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Stability of fronts and transient behaviour in KPP systems

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We consider a system of two reaction diffusion equations with the Kolmogorov–Petrovsky–Piskunov (KPP) type nonlinearity which describes propagation of pressure-driven flames. It is known that the system admits a family of travelling wave solutions parameterized by their velocity. In this paper, we show that these travelling fronts are stable under the assumption that perturbations belong to an appropriate weighted L^2 space. We also discuss an interesting meta-stable pattern the system exhibits in certain cases.

Keywords: Kolmogorov–Petrovsky–Piskunov (KPP) systems; travelling fronts; stability; meta-stable regimes

1. Introduction

In this paper, we consider the following system of reaction diffusion equations:

$$\left.\begin{aligned} u_t &= u_{zz} + (1-v)(hu + (1-h)v) \\ \text{and}\qquad v_t &= \varepsilon v_{zz} + (1-v)(hu + (1-h)v), \quad x \in \mathbb{R}, \quad t > 0, \end{aligned}\right\} \tag{1.1}$$

where $h \in [0, 1]$, $\varepsilon \in (0, 1]$ are parameters of the system.

The model (1.1) describes propagation of pressure-driven flames in porous media under assumption of linear reaction kinetics. In this case, functions v and u are certain combinations of the normalized temperature and pressure. The model was originally derived in Brailovsky *et al.* (1997) and details can be found in Gordon (2007) and Ghazaryan & Gordon (2008).

There is also a different perspective that makes the system (1.1) interesting. When $h = 0$, the second equation of the system degenerates to the classic Kolmogorov–Petrovsky–Piskunov (KPP) equation (Fisher 1937; Kolmogorov *et al.* 1937), whereas when $h = 1$, the system (1.1) takes the form of the system describing quadratic autocatalysis (Billingham & Needham 1991). Thus, we see the system (1.1) as a system that provides a natural link between two well-known and well-studied equations.

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In this paper, we continue our work on the analysis of dynamical features of the system (1.1). In our recent paper (Ghazaryan & Gordon 2008), we studied travelling front solutions for the system (1.1) that invade the unstable equilibrium $(u, v) = (0, 0)$. These are special solutions of the form $(u, v)(t, z) = (U, V)(z - ct)$, where c is an *a priori* unknown speed of the travelling front. Substituting the travelling front ansatz into equation (1.1), we obtain the following nonlinear eigenvalue problem:

$$\left.\begin{aligned} U'' + cU' + (1 - V)(hU + (1 - h)V) = 0 \\ \text{and} \qquad \varepsilon V'' + cV' + (1 - V)(hU + (1 - h)V) = 0, \end{aligned}\right\} \tag{1.2}$$

where the derivatives are taken with respect to $x = z - ct$, together with the boundary-like condition,

$$(U, V) \longrightarrow (1, 1) \quad \text{as } x \longrightarrow -\infty, \quad (U, V) \longrightarrow (0, 0) \quad \text{as } x \longrightarrow \infty. \tag{1.3}$$

In Ghazaryan & Gordon (2008) it was shown that the existence of a solution for the problem (1.2)–(1.3) is fully determined by the existence of positive, exponentially decaying to the equilibrium, solutions of the linearization of the system equation (1.2) about the unstable equilibrium $(U, V) = (0, 0)$. Specifically, linearizing equation (1.2) about $(U, V) = (0, 0)$ and substituting $(U, V) \sim \exp(-\lambda x)$, one obtains the solvability condition,

$$(\lambda^2 - c\lambda + h)(\varepsilon\lambda^2 - c\lambda + (1 - h)) = h(1 - h). \tag{1.4}$$

Equation (1.4) has two solutions that provide a relation between the velocity of travelling wave c and the rate of exponential decay λ. However, only one of the solutions of equation (1.4) corresponds to non-negative travelling waves. This solution reads

$$c(\lambda) = \frac{1}{2}\left[\frac{1}{\lambda} + (1 + \varepsilon)\lambda + \sqrt{\left(\frac{1}{\lambda} + (1 - \varepsilon)\lambda\right)^2 - 4(1 - \varepsilon)(1 - h)}\,\right]. \tag{1.5}$$

It is not hard to see that there exists a unique $0 < c^* < \infty$ such that equation (1.5) has no solution when $c < c^*$, one solution λ^* when $c = c^*$, and two distinct solutions λ_+ and λ_- $(\lambda_+ < \lambda_-)$ when $c > c^*$. We call this critical value c^* of the front's velocity the critical velocity. The following theorem has been proved in Ghazaryan & Gordon (2008).

Theorem 1.1. *For any $c \geq c^*$, the system (1.2) has a unique, up to translation, positive travelling wave solution (front) $U(x), V(x)$ asymptotically connecting the equilibrium $(U, V) = (1, 1)$ as $x \to -\infty$ to the equilibrium $(U, V) = (0, 0)$ as $x \to \infty$. The solution converges to both equilibria exponentially fast and the rate of convergence to $(U, V) = (0, 0)$ is $\lambda = \lambda_+$ where λ_+ is the minimal solution of equation (1.5). The system (1.2) has no positive travelling wave solutions for $c < c^*$.*

The question of the existence of the fronts is resolved in theorem 1.1. The next immediate question is the stability of the fronts. Raugel & Kirchgässner (1996, 1998) investigated a system similar to the one considered in this paper, but with a small h; in other words, they considered a system that is a singular perturbation of the system (1.1) with $h = 0$. It was shown that a reduction principle can be used

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to prove that the stability properties of the scalar KPP equation dominate the properties of the fronts in the perturbed system. In the next section, we will show that for the system (1.1), the stability properties of fronts are inherited from the scalar KPP equation even for h that is not so small: each travelling wave in the co-moving coordinate frame is stable provided that the initial perturbation decays faster than the travelling wave itself for x large enough and parameters of the system satisfy $h \in (0, 3/4]$, $\varepsilon \in (0, 1]$. More precisely, for any given front, we will consider a space of perturbations that decay at $+\infty$ faster than the front and show that in the co-moving frame the front is stable with respect to the weighted L^2-norm. Moreover, we will show that this result translates into convective behaviour in a sense that, for any fixed $L \in \mathbb{R}$, within the interval $[L, \infty)$ initial perturbation will decay at every point of the interval. In the case when $h > 3/4$, we prove a similar stability result for fronts that move at sufficiently fast speeds. The case of $h = 0$ will be discussed separately in §3.

System (1.1), unlike the system considered in Raugel & Kirchgässner (1996, 1998), also allows the diffusion coefficients to be quite different ($\varepsilon \in (0, 1]$). This additional multi-scale structure also serves as a complication. Instead of exploiting smallness of parameters, i.e. their closeness to singular limits, we try to obtain results that are valid for more general parameter regimes. This is done by employing energy type estimates conceptually similar to those used by in Focant & Gallay (1998) for analysis of stability of fronts for reaction diffusion systems with more general reaction term, but diffusion coefficients which are close to each other.

We also use a technique that works for both the cases of supercritical fronts ($c > c^*$) and the critical front ($c = c^*$). Raugel & Kirchgässner (1996) prove that admissible perturbations to the fronts with non-critical velocities decay exponentially in time, whereas perturbations to the critical front decay algebraically (Raugel & Kirchgässner 1998). These two cases are quite different (the reasons are discussed below) and, in general, require different techniques. Our approach for special classes of perturbations yields the information about the exponential decay of perturbations to the fronts with $c > c^*$, but provides weaker information about the stability of the critical front: the perturbations decay, but the rate is unspecified.

Exponentially weighted L^∞ spaces were used to prove the orbital stability of supercritical fronts by Sattinger (1976, 1977). For our technique, and as in Raugel & Kirchgässner (1996, 1998), exponentially weighted Hilbert spaces are more suitable. The necessity of using weights is determined by the presence of the unstable essential spectrum of the linearization of the system about the given front.

To stabilize the front, at least on the linear level, one uses an exponential weight, in our case with a positive rate. This works well for the supercritical fronts: the essential spectrum is shifted to the open left-hand side of the complex plane. On the other hand, the nonlinearity is not well defined in the exponentially weighted spaces; therefore, the nonlinear stability of the non-critical fronts cannot be deduced from the linear stability using Henry's (1980) theory, and a different approach is needed.

The situation is even more complicated for the critical front. The essential spectrum cannot be separated from the imaginary axis, and therefore even on the linear level, the completely stable spectrum cannot be achieved. As it is

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mentioned in Raugel & Kirchgässner (1996, 1998), the Evans function approach that has been successfully used for some parabolic equations (e.g. Gardner *et al.* 1993; Kapitula 1994) is not applicable for KPP-type equations as the condition of 'consistent splitting' necessary for the construction of the Evans function does not hold.

In addition to stabilizing (or almost stabilizing in the critical case) the essential spectrum, the weights that one has to use here remove an eigenvalue at 0 as the derivative of the front does not belong to that weighted space. The family of the fronts that we are considering is parameterized by two parameters: the velocity of the fronts and the translation for a front with each fixed velocity. Using an exponential weight associated with the rate of decay as $x \to \infty$ of each individual front allows us to concentrate on a front with a fixed velocity—no shifts in the velocity are allowed, and fixes the translation invariance of the system—the stability is not orbital, i.e. it is of an individual front not within the family.

2. Stability of travelling fronts

In this section, we study stability of the travelling fronts for the system (1.1). For brevity, we will denote

$$w = hu + (1-h)v, \quad W = hU + (1-h)V. \tag{2.1}$$

For further analysis, we consider the problem in the system of coordinates that moves with the velocity of the corresponding front, that is, $(t, z) \to (t, x) = (t, z - ct)$. In the new system of coordinates, the system (1.1) reads

$$\left.\begin{aligned} u_t &= u_{xx} + cu_x + (1-v)w \\ \text{and} \qquad v_t &= \varepsilon v_{xx} + cv_x + (1-v)w. \end{aligned}\right\} \tag{2.2}$$

We now consider the problem (2.2) with the initial data $u_0(x), v_0(x)$ satisfying the following conditions:

$$0 \le u_0(x) \le K, \quad 0 \le v_0(x) \le 1, \tag{2.3}$$

where $K > 0$ is some constant. Moreover, we assume that initially the solution is a perturbation of the travelling front in the following sense:

$$u_0(x) = U(x) + \tilde{u}_0(x), \quad v_0(x) = V(x) + \tilde{v}_0(x), \tag{2.4}$$

where (U, V) is a travelling wave solution corresponding to $c(\lambda)$ and

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \left(\tilde{u}_0(x)\,\mathrm{e}^{sx}\right)^2 \mathrm{d}x < \infty, \quad \int_{-\infty}^{\infty} \left(\tilde{v}_0(x)\,\mathrm{e}^{sx}\right)^2 \mathrm{d}x < \infty \\ \text{and} \quad &\int_{-\infty}^{\infty} \left(\frac{\mathrm{d}\tilde{u}_0(x)}{\mathrm{d}x}\,\mathrm{e}^{sx}\right)^2 \mathrm{d}x < \infty, \quad \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}\tilde{v}_0(x)}{\mathrm{d}x}\,\mathrm{e}^{sx}\right)^2 \mathrm{d}x < \infty, \end{aligned}\right\} \tag{2.5}$$

where it is assumed that either $s = \lambda$ or $s = \lambda + \delta$ with $\delta > 0$ sufficiently small. In other words, we assume that the initial perturbation to the front decays slightly faster at $x \to \infty$ than the front itself.

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We now seek the solution of the problem (2.2) in the following form:

$$u(t,x) = U(x) + \tilde{u}(t,x), \quad v(t,x) = V(x) + \tilde{v}(t,x). \tag{2.6}$$

Substituting equation (2.6) into equation (2.2) and taking into account equation (1.2), we have the following system describing dynamics of perturbation $(\tilde{u}, \tilde{v})$:

$$\left.\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + c\tilde{u}_x + h(1 - V(x) - \tilde{v}(t,x))\tilde{u} \\ &\quad + [(1-h)(1 - V(x) - \tilde{v}(t,x)) - W]\tilde{v} \\ \text{and} \qquad \tilde{v}_t &= \varepsilon\tilde{v}_{xx} + c\tilde{v}_x + h(1 - V(x) - \tilde{v}(t,x))\tilde{u} \\ &\quad + [(1-h)(1 - V(x) - \tilde{v}(t,x)) - W]\tilde{v}. \end{aligned}\right\} \tag{2.7}$$

For brevity, and technical convenience (see equation (2.15) below), we will use the notation $v(t,x) = V(x) + \tilde{v}(t,x)$ when working with equation (2.7).

Now, we can formulate the main result of this section.

Theorem 2.1. *Consider problem (2.2) with $\varepsilon \in (0,1]$ and $h \in (0, 3/4]$. Assume that initial data satisfy equations (2.3)–(2.5) with $s = \lambda$. Then, for any $c \geq c^*$ and $L \in \mathbb{R}$, one has*

$$\sup_{x \in [L,\infty)} |U(x) - u(t,x)| \longrightarrow 0, \quad \sup_{x \in [L,\infty)} |V(x) - v(t,x)| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{2.8}$$

Moreover, if condition (2.5) is imposed with $s = \lambda + \delta$ with $\delta > 0$ and $c > c^*$, then there exist two positive constants M and ω such that

$$\sup_{x \in [L,\infty)} |U(x) - u(t,x)| < M\,\mathrm{e}^{-\omega t}, \quad \sup_{x \in [L,\infty)} |V(x) - v(t,x)| < M\,\mathrm{e}^{-\omega t}, \quad \text{for all } t > 0. \tag{2.9}$$

Remark 2.2. It is important to note that the result of theorem 2.1 is optimal. In other words, any perturbation with an exponential rate of decay as $x \to \infty$ slower than the one of the corresponding travelling front leads to an instability.

For the sake of convenience, we first make the following change of variables $(\tilde{u}, \tilde{v})(t,x) = (\phi, \psi)(t,x)\,\mathrm{e}^{-sx}$. In the new variables, the system (2.7) reads

$$\left.\begin{aligned} \phi_t &= \phi_{xx} + (c - 2s)\phi_x + [s^2 - cs + h(1-v)]\phi + [(1-h)(1-v) - W]\psi \\ \text{and} \quad \psi_t &= \varepsilon\psi_{xx} + (c - 2\varepsilon s)\psi_x + h(1-v)\phi + [\varepsilon s^2 - cs + (1-h)(1-v) - W]\psi. \end{aligned}\right\} \tag{2.10}$$

The proof of theorem 2.1 uses energy-type estimates and we will need the following non-negative functionals evaluated on solutions of equation (2.10)

$$\left.\begin{aligned} &E_0(t) = h\|\phi\|^2 + (1-h)\|\psi\|^2, \quad E_0^x = h\|\phi_x\|^2 + \varepsilon(1-h)\|\psi_x\|^2, \\ &E_1(t) = \langle W\psi, \psi\rangle, \quad E_1^x(t) = \langle W\psi_x, \psi_x\rangle \\ \text{and} \quad &E_2(t) = \langle W\phi, \phi\rangle, \quad E_2^x(t) = \langle W\phi_x, \phi_x\rangle. \end{aligned}\right\} \tag{2.11}$$

Here, $\|\psi\|(t) = \left[\int_{-\infty}^{\infty} (\psi)^2(t,\cdot)\right]^{1/2}$ and $\langle\psi, \phi\rangle(t) = \int_{-\infty}^{\infty} \psi(t,\cdot)\phi(t,\cdot)$ are the standard norm and inner products in $L^2(\mathbb{R})$. We also note that, thanks to equation (2.5), all functionals in equation (2.11) are well defined at time $t = 0$ provided s satisfies the assumption of theorem 2.1.

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Conceptually, our construction is similar to the one used in Focant & Gallay (1998). Specifically, we show that under the assumptions of the first part of theorem 2.1, the functionals $E_1(t)$, $E_1^x(t)$, $E_2(t)$, $E_2^x(t) \to 0$ as $t \to \infty$ (lemmas 2.5 and 2.6) which by lemma 2.7 imply equation (2.8). The proof of the second part of theorem 2.1 is based on the observation that under the assumptions of the second part of the theorem, the functionals $E_0(t)$ and $E_0^x(t)$ approach zero exponentially fast as $t \to \infty$ (lemma 2.8). This estimate together with lemma 2.7 gives equation (2.9).

Lemma 2.3. *Let the assumption of theorem* 2.1 *be satisfied, then*

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_0(t) = h(s^2 - cs + h)\|\phi\|^2 + (1-h)(\varepsilon s^2 - cs + 1 - h)\|\psi\|^2 \\ + h\langle[2(1-h) - W]\phi, \psi\rangle - E_0^x(t) - (1-h)E_1(t) \\ - \langle v(h\phi + (1-h)\psi), (h\phi + (1-h)\psi)\rangle\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_0(t) \le h(s^2 - cs + h)\|\phi\|^2 + (1-h)(\varepsilon s^2 - cs + 1 - h)\|\psi\|^2 + 2h(1-h)\langle|\phi|, |\psi|\rangle \\ - E_0^x(t) - (1-h)E_1(t) - \langle v(h\phi + (1-h)\psi), (h\phi + (1-h)\psi)\rangle.\end{aligned} \tag{2.13}$$

Proof. Multiplying the first and the second equations of equation (2.10) by ϕ and ψ, respectively, and integrating by parts, we have

$$\left.\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|^2 = -\|\phi_x\|^2 + (s^2 - cs + h)\|\phi\|^2 \\ - h\langle v\phi, \phi\rangle + \langle[(1-h)(1-v) - W]\psi, \phi\rangle \\ \text{and} \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2 = -\varepsilon\|\psi_x\|^2 + (\varepsilon s^2 - cs + 1 - h)\|\psi\|^2 \\ + \langle h(1-v)\phi, \psi\rangle - \langle[(1-h)v + W]\psi, \psi\rangle.\end{aligned}\right\} \tag{2.14}$$

Multiplying the first and the second equations in equation (2.14) by h and $1-h$, respectively, and adding the results, we obtain equation (2.12).

It was shown in Ghazaryan & Gordon (2008) that for the component $v(t,x)$ of the solution of the problem (2.2) satisfying equation (2.3)

$$0 \le v(t,x) \le 1, \quad \text{for all } x \in \mathbb{R} \quad \text{and} \quad t \ge 0, \tag{2.15}$$

and the travelling wave solution W satisfies

$$0 < W(x) < 1, \quad \text{for all } x \in \mathbb{R}. \tag{2.16}$$

These bounds imply that the last term in equation (2.12) is non-positive. Moreover, one can see that

$$h|2(1-h) - W(x)| \le 2h(1-h), \quad \text{for each } x \in \mathbb{R}, \quad \text{provided that } h \in [0, 3/4].$$

Incorporating these observations in equation (2.12), we obtain equation (2.13). ■

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Lemma 2.4. *Let the assumption of theorem* 2.1 *be satisfied, then*

$$\varepsilon\lambda^2 - c\lambda + 1 - h < 0, \quad \lambda^2 - c\lambda + h < 0, \tag{2.17}$$

and the quadratic form

$$\boldsymbol{A}(\boldsymbol{s}) = \begin{pmatrix} h(s^2 - cs + h) & h(1-h) \\ h(1-h) & (1-h)(\varepsilon s^2 - cs + 1 - h) \end{pmatrix} \tag{2.18}$$

is negatively defined for $s = \lambda + \delta$ *with* $\delta > 0$ *sufficiently small and* $c > c^*$.

Proof. First, observe that Trace $\boldsymbol{A}(\lambda) < 0$. Indeed, let

$$p_1 = \varepsilon\lambda^2 - c\lambda + 1 - h, \quad p_2 = \lambda^2 - c\lambda + h, \tag{2.19}$$

then

$$\text{Trace } \boldsymbol{A}(\lambda) = (1-h)p_1 + hp_2. \tag{2.20}$$

By equation (1.4) $p_1 p_2 = (1-h)h > 0$. On the other hand,

$$p_1 + p_2 = (1+\varepsilon)\lambda^2 + 1 - 2c\lambda. \tag{2.21}$$

Substituting definition of $c(\lambda)$ given by equation (1.5) into equation (2.21), we have

$$p_1 + p_2 = -\lambda\sqrt{\left(\frac{1}{\lambda} + (1-\varepsilon)\lambda\right)^2 - 4(1-\varepsilon)(1-h_\varepsilon)} < 0. \tag{2.22}$$

Therefore, $p_1, p_2 < 0$ and thus

$$\text{Trace } \boldsymbol{A}(\lambda) < 0. \tag{2.23}$$

By continuity, we thus have Trace $\boldsymbol{A}(\lambda + \delta) < 0$ for a sufficiently small δ.

Next, by straightforward computations, we obtain

$$\begin{aligned} \text{Det } \boldsymbol{A}(\lambda+\delta) &= h(1-h)[(\lambda^2 - c\lambda + h)(\varepsilon\lambda^2 - c\lambda + (1-h)) - h(1-h)] \\ &\quad + \delta[(2\lambda - c)(\varepsilon\lambda^2 - c\lambda + (1-h)) + (2\varepsilon\lambda - c)(\lambda^2 - c\lambda + h)] + O(\delta^2). \end{aligned} \tag{2.24}$$

Observe that Det $\boldsymbol{A}(\lambda) = 0$ by equation (1.4). Differentiating equation (1.4) with respect to λ, we have

$$(2\lambda - c)p_2 + (2\varepsilon\lambda - c)p_1 = \lambda\frac{\mathrm{d}c}{\mathrm{d}\lambda}(p_1 + p_2). \tag{2.25}$$

It was shown in Ghazaryan & Gordon (2008) that for supercritical waves ($\lambda < \lambda^*$), one has $\mathrm{d}c/\mathrm{d}\lambda < 0$. Therefore,

$$(2\lambda - c)p_2 + (2\varepsilon\lambda - c)p_1 = k^2 \tag{2.26}$$

for some $k^2 > 0$. So that

$$\text{Det } \boldsymbol{A}(\lambda+\delta) = k^2\delta + O(\delta^2) \tag{2.27}$$

and, thus, Det $\mathbf{A}(\lambda + \delta) > 0$ for $\delta > 0$ sufficiently small. ■

Lemma 2.5. *Let the assumptions of theorem* 2.1 *be satisfied, then* $E_1(t)$, $E_1^x(t) \to 0$ *as* $t \to 0$.

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Proof. First, we set $s=\lambda$ in equations (2.12) and (2.13) and obtain estimates on evolution of the functional $E_0(t)$. By straightforward computations and taking into account equation (1.4), we have

$$h(\lambda^2-c\lambda+h)\|\phi\|^2+(1-h)(\varepsilon\lambda^2-c\lambda+1-h)\|\psi\|^2+2h(1-h)\langle\psi,\phi\rangle \\ =-\mu^2\langle\phi-\nu^2\psi,\phi-\nu^2\psi\rangle$$

and

$$h(\lambda^2-c\lambda+h)\|\phi\|^2+(1-h)(\varepsilon\lambda^2-c\lambda+1-h)\|\psi\|^2+2h(1-h)\langle|\psi|,|\phi|\rangle \\ =-\mu^2\langle|\phi|-\nu^2|\psi|,|\phi|-\nu^2|\psi|\rangle,$$

where

$$\mu^2=-h(\lambda^2-c\lambda+h)>0,\quad \nu^4=\frac{(1-h)(\varepsilon\lambda^2-c\lambda+1-h)}{h(\lambda^2-c\lambda+h)}>0, \tag{2.28}$$

based on lemma 2.4. Hence, by equations (2.12), (2.13) and (2.28), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_0(t)=-E_0^x(t)-(1-h)E_1(t)-h\langle W\phi,\psi\rangle \\ -\langle v(h\phi+(1-h)\psi),(h\phi+(1-h)\psi)\rangle-\mu^2\langle\phi-\nu^2\psi,\phi-\nu^2\psi\rangle \tag{2.29}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_0(t)\le-E_0^x(t)-(1-h)E_1(t) \\ -\langle v(h\phi+(1-h)\psi),(h\phi+(1-h)\psi)\rangle-\mu^2\langle|\phi|-\nu^2|\psi|,|\phi|-\nu^2|\psi|\rangle. \tag{2.30}$$

Next, we obtain equation for the evolution of $E_1(t)$. Multiplying the second equation of equation (2.10) by $W\psi$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)=-\varepsilon\langle W\psi_x,\psi_x\rangle+\langle\rho W\psi,\psi\rangle+h\langle W\phi,\psi\rangle-h\langle vW\phi,\psi\rangle, \tag{2.31}$$

where

$$\rho(x)=\left[\frac{\varepsilon}{2}\frac{W_{xx}}{W}-\frac{(c-2\varepsilon\lambda)}{2}\frac{W_x}{W}+\varepsilon\lambda^2-c\lambda+(1-h)(1-v)-W\right]. \tag{2.32}$$

It was shown in Ghazaryan & Gordon (2008) that W is a monotone C^2 function connecting equilibrium points $(0,0)$ and $(1,1)$ as $x\to\pm\infty$ with bounded derivatives. Standard regularity results imply that $|W_{xx}/W|$ and $|W_x/W|$ are uniformly bounded and $C_1=\sup_{x\in\mathbb{R}}|\rho(x)|$ is finite. Here and below C_i are positive constants. Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)\le-\varepsilon E_1^x(t)+C_1E_1(t)+h\langle W\phi,\psi\rangle-h\langle vW\phi,\psi\rangle. \tag{2.33}$$

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We differentiate the second equation of equation (2.10) with respect to x,

$$\psi_{xt} = \varepsilon\psi_{xxx} + (c - 2\varepsilon s)\psi_{xx} + h(1-v)\phi_x + [\varepsilon s^2 - cs + (1-h)(1-v) - W]\psi_x \\ - hv_x\phi - [(1-h)v_x + W_x]\psi,$$

multiply the result by $W\psi_x$,

$$\begin{aligned}\psi_{xt}W\psi_x = \varepsilon\psi_{xxx}W\psi_x &+ (c-2\varepsilon s)\psi_{xx}W\psi_x + h(1-v)\phi_x W\psi_x \\ &+ [\varepsilon s^2 - cs + (1-h)(1-v) - W]\psi_x W\psi_x \\ &- hv_x\phi W\psi_x - [(1-h)v_x + W_x]\psi W\psi_x,\end{aligned}$$

and integrate with respect to x. We then integrate some of the terms by parts and obtain the equation for the evolution of $E_1^x(t)$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1^x(t) = &-\varepsilon\langle W\psi_{xx}, \psi_{xx}\rangle + \langle vW\phi, \psi_{xx}\rangle + \langle W^2\psi, \psi_{xx}\rangle \\ &+ \langle[\rho + (1-h)v + W]W\psi_x, \psi_x\rangle + h\langle W\phi_x, \psi_x\rangle + h\langle vW_x\phi, \psi_x\rangle \\ &+ (1-h)\langle vW_x\psi, \psi_x\rangle + \langle WW_x\psi, \psi_x\rangle.\end{aligned} \tag{2.34}$$

Using Young's inequality and the fact that $0 \le v,\ W \le 1$ and $|W_x| < 1 + h/2c$ (see Ghazaryan & Gordon (2008)) we have

$$\left.\begin{aligned}&-\varepsilon\langle W\psi_{xx}, \psi_{xx}\rangle + \langle vW\phi, \psi_{xx}\rangle + \langle W^2\psi, \psi_{xx}\rangle \le \frac{1}{2\varepsilon}\langle v\psi, \psi\rangle + \frac{1}{2\varepsilon}\langle W\psi, \psi\rangle, \\ &\langle[\rho+(1-h)v+W]W\psi_x, \psi_x\rangle + h\langle W\phi_x, \psi_x\rangle \le \left(C_1 + 2 - \frac{h}{2}\right)\|\psi_x\|^2 + \frac{h}{2}\|\phi_x\|^2 \\ &\text{and } h\langle vW_x\phi, \psi_x\rangle + (1-h)\langle vW_x\psi, \psi_x\rangle + \langle WW_x\psi, \psi_x\rangle \\ &\quad \le \frac{1+h}{4c}(2\langle v\phi, \phi\rangle + (1-h)\langle v\psi, \psi\rangle + \langle W\psi, \psi\rangle).\end{aligned}\right\} \tag{2.35}$$

These estimates imply that there exists a constant $\tilde{C}_2 > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1^x(t) \le \tilde{C}_2(E_0^x(t) + E_1(t) + \langle v\phi, \phi\rangle + \langle v\psi, \psi\rangle).$$

It is also possible to choose a constant $C_2 > \tilde{C}_2 > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1^x(t) \le C_2(E_0^x(t) + E_1(t) + h^2\langle v\phi, \phi\rangle + \langle v((1-h) + W/2)^2\psi, \psi\rangle). \tag{2.36}$$

Next, we add equations (2.29) and (2.33) together and obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(E_0(t) + E_1(t)) \le C_3E_1(t) &- \mu^2\langle\phi - \nu^2\psi, \phi - \nu^2\psi\rangle \\ &- \langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi)\rangle,\end{aligned} \tag{2.37}$$

with

$$C_3 = C_1 - 1 + h + \sup_{t>0, x\in\mathbb{R}} v(t,x)[(1-h)W + W^2/4] = C_1 + \frac{1}{4}.$$

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We multiply equation (2.36) by σ/C_2 and add the result to equation (2.37) to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left(E_0(t)+E_1(t)+\frac{\sigma}{C_2}E_1^x(t)\right)\\ &\le \sigma E_0^x(t)+(C_3+\sigma)E_1(t)+\sigma(h^2\langle v\phi,\phi\rangle+\langle v[(1-h+W/2)^2]\psi,\psi\rangle)\\ &\quad -\langle v(h\phi+[(1-h)+W/2]\psi),(h\phi+[(1-h)+W/2]\psi)\rangle\\ &\quad -\mu^2\langle\phi-\nu^2\psi,\phi-\nu^2\psi\rangle.\end{aligned}$$

We then rewrite the last three terms on the right as follows:

$$\begin{aligned}&\sigma(h^2\langle v\phi,\phi\rangle+\langle v[(1-h+W/2)^2]\psi,\psi\rangle)\\ &\qquad -\langle v(h\phi+[(1-h)+W/2]\psi),(h\phi+[(1-h)+W/2]\psi)\rangle\\ &\qquad -\mu^2\langle\phi-\nu^2\psi,\phi-\nu^2\psi\rangle-2\sigma h\langle v[1-h+W/2]\phi,\psi\rangle\\ &\qquad -\mu^2(\|\phi\|^2+\nu^4\|\psi\|^2-2\nu^2\langle\psi,\phi\rangle)\\ &\quad =-(1-\sigma)\langle v(h\phi+[(1-h)+W/2]\psi),(h\phi+[(1-h)+W/2]\psi)\rangle\\ &\qquad -\mu^2\|\phi\|^2-\mu^2\nu^4\|\psi\|^2+2\mu^2\nu^2\langle\phi,\psi\rangle-2\sigma h\langle v[1-h+W/2]\phi,\psi\rangle.\end{aligned}$$

We choose σ so small that the following bound holds:

$$2\mu^2\nu^2\langle\phi,\psi\rangle-2\sigma h\langle v[1-h+W/2]\phi,\psi\rangle\le 2\mu^2\nu^2\langle|\phi|,|\psi|\rangle,$$

and arrive at

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left(E_0(t)+E_1(t)+\frac{\sigma}{C_2}E_1^x(t)\right)\\ &\le\sigma E_0^x(t)+(C_3+\sigma)E_1(t)-\mu^2\langle|\phi|-\nu^2|\psi|,|\phi|-\nu^2|\psi|\rangle\\ &\quad-(1-\sigma)\langle v(h\phi+[(1-h)+W/2]\psi),(h\phi+[(1-h)+W/2]\psi)\rangle. \end{aligned}\tag{2.38}$$

From equation (2.30),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_0(t)\le -E_0^x(t)-(1-h)E_1(t).\tag{2.39}$$

We denote

$$C_4=\frac{C_3+\sigma}{1-h}+1.$$

Multiplying equation (2.39) by C_4-1 and adding the result to equations (2.37) and (2.38), we obtain

$$\left.\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(C_4E_0(t)+E_1(t))\le 0\\ \text{and}\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(C_4E_0(t)+E_1(t)+\frac{\sigma}{C_2}E_1^x(t)\right)\le 0.\end{aligned}\right\}\tag{2.40}$$

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We observe that non-negative functionals $E_0(t)$ and $C_4E_0(t)+E_1(t)$ and $C_4E_0(t)+E_1(t)+(\sigma/C_2)E_1^x(t)$ are non-increasing by equations (2.39) and (2.40). Hence, $E_0(t)$, $C_3E_0(t)+E_1(t)$, $C_4E_0(t)+E_1(t)+(\sigma/C_2)E_1^x(t)$ converge as $t\to\infty$. This implies that $E_1(t)$ and $E_1^x(t)$ also converge as $t\to\infty$.

We integrate equation (2.39) to obtain

$$\int_0^\infty E_0^x(t)\,\mathrm{d}t+(1-h)\int_0^\infty E_1(t)\,\mathrm{d}t\le\frac{1}{2}(E_0(0)-E_0(\infty))<\infty. \tag{2.41}$$

As $E_0^x(t)\ge\varepsilon(1-h)E_1^x(t)+hE_2^x(t)$ for all t, we have

$$\left.\begin{aligned}&\int_0^\infty E_1(t)\,\mathrm{d}t\le\frac{1}{2(1-h)}(E_0(0)-E_0(\infty))<\infty,\\ &\int_0^\infty E_1^x(t)\,\mathrm{d}t\le\frac{1}{2\varepsilon(1-h)}(E_0(0)-E_0(\infty))<\infty\\ \text{and}\qquad&\int_0^\infty E_2^x(t)\,\mathrm{d}t\le\frac{1}{2h}(E_0(0)-E_0(\infty))<\infty.\end{aligned}\right\} \tag{2.42}$$

The first two inequalities in equation (2.42) guarantee the integrability of $E_1(t)$ and $E_1^x(t)$ and thus, as $E_1(t)$ and $E_1^x(t)$ converge as $t\to\infty$, they must converge to zero. The last inequality in equation (2.42) will be used in the next lemma which establishes the same result for $E_2(t)$ and $E_2^x(t)$. ■

Lemma 2.6. *Under the assumptions of theorem* 2.1, $E_2(t)$, $E_2^x(t)\to 0$ *as* $t\to 0$

Proof. We multiply the first equation of equation (2.10) by $W\phi$ and integrate some of the terms by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_2(t)=-\langle W\phi_x,\phi_x\rangle+\langle\gamma W\phi,\phi\rangle+\langle W[(1-h)(1-v)-W]\psi,\phi\rangle, \tag{2.43}$$

where

$$\gamma(x)=\left[\frac{1}{2}\frac{W_{xx}}{W}-\frac{(c-2\lambda)}{2}\frac{W_x}{W}+\lambda^2-c\lambda+h(1-v)\right]. \tag{2.44}$$

As $|W_{xx}/W|$ and $|W_x/W|$ are uniformly bounded, we have $C_5=\sup_{x\in\mathbb{R}}|\gamma(x)|<\infty$. From equation (2.43), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_2(t)\le C_5E_2(t)+\langle W|\psi|,|\phi|\rangle. \tag{2.45}$$

The following two inequalities are easy to check:

$$\left.\begin{aligned}\phi^2&=(\phi-\nu^2\psi+\nu^2\psi)^2=(\phi-\nu^2\psi)^2+\nu^4\psi^2+2\nu^2(\phi-\nu^2\psi)\psi\\ &\le(\nu^4+\nu^2)\psi^2+(1+\nu^2)(\phi-\nu^2\psi)^2\\ \text{and}\qquad|\phi||\psi|&=|\psi||\phi-\nu^2\psi+\nu^2\psi|\le\nu^2\psi^2+|\psi||\phi-\nu^2\psi|\\ &\le(\nu^2+1/2)\psi^2+\frac{1}{2}(\phi-\nu\psi)^2.\end{aligned}\right\} \tag{2.46}$$

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Therefore,

$$\left.\begin{aligned} &E_2(t) \le (\nu^4+\nu^2)E_1(t) + (1+\nu^2)\langle W(\phi-\nu^2\psi),(\phi-\nu^2\psi)\rangle \\ \text{and}\qquad &\langle W|\phi|,|\psi|\rangle \le (\nu^2+1/2)E_1(t) + \frac{1}{2}\langle W(\phi-\nu^2\psi),(\phi-\nu^2\psi)\rangle. \end{aligned}\right\} \tag{2.47}$$

From equations (2.45) and (2.47), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) \le -E_2(t) + C_6E_1(t) + C_7\langle W(\phi-\nu^2\psi),(\phi-\nu^2\psi)\rangle, \tag{2.48}$$

where

$$C_6 = (C_5+1)(\nu^2+\nu^4)+\nu^2+\frac{1}{2}, \quad C_7 = (C_5+1)(1+\nu^2)+\frac{1}{2}.$$

We multiply equation (2.37) by C_7/μ^2, equation (2.39) by $C_8 = (C_2 + C_2C_6/\mu^2)/(1-h)$ and add the results to equation (2.48). Taking into account that $\langle W(\phi-\nu^2\psi),(\phi-\nu^2\psi)\rangle \le \langle(\phi-\nu^2\psi),(\phi-\nu^2\psi)\rangle$ for all $t\ge 0$, we then obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(E_2 + \frac{C_7}{\mu^2}E_1(t) + \left(C_8+\frac{C_7}{\mu^2}\right)E_0(t)\right) \le -E_3(t). \tag{2.49}$$

From equation (2.49), the functional $E_2 + (C_7/\mu^2)E_1(t) + (C_8+(C_7/\mu^2))E_0(t)$ is not increasing and thus converge as $t\to\infty$. As functionals E_0 and E_1 converge as $t\to\infty$, we conclude that E_2 also converges as $t\to\infty$.

Integrating equation (2.49), we have

$$\begin{aligned} \int_0^\infty E_2(t)\mathrm{d}t \le E_2(0) - E_2(\infty) + \frac{C_7}{\mu^2}(E_1(0)-E_1(\infty)) \\ + \left(C_8+\frac{C_7}{\mu^2}\right)(E_0(0)-E_0(\infty)) < \infty. \end{aligned} \tag{2.50}$$

Therefore, $E_2(t)$ converges to zero as $t\to\infty$.

Next, differentiating the first equation of equation (2.10) with respect to x, multiplying by $W\phi_x$, and integrating by parts, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_2^x(t) = &-\langle W\phi_{xx},\phi_{xx}\rangle + h\langle vW\phi,\phi_{xx}\rangle + (1-h)\langle vW\psi,\phi_{xx}\rangle \\ &+ \langle W^2\psi,\phi_{xx}\rangle + \langle(\gamma+hv)W\phi_x,\phi_x\rangle + (1-h)\langle W\phi_x,\psi_x\rangle + h\langle vW_x\phi,\phi_x\rangle \\ &+ (1-h)\langle vW_x\psi,\phi_x\rangle + \langle WW_x\psi,\phi_x\rangle. \end{aligned} \tag{2.51}$$

Applying Young's inequality to the right-hand side of equation (2.51) in a way similar to equation (2.35), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_2^x(t) \le C_9(E_0^x(t) + E_1(t) + h^2\langle v\phi,\phi\rangle + ((1-h)+W/2)^2\langle v\psi,\psi\rangle), \tag{2.52}$$

where $C_9 > 0$ is sufficiently large.

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Multiplying equation (2.52) by σ/C_9 with a sufficiently small $\sigma > 0$ and adding the result to equation (2.37), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left(E_0(t) + E_1(t) + \frac{\sigma}{C_9}E_2^x(t)\right)\\ &\leq \sigma E_0^x(t) + (C_3+\sigma)E_1(t) - \mu^2\langle|\phi| - \nu^2|\psi|, |\phi| - \nu^2|\psi|\rangle\\ &\quad - (1-\sigma)\langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi)\rangle. \qquad (2.53)\end{aligned}$$

Multiplying equation (2.39) by $(C_3+\sigma)/1-h$ and adding the result to equation (2.53), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(C_4E_0(t) + E_1(t) + \frac{\sigma}{C_9}E_2^x(t)\right) \leq 0, \qquad (2.54)$$

which implies that the non-negative functional $C_4E_0(t) + E_1(t) + (\sigma/C_9)E_2^x(t)$ is non-increasing. This, together with the fact that non-increasing functionals $E_0(t)$ and $E_1(t)$ converge as $t \to \infty$, immediately yields convergence of $E_2^x(t)$ as $t \to \infty$.

In lemma 2.5, we already established that $\int_0^\infty E_2^x(t)\,\mathrm{d}t < \infty$ (see equation (2.42)), therefore $E_2^x(t) \to 0$ as $t \to \infty$. ■

Lemma 2.7. *Let f be such that $\langle Wf, f\rangle(t) + \langle Wf_x, f_x\rangle(t)$ is well defined and set $g(x) = f(x)\mathrm{e}^{-\lambda x}$. Then, for any given $L \in \mathbb{R}$, there exists a constant $C_{10} > 0$ independent of t and L such that*

$$\sup_{x\in[L,\infty)} |g(t,x)| \leq C_{10}\,\mathrm{e}^{-\lambda L/2}(\langle Wf, f\rangle(t) + \langle Wf_x, f_x\rangle(t)))^{1/2}. \qquad (2.55)$$

Proof. First, we observe that

$$\begin{aligned}|W(x)f^2(x)| &= \left|\int_x^\infty [W(z)f^2(t,z)]_z\,\mathrm{d}z\right|\\ &\leq \int_x^\infty |W_z|f^2(t,z)dz + 2\int_x^\infty |Wf(t,z)f(t,z)_z|\,\mathrm{d}z. \qquad (2.56)\end{aligned}$$

As $|W_z/W| < C_{11}$ for some finite constant $C_{11} > 0$, we have

$$\begin{aligned}|W(x)f^2(x)| &\leq C_{11}\int_x^\infty Wf^2(t,z)\,\mathrm{d}z + 2\int_x^\infty |Wf(t,z)f(t,z)_z|\,\mathrm{d}z\\ &\leq (C_{11}+1)\langle Wf, f\rangle + \langle Wf_x, f_x\rangle. \qquad (2.57)\end{aligned}$$

Thus,

$$|g(x)| \leq \left((C_{11}+1)\frac{\mathrm{e}^{-2\lambda x}}{W(x)}(\langle Wf, f\rangle + \langle Wf_x, f_x\rangle)\right)^{1/2}. \qquad (2.58)$$

W is a monotone decreasing function approaching unity as $x \to -\infty$ and zero as $x \to \infty$. Moreover, $W \sim O(\mathrm{e}^{\lambda x})$ for large x. Therefore, given $L \in \mathbb{R}$

$$\sup_{x\in[L,\infty)} \frac{\mathrm{e}^{-2\lambda x}}{W(x)} < C_{12}\mathrm{e}^{-\lambda L}, \qquad (2.59)$$

where C_{12} is independent of L. Combining equations (2.58) and (2.59), we have equation (2.55). ■

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Lemma 2.8. *Let the assumption of theorem* 2.1 *be satisfied with* $s=\lambda+\delta$ *and* $c>c^*$, *then there exist two constants* $N>0$ *and* $\alpha>0$ *such that*

$$E_0(t)<N\,\mathrm{e}^{-\alpha t},\quad E_0^x<N\,\mathrm{e}^{-\alpha t}. \tag{2.60}$$

Proof. It was shown in lemma 2.4 that the quadratic form

$$\begin{aligned}&h(s^2-cs+h)\|\phi\|^2+(1-h)(\varepsilon s^2-cs+1-h)\|\psi\|^2+2h(1-h)\|\phi\|\|\psi\|\\ &\quad=\boldsymbol{J}^{\mathrm{T}}(t)\cdot\boldsymbol{A}(s)\cdot\boldsymbol{J}(t),\end{aligned} \tag{2.61}$$

where $\boldsymbol{J}(t)=(\|\phi\|,\|\psi\|)(t)$ and

$$\boldsymbol{A}(\mathrm{s})=\begin{pmatrix}h(s^2-cs+h) & h(1-h)\\ h(1-h) & (1-h)(\varepsilon s^2-cs+1-h)\end{pmatrix} \tag{2.62}$$

is negatively defined for $s=\lambda+\delta$ and a small enough $\delta>0$. Therefore, there exists $\alpha>0$ such that

$$\boldsymbol{J}^{\mathrm{T}}(t)\cdot\boldsymbol{A}(\lambda+\delta)\cdot\boldsymbol{J}(t)<-\alpha E_0(t). \tag{2.63}$$

Thus, from equation (2.13), we have

$$\frac{1}{2}\frac{\mathrm{d}E_0(t)}{\mathrm{d}t}\le-\alpha E_0(t)-E_0^x(t). \tag{2.64}$$

We differentiate the system (2.10) with respect to x, multiply the first and the second equation by ϕ_x and ψ_x, respectively, and integrate some of the terms by parts to obtain

$$\left.\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\|\phi_x\|^2}{\mathrm{d}t}&=-\|\phi_{xx}\|^2+h\langle v\phi,\phi_{xx}\rangle+\langle[(1-h)v+W]\psi,\phi_{xx}\rangle\\ &\quad+(s^2-cs+h)\|\phi_x\|^2+(1-h)\langle\psi_x,\phi_x\rangle\\ \text{and}\qquad\frac{1}{2}\frac{\mathrm{d}\|\psi_x\|^2}{\mathrm{d}t}&=-\varepsilon\|\psi_{xx}\|^2+h\langle v\phi,\psi_{xx}\rangle+\langle[(1-h)v+W]\psi,\psi_{xx}\rangle\\ &\quad+h\langle\phi_x,\psi_x\rangle+(\varepsilon s^2-cs+1-h)\|\psi_x\|^2.\end{aligned}\right\} \tag{2.65}$$

Applying Young's inequality to the right-hand side of equation (2.65)

$$\left.\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\|\phi_x\|^2}{\mathrm{d}t}&\le C_{13}(E_0(t)+E_0^x(t))\\ \text{and}\qquad\frac{1}{2}\frac{\mathrm{d}\|\psi_x\|^2}{\mathrm{d}t}&\le C_{14}(E_0(t)+E_0^x(t)),\end{aligned}\right\} \tag{2.66}$$

for sufficiently large C_{13} and C_{14}. Multiplying the first and the second equations of equation (2.66) by h and $\varepsilon(1-h)$, respectively, and adding the results together, we have

$$\frac{1}{2}\frac{\mathrm{d}E_0^x(t)}{\mathrm{d}t}\le C_{15}(E_0(t)+E_0^x(t)), \tag{2.67}$$

with $C_{15}=hC_{13}+\varepsilon(1-h)C_{14}$. Multiplying equation (2.67) by $\alpha/2C_{15}$ and adding the result to equation (2.64), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0(t)+\frac{\alpha}{2C_{15}}E_0^x(t)\right)\le-\frac{\alpha}{2}(E_0(t)+E_0^x(t)). \tag{2.68}$$

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If we take C_{15} sufficiently large so that $\alpha/2C_{15} \le 1$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0(t) + \frac{\alpha}{2C_{15}}E_0^x(t)\right) \le -\frac{\alpha}{2}\left(E_0(t) + \frac{\alpha}{2C_{15}}E_0^x(t)\right). \tag{2.69}$$

Integrating this equation, we get

$$\left(E_0(t) + \frac{\alpha}{2C_{15}}E_0^x(t)\right) \le \left(E_0(0) + \frac{\alpha}{2C_{15}}E_0^x(0)\right)\mathrm{e}^{-\alpha t/2}, \tag{2.70}$$

which immediately implies equation (2.60). ■

Proof (*Theorem* 2.1.). By setting $f = (\phi, \psi)$ and $g = (\tilde{u}, \tilde{v})$ in lemma 2.7, we have

$$\left.\begin{aligned} &\sup_{x\in[L,\infty)} |\tilde{v}(t,x)| \le C_{16}\,\mathrm{e}^{-\lambda L/2}(E_1(t) + E_1^x(t)\\ \text{and}\quad &\sup_{x\in[L,\infty)} |\tilde{u}(t,x)| \le C_{17}\,\mathrm{e}^{-\lambda L/2}(E_2(t) + E_2^x(t)).\end{aligned}\right\} \tag{2.71}$$

Let $s = \lambda$. Then, according to lemmas 2.5 and 2.6, $E_1(t)$, $E_1^x(t)$, $E_2(t)$, $E_2^x(t) \to 0$ as $t \to \infty$ which proves equation (2.8).

We set $s = \lambda + \delta$ and observe that $E_1(t) + E_2(t) + E_1^x(t) + E_2^x(t) < C_{18}(E_0(t) + E_0^x(t))$, therefore from equation (2.71), we obtain

$$\left.\begin{aligned} &\sup_{x\in[L,\infty)} |\tilde{v}(t,x)| \le C_{19}\mathrm{e}^{-\lambda L/2}(E_0(t) + E_0^x(t))\\ \text{and}\quad &\sup_{x\in[L,\infty)} |\tilde{u}(t,x)| \le C_{20}\mathrm{e}^{-\lambda L/2}(E_0(t) + E_0^x(t)).\end{aligned}\right\} \tag{2.72}$$

By lemma 2.8 $E_0(t)$ and $E_0^x(t)$ converge to zero exponentially fast. The proof of equation (2.9) is completed. ■

Remark 2.9. For $h \in (3/4, 1]$, we will show that sufficiently fast supercritical fronts are stable under perturbations that decay at $x \to \infty$ exponentially at faster rates, although the obtained bounds on the rates of convergence are not optimal.

Indeed, by adding the first and the second equations in equation (2.10), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_0(t) \le (s^2 - cs + h)\|\phi\|^2 + (\varepsilon s^2 - cs + 1 - h)\|\psi\|^2 + \|\phi\|\,\|\psi\| - \tilde{E}_0^x(t), \tag{2.73}$$

where

$$\tilde{E}_0(t) = \|\phi\|^2 + \|\psi\|^2 \ge 0, \quad \tilde{E}_0^x(t) = \|\phi_x\|^2 + \varepsilon\|\psi_x\|^2 \ge 0. \tag{2.74}$$

The first term in the right-hand side of equation (2.73) is associated with the quadratic form $\boldsymbol{J}^{\mathrm{T}}(t)\cdot\tilde{\boldsymbol{A}}(s)\cdot\boldsymbol{J}(t)$, where $\boldsymbol{J}(t) = (\|\phi\|, \|\psi\|)(t)$ and

$$\tilde{\boldsymbol{A}}(s) = \begin{pmatrix} (s^2 - cs + h) & 1/2 \\ 1/2 & (\varepsilon s^2 - cs + 1 - h)\end{pmatrix}. \tag{2.75}$$

Arguments similar to ones in the proof of lemma 2.4 can be used to show that the quadratic form $\boldsymbol{J}^{\mathrm{T}}(t)\cdot\tilde{\boldsymbol{A}}(\lambda+\delta)\cdot\boldsymbol{J}(t)$ is negatively definite provided $c > c^{\dagger}$

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and perturbations satisfy equation (2.5) with $s>\lambda$, where

$$c^{\dagger}=\min_{s\in(0,\infty)} c(s)=\frac{1}{2}\left[\frac{1}{\lambda}+(1+\varepsilon)\lambda+\sqrt{\left(\frac{1}{\lambda}+(1-\varepsilon)\lambda\right)^{2}+1-4(1+h-\varepsilon)(1-h)}\right], \tag{2.76}$$

and λ is the smallest solution of

$$c(s)=\frac{1}{2}\left[\frac{1}{s}+(1+\varepsilon)s+\sqrt{\left(\frac{1}{s}+(1-\varepsilon)s\right)^{2}+1-4(1+h-\varepsilon)(1-h)}\right]. \tag{2.77}$$

In this case,

$$\boldsymbol{J}^{\mathrm{T}}(t)\cdot\tilde{\boldsymbol{A}}(s)\cdot\boldsymbol{J}(t)\leq-\beta\tilde{E}_0(t), \tag{2.78}$$

for some $\beta>0$, and thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_0(t)\leq-\beta\tilde{E}_0(t)-\tilde{E}_0^{x}(t). \tag{2.79}$$

Computations similar to those of lemma 2.8 also give

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_0^{x}(t)\leq C_{21}(\tilde{E}_0(t)+\tilde{E}_0^{x}(t)). \tag{2.80}$$

Combining equations (2.79) and (2.80) as in lemma 2.8, we obtain $\tilde{E}_0(t)$, $\tilde{E}_0^{x}(t)\to 0$ as $t\to\infty$, and arguments identical to those in the proof of theorem 2.1 give equation (2.9).

3. Transient behaviour in the limit of weak coupling

In this section, we will discuss an interesting dynamics that the system (1.1) exhibits when $\varepsilon<1/2$ and h is sufficiently small. When $h=0$, the system (1.1) decouples. The second equation of equation (1.1) becomes the classic KPP equation and the first equation of the system (1.1) is driven by the second one. This situation is rather degenerate; however, it is worth some additional discussion. When $h=0$, the system (1.2) that describes travelling fronts takes the form

$$\left.\begin{aligned}U_0''+cU_0'+(1-V_0)V_0&=0\\ \text{and}\qquad \varepsilon V_0''+cV_0'+(1-V_0)V_0&=0,\end{aligned}\right\} \tag{3.1}$$

with the boundary conditions

$$(U_0,V_0)\to(1,1)\quad\text{as } x\to-\infty,\quad (U_0,V_0)\to(0,0)\quad\text{as } x\to\infty. \tag{3.2}$$

Linearizing the system (3.1) about the unstable equilibrium $(U_0,V_0)=(0,0)$ and substituting $(U_0,V_0)=(M,N)\exp(-\lambda x)$, we obtain

$$\left.\begin{aligned}(\lambda^2-c\lambda)M+N&=0\\ \text{and}\qquad (\varepsilon\lambda^2-c\lambda+1)N&=0.\end{aligned}\right\} \tag{3.3}$$

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In order to have a non-trivial solution of the second equation, we must have

$$c = \varepsilon\lambda + \frac{1}{\lambda}. \tag{3.4}$$

In order to have a non-trivial positive solution of the system one needs, in addition to equation (3.4), to satisfy

$$c \geq \lambda. \tag{3.5}$$

If $\varepsilon \geq 1/2$, the condition (3.5) is satisfied if equation (3.4) holds. However, for smaller ε, the condition (3.5) is restrictive. Elementary algebraic calculations can be used to conclude that the system (3.3) has a non-trivial positive solution for

$$c \geq c^* = \begin{cases} 2\sqrt{\varepsilon} & \text{if } \varepsilon \geq \frac{1}{2}, \\ \frac{1}{\sqrt{1-\varepsilon}} & \text{if } \varepsilon < \frac{1}{2}. \end{cases} \tag{3.6}$$

It is not hard to verify that the equilibrium point $(U_0, V_0) = (0, 0)$ is hyperbolic, and thus all solutions of the system (3.1) and (3.2), if they exist, approach this equilibrium at an exponential rate. Therefore, there exist no positive travelling wave solutions for the system (3.1)–(3.2) for $c < c^*$. Moreover, following steps of theorem 2.1 of Ghazaryan & Gordon (2008), one can show that for any $c \geq c^*$, the system (3.1) has a unique, up to translation, positive monotone travelling wave solution (front) $U_0(x)$, $V_0(x)$ asymptotically connecting the equilibrium $(U_0, V_0) = (1, 1)$ as $x \to -\infty$ to the equilibrium $(U_0, V_0) = (0, 0)$ as $\xi \to \infty$. The solution converges to both of the equilibria exponentially fast.

As noted above, when $h = 0$ the system (1.1) decouples and the second equation of the system (3.1) can be considered independently, that is

$$\varepsilon \hat{V}_0'' + c\hat{V}_0' + (1 - \hat{V}_0)\hat{V}_0 = 0, \tag{3.7}$$

with the boundary-like conditions

$$\hat{V}_0 \to 1 \quad \text{as } \xi \to -\infty, \qquad \hat{V}_0 \to 0 \quad \text{as } \xi \to \infty. \tag{3.8}$$

It is well known (Kolmogorov *et al.* 1937) that the problem (3.7)–(3.8) has a family of positive travelling wave solutions parameterized by the speed of propagation c. The smallest, critical, speed of propagation is $c^{**} = 2\sqrt{\varepsilon}$. It is the minimal value of c for which the expression (3.4), which comes from the formal linearization of equation (3.7) at $V_0 = 0$, has a real solution. All fronts converge to the equilibria exponentially fast. The rate of convergence of the critical front to the unstable equilibrium is $\lambda^{**} = 1/\sqrt{\varepsilon}$, whereas the rate of convergence to zero for supercritical fronts is defined as the smallest solution of equation (3.4). Therefore, when $\varepsilon < 1/2$, along with the family of travelling wave solutions for the system (3.1) with the speed $c \in [c^*, \infty) = [(1/\sqrt{1-\varepsilon}), \infty)$, there exists an interval of velocities $[c^{**}, c^*) = [2\sqrt{\varepsilon}, (1/\sqrt{1-\varepsilon}))$ for which there exist travelling wave solutions of the second equation of the system considered separately from the full system as it is. We call such solutions semi-fronts to emphasize this 'semi-existence' result.

Thus, in relation with existence of semi-fronts and fronts, for the system (3.1) with $\varepsilon < 1/2$, the following is true:

— There exists a critical semi-front that propagates with the speed $c^{**} = 2\sqrt{\varepsilon}$, with the rate of decay to the equilibrium at $V_0 = 0$ given by $\lambda^{**} = 1/\sqrt{\varepsilon}$.

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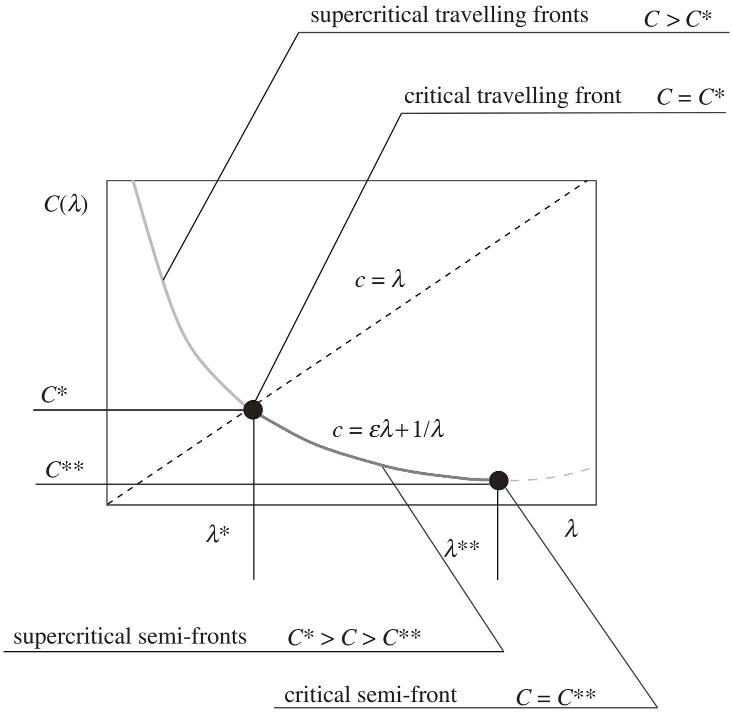

Figure 1. Propagation velocity of travelling fronts and travelling semi-fronts versus their rate of decay.

— There exist supercritical semi-fronts propagating with speeds $c \in (c^{**}, c^{*})$.
— There exists a critical front $c^* = 1/\sqrt{1-\varepsilon}$. with the rate of decay to the equilibrium at $(U_0, V_0) = (0, 0)$ given by $\lambda^* = 1/\sqrt{1-\varepsilon}$.
— There exist supercritical fronts propagating with speeds $c \in (c^*, \infty)$.

The situation described above is illustrated in figure 1.

Note that when ε is sufficiently small, the critical semi-front is very slow and decays to zero very fast, whereas the critical front propagates with the velocity of order unity which accidentally coincides with its rate of exponential decay. Thus, there is a transparent separation of scales in the problem. It is clear that for any $h > 0$, semi-fronts fail to persist; however, we expect that, when h is small, propagation of disturbances in the problem (1.1) with initial conditions which resemble a semi-front will have a scaling of the velocity of that semi-front for a time interval $[0, \tau(h))$ such that $\tau(h) \to \infty$ as $h \to 0$. In order to validate this conjecture, we performed a number of numerical experiments with the system (1.1) and initial data

$$u_0(x) = 0, \quad v_0(x) = \chi_{(-\infty,0]}(x), \tag{3.9}$$

where $\chi_{(-\infty,0]}(x)$ is characteristic function of $(-\infty, 0]$. In order to measure the characteristic velocity of propagation in the system (1.1), we use the bulk burning

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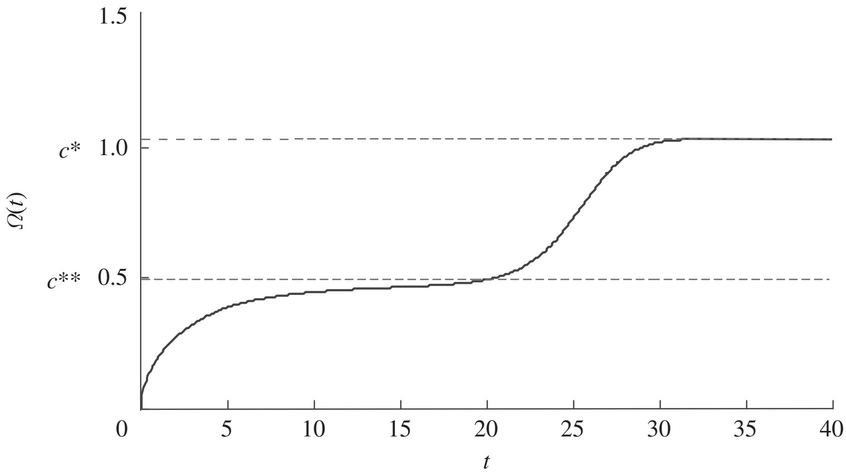

Figure 2. Bulk burning rate Ω evaluated on the solution of the problem (1.1) with initial conditions (3.9). Here, $\varepsilon = 0.06$ and $h = 10^{-5}$, c^* and c^{**} are the velocities of the critical front and the semi-front, respectively.

rate (Constantin *et al.* 2000) that is the average of the reaction rate,

$$\Omega(t) = \int (1 - v(t, \cdot))w(t, \cdot). \tag{3.10}$$

This quantity is fairly standard tool for measuring velocity of propagation of disturbances in reactive systems.

Numerical simulations of equations (1.1) and (3.9) showed that, as expected, the bulk burning rate $\Omega(t)$ initially approaches the velocity of critical semi-front c^{**} and after some time that scales as $\tau(h) \sim \log(1/h)$ jumps to the velocity of the critical front c^*. The typical dynamics of the bulk burning rate is shown in figure 2. This behaviour and scaling of $\tau(h)$ are observed to be very robust and rather insensitive to the initial data v_0 provided it is squeezed between the characteristic function of the interval and the critical semi-front.

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References

Billingham, J. & Needham, D. J. 1991 The development of travelling waves in quadratic and cubic autocatalysis with unequal diffusion rates. I. Permanent form travelling waves. *Phil. Trans. R. Soc. Lond. A* **334**, 1–24. (doi:10.1098/rsta.1991.0001)

Brailovsky, I., Goldshtein, V., Shreiber, I. & Sivashinsky G. 1997 On combustion waves driven by diffusion of pressure. *Combust. Sci. Tech.* **124**, 145–165. (doi:10.1080/00102209708935643)

Constantin, P., Kiselev, A., Oberman, A. & Ryzhik L. 2000 Bulk burning rate in passive-reactive diffusion. *Arch. Ration. Mech. Anal.* **154**, 53–91. (doi:10.1007/s002050000090)

Fisher, R. A. 1937 The advance of advantageous genes. *Ann. Eugenics* **7**, 355–369.

Focant, S. & Gallay, Th. 1998 Existence and stability of propagating fronts for an autocatalytic reaction–diffusion system. *Physica D* **120**, 346–368. (doi:10.1016/S0167-2789(98)00096-7)

Gardner, R., Jones C. K. R. T. & Kapitula T. 1993 Stability of travelling waves for nonconvex scalar viscous conservation laws. *Comm. Pure Appl. Math.* **XLVI**, 505–526.

Downloaded from rspa.royalsocietypublishing.org on January 9, 2014

Ghazaryan, A. & Gordon, P. 2008 KPP type flame fronts in porous media. *Nonlinearity* **21**, 973–992. (doi:10.1088/0951-7715/21/5/004)

Gordon, P. 2007 Recent mathematical results on combustion in hydraulically resistant porous media. *Math. Model. Nat. Phenom.* **2**, 56–76. (doi:10.1051/mmnp:2008019)

Henry, D. 1980 *Geometric theory of semilinear parabolic equations.* Berlin, New York: Springer.

Kapitula, T. 1994 On the stability of travelling waves in weighted L^∞ spaces. *J. Differ. Eq.* **112**, 179–215. (doi:10.1006/jdeq.1994.1100)

Kolmogorov, A. N., Petrovsky, I. G. & Piskunov, N. S. 1937 A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem. *Bull. Moscow Univ. Math. Mech.* **1**, 1–26.

Raugel, G. & Kirchgässner, K. 1996 Stability of fronts for a KPP-system: the noncritical case. In *Dynamics of nonlinear waves in dissipative systems: reduction, bifurcation and stability* (eds G. Dangelmayr, B. Fiedler, K. Kirchgässner & A. Mielke). Pitman Research Notes in Mathematics Series, no. 352, pp. 147–208. Harlow, UK: Addison Wesley Longman Ltd.

Raugel, G. & Kirchgässner, K. 1998 Stability of fronts for a KPP-system, II: the critical case, *J. Differ. Eq.* **146**, 399–456. (doi:10.1006/jdeq.1997.3391)

Sattinger, D. H. 1976 On the stability of waves of nonlinear parabolic systems. *Adv. Math.* **22**, 312–355. (doi:10.1016/0001-8708(76)90098-0)

Sattinger, D. H. 1977 Weighted norms for the stability of travelling waves. *J. Differ. Eq.* **25**, 130–144. (doi:10.1016/0022-0396(77)90185-1)