Particle tracking in multiphase systems by radioactive detection

April 23, 2021

1 Physical model and equations

The algorithm of the radioactive particle tracking (RPT) technique aims at detecting the position of a tracer particle in a vessel of interest by measuring the number of recorded photon counts, C. Beam et al.(1978) proposed a reconstruction algorithm based on solving a minimization problem between a precise phenomenological model and measured events. The proposed model works based on the relation between the number of recorded photon counts by detectors and the tracer particle position,$\overrightarrow{X}(x, y, z)$ in a domain $V \subset R^3$. Depending on the position of the tracer particle, each detector senses a different amount of radiation intensity from that tracer. Equation 1 shows the relation between the number of recorded photon counts by each detector and the tracer particle position.

$$C = \frac{T\upsilon R\phi\xi_i(\overrightarrow{X})}{1 + \tau\upsilon R\phi\xi_i(\overrightarrow{X})} \tag{1}$$

In Equation 1, T is the sampling time (s), ν is the number of γ-rays emitted by each disintegration, R is the activity of the tracer (Beq), ϕ is the peak-to-total (photo peak) ratio, τ is the dead time of the detector (s) per accepted pulse, and $\xi_i(\overrightarrow{r})$ is the efficiency of the i_{th} detector related to the position $\overrightarrow{X}$. Each of these variables will be discussed in detail in the next sections.

2 Calculation of detector efficiency

The efficiency for each detector can be expressed as:

$$\xi_i(\overrightarrow{X}) = \oiint_{\Omega} \frac{\overrightarrow{r}.d\overrightarrow{A}}{\|\overrightarrow{r}\|^3} f_a(\alpha, \theta) f_d(\alpha, \theta) \tag{2}$$

where, $\overrightarrow{r}$ is a vector from the position of the tracer particle to a variable point, $\overrightarrow{P}$ on the exposed surface of the detector. $\overrightarrow{P}$ is considered as a variable point since it is the location that the γ-rays enters to the detector which depends on the direction of the traced ray, $d\overrightarrow{A}$ is the external surface vector normal to the surface at the contact point on the detector crystal. For a certain position of the tracer particle, Monte Carlo algorithm randomly selects several thousands of photon path directions to estimate the surface integral of Equation 2 and thus obtain ξ.

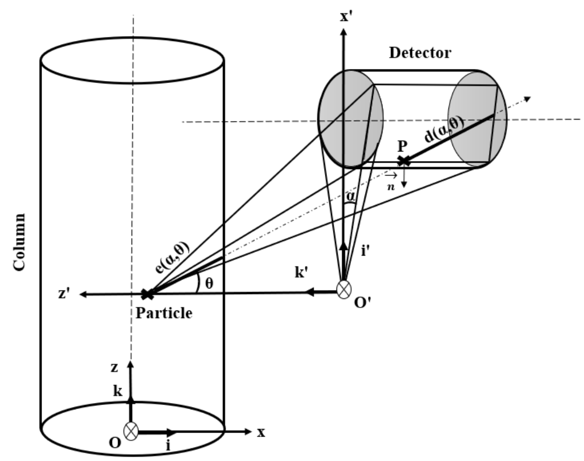

Figure 1: The geometrical structure for determining the path length travelled by the photons inside the column and the detectors

The angles α and θ as they are shown in Figure 1, specify the direction of each traced ray from the particle position. An appropriate weighting factor, $\omega(\alpha)$ and $\omega(\theta)$, should be assigned for each selection of α and θ. Finally, the efficiency of the detector i is calculated as follows:

$$\xi_i(X) = \frac{1}{N}\sum_{1}^{N}\omega(\alpha)\omega(\theta)f_a(\alpha,\theta)f_d(\alpha,\theta) \tag{3}$$

Where N is the number of randomly generated photons. The following factors must be considered for the Monte Carlo calculation:

- The solid angle, Ω, subtended by the radiation.
- f_a is The probability of non-interaction between the γ-rays emitted within Ω and the material inside the column. It can be calculated as:

$$f_a(\alpha,\theta) = exp(-\mu_r e(\alpha,\theta)) \tag{4}$$

In Equation 4, μ_r is the total linear attenuation coefficient of the medium inside the vessel of interest and $e(\alpha,\theta)$ is the travelled path length by the photons inside the reactor which is dependent of the angles α and θ.

- f_d is the probability functions of the $\gamma - rays$ interaction with the detector. It is given by:

$$f_d(\alpha, \theta) = 1 - exp(-\mu_d d(\alpha, \theta)) \tag{5}$$

  where μ_d is the total linear attenuation coefficient of the detector and $d(\alpha, \theta)$ is the effective distance travelled by undistributed $\gamma - ray$ inside the crystal along the direction of α and θ angles.

2.1 Solid angle calculation

For each emitted photon, three distinct scenarios are possible according to the relative position of the particle and the detector. To indicate the relative position of the particle regarding each detector, the parameters h and ρ should be defined. The parameter ρ is the distance from the centre of the detector to a line parallel to the detector axis containing the tracer particle and h shows the distance between the point source and the detector surface.

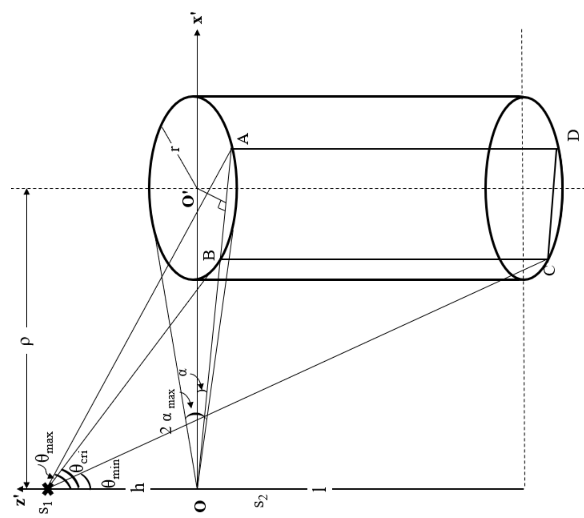

Figure 2: Notations used for angles selection in Monte Carlo calculation.

To calculate the solid angle, Ω, subtended by a cylindrical detector from an arbitrary position of the tracer particle considering the parameters h and ρ, three different cases may happen:

- For $h > 0$ and $\rho > r$, the tracer position is S_1:
  The tracer views the detector from both top and the lateral side of the

crystal.

- For $h < 0$ and $\rho > r$ the tracer position is S_2:
  The tracer views the detector from the lateral side of the crystal.

- For $h > 0$ and $\rho < r$ the tracer position is S_3:
  The tracer views the detector only from the top surface.

where r is the detector radius. These three possibilities for the tracer location are illustrated in Figures 2 and 3.

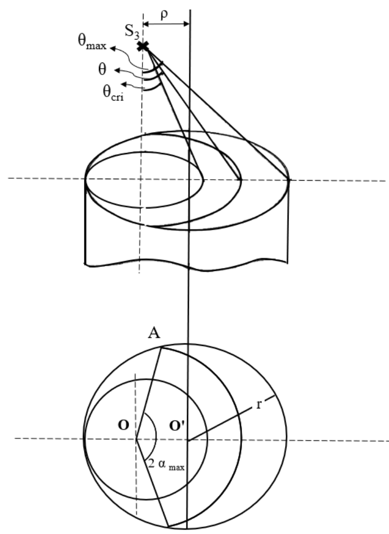

Figure 3: Notations for the case of a tracer located above the circular face of the detector.

For S_1 and S_2 by randomly choosing n from a uniform distribution between 0 and 1, α is defined as:

$$\alpha = \alpha_{max}(2n - 1) \tag{6}$$

$$-\alpha_{max} \leq \alpha \leq \alpha_{max}$$

The weighting factor associated with this selection of α, $\omega(\alpha)$, is given by:

$$\omega(\alpha) = \frac{\alpha_{max}}{\pi} \tag{7}$$

where α_{max} is given by:

$$\alpha_{max} = arcsin(\frac{r}{\rho}) \tag{8}$$

Once α is selected, the points A, B and consequently C, D, θ_{min} and θ_{max} will be obtained. A, B, C and D represent the plane that photon enters the detector from the locations S_1 and S_2. To verify the exact position that the photon enters the detector, θ can be calculated by the following steps. Referring to Figure 2, the line segment $\overline{OA}$ and $\overline{OB}$ are defined as:

$$\overline{OA} = \rho \cos \alpha + (r^2 - \rho^2 \sin^2 \alpha)^2 \tag{9}$$

$$\overline{OB} = \rho \cos \alpha - (r^2 - \rho^2 \sin^2 \alpha)^2 \tag{10}$$

For a particle at S_1 for $h > 0$, θ_{min}, θ_{max} and θ_{cri} can be obtained by the following equations:

$$\theta_{max} = \arctan(\frac{\overline{OA}}{h}) \tag{11}$$

$$\theta_{min} = \arctan(\frac{\overline{OB}}{h+l}) \tag{12}$$

$$\theta_{cri} = \arctan(\frac{\overline{OB}}{h}) \tag{13}$$

where l is the detector length.
For $h = 0$, where the tracer is located at O:

$$\theta_{max} = \frac{\pi}{2} \tag{14}$$

$$\theta_{min} = \arctan(\frac{\overline{OB}}{l}) \tag{15}$$

$$\theta_{cri} = \frac{\pi}{2} \tag{16}$$

For $h < 0$ which is considered as S_2:

$$\theta_{max} = \frac{\pi}{2} + \arctan(\frac{|h|}{\overline{OB}}) \tag{17}$$

$$\theta_{min} = \arctan(\frac{\overline{OB}}{l - |h|}) \tag{18}$$

$$\theta_{cri} = \frac{\pi}{2} + \arctan(\frac{|h|}{\overline{OB}}) \tag{19}$$

Then the angle θ at which this photon strike the detector is chosen by another random number between 0 and 1:

$$\theta = \arccos\{\cos(\theta_{min}) - n[\cos(\theta_{min}) - \cos(\theta_{max})]\} \tag{20}$$

The associated weighting factor, $\omega(\theta)$ is given by:

$$\omega(\theta) = \frac{\cos(\theta_{min}) - \cos(\theta_{max})}{2} \tag{21}$$

For the last case in which ρ is less than the detector radius, according to the geometric configuration, θ_{max} remains constant. Consequently, θ is calculated

first and based on that α will be determined. In this case, when $\theta < \theta_{cri}$, α varies over 2π. However, for $\theta > \theta_{cri}$, α will be restricted by $2\alpha_{max}$.
At S_3, θ_{min} is always equal to zero and θ_{max} and θ_{cri} are defined as:

$$\theta_{max} = \arctan(\frac{r+\rho}{h}) \tag{22}$$

$$\theta_{cri} = \arctan(\frac{r-\rho}{h}) \tag{23}$$

The angle θ is defined by Equation 20 and the angle α can be obtained from the Equation 6. In this case (when $\theta < \theta_{cri}$) the associated weight factor is equal to one. Otherwise for $\theta > \theta_{cri}$, it can be determined from Equation 7.
Where:

$$\alpha_{max} = \arccos\left[\frac{(\rho^2 + h^2\tan^2\theta - r^2)}{2h\rho\tan\theta}\right] \tag{24}$$

The total weighting factor for any selection of α and θ is calculated as:

$$\omega_i = \omega(\alpha)\omega(\theta) \tag{25}$$

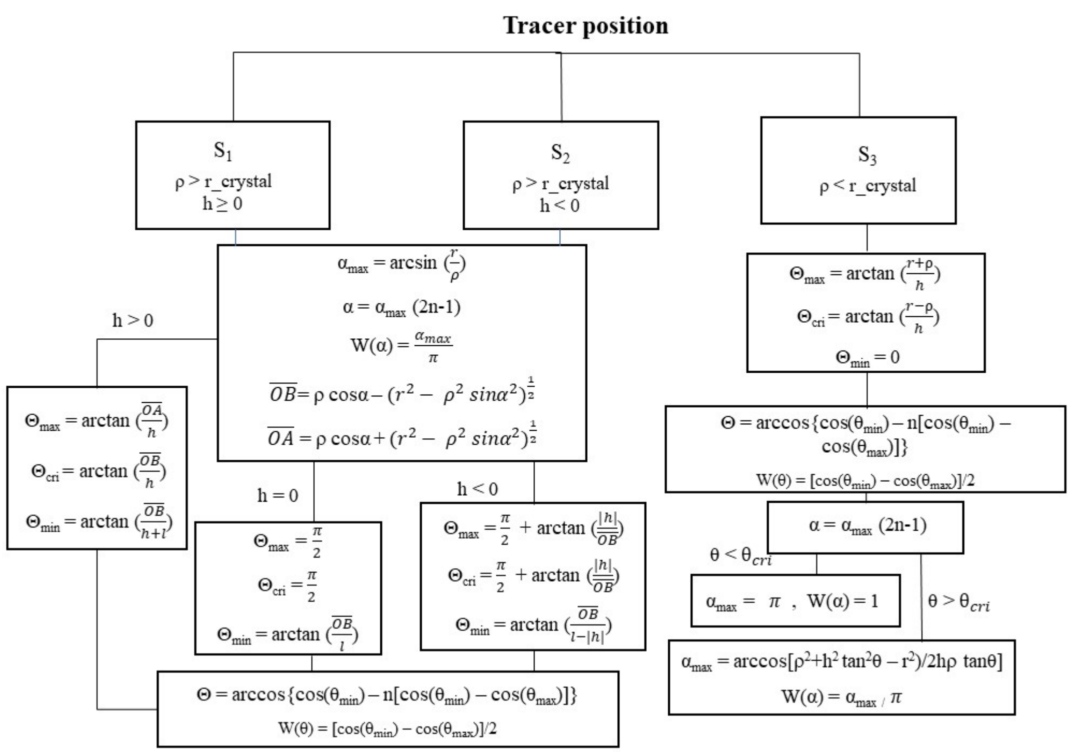

Figure 4: The algorithm of solid angle calculation

2.2 Photon's track length in the detector

Based on the particle position with respect to each detector a new coordinate system, namely "particle-detector" coordinate, must be defined. This new coordinate system changes with the particle movements with respect to each detector.

2.3 ρ and h calculation

Considering point $\overrightarrow{P}$ as the source position, point d as the position of the detector face center and a as a point inside the detector on the axis as it is shown in Figure 5. The vector from the detector face center to the point source is given by :

$$\vec{X_{dp}} = \vec{X_p} - \vec{X_d} \tag{26}$$

The detector orientation can be specified by the vector between points d and a by Equation 27. Detector orientation vector is parallel to the z' axis of the particle-detector coordinate.

$$\vec{e'_z} = \frac{\vec{X_d} - \vec{X_a}}{||\vec{X_d} - \vec{X_a}||} \tag{27}$$

In particle-detector coordinate frame, particle location on the z' axis and its coordinate z'_p can be obtained by the projection of $\vec{X_{dp}}$ on the z' axis as:

$$z'_P = \vec{X_{dp}}.\vec{e'_z} \tag{28}$$

As it is illustrated in Figure 5, z'_P is the distance between the particle and detector face which is equal to parameter h.

The x' coordinate of the particle-detector frame can be defined using $\vec{X_{dp}}$ and $\vec{Z'}$ as:

$$\vec{X'} = \vec{Z'} - \vec{X_{dp}} \tag{29}$$

$$\vec{e'_x} = \frac{\vec{X'}}{||\vec{X'}||} \tag{30}$$

where:

$$\vec{Z'} = z'_P.\vec{e_{z'}} \tag{31}$$

As it is shown in Figure 5, the parameter ρ can be obtained as:

$$\rho = ||\vec{X'}|| \tag{32}$$

Consequently, the last basic axis, e'_y, can be defined using the cross product of two other coordinates as:

$$\vec{e'_y} = e'_z \times e'_x \tag{33}$$

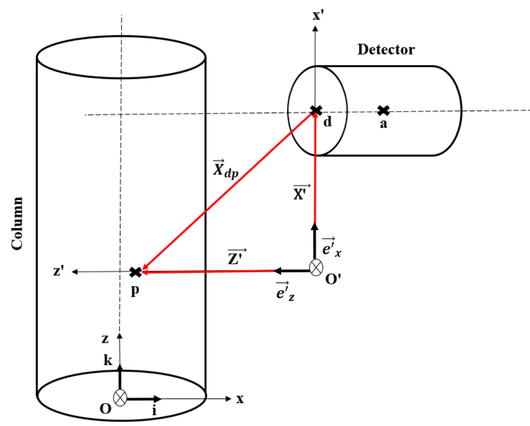

Figure 5: Particle-detector frame configuration

2.4 Photon's track length in the detector

Knowing the angles α, θ and the location of the tracer particle, basic trigonometric relations can be implemented to find the entrance location of γ-ray to the detector as well as the exit point. Figure 6 shows the four possible cases that a photon can travel through the detector. In general, two different scenario may happen. The photon source views the detector either from the top or the lateral surface which is shown as S_1 in Figure 2 or it views the detector only from the top of the detector, S_3 in Figure 3.

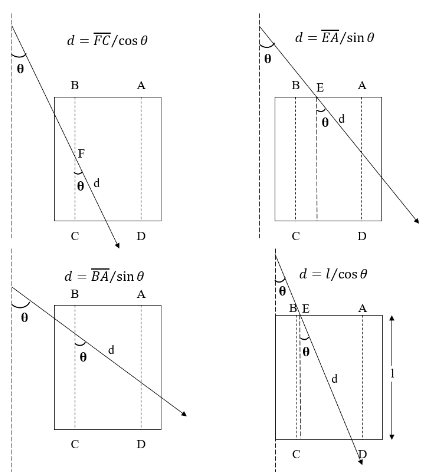

Figure 6: The four possible cases of photon travel through the detector.

2.4.1 Point source viewing the top and the lateral surface of a detector

In this case for $\theta < \theta_{cri}$, the photon strikes the lateral surface of the crystal and it is possible to leave the detector from bottom or lateral wall. To determine the exit point, the angle θ_1 as it is shown in Figure 7 should be determined as:

$$\theta_1 = \arctan \frac{\overline{OA}}{h + l} \tag{34}$$

if $\theta < \theta_1$, the photon leaves the detector from the bottom of the cylinder and the path length can be obtained from:

$$x(\alpha, \theta) = \frac{(h + l)}{cos\theta} - \frac{\overline{OB}}{sin\theta} \tag{35}$$

where $\overline{OB}$ can be obtained from Equation 10.
Else, for $\theta > \theta_1$:

$$x(\alpha, \theta) = \frac{(\overline{OA} - \overline{OB})}{sin\theta} \tag{36}$$

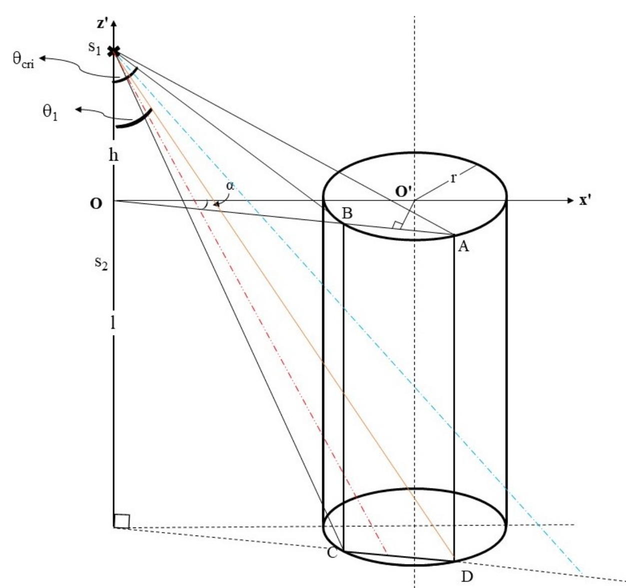

Figure 7: Geometrical configuration to find θ_1 for $\theta < \theta_{cri}$

For the case that $\theta > \theta_{cri}$, the photon strikes the detector's top and it can leave from bottom or the lateral wall of the reactor.
If $\theta < \theta_1$ as it is illustrated in Figure 8, the photo enters from the top and leaves

from the bottom of the detector,the path length can be calculated as:

$$x(\alpha,\theta) = \frac{l}{cos\theta} \tag{37}$$

Else, for $\theta > \theta_1$:

$$x(\alpha,\theta) = \frac{\overline{OA}}{sin\theta} - \frac{h}{cos\theta} \tag{38}$$

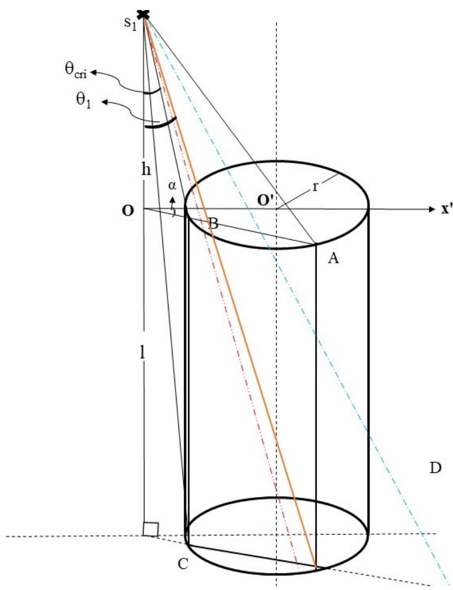

Figure 8: Geometrical configuration to find θ_1 for $\theta > \theta_{cri}$

2.4.2 Point source viewing only the top of a detector

To calculate the value of $x(\alpha,\theta)$, θ_1 and θ_2 should be determined:

$$\theta_1 = \arctan(\frac{\overline{TI}}{h+l}) \tag{39}$$

$$\theta_2 = \arctan(\frac{\overline{TI}}{h}) \tag{40}$$

and $\overline{TI}$ can be determined geometrically as:

$$\overline{TI} = \rho cos\alpha + (r^2 - \rho^2 sin^2\alpha)^{0.5} \tag{41}$$

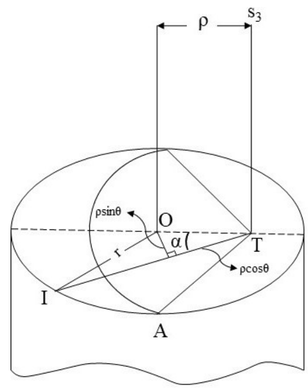

Figure 9: Geometrical configuration for $\overline{TI}$ calculation

If $\theta \leq \theta_1$, photon enters from the top and leaves from the bottom and the path length can be calculated as:

$$x(\alpha, \theta) = \frac{l}{cos\theta} \tag{42}$$

If $\theta_1 < \theta \leqslant \theta_2$, photon enters from the top and leaves from the lateral wall and the path length can be calculated as:

$$x(\alpha, \theta) = \frac{\overline{TI}}{sin\theta} - \frac{h}{cos\theta} \tag{43}$$

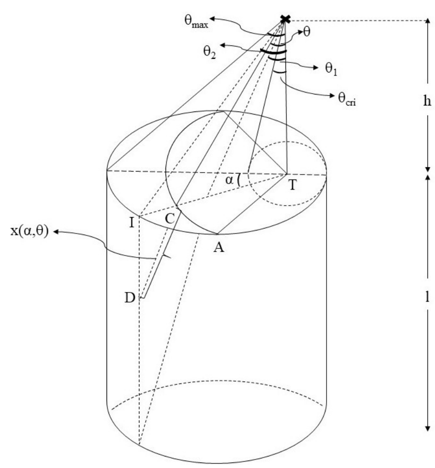

Figure 10: Geometrical configuration of a point source viewing only the top of a cylindrical detector

2.5 Photon's track length in the reactor

The pathway length that each photon travel inside the column can be obtained as:

$$x(\alpha, \theta) = \|\mathbf{X}_{particle} - \mathbf{X}_{intersection}\| \tag{44}$$

In Equation 44, the coordinate of the intersection point can be calculated by solving the circle equation with equation of the line from the tracer location to the point on the detector surface that γ-ray enter. The equation of the circle representing the perimeter of the column can be defined as:

$$x^2 + y^2 = R^2 \tag{45}$$

Equation 45 represents the cylinder perimeter in lab coordinate frame work. As α and θ are defined regarding the detector-particle coordinate, it's more straightforward to transfer the point source and reactor position from the lab coordinate to detector coordinate to calculate the photon path length inside the reactor. The origin of the lab coordinate is the center of the bottom of the reactor with the $x - y$ plane forming the horizontal cross-section and the z-axis pointing to the upward direction. As is illustrated in Figure 1, for the point source x' and y' are equal to zero. The point source only possess a non-zero amount for the z' direction,like $(0, 0, z'_p)$.

2.5.1 Line equation

The equation of a straight line in parametric form can be written as:

$$x = x_1 + t\cos\alpha \tag{46}$$

$$y = y_1 + t\cos\beta \tag{47}$$

$$z = z_1 + t\cos\gamma \tag{48}$$

where the point $P_1(x_1, y_1, z_1)$ stands on the line and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the direction vector of the line. The parameter t can be any real number. Therefore by having the point source coordinate and the direction cosines of the line direction vector it is possible to write the equation of the line that shows the γ-ray path. At each position based on the illustrated geometry in Figure 11, the traced ray has the angle $\pi - \theta$ with $z' - axis$. Based on the selected α and θ by the Monte Carlo algorithm, each ray has a unit direction. The direction is given by its unit direction vector, r, whose Cartesian components are:

$$r = (\cos\alpha, \cos\beta, \cos\gamma) \tag{49}$$

where:

$$|r|^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{50}$$

The three direction cosines are not independent and one of them is taken to be dependent implicitly on the other two. Here as α and θ are selected by the

Monte Carlo algorithm, the traced ray angle with the $x' - axis$, considering it as γ, can be obtained as:

$$\cos\gamma = \frac{\overline{EF}}{\overline{EG}} \tag{51}$$

where the length of $\overline{EF}$ is equal to the length of $\overline{O''H}$. Based on the relation between the angles α and θ, referring to Figure 11:

$$\cos\gamma = \sin\theta . \cos\alpha \tag{52}$$

where:

$$\sin\theta = \frac{\overline{O''G}}{\overline{EG}} \tag{53}$$

$$\cos\alpha = \frac{\overline{O''H}}{\overline{O''G}} \tag{54}$$

Considering Equation 50 the traced ray angle with the third axis (y') can be calculated as:

$$\cos\beta = \sin\alpha . \sin\theta \tag{55}$$

Accordingly, the line equation which shows the path of the traced ray from point source $P(x''_p, y''_p, z''_p)$ (in particle-detector frame) with angles of α and θ is written as:

$$x'' = x''_p + t\sin\theta . \cos\alpha \tag{56}$$

$$y'' = y''_p + t\sin\alpha . \sin\theta \tag{57}$$

$$z'' = z''_p + t\cos(\pi - \theta) \tag{58}$$

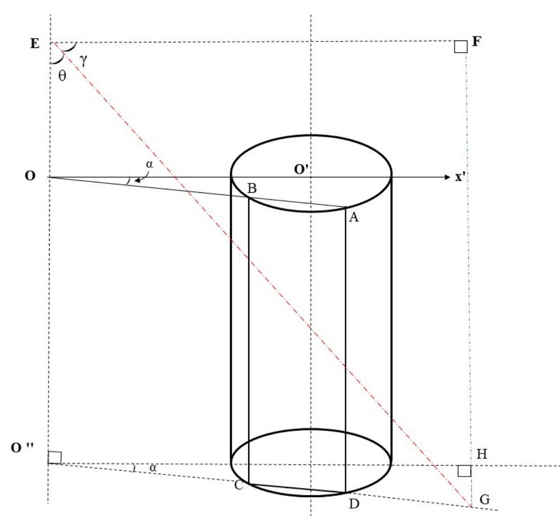

Figure 11: Geometrical for calculation of the direction cosines

2.5.2 Transfer the reactor perimeter equation from lab coordinate to detector coordinate

-Translation
Before performing the rotation, we transfer the lab coordinate origin to detector-particle coordinate origin. To determine each point in detector-particle coordinate the first origin in each direction should move toward the new origin.
The new x,y,z coordinate of each point can be calculated as:

$$x' = x - (x_{Detector-Particle-Origin}) \tag{59}$$

$$y' = y - (y_{Detector-Particle-Origin}) \tag{60}$$

$$z' = z - (z_{Detector-Particle-Origin}) \tag{61}$$

Where x, y, z of the "Detector-Particle-Origin" can be obtained as:

$$X_{Detector-Particle-Origin} = X_{Particle} - \vec{h} \tag{62}$$

where:

$$\vec{h} = (\vec{X_{dp}}.\vec{e'_z})\vec{e'_z} \tag{63}$$

-Rotation
After implementing the translation, the translated coordinate of each point is considered as (x', y', z'). Next, using the following rotation matrix, the new coordinate of each translated point in "particle-detector" coordinate is considered as (x'', y'', z''):

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \vec{e'_x} \\ \vec{e'_y} \\ \vec{e'_z} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{64}$$

Where $\vec{e'_x}$, $\vec{e'_y}$ and $\vec{e'_z}$ are the unit vectors forming the "detector-particle" frame which can be determined using Equations 26 to 33. To find the Equation 45 in particle-detector coordinate it is necessary to substitute parameters x and y with their equivalent in particle-detector frame. Therefore by multiplication of the Equation 65 in inverse of the transformation matrix ($\begin{bmatrix} \vec{e'_x} \\ \vec{e'_y} \\ \vec{e'_z} \end{bmatrix}^{-1}$) the equivalent of x and y van be obtained as:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \vec{e'_x} \\ \vec{e'_y} \\ \vec{e'_z} \end{bmatrix}^{-1} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \tag{65}$$

where:

$$x = x' + (x_{Detector-Particle-Origin}) \tag{66}$$

$$y = y' + (y_{Detector-Particle-Origin}) \tag{67}$$

$$z = z' + (z_{Detector-Particle-Origin}) \tag{68}$$

$$min_{R,\tau,\mu_s} = \sum_{k=1}^{n} \left(\frac{\Phi_j(x_k) - \Phi_j^m}{\Phi_j(x_k) + \Phi_j^m} \right)^2 \tag{69}$$