SPECTRAL STABILITY OF COMBUSTION FRONTS: TRANSITION FROM HIGH TO INFINITE LEWIS NUMBER

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We use a geometric approach that involves construction of the Stability Index Bundles to relate the spectral stability of the wavefronts with high Lewis numbers to the spectral stability of wavefront with the infinite Lewis number.

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We consider a combustion model for which the evolution of the temperature u and concentration of the fuel y is described by

$$\begin{aligned} u_t &= u_{xx} + y\Omega(u), \qquad t > 0, \;\; x \in \mathbb{R}, \\ y_t &= \varepsilon y_{xx} - \beta y \Omega(u), \end{aligned} \tag{1}$$

where $\Omega(u) = e^{-1/u}$ for $u > 0$ and $\Omega(u) = 0$ otherwise, $\beta > 0$ is the exothermicity and ε is the reciprocal of the Lewis number. The physical prototype of Eq. (1) with $\varepsilon = 0$ is the combustion of solid fuels. Equation (1) with $0 < \varepsilon \ll 1$ describes burning of high density fluids at high temperatures. Moreover, during burning of solid fuels some liquefaction of the fuel might occur in the reaction zone, thus causing a non-zero value of ε.

System (1) supports traveling waves that asymptotically connect the completely burned state $(u, y) = (1/\beta, 0)$ at $-\infty$, and the unburned state $(u, y) = (0, 1)$ at ∞, and approach these rest states at exponential rates. The existence and uniqueness (up to translation) of such fronts is known from Refs. 1, 2 for $\varepsilon = 0$ and from Refs. 3, 4, 5 for $0 < \varepsilon < 1$.

We analytically investigate the relationship between the $\varepsilon = 0$ and $\varepsilon > 0$ cases and show that, for most values of β, combustion fronts with sufficiently small $\varepsilon > 0$ inherit stability properties of the front with $\varepsilon = 0$.

We denote the front by $(u_f(\xi), y_f(\xi))$, where $\xi = x - c_f t$ and $c_f > 0$ is its speed: (u_f, y_f, c_f) is (u_0, y_0, c_0) when $\varepsilon = 0$, and $(u_\varepsilon, y_\varepsilon, c_\varepsilon)$ when $\varepsilon > 0$. The eigenvalue problem for the linearization of (1) about (u_f, y_f) reads

$$\begin{aligned} \lambda p &= p_{\xi\xi} + c_f p_\xi + \Omega(u_f) r + y_f \Omega_u(u_f) p, \\ \lambda r &= \varepsilon r_{\xi\xi} + c_f r_\xi - \beta\Omega(u_f) r - \beta y_f \Omega_u(u_f) p. \end{aligned} \tag{2}$$

To discuss the point spectrum and its robustness under perturbations, we use the concept of the Evans function as described in Ref. 6. Evans function is an analytic function of $\lambda \in \mathbb{C}$ defined to the right of the essential spectrum. Its zeroes coincide with the eigenvalues with the order of a zero being equal to the multiplicity of the eigenvalue.

The essential spectrum σ_e of the front with any $\varepsilon \geq 0$ reaches the imaginary axis. The translational eigenvalue at the origin is embedded in σ_e. To include the origin in the consideration we use a generalization of the approach developed in Ref. 7 and analytically continue the Evans function across the rightmost boundary $\{\lambda = -\varepsilon\nu^2 + c_f i\nu;\ \nu \in \mathbb{R}\}$ of σ_e. Eigenvalues that are embedded in σ_e are still zeroes of the extended Evans function, but not necessarily vice versa. We denote the extended Evans function E_0 when $\varepsilon = 0$, and E_ε when $\varepsilon > 0$. For E_0 we distinguish three situations:

(a) There is at least one λ such that $E_0(\lambda) = 0$ and $\mathrm{Re}\,\lambda > 0$.

(b) The only λ with $\mathrm{Re}\,\lambda \geq 0$ such that $E_0(\lambda) = 0$ is $\lambda = 0$.

(c) Neither (b) — there exist $\lambda \neq 0$, $\mathrm{Re}\,\lambda = 0$, $E(\lambda) = 0$, nor (a).

Cases (a) and (b) are robust under perturbations with sufficiently small ε:

Theorem 1.1. *If the front (u_0, y_0) is spectrally unstable ((a) holds) then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the front $(u_\varepsilon, y_\varepsilon)$ is unstable.*

If the front (u_0, y_0) is spectrally stable and (b) holds then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the front $(u_\varepsilon, y_\varepsilon)$ is spectrally stable.

Theorem 1.1 is a consequence of the following key proposition.

Proposition 1.1. *Assume that λ_0 is a zero of order m of E_0. There exists $\varepsilon_0 > 0$, independent of λ_0, such that for any $0 < \varepsilon < \varepsilon_0$ there is an ε–neighborhood of λ_0 with exactly m zeroes (counting multiplicity) of E_ε.*

To prove Prop. 1.1 we apply the Stability Index technique developed in Ref. 6. Stability Index is a topological invariant which counts the eigenvalues inside a given closed contour K that does not go through any of the zeros of the extended Evans function. More precisely, it is the first Chern number of the Stability Index Bundle (SIB). SIB is a bundle with fibers formed by certain invariant manifolds in the phase space of Eq. (2) and the base given by a compactification of an infinite cylinder $(\xi, \lambda) \in \mathbb{R} \times K$ capped at $\pm\infty$ by $\{K \cup \text{Interior}\, K\}$. Construction of SIB is performed in the framework of exterior powers $\Lambda^k(\mathbb{C}^4)$. The first Chern number of SIB coincides with the winding number of Evans function over K and therefore counts eigenvalues inside K. If K encloses all of the unstable eigenvalues then the spectral stability of the wave can be concluded from the stability index of K.

Because of the slow-fast structure of Eq. (2), generally speaking, SIB is the Whitney sum of the associated slow and fast subbundles. Within the slow subbundle the perturbation is not singular and the Stability Index is preserved under the limit $\varepsilon \to 0$. In our case, it follows from the construction that the fibers of SIB are determined only by the slow dynamics of the flow, *i.e.* the slow bundle is, in fact, the full bundle. Therefore the first Chern numbers for the full ($\varepsilon > 0$) and reduced ($\varepsilon = 0$) systems coincide.

Using Prop. 1.1 and the estimates[2] on the moduli of the unstable eigenvalues we build a contour K that encloses the unstable eigenvalues of (u_0, y_0) and $(u_\varepsilon, y_\varepsilon)$ and does not go through any of zeroes of either E_0 or E_ε. Theorem 1.1 follows if we take into account that eigenvalue $\lambda = 0$ is simple when $\varepsilon = 0$,[8] and by of Prop. 1.1 persists as such for small enough ε. Therefore no eigenvalue can cross into the right half-plane through the origin when a non-zero ε is introduced in the problem.

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