preference shock