NUCLEAR INSTRUMENTS AND METHODS 143 (1977) 577-581 ;

THE MONTE CARLO CALCULATION OF THE AVERAGE SOLID ANGLE SUBTENDED BY A RIGHT CIRCULAR CYLINDER FROM DISTRIBUTED SOURCES

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Received 16 December 1976

A Monte Carlo method utilizing total variance reduction is applied to the calculation of the solid angle subtended by a right circular cylindrical detector from a point of arbitrary position. This method should be useful in those cases where the average solid angle from a distributed source of unusual shape is desired. Calculated values for various cases of interest compared to values given in the literature indicate that the method is correct.

1. Introduction

Knowledge of the solid angle subtended by a detector is required in a variety of problems involving the measurement of nuclear radiation. The general definition of solid angle Ω subtended by an object at a point P whose position vector is $\boldsymbol{r}_\mathrm{p}$ is

$$\Omega_\mathrm{p} = \int_S \frac{\boldsymbol{n}\cdot(\boldsymbol{r}-\boldsymbol{r}_\mathrm{p})}{|\boldsymbol{r}-\boldsymbol{r}_\mathrm{p}|^3}\,\mathrm{d}S, \qquad (1)$$

where $\boldsymbol{r}$ is the variable position vector of the surface element dS "visible" at P, $\boldsymbol{n}$ is the unit vector normal to dS pointing away from P (the angle between $\boldsymbol{n}$ and $\boldsymbol{r}-\boldsymbol{r}_\mathrm{p}$ is always less than or equal to $\frac{1}{2}\pi$) and S is the surface over which the integration is to be carried out (for example, the detector surface).

Eq. (1) cannot be solved analytically but for the simple cases. The solutions provided consist of numerical integration of eq. (1) or a series expansion of the integral. There are closed form solutions which approximate the integral or replace the original geometry by an approximate one for which eq. (1) is integrable. Gardner[1]), in addition to his model, has included a short review of the different analytical and numerical methods for solving eq. (1). A completely different approach is to estimate the solid angle by a Monte Carlo method. Carchon et al.[2]) have evaluated the average solid angle subtended by one circular disk to another not in parallel by a rejection scheme proposed by Williams[3]), which was originally devised for parallel planes and consists of counting the number of hits and tries. In addition, they have calculated the average solid angle subtended by a disk at a thick source, applying Bonnet's[4]) procedure, which is a Monte Carlo evaluation of a multiple integral. The methods presented so far suffer from a substantial drawback when a lateral side of the detector has to be considered. Verghese et al.[5]) and Masket et al.[6]) have treated this case for right circular cylinders. The calculation complicates furthermore when distributed sources of an unusual shape are to be considered.

The aim of this publication is to present an efficient Monte Carlo scheme to estimate the average solid angle subtended by a right circular cylinder from a distributed source. Values thus obtained (including their errors) are compared with

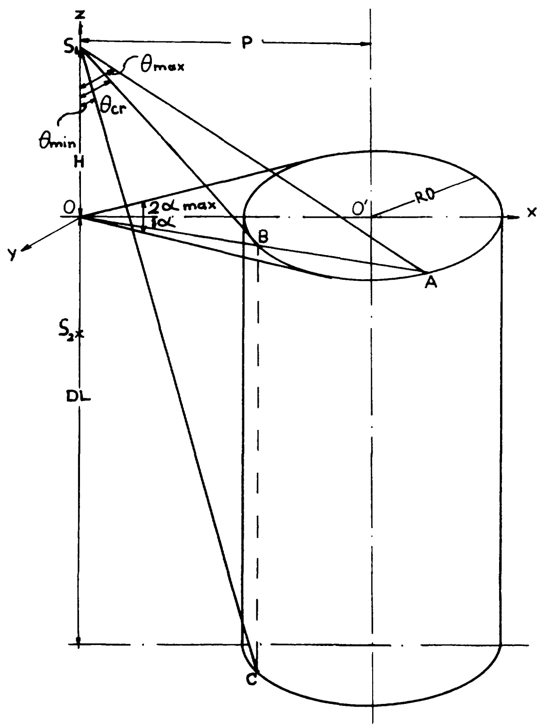

Fig. 1. Point located on the side of the detector.

those found in the literature. Finally, a computer program written in FORTRAN is included in the appendix.

2. Solid angle subtended by a circular cylinder at a point P

Applying the notation of figs. 1 and 2, there are three distinctive cases. In the first one, the point under consideration is located at S_1 in fig. 1. In this case the point "views" the top and the side of the detector and it can be described as the case for which $P > \mathrm{R0}$ and $H > 0$. In the second case when the point is located at S_2 in fig. 1, only the lateral side of the detector can be "seen"; this corresponds to $P > \mathrm{R0}$ and $H < 0$. And the last case when the point is above the detectors at S_3 in fig. 2, only the top of the detector can be "seen", and for this case we have $P < \mathrm{R0}$ and $H > 0$.

The isotropic emission into a unit sphere can be stated as

$$d\Omega = \sin\theta \, d\theta d\alpha, \tag{2}$$

and

$$p(\theta, \alpha) \, d\theta d\alpha = d\Omega/4\pi, \tag{3}$$

where θ and α are the longitudinal and horizontal angles respectively, $p(\theta, \alpha)$ is the joint probability density distribution for isotropic emission. To obtain the marginal distributions of θ and α separately eq. (3) must be integrated over α and θ respectively, as follows:

$$p(\theta) \, d\theta = \int_0^{2\pi} d\alpha \frac{\sin\theta}{4\pi} \, d\theta,$$

$$p(\theta) = \tfrac{1}{2}\sin\theta, \qquad 0 \le \theta \le \pi, \tag{4}$$

and

$$p(\alpha) \, d\alpha = \int_0^{\pi} \sin\theta \, d\theta \, \frac{d\alpha}{4\pi},$$

$$p(\alpha) = 1/2\pi, \qquad 0 \le \alpha \le 2\pi. \tag{5}$$

Now the random direction can be chosen by sampling eqs. (4) and (5). However, before this is done we want to restrict the free selection of θ and α in a way that each direction will intercept the detector. Such a restriction has to be associated with a proper weighting factor, W_i.

Let us consider now case number one. Referring to fig. 1 we notice that $\alpha_{\max}$ remains constant for any selected direction which hits the detector. Therefore, defining x as a uniformly distributed random number between zero and one, the angle α is derived from

$$x = \frac{\int_{-\alpha_{\max}}^{\alpha} d\alpha/2\pi}{\int_{-\alpha_{\max}}^{\alpha_{\max}} d\alpha/2\pi},$$

and

$$\alpha = \alpha_{\max}(2x - 1), \qquad -\alpha_{\max} \le \alpha \le \alpha_{\max}. \tag{6}$$

The weighting factor associated with this selection of α is w1

$$\mathrm{w1} = \frac{\int_{-\alpha_{\max}}^{\alpha_{\max}} d\alpha/2\pi}{\int_0^{2\pi} d\alpha/2\pi},$$

$$\mathrm{w1} = \alpha_{\max}/\pi, \tag{7}$$

where $\alpha_{\max}$ is given by

$$\alpha_{\max} = \arcsin(\mathrm{R0}/P). \tag{8}$$

Once α is determined it is possible to calculate $\theta_{\max}$ and $\theta_{\min}$ since the points S_1, A, B and C are located in the same plane

$$\theta_{\max} = \arctan\{(P\cos\alpha + (\mathrm{R0}^2 - P^2\sin^2\alpha)^{\frac{1}{2}})/H\}, \tag{9}$$

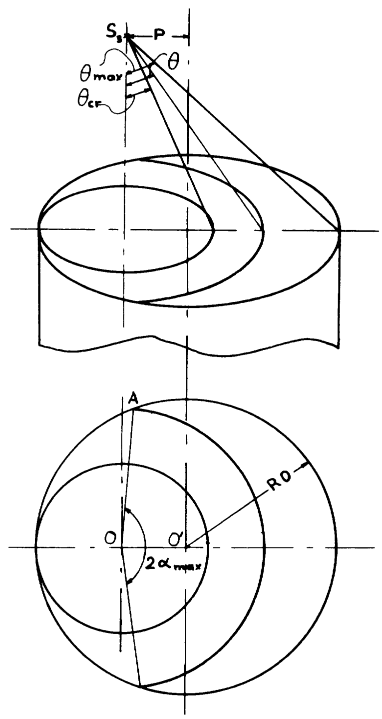

Fig. 2. Point located above the detector.

$$\theta_{\min} = \arctan \times \times \{(P\cos\alpha - (\mathrm{R0}^2 - P^2\sin^2\alpha)^{\frac{1}{2}})/(H+\mathrm{DL})\}. \tag{10}$$

When this scheme is applied, togeher with the detector efficiency calculation, angle θ has to be selected. As we are concerned here with the solid angle only, there is no need to calculate any particular angle θ; however, we need to evaluate the weighting factor, w2, associated with this selection. The factor w2 is given by

$$\mathrm{w2} = \frac{\int_{\theta_{\min}}^{\theta_{\max}} \frac{1}{2}\sin\theta\, \mathrm{d}\theta}{\int_0^{\pi} \frac{1}{2}\sin\theta\, \mathrm{d}\theta},$$

$$\mathrm{w2} = \tfrac{1}{2}\{\cos(\theta_{\min}) - \cos(\theta_{\max})\}, \tag{11}$$

and the total weighting factor for this selection of α and θ, W_i is

$$W_i = \mathrm{w1w2}. \tag{12}$$

W_i represents the solid angle subtended for this particular selection. The mean weighting factor over all possible selections represents the solid angle of the configuration. Therefore, the estimate of the solid angle, Ω_p is given by the expectation value, $E(W_i)$,

$$\Omega_\mathrm{p} = \frac{1}{N}\sum_{i=1}^{N} W_i, \tag{13}$$

and the standard deviation of Ω_p is obtained from

$$\sigma_{\Omega_\mathrm{p}} = \left[\frac{1}{N(N-1)}\left(\sum_{i=1}^{N} W_i^2 - N\Omega_\mathrm{p}^2\right)\right]^{\frac{1}{2}}. \tag{14}$$

In the second case, when the point is located below the top surface at S_2 in fig. 1, the height, H, is replaced by the depth, $-H$. The selection of α does not change, and the only thing which changes is $\theta_{\max}$. Therefore, eq. (9) becomes:

$$\theta_{\max} = \tfrac{1}{2}\pi + \arctan \times \times \{-H/(P\cos\alpha - (\mathrm{R0}^2 - P^2\sin^2\alpha)^{\frac{1}{2}})\}, \tag{15}$$

for $H = 0$, $\theta_{\max} = \frac{1}{2}\pi$.

To consider the last case when the point is located at S_3 in fig. 2, we notice that $\theta_{\max}$ remains constant. In addition there is a critical angle θ_cr below which angle α may vary over 2π and above which the variation of α is limited to $2\alpha_{\max}$. $\theta_{\max}$ according to fig. 2 is given by

$$\theta_{\max} = \arctan\{(\mathrm{R0}+P)/H\}, \tag{16}$$

and

$$\theta_\mathrm{cr} = \arctan\{(\mathrm{R0}-P)/H\}, \tag{17}$$

in this case $\theta_{\min} = 0$. A particular selection of θ at first will allow one to determine $\alpha_{\max}$. Therefore,

$$x = \frac{\int_0^{\theta} \frac{1}{2}\sin\theta\, \mathrm{d}\theta}{\int_0^{\theta_{\max}} \frac{1}{2}\sin\theta\, \mathrm{d}\theta},$$

$$\theta = \arccos \times \times \{\cos(\theta_{\min}) - x[\cos(\theta_{\min}) - \cos(\theta_{\max})]\}, \tag{18}$$

and the weighting factor is given according to eq. (11). Once θ is selected we can check whether θ is lower or higher than θ_cr. In the first case angle α varies over 2π, and the weighting factor associated with it is one. In the second case α is restricted to $2\alpha_{\max}$, and $\alpha_{\max}$ is given by

$$\alpha_{\max} = \arccos\{(P^2 + H^2\tan^2\theta - \mathrm{R0}^2)/2HP\tan\theta\}, \tag{19}$$

and the weighting factor is given according to eq. (7). A computer program written in FORTRAN, which employs this scheme of calculation is given in the appendix.

3. Surface and volumetric sources

In the previous section a scheme for calculating a solid angle subtended by a circular cylinder from a point was presented. There were only four parameters required by the scheme: detector length, DL, detector radius, R0, and two more which describe the point location with respect to the center of the top plane of the detector, namely, the height, H, of the point above the plane and the distance, P, from the point to the detector axis. For a point source these parameters remain constant. However, when the average solid angle from a surface or a volumetric source is regarded, a new point sampled properly from the source has to be determined separately for each history. P and H, therefore, do not remain constant any more and their values will depend upon the shape of the source. Naturally, additional parameters of the source have to be specified. There are cases in which the number of independent parameters can be reduced to three.

Let us consider as an example a disk source of radius RS, whose center is located on the axis of the detector, H0 above it, and which is perpendicular to a disk detector. In this case the variable

TABLE 1

Comparison between presently calculated solid angles and those in the literature.

Case[a]	Solid angle by Monte Carlo	Relative standard error (%)	Execution time[b] (sec)	Solid angle[c] literature)	Relative standard error (%)
1	1.6572	1.6	0.7	1.6371[6]	0.1
2	0.3775	0.49	1.26	0.3791[6]	0.1
3	1.2602	0.2	0.75	1.2624[6]	0.1
4	0.00468	3.9	0.83	0.00477[1]	0.1
5	0.0400	0.7	1.45	—	—
6	0.0565	7.18	0.98	0.0512[2]	1.0[d]
7	0.0333	5.03	1.11	0.0335[2]	1.0[d]

[a] Identification of the cases:
(1) Point source and a disk detector.
(2) Point source at S_1 above the cylindrical detector.
(3) Point source at S_2 on the side of a cylindrical detector.
(4) Disk source in parallel to a disk detector.
(5) Disk source parallel to a cylindrical detector.
(6) Disk source perpendicular to a disk detector.
(7) Cylindrical source with an axis parallel to a disk detector.

[b] The execution time refers to an IBM-360 computer and WATFIV compiler.

[c] The upper number refers to the reference number.

[d] Point taken from a graph, therefore this increased error.

TABLE 2

Dimensions, in arbitrary units, used in the calculation of table 1.

Case	R0	DL	P	H	RS	D
1	1.0	2.0	0.5	1.0	0.0	0.0
2	1.0	2.0	2.0	2.0	0.0	0.0
3	1.0	2.0	2.0	−0.5	0.0	0.0
4	0.1	0.0	—	1.0	3.0	0.0
5	0.1	2.0	—	1.0	3.0	0.0
6	1.0	0.0	—	10.0[a]	9.0	0.0
7	1.0	0.0	—	10.0[a]	9.0	10.0

[a] Actually, it is the height of the center of the disk source above the detector.

point position is given by

$$P = |\mathrm{R1}\cos(\beta)|, \tag{20}$$

$$H = \mathrm{H0}+\mathrm{R1}\sin(\beta), \tag{21}$$

where $\mathrm{R1} = \mathrm{RS}\sqrt{x}$ and $\beta = 2\pi x$. The attached computer program in the appendix includes the same case except it considers a thick source of thickness D.

4. Results and discussion

The results according to the scheme presented so far are summarized in table 1. There were seven cases analyzed; the cases are explained underneath the table. The dimensions for each case are specified in table 2. All the cases were analyzed using one thousand histories. It is evident from the results that the scheme proposed provides rather good estimates of the solid angle with a relative small error in a significantly short time. The execution time is directly proportional to the number of histories, N, which controls the error. Thus to reduce the error by a factor of $\sqrt{10}$ it is necessary to increase the number of histories by a factor of 10 and, therefore, the execution time by the same factor. It is difficult to establish a sharp criterion for the magnitude of the error. First of all, it is controlled by the number of histories, N, and secondly, it depends on the spread of the calculated weights around the mean value. For example, using one thousand histories in two cases of a point source at different locations with respect to a detector, one case yielded a solid angle of $0.031 \pm 1.37\%$, the other, $4.51 \pm 0.054\%$. For a given configuration it is possible to determine an upper limit to the error only by looking at a worse case. Since the standard deviation is always available from the computed results, an estimate of the number of histories required for a desired accuracy can be made on the basis of a preliminary run with a hundred histories.

The main advantage of the proposed scheme is its adaptability to different geometrical configurations. In order to change from a cylindrical detector to a disk detector, it suffices to set detector length, DL = 0. Distributed sources are handled by uniform sampling throughout the space occupied by the source and feeding the information about a selected point into the main stream of the calculations. The suggested scheme might be found valuable in the calculation of efficiency of detectors when distributed sources of unusual shapes and non-homogeneous sources are considered; furthermore, non-isotropic sources can be handled by replacing eqs. (2), (3), (4) and (5) by their proper distribution. Whenever it is possible to set properly the ranges of the angles, the method should be suitable for non-conventional detectors as well.

Finally, when setting up all these functions with their ranges, one must take care not to introduce any bias in the calculated solid angle. It should be pointed out that the scheme presented here does not include self-absorption and self-scattering in-side the source, although these can be included when selecting a point in the source[7]).

The author wishes to thank Dr. R.P. Gardner and Dr. K. Verghese for their constructive comments.

Appendix

The program is written in a FORTRAN language and employs the same symbols as in the text. It is, therefore, self-explanatory. The output includes the variable SOLAN, which is the solid angle and the variable SERR which is the standard error.

```
      DIMENSION W(N)
      N=1000
      PI=3.1415926
      SUM=0.0
      SUMS=0.0
      IX=111111
      HO=10.0
      DL=0.0
      RS=9.0
      RO=1.0
      D=10.0
      DO 8 I=1,N
C
C         POINT SELECTION
C
      CALL RANDU (IX,IY,X)
      IX=IY
      Z=D*(X-0.5)
      CALL RANDU (IX, IY,X)
      IX=IY
      R1=RS*SQRT(X)
      CALL RANDU (IX, IY, X)
      IX=IY
      BETHA=2.0*PI*X
      P=SQRT(Z*Z+R1*R1*COS(BETHA)*COS(BETHA))
      H=H0+R1*SIN(BETHA)
C
C         CALCULATION OF THE SOLID ANGLE
C         FOR THE GIVEN POINT.
C
      IF (P.GT.RO) GO TO 2
      TMIN=0.0
      TMAX=ATAN((R0+P)/H)
      TCRI=ATAN((R0-P)/H)
      CALL RANDU (IX, IY, X)
      IX=IY
      TETA=AR-
      COS(COS(TMIN)-X*(COS(TMIN)-COS(TMAX)))
      TT=TAN(TETA)
      W1=(COS(TMIN)-COS(TMAX))/2.0
      IF (TETA.GT.TCRI) GO TO1
      W2=1.0
      GO TO 7
    1 ALMX=ARCOS((P*P+H*H*TT*TT
     -R0*R0)/(2.0*H*TT*P))
      W2=ALMX/PI
      GO TO 7
    2 ALMX=ARSIN(R0/P)
      ALMN=-ALMX
      CALL RANDU (IX, IY, X)
      IX=IY
      ALPHA=ALMX*(2.0*X-1.0)
      W1=ALMX/PI
      SI=SIN(ALPHA)
      XL=P*COS(ALPHA)-SQRT(R0*R0*
     -P*P*SI*SI)
      IF (H) 5,4,3
    3 XLL=P*COS(ALPHA)+SQRT(R0*R0
     -P*P*SI*SI)
      TMAX=ATAN(XLL/H)
      GO TO 6
    4 TMAX=PI/2.0
      GO TO 6
    5 TMAX=PI/2.0+ATAN(-H/XL)
    6 TMIN=ATAN(XL/(H+DL))
      W2=(COS(TMIN)-COS(TMAX))/2.0
    7 W(I)=W1*W2
      SUMSQ=SUMSQ+W(I)*W(I)
    8 SUM=SUM+W(I)
      OMEGA=SUM/N
      S=(SUMSQ-N*OMEGA*OMEGA)/(N-1)
      SOLAN=4.0*PI*OMEGA
      SERR=4.0*PI*SQRT(S/N)
      PRINT, SOLAN, SERR
      STOP
      END
```

RANDU[8]) is an IBM routine to generate a random number rectangularly distributed between zero and one.

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