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On the convective nature of the instability of a front undergoing a supercritical Turing bifurcation

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Abstract

Fronts are traveling waves in spatially extended systems that connect two different spatially homogeneous rest states. If the rest state behind the front undergoes a supercritical Turing instability, then the front will also destabilize. On the linear level, however, the front will be only convectively unstable since perturbations will be pushed away from the front as it propagates. In other words, perturbations may grow but they can do so only behind the front. It is of interest to show that this behavior carries over to the full nonlinear system. It has been successfully done in a case study by Ghazaryan and Sandstede [A. Ghazaryan, B. Sandstede, Nonlinear convective instability of Turing-unstable fronts near onset: a case study, SIAM J. Appl. Dyn. Syst. 6 (2007) 319–347]. In the present paper, analogous results are obtained for the same system as in Ghazaryan and Sandstede (2007), but for a different parameter regime.

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1. Introduction

A specific feature of the dynamics of partial differential equations on unbounded domains is that instability mechanisms involve propagation of perturbations as well as their growth. If an initial perturbation does not decay in time in a translation invariant norm but is transported to infinity and eventually dies out at every fixed point in the space then the instability is called convective, as opposed to absolute.

This study concerns fronts—traveling waves that asymptotically connect two different spatially homogeneous rest states. By stability of a front we understand its orbital stability, when each solution close to the front is attracted to some fixed translate of it. A front becomes unstable if a subset of the spectrum of the linearization of the system about the front crosses the imaginary axis. Instability mechanisms for fronts in reaction-diffusion systems are classified according to the type of the spectrum on the imaginary axis. Situations when the essential spectrum crosses the imaginary axis cannot be analyzed using the center manifold theorem or Lyapunov–Schmidt method. The essential spectrum is defined by the two asymptotic rest states of the front. If the rest state ahead of the front triggers the instability, then there exists [11] a continuum of modulated fronts (time-periodic in some moving coordinate frame solutions) which connect the rest state behind the front with small spatially periodic patterns ahead of the front.

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A completely different situation arises when the rest state behind the front destabilizes. Modulated fronts which leave a spatially periodic pattern behind do not exist (see Ref. [11] for the proof, and Ref. [13] for formal results). The situations when the bifurcation at the rest state behind the front is supercritical are of a particular interest. Typically, the small amplitude spatially periodic pattern is outrun by the front.

In the co-moving frame, on the linear level, the front is only convectively unstable since perturbations are pushed away from the interface of the front. On the nonlinear level the convective nature of the instability has been observed numerically in Refs. [11] and [6]. It is expected to be a general phenomenon [6]. Nevertheless, beyond the case study [6], the author does not know about rigorous analytic results capturing such behavior.

We consider the model proposed in Ref. [6]. It consists of the Chafee–Infante equation coupled to the Swift–Hohenberg equation, and is similar to one constructed in Ref. [5] to yield modulated fronts. The system in our case produces a front with wanted properties. More precisely, the system reads

$$\begin{aligned}\partial_t u_1 &= \partial_x^2 u_1 + \frac{1}{2}(u_1 - c)(1 - u_1^2) + \gamma_1 u_2^2,\\ \partial_t u_2 &= -(1 + \partial_x^2)^2 u_2 + \alpha u_2 - u_2^3 - \gamma_2 u_2(1 + u_1),\end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}$, $t \geq 0$, and $U = (u_1, u_2)$. The parameters $\gamma_1 \in \mathbb{R}$, $\gamma_2 > 0$ and $c \in (0, 1)$ are fixed, while the parameter α is a bifurcation parameter which varies near zero. For every α, the system (1.1) admits the front $U_h(x - ct) = (h(x - ct), 0)$, where $h(x) = \tanh(x/2)$, which connects the rest state $U_- = (-1, 0)$ at $x = -\infty$ with the rest state $U_+ = (1, 0)$ at $x = \infty$. In the coordinate frame $\xi = x - ct$, when $\alpha > 0$, the rest state ahead of the front is asymptotically stable, but the rest state behind the front destabilizes, thus generating spatially periodic (Turing) patterns.

In addition to a zero eigenvalue due to the translational invariance, the essential spectrum of the linearization of the system about the front crosses the imaginary axis. Exponential weights can be used [2–4,12], to handle the essential spectrum. The spectrum of the front U_h can actually be moved [6] into the left half-plane in the co-moving frame $\xi = x - ct$, provided it is computed in a weighted space $\mathrm{L}^2_\beta = \{u | \rho_\beta u \in \mathrm{L}^2(\mathbb{R})\}$ with norm $||\rho_\beta(\xi)U(\xi)||$, where, for some appropriate $\beta > 0$,

$$\rho_\beta(x) := \mathrm{e}^{\beta x}, \quad \text{for}\, x \leq -1, \qquad \rho_\beta(x) := 1, \quad \text{for}\, x \geq 1, \text{ and} \qquad \rho'_\beta(x) \geq 0, \qquad \text{for all } x. \tag{1.2}$$

In the weighted space L^2_β the front is linearly stable. Thus, in the coordinate frame which moves with the front, and on the linear level, the front pushes perturbations toward $\xi = -\infty$.

Nonlinear stability of the front U_h cannot be concluded [10] from the linear stability because the nonlinearity does not map the weighted spaces into themselves. Indeed, if we define $W = \mathrm{e}^{\beta x} V$ and use $W = (w_1, w_2)$ as the new dependent variable, then in the equation for W the nonlinear term $u_1^n (n > 1)$ becomes $\mathrm{e}^{\beta x}[\mathrm{e}^{-\beta x} w_1]^n = \mathrm{e}^{(1-n)\beta x} w_1^n$ which is unbounded as $x \to -\infty$. To overcome this difficulty, a method introduced originally in Ref. [10] in the Hamiltonian context has been used in Ref. [6]. If a priori estimates are available for the solution in the space without weight, for instance in C^0, which show that it stays sufficiently small, then the nonlinear terms in the weighted norm can be controlled as linear in the weighted variable: $e^{\beta x} u_j^n = u_j^{n-1} w_j$. The interplay of the spatially uniform norm and the exponentially weighted norm is the key for the proof of nonlinear stability of the front. An example of a successful application of this technique in a different situation can be found in Ref. [1]. In Ref. [6], suitable a priori estimates has been obtained in the case of $\gamma_1 \geq 0$ and it has been shown how these estimates can be used to prove that the front is asymptotically stable in the orbital sense, when considered in a carefully chosen weighted space and in the co-moving frame. In other words, in the co-moving frame the front is convectively unstable (Fig. 1).

Fig. 1. In the co-moving frame, the Turing patterns are pushed to the left from the interface of the front.

Nevertheless, the bifurcation of the periodic patterns at the rest state behind the front is supercritical not only for $\gamma_1 \geq 0$, but also for any γ_1, γ_2 such that

$$\gamma_1\gamma_2 > -\frac{3(1+c)(5+c)}{11+3c}, \quad \gamma_2 > 0. \tag{1.3}$$

In this paper, we present a proof of the convective nature of the nonlinear stability for the case when $\gamma_2 - \sqrt{2} < \gamma_1 < 0$. We refer to [6] for statements that are valid for both γ_1-regimes, and present only proofs that are essentially different from $\gamma_1 \geq 0$ case. To be more specific, the value of γ_1 strongly influences the way the a priori estimates are obtained. It is important to mention that the proofs of a priori estimates in both cases $\gamma_1 \geq 0$ and $\gamma_2 - \sqrt{2} < \gamma_1 < 0$ are also based on the interplay of the spatially uniform norm and the exponentially weighted norm. According to the author's knowledge, this constitutes a new approach potentially useful for larger classes of problems.

2. Nonlinear convective instability

2.1. Turing patterns

We recall known from [6] facts about the stability of the rest state behind the front. Spatially periodic equilibria bifurcate at $\alpha = 0$ from the rest state U_-. If γ_1 and γ_2 satisfy (1.3), then Eq. (1.1) has spatially periodic equilibria U_{per} for $\alpha > 0$ sufficiently close to zero which are given by

$$U_{\text{per}}(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sqrt{\frac{\alpha}{a_0}} \begin{pmatrix} 0 \\ \cos x \end{pmatrix} + \mathrm{O}(\alpha), \quad a_0 = \frac{3}{4} + \frac{\gamma_1\gamma_2}{2}\left(\frac{1}{1+c} + \frac{1}{2(5+c)}\right). \tag{2.1}$$

The bifurcation is supercritical if (1.3) holds. The patterns (2.1) are nonlinearly stable with respect to perturbations in the space $H^2(2) = \{U \in L^2 : ||U||_{H^2(2)}\} = \left(\sum_{j=0}^{2} \int_{\mathbb{R}} |\partial_x^j U(x)|^2 (1+x^2)^2 \,\mathrm{d}x\right)^{(1/2)} < \infty$:

Theorem 1. [6, Theorem 1] *Assume that $\gamma_2 > 0$ and $c \in (0, 1)$ are fixed and that (1.3) is met. For each $\alpha > 0$ sufficiently small, there are positive numbers K and δ such that, for every $V_0 \in H^2(2)$ with $||V_0||_{H^2(2)} \leq \delta$,* Eq. (1.1) *with initial data $U_{\text{per}} + V_0$ has a unique global solution $U(t) = U_{\text{per}} + V(t)$, and $||V(t)||_{C^0} \leq K(1+t)^{-1/2}$ for $t \geq 0$.*

2.2. Spectrum of the front

Upon transforming (1.1) into the co-moving coordinate $\xi = x - ct$, we obtain the system

$$\begin{aligned} \partial_t u_1 &= \partial_\xi^2 u_1 + c\partial_\xi u_1 + \frac{1}{2}(u_1 - c)(1 - u_1^2) + \gamma_1 u_2^2, \\ \partial_t u_2 &= -(1+\partial_\xi^2)^2 u_2 + c\partial_\xi u_2 + \alpha u_2 - u_2^3 - \gamma_2 u_2(1+u_1). \end{aligned} \tag{2.2}$$

The linearization of (2.2) about a stationary solution of the form $U_* = (u_*, 0)$ is given by the operator

$$\mathcal{L}_0[U_*] := \begin{pmatrix} \partial_\xi^2 + c\partial_\xi + \frac{1}{2}(1 + 2cu_* - 3u_*^2) & 0 \\ 0 & -[1+\partial_\xi^2]^2 + c\partial_\xi + \alpha - \gamma_2(1+u_*) \end{pmatrix}.$$

The operator $\mathcal{L}_0[U_*]$ is sectorial on $\mathcal{X}_0 := H^1_{\text{ul}} \times H^1_{\text{ul}}$ with dense domain $H^3_{\text{ul}} \times H^5_{\text{ul}}$.

We set $W(\xi) := \rho_\beta(\xi)V(\xi)$, where $\rho_\beta(\xi)$ is as defined in (1.2), so that W satisfies $W_t = \mathcal{L}_\beta[U_*]W$, where $\mathcal{L}_\beta[U_*] = \rho_\beta\mathcal{L}_0[U_*]\rho_\beta^{-1}$, whenever V satisfies $V_t = \mathcal{L}_0[U_*]V$. Operator $\mathcal{L}_\beta[U_*]$ is again sectorial on $\mathcal{X}_0$.

We denote

$$\mathcal{L}_\beta[U_h] := \mathcal{L}_\beta, \quad \mathcal{L}_0[U_h] := \mathcal{L}_0. \tag{2.3}$$

The following proposition describes the location of the spectra of $\mathcal{L}_\beta$ and $\mathcal{L}_0$ (Fig. 2).

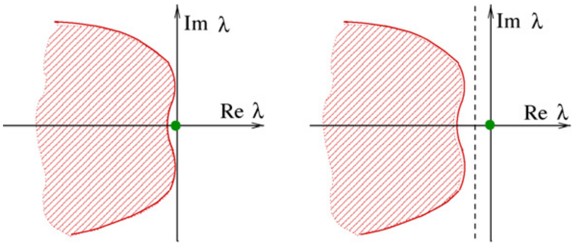

Fig. 2. A schematic illustration of the spectrum in the case $\alpha = 0$. Weight ρ_β opens a gap between the essential spectrum and the imaginary axis by pushing the essential spectrum to the left.

Proposition 2. [6, Proposition 2.2] *Given $\gamma_2 > 0$ and $c \in (0, 1)$, there are positive numbers α_0 and β_0 and a strictly positive function $\Lambda_0(\beta)$ defined for $0 < \beta < \beta_0$ so that the following holds for $|\alpha| \leq \alpha_0$: The spectrum of $\mathcal{L}_\beta$ satisfies*

$$\mathrm{spec}(\mathcal{L}_\beta) = \{0\} \cup \Sigma \quad \text{with} \quad \mathrm{Re}\Sigma \leq -\Lambda_0(\beta),$$

and $\lambda = 0$ is simple eigenvalue of $\mathcal{L}_\beta$ and $\mathcal{L}_0$. Furthermore, the spectrum of $\mathcal{L}_0$ satisfies

(i) *if $\alpha < 0$ then $\mathrm{spec}(\mathcal{L}_0) = \{0\} \cup \Sigma$ with $\mathrm{Re}\Sigma < 0$;*
(ii) *if $c\,\alpha = 0$ then $\mathrm{spec}(\mathcal{L}_0) \cap \{\lambda : \mathrm{Re}\lambda \geq 0\} = \{0\} \cup \{\pm i\}$;*
(iii) *if $\alpha > 0$ then $\mathrm{spec}(\mathcal{L}_0) \cap \{\lambda : \mathrm{Re}\lambda > 0\} \neq \varnothing$.*

According to Proposition 2 the front U_h is orbitally stable for $\alpha < 0$ due to [7, §5.1], while it is spectrally unstable for $\alpha > 0$, since part of its essential spectrum lies in the open right half-plane.

2.3. Formulation of the results

The results on nonlinear convective instability of the front U_h in [6]($\gamma_1 > 0$) and here ($\gamma_1 < 0$) are formulated in the spaces $H^1_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^2)$ of uniformly local functions (see [9, §3.1]). These spaces contain in particular all differentiable bounded functions such as fronts or periodic solutions. To define $H^1_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^2)$, pick any positive and bounded function $\sigma \in C^2(\mathbb{R})$ for which $\int_{\mathbb{R}} \sigma(x)\,\mathrm{d}x = 1$ and $|\sigma'(x)|, |\sigma''(x)| \leq \sigma(x)$ for $x \in \mathbb{R}$. For instance, set $\sigma(x) = \frac{1}{\pi}\mathrm{sech}x$. For each $0 < b < 1$, denote $\sigma_b(x) := \sigma(bx)$ and record that $\int_{\mathbb{R}} \sigma_b(x)\,\mathrm{d}x = 1/b$.

The Banach space L^2_{ul} of uniformly local weighted L^2 functions then is defined as

$$L^2_{\mathrm{ul}}(\mathbb{R}) = \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}) : \ ||u||^2_{L^2_{\mathrm{ul}}} := \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} \sigma(x + y)|u(x)|^2\,\mathrm{d}x < \infty \ \text{ and } \ ||T_y u - u||_{L^2_{\mathrm{ul}}} \to 0 \ \text{ as } \ y \to 0 \right\},$$

where $[T_y u](x) := u(x + y)$ is the translation operator. The associated Sobolev spaces are denoted $H^k_{\mathrm{ul}}(\mathbb{R})$. Different choices for σ result in the same spaces with equivalent norms. Properties of these spaces are described in the following lemma:

Lemma 3. [9, Lemma 3.1 and 3.8] *There is a constant K_0 with the following properties*:

(i) *H^1_{ul} is an algebra and embeds continuously into C^0_{unif} with $||u||^2_{C^0} \leq K_0 ||u||_{L^2_{\mathrm{ul}}} ||u||_{H^1_{\mathrm{ul}}}$ for all $u \in H^1_{\mathrm{ul}}$.*
(ii) *For each $0 < b < 1$, let $\sigma_b(x) := \sigma(bx)$, then $||u||^2_{L^2_{\mathrm{ul}}(\sigma)} \leq K_0(1 + b)||u||^2_{L^2_{\mathrm{ul}}(\sigma_b)}$ for all $u \in L^2_{\mathrm{ul}}(\sigma)$.*
(iii) *We have $-\int_{\mathbb{R}} \sigma_b u(1 + \partial_x^2)^2 u\,\mathrm{d}x \leq (7b^2/2)\int_{\mathbb{R}} \sigma_b u^2\,\mathrm{d}x$ for all $u \in H^4_{\mathrm{ul}}$.*

The following theorem shows that perturbations to the front stay bounded in the H^1_{ul} and thus C^0 norm and decay exponentially to zero as $t \to \infty$ when they are multiplied by $\mathrm{e}^{\beta_*(x-ct)}$ for some appropriate $\beta_* > 0$, so that the front is nonlinearly stable in this norm for all values of α near zero. The proof of similar results for the case of $\gamma_1 > 0$ is available in [6, Theorem 1].

Theorem 4. *Assume that $c \in (0, 1)$, $\gamma_2 > 0$, $\gamma_2 - \sqrt{2} < \gamma_1 < 0$, and the condition (1.3) is met. There are then positive constants α_*, β_*, K and Λ_* so that the following is true for all α with $0 < \alpha < \alpha_*$: For every function $V_0 = (v_1^0, v_2^0)$ with*

$$||V^0||_{H^1_{\rm ul}} \leq \alpha, \quad ||\rho_{\beta_*} V_0||_{H^1_{\rm ul}} \leq \alpha, \tag{2.4}$$

Eq. (1.1) *with initial data $U_0 = U_h + V_0$ has a unique global solution $U(t)$, which can be expressed as*

$$U(x, t) = U_h(x - ct - q(t)) + V(x, t)$$

for an appropriate real-valued function $q(t)$, and there is a $q_* \in \mathbb{R}$ so that for $t \geq 0$

$$||V(\cdot, t)||_{H^1_{\rm ul}} \leq K\alpha^{1/4}, \quad |q(t)| \leq K\alpha, \quad ||\rho_{\beta_*}(\cdot - ct)V(\cdot, t)||_{H^1_{\rm ul}} + |q(t) - q_*| \leq K\mathrm{e}^{-\Lambda_* t}.$$

The proof of Theorem 4 consists of two parts. In the first part, under the assumption of suitable a priori estimates the nonlinear stability of the front is shown in an appropriate exponentially weighted norm imposed in the co-moving frame. This part does not depend on the value of γ_1, unlike the second part where a boot-strapping argument is used to establish these a priori estimates.

2.4. A priori estimates imply nonlinear stability

We here will briefly recall the independent of the sign of γ_1 arguments from [6]. To capture orbital stability, a time-dependent spatial shift function $q(t)$ is introduced in the argument of the front U_h and solutions to (1.1) are written as

$$U(\xi, t) = \begin{pmatrix} u_1(\xi, t) \\ u_2(\xi, t) \end{pmatrix} = \begin{pmatrix} h(\xi - q(t)) \\ 0 \end{pmatrix} + \begin{pmatrix} v_1(\xi, t) \\ v_2(\xi, t) \end{pmatrix}, \quad \text{where } h(\xi) = \tanh \frac{\xi}{2}. \tag{2.5}$$

On the account of the translational invariance, we may assume that $q(0) = 0$. The decomposition (2.5) is unique if the perturbation $V = (v_1, v_2)$ to the front is perpendicular to the one-dimensional subspace spanned by the derivative of the front. More precisely, $V = (v_1, v_2)$ satisfies the system

$$\begin{aligned} \partial_t v_1 &= \partial_\xi^2 v_1 + c\partial_\xi v_1 + \frac{1}{2}[1 - 3h^2(\xi - q(t)) + 2ch(\xi - q(t))]v_1 + \frac{1}{2}[c - 3h(\xi - q(t))]v_1^2 \\ &\quad -\frac{1}{2}v_1^3 + \dot{q}(t)h_\xi(\xi - q(t)) + \gamma_1 v_2^2, \\ \partial_t v_2 &= -(1 + \partial_\xi^2)^2 v_2 + c\partial_\xi v_2 + \alpha v_2 - v_2^3 - \gamma_2(1 + h(\xi - q(t)))v_2 - \gamma_2 v_1 v_2 \end{aligned} \tag{2.6}$$

with initial data $v_1(\xi, 0) = v_1^0(\xi)$, $v_2(\xi, 0) = v_2^0(\xi)$ and $q(0) = 0$. In the notation

$$\mathcal{A} = \begin{pmatrix} \partial_\xi^2 + c\partial_\xi & 0 \\ 0 & -(1 + \partial_\xi^2)^2 + c\partial_\xi + \alpha \end{pmatrix},$$

$$\mathcal{R}(\xi) = \begin{pmatrix} \mathcal{R}_1 & 0 \\ 0 & \mathcal{R}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}[1 - 3h^2(\xi) + 2ch(\xi)] & 0 \\ 0 & -\gamma_2(1 + h(\xi)) \end{pmatrix},$$

$$\mathcal{N}(V) = \begin{pmatrix} \mathcal{N}_1(V) \\ \mathcal{N}_2(V) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}[c - 3h(\xi - q(t))]v_1(\xi, t) - \frac{1}{2}v_1^2(\xi, t) & \gamma_1 v_2(\xi, t) \\ 0 & -v_2^2(\xi, t) - \gamma_2 v_1(\xi, t) \end{pmatrix},$$

system (2.6) becomes

$$\partial_t V = \mathcal{A}V + \mathcal{R}(\xi - q(t))V + \mathcal{N}(V)V + \dot{q}(t)h_\xi(\xi - q(t))e_1, \quad e_1 = (1, 0). \tag{2.7}$$

The weighted solution $W = (w_1, w_2)$, $W(\xi, t) = \rho_\beta(\xi)V(\xi, t)$, with ρ_β as in (1.2), then satisfies the system

$$\partial_t W = \mathcal{L}_\beta W + [\mathcal{R}(\xi - q(t)) - \mathcal{R}(\xi)]W + \mathcal{N}(V)W + \dot{q}(t)h_\xi(\xi - q(t))\rho_\beta(\xi)e_1 \tag{2.8}$$

with $\mathcal{L}_\beta = \rho_\beta \mathcal{A}\rho_\beta^{-1} + \mathcal{R}(\xi)$ as in (2.3), whenever $V(\xi, t)$ satisfies (2.7).

If we fix β with $0 < \beta < \beta_0$ as in Proposition 2, then $\lambda = 0$ is a simple isolated eigenvalue of $\mathcal{L}_\beta$ with eigenfunction $\rho_\beta(\xi)\partial_\xi U_h$ and the rest of the spectrum has real part less than Λ_0. In the following lemma $\mathcal{P}^c_\beta : H^1_{\mathrm{ul}} \times H^1_{\mathrm{ul}} \to H^1_{\mathrm{ul}} \times H^1_{\mathrm{ul}}$ is the spectral projection onto the one-dimensional eigenspace of $\mathcal{L}_\beta$ corresponding to the zero eigenvalue and $\mathcal{P}^s_\beta = 1 - \mathcal{P}^c_\beta$ is the complementary projection onto the stable eigenspace.

Lemma 5. [6, Lemma 3.1] *For $0 < \beta < \beta_0$, there are constants $K_0 > 0$ and $\alpha_0 > 0$ such that the following is true for any α with $|\alpha| < \alpha_0$. The spectral projection $\mathcal{P}^c_\beta$ is given by*

$$\mathcal{P}^c_\beta W = \begin{pmatrix} P^c_\beta & 0 \\ 0 & 0 \end{pmatrix} W = \langle \psi^c_1, W_1 \rangle_{L^2} \rho_\beta \partial_\xi U_h,$$

where $\psi^c_1(\xi) = \mathrm{e}^{c\xi}\rho_\beta(\xi)h_\xi(\xi)\left[\int_{\mathbb{R}} \mathrm{e}^{c\zeta}h_\xi(\zeta)^2\,\mathrm{d}\zeta\right]^{-1}$, and, with Λ_0 as in Proposition 2, we have

$$||\mathrm{e}^{\mathcal{P}^s_\beta \mathcal{L}_\beta t}||_{H^1_{\mathrm{ul}}} \le K_0 \mathrm{e}^{-\Lambda_0 t}, \quad t \ge 0.$$

Condition $\mathcal{P}^c_\beta W(t) = 0$, for all t for which the decomposition (2.5) exists, defines $q(t)$ in a unique way. Applying $\mathcal{P}^c_\beta$ and $\mathcal{P}^s_\beta$ to (2.8), one obtains the evolution system for $V = (v_1, v_2)$, $W = (w_1, w_2)$ and —it q

$$\partial_t V = \mathcal{A}V + \mathcal{R}(\xi - q(t))V + \mathcal{N}(V)V + \dot{q}(t)h_\xi(\xi - q(t))e_1, \tag{2.9}$$

$$\partial_t W = \mathcal{P}^s_\beta \mathcal{L}_\beta W + \mathcal{P}^s_\beta \left([\mathcal{R}(\xi - q(t)) - \mathcal{R}(\xi)]W + \mathcal{N}(V)W + \dot{q}(t)h_\xi(\xi - q(t))\rho_\beta(\xi)e_1\right), \tag{2.10}$$

$$\dot{q}(t) = -\frac{\langle \psi^c_1, [\mathcal{R}_1(\xi - q(t)) - \mathcal{R}_1(\xi)]W_1 + \mathcal{N}_1(V)W \rangle_{L^2}}{\langle \psi^c_1, h_\xi(\xi - q(t))\rho_\beta(\xi) \rangle_{L^2}} \tag{2.11}$$

The linear parts of the right-hand sides in (2.9) and (2.10) are sectorial operators on $H^1_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^2)$ with dense domain $H^3_{\mathrm{ul}} \times H^5_{\mathrm{ul}}$. The nonlinearity is smooth from $\mathcal{Y} := H^1_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^4) \times \mathbb{R}$ into itself, and there is a constant K_1 such that $||\mathcal{R}_1(\cdot - q) - \mathcal{R}_1(\cdot)||_{H^1_{\mathrm{ul}}} + ||\mathcal{N}(V)||_{H^1_{\mathrm{ul}}} \le K_1(|q| + ||V||_{H^1_{\mathrm{ul}}})$. Therefore

$$|\dot{q}| \le K_1(|q| + ||V||_{H^1_{\mathrm{ul}}})||W||_{H^1_{\mathrm{ul}}}$$

for all $(V, W, q) \in \mathcal{Y}$ with norm less than one, say. The methods developed in [7] for sectorial operators imply local existence and uniqueness of solutions for initial data in $\mathcal{Y}$ as well as continuous dependence on initial conditions, thus proving local existence and uniqueness of the decomposition (2.5). For each given $0 < \eta_0 \le 1$, then there exists a $\delta_0 > 0$ and a time $T > 0$ such that (2.5) exists for $0 \le t < T$ with

$$|q(t)| + ||V(t)||_{H^1_{\mathrm{ul}}} \le \eta_0 \tag{2.12}$$

provided $||V(0)||_{H^1_{\mathrm{ul}}} \le \delta_0$. Let $T_{\max} = T_{\max}(\eta_0)$ be the maximal time for which (2.12) holds.

Lemma 6. [6, Lemma 3.1] *Pick Λ with $0 < \Lambda < \Lambda_0$ and $\hat{\eta}_0 > 0$ so that*

$$\frac{2K_0K_1(1 + K_0)}{\Lambda_0 - \Lambda}\hat{\eta}_0 < 1, \tag{2.13}$$

then there are positive constants K_2, K_3 and K_4 that are independent of α such that for any $0 < \eta_0 \le \hat{\eta}_0$

$$||W(t)||_{H^1_{\mathrm{ul}}} \le K_2 \mathrm{e}^{-\Lambda t}||W(0)||_{H^1_{\mathrm{ul}}}, \quad |q(t)| \le K_3||W(0)||_{H^1_{\mathrm{ul}}}, \quad |\dot{q}| \le K_4 \mathrm{e}^{-\Lambda t}||W(0)||_{H^1_{\mathrm{ul}}}, \tag{2.14}$$

for all $0 \le t < T_{\max}(\eta_0)$ and any solution that satisfies (2.12). If $T_{\max}(\eta_0) = \infty$, then there is a $q_* \in \mathbb{R}$ with

$$|q(t) - q_*| \le \frac{K_1K_2}{\Lambda}\mathrm{e}^{-\Lambda t}||W(0)||_{H^1_{\mathrm{ul}}} \text{ for } t \ge 0.$$

To complete the proof of Theorem 4, it suffices to prove that, for sufficiently small $0 < \eta_0 \leq 1$, there exists a $\delta_0 > 0$ such that (2.12) holds for all $t \geq 0$ provided $||V(0)||_{H^1_{\rm ul}} \leq \delta_0$.

2.5. Establishing the necessary a priori estimates

Throughout this section, we consider initial data $q(0) = 0$ and $V(0) \in H^1_{\rm ul}$ for which $W(0) = \rho_\beta V(0) \in H^1_{\rm ul}$. We also assume that $\gamma_2 - \sqrt{2} < \gamma_1 < 0, \quad \gamma_2 > 0$.

Proposition 7. *There exists a constant $\alpha_0 > 0$ such that, if $0 < \alpha \leq \alpha_0$ and*

$$||V(0)||_{H^1_{\rm ul}} \leq \alpha, \quad ||W(0)||_{H^1_{\rm ul}} \leq \alpha, \tag{2.15}$$

then there exists a constant $K_5 > 0$ independent of α such that

$$||V(t)||_{H^1_{\rm ul}} \leq K_5\alpha^{1/4}, \quad |q(t)| \leq K_5\alpha,$$

for $t \geq 0$ and, in particular, $T_{\max}(\eta_0) = \infty$ for $\eta_0 > 0$ sufficiently small.

Theorem 4 follows now from Proposition 7. Indeed, the proposition implies that (2.12) holds for all $t > 0$ so that (2.14) are valid for all positive times. In the remainder of this section, we prove Proposition 7.

First of all, we note that the estimate for $q(t)$ follows from (2.14). Since the $H^1_{\rm ul}$-norm is invariant under translations, we may consider (2.6) in the original frame (x, t):

$$\begin{aligned} \partial_t v_1 &= \partial_x^2 v_1 + \frac{1}{2}[1 - 3h^2(x - ct - q(t)) + 2ch(x - ct - q(t))]v_1 + \frac{1}{2}[c - 3h(x - ct - q(t))]v_1^2 \\ &\quad -\frac{1}{2}v_1^3 + \dot{q}(t)h_\xi(x - ct - q(t)) + \gamma_1 v_2^2, \end{aligned} \tag{2.16}$$

$$\partial_t v_2 = -(1 + \partial_x^2)^2 v_2 + \alpha v_2 - v_2^3 - \gamma_2(1 + h(x - ct - q(t)))v_2 - \gamma_2 v_1 v_2. \tag{2.17}$$

Eq. (2.16) for v_1 we rewrite in the following form

$$\partial_t v_1 = \tilde{\mathcal{A}}_1 v_1 + \tilde{\mathcal{G}}_1(x - ct, q(t), W(t)) + \tilde{\mathcal{N}}_1(x - ct, q(t), v_1) + \gamma_1 v_2^2, \tag{2.18}$$

where $\tilde{\mathcal{A}}_1 = \partial_x^2 - (1 + c)$, $\tilde{\mathcal{N}}_1(\xi, q, v_1) = (1/2)[c - 3h(\xi - q)]v_1^2 - \frac{1}{2}v_1^3$, and

$$\tilde{\mathcal{G}}_1(\xi, q, W) := \left[\frac{3}{2}(1 - h(\xi - q)) + c\right][1 + h(\xi - q)]\rho_\beta(\xi)^{-1}w_1 + \dot{q}h_x(\xi - q).$$

We also consider the equation which is obtained from (2.18) by omitting the coupling term $\gamma_1 v_2^2$:

$$\partial_t \bar{v}_1 = \tilde{\mathcal{A}}_1 \bar{v}_1 + \tilde{\mathcal{G}}_1(x - ct, q(t), W(t)) + \tilde{\mathcal{N}}_1(q(t), \bar{v}_1). \tag{2.19}$$

We wish to compare solution v_1 of (2.18) to the solution $\bar{v}_1$ of (2.19) with the same initial condition $\bar{v}_1(x, 0) = v_1(x, 0)$. We shall use a suitable estimate for $\bar{v}_1$ to obtain estimates for the solution of (2.16) and (2.17) on the interval $[0, T_{\max}]$, where $T_{\max}$ is the maximal time for which the inequality (2.12) holds for some η_0 satisfying (2.13) and for all small enough initial conditions satisfying (2.15).

There is no explicit dependence on γ_1 in (2.19) and the following estimate holds [6, Lemma 3.4]: There exists a constant $K_6 > 0$ with the following property. If $||\bar{v}_1(0)||_{H^1_{\rm ul}} \leq (1/20K_0^2)$, then for $t \in [0, T_{\max})$ the solution $\bar{v}_1$ of (2.19) exists and

$$||\bar{v}_1(t)||_{H^1_{\rm ul}} \leq K_6(||v_1(0)||_{H^1_{\rm ul}} + ||W(0)||_{H^1_{\rm ul}}). \tag{2.20}$$

Lemma 8. *There are positive constants K_{11} and α_0, such that the following is true for all α with $0 < \alpha < \alpha_0$: If $(V, W, q) = (v_1, v_2, w_1, w_2, q)$ satisfies (2.9)–(2.11)with initial data for which (2.15) holds, then*

$$||V(t)||_{L^2_{\rm ul}} \leq K_{11}\alpha^{1/4}, \quad \textit{for all } t \textit{ with } 0 < t < T_{\max}.$$

Proof. Using (2.15), we infer from (2.20) that there is an independent of ε constant $K_7 > 0$ such that $||\bar{v}_1(t)||_{C^0} \leq K_0||\bar{v}_1(t)||_{H^1_{\rm ul}} \leq K_7\alpha$, and therefore

$$\bar{v}_1(x,t) \leq K_7\alpha, \quad \text{for all } x \in \mathbb{R} \text{ and } 0 < t < T_{\max}. \tag{2.21}$$

Eqs. (2.18) and (2.19) together with the assumption $\gamma_1 < 0$ show that

$$\partial_t\bar{v}_1 - \mathcal{A}_1\bar{v}_1 - \tilde{G}_1(x-ct, q(t), W(t)) - \tilde{\mathcal{N}}_1(q(t), \bar{v}_1) \overset{(2.19)}{=} 0 \geq$$

$$\geq \gamma_1 v_2^2 \overset{(2.18)}{=} \partial_t v_1 - \mathcal{A}_1 v_1 - \tilde{G}_1(x-ct, q(t), W(t)) - \tilde{\mathcal{N}}_1(q(t), v_1).$$

The comparison principle [2, Theorem 25.1 in §VII] together with (2.21) then give

$$v_1(x,t) \leq \bar{v}_1(x,t) \leq K_7\alpha \quad \text{for} \quad 0 \leq t < T_{\max} \quad \text{and} \quad x \in \mathbb{R}. \tag{2.22}$$

Having established the upper point-wise bound (2.22) for v_1, we return to (2.16) and (2.17). We consider V in the space $L^2_{\rm ul}(\sigma_b)$ with $b = \sqrt{\alpha}$. In what follows, for briefness, $h = h(x - ct - q(t))$. Using Lemma 3(iii) and the bounds (2.22), $\gamma_2 \geq 0$ and $1 + h \geq 0$, we obtain from (2.17) that

$$\begin{aligned} \frac{1}{2}\partial_t||v_2||^2_{L^2_{\rm ul}(\sigma_b)} &\leq \left[\frac{7}{2}b^2 + \alpha\right]\int_{\mathbb{R}}\sigma_b v_2^2\,\mathrm{d}x - \int_{\mathbb{R}}\sigma_b v_2^4\,\mathrm{d}x - \int_{\mathbb{R}}\sigma_b\gamma_2[1+h]v_2^2\,\mathrm{d}x - \gamma_2\int_{\mathbb{R}}\sigma_b v_1 v_2^2\,\mathrm{d}x \\ &\leq \frac{9}{2}\alpha\int_{\mathbb{R}}\sigma_b v_2^2\,\mathrm{d}x - \int_{\mathbb{R}}\sigma_b v_2^4\,\mathrm{d}x - \gamma_2\int_{\mathbb{R}}\sigma_b v_1 v_2^2. \end{aligned} \tag{2.23}$$

Next, we turn to (2.16):

$$\begin{aligned} \frac{1}{2}\partial_t||v_1||^2_{L^2_{\rm ul}(\sigma_b)} &\leq \left(\frac{\alpha}{2} - 1 - c\right)\int\sigma_b v_1^2 + \gamma_1\int\sigma_b v_1 v_2^2 + \frac{1}{2}\int\sigma_b(1 - 3h^2 + 2ch)v_1^2 \\ &\quad + \frac{1}{2}\int\sigma_b(c - 3h)v_1^3 - \frac{1}{2}\int\sigma_b v_1^4 + \int\sigma_b\dot{q}h_x v_1. \end{aligned}$$

For the terms containing h, it is easy to see that

$$\frac{1}{2}\int\sigma_b(1 - 3h^2 + 2ch)v_1^2 \leq (\frac{1}{2} + \frac{c^2}{6})\int\sigma_b v_1^2, \tag{2.24}$$

and

$$\frac{1}{2}\int\sigma_b(c - 3h)v_1^3 \leq \frac{1}{2}\int\sigma_b(c + 3 - 3(h+1))v_1^3 \leq \frac{(c+3)}{2}\int\sigma_b v_1^3 - \frac{3}{2}\int\sigma_b(1+h)v_1^3.$$

In the last inequality, we rewrite v_1^3 as $w_1\rho_b^{-1}v_1^2$ and use the estimate (2.22) to obtain

$$\frac{1}{2}\int\sigma_b(c - 3h)v_1^3 \leq 2K_7\alpha\int\sigma_b v_1^2 - \frac{3}{2}\int\sigma_b(1+h)\rho_b^{-1}w_1 v_1^2.$$

The function $(1+h)\rho_b^{-1}$ is bounded. Moreover, for w_1 the estimate (2.14) holds, and therefore, on the account of (2.15), there exists independent of α constant $K_8 > 0$ such that

$$\frac{1}{2}\int\sigma_b(c - 3h)v_1^3 \leq K_8\alpha\int\sigma_b v_1^2. \tag{2.25}$$

We note the inequality $\dot{q}v_1 \leq (a_1/2)v_1^2 + (1/2a_1)\dot{q}^2$, where we set with $a_1 = 1/4$. Taking into account $h_x < 2$, inequality (2.14), and the condition on the initial data (2.15), we then get

$$\int\sigma_b\dot{q}h_x v_1 \leq 2\int\sigma_b(\frac{1}{8}v_1^2 + 2\dot{q}^2) \leq \frac{1}{4}\int\sigma_b v_1^2 + \frac{4K_4^2}{\sqrt{\alpha}}||W(0)||^2_{\mathrm{H}^1_{\rm ul}} \leq \frac{1}{4}\int\sigma_b v_1^2 + 4K_4^2\alpha^{3/2}. \tag{2.26}$$

We combine (2.24), (2.25) and (2.26) to obtain

$$\frac{1}{2}\partial_t||v_1||^2_{L^2_{\mathrm{ul}}(\sigma_b)} \leq \left(-\frac{1}{4}+\frac{\alpha}{2}+K_8\alpha\right)\int\sigma_b v_1^2+\gamma_1\int\sigma_b v_1 v_2^2+4K_4^2\alpha^{3/2}. \tag{2.27}$$

We denote $\partial_t(||V||^2_{\sigma_b})=\partial_t(||v_1||^2_{\sigma_b}+||v_2||^2_{\sigma_b})$, and add estimates (2.27) for v_1 and (2.23) for v_2:

$$\frac{1}{2}\partial_t||V||^2_{L^2_{\mathrm{ul}}(\sigma_b)} \leq \left(-\frac{1}{4}+\left(\frac{1}{2}+K_8\right)\alpha\right)\int\sigma_b v_1^2+\frac{9\alpha}{2}\int\sigma_b v_2^2-\int\sigma_b v_2^4+4K_4^2\alpha^{3/2}+(\gamma_1-\gamma_2)\int\sigma_b v_1 v_2^2.$$

In the last term we use the inequality $(\gamma_1-\gamma_2)v_2^2 v_1 \leq ((\gamma_2-\gamma_1)^2/2a_2)v_1^2+(a_2/2)v_2^4$, where $0<a_2<2$,

$$\frac{1}{2}\partial_t||V||^2_{L^2_{\mathrm{ul}}(\sigma_b)} \leq \left(-\frac{1}{4}+\left(\frac{1}{2}+K_8\right)\alpha+\frac{(\gamma_2-\gamma_1)^2}{2a_2}\right)\int\sigma_b v_1^2+\frac{9\alpha}{2}\int\sigma_b v_2^2-(1-\frac{a_2}{2})\int\sigma_b v_2^4+4K_4^2\alpha^{3/2}.$$

Next, we record (see [6]) that for any constant $a>0$

$$\int_{\mathbb{R}} a\sigma_b v_2^2\,\mathrm{d}x-\int_{\mathbb{R}}\sigma_b v_2^4\,\mathrm{d}x \leq \frac{a^2}{b}-\int_{\mathbb{R}} a\sigma_b v_2^2\,\mathrm{d}x.$$

The terms containing v_2 then can be estimated as

$$\frac{9\alpha}{2}\int\sigma_b v_2^2-\left(1-\frac{a_2}{2}\right)\int\sigma_b v_2^4 \leq \frac{(9\alpha)^2}{2\sqrt{\alpha}(2-a_2)}-\frac{9\alpha}{2}\int\sigma_b v_2^2,$$

and therefore

$$\frac{1}{2}\partial_t||V||^2_{L^2_{\mathrm{ul}}(\sigma_b)} \leq \left(-\frac{1}{4}+\left(\frac{1}{2}+K_8\right)\alpha+\frac{(\gamma_2-\gamma_1)^2}{2a_2}\right)\int\sigma_b v_1^2+\frac{81\alpha^{\frac{3}{2}}}{2(2-a_2)}-\frac{9\alpha}{2}\int\sigma_b v_2^2+4K_4^2\alpha^{4+2\delta_2}.$$

We want to fix a_2 such that $(\gamma_2-\gamma_1)^2<a_2<2$. For example, we may choose $a_2=1+\frac{(\gamma_2-\gamma_1)^2}{2}$. For any

$$\alpha \leq \alpha_0=\left(-\frac{1}{4}+\frac{2(\gamma_2-\gamma_1)^2}{2+(\gamma_2-\gamma_1)^2}\right)\frac{1}{5+K_8}$$

then we have

$$-\frac{1}{4}+\left(\frac{1}{2}+K_8\right)\alpha+\frac{2(\gamma_2-\gamma_1)^2}{2+(\gamma_2-\gamma_1)^2} \leq -\frac{9\alpha}{2},$$

and therefore

$$\frac{1}{2}\partial_t||V||^2_{L^2_{\mathrm{ul}}(\sigma_b)} \leq -\frac{9\alpha}{2}||V||^2_{L^2_{\mathrm{ul}}(\sigma_b)}+\left(\frac{81}{(2-(\gamma_2-\gamma_1)^2)}+4K_4^2\right)\alpha^{3/2}.$$

This is a differential inequality of the form $(1/2)f'(t)\leq d_1-d_2 f(t)$ for which Gronwall's estimate [8, Theorem 1.5.7] gives

$$f(t)\leq \mathrm{e}^{-2d_2 t}f(0)+\frac{d_1}{d_2}(1-\mathrm{e}^{-2d_2 t})\leq f(0)+\frac{d_1}{d_2}$$

for $d_2>0$. In our case, this estimate becomes

$$||V(t)||^2_{L^2_{\mathrm{ul}}(\sigma_b)} \leq ||V(0)||^2_{L^2_{\mathrm{ul}}(\sigma_b)}+K_9\alpha^{1/2}, \quad \text{where} \quad K_9=\frac{2}{9}\left(\frac{81}{(2-(\gamma_2-\gamma_1)^2)}+4K_4^2\right).$$

In the uniformly local norm using Lemma 3(ii) and (2.15) we get

$$||V(t)||^2_{L^2_{\mathrm{ul}}} \leq K_0(1+\sqrt{\alpha})\left(\frac{||V(0)||^2_{\mathrm{H}^1_{\mathrm{ul}}}}{\sqrt{\alpha}}+K_9\alpha^{1/2}\right).$$

Finally, after referring to (2.15), we conclude that there is an independent of $\alpha < 1$ constant K_{10} such that

$$||V(t)||^2_{L^2_{\text{ul}}} \leq K_{10}\left[\alpha^{3/2} + \alpha^{1/2}\right] \leq 2K_{10}\alpha^{1/2}$$

which implies the inequality in Lemma 8. □

We note that the proof of Lemma 3.6 in [6] is based on norm estimates applied to the variation-of-constants formula and does not depend on the sign or value of γ_1 explicitly. It uses a bound on $||v_2||_{L^2_{\text{ul}}}$, which in our case follows from Lemma 8, and produces a bound on $||V||^1_{H^2_{\text{ul}}}$. We formulate this result as a lemma:

Lemma 9. *There are positive numbers K_{12} and α_0 such that the following is true for all α with $0 < \alpha < \alpha_0$: If $(V, W, q) = (v_1, v_2, w_1, w_2, q)$ satisfies (2.9)–(2.11)with initial data for which (2.4) holds, then*

$$||V(t)||_{H^1_{\text{ul}}} \leq K_{12}\alpha^{1/4}, \quad \textit{for all } t \textit{ with } 0 < t < T_{\max}.$$

We are now ready to complete the proof of Proposition 7.

Proof of Proposition 7. For sufficiently small $\varepsilon > 0$, it follows from Lemma 9 that $|q(t)| + ||V(t)||^1_{H^2_{\text{ul}}}|| \leq (1/2)\eta_0$ for $0 \leq t < T_{\max}$, which contradicts the maximality of $T_{\max}$, see (2.12), if $T_{\max}$ is finite. Thus, (2.12) holds for any t, which in turn implies that (2.14) is valid for all times. Therefore, (2.12) holds with $T_{\max} = \infty$. The proof of Proposition 7 is complete. □

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