STABILITY OF GASLESS COMBUSTION FRONTS IN ONE-DIMENSIONAL SOLIDS

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Abstract. For gasless combustion in a one-dimensional solid, we show a type of nonlinear stability of the physical combustion front: if a perturbation of the front is small in both a spatially uniform norm and an exponentially weighted norm, then the perturbation stays small in the spatially uniform norm and decays in the exponentially weighted norm, provided the linearized operator has no eigenvalues in the right half-plane other than zero. Using the Evans function, we show that the zero eigenvalue must be simple. Factors that complicate the analysis are: (1) the linearized operator is not sectorial, and (2) the linearized operator only has good spectral properties when the weighted norm is used, but then the nonlinear term is not Lipschitz. The result is nevertheless physically natural. To prove it, we first show that when the weighted norm is used, the semigroup generated by the linearized operator decays on a subspace complementary to the operator's kernel, by showing that it is a compact perturbation of the semigroup generated by a more easily analyzed triangular operator. We then use this result to help establish that solutions stay small in the spatially uniform norm, which in turn helps establish nonlinear convergence in the weighted norm.

1. Introduction

We consider the system

$$\partial_t u_1 = \partial_{xx} u_1 + \omega(u_1, u_2), \tag{1.1}$$
$$\partial_t u_2 = -\beta\omega(u_1, u_2), \tag{1.2}$$

with $\beta > 0$, $\omega(u_1, u_2) = u_2\rho(u_1)$, and

$$\rho(u_1) = \begin{cases} e^{-\frac{1}{u_1}} & \text{if } u_1 > 0, \\ 0 & \text{if } u_1 \leq 0. \end{cases} \tag{1.3}$$

This system is a simple model for combustion of a solid fuel in one space dimension: u_1 is temperature, u_2 is concentration of unburned fuel, ρ is the unit reaction rate, and β is the "exothermicity" parameter; the larger β is, the more fuel one must burn to achieve a given increase in temperature. The value $u_1 = 0$ represents a background temperature at which the reaction does not take place. The unit reaction rate $\rho(u_1)$ could be replaced by any C^2

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function that equals 0 for $u_1 \leq 0$ and is positive for $u_1 > 0$; we have chosen to use the one that is most common on combustion theory.

We are interested in traveling combustion fronts $(u_1, u_2)(\xi)$, $\xi = x - \sigma t$, where σ is the speed of the front. Behind the front we require that $(u_1, u_2) = (u_1^\sharp, 0)$, where $u_1^\sharp > 0$ is to be determined. Ahead of the front we require that $(u_1, u_2) = (0, u_2^\sharp)$, where $u_2^\sharp > 0$ is the concentration of fuel in the medium. We will normalize so that $u_2^\sharp = 1$.

The system (1.1)–(1.2) is symmetric with respect to translation and reflection in x. Hence, any translate of a combustion front is a combustion front, and if the system admits traveling waves moving to the right, it also admits traveling waves moving to the left. Without loss of generality, we will consider only traveling waves moving to the right, i.e., with wave speed $\sigma > 0$. (There are no nonconstant standing waves.)

In the literature one finds numerical simulations of this system [3], study of simplified model equations [26], and some rigorous results. Proofs of existence of traveling wave solutions by phase-plane analysis have been given by Billingham [8] and by Varas and Vega [39]. It turns out that the traveling waves have $(u_1, u_2)(-\infty) = (\frac{1}{\beta}, 0)$ and, of course, $(u_1, u_2)(\infty) = (0, 1)$. There is a positive number c such that for each $\sigma \geq c$, there is a unique (up to translation) traveling wave with speed σ. The one with speed c, which we denote $(h_1, h_2)(\xi)$, $\xi = x - ct$, approaches both end states exponentially. The others approach the burned end state $(u_1, u_2) = (\frac{1}{\beta}, 0)$ exponentially and the unburned end state $(u_1, u_2) = (0, 1)$ very slowly (slower than algebraically, i.e., slower that $|\xi|^{-a}$ for any a). Varas and Vega argue that all traveling waves but (h_1, h_2) disappear if one perturbs the problem by allowing heat loss to the environment, and hence should be ignored. We will shortly suggest a different argument for why (h_1, h_2), is the most important of the traveling waves.

Stability of a traveling wave, sometimes called nonlinear stability, means that a small perturbation of the wave converges to some translate of the wave. (More precisely, this notion is called "stability with asymptotic phase.") Stability is studied using a spatial variable moving at the speed of the wave, so that the wave becomes a stationary solution. If one writes the PDE (1.1)–(1.2) in the moving coordinate and linearizes at the traveling wave, one obtains a linear differential equation $\partial_t V = \mathcal{B}V$, with V in the function space $\mathcal{X}$ in which one chooses to allow perturbations. We shall always use a space $\mathcal{X}$ that includes the difference between any two translates of the traveling wave, and the derivative of the traveling wave with respect to ξ.

Typically one approaches the issue of nonlinear stability by first studying spectral stability and then linearized stability. Because the traveling wave can be shifted, 0 is an eigenvalue of $\mathcal{B}$ with eigenfunction (h_1', h_2').

In this paper, we shall say that a traveling wave is *spectrally stable* in $\mathcal{X}$ if

(S1) 0 is a simple eigenvalue of $\mathcal{B}$, and
(S2) the rest of the spectrum of $\mathcal{B}$ lies in Re $\lambda < -\nu$ for some $\nu > 0$.

We shall say that a traveling wave is *linearly stable* in $\mathcal{X}$ if $\mathcal{B}$ generates a C_0-semigroup $e^{t\mathcal{B}}$ such that

(L1) $e^{t\mathcal{B}}$ has a simple eigenvalue 1, and
(L2) $e^{t\mathcal{B}}$ has a codimension-one invariant subspace on which $\|e^{t\mathcal{B}}\| \leq Ke^{-\nu t}$ for some $K > 0$ and $\nu > 0$.

If $\mathcal{B}$ is sectorial, then spectral stability implies linearized stability; if, in addition, the nonlinear terms satisfy a certain Lipschitz condition, then spectral stability also implies nonlinear stability. If $\mathcal{B}$ is not sectorial, a result of Bates and Jones [2] says that if the nonlinear terms yield a map from $\mathcal{X}$ to itself with sufficiently small Lipschitz number, then linearized stability implies nonlinear stability. For operators that are not sectorial, however, spectral stability does not in general imply linearized stability.

We will work in spaces based on $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, and $BUC(\mathbb{R})$ (bounded, uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the L^∞ norm). Spaces based on $L^2(\mathbb{R})$ are not suitable to the study of nonlinear stability; we include them because it is convenient to study linear operators on $H^1(\mathbb{R})$ by first studying them on $L^2(\mathbb{R})$.

Let $\mathcal{E}_0$ denote $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$. The linearization of the PDE at (h_1, h_2) defines an operator on $\mathcal{E}_0^2 = \mathcal{E}_0 \times \mathcal{E}_0$ that we denote $\mathcal{A}_0$.

There are several previous studies of the stability of (h_1, h_2). In [39] Varas and Vega obtain a bound on the possible size of eigenvalues with nonnegative real part and square-integrable eigenfunctions. Balasuriya et. al. [1] performed numerical Evans function computations that indicate that the 0 eigenvalue is simple. Their computations also indicate that there are no positive real eigenvalues. On the other hand, numerical simulations by Bayliss and Matkowsky [3] suggest that as β increases, the traveling wave loses stability due to a pair of complex eigenvalues crossing the imaginary axis.

Of course, eigenvalue information, even if complete, cannot by itself yield even spectral stability. In fact, the essential spectrum of $\mathcal{A}_0$ includes the imaginary axis, so spectral stability does not hold. In any event, nonlinear stability certainly does not hold in $BUC(\mathbb{R})^2$, because of the existence of a family of nearby traveling waves with the same end states.

What sort of stability is it physically reasonable to expect? Consider, for example an initial state with $(u_1, u_2) = (u_1^\flat, 0)$, $0 < u_1^\flat < \frac{1}{\beta}$, for $x \leq -\frac{1}{2}$; $(u_1, u_2) = (0, 1)$ for $\frac{1}{2} \leq x$; and some interpolation between these values for $-\frac{1}{2} < x < \frac{1}{2}$. Such an initial state is physically important: at the left the temperature is positive but below combustion temperature, and there is no fuel; at the right the temperature is 0 and there is plenty of fuel. If the temperature $u_1^\flat$ is great enough, one expects a combustion front to form, with temperature $u_1 = \frac{1}{\beta}$, and propagate to the right. A growing region of temperature near $\frac{1}{\beta}$ remains behind the front and diffuses. See Figure 1.1. Numerical simulations indicate that the front that forms is close to a translate of (h_1, h_2) at the right and moves with the speed of (h_1, h_2). See [10] for numerical simulations of a related equation. In the spatial variable $\xi = x - ct$, in which any translate of (h_1, h_2) is fixed, such a solution becomes very close to a translate of (h_1, h_2) on a region $-a(t) < \xi < \infty$, where $a(t) \to \infty$ as $t \to \infty$; nevertheless, at any time, the solution remains far from the translate of (h_1, h_2) far to the left, where the temperatures are approximately $u_1^\flat$ and $\frac{1}{\beta}$ respectively.

A mathematical notion that captures this kind of stability of the combustion front is stability with respect to a norm with weight function $e^{\alpha\xi}$, $\alpha > 0$ small. The norm is applied to the difference of two solutions, not to the solutions themselves. A solution that approaches a translate of the traveling wave in this norm becomes very close to it at the right, but may continue to be far from it far to the left. In this norm, the solution just considered approaches a translate of the traveling wave. The same norm is used in the study of convective instability [35], which occurs when perturbations are convected to the left without decreasing in size.

For a small $\alpha > 0$, let

$$\mathcal{E}_\alpha = \{v(\xi) : w(\xi) = e^{\alpha\xi} v(\xi) \in \mathcal{E}_0\},$$

with norm $\|v\|_\alpha = \|e^{\alpha\xi} v\|_0 = \|w\|_0$. The considerations just given suggest studying stability of the combustion front to perturbations in the space $\mathcal{E}_\alpha^2$.

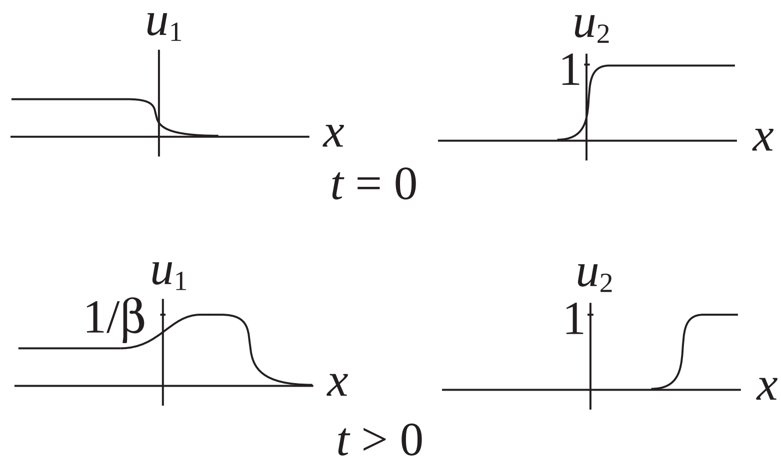

FIGURE 1.1. A combustion front forms and propagates toward the right, where there is fuel, leaving behind high-temperature region.

The linearization of the PDE at (h_1, h_2) defines an operator $\mathcal{A}_\alpha$ on $\mathcal{E}_\alpha^2 = \mathcal{E}_\alpha \times \mathcal{E}_\alpha$. Working in $\mathcal{E}_\alpha^2$ rather than $\mathcal{E}_0^2$ shifts the essential spectrum to the left; the essential spectrum of $\mathcal{A}_\alpha$ lies in $\operatorname{Re}\lambda < -\nu_1 < 0$, thus allowing the possibility of spectral stability. However, $\mathcal{A}_\alpha$ has a vertical line in its spectrum, so that the C_0-semigroup generated by $\mathcal{A}_\alpha$ is not an analytic semigroup. An additional difficulty is that the nonlinear terms in the PDE do not yield a Lipschitz map from $\mathcal{E}_\alpha^2$ to itself, so that standard results on well-posednes of the PDE (as opposed to its linearization at (h_1, h_2)) cannot be used in $\mathcal{E}_\alpha^2$.

In this paper we resolve these issues and obtain a physically natural stability result. We do the following:

(1) For each of the three choices for $\mathcal{E}_0$, we prove that 0 is a simple eigenvalue of $\mathcal{A}_\alpha$ that is isolated in its spectrum. The eigenvalues of $\mathcal{A}_\alpha$ are the zeros of the Evans function $D(\lambda)$, which is independent of α. Of course, $D(0) = 0$. We prove that $D'(0)$ is positive, which implies the result.

(2) We prove that $D(\lambda)$ is positive for large positive real λ. This result is consistent with spectral stability.

(3) For each of the three choices for $\mathcal{E}_0$, we prove that if the only zero of $D(\lambda)$ in $\operatorname{Re}\lambda \geq 0$ is $\lambda = 0$, then in $\mathcal{E}_\alpha^2$ the combustion front is both spectrally stable and linearly stable.

(4) Let $\mathcal{E}_0$ be either $H^1(\mathbb{R})$ or $BUC(\mathbb{R})$. Let $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ with $\|v\| = \max(\|v\|_0, \|v\|_\alpha)$. The nonlinear terms in the PDE yield a Lipschitz map from $\mathcal{E}^2$ to itself, so the PDE is well posed in $\mathcal{E}^2$. However, in $\mathcal{E}^2$ the essential spectrum of the linearization of the PDE at (h_1, h_2) contains the origin. Suppose the only zero of $D(\lambda)$ in $\operatorname{Re}\lambda \geq 0$ is $\lambda = 0$. We prove that if (1.1)–(1.2) is solved for the initial condition (u_1^0, u_2^0) with $\|(u_1^0, u_2^0) - (h_1, h_2)\|$ small, then there is a small number q^* such that (i) $\|(u_1, u_2)(t, \xi) - (h_1, h_2)(\xi - q^*)\|_0$ stays small, and (ii) $\|(u_1, u_2)(t, \xi) - (h_1, h_2)(\xi - q^*)\|_\alpha \to 0$ as $t \to \infty$. Thus if the perturbation is small in two norms, it stays small in one and decays in the other.

The fourth result, for $\mathcal{E}_0 = BUC(\mathbb{R})$, implies that in the example just considered, if the condition on $D(\lambda)$ holds and $u_1^\flat$ is not too far below the combustion temperature $\frac{1}{\beta}$, then the solution will converge to the traveling wave in the weighted norm, as we have described.

The first and second results are proved by fairly standard Evans function calculations. In the terminology of [5], we use a "mixed" Evans function rather than the "classical" Evans function used by Balasuriya, et. al. This choice, plus a simple change of variables, makes the calculations easy. Some care must be taken because the equilibria of the traveling wave equation are not hyperbolic.

To check whether there are zeros of $D(\lambda)$ in $\operatorname{Re}\lambda \geq 0$ other than $\lambda = 0$, one must do a numerical Evans function calculation of a winding number, taking advantage of the fact that the Evans function is analytic. It turns out that if $D(\lambda)$ has a zero in $\operatorname{Re}\lambda \geq 0$, then the corresponding eigenfunction decays exponentially at $\xi = \pm\infty$, so λ must satisfy the bound of Varas and Vega. This remark fixes the size of the contour on which the winding number should be calculated.

The spectral stability result just says that if $\mathcal{A}_\alpha$ has no eigenvalues other than 0 in $\operatorname{Re}\lambda \geq 0$, then there is a number $\nu > 0$ such that $\mathcal{A}_\alpha$ has no spectrum other than 0 in $\operatorname{Re}\lambda \geq -\nu$. It is necessary to prove this because the operator $\mathcal{A}_\alpha$ does not belong to a category for which it is known that if the essential spectrum lies in $\operatorname{Re}\lambda < \omega_1$ and $\omega_1 < \omega$, then there are only a finite number of eigenvalues with real part greater than ω. The linear stability result is necessary because the operator $\mathcal{A}_\alpha$ does not belong to a category for which it is known that information about its spectrum yields information about the semigroup it generates. The proof of spectral and linear stability relies on the fact that, because $\partial_{u_1}\omega(h_1, h_2)(\xi)$ approaches 0 as $\xi \to \pm\infty$, the semigroup generated by $\mathcal{A}_\alpha$ is a compact perturbation of the semigroup generated by a more easily analyzed triangular operator. A recent preprint that uses related ideas is [7].

The nonlinear stability result uses an approach that originated in [32] in the Hamiltonian context. Other examples of this approach are [27], [16], [17], [20], [18], [4], and [22]. In [19] the same approach is used to prove nonlinear stability of the combustion front when a small diffusion term $\epsilon\partial_{xx}u_2$ is added to (1.2).

Let us briefly explain, using the spaces $BUC(\mathbb{R})$ and $BUC(\mathbb{R})_\alpha$, the need to work in two norms to prove nonlinear stability. Consider the Taylor expansion of $\omega(u_1, u_2) = u_2\rho(u_1)$ about the traveling wave (h_1, h_2):

$$\omega(h_1 + v_1, h_2 + v_2) = h_2\rho(h_1) + v_2\rho(h_1) + h_2\rho'(h_1)v_1 + v_2\rho'(h_1)v_1 + \dots,$$

where we have omitted one quadratic term and all higher order terms. The norm of the term $v_2\rho'(h_1)v_1$ in $BUC(\mathbb{R})_\alpha$ is

$$\sup_\xi |e^{\alpha\xi}v_2(\xi)\rho'(h_1(\xi))v_1(\xi)|. \tag{1.4}$$

If v_1 and v_2 are in $BUC(\mathbb{R})_\alpha$, then $e^{\alpha\xi}v_2(\xi)$ and $\rho'(h_1(\xi))$ are bounded, but $v_1(\xi)$ may not be bounded. This illustrates the difficulty of dealing with the nonlinear terms in the space $BUC(\mathbb{R})_\alpha$. On the other hand, if we work in $BUC(\mathbb{R}) \cap BUC(\mathbb{R})_\alpha$, then $v_1(\xi)$ *is* bounded. Similarly, to prove the decay of (1.4) over time in a solution of the PDE, we can try to show, as a step in the analysis, that v_1 does not grow in $BUC(\mathbb{R})$.

In [32], [27], [16], and [22], boundedness in the spatially uniform norm is related to the Hamiltonian structure. In [17], [20], [18], and [4] it is related to the stability of the bifurcating patterns that are connected by the front. Here we use the fact that perturbations of the traveling wave, after a component along the family of shifted traveling waves is subtracted, satisfy a system of the form

$$\partial_t v_1 = \partial_{\xi\xi}v_1 + c\partial_\xi v_1 + \dots, \quad \partial_t v_2 = (c\partial_\xi + a(t, \xi))v_2 + \dots,$$

with $a(t,\xi) < -\nu$ for some $\nu > 0$. Ignoring the omitted terms, solutions of the second equation satisfy the estimate $\|v_2(t,\xi)\|_{BUC(\mathbb{R})} \le e^{-\nu t}\|v_2(0,\xi)\|_{BUC(\mathbb{R})}$, and the first equation generates a bounded semigroup in $BUC(\mathbb{R})$.

There is a physical reason why, in a spatially uniform norm at the linear level, the first equation, for the perturbation of the temperature v_1, has solutions that are only bounded, while the second equation, for the perturbation of the fuel v_2, has solutions that decay. Note that the perturbation of the temperature of the front at the left that we have already discussed does not decay in the sup norm. On the other hand, suppose we add some fuel to the front at the left: let $v_1 = 0$, and let $v_2 = v_2^\flat$, with $v_2^\flat$ small and positive, for $\xi \le -\frac{1}{2}$; $v_2 = 0$ for $\frac{1}{2} \le \xi$; and v_2 equals some interpolation between these values for $-\frac{1}{2} < \xi < \frac{1}{2}$. Because of the high temperature of the front at the left, the added fuel will all burn, so v_2 will decay to 0 in the sup norm.

In fact, this difference between the equations for v_1 and v_2 at the linear level persists at the nonlinear level: we show that $\|u_1(t,\xi) - h_1(\xi - q^*)\|_0$ stays small, as we have indicated, but $\|u_2(t,\xi) - h_2(\xi - q^*)\|_0 \to 0$ as $t \to \infty$.

The remainder of the paper is organized as follows. In Section 2 we rewrite the PDE (1.1)–(1.2) in moving coordinates, and review the construction of the traveling waves. In Section 3 we prove the Evans function results. In Section 4 we study the linear operator $\mathcal{A}_\alpha$. In Section 5 we prove the nonlinear stability results; an outline of the proof is given at the start of that section. We give some extensions of our results in Section 6.

In Appendix A we do a numerical Evans function calculation that indicates that, for $\beta = 1$, there are no zeros of $D(\lambda)$ with $0 \le \max(\operatorname{Re}\lambda, \operatorname{Im}\lambda) \le 1000$ other than $\lambda = 0$. (We have not, however, calculated the bound given by Varas and Vega in order to justify using a contour of this size.) The method of computation may be of some independent interest: it uses the boundary-value-problem continuation routines of AUTO (available from http://cmvl.cs.concordia.ca/auto) to compute solutions of linear differential equations, rather than the initial-value-problem solvers that are usually used.

Finally, let us make several remarks.

(1) The fact that $\mathcal{A}_\alpha$ has a vertical line in its spectrum, and hence is not sectorial, is a consequence of the fact that the system (1.1)–(1.2) includes both a PDE and an ODE. For traveling *pulses* (left and right states are the same) in systems such as (1.1)–(1.2), Evans [15] showed that, for $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$, if the linearization has its essential spectrum in $\operatorname{Re}\lambda < -\nu_1 < 0$ and has no eigenvalues other than a simple eigenvalue 0 in $\operatorname{Re}\lambda \ge 0$, then the traveling pulse is spectrally stable, linearly stable, and nonlinearly stable in $\mathcal{E}_0^2$. His argument was later simplified by Bates and Jones [2]. However, as far as we know, there are no analogous results in the literature for traveling *fronts* (left and right states different) in such systems. As mentioned earlier, our proof of spectral and linear stability uses the fact that $\partial_{u_1}\omega(h_1,h_2)(\xi)$ approaches 0 as $\xi \to \pm\infty$, which is a special feature of our problem. Thus our work does not yield a generally applicable result about stability of fronts in systems that include both ODEs and PDEs. Such a result will be the subject of a later paper.

(2) A consequence of simplicity of the 0 eigenvalue, which holds for all β, is that there is no bifurcation from the eigenvalue 0 as β increases.

(3) Why do numerical simulations with the physically important initial conditions described earlier always converge at the right to a translate of (h_1,h_2), and not to a faster combustion front? Note that such an initial condition is, at the right, an exponentially small perturbation of (h_1,h_2), which is, we recall, the only traveling wave (up to translation)

that approaches the right end state exponentially. It is possible that the traveling waves with faster wave speeds, which approach $(u_1, u_2) = (0, 1)$ very slowly, are similarly stable to sufficiently small perturbations of themselves. This is what occurs for certain scalar reaction-diffusion equations [42]. This analogy suggests that (h_1, h_2) derives its importance from the fact that physically relevant initial conditions, which have $(u_1, u_2) = (0, 1)$ for ξ greater than some value, are exponentially small perturbations of it, not small perturbations of one of the other waves.

2. Traveling waves

In (1.1)–(1.2), we replace the spatial coordinate x with one ξ that is moving with speed σ: $\xi = x - \sigma t$. We obtain

$$\partial_t u_1 = \partial_{\xi\xi} u_1 + \sigma \partial_\xi u_1 + \omega(u_1, u_2), \tag{2.1}$$

$$\partial_t u_2 = \sigma \partial_\xi u_2 - \beta \omega(u_1, u_2). \tag{2.2}$$

A steady solution of (2.1)–(2.2) is a traveling wave solution of (1.1)–(1.2) with speed σ. Steady solutions of (2.1)–(2.2) satisfy the system of ODEs

$$0 = \partial_{\xi\xi} u_1 + \sigma \partial_\xi u_1 + \omega(u_1, u_2), \tag{2.3}$$

$$0 = \sigma \partial_\xi u_2 - \beta \omega(u_1, u_2). \tag{2.4}$$

We are interested in solutions of (2.3)–(2.4) that satisfy the boundary conditions

$$(u_1, u_2, \partial_\xi u_1)(-\infty) = (u_1^\sharp, 0, 0), \quad (u_1, u_2, \partial_\xi u_1)(\infty) = (0, 1, 0). \tag{2.5}$$

Such solutions represent traveling combustion fronts. The speed σ and the left temperature $u_1^\sharp$ are, at this stage, unknowns to be determined.

In the system (2.3)–(2.4) we set $u_3 = \partial_\xi u_1$ and use prime to denote derivative with respect to ξ. We obtain the first-order system

$$u_1' = u_3, \tag{2.6}$$

$$u_2' = \frac{\beta}{\sigma} \omega(u_1, u_2), \tag{2.7}$$

$$u_3' = -\sigma u_3 - \omega(u_1, u_2). \tag{2.8}$$

We write $\mathbf{u} = (u_1, u_2, u_3)$, treat β as fixed, and write the system (2.6)–(2.8) as $\mathbf{u}' = f(\mathbf{u}, \sigma)$. We restrict our attention to $\sigma > 0$. A solution of (2.3)–(2.4) that satisfies the boundary conditions (2.5) corresponds to a solution of (2.6)–(2.8) that goes from an equilibrium $(u_1^\sharp, 0, 0)$ (each such point is an equilibrium) to the equilibrium $(0, 1, 0)$.

The function $I_\sigma(u_1, u_2, u_3) = \sigma u_1 + \frac{\sigma}{\beta} u_2 + u_3$ is a first integral of (2.6)–(2.8); one easily checks that $I_\sigma' = 0$. To take advantage of this fact, we define new variables $\mathbf{y} = (u_1, u_2, y_3)$ by $\mathbf{y} = P(\sigma)\mathbf{u}$,

$$P(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma & \frac{\sigma}{\beta} & 1 \end{pmatrix}.$$

In the new variables, the differential equation $\mathbf{u}' = f(\mathbf{u}, \sigma)$ becomes $\mathbf{y}' = g(\mathbf{y}, \sigma)$, given by

$$u_1' = -\sigma u_1 - \frac{\sigma}{\beta} u_2 + y_3, \tag{2.9}$$

$$u_2' = \frac{\beta}{\sigma}\omega(u_1, u_2), \tag{2.10}$$

$$y_3' = 0. \tag{2.11}$$

The system (2.9)–(2.11) has two sets of equilibria: low-temperature equilibria (u_1, u_2, y_3) with $u_1 \leq 0$, u_2 arbitrary, and $y_3 = \sigma u_1 + \frac{\sigma}{\beta} u_2$; and 0-reactant equilibria $(u_1, 0, y_3)$ with u_1 arbitrary and $y_3 = \sigma u_1$. The equilibrium $(0, 1, 0)$ of (2.6)–(2.8) corresponds to the equilibrium $(0, 1, \frac{\sigma}{\beta})$ of (2.9)–(2.11), which is low-temperature. Since y_3 is constant on solutions, the equilibrium $(u_1^\sharp, 0, 0)$ of (2.6)–(2.8) must correspond to the equilibrium $(\frac{1}{\beta}, 0, \frac{\sigma}{\beta})$ of (2.9)–(2.11), which is 0-reactant. We set $y_3 = \frac{\sigma}{\beta}$ in (2.9)–(2.10), and obtain the system

$$u_1' = g_1(u_1, u_2, \sigma) = -\sigma u_1 - \frac{\sigma}{\beta} u_2 + \frac{\sigma}{\beta}, \tag{2.12}$$

$$u_2' = g_2(u_1, u_2, \sigma) = \frac{\beta}{\sigma}\omega(u_1, u_2), \tag{2.13}$$

a system in the plane with a parameter σ and equilibria $(\frac{1}{\beta}, 0)$ and $(0, 1)$ that we wish to connect.

The linearization of (2.12)–(2.13) is

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -\sigma & -\frac{\sigma}{\beta} \\ \frac{\beta}{\sigma}\partial_{u_1}\omega(u_1, u_2) & \frac{\beta}{\sigma}\partial_{u_2}\omega(u_1, u_2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2.14}$$

Note that

$$\partial_{u_1}\omega(\frac{1}{\beta}, 0) = 0, \quad \partial_{u_2}\omega(\frac{1}{\beta}, 0) = e^{-\beta}, \quad \partial_{u_1}\omega(0, 1) = 0, \quad \partial_{u_2}\omega(0, 1) = 0. \tag{2.15}$$

Therefore the equilibrium $(\frac{1}{\beta}, 0)$ is a hyperbolic saddle, and the equilibrium $(0, 1)$ has one negative eigenvalue and one 0 eigenvalue. According to [39], there is a unique value $\sigma = c > 0$ for which the system (2.12)–(2.13) has a solution that approaches $(\frac{1}{\beta}, 0)$ exponentially as $\xi \to -\infty$, and approaches $(0, 1)$ exponentially as $\xi \to \infty$. See Figure 2.1. We denote the solution $(h_1, h_2)(\xi)$.

The connection between the unstable manifold of $(\frac{1}{\beta}, 0)$ and the stable manifold of $(0, 1)$ breaks in a nondegenerate manner as σ varies provided a certain Melnikov integral M is nonzero.

To define the integral, we first note that the linearization of (2.12)–(2.13) along the solution $(h_1, h_2)(\xi)$ is (2.14) with $\sigma = c$ and $(u_1, u_2) = (h_1, h_2)(\xi)$. The adjoint equation is

$$\begin{pmatrix} \phi_1' & \phi_2' \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} c & \frac{c}{\beta} \\ -\frac{\beta}{c}\partial_{u_1}\omega(h_1, h_2)(\xi) & -\frac{\beta}{c}\partial_{u_2}\omega(h_1, h_2)(\xi) \end{pmatrix}. \tag{2.16}$$

Up to scalar multiplication, (2.16) has the unique bounded solution $(h_1', h_2')(\xi)$. Let

$$\begin{pmatrix} \phi_1^*(\xi) & \phi_2^*(\xi) \end{pmatrix} = \exp\left(-\int_0^\xi a(\eta)\, d\eta\right) \begin{pmatrix} -h_2'(\xi) & h_1'(\xi) \end{pmatrix}, \tag{2.17}$$

with $a(\xi) = -c + \frac{\beta}{c}\partial_{u_2}\omega(h_1, h_2)(\xi)$, the trace of (2.14) with $\sigma = c$ and $(u_1, u_2) = (h_1, h_2)(\xi)$. The facts gathered in the following proposition are easily shown.

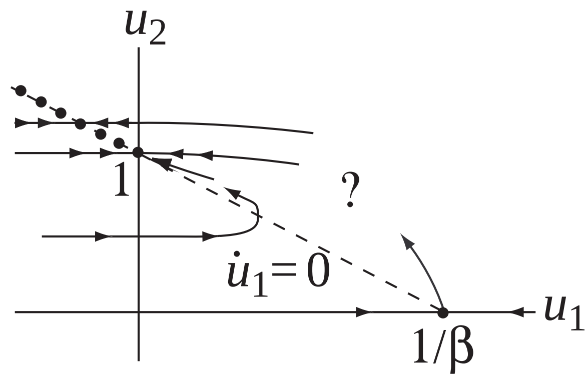

FIGURE 2.1. Phase portrait of (2.12)–(2.13). There is a unique $\sigma = c > 0$ for which the unstable manifold of $(\frac{1}{\beta}, 0)$ meets the stable manifold of $(0, 1)$. For smaller σ, the unstable manifold of $(\frac{1}{\beta}, 0)$ misses above; for larger σ, the unstable manifold of $(\frac{1}{\beta}, 0)$ approaches $(0, 1)$ along its center direction.

Proposition 2.1. *Up to scalar multiplication,* (2.17) *is the unique solution of* (2.16) *that approaches* $(0,0)$ *as* $\xi \to -\infty$. *For all* ξ, $\phi_1^*(\xi) < 0$ *and* $\phi_2^*(\xi) < 0$. *As* $\xi \to -\infty$, (2.17) *approaches* $(0,0)$ *like* $e^{c\xi}$. *There is a number* $k > 0$ *such that as* $\xi \to \infty$, (2.17) *approaches* $(0, -k)$ *exponentially.*

From [43] we have

$$
\begin{aligned}
M &= \int_{-\infty}^{\infty} \begin{pmatrix} \phi_1^*(\xi) & \phi_2^*(\xi) \end{pmatrix} \begin{pmatrix} \partial_\sigma g_1(h_1(\xi), h_2(\xi), c) \\ \partial_\sigma g_2(h_1(\xi), h_2(\xi), c) \end{pmatrix} dt \\
&= \int_{-\infty}^{\infty} \exp\left(-\int_0^{\xi} a(\eta)\, d\eta\right) \begin{pmatrix} -h_2'(\xi) & h_1'(\xi) \end{pmatrix} \begin{pmatrix} -h_1(\xi) - \frac{1}{\beta} h_2(\xi) + \frac{1}{\beta} \\ -\frac{\beta}{c^2}\omega(h_1(\xi), h_2(\xi)) \end{pmatrix} dt \\
&= \int_{-\infty}^{\infty} \exp\left(-\int_0^{\xi} a(\eta)\, d\eta\right) \begin{pmatrix} -h_2'(\xi) & h_1'(\xi) \end{pmatrix} \begin{pmatrix} \frac{1}{c} h_1'(\xi) \\ -\frac{1}{c} h_2'(\xi) \end{pmatrix} dt \\
&= -\frac{2}{c} \int_{-\infty}^{\infty} \exp\left(-\int_0^{\xi} a(\eta)\, d\eta\right) h_1'(\xi) h_2'(\xi)\, dt > 0
\end{aligned}
$$

because $h_1'(\xi) < 0$ and $h_2'(\xi) > 0$.

The formula for M that we have used is usually given for the case in which both equilibria are hyperbolic. It is also correct for equilibria with a 0 eigenvalue if the position of the equilibrium does not change as the parameter changes [37], which is the case here: for all σ the equilibrium with the 0 eigenvalue is $(0, 1)$.

3. Eigenvalues and Evans function

Throughout this section we consider only $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, since spectral theory is better developed for these spaces.

3.1. **Region of consistent splitting.** Let $\mathcal{X}$ be a Banach space, and let $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ be a closed, densely defined linear operator. Its *resolvent set* $\rho(\mathcal{L})$ is the set of $\lambda \in \mathbb{C}$ such that $\mathcal{L} - \lambda\mathcal{I}$ has a bounded inverse. The complement of $\rho(\mathcal{L})$ is the *spectrum* $\mathrm{Sp}(\mathcal{L})$. It is the union of the *discrete spectrum* $\mathrm{Sp_d}(\mathcal{L})$, which is the set of isolated eigenvalues of $\mathcal{L}$ of finite algebraic multiplicity, and the *essential spectrum* $\mathrm{Sp_{ess}}(\mathcal{L})$, which is the rest.

$\mathcal{L} - \lambda I$ is *Fredholm of index zero* if its range is closed, its kernel has finite dimension n, its range has finite codimension m, and $n = m$. The *Fredholm resolvent set* $\rho_{\mathrm{F}}(\mathcal{L})$ is the set of all $\lambda \in \mathbb{C}$ such that $\mathcal{L} - \lambda\mathcal{I}$ is Fredholm of index zero. The set $\rho_{\mathrm{F}}(\mathcal{L})$ is open, and its complement is contained in $\mathrm{Sp_{ess}}(\mathcal{L})$.

Let $BUC^k(\mathbb{R})$ be the space of functions v on $\mathbb{R}$ such that $v, v', \dots, v^{(k)} \in BUC(\mathbb{R})$, with the norm of v equal to $\|v\|_{L^\infty} + \|v'\|_{L^\infty} + \ldots + \|v^{(k)}\|_{L^\infty}$.

Let $V = (v_1, v_2)$. The linearization of (2.1)–(2.2), with $\sigma = c$, at $(h_1, h_2)(\xi)$ is

$$\partial_t V = AV, \quad A = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi + \partial_{u_1}\omega(h_1,h_2)(\xi) & \partial_{u_2}\omega(h_1,h_2)(\xi) \\ -\beta\partial_{u_1}\omega(h_1,h_2)(\xi) & c\partial_\xi - \beta\partial_{u_2}\omega(h_1,h_2)(\xi) \end{pmatrix}. \tag{3.1}$$

The mapping $V \mapsto AV$ yields a closed, densely defined linear operator $\mathcal{A}_\alpha$ on $\mathcal{E}_\alpha^2$. The domain of $\mathcal{A}_\alpha$ is the direct sum of the domains of the operators $\partial_{\xi\xi}$ and ∂_ξ. For $\mathcal{E}_0 = L^2(\mathbb{R})$, the domain of ∂_ξ (respectively $\partial_{\xi\xi}$) is the set of functions v such that $e^{\alpha\xi}v$ belongs to $H^1(\mathbb{R})$ (respectively $H^2(\mathbb{R})$). For $\mathcal{E}_0 = BUC(\mathbb{R})$, $H^1(\mathbb{R})$ and $H^2(\mathbb{R})$ should be replaced by $BUC^1(\mathbb{R})$ and $BUC^2(\mathbb{R})$ respectively.

The complex number λ is an eigenvalue of $\mathcal{A}_\alpha$ if there is a nontrivial V in the domain of $\mathcal{A}_\alpha$ such that

$$\lambda V = AV. \tag{3.2}$$

Let $\mathbf{v} = (v_1, v_2, v_3)$, and let

$$B(\xi) = \begin{pmatrix} 0 & 0 & 1 \\ \frac{\beta}{c}\partial_{u_1}\omega(h_1,h_2)(\xi) & \frac{\beta}{c}\partial_{u_2}\omega(h_1,h_2)(\xi) & 0 \\ -\partial_{u_1}\omega(h_1,h_2)(\xi) & -\partial_{u_2}\omega(h_1,h_2)(\xi) & -c \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then (3.2) can be written as the first-order system $\mathbf{v}_\xi = (B(\xi) + \lambda C)\mathbf{v}$. The complex number λ is an eigenvalue of $\mathcal{A}_\alpha$ provided $\mathbf{v}_\xi = (B(\xi) + \lambda C)\mathbf{v}$ has a solution $\mathbf{v}(\xi)$ in $\mathcal{E}_\alpha^3$. We remark that $v_{1\xi}$ is in the same space as v_1 because V is in the domain of $\mathcal{A}_\alpha$ on $\mathcal{E}_\alpha^2$.

We note that the system $\mathbf{u}' = f(\mathbf{u}, c)$, given by (2.6)–(2.8) with $\sigma = c$, has the solution $\mathbf{h}(\xi) = (h_1, h_2, h_1')(\xi)$, and the linearization of $\mathbf{u}' = f(\mathbf{u}, c)$ along $\mathbf{h}(\xi)$ is

$$\mathbf{v}' = \partial_{\mathbf{u}} f(\mathbf{h}(\xi), c)\mathbf{v} = B(\xi)\mathbf{v}.$$

Define a closed, densely defined linear operator $\mathcal{T}_\alpha^\lambda$ on $\mathcal{E}_\alpha^3$ by $\mathbf{v} \mapsto \mathbf{v}_\xi - (B(\xi) + \lambda C)\mathbf{v}$.

The *region of consistent splitting for the weight function* $e^{\alpha\xi}$ is the set of λ such that the matrices $B(\infty) + \lambda C$ and $B(-\infty) + \lambda C$ have no eigenvalue with real part $-\alpha$, and have the same number of eigenvalues with real part greater than $-\alpha$.

Proposition 3.1. *Let $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. Then:*

(1) *$\mathcal{A}_\alpha - \lambda I$ is Fredholm of index zero if and only if $\mathcal{T}_\alpha^\lambda$ is Fredholm of index zero.*
(2) *$\mathcal{T}_\alpha^\lambda$ is Fredholm of index zero if and only if λ is in the* region of consistent splitting for the weight function $e^{\alpha\xi}$.
(3) *The null spaces and generalized eigenspaces of $\mathcal{A}_\alpha - \lambda I$ and $\mathcal{T}_\alpha^\lambda$ have the same dimension.*

Proof. (1) is proved in [36]. (Sandstede and Scheel do not discuss weighted spaces or systems in which some equations do not have a second-derivative term, but the argument would be similar.) (2) follows from results of Palmer [29, 30]. For (3), see [33]. □

The first two results imply that $\rho_{\mathrm{F}}(\mathcal{A}_\alpha)$ is precisely the region of consistent splitting for the weight function $e^{\alpha\xi}$. We denote by Ω_α the component of $\rho_{\mathrm{F}}(\mathcal{A}_\alpha)$ that is unbounded at the right. The boundary of Ω_α is contained in the set of λ for which $B(\infty)+\lambda C$ or $B(-\infty)+\lambda C$ has an eigenvalue with real part $-\alpha$. This set is contained in the essential spectrum of $\mathcal{A}_\alpha$. We shall identify it using (2.15).

In the remainder of the paper we shall always assume, usually without mention, that

$$0 < \alpha < \frac{1}{2}c. \tag{3.3}$$

This assumption implies that

$$(h_1', h_2') \in \mathcal{E}_\alpha^2, \tag{3.4}$$

and that $\alpha^2 - c\alpha < 0$, which we shall need. It also achieves some ease of exposition, as we shall see.

The matrix $B(\infty) + \lambda C$ has the eigenvalues $\frac{\lambda}{c}$ and $-\frac{1}{2}(c \pm \sqrt{c^2 + 4\lambda})$. Let $\lambda = \gamma + i\theta$. One of these eigenvalues has real part $-\alpha$ provided $\gamma = -c\alpha$ or $\gamma = \alpha^2 - c\alpha - \frac{\theta^2}{(c-2\alpha)^2}$. (Here assumption (3.3) has been used to avoid the case $\alpha = \frac{1}{2}c$.) Let $\gamma_+^\alpha(\theta) = \max(-c\alpha, \alpha^2 - c\alpha - \frac{\theta^2}{(c-2\alpha)^2})$. If $\gamma > \gamma_+^\alpha(\theta)$, then $B(\infty) + \lambda C$ has one eigenvalue with real part smaller than $-\alpha$ and two eigenvalues with real part greater than $-\alpha$.

The matrix $B(-\infty) + \lambda C$ has the eigenvalues $\frac{\beta}{c}e^{-\beta} + \frac{\lambda}{c}$ and $-\frac{1}{2}(c \pm \sqrt{c^2 + 4\lambda})$. Again let $\lambda = \gamma + i\theta$, and let $\gamma_-^\alpha(\theta) = \max(-c\alpha - \beta e^{-\beta}, \alpha^2 - c\alpha - \frac{\theta^2}{(c-2\alpha)^2})$. If $\gamma > \gamma_-^\alpha(\theta)$, then $B(-\infty) + \lambda C$ also has one eigenvalue with real part smaller than $-\alpha$ and two eigenvalues with real part greater than $-\alpha$.

Let $\gamma^\alpha(\theta) = \max(\gamma_+^\alpha(\theta), \gamma_-^\alpha(\theta)) = \gamma_+^\alpha(\theta)$. See Figure 3.1. We have shown:

Proposition 3.2. *Let $\lambda = \gamma + i\theta$. For $0 < \alpha < \frac{1}{2}c$, the graph of $\gamma = \gamma^\alpha(\theta)$ is contained in $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{A}_\alpha)$, and $\Omega_\alpha = \{\lambda = \gamma + i\theta : \gamma > \gamma^\alpha(\theta)\}$. The complement of Ω_α is contained in $\mathrm{Re}\,\lambda \leq \alpha^2 - c\alpha < 0$.*

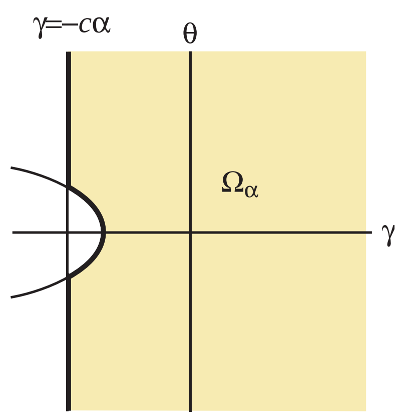

FIGURE 3.1. The curve $\gamma = \gamma^\alpha(\theta)$ (thick) with $0 < \alpha < \frac{1}{2}c$, and the region Ω_α.

The same calculations show that the imaginary axis is in $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{A}_0)$, and Ω_0 is just the set of λ with $\mathrm{Re}\,\lambda > 0$.

In Ω_α the matrices $B(\infty) + \lambda C$ and $B(-\infty) + \lambda C$ both have one eigenvalue with real part smaller than $-\alpha$ and two eigenvalues with real part greater than $-\alpha$. For $\lambda \in \Omega_\alpha$, we denote the eigenvalue of $B(\infty)+\lambda C$ with real part less than $-\alpha$ by $\mu(\lambda) = -\frac{1}{2}(c+\sqrt{c^2+4\lambda})$, and we denote the corresponding 1-dimensional eigenspace $E_+(\lambda)$. We denote the 2-dimensional eigenspace of $B(-\infty)+\lambda C$ for the eigenvalues with real part greater than $-\alpha$ by $E_-(\lambda)$. If $\lambda \in \Omega_\alpha$, then λ is an eigenvalue of $\mathcal{A}_\alpha$ if and only if there is a solution of $\mathbf{v}' = (B(\xi)+\lambda C)\mathbf{v}$ such that $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ lies in $E_-(\lambda)$ at $\xi=-\infty$ and in $E_+(\lambda)$ at $\xi=\infty$. For $\lambda = 0$, $\mathbf{h}'$ satisfies this condition.

3.2. **Change of variables.** Because of the previous section, we are interested in the linear system $\mathbf{v}' = (B(\xi)+\lambda C)\mathbf{v}$. We shall also need to consider its adjoint system $\boldsymbol{\phi}' = -\boldsymbol{\phi}(B(\xi)+\bar\lambda C)$, and the product $\bar{\boldsymbol{\phi}}(\xi,\bar\lambda)\mathbf{v}(\xi,\lambda)$, where $\mathbf{v}(\xi,\lambda)$ and $\boldsymbol{\phi}(\xi,\bar\lambda)$ are solutions of these equations. (We use an overbar to denote complex conjugation.)

Recall that the change of variable $\mathbf{y} = P(c)\mathbf{u}$ used in Section 2 converts the system $\mathbf{u}' = f(\mathbf{u},c)$ to $\mathbf{y}' = g(\mathbf{y},c)$, with $g(\mathbf{y},c) = P(c)f(P(c)^{-1}\mathbf{y},c)$. The system $\mathbf{y}' = g(\mathbf{y},c)$ has the solution

$$\mathbf{k}(\xi) = P(c)\mathbf{h}(\xi) = (h_1(\xi), h_2(\xi), \frac{c}{\beta}).$$

It approaches $(\frac{1}{\beta}, 0, \frac{c}{\beta})$ exponentially as $\xi \to -\infty$, and approaches $(0,1,\frac{c}{\beta})$ exponentially as $\xi \to \infty$.

Let

$$E(\xi) = P(c)B(\xi)P(c)^{-1} = \begin{pmatrix} -c & -\frac{c}{\beta} & 1 \\ \frac{\beta}{c}\partial_{u_1}\omega(h_1,h_2)(\xi) & \frac{\beta}{c}\partial_{u_2}\omega(h_1,h_2)(\xi) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$F = P(c)CP(c)^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 1 & \frac{1}{\beta} & 0 \end{pmatrix}.$$

Then the linearization of $\mathbf{y}' = g(\mathbf{y},c)$ along $\mathbf{k}(\xi)$ is $\mathbf{z}' = E(\xi)\mathbf{z}$, with $\mathbf{z} = (z_1,z_2,z_3)$.

The system $\mathbf{z}' = E(\xi)\mathbf{z}$ can also be obtained by applying the change of variables $\mathbf{z} = P(c)\mathbf{v}$ to the linear system $\mathbf{v}' = B(\xi)\mathbf{v}$. Applying this change of variable to the linear system $\mathbf{v}' = (B(\xi)+\lambda C)\mathbf{v}$, we obtain $\mathbf{z}' = (E(\xi)+\lambda F)\mathbf{z}$. Making the corresponding change of variables $\boldsymbol{\psi} = \boldsymbol{\phi}P(c)^{-1}$ in the adjoint system $\boldsymbol{\phi}' = -\boldsymbol{\phi}(B(\xi)+\bar\lambda C)$, we obtain $\boldsymbol{\psi}' = -\boldsymbol{\psi}(E(\xi)+\bar\lambda F)$, which is the adjoint system to $\mathbf{z}' = (E(\xi)+\lambda F)\mathbf{z}$. The product of solutions is unchanged:

$$\bar{\boldsymbol{\psi}}(\xi)\mathbf{z}(\xi) = \bar{\boldsymbol{\phi}}(\xi)P(c)^{-1}P(c)\mathbf{v}(\xi) = \bar{\boldsymbol{\phi}}(\xi)\mathbf{v}(\xi).$$

Thus, instead of studying $\mathbf{v}' = (B(\xi)+\lambda C)\mathbf{v}$ and its adjoint, we will, when convenient, study $\mathbf{z}' = (E(\xi)+\lambda F)\mathbf{z}$ and its adjoint. The asymptotic behavior of corresponding solutions is of course the same.

3.3. **Evans function.** For $\lambda \in \Omega_\alpha$, a right eigenvector of $E(\infty)+\lambda F$ for the eigenvalue $\mu(\lambda)$ is

$$\hat{\mathbf{z}}(\lambda) = \begin{pmatrix} -1 \\ 0 \\ -(c+\mu(\lambda)) \end{pmatrix}.$$

Let $\mathbf{z}(\xi,\lambda)$ be the unique solution of $\mathbf{z}' = (E(\xi)+\lambda F)\mathbf{z}$ such that

$$\lim_{\xi\to\infty} e^{-\mu(\lambda)\xi}\mathbf{z}(\xi,\lambda) = \hat{\mathbf{z}}(\lambda).$$

$\mathbf{z}(\xi,0)$ is a positive multiple of $\mathbf{k}'(\xi)=(h_1'(\xi),h_2'(\xi),0)$, which is a solution of $\mathbf{z}'=E(\xi)\mathbf{z}$. Note that $h_1'(\xi)<0$; that's why we chose $\hat{\mathbf{z}}(\lambda)$ to have its first component negative.

If $\lambda\in\Omega_\alpha$, then $\bar\lambda\in\Omega_\alpha$, and the unique eigenvalue of $-(E(-\infty)+\bar\lambda F)$ with real part greater than $-\alpha$ is $-\mu(\bar\lambda)=-\overline{\mu(\lambda)}=\frac{1}{2}(c+\sqrt{c^2+4\bar\lambda})$. A corresponding left eigenvector is $\hat{\boldsymbol\psi}(\bar\lambda)$, where

$$\hat{\boldsymbol\psi}(\lambda)=\begin{pmatrix}\mu(\lambda) & \frac{\mu(\lambda)^2c}{\beta\left(\mu(\lambda)c-\beta e^{-1/\beta}-\lambda\right)} & 1\end{pmatrix}.$$

Let $\boldsymbol\psi(\xi,\bar\lambda)$ be the unique solution of $\boldsymbol\psi'=-\boldsymbol\psi(E(\xi)+\bar\lambda F)$ such that

$$\lim_{\xi\to-\infty}e^{\mu(\bar\lambda)\xi}\boldsymbol\psi(\xi,\bar\lambda)=\hat{\boldsymbol\psi}(\bar\lambda).$$

Let $\boldsymbol\psi^*(\xi)=\boldsymbol\psi(\xi,0)$.

Recall $\begin{pmatrix}\phi_1^*(\xi) & \phi_2^*(\xi)\end{pmatrix}$ defined by (2.17), and define

$$\phi_3^*(\xi)=-\int_{-\infty}^{\xi}\phi_1^*(\eta)\,d\eta.$$

Proposition 3.3. *As $\xi\to-\infty$, $\phi_3^*(\xi)\to0$ like $e^{c\xi}$; and there is a number $d>0$ such that as $\xi\to\infty$, $\phi_3^*(\xi)\to d$ exponentially. $\boldsymbol\psi^*(\xi)$ is a positive multiple of $\begin{pmatrix}\phi_1^*(\xi) & \phi_2^*(\xi) & \phi_3^*(\xi)\end{pmatrix}$.*

Proof. $\boldsymbol\psi^*(\xi)$ satisfies $\boldsymbol\psi'=-\boldsymbol\psi E(\xi)$. Therefore $\begin{pmatrix}\psi_1^*(\xi) & \psi_2^*(\xi)\end{pmatrix}$ satisfies (2.16) and, of course, approaches $\begin{pmatrix}0 & 0\end{pmatrix}$ as $\xi\to-\infty$. Hence $\begin{pmatrix}\psi_1^*(\xi) & \psi_2^*(\xi)\end{pmatrix}$ is a scalar multiple of $\begin{pmatrix}\phi_1^*(\xi) & \phi_2^*(\xi)\end{pmatrix}$. Defining $\phi_3^*(\xi)$ as above, we see that $\begin{pmatrix}\phi_1^*(\xi) & \phi_2^*(\xi) & \phi_3^*(\xi)\end{pmatrix}$ solves $\boldsymbol\psi'=-\boldsymbol\psi E(\xi)$ and approaches $\begin{pmatrix}0 & 0 & 0\end{pmatrix}$ as $\xi\to-\infty$. Therefore $\boldsymbol\psi^*(\xi)$ is a multiple of $\begin{pmatrix}\phi_1^*(\xi) & \phi_2^*(\xi) & \phi_3^*(\xi)\end{pmatrix}$. The formula for $\phi_3^*(\xi)$ and Proposition 2.1 show that $\phi_3^*(\xi)\to d>0$ as $\xi\to\infty$. Since $\psi_3^*(\xi)$ is also positive, the multiple is positive. □

For $\lambda\in\Omega_\alpha$, solutions $\mathbf{z}(\xi)$ of $\mathbf{z}'=(E(\xi)+\lambda F)\mathbf{z}$ have $e^{\alpha\xi}\mathbf{z}(\xi)$ bounded as $\xi\to-\infty$ if and only if $\bar{\boldsymbol\psi}(\xi,\bar\lambda)\mathbf{z}(\xi)=0$ (a property that does not depend on ξ).

On Ω_α we define the (mixed) Evans function

$$D(\lambda)=\bar{\boldsymbol\psi}(\xi,\bar\lambda)\mathbf{z}(\xi,\lambda).\tag{3.5}$$

(The product is independent of ξ.) Note that $\{\lambda:\operatorname{Re}\lambda\ge0\}$ is contained in Ω_α for any choice of α, and for $\operatorname{Re}\lambda\ge0$, the definition of $D(\lambda)$ is independent of the choice of α.

For $\lambda\in\Omega_\alpha$, λ is an eigenvalue of (3.1) with eigenfunction in $\mathcal{E}_\alpha$ if and only if $D(\lambda)=0$. It is known that $D(\lambda)$ is analytic; $D(\lambda)\neq0$ if and only if $\lambda\in\rho(\mathcal{A}_\alpha)\cap\Omega_\alpha$; and the algebraic multiplicity of λ as an eigenvalue of $\mathcal{A}_\alpha$ equals its multiplicity as a root of $D(\lambda)$ [33]. Since $D(\lambda)$ is not identically 0 (by, for example, Theorem 3.4 below), it follows that every point of Ω_α is in either $\rho(\mathcal{A}_\alpha)$ or $\mathrm{Sp_d}(\mathcal{A}_\alpha)$.

Since $\begin{pmatrix}\phi_1^*(\xi) & \phi_2^*(\xi)\end{pmatrix}\begin{pmatrix}h_1'(\xi)\\ h_2'(\xi)\end{pmatrix}=0$, and $D(0)$ is a positive multiple of the product of $\begin{pmatrix}\phi_1^*(\xi) & \phi_2^*(\xi) & \phi_3^*(\xi)\end{pmatrix}$ and $\mathbf{k}'(\xi)=(h_1'(\xi),h_2'(\xi),0)$, we have $D(0)=0$.

3.4. **Formula for $D'(0)$.** Sandstede [33] gives the formula: up to multiplication by a positive number,

$$D'(0)=-\int_{-\infty}^{\infty}\boldsymbol\psi^*(\xi)F\mathbf{k}'(\xi)\,d\xi.\tag{3.6}$$

He states this formula for the case in which both equilibria are hyperbolic. Since our equilibria have 0 eigenvalues, we shall rederive the formula for our case.

We denote the solution of $\mathbf{z}' = E(\xi)\mathbf{z}$ by $\mathbf{z}(\xi) = \Phi(\xi,\eta)\mathbf{z}(\eta)$. Then the solution of the adjoint equation $\boldsymbol{\psi}' = -\boldsymbol{\psi}E(\xi)$ is $\boldsymbol{\psi}(\eta) = \boldsymbol{\psi}(\xi)\Phi(\xi,\eta)$.

Let $D(\lambda)$ be given by (3.5) with $\xi = 0$. Then for λ real we have

$$D'(\lambda) = \partial_\lambda\boldsymbol{\psi}(0,\lambda)\mathbf{z}(0,\lambda) + \boldsymbol{\psi}(0,\lambda)\partial_\lambda\mathbf{z}(0,\lambda). \tag{3.7}$$

To calculate $\partial_\lambda\boldsymbol{\psi}(0,0)$, we note that

$$\begin{aligned}\partial_{\xi\lambda}\boldsymbol{\psi}(\xi,\lambda) = \partial_{\lambda\xi}\boldsymbol{\psi}(\xi,\lambda) &= \partial_\lambda(-\boldsymbol{\psi}(\xi,\lambda)(E(\xi)+\lambda F))\\ &= -\partial_\lambda\boldsymbol{\psi}(\xi,\lambda)(E(\xi)+\lambda F) - \boldsymbol{\psi}(\xi,\lambda)F.\end{aligned}$$

In other words, $\partial_\lambda\boldsymbol{\psi}(\xi,\lambda)$, satisfies the linear differential equation

$$\boldsymbol{\psi}' = -\boldsymbol{\psi}(E(\xi)+\lambda F) - \boldsymbol{\psi}(\xi,\lambda)F.$$

Therefore, by the variation of constants formula,

$$\partial_\lambda\boldsymbol{\psi}(0,0) = \partial_\lambda\boldsymbol{\psi}(-T,0)\Phi(-T,0) - \int_{-T}^{0}\boldsymbol{\psi}^*(\eta)F\Phi(\eta,0)\,d\eta. \tag{3.8}$$

By an analogous argument,

$$\partial_\lambda\mathbf{z}(0,0) = \Phi(0,T)\partial_\lambda\mathbf{z}(T,0) + \int_{T}^{0}\Phi(0,\eta)F\mathbf{z}(\eta,0)\,d\eta. \tag{3.9}$$

We multiply (3.8) on the right by $\mathbf{z}(0,0)$, (3.9) on the left by $\boldsymbol{\psi}(0,0) = \boldsymbol{\psi}^*(0)$, and add. Using (3.7) with $\lambda = 0$, we obtain

$$D'(0) = \partial_\lambda\boldsymbol{\psi}(-T,0)\mathbf{z}(-T,0) + \boldsymbol{\psi}^*(T)\partial_\lambda\mathbf{z}(T,0) - \int_{-T}^{T}\boldsymbol{\psi}^*(\eta)F\mathbf{z}(\eta,0)\,d\eta. \tag{3.10}$$

Let $T\to\infty$ in (3.10). Then $\partial_\lambda\boldsymbol{\psi}(-T,0)\to 0$, $\mathbf{z}(-T,0)\to 0$, $\boldsymbol{\psi}^*(T)$ is bounded, and $\partial_\lambda\mathbf{z}(T,0)\to 0$. We obtain (3.6) up to multiplication by a positive number.

3.5. **Calculation of $D'(0)$.**

Theorem 3.4. *Let D be given by* (3.5)*. Then $D'(0) > 0$.*

Proof. Up to multiplication by a positive number, we calculate:

$$\begin{aligned}D'(0) &= -\int_{-\infty}^{\infty}\boldsymbol{\psi}^*(\xi)F\mathbf{k}'(\xi)\,d\xi\\ &= -\int_{-\infty}^{\infty}\begin{pmatrix}\psi_1^*(\xi) & \psi_2^*(\xi) & \psi_3^*(\xi)\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{1}{c} & 0\\ 1 & \frac{1}{\beta} & 0\end{pmatrix}\begin{pmatrix}h_1'(\xi)\\ h_2'(\xi)\\ 0\end{pmatrix}d\xi\\ &= -\frac{1}{c}\int_{-\infty}^{\infty}\psi_2^*(\xi)h_2'(\xi)\,d\xi - \int_{-\infty}^{\infty}\psi_3^*(\xi)(h_1'(\xi) + \frac{1}{\beta}h_2'(\xi))\,d\xi.\\ &= -\frac{1}{c}\int_{-\infty}^{\infty}\psi_2^*(\xi)h_2'(\xi)\,d\xi + \frac{1}{c}\int_{-\infty}^{\infty}\psi_3^*(\xi)h_1''(\xi)\,d\xi.\end{aligned}$$

We integrate the second integral by parts:

$$\int_{-\infty}^{\infty}\psi_3^*(\xi)h_1''(\xi)\,d\xi = \psi_3^*(\infty)h_1'(\infty) - \psi_3^*(-\infty)h_1'(-\infty) - \int_{-\infty}^{\infty}(\psi_3^*)'(\xi)h_1'(\xi)\,d\xi.$$

We have $\psi_3^*(\infty)$ finite, $h_1'(\infty)=0$, $\psi_3^*(-\infty)=0$, and $h_1'(-\infty)=0$. Therefore the boundary terms vanish. We conclude that, up to multiplication by a positive number,

$$D'(0)=\frac{1}{c}\int_{-\infty}^{\infty}-\psi_2^*(\xi)h_2'(\xi)-(\psi_3^*)'(\xi)h_1'(\xi)\,d\xi=\frac{1}{c}\int_{-\infty}^{\infty}-\psi_2^*(\xi)h_2'(\xi)+\psi_1^*(\xi)h_1'(\xi)\,d\xi$$
$$=-\frac{2}{c}\int_{-\infty}^{\infty}\exp\left(-\int_0^{\xi}a(\eta)\,d\eta\right)h_1'(\xi)h_2'(\xi)\,d\xi>0. \quad (3.11)$$

□

Note that $D'(0)$ equals the Melnikov integral calculated in Section 2. We discuss this fact in the next section.

3.6. **Relation to separation functions.** According to Sandstede [33], an alternative expression for $D'(0)$, up to multiplication by a positive number, is

$$D'(0)=\int_{-\infty}^{\infty}\boldsymbol{\psi}^*(\xi)\partial_\sigma g(\mathbf{k}(\xi),c)\,d\xi. \quad (3.12)$$

To check this in our problem, we note that

$$\boldsymbol{\psi}^*(\xi)\partial_\sigma g(\mathbf{k}(\xi),c)=\begin{pmatrix}\psi_1^*(\xi) & \psi_2^*(\xi) & \psi_3^*(\xi)\end{pmatrix}\begin{pmatrix}-h_1(\xi)-\frac{1}{c}h_2(\xi)\\ -\frac{\beta}{c^2}\omega(h_1,h_2)(\xi)\\ 0\end{pmatrix}=\frac{1}{c}(\psi_1^*(\xi)h_1'(\xi)-\psi_2^*(\xi)h_2'(\xi))$$

and compare (3.11).

The right-hand side of (3.12) can be interpreted as the derivative of the separation function between two invariant manifolds. In fact, for the system (2.9)–(2.11), with parameter $\sigma>0$, consider the line of equilibria $L_\sigma=\{(u_1,0,y_3):\ u_1 \text{ arbitrary}, y_3=\sigma u_1\}$. For each σ, $W^s_\sigma(0,1,\frac{\sigma}{\beta})$ is one-dimensional, and $W^u_\sigma(L_\sigma)$ is two-dimensional. One can define a separation function $S(\sigma)$ between these manifolds such that $S(c)=0$ and

$$S'(c)=\int_{-\infty}^{\infty}\boldsymbol{\psi}^*(\xi)\partial_\sigma g(\mathbf{k}(\xi),c)\,d\xi.$$

On the other hand, the intersection of $W^s_\sigma(0,1,\frac{\sigma}{\beta})$ and $W^u_\sigma(L_\sigma)$, if it is nonempy, must lie in the plane $w=\frac{\sigma}{\beta}$. Restricting (2.9)–(2.11) to this plane, and using u_1 and u_2 as coordinates, we obtain the system (2.12)–(2.13), in which $\sigma>0$ is a parameter. The points $(0,1)$ and $(\frac{1}{\beta},0)$ are equilibria, with one-dimensional invariant manifolds $W^s_\sigma(0,1)$ and $W^u_\sigma(\frac{1}{\beta},0)$. $W^s_c(0,1)$ and $W^u_c(\frac{1}{\beta},0)$ meet along the curve $(h_1,h_2)(\xi)$. One can define a separation function $\hat{S}(\sigma)$ between $W^s_\sigma(0,1)$ and $W^u_\sigma(\frac{1}{\beta},0)$ such that $\hat{S}(c)=0$. $\hat{S}'(c)$ is the Melnikov integral M calculated in Section 2.

3.7. $D(\lambda)$ **for large positive** λ**.** To treat large positive λ, it is more convenient to consider the system $\mathbf{v}'=(B(\xi)+\lambda C)\mathbf{v}$ from Subsection 3.2. We consider this system with λ restricted to be real and positive. We write the system as

$$\mathbf{v}'=(B(\xi)+\lambda C)\mathbf{v}, \quad (3.13)$$
$$\xi'=1. \quad (3.14)$$

Let $\lambda = \frac{1}{\delta}$, $\delta > 0$, and let $\xi = \delta\eta$. We obtain

$$\frac{d\mathbf{v}}{d\eta} = (\delta B(\xi) + C)\mathbf{v}, \tag{3.15}$$

$$\frac{d\xi}{d\eta} = \delta. \tag{3.16}$$

The eigenvalues of C are 0, with algebraic multiplicity two, and $\frac{1}{c}$. The generalized eigenspace for the eigenvalue 0 is v_1v_3-space. Therefore, for $\delta = 0$, the product of v_1v_3-space and ξ-space is a normally hyperbolic invariant manifold for (3.15)–(3.16).

Hence (3.15)–(3.16) has, for each small $\delta > 0$, a linear normally hyperbolic invariant manifold near the product of v_1v_3-space and ξ-space, given by

$$v_2 = (a(\xi)v_1 + b(\xi)v_3)\delta + \mathcal{O}(\delta^2). \tag{3.17}$$

It will not be necessary to calculate $a(\xi)$ and $b(\xi)$.

The system (3.15)–(3.16) restricted to this manifold is

$$\begin{pmatrix} \frac{dv_1}{d\eta} \\ \frac{dv_3}{d\eta} \end{pmatrix} = \begin{pmatrix} 0 & \delta \\ 1 + \mathcal{O}(\delta) & \mathcal{O}(\delta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_3 \end{pmatrix}, \tag{3.18}$$

$$\frac{d\xi}{d\eta} = \delta. \tag{3.19}$$

Let $\delta = \epsilon^2$ with $\epsilon > 0$, $v_1 = \epsilon w_1$, and $\zeta = \epsilon\eta$:

$$\begin{pmatrix} \frac{dw_1}{d\eta} \\ \frac{dv_3}{d\eta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 + \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) \end{pmatrix} \begin{pmatrix} w_1 \\ v_3 \end{pmatrix}, \tag{3.20}$$

$$\frac{d\xi}{d\zeta} = \epsilon. \tag{3.21}$$

The eigenvalues of (3.20) with $\epsilon = 0$ are ± 1. Therefore, for $\epsilon = 0$, ξ-space is a normally hyperbolic invariant manifold for (3.20)–(3.21).

For (3.20) with $\epsilon = 0$, eigenvectors corresponding to the eigenvalues ± 1 are $(1, \pm 1)$. Let $(w_1(\zeta, \epsilon), v_3(\zeta, \epsilon))$ be a solution of (3.20) that approaches 0 as $\zeta \to \infty$. Then, up to scalar multiplication,

$$(w_1(0, \epsilon), v_3(0, \epsilon)) = (1, -1 + \mathcal{O}(\epsilon)).$$

Substituting $v_1 = \epsilon w_1$ yields

$$(v_1(0, \epsilon), v_3(0, \epsilon)) = (\epsilon, -1 + \mathcal{O}(\epsilon)).$$

Then, using (3.17), we see that if $\mathbf{v}(\eta, \epsilon)$ is a solution of (3.15), with $\delta = \epsilon^2$, that approaches 0 as $\eta \to \infty$, then $\mathbf{v}(0, \epsilon)$ is a multiple of $(\epsilon, \mathcal{O}(\epsilon^2), -1 + \mathcal{O}(\epsilon))$. We choose $\tilde{\mathbf{v}}(0, \epsilon)$ to be -1 times this vector:

$$\tilde{\mathbf{v}}(0, \epsilon) = (-\epsilon, \mathcal{O}(\epsilon^2), 1 + \mathcal{O}(\epsilon)). \tag{3.22}$$

The reason for this choice is to achieve consistency with the earlier choice of $\mathbf{z}(\xi, \lambda)$, which had its first component negative. Then the corresponding solution $\mathbf{v}(\xi, \lambda)$ of $\mathbf{v}' = (B(\xi) + \lambda C)\mathbf{v}$ also has its first component negative. Therefore, up to multiplication by a positive constant,

$$\mathbf{v}(0, \frac{1}{\epsilon^2}) = \tilde{\mathbf{v}}(0, \epsilon). \tag{3.23}$$

The adjoint equation $\boldsymbol{\phi}' = -\boldsymbol{\phi}(B(\xi)+\lambda C)$ (recall that λ is real) can be treated analogously. In the system

$$\boldsymbol{\phi}' = -\boldsymbol{\phi}(B(\xi) + \lambda C), \tag{3.24}$$

$$\xi' = 1, \tag{3.25}$$

let $\lambda = \frac{1}{\delta}$ with $\delta > 0$ and $\xi = \delta\eta$:

$$\frac{d\boldsymbol{\phi}}{d\eta} = -\boldsymbol{\phi}(\delta B(\xi) + C), \tag{3.26}$$

$$\frac{d\xi}{d\eta} = \delta. \tag{3.27}$$

On the normally hyperbolic invariant manifold

$$\phi_2 = (a(\xi)\phi_1 + b(\xi)\phi_3)\delta + \mathcal{O}(\delta^2),$$

where $a(\xi)$ and $b(\xi)$ need not be computed, we obtain

$$\begin{pmatrix} \frac{d\phi_1}{d\eta} & \frac{d\phi_3}{d\eta} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_3 \end{pmatrix} \begin{pmatrix} \mathcal{O}(\delta^2) & -\delta \\ -1 + \mathcal{O}(\delta) & \delta c \end{pmatrix}, \tag{3.28}$$

$$\frac{d\xi}{d\eta} = \delta. \tag{3.29}$$

Let $\delta = \epsilon^2$ with $\epsilon > 0$, $\epsilon\phi_1 = \chi_1$, and $\zeta = \epsilon\eta$:

$$\begin{pmatrix} \frac{d\chi_1}{d\zeta} & \frac{d\phi_3}{d\zeta} \end{pmatrix} = \begin{pmatrix} \chi_1 & \phi_3 \end{pmatrix} \begin{pmatrix} \mathcal{O}(\epsilon^3) & -1 \\ -1 + \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) \end{pmatrix}, \tag{3.30}$$

$$\frac{d\xi}{d\zeta} = \epsilon. \tag{3.31}$$

The eigenvalues of (3.30) with $\epsilon = 0$ are ± 1. Therefore, for $\epsilon = 0$, ξ-space is a normally hyperbolic invariant manifold for (3.30).

For (3.30) with $\epsilon = 0$, left eigenvectors corresponding to the eigenvalues 1 and -1 are $\begin{pmatrix} 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \end{pmatrix}$ respectively. Let $(\chi_1(\zeta, \epsilon), \phi_3(\zeta, \epsilon))$ be a solution of (3.30) that approaches 0 as $\zeta \to -\infty$. Then, up to scalar multiplication,

$$\begin{pmatrix} \chi_1(0, \epsilon) & \phi_3(0, \epsilon) \end{pmatrix} = \begin{pmatrix} 1 & -1 + \mathcal{O}(\epsilon) \end{pmatrix}.$$

Substituting $\epsilon\phi_1 = \chi_1$ yields

$$\epsilon \begin{pmatrix} \phi_1(0, \epsilon) & \phi_3(0, \epsilon) \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon + \mathcal{O}(\epsilon^2) \end{pmatrix}.$$

Hence if $\boldsymbol{\phi}(\eta, \epsilon)$ is a solution of (3.26), with $\delta = \epsilon^2$, that approaches 0 as $\eta \to -\infty$, then $\boldsymbol{\phi}(0, \epsilon)$ is a multiple of $\begin{pmatrix} 1 & \mathcal{O}(\epsilon^2) & -\epsilon + \mathcal{O}(\epsilon^2) \end{pmatrix}$ We choose $\tilde{\boldsymbol{\phi}}(0, \epsilon)$ to be -1 times this vector:

$$\tilde{\boldsymbol{\phi}}(0, \epsilon) = \begin{pmatrix} -1 & \mathcal{O}(\epsilon^2) & \epsilon + \mathcal{O}(\epsilon^2) \end{pmatrix}. \tag{3.32}$$

The reason for this choice to achieve consistency with the earlier choice of $\boldsymbol{\psi}(\xi, \lambda)$, which had its third component positive. Then the corresponding solution $\boldsymbol{\phi}(\xi, \lambda)$ of $\boldsymbol{\phi}' = -\boldsymbol{\phi}(B(\xi)+\lambda C)$ also has its third component positive. Therefore, up to multiplication by a positive constant,

$$\boldsymbol{\phi}(0, \frac{1}{\epsilon^2}) = \tilde{\boldsymbol{\phi}}(0, \epsilon). \tag{3.33}$$

Using (3.33), (3.23), (3.32), and (3.22), we conclude that, for small $\epsilon > 0$,

$$D(\frac{1}{\epsilon^2}) = \boldsymbol{\phi}(0, \frac{1}{\epsilon^2})\mathbf{v}(0, \frac{1}{\epsilon^2}) = \tilde{\boldsymbol{\phi}}(0, \epsilon)\tilde{\mathbf{v}}(0, \epsilon) = 2\epsilon + \mathcal{O}(\epsilon^2) > 0.$$

Thus $D(\lambda) > 0$ for large positive real λ.

Since $D(0) = 0$ and $D'(0) > 0$, this result is consistent with stability and with the numerical computations of [1].

4. Spectral and linearized stability

Our goal in this section is to prove the following result. We recall the standing assumption that $0 < \alpha < \frac{1}{2}c$. We return to allowing $\mathcal{E}_0$ to be any of $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$.

Theorem 4.1. *Let $\mathcal{E}_0$ be $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$. Suppose the only zero of $D(\lambda)$ in $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ is $\lambda = 0$. Then the traveling wave is spectrally stable and linearly stable in $\mathcal{E}^2_\alpha$.*

If $V(\xi)$ is an element of $\mathcal{E}^2_\alpha$, then $W(\xi) = e^{\alpha\xi}V(\xi)$ is an element of $\mathcal{E}^2_0$, and $\|V\|_\alpha = \|W\|_0$. Let

$$\tilde{A} = \begin{pmatrix} \partial_{\xi\xi} + (c - 2\alpha)\partial_\xi + \alpha^2 - c\alpha + \partial_{u_1}\omega(h_1, h_2) & \partial_{u_2}\omega(h_1, h_2) \\ -\beta\partial_{u_1}\omega(h_1, h_2) & c\partial_\xi - c\alpha - \beta\partial_{u_2}\omega(h_1, h_2) \end{pmatrix}. \tag{4.1}$$

Proposition 4.2. *$V(t, \xi)$ is a solution of $\partial_t V = AV$ with values in $\mathcal{E}^2_\alpha$ if and only if $W(t, \xi) = e^{\alpha\xi}V(t, \xi)$ is a solution of $\partial_t W = \tilde{A}W$ with values in $\mathcal{E}^2_0$.*

The proof is a simple calculation.

Let $\tilde{\mathcal{A}}$ be the closed, densely defined operator on $\mathcal{E}^2_0$ defined by $W \mapsto \tilde{A}W$, with the maximal domain. The operator $\mathcal{M}_\alpha$ defined by $V(\xi) \mapsto e^{\alpha\xi}V(\xi)$ is an isometric isomorphism from $\mathcal{E}^2_\alpha$ to $\mathcal{E}^2_0$ that conjugates $\mathcal{A}_\alpha$ to $\tilde{\mathcal{A}}$: $\mathcal{M}_\alpha\mathcal{A}_\alpha\mathcal{M}_\alpha^{-1} = \tilde{\mathcal{A}}$. Thus we can prove Theorem 4.1 by proving that $\tilde{\mathcal{A}}$ satisfies (S1), (S2), (L1), and (L2) from the Introduction.

4.1. **Semigroup background.** We recall some definitions. A collection of linear operators $\mathcal{T}(t)$, $t \geq 0$, on a Banach space $\mathcal{X}$ is a C_0-*semigroup* provided

1. Each $\mathcal{T}(t)$ is a bounded linear operator from $\mathcal{X}$ to $\mathcal{X}$.
2. $\mathcal{T}(0) = \mathcal{I}$.
3. For all $t, s \geq 0$, $\mathcal{T}(t + s) = \mathcal{T}(t)\mathcal{T}(s)$.
4. For each fixed $u \in \mathcal{X}$, $\mathcal{T}(t)u$ is a continuous function from $[0, \infty)$ into $\mathcal{X}$.

Define a linear operator $\mathcal{L}$ from $\mathcal{X}$ to $\mathcal{X}$ by $\mathcal{L}u = \lim_{t\to 0+} \frac{1}{t}(\mathcal{T}(t)u - u)$. The set of u for which the limit exists is $\mathrm{D}(\mathcal{L})$, the domain of $\mathcal{L}$. $\mathrm{D}(\mathcal{L})$ is a dense subset of $\mathcal{X}$, and $\mathcal{L}$ is a closed linear operator, but $\mathcal{L}$ is typically unbounded. One says that $\mathcal{L}$ *generates* the C_0-semigroup $\mathcal{T}(t)$, and we write $\mathcal{T}(t) = e^{t\mathcal{L}}$, $t \geq 0$.

Suppose λ_0 is an isolated point of $\mathrm{Sp}(\mathcal{L})$, i.e., $\lambda_0 \in \mathrm{Sp}(\mathcal{L})$ and there is a number $d > 0$ such that for $0 < |\lambda - \lambda_0| < d$, $\lambda \in \rho(\mathcal{L})$. Define the (Riesz) spectral projection

$$\mathcal{P}_{\lambda_0}(\mathcal{L}) = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{L} - \lambda\mathcal{I})^{-1}\, d\lambda, \tag{4.2}$$

where Γ is a simple closed curve in $\{\lambda : 0 < |\lambda - \lambda_0| < d\}$ that surrounds λ_0. Then the range and kernel of $\mathcal{P}_{\lambda_0}(\mathcal{L})$ are both closed and invariant under $\mathcal{L}$; the restriction of $\mathcal{L}$ to the range of $\mathcal{P}_{\lambda_0}(\mathcal{L})$ has spectrum equal to $\{\lambda_0\}$; and the restriction of $\mathcal{L}$ to the kernel of $\mathcal{P}_{\lambda_0}(\mathcal{L})$ has spectrum equal to $\mathrm{Sp}(\mathcal{L}) \setminus \{\lambda_0\}$. We have

$$(\mathcal{L} - \lambda\mathcal{I})^{-1} = -\mathcal{P}_{\lambda_0}(\mathcal{L})(\lambda - \lambda_0)^{-1} - \sum_{n=1}^{\infty} \mathcal{D}^n(\lambda - \lambda_0)^{-(n+1)} + \sum_{n=0}^{\infty} \mathcal{S}^{n+1}(\lambda - \lambda_0)^n, \tag{4.3}$$

where $\mathcal{D}$ and $\mathcal{S}$ are bounded operators on the range and kernel of $\mathcal{P}_{\lambda_0}(\mathcal{L})$ respectively, both commute with $\mathcal{P}_{\lambda_0}(\mathcal{L})$, and $\mathrm{Sp}(\mathcal{D}) = \{0\}$. The number λ_0 is called an *isolated eigenvalue of $\mathcal{L}$ of finite algebraic multiplicity* n provided the range of $\mathcal{P}_{\lambda_0}(\mathcal{L})$ has dimension n [23, Sec. III.6.5]. This term was used in Section 3 to define the discrete spectrum $\mathrm{Sp_d}(\mathcal{L})$, but we did not give a precise definition there.

In order to discuss bounds on the spectrum of a linear operator $\mathcal{L}$, the following definitions are useful. For $\omega \in \mathbb{R}$ let $\mathbb{C}_\omega = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > \omega\}$. Then we define:

- The *spectral bound* $\mathrm{s}(\mathcal{L}) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L})\}$.
- The *essential spectral bound* $\mathrm{s_{ess}}(\mathcal{L})$, the infimum of all real ω such that $\mathrm{Sp}(\mathcal{L}) \cap \mathbb{C}_\omega$ is a subset of $\mathrm{Sp_d}(\mathcal{L})$ and has only finitely many points.

For a *bounded* linear operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$, we define:

- The *spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \mathrm{Sp}(\mathcal{T})\}$.
- The *essential spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \mathrm{Sp_{ess}}(\mathcal{T})\}$.
- The seminorm
$$\|\mathcal{T}\|_C = \inf_{\mathcal{K}} \|\mathcal{T} + \mathcal{K}\|,$$
where the infimum is over the set of all compact operators $\mathcal{K} : \mathcal{X} \to \mathcal{X}$.

If $\mathcal{L}$ generates a C_0-semigroup $\mathcal{T}(t)$, $t \geq 0$, we define:

- The *growth bound* $\omega(\mathcal{L}) = \lim_{t\to\infty} t^{-1} \log \|\mathcal{T}(t)\|$.
- The *essential growth bound* $\omega_{\mathrm{ess}}(\mathcal{L}) = \lim_{t\to\infty} t^{-1} \log \|T(t)\|_C$.

Proposition 4.3. *Suppose $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ generates the C_0-semigroup $e^{t\mathcal{L}}$, $t \geq 0$. Then*

(1) *For each $t > 0$, $e^{t\,\mathrm{Sp}(\mathcal{L})} \subset \mathrm{Sp}(e^{t\mathcal{L}})$ and $e^{t\,\mathrm{Sp_{ess}}(\mathcal{L})} \subset \mathrm{Sp_{ess}}(e^{t\mathcal{L}})$.*
(2) $\mathrm{s}(\mathcal{L}) \leq \omega(\mathcal{L})$ *and* $\mathrm{s_{ess}}(\mathcal{L}) \leq \omega_{\mathrm{ess}}(\mathcal{L})$.
(3) *For each $t > 0$, $e^{t\omega(\mathcal{L})}$ is the spectral radius of $e^{t\mathcal{L}}$, and $e^{t\omega_{\mathrm{ess}}(\mathcal{L})}$ is the essential spectral radius of $e^{t\mathcal{L}}$.*
(4) *If $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a compact operator, then $\omega_{\mathrm{ess}}(\mathcal{L} + \mathcal{K}) = \omega_{\mathrm{ess}}(\mathcal{L})$,*
(5) *Let $\omega > \omega_{\mathrm{ess}}(\mathcal{L})$ be a number such that no isolated eigenvalue of $\mathcal{L}$ has real part ω. Then there is a finite set $\{\lambda_1, \dots, \lambda_k\} \subset \mathbb{C}$ such that*
$$\mathrm{Sp}(\mathcal{L}) \cap \mathbb{C}_\omega = \mathrm{Sp_d}(\mathcal{L}) \cap \mathbb{C}_\omega = \{\lambda_1, \dots, \lambda_k\}.$$
Let $E_1, \dots, E_k$ be the generalized eigenspaces of $\lambda_1, \dots, \lambda_k$ respectively; they are finite-dimensional. Then there is a closed subspace E_0 of $\mathcal{X}$ such that $\mathcal{X} = E_0 \oplus E_1 \oplus \cdots \oplus E_k$ and E_0 is invariant under $\mathcal{L}$. Moreover, there is a number $M > 0$ such that $\|e^{t\mathcal{L}}|E_0\| \leq Me^{\omega t}$.

Proof. For (1), see [25], p. 113, Theorem 3.5, and p. 150, Proposition 3.49. For (2) see [25], p. 114 and p. 152, Theorem 3.51, or else [14, Sec. IV.3] and [14, Cor. IV.2.11] for the last inequality. For (3) see [14, Sec. IV.1,2]. For (4) see [14, Prop. IV.2.12]. To prove (5), note that the fact that the finite set $\{\lambda_1, \dots, \lambda_k\}$ exists follows from the second inequality in (2). For the rest of (5), see [40], Section 4.3, statement 4.65, p. 181. Also, see [14, Cor. IV.2.11]. [23, Thm. IV.5.28]. Thus, $\lambda_0 \in \mathrm{Sp_d}(\mathcal{L})$. Compare [13, Thm. IX.1.5]. □

We remark that the inclusions in (1) can be proper, and the inequalities in (2) can be strict [14, Chapter IV].

4.2. **Proof of spectral and linearized stability.** We can now give the proof of Theorem 4.1.

Proof. The assumption of the theorem implies that the only element of $\mathrm{Sp}(\tilde{\mathcal{A}})$ with nonnegative real part is 0. By Theorem 4.4, to be proved below, $\tilde{\mathcal{A}}$ generates a C_0-semigroup, and $\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}) < 0$. Therefore by Proposition 4.3 (5) we can choose $\nu > 0$ such that $\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}) < -\nu$ and the only element of $\mathrm{Sp}(\tilde{\mathcal{A}})$ with real part greater than or equal to $-\nu$ is 0. The eigenvalue 0 is simple by Theorem 3.4. Thus $\tilde{\mathcal{A}}$ is spectrally stable. Using Proposition 4.3 (5) again, we see that $\mathcal{E}_0^2$ can be decomposed as the direct sum of two closed subspaces invariant under $\tilde{\mathcal{A}}$: the first has codimension 1, and the second, with dimension 1, is the eigenspace of $\tilde{\mathcal{A}}$ for the eigenvalue 0. Furthermore, there exists $K > 0$ such that the restriction of $e^{t\tilde{\mathcal{A}}}$, $t \geq 0$, to the first subspace has norm at most $Ke^{-\nu t}$. Thus $\tilde{\mathcal{A}}$ is linearly stable. $\square$

The remainder of this section is devoted to the statement and proof of Theorem 4.4.

We write

$$\tilde{\mathcal{A}} = \begin{pmatrix} \tilde{\mathcal{A}}_{11} & \tilde{\mathcal{A}}_{12} \\ \tilde{\mathcal{A}}_{21} & \tilde{\mathcal{A}}_{22} \end{pmatrix},$$

where:

- $\tilde{\mathcal{A}}_{11}v(\xi) = (\partial_{\xi\xi} + (c-2\alpha)\partial_\xi + \alpha^2 - c\alpha + \partial_{u_1}\omega(h_1,h_2)(\xi))v(\xi)$. $\tilde{\mathcal{A}}_{11}$ is an operator on $\mathcal{E}_0$. We denote its domain $\mathcal{E}_{02}$. However, when we discuss $\mathcal{E}_{02}$, we will usually consider it to be a Banach space with the graph norm it acquires from $\tilde{\mathcal{A}}_{11}$. Thus, if $\mathcal{E}_0 = L^2(\mathbb{R})$, then $\mathcal{E}_{02} = H^2(\mathbb{R})$; if $\mathcal{E}_0 = H^1(\mathbb{R})$, then $\mathcal{E}_{02} = H^3(\mathbb{R})$; and, if $\mathcal{E}_0 = BUC(\mathbb{R})$, then $\mathcal{E}_{02} = BUC^2(\mathbb{R})$.
- $\tilde{\mathcal{A}}_{12}$ is multiplication by $\partial_{u_2}\omega(h_1,h_2)(\xi)$.
- $\tilde{\mathcal{A}}_{21}$ is multiplication by $-\beta\partial_{u_1}\omega(h_1,h_2)(\xi)$.
- $\tilde{\mathcal{A}}_{22}v(\xi) = (c\partial_\xi - c\alpha - \beta\partial_{u_2}\omega(h_1,h_2)(\xi))v(\xi)$. $\tilde{\mathcal{A}}_{22}$ is an operator on $\mathcal{E}_0$. We denote its domain $\mathcal{E}_{01}$. However, when we discuss $\mathcal{E}_{01}$, we will usually consider it to be a Banach space with the graph norm it acquires from $\tilde{\mathcal{A}}_{22}$. Thus, if $\mathcal{E}_0 = L^2(\mathbb{R})$, then $\mathcal{E}_{01} = H^1(\mathbb{R})$; if $\mathcal{E}_0 = H^1(\mathbb{R})$, then $\mathcal{E}_{01} = H^2(\mathbb{R})$; and, if $\mathcal{E}_0 = BUC(\mathbb{R})$, then $\mathcal{E}_{01} = BUC^1(\mathbb{R})$.

The domain of $\tilde{\mathcal{A}}$ is the direct sum of the domains of $\tilde{\mathcal{A}}_{11}$ and $\tilde{\mathcal{A}}_{22}$. We also define operators

$$\mathcal{J}_1 = \begin{pmatrix} \tilde{\mathcal{A}}_{11} & \tilde{\mathcal{A}}_{12} \\ 0 & \tilde{\mathcal{A}}_{22} \end{pmatrix} \text{ and } \mathcal{J}_2 = \begin{pmatrix} \tilde{\mathcal{A}}_{11} & 0 \\ 0 & \tilde{\mathcal{A}}_{22} \end{pmatrix},$$

with the same domain.

Theorem 4.4. *Let $0 < \alpha < \frac{1}{2}c$, and let $\mathcal{E}_0$ be $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$. Then $\tilde{\mathcal{A}}$, $\mathcal{J}_1$, and $\mathcal{J}_2$ generate C_0-semigroups. We have*

$$\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}) = \omega_{\mathrm{ess}}(\mathcal{J}_1) = \omega_{\mathrm{ess}}(\mathcal{J}_2) = \alpha^2 - c\alpha < 0. \tag{4.4}$$

Proof. We shall begin by giving the proof with $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. The reason for working with $\mathcal{E}_0 = L^2(\mathbb{R})$ rather than $\mathcal{E}_0 = H^1(\mathbb{R})$ is that the spectral theory for the differential operators is better developed in $L^2(\mathbb{R})$ than in $H^1(\mathbb{R})$. At the end of the proof we will show that (4.4) with $\mathcal{E}_0 = L^2(\mathbb{R})$ implies (4.4) with $\mathcal{E}_0 = H^1(\mathbb{R})$.

Let $\mathcal{E}_0$ be either $L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. The operator $\partial_{\xi\xi}+(c-2\alpha)\partial_\xi$ on $\mathcal{E}_0$ is sectorial (see [21], pp. 136–137, and [31], Section 3.2, Corollary 2.3) and hence generates an analytic semigroup ([21], Theorem 1.3.4). The operator $c\partial_\xi$ generates the semigroup $P(t)u(\xi) = u(\xi + ct)$, which is clearly a C_0-semigroup. Therefore the diagonal operator $\begin{pmatrix} \partial_{\xi\xi} + (c-2\alpha)\partial_\xi & 0 \\ 0 & c\partial_\xi \end{pmatrix}$ generates a C_0-semigroup. Since $\tilde{\mathcal{A}}$, $\mathcal{J}_1$, and $\mathcal{J}_2$ are bounded perturbations of this operator, each generates a C_0-semigroup ([31], Section 3.1, Theorem 1.1).

To prove (4.4), we note that by Proposition 3.2 and Proposition 4.3 (2),

$$\alpha^2 - c\alpha \le \omega_{\rm ess}(\tilde{\mathcal{A}}). \tag{4.5}$$

We shall prove

$$\omega_{\rm ess}(\tilde{\mathcal{A}}) = \omega_{\rm ess}(\mathcal{J}_1) \le \omega_{\rm ess}(\mathcal{J}_2) = \alpha^2 - c\alpha < 0. \tag{4.6}$$

Then (4.4) follows from (4.5) and (4.6).

To prove (4.6), we first discuss the spectra of the operators $\tilde{\mathcal{A}}_{11}$ and $\tilde{\mathcal{A}}_{22}$. The proof then goes as follows. We first show that $\omega_{\rm ess}(\tilde{\mathcal{A}}_{11}) = \alpha^2 - c\alpha < 0$ and $\omega_{\rm ess}(\tilde{\mathcal{A}}_{22}) = -c\alpha$. Since $\mathcal{J}_2$ is diagonal, we have $\omega_{\rm ess}(\mathcal{J}_2) = \max(\omega_{\rm ess}(\tilde{\mathcal{A}}_{11}), \omega_{\rm ess}(\tilde{\mathcal{A}}_{22})) = \omega_{\rm ess}(\tilde{\mathcal{A}}_{11}) = \alpha^2 - c\alpha$. We then show from the triangularity of $\mathcal{J}_1$ that $\omega_{\rm ess}(\mathcal{J}_1) \le \omega_{\rm ess}(\mathcal{J}_2)$. Finally, we show that $e^{t\tilde{\mathcal{A}}}$ is a compact perturbation of $e^{t\mathcal{J}_1}$, so $\omega_{\rm ess}(\tilde{\mathcal{A}}) = \omega_{\rm ess}(\mathcal{J}_1)$ by definition.

In order to discuss the spectrum of $\tilde{\mathcal{A}}_{11}$, we note that since $\tilde{\mathcal{A}}_{11}$ is a bounded perturbation of the sectorial operator $\partial_{\xi\xi} + (c-2\alpha)\partial_\xi$, it is sectorial ([31], Section 3.2, Corollary 2.2). For $\tilde{\mathcal{A}}_{11}$, as for any sectorial operator, it is easy to see that $\mathrm{s}_{\rm ess}(\tilde{\mathcal{A}}_{11}) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}_{\rm ess}(\tilde{\mathcal{A}}_{11})\}$. Since $\lim_{\xi\to\pm\infty} \partial_{u_1}\omega(h_1,h_2)(\xi) = 0$, the essential spectra of $\tilde{\mathcal{A}}_{11}$ and the constant-coefficient operator $\partial_{\xi\xi}+(c-2\alpha)\partial_\xi-(\alpha^2-c\alpha)$ have the same right boundary ([21], appendix to Chapter 5). The essential spectrum of the constant-coefficient operator is easily computed to be the parabola

$$\{\lambda = \gamma + i\theta : \gamma = \alpha^2 - c\alpha - \frac{\theta^2}{(c-2\alpha)^2}\}. \tag{4.7}$$

Therefore $\mathrm{s}_{\rm ess}(\tilde{\mathcal{A}}_{11}) = \alpha^2 - c\alpha$.

To discuss the spectrum of $\tilde{\mathcal{A}}_{22}$, we note that the operator $\tilde{\mathcal{A}}_{22} - \lambda\mathcal{I}$, given by $v(\xi) \mapsto (c\partial_\xi - c\alpha - \beta\partial_{u_2}\omega(h_1,h_2)(\xi) - \lambda)v(\xi)$, is invertible if and only if the linear ODE

$$v_\xi = c^{-1}(c\alpha + \beta\partial_{u_2}\omega(h_1,h_2)(\xi) + \lambda)v(\xi) \tag{4.8}$$

has an exponential dichotomy on $\mathbb{R}$. (To see this, note that $-c^{-1}\tilde{\mathcal{A}}_{22}$ belongs to the well-studied class of differential operators generating *evolution semigroups* [9], and apply the Dichotomy Theorem for evolution semigroups [9, Thm. 3.17]. This occurs if and only if the numbers $c^{-1}(c\alpha + \beta\partial_{u_2}\omega(h_1,h_2)(\pm\infty) + \lambda)$ have the same sign, which occurs if and only if $\mathrm{Re}\,\lambda < -c\alpha - \beta e^{-\beta}$ or $-c\alpha < \mathrm{Re}\,\lambda$. Therefore

$$\mathrm{Sp}(\tilde{\mathcal{A}}_{22}) = \{\lambda \in \mathbb{C} : -c\alpha - \beta e^{-\beta} \le \mathrm{Re}\,\lambda \le -c\alpha\}. \tag{4.9}$$

We conclude that $\mathrm{Sp}(\tilde{\mathcal{A}}_{22}) = \mathrm{Sp}_{\rm ess}(\tilde{\mathcal{A}}_{22})$ and $\mathrm{s}(\tilde{\mathcal{A}}_{22}) = \mathrm{s}_{\rm ess}(\tilde{\mathcal{A}}_{22}) = -c\alpha$.

The remainder of the proof of (4.6) with $\mathcal{E}_0$ either $L^2(\mathbb{R})$ or $BUC(\mathbb{R})$ goes as follows.

(1) Since $\tilde{\mathcal{A}}_{11}$ is sectorial, it generates an analytic semigroup which enjoys the spectral mapping property: $\mathrm{Sp}(e^{t\tilde{\mathcal{A}}_{11}}) \setminus \{0\} = e^{t\,\mathrm{Sp}(\tilde{\mathcal{A}}_{11})}$, $t > 0$ [14, Cor.IV.3.12]. It follows that

$$\mathrm{s}_{\rm ess}(\tilde{\mathcal{A}}_{11}) = \omega_{\rm ess}(\tilde{\mathcal{A}}_{11}) = \alpha^2 - c\alpha. \tag{4.10}$$

(2) The semigroup generated by $\tilde{\mathcal{A}}_{22}$ also enjoys the spectral mapping property [9, Thm. 3.13]. Then from the description of $\mathrm{Sp}(\tilde{\mathcal{A}}_{22})$ above, we see that

$$s(\tilde{\mathcal{A}}_{22}) = s_{\mathrm{ess}}(\tilde{\mathcal{A}}_{22}) = \omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{22}) = \omega(\tilde{\mathcal{A}}_{22}) = -c\alpha. \tag{4.11}$$

(3) The semigroup $e^{t\mathcal{J}_2}(t)$ is the direct sum of the semigroups generated by $e^{t\tilde{\mathcal{A}}_{11}}$ and $e^{t\tilde{\mathcal{A}}_{22}}$. Hence (1) and (2) imply that $\omega_{\mathrm{ess}}(\mathcal{J}_2) = \max\{\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{11}), \omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{22})\} = \alpha^2 - c\alpha$.

(4) We claim that the semigroup $e^{t\mathcal{J}_1}$, $t \geq 0$, is given by

$$e^{t\mathcal{J}_1} = \begin{pmatrix} e^{t\tilde{\mathcal{A}}_{11}} & \mathcal{R}_{12}(t) \\ 0 & e^{t\tilde{\mathcal{A}}_{22}} \end{pmatrix}, \quad \mathcal{R}_{12}(t) = \int_0^t e^{(t-\tau)\tilde{\mathcal{A}}_{11}} \tilde{\mathcal{A}}_{12} e^{\tau\tilde{\mathcal{A}}_{22}}\, d\tau. \tag{4.12}$$

To see this, note first that $\mathcal{J}_1 = \mathcal{J}_2 + \begin{pmatrix} 0 & \tilde{\mathcal{A}}_{12} \\ 0 & 0 \end{pmatrix}$. Then, using the variation of constants formula [14, Sec. III.1.7(IE*)], we obtain

$$e^{t\mathcal{J}_1} = e^{t\mathcal{J}_2} + \int_0^t e^{(t-\tau)\mathcal{J}_1} \begin{pmatrix} 0 & \tilde{\mathcal{A}}_{12} \\ 0 & 0 \end{pmatrix} e^{\tau\mathcal{J}_2}\, d\tau. \tag{4.13}$$

Writing $e^{t\mathcal{J}_1}$ in the block-operator form $e^{t\mathcal{J}_1} = (\mathcal{R}_{ij}(t))_{i,j=1}^2$, (4.13) yields

$$\begin{aligned} e^{t\mathcal{J}_1} &= \begin{pmatrix} \mathcal{R}_{11}(t) & \mathcal{R}_{12}(t) \\ \mathcal{R}_{21}(t) & \mathcal{R}_{22}(t) \end{pmatrix} \\ &= \begin{pmatrix} e^{t\tilde{\mathcal{A}}_{11}} & 0 \\ 0 & e^{t\tilde{\mathcal{A}}_{22}} \end{pmatrix} + \int_0^t \begin{pmatrix} \mathcal{R}_{11}(t-\tau) & \mathcal{R}_{12}(t-\tau) \\ \mathcal{R}_{21}(t-\tau) & \mathcal{R}_{22}(t-\tau) \end{pmatrix} \begin{pmatrix} 0 & \tilde{\mathcal{A}}_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{\tau\tilde{\mathcal{A}}_{11}} & 0 \\ 0 & e^{\tau\tilde{\mathcal{A}}_{22}} \end{pmatrix} d\tau \\ &= \begin{pmatrix} e^{t\tilde{\mathcal{A}}_{11}} & 0 \\ 0 & e^{t\tilde{\mathcal{A}}_{22}} \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & \mathcal{R}_{11}(t-\tau)\tilde{\mathcal{A}}_{12} e^{\tau\tilde{\mathcal{A}}_{22}} \\ 0 & \mathcal{R}_{21}(t-\tau)\tilde{\mathcal{A}}_{12} e^{\tau\tilde{\mathcal{A}}_{22}} \end{pmatrix} d\tau. \end{aligned}$$

Therefore $\mathcal{R}_{11}(t) = e^{t\tilde{\mathcal{A}}_{11}}$, $\mathcal{R}_{21}(t) = 0$, $\mathcal{R}_{22}(t) = e^{t\tilde{\mathcal{A}}_{22}}$, and $\mathcal{R}_{12}(t)$ is given by (4.12).

(5) Since $e^{t\mathcal{J}_1}$ is triangular, we see by setting $t = 1$ that if a complex number λ belongs to any two of the three resolvent sets $\rho(e^{\mathcal{J}_1})$, $\rho(e^{\tilde{\mathcal{A}}_{11}})$, and $\rho(e^{\tilde{\mathcal{A}}_{22}})$, then λ belongs to all three resolvent sets. Indeed, if $\lambda \in \rho(e^{\tilde{\mathcal{A}}_{11}}) \cap \rho(e^{\tilde{\mathcal{A}}_{22}})$, then

$$(e^{\mathcal{J}_1} - \lambda\mathcal{I})^{-1} = \begin{pmatrix} (e^{\tilde{\mathcal{A}}_{11}} - \lambda\mathcal{I})^{-1} & -(e^{\tilde{\mathcal{A}}_{11}} - \lambda\mathcal{I})^{-1}\mathcal{R}_{12}(1)(e^{\tilde{\mathcal{A}}_{22}} - \lambda\mathcal{I})^{-1} \\ 0 & (e^{\tilde{\mathcal{A}}_{22}} - \lambda\mathcal{I})^{-1} \end{pmatrix}, \tag{4.14}$$

so $\lambda \in \rho(e^{\mathcal{J}_1})$. If $\lambda \in \rho(e^{\mathcal{J}_1}) \cap \rho(e^{\tilde{\mathcal{A}}_{11}})$, respectively $\lambda \in \rho(e^{\mathcal{J}_1}) \cap \rho(e^{\tilde{\mathcal{A}}_{22}})$, then one can directly check that the operator $e^{\tilde{\mathcal{A}}_{22}} - \lambda\mathcal{I}$, respectively $e^{\tilde{\mathcal{A}}_{11}} - \lambda\mathcal{I}$, is both injective and surjective.

(6) We claim that $\omega_{\mathrm{ess}}(\mathcal{J}_1) \leq \max\{\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{11}), \omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{22})\}$. Recall that if $\mathcal{L}$ is the generator of a C_0-semigroup, then by Proposition 4.3 (5), $\omega_{\mathrm{ess}}(\mathcal{L})$ is the log of the radius of $\mathrm{Sp}_{\mathrm{ess}}(e^{\mathcal{L}})$. For $\omega \in \mathbb{R}$, let $\Pi_\omega = \{z \in \mathbb{C} : |z| > e^\omega\}$. Then $\omega_{\mathrm{ess}}(\mathcal{L})$ is the infimum of the set of $\omega \in \mathbb{R}$ with the property that $\mathrm{Sp}(e^{\mathcal{L}}) \cap \Pi_\omega$ is a finite subset of $\mathrm{Sp}_{\mathrm{d}}(e^{\mathcal{L}})$.

Fix $\omega > \max\{\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{11}), \omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{22})\}$. We must show that $\Pi_\omega \cap \mathrm{Sp}(e^{\mathcal{J}_1}) \subset \mathrm{Sp}_{\mathrm{d}}(e^{\mathcal{J}_1})$ and consists of finitely many points. We know that $\Pi_\omega \cap \mathrm{Sp}(e^{\tilde{\mathcal{A}}_{11}}) \subset \mathrm{Sp}_{\mathrm{d}}(e^{\tilde{\mathcal{A}}_{11}})$ and consists of finitely many points. Similarly, $\Pi_\omega \cap \mathrm{Sp}(e^{\tilde{\mathcal{A}}_{22}}) \subset \mathrm{Sp}_{\mathrm{d}}(e^{\tilde{\mathcal{A}}_{22}})$; however, by (2) the latter is empty, so $\Pi_\omega \cap \mathrm{Sp}(e^{\tilde{\mathcal{A}}_{22}})$ is empty. Since $\mathrm{Sp}(e^{\mathcal{J}_1}) \subset \mathrm{Sp}(e^{\tilde{\mathcal{A}}_{11}}) \cup \mathrm{Sp}(e^{\tilde{\mathcal{A}}_{22}})$ by (5), we have $\Pi_\omega \cap \mathrm{Sp}(e^{\mathcal{J}_1}) \subset \Pi_\omega \cap \mathrm{Sp}(e^{\tilde{\mathcal{A}}_{11}})$, which consists of finitely many points in $\mathrm{Sp}_{\mathrm{d}}(e^{\tilde{\mathcal{A}}_{11}})$. It remains to show that if $\lambda_0 \in \Pi_\omega \cap \mathrm{Sp}(e^{\mathcal{J}_1})$ then the Riesz projection $\mathcal{P}_{\lambda_0}(e^{\mathcal{J}_1})$ has finite dimensional range. From (4.2) and (4.14), we have

$$\begin{aligned}\mathcal{P}_{\lambda_0}(e^{\mathcal{J}_1}) &= -\frac{1}{2\pi i}\int_\gamma (e^{\mathcal{J}_1} - \lambda\mathcal{I})^{-1}\,d\lambda \\ &= -\frac{1}{2\pi i}\int_\gamma \begin{pmatrix} (e^{\tilde{\mathcal{A}}_{11}} - \lambda\mathcal{I})^{-1} & (e^{\tilde{\mathcal{A}}_{11}} - \lambda\mathcal{I})^{-1}\mathcal{R}_{12}(1)(e^{\tilde{\mathcal{A}}_{22}} - \lambda\mathcal{I})^{-1} \\ 0 & (e^{\tilde{\mathcal{A}}_{22}} - \lambda\mathcal{I})^{-1}\end{pmatrix} d\lambda. \quad (4.15)\end{aligned}$$

Now the function $\lambda \mapsto (e^{\tilde{\mathcal{A}}_{22}} - \lambda\mathcal{I})^{-1}$ is analytic at $\lambda = \lambda_0$, and $\mathcal{P}_{\lambda_0}(e^{\tilde{\mathcal{A}}_{11}})$ is the residue of the function $\lambda \mapsto (e^{\tilde{\mathcal{A}}_{11}} - \lambda\mathcal{I})^{-1}$ at $\lambda = \lambda_0$. Using (4.15) and (4.3) with $\mathcal{L}$ replaced by $e^{\tilde{\mathcal{A}}_{11}}$, we then infer that $\mathcal{P}_{\lambda_0}(e^{\mathcal{J}_1}) = \begin{pmatrix} \mathcal{P}_{\lambda_0}(e^{\tilde{\mathcal{A}}_{11}}) & \mathcal{P}_{\lambda_0}(e^{\tilde{\mathcal{A}}_{11}})\mathcal{B} \\ 0 & 0\end{pmatrix}$, where $\mathcal{B}$ is a bounded operator. Therefore the range of $\mathcal{P}_{\lambda_0}(e^{\mathcal{J}_1})$ is finite dimensional if and only if the range of $\mathcal{P}_{\lambda_0}(e^{\tilde{\mathcal{A}}_{11}})$ is finite dimensional. Since the range of $\mathcal{P}_{\lambda_0}(e^{\tilde{\mathcal{A}}_{11}})$ is in fact finite dimensional, we are done.

(7) Let $\mathcal{I}_0 : \mathcal{E}_{02} \to \mathcal{E}_0$ denote the imbedding operator. We claim that $\tilde{\mathcal{A}}_{21}\mathcal{I}_0 : \mathcal{E}_{02} \to \mathcal{E}_0$ is a compact operator. To prove this, let χ_n be an appropriately chosen sequence of C^∞ scalar cut-off functions supported in the interval $[-n, n]$. Note that $\partial_{u_1}\omega(h_1, h_2)(\xi)$, along with its derivatives of all orders with respect to ξ, approaches 0 as $\xi \to \pm\infty$. It follows that each operator $\chi_n\tilde{\mathcal{A}}_{21}\mathcal{I}_0 : \mathcal{E}_{02} \to \mathcal{E}_0$ is compact. Since $\chi_n\tilde{\mathcal{A}}_{21}\mathcal{I}_0 \to \tilde{\mathcal{A}}_{21}\mathcal{I}_0$ in the operator norm as $n \to \infty$, the result follows.

(8) We will need the following abstract Voigt's Lemma [14, Thm. C7] that establishes compactness of integrals of strongly continuous operator-valued functions. (We note that this lemma also shortens some arguments in [2, Sec. 4].) Assume that $\mathcal{K}$ is a strongly continuous function from a closed interval $[\alpha, \beta]$ into the set of bounded linear operators between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$; that is, for each $x \in \mathcal{X}$, the function $t \to \mathcal{K}(t)x$ is continuous. Voigt's Lemma says that if $\mathcal{K}(t)$ is a compact operator for each $t \in (\alpha, \beta)$, then the operator $\int_\alpha^\beta \mathcal{K}(t)\,dt$ is compact as well.

(9) We claim that $\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}) = \omega_{\mathrm{ess}}(\mathcal{J}_1)$. This follows from the definition of ω_{ess} provided we can show that the difference $e^{t\tilde{\mathcal{A}}} - e^{t\mathcal{J}_1}$ is a compact operator for each $t > 0$. Using the variation of constants formula and (4.12), we have

$$\begin{aligned}e^{t\tilde{\mathcal{A}}} - e^{t\mathcal{J}_1} &= \int_0^t e^{(t-\tau)\tilde{\mathcal{A}}}\begin{pmatrix} 0 & 0 \\ \tilde{\mathcal{A}}_{21} & 0\end{pmatrix} e^{\tau\mathcal{J}_1}\,d\tau = \int_0^t e^{(t-\tau)\tilde{\mathcal{A}}}\begin{pmatrix} 0 & 0 \\ \tilde{\mathcal{A}}_{21} & 0\end{pmatrix}\begin{pmatrix} e^{\tau\tilde{\mathcal{A}}_{11}} & \mathcal{R}_{12}(\tau) \\ 0 & e^{\tau\tilde{\mathcal{A}}_{22}}\end{pmatrix} d\tau \\ &= \int_0^t e^{(t-\tau)\tilde{\mathcal{A}}}\begin{pmatrix} 0 & 0 \\ \tilde{\mathcal{A}}_{21}e^{\tau\tilde{\mathcal{A}}_{11}} & \tilde{\mathcal{A}}_{21}\mathcal{R}_{12}(\tau)\end{pmatrix} d\tau.\end{aligned}$$

In view of (8), to finish the proof of the claim it remains to show, for each $\tau \in (0, t)$, the following two assertions: First, $\tilde{\mathcal{A}}_{21}e^{\tau\tilde{\mathcal{A}}_{11}}$ is compact as an operator from $\mathcal{E}_0$ to $\mathcal{E}_0$, and, second, the operator

$$\tilde{\mathcal{A}}_{21}\mathcal{R}_{12}(\tau) = \int_0^\tau \tilde{\mathcal{A}}_{21}e^{(\tau-\sigma)\tilde{\mathcal{A}}_{11}}\tilde{\mathcal{A}}_{12}e^{\sigma\tilde{\mathcal{A}}_{22}}\,d\sigma \quad (4.16)$$

is compact as an operator from $\mathcal{E}_0$ to $\mathcal{E}_0$. Applying Voigt's Lemma in (4.16) once again, the second assertion holds as soon as we know that $\tilde{\mathcal{A}}_{21}e^{(\tau-\sigma)\tilde{\mathcal{A}}_{11}}$ is compact for each $\sigma \in (0, \tau)$; this, however, follows from the first assertion. To prove the first assertion, we recall that the semigroup generated by $\tilde{\mathcal{A}}_{11}$ is analytic. Therefore, for each $\tau > 0$ the operator $e^{\tau\tilde{\mathcal{A}}_{11}}$ can be viewed as a bounded operator from $\mathcal{E}_0$ into the domain of $\tilde{\mathcal{A}}_{11}$ with the norm inherited from $\mathcal{E}_0$ ([21], p. 23), and hence as a bounded operator into the domain of $\tilde{\mathcal{A}}_{11}$ equipped with the graph norm it acquires from $\tilde{\mathcal{A}}_{11}$. This space is just $\mathcal{E}_{02}$. The first assertion then follows from (7).

Equality (4.6) for $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$ now follows from (9), (6), and (3).

Finally, we show that (4.4) for $\mathcal{E}_0 = L^2(\mathbb{R})$ implies (4.4) for $\mathcal{E}_0 = H^1(\mathbb{R})$.

Write $\tilde{\mathcal{A}} = \mathcal{C} + \mathcal{B}$, where, for $V = (v_1, v_2)$,

$$\mathcal{C}V = \begin{pmatrix} \partial_{\xi\xi} + (c-2\alpha)\partial_\xi + \alpha^2 - c\alpha & 0 \\ 0 & c\partial_\xi - c\alpha \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

$$\mathcal{B}V = \begin{pmatrix} \partial_{u_1}\omega(h_1,h_2)(\xi) & \partial_{u_2}\omega(h_1,h_2)(\xi) \\ -\beta\partial_{u_1}\omega(h_1,h_2)(\xi) & -\beta\partial_{u_2}\omega(h_1,h_2)(\xi) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Define

$$\mathcal{B}'V = \begin{pmatrix} \partial_{\xi u_1}\omega(h_1,h_2)(\xi) & \partial_{\xi u_2}\omega(h_1,h_2)(\xi) \\ -\beta\partial_{\xi u_1}\omega(h_1,h_2)(\xi) & -\beta\partial_{\xi u_2}\omega(h_1,h_2)(\xi) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

We shall use the notation $\tilde{\mathcal{A}}_{H^1}$ (respectively $\tilde{\mathcal{A}}_{L^2}$) to indicate that $\tilde{\mathcal{A}}$ is acting on the space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ (respectively $L^2(\mathbb{R}) \times L^2(\mathbb{R})$), $\partial_{\xi\, H^1}$ (respectively $\partial_{\xi\, L^2}$) to indicate that ∂_ξ is acting on the space $H^1(\mathbb{R})$ (respectively $L^2(\mathbb{R})$), etc.

We claim that

$$\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{H^1}) = \omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{L^2}), \quad \omega_{\mathrm{ess}}(\mathcal{J}_{1\,H^1}) = \omega_{\mathrm{ess}}(\mathcal{J}_{1\,L^2}), \quad \omega_{\mathrm{ess}}(\mathcal{J}_{2\,H^1}) = \omega_{\mathrm{ess}}(\mathcal{J}_{2\,L^2}); \tag{4.17}$$

these equalities imply the result. We shall prove only the first equality in (4.17); the other two are proved similarly.

We recall that the operator $\partial_{\xi\, L^2}$ has domain $H^1(\mathbb{R})$ and spectrum $i\mathbb{R}$. Therefore the operator

$$\mathcal{D} = \begin{pmatrix} \partial_{\xi\, L^2} + I & 0 \\ 0 & \partial_{\xi\, L^2} + I \end{pmatrix} : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$$

is an isomorphism. Let $\mathcal{I}_0 : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ be the imbedding operator. Using the identities

$$\mathcal{D}\mathcal{C}_{H^1} = \mathcal{C}_{L^2}\mathcal{D}, \quad \mathcal{D}\mathcal{B}_{H^1} = \mathcal{B}_{L^2}\mathcal{D} + \mathcal{B}'_{L^2}\mathcal{I}_0,$$

we obtain

$$\mathcal{D}\,\tilde{\mathcal{A}}_{H^1}\,\mathcal{D}^{-1} = \mathcal{D}\,(\mathcal{C}_{H^1} + \mathcal{B}_{H^1})\,\mathcal{D}^{-1} = \mathcal{C}_{L^2} + \mathcal{B}_{L^2} + \mathcal{B}'_{L^2}\mathcal{I}_0\mathcal{D}^{-1} = \tilde{\mathcal{A}}_{L^2} + \mathcal{K}, \tag{4.18}$$

where $\mathcal{K} = \mathcal{B}'_{L^2}\mathcal{I}_0\mathcal{D}^{-1}$.

Note that for $i = 1, 2$, $\partial_{\xi u_i}\omega(h_1,h_2)(\xi)$, along with its derivatives of all orders with respect to ξ, approaches 0 as $\xi \to \pm\infty$. It follows that $\mathcal{B}'_{L^2}\mathcal{I}_0 : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ is compact; the argument is the same as that in step 7 above. Since $\mathcal{D}^{-1} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to H^1(\mathbb{R}) \times H^1(\mathbb{R})$ is a bounded operator, it follows that $\mathcal{K}$ is compact.

Finally, we show the first equality in (4.17). From (4.18) and Proposition 4.3 (4), we obtain

$$\omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{H^1}) = \omega_{\mathrm{ess}}(\mathcal{D}\,\tilde{\mathcal{A}}_{H^1}\,\mathcal{D}^{-1}) = \omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{L^2} + K) = \omega_{\mathrm{ess}}(\tilde{\mathcal{A}}_{L^2}).$$

□

5. Nonlinear stability

In this section we only consider $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$. We recall that $\mathcal{E}_\alpha = \{v(\xi) : w(\xi) = e^{\alpha\xi}v(\xi) \in \mathcal{E}_0\}$, with norm $\|v\|_\alpha = \|e^{\alpha\xi}v\|_0 = \|w\|_0$, and we recall from the Introduction that $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ with norm $\|v\| = \max(\|v\|_0, \|v\|_\alpha)$.

We shall prove nonlinear stability by studying the system (2.1)–(2.2) in the space $\mathcal{E}^2$, where short-time existence and uniqueness of solutions hold. We decompose the solution into a shifted traveling wave $(h_1, h_2)(\xi - q(t))$, with $q(t)$ to be determined, and a component in $\mathrm{R}(\mathcal{A}_\alpha) \cap \mathcal{E}^2$. We then use Theorem 4.1 and Gronwall's inequality to show that over the short time, solutions decay exponentially in $\| \, \|_\alpha$. Next we use this result and a linear analysis in $\| \, \|_0$ to show that over that time, solutions that start small in $\| \, \|_0$ stay small in $\| \, \|_0$, and in fact the second component decays. Finally we combine these estimates to extend the time to infinity and prove the nonlinear stability results.

After some preliminaries in Subsections 5.1 and 5.2, we formulate the system to be studied in Subsection 5.3 and derive some estimates on the nonlinear operators that appear in it in Subsection 5.4. We prove the nonlinear stability results in Subsection 5.5. In order to allow the outline of the argument to stand out more clearly, the proofs of three propositions are delayed until Subsections 5.6–5.8.

5.1. **Preliminaries.** We gather in the following lemma some facts we will need about the functions ρ and ω in (1.1)–(1.3).

Let $Y = (y_1, y_2)$, $Z = (z_1, z_2)$, etc.

Lemma 5.1. (1) *$\rho(y + z) = \rho(y) + \rho_1(y, z)z$ where*

$$\rho_1(y, z) = \int_0^1 \rho'(y + tz)dt. \tag{5.1}$$

(2) *$\rho(y + z) = \rho(y) + \rho'(y)z + \rho_2(y, z)z^2$ where*

$$\rho_2(y, z) = \int_0^1 \int_0^1 \rho''(y + stz)t \, ds \, dt. \tag{5.2}$$

(3) *$D\omega(Y) = \begin{pmatrix} y_2\rho'(y_1) & \rho(y_1) \end{pmatrix}$.*

(4) *$\omega(Y + Z) = \omega(Y) + D\omega(Y)Z + n(Y, Z)z_1$ where*

$$n(Y, Z) = y_2\rho_2(y_1, z_1)z_1 + \rho_1(y_1, z_1)z_2. \tag{5.3}$$

Proof. We only prove (4):

$$\begin{aligned}
&\omega(Y + Z) - \omega(Y) - D\omega(Y)Z \\
&\qquad = (y_2 + z_2)\rho(y_1 + z_1) - y_2\rho(y_1) - y_2\rho'(y_1)z_1 - \rho(y_1)z_2 \\
&\qquad = y_2(\rho(y_1 + z_1) - \rho(y_1) - \rho'(y_1)z_1) + (\rho(y_1 + z_1) - \rho(y_1)z_2) \\
&\qquad = y_2\rho_2(y_1, z_1)z_1^2 + \rho_1(y_1, z_1)z_1z_2 \\
&\qquad = (y_2\rho_2(y_1, z_1)z_1 + \rho_1(y_1, z_1)z_2)z_1.
\end{aligned}$$

□

Let

$$L = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi & 0 \\ 0 & c\partial_\xi \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -\beta \end{pmatrix}.$$

With this notation, we can rewrite (2.1)–(2.2) as

$$\partial_t U = LU + B\omega(U). \tag{5.4}$$

Let $H(\xi) = (h_1, h_2)(\xi)$, the traveling wave. In order to consider perturbations of the traveling wave, we substitute $U = H + V$ into (5.4). Using $LH + B\omega(H) = 0$, we obtain

$$\partial_t V = LV + BD\omega(H)V + Bn(H, V)v_1. \tag{5.5}$$

We introduce the notation:

$$R(\xi) = \begin{pmatrix} r_1(\xi) & r_2(\xi) \end{pmatrix} = D\omega(H(\xi)) = \begin{pmatrix} h_2(\xi)\rho'(h_1(\xi)) & \rho(h_1(\xi)) \end{pmatrix}. \tag{5.6}$$

Then A given by (3.1) is just $L + BR(\xi)$, and the linearization of (5.4) at H is

$$\partial_t V = AV = LV + BR(\xi)V. \tag{5.7}$$

Equation (5.5) becomes

$$\partial_t V = AV + Bn(H, V)v_1. \tag{5.8}$$

5.2. **Linear operators.** We recall the standing assumption that $0 < \alpha < \frac{1}{2}c$, so that $H' \in \mathcal{E}^2_\alpha$.

5.2.1. *Three spaces.* Recall that $\mathcal{A}_0$ and $\mathcal{A}_\alpha$ are the linear operators on $\mathcal{E}^2_0$ and $\mathcal{E}^2_\alpha$ respectively given by $V \to AV$. Each of these operators is a bounded perturbation of the operator given by $V \to LV$ on the same space, and the domain is just the domain of the operator given by $V \to LV$ on the same space. On $\mathcal{E}^2_0$ and $\mathcal{E}^2_\alpha$, the operator given by $V \to LV$ generates a strongly continuous semigroup, and these semigroups agree on $\mathcal{E}^2$, the intersection of the two spaces. Since $\mathcal{A}_0$ and $\mathcal{A}_\alpha$ are bounded perturbations of the operator given by $V \to LV$, the semigroups generated by $\mathcal{A}_0$ and $\mathcal{A}_\alpha$ agree on $\mathcal{E}^2$. Since the norm on $\mathcal{E}^2$ is the maximum of the two norms, the restriction of either semigroup on the intersection space is strongly continuous in the norm of $\mathcal{E}^2$. Let $\mathcal{A}$ denote the generator of the restricted semigroup on $\mathcal{E}^2$. Again, since the norm on $\mathcal{E}^2$ is the maximum of two norms, it follows that the domain of $\mathcal{A}$ is the intersection of the domains of $\mathcal{A}_0$ and $\mathcal{A}_\alpha$, and thus $\mathcal{A}$ is given by $V \to AV$ for V in its domain. We summarize this discussion in the following proposition.

Proposition 5.2. $\mathrm{D}(\mathcal{A}) = \mathrm{D}(\mathcal{A}_0) \cap \mathrm{D}(\mathcal{A}_\alpha)$. *(1) If $V \in \mathrm{D}(\mathcal{A})$, then $\mathcal{A}_0 V = \mathcal{A}_\alpha V = \mathcal{A}V$, and (2) if $V \in \mathcal{E}^2$ then $e^{t\mathcal{A}_0}V = e^{t\mathcal{A}_\alpha}V = e^{t\mathcal{A}}V$.*

As we have mentioned in Subsection 3.1, $0 \in \mathrm{Sp}_{\mathrm{ess}}(\mathcal{A}_0)$. Since $\|v\| = \max(\|v\|_0, \|v\|_\alpha)$ and $\mathrm{D}(\mathcal{A}) = \mathrm{D}(\mathcal{A}_0) \cap \mathrm{D}(\mathcal{A}_\alpha)$, we have $\rho(\mathcal{A}) = \rho(\mathcal{A}_0) \cap \rho(A_\alpha)$ and $\mathrm{Sp}_{\mathrm{d}}(\mathcal{A}) = \mathrm{Sp}_{\mathrm{d}}(\mathcal{A}_0) \cap \mathrm{Sp}_{\mathrm{d}}(A_\alpha)$. Therefore $0 \in \mathrm{Sp}_{\mathrm{ess}}(\mathcal{A})$.

On the other hand, since we saw in Subsection 3.1 that $0 \in \rho_{\mathrm{F}}(\mathcal{A}_\alpha)$, $\mathcal{A}_\alpha$ is Fredholm of index zero. We have seen that 0 is a simple eigenvalue of $\mathcal{A}_\alpha$ and of $\tilde{\mathcal{A}}$. The kernel of $\mathcal{A}_\alpha$ is spanned by $H' = (h_1', h_2')$.

5.2.2. *Projections.* We shall denote the kernel and range of an operator $\mathcal{L}$ by $\mathrm{N}(\mathcal{L})$ and $\mathrm{R}(\mathcal{L})$ respectively.

Since 0 is isolated in the spectrum of $\mathcal{A}_\alpha$, we can define the Riesz spectral projection $\mathcal{P}^c_\alpha = \mathcal{P}_0(\mathcal{A}_\alpha)$ onto the one-dimensional space $N(\mathcal{A}_\alpha)$. $\mathcal{P}^c_\alpha$ commutes with $\mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha$ is Fredholm of index zero and 0 is a simple eigenvalue of $\mathcal{A}_\alpha$, $\mathcal{E}^2_\alpha = R(\mathcal{A}_\alpha) \oplus N(\mathcal{A}_\alpha)$, and $\mathrm{N}(\mathcal{P}^c_\alpha) = \mathrm{R}(\mathcal{A}_\alpha)$. Since $\mathrm{R}(\mathcal{P}^c_\alpha) = \mathrm{N}(\mathcal{A}_\alpha)$ is spanned by H', we write $\mathcal{P}^c_\alpha V = \pi_\alpha(V)H'$, where $\pi_\alpha : \mathcal{E}^2_\alpha \to \mathbb{R}$ is a bounded linear functional such that $\pi_\alpha(cH') = c$.

Let $\mathcal{P}^s_\alpha = I - \mathcal{P}^c_\alpha$. $\mathcal{P}^s_\alpha$ is projection onto $\mathrm{R}(\mathcal{A}_\alpha)$, with kernel $\mathrm{N}(\mathcal{A}_\alpha)$. It also commutes with $\mathcal{A}_\alpha$. From Theorem 4.1 we have:

Corollary 5.3. There are numbers $K > 0$ and $\nu > 0$ such that $\|e^{t\mathcal{A}_\alpha \mathcal{P}^s_\alpha}\| \leq Ke^{-\nu t}$.

We may assume

$$\nu < \frac{\beta}{2}e^{-\beta}; \tag{5.9}$$

we recall that β is the exothermicity parameter in (1.2).

Unfortunately the formula $V \mapsto Bn(H, V)v_1$ does not define a Lipschitz mapping from $\mathcal{E}^2_\alpha$ to itself. It *does* define Lipschitz mappings from $\mathcal{E}^2_0$ to itself and from $\mathcal{E}^2$ to itself, as we shall see.

5.3. **System to be studied.** Using the notation of Subsection 5.1, for $(\xi, q) \in \mathbb{R}^2$, let

$$S(\xi, q) = R(\xi - q) - R(\xi). \tag{5.10}$$

Then

$$L + BR(\xi - q) = L + BR(\xi) + BS(\xi, q) = A + BS(\xi, q). \tag{5.11}$$

We consider (5.4) on $\mathcal{E}_\alpha$; *our considerations at this point are formal, since the PDE is not known to be well-posed on* $\mathcal{E}_\alpha$. Let

$$U(t, \xi) = H(\xi - q(t)) + V(t, \xi)$$

with $V(t, \xi)$ in $\mathrm{R}(\mathcal{A}_\alpha)$ for each t. Using the fact that

$$LH(\xi - q) + B\omega(H(\xi - q)) = 0,$$

we obtain

$$-H'(\xi - q(t))\dot{q}(t) + \partial_t V = \Big(A + BS(\xi, q(t))\Big)V + Bn(H(\xi - q(t)), V)v_1. \tag{5.12}$$

Applying $\mathcal{P}^s_\alpha$ and $\mathcal{P}^c_\alpha$ to (5.12), we obtain:

$$\partial_t V = AV + \mathcal{P}^s_\alpha\Big(BS(\xi, q(t))V + Bn(H(\xi - q(t)), V)v_1 + H'(\xi - q(t))\dot{q}(t)\Big), \tag{5.13}$$

$$-\mathcal{P}^c_\alpha H'(\xi - q(t))\dot{q}(t) = \mathcal{P}^c_\alpha\Big(BS(\xi, q(t))V + Bn(H(\xi - q(t)), V)v_1\Big). \tag{5.14}$$

From (5.14) we obtain

$$-\dot{q}(t)\pi(H'(\xi - q(t))) = \pi\Big(BS(\xi, q(t))V + Bn(H(\xi - q(t)), V)v_1\Big) \tag{5.15}$$

Lemma 5.4. *There is a number $\delta_1 > 0$ such that if $|q| \leq \delta_1$, then*

$$\frac{1}{2} \leq |\pi(H'(\xi - q))| \leq \frac{3}{2}.$$

Proof. The mapping $q \to H'(\xi - q)$ is continuous from $\mathbb{R}$ to $\mathcal{E}_\alpha$, and $\pi(H'(\xi)) = 1$. □

Assuming $|q| \leq \delta_1$, we introduce the notation

$$N(V, q) = Bn(H(\xi - q), V)v_1, \tag{5.16}$$

$$G(V, q) = BS(\xi, q)V + N(V, q), \tag{5.17}$$

$$\kappa(V, q) = -(\pi(H'(\xi - q)))^{-1}\pi(G(V, q)). \tag{5.18}$$

We have

$$\frac{2}{3} \leq |(\pi(H'(\xi - q)))^{-1}| \leq 2. \tag{5.19}$$

Since $\kappa(V, q)$ has been chosen to make

$$\mathcal{P}^c_\alpha\Big(G(V, q) + \kappa(V, q)H'(\xi - q)\Big) = 0,$$

we may rewrite (5.13)–(5.14) as the following system on $\mathrm{R}(\mathcal{A}_\alpha) \times \mathbb{R}$:

$$\partial_t V = AV + G(V, q) + \kappa(V, q)H'(\xi - q), \tag{5.20}$$

$$\dot{q} = \kappa(V, q). \tag{5.21}$$

5.4. **Nonlinear operators.** Let $U \subset \mathbb{R}^l$ and let $C^0(U)$ denote the space of bounded C^0 functions $m : U \to \mathbb{R}$ with the sup norm, which we now denote $\|\cdot\|_{C^0}$. More generally, let $C^k(U)$ denote the space of C^k functions $m : U \to \mathbb{R}$ such that m, Dm, $\ldots$, $D^k m$ are all bounded functions, with the C^k norm:

$$\|m\|_{C^k} = \|m\|_{C^0} + \|Dm\|_{C^0} + \ldots + \|D^k m\|_{C^0}.$$

Recall that $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$.

Proposition 5.5. (1) *If $u \in \mathcal{E}_0$ then $u \in C^0(\mathbb{R})$ and there is a constant $C > 0$ such that $\|u\|_{C^0} \leq C\|u\|_0$.*
(2) *There is a constant $C > 0$ such that if $u, v \in \mathcal{E}_0$, then $uv \in \mathcal{E}_0$ and $\|uv\|_0 \leq C\|u\|_0\|v\|_0$.*
(3) *If $u \in \mathcal{E}$ then $u \in C^0(\mathbb{R})$ and there is a constant $C > 0$ such that $\|u\|_{C^0} \leq C\|u\|$.*
(4) *There is a constant $C > 0$ such that if $u, v \in \mathcal{E}$, then $uv \in \mathcal{E}_\alpha$ and $\|uv\|_\alpha \leq C\|u\|_0\|v\|_\alpha$.*
(5) *There is a constant $C > 0$ such that if $u, v \in \mathcal{E}$, then $uv \in \mathcal{E}$ and $\|uv\| \leq C\|u\|\|v\|$.*

Proof. (1) is obvious for $\mathcal{E}_0 = BUC(\mathbb{R})$ and well-known for $\mathcal{E}_0 = H^1(\mathbb{R})$; the same is true for (2). Since $\|u\|_0 \leq \|u\|$, (3) follows from (1). To show (4), let $u, v \in \mathcal{E}$. Then, using (2),

$$\|uv\|_\alpha = \|e^{\alpha\xi}uv\|_0 = \|uw\|_0 \leq C\|u\|_0\|w\|_0 = C\|u\|_0\|v\|_\alpha.$$

To show (5), let $u, v \in \mathcal{E}$. Then by (2), $\|uv\|_0 \leq C\|u\|_0\|v\|_0 \leq C\|u\|\|v\|$, and by (4), $\|uv\|_\alpha \leq C\|u\|_0\|v\|_\alpha \leq C\|u\|\|v\|$. Therefore $uv \in \mathcal{E}$ and $\|uv\| \leq C\|u\|\|v\|$. ☐

Proposition 5.6. *Let $m(\xi, q, u) \in C^2(R^3)$. Consider the formula*

$$(q, u(\xi), v(\xi)) \mapsto m(\xi, q, u(\xi))v(\xi). \tag{5.22}$$

(1) *Formula (5.22) defines a mapping from $\mathbb{R} \times \mathcal{E}_0^2$ to $\mathcal{E}_0$ that is Lipschitz on any set of the form $\{(q, u, v) : |q| + \|u\|_0 + \|v\|_0 \leq K\}$. If $m(\xi, 0, u)$ is identically 0, then there is a constant C such that on this set, $\|m(\xi, q, u(\xi))v(\xi)\|_0 \leq C|q|\|v\|_0$.*
(2) *Formula (5.22) defines a mapping from from $\mathbb{R} \times \mathcal{E}^2$ to $\mathcal{E}$ that is Lipschitz on any set of the form $\{(q, u, v) : |q| + \|u\| + \|v\| \leq K\}$. If $m(\xi, 0, u)$ is identically 0, then there is a constant C such that on this set, $\|m(\xi, q, u(\xi))v(\xi)\|_\alpha \leq C|q|\|v\|_\alpha$ and $\|m(\xi, q, u(\xi))v(\xi)\| \leq C|q|\|v\|$.*

The straightforward proof is given in Section 5.6.

Corollary 5.7. Let $m(\xi, q, v) \in C^2(R^3)$. Then the formula

$$(q, v(\xi)) \mapsto m(\xi, q, v(\xi))v(\xi)$$

defines mappings from $\mathbb{R} \times \mathcal{E}_0$ to $\mathcal{E}_0$ and from $\mathbb{R} \times \mathcal{E}$ to $\mathcal{E}$. The first is Lipschitz on any set of the form $\{(q, v) : |q| + \|v\|_0 \le K\}$; the second is Lipschitz on any set of the form $\{(q, v) : |q| + \|v\| \le K\}$.

We remark that in both Proposition 5.6 and Corollary 5.7, it is enough to assume that $m \in C^2(U)$ for any set U of the form $\{(\xi, q, u) : |q| + |u| \le K\}$.

Proposition 5.8. (1) *The formula* $(V(\xi), q) \mapsto S(\xi, q)V(\xi)$ *defines a mapping from* $\mathcal{E}_0^2 \times \mathbb{R}$ *to* $\mathcal{E}_0$ *that is Lipschitz on any set of the form* $\{(V, q) : \|V\|_0 + |q| \le K\}$. *On such a set there is a constant* C *such that* $\|S(\xi, q)V(\xi)\|_0 \le C|q|\|V\|_0$ *.*

(2) *The formula* $(V(\xi), q) \mapsto S(\xi, q)V(\xi)$ *defines a mapping from* $\mathcal{E}^2 \times \mathbb{R}$ *to* $\mathcal{E}$ *that is Lipschitz on any set of the form* $\{(V, q) : \|V\| + |q| \le K\}$. *On such a set there is a constant* C *such that* $\|S(\xi, q)V(\xi)\|_\alpha \le C|q|\|V\|_\alpha$ *and* $\|S(\xi, q)V(\xi)\| \le C|q|\|V\|$*.*

Proof. Just apply Proposition 5.6 to each component of $S(\xi, q)V(\xi)$; clearly $S(\xi, 0) = 0$. (In this case the function m depends only on ξ and q). □

Proposition 5.9. (1) *The formula* $(V, q) \mapsto n(H(\xi - q), V)$ *defines a mapping from* $\mathcal{E}_0^2 \times \mathbb{R}$ *to* $\mathcal{E}_0$ *that is Lipschitz and* $\mathcal{O}(\|V\|_0)$ *on any bounded neighborhood of* $(0, 0)$ *in* $\mathcal{E}_0^2 \times \mathbb{R}$*.*

(2) *The formula for* $N(V, q)$ *defines a mapping from* $\mathcal{E}_0^2 \times \mathbb{R}$ *to* $\mathcal{E}_0^2$ *that is Lipschitz and* $\mathcal{O}(\|V\|_0^2)$ *on any bounded neighborhood of* $(0, 0)$ *in* $\mathcal{E}_0^2 \times \mathbb{R}$*.*

(3) *The formula for* $G(V, q)$ *defines a mapping from* $\mathcal{E}_0^2 \times \mathbb{R}$ *to* $\mathcal{E}_0^2$ *that is Lipschitz and* $\mathcal{O}((\|V\|_0 + |q|)\|V\|_0)$ *on any bounded neighborhood of* $(0, 0)$ *in* $\mathcal{E}_0^2 \times \mathbb{R}$*.*

Proof. (1) The Lipschitz property follows from Corollary 5.7. The mapping is $\mathcal{O}(\|V\|_0)$ on the given set because it is Lipschitz and $n(H(\xi - q), 0) = 0$.

(2) Since $N(V, q) = Bn(H(\xi - q), V)v_1$, (2) follows from (1).

(3) This follows from the formula for G, Proposition 5.8, and (2). □

Proposition 5.10. (1) *If* $V \in \mathcal{E}^2$ *then* $N(V, q) \in \mathcal{E}_\alpha^2$*, and on any bounded neighborhood of* $(0, 0)$ *in* $\mathcal{E}^2 \times \mathbb{R}$ *there is a constant* $C > 0$ *such that* $\|N(V, q)\|_\alpha \le C\|V\|_0\|V\|_\alpha$*.*

(2) *The formula* $(V, q) \mapsto n(H(\xi - q), V)$ *defines a mapping from* $\mathcal{E}^2 \times \mathbb{R}$ *to* $\mathcal{E}$ *that is Lipschitz and* $\mathcal{O}(\|V\|)$ *on any bounded neighborhood of* $(0, 0)$ *in* $\mathcal{E}^2 \times \mathbb{R}$*.*

(3) *The formula for* $N(V, q)$ *defines a mapping from* $\mathcal{E}^2 \times \mathbb{R}$ *to* $\mathcal{E}^2$ *that is Lipschitz and* $\mathcal{O}(\|V\|^2)$ *on any bounded neighborhood of* $(0, 0)$ *in* $\mathcal{E}^2 \times \mathbb{R}$*.*

Proof. (1) $\|N(V, q)\|_\alpha = \|e^{\alpha\xi}N(V, q)\|_0 = \|e^{\alpha\xi}Bn(H(\xi - q), V)v_1\|_0 \le C\|n(H(\xi - q), V)\|_0$ $\|Be^{\alpha\xi}v_1\|_0 \le C\|V\|_0\|V\|_\alpha$.

(2) and (3) are proved like Proposition 5.9 (1) and (2). □

Proposition 5.11. *The formulas* (5.17) *and* (5.18) *for* $G(V, q)$ *and* $\kappa(V, q)$ *define mappings from* $\mathcal{E}^2 \times \mathbb{R}$ *to* $\mathcal{E}^2$ *and to* $\mathbb{R}$ *respectively. On any bounded neighborhood of* $(0, 0)$ *in* $\mathcal{E}^2 \times \mathbb{R}$*, the mappings are Lipschitz, and there is a constant* C *such that:*

(1) $\|G(V, q)\|_\alpha \le C(\|V\|_0 + |q|)\|V\|_\alpha$.

(2) $\|G(V, q)\| \le C(\|V\|_0 + |q|)\|V\|$.

(3) $|\kappa(V, q)| \le Ce^{-\alpha q}(\|V\|_0 + |q|)\|V\|_\alpha$.

Proof. (1) follows from $\|G(V,q)\|_\alpha = \|BS(\xi,q)V + N(V,q)\|_\alpha$ together with Proposition 5.8 (2) and Proposition 5.10 (1). (2) follows from (1) and Proposition 5.9 (3). For (3), note that

$$|\kappa(V,q| = |\pi(H'(\xi-q))|^{-1}|\pi(G(V,q))|.$$

By Lemma 5.4, $|\pi(H'(\xi-q))|^{-1} \le 2$, and $|\pi(G(V,q))|$ is bounded by a constant times the bound on $\|G(V,q)\|_\alpha$ given by (1). ☐

5.5. **Proof of nonlinear stability.** Since $H' \in \mathcal{E}^2$, we see that if $V \in \mathcal{E}^2 \subset \mathcal{E}^2_\alpha$, then $\mathcal{P}^c_\alpha V \in \mathcal{E}^2$, and therefore $\mathcal{P}^s_\alpha V = V - \mathcal{P}^c_\alpha V \in \mathcal{E}^2$. Hence we can define $\mathcal{P}^c$ and $\mathcal{P}^s$ to be operators from $\mathcal{E}^2$ to itself given by restricting $\mathcal{P}^c_\alpha$ and $\mathcal{P}^s_\alpha$ respectively to $\mathcal{E}^2$. We also define π to be the restriction of π_α to $\mathcal{E}^2$. For $V \in \mathcal{E}^2$, $\|V\|_\alpha \le \|V\|$; it follows that π is a bounded linear functional. Therefore $\mathcal{P}^c$ is a bounded operator, so $\mathcal{P}^s = I - \mathcal{P}^c$ is also bounded. It is easy to see that $\mathcal{P}^c$ and $\mathcal{P}^s$ are projections, and the range of one is the kernel of the other. It follows that $\mathrm{R}(\mathcal{P}^{\mathrm{s}})$ is a closed subspace of $\mathcal{E}^2$, and $\mathcal{E}^2 = \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \oplus \mathrm{R}(\mathcal{P}^{\mathrm{c}})$.

5.5.1. *Existence of solutions and* a priori *bound for* $\|V(t)\| + |q(t)|$. We shall study solutions the system (5.20)–(5.21) on $\mathcal{E}^2 \times \mathbb{R}$.

The operator $(\mathcal{A}, 0)$ generates a strongly continuous semigroup on $\mathcal{E}^2 \times \mathbb{R}$. The nonlinearity is locally Lipschitz by Proposition 5.11. Therefore, given initial data $(V^0, q^0) \in \mathcal{E}^2 \times \mathbb{R}$, the system (5.20)–(5.21) has a unique mild solution $(V,q)(t,V^0,q^0)$ with $(V,q)(0,V^0,q^0) = (V^0,q^0)$. The solution is defined for t in the maximal interval $0 \le t < t_{\max}(V^0,q^0)$, where $0 < t_{\max}(V^0,q^0) \le \infty$; see, e.g., [31, Theorem 6.1.4]. The set $\{(t,V^0,q^0) \in \mathbb{R}_+ \times \mathcal{E}^2 \times \mathbb{R} : 0 \le t < t_{\max}(V^0,q^0)\}$ is open in $\mathbb{R}_+ \times \mathcal{E}^2 \times \mathbb{R}$, and the map $(t,V^0,q^0) \mapsto (V,q)(t,V^0,q^0)$ from this set to $\mathcal{E}^2 \times \mathbb{R}$ is continuous; see, e.g., [38, Theorem 46.4].

Moreover, if $(V,q) \in \mathcal{E}^2 \times \mathbb{R}$, then we recall from Section 5.3 that the right hand side of (5.20) belongs to $\mathrm{R}(\mathcal{P}^{\mathrm{s}})$, and $\mathcal{P}^s$ commutes with $\mathcal{A}$ and $e^{t\mathcal{A}}$. We may therefore consider (5.20)–(5.21) on $\mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$. We conclude:

Proposition 5.12. *For each $\delta > 0$, if $0 < \rho < \delta$, then there exists $T_{\max}$, with $0 < T_{\max} \le \infty$, such that the following is true: if $(V^0,q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$ satisfies*

$$\|(V^0,q^0)\|_{\mathcal{E}^2\times\mathbb{R}} = \|V^0\| + |q^0| \le \rho \tag{5.23}$$

and $0 \le t < T_{\max}$, then $(V,q)(t,V^0,q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$ is defined and satisfies

$$\|V(t,V^0,q^0)\| + |q(t,V^0,q^0)| \le \delta. \tag{5.24}$$

We remark that if V^0 is in the domain of the operator $\mathcal{A}$, then the mild solution is in fact a classical solution; see, e.g., [31, Theorem 6.1.5].

Let $T_{\max}(\delta,\rho)$ denote the supremum of all T such that (5.24) holds for all $0 \le t < T$ whenever (5.23) is satisfied.

5.5.2. *Decay of* $\|V(t)\|_\alpha$. Let $\delta_1 < 1$ be chosen as in Lemma 5.4. Let $K > 0$ and $\nu > 0$ be the numbers given by Corollary 5.3, with ν satisfying (5.9).

Proposition 5.13. *There exist δ_2 in $(0,\delta_1)$ and $C > 0$ such that for every $\delta \in (0,\delta_2)$ and every ρ with $0 < \rho < \delta$, the following is true. Let $(V^0,q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$ satisfy (5.23), so that $(V,q)(t,V^0,q^0)$ satisfies (5.24) for $0 \le t < T_{\max}(\delta,\rho)$. Then:*

$$\|V(t)\|_\alpha \le Ke^{-\nu t/2}\|V^0\|_\alpha \text{ and } |q(t)-q^0| \le C\|V^0\|_\alpha \text{ for } 0 \le t < T_{\max}(\delta,\rho). \tag{5.25}$$

Moreover, if $T_{\max}(\delta,\rho) = \infty$, then there is $q^* \in \mathbb{R}$ such that

$$|q(t)-q^*| \le Ce^{-\nu t/2}\|V^0\|_\alpha \text{ for all } t \ge 0. \tag{5.26}$$

The proof is given in Subsection 5.7. It uses the *a priori* bound (5.24), Theorem 5.3 about the linear operator $\mathcal{A}_\alpha$, and Gronwall's inequality.

5.5.3. *Bounds for* $\|V(t)\|_0$.

Proposition 5.14. *There exist* δ_3 *in* $(0, \delta_2)$ *and* $C > 0$ *such that for every* $\delta \in (0, \delta_3)$ *and every and every* ρ *with* $0 < \rho < \delta$, *the following is true. Let* $(V^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$ *satisfy* (5.23)*. Then* $(V, q)(t, V^0, q^0)$ *satisfies* (5.24) *for* $0 \le t < T_{\max}(\delta, \rho)$*, and the following estimates hold for* $0 \le t < T_{\max}(\delta, \rho)$*):*

$$\|v_1(t)\|_0 \le C(\|V^0\| + |q^0|), \tag{5.27}$$

$$\|v_2(t)\|_0 \le C(\|V^0\| + |q^0|)e^{-\nu t/2}. \tag{5.28}$$

This proposition establishes the required boundedness in the uniform norm. The proof is given in Subsection 5.8. It uses the *a priori* bound (5.24) and Proposition 5.13 to make estimates.

5.5.4. *Nonlinear stability.*

Lemma 5.15. *Define* $\mathcal{F} : \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R} \to \mathcal{E}^2$ *by* $\mathcal{F}(V, q) = V + H(\xi - q)$*. Then* $D\mathcal{F}(0, 0)$ *is an isomorphism, so* $\mathcal{F}$ *maps a neighborhood* $\mathcal{V}$ *of* $(0, 0)$ *in* $\mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$ *diffeomorphically onto a neighborhood* $\mathcal{U}$ *of* H *in* $\mathcal{E}^2$*.*

Proof. The mapping $q \to H(\xi - q)$ is C^1 as a map from $\mathbb{R}$ to $\mathcal{E}^2$, so $\mathcal{F}$ is C^1. $R(\mathcal{P}^s)$ is a codimension-one subspace of $\mathcal{E}^2$, and $\frac{\partial \mathcal{F}}{\partial q}(0, 0) = -H'(\xi)$ is not in it. Therefore $D\mathcal{F}(0, 0)$ is an isomorphism. The rest of the result is a consequence of the Inverse Function Theorem. □

Assume that $\mathcal{V}$ is chosen small enough so $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Lipschitz. Let Q denote the Lipschitz constant of $\mathcal{F}^{-1}$.

Choose $\rho_{\mathcal{U}} > 0$ so that the closed ball of radius $\rho_{\mathcal{U}}$ about H in $\mathcal{E}^2$ is contained in $\mathcal{U}$. We have:

Lemma 5.16. *Let* $U \in \mathcal{E}^2$ *with* $\|U - H\| \le \rho_{\mathcal{U}}$*. Then:*

(1) $(V, q) = \mathcal{F}^{-1}(U) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$ *is defined, so* $U = V + H(\xi - q)$*.*
(2) $\|V\| + |q| \le Q\|U - H\|$*.*

Given $U^0 \in \mathcal{E}^2$, let $U(t) = U(t, U^0)$ be the solution of (5.4) in $\mathcal{E}^2$ with $U(0) = U^0$. If $\|U^0 - H\| \le \rho_{\mathcal{U}}$, we can use Lemma 5.16 to write

$$U^0 = V^0 + H(\xi - q^0) \text{ with } (V^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}. \tag{5.29}$$

If $\|U(t) - H\| \le \rho_{\mathcal{U}}$, we can use Lemma 5.16 to write

$$U(t) = V(t) + H(\xi - q(t)) \text{ with } (V(t), q(t)) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}. \tag{5.30}$$

The following theorem gathers all of our nonlinear stability results.

Theorem 5.17. *There is a constant* $C > 0$ *such that for each* $\delta \in (0, \max(\delta_3, \rho_{\mathcal{U}}))$*, there exists* ρ *with* $0 < \rho \le \rho_{\mathcal{U}}$ *such that the following is true. Let* $U^0 \in \mathcal{E}^2$ *with* $\|U^0 - H\| < \rho$*, and let* (V^0, q^0) *be given by* (5.29)*. Let* $U(t)$ *be the solution of* (5.4) *in* $\mathcal{E}^2$ *with* $U(0) = U^0$*. Then:*

(1) $U(t)$ *is defined for all* $t \ge 0$*.*
(2) *For all* $t \ge 0$, $U(t) \in \mathcal{U}$, *so we can define* $(V(t), q(t))$ *by* (5.30)*.*
(3) $\|V(t)\| + |q(t)| < \delta$*.*

(4) $\|V(t)\|_\alpha \leq Ke^{-\nu t/2}\|V^0\|_\alpha$.
(5) *There exists* q^* *such that* $|q(t) - q^*| \leq Ce^{-\nu t/2}\|V^0\|_\alpha$.
(6) $\|v_1(t)\|_0 \leq C(\|V^0\| + |q^0|)$.
(7) $\|v_2(t)\|_0 \leq C(\|V^0\| + |q^0|)e^{-\nu t/2}$.

Note that (4) and (5) imply easily that for a larger constant $\tilde{C}$, $\|U(t) - H(\xi - q^*)\|_\alpha \leq \tilde{C}e^{-\nu t/2}\|\tilde{V}^0\|_\alpha$.

Proof. Let $\delta \in (0, \delta_3)$. Let ρ_1 be given by Proposition 5.12. Let $\rho_2 = \frac{\rho_1}{\max(1,C)}$, where C is the larger of the constants appearing in Propositions 5.13 and 5.14. Let $\rho_3 = \frac{\rho_2}{Q}$. Let $\rho = \min(\rho_3, \rho_{\mathcal{U}})$.

Let $U^0 \in \mathcal{E}_2$ with $\|U^0 - H\| \leq \rho$. By Lemma 5.16, there exist $(V^0, q^0) \in \mathrm{R}(\mathcal{P}^s) \times \mathbb{R}$ with $U^0 = V^0 + H(\xi - q^0)$ and $\|V_0\| + |q^0| \leq Q\rho \leq \rho_2 \leq \rho_1$. By Proposition 5.12, $(V, q)(t, V^0, q^0)$ is defined for $0 \leq t < T_{\max}(\delta, \rho_2)$; by Propositions 5.12, 5.13, and 5.14, it satisfies (5.24), (5.25), (5.27), and (5.28).

We claim that $T_{\max}(\delta, \rho_2) = \infty$. If $T_{\max}(\delta, \rho_1) = \infty$, then this is clearly true, so suppose that $T_{\max}(\delta, \rho_1)$ is finite. Let $(V^0, q^0) \in \mathrm{R}(\mathcal{P}^s) \times \mathbb{R}$ with $\|V_0\| + |q^0| \leq \rho_2$. For any T in $(0, T_{\max}(\delta, \rho_2))$, the inequalities (5.25), (5.27), and (5.28) yield

$$\|V(T, V^0)\| + |q(T, q^0)| \leq C(\|V^0\| + |q^0|) \leq C\rho_2 \leq \rho_1. \tag{5.31}$$

Consider the solution with initial data $(V^1, q^1) = (V, q)(T, V^0, q^0)$. Since $\|V^1\| + |q^1| \leq \rho_1$, Proposition 5.12 applies to this solution. Therefore, for all $t \in [0, T_{\max}(\delta, \rho_1))$, we have

$$\|V(t + T, V^0)\| + |q(t + T, q^0)| = \|V(t, V^1)\| + |q(t, q^1)| \leq \delta. \tag{5.32}$$

This shows that the *a priori* bound (5.24) for the solution with any initial data satisfying $\|V^0\| + |q^0| \leq \rho_2$ holds for all $t \in [0, T + T_{\max}(\delta, \rho_1))$. Therefore $T_{\max}(\delta, \rho_2) \geq T + T_{\max}(\delta, \rho_1)$ and, thus, $T_{\max}(\delta, \rho_2) \geq T_{\max}(\delta, \rho_2) + T_{\max}(\delta, \rho_1)$. Hence $T_{\max}(\delta, \rho_2) = \infty$.

(1) follows from $T_{\max}(\delta, \rho_2) = \infty$. For all $t \geq 0$, $\|V(t))\| + |q(t)| \leq \delta < \rho_{\mathcal{U}}$, so $(V(t), q(t) \in \mathcal{V}$, so $U(t) = V(t) + H(\xi - q(t))$ is in $\mathcal{U}$; thus (2) and (3) hold. (4) is just (5.25); (5) is (5.26); (6) and (7) are (5.27) and (5.28). ☐

5.6. **Proof of Proposition 5.6.** We will only consider the case $\mathcal{E}_0 = H^1(\mathbb{R})$; the case $\mathcal{E}_0 = BUC(\mathbb{R})$ is easier. First we show that the mappings go into the correct spaces. We have

$$\|m(\xi, q, u)v\|_{L^2} \leq \|m\|_{C^0}\|v\|_{L^2} \tag{5.33}$$

and

$$\begin{aligned}\|(mv)_\xi\|_{L^2} &\leq \|m_\xi v\|_{L^2} + \|m_u u_\xi v\|_{L^2} + \|m v_\xi\|_{L^2} \\ &\leq \|m\|_{C^1}\|v\|_{L^2} + \|m\|_{C^1}\|u_\xi\|_{L^2}\|v\|_{L^2} + \|m\|_{C^0}\|v_\xi\|_{L^2}.\end{aligned} \tag{5.34}$$

Therefore if $(q, u, v) \in \mathbb{R} \times H^1(\mathbb{R})^2$ then $m(\xi, q, u)v \in H^1(\mathbb{R})$. Next, we have

$$\|e^{\alpha\xi}m(q, u, v)v\|_{L^2} \leq \|m\|_{C^0}\|e^{\alpha\xi}v\|_{L^2} \tag{5.35}$$

and

$$\begin{aligned}\|e^{\alpha\xi}(mv)_\xi\|_{L^2} &\leq \|e^{\alpha\xi}m_\xi v\|_{L^2} + \|e^{\alpha\xi}m_u u_\xi v\|_{L^2} + \|e^{\alpha\xi}m v_\xi\|_{L^2} \\ &\leq \|m\|_{C^1}\|e^{\alpha\xi}v\|_{L^2} + \|m\|_{C^1}\|u_\xi\|_{L^2}\|e^{\alpha\xi}v\|_{L^2} + \|m\|_{C^0}\|e^{\alpha\xi}v_\xi\|_{L^2}.\end{aligned} \tag{5.36}$$

Therefore if $(q, u, v) \in \mathbb{R} \times H^1(\mathbb{R})^2_\alpha$ then $m(\xi, q, u)v \in H^1(\mathbb{R})_\alpha$.

Now we show the Lipschitz properties.

First we consider variations in q. We have

$$m(\xi,q+\bar q,u(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi)=\int_0^1 m_q(\xi,q+t\bar q,u(\xi))\,dt\,\bar q v(\xi).$$

Therefore

$$\|m(\xi,q+\bar q,u)v-m(\xi,q,u)v\|_{L^2}\le \|m\|_{C^1}\|v\|_{L^2}|\bar q|$$

and

$$\|e^{\alpha\xi}(m(\xi,q+\bar q,u)v-m(\xi,q,u)v)\|_{L^2}\le \|m\|_{C^1}\|e^{\alpha\xi}v\|_{L^2}|\bar q|.$$

Also,

$$\begin{aligned}(m(\xi,q+\bar q,u(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi))_\xi&=\int_0^1 m_{q\xi}(\xi,q+t\bar q,u(\xi))\,dt\,\bar q v(\xi)\\ &+\int_0^1 m_{qu}(\xi,q+t\bar q,u(\xi))\,dt\,\bar q u_\xi v(\xi)+\int_0^1 m_q(\xi,q+t\bar q,u(\xi))\,dt\,\bar q v_\xi.\end{aligned}$$

Therefore

$$\begin{aligned}&\|(m(\xi,q+\bar q,u(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi))_\xi\|_{L^2}\\ &\qquad\le(\|m\|_{C^2}\|v\|_{L^2}+\|m\|_{C^2}\|u_\xi\|_{L^2}\|v\|_{L^2}+\|m\|_{C^1}\|v_\xi\|_{L^2})|\bar q|\end{aligned}$$

and

$$\begin{aligned}&\|e^{\alpha\xi}(m(\xi,q+\bar q,u(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi))_\xi\|_{L^2}\\ &\qquad\le(\|m\|_{C^2}\|e^{\alpha\xi}v\|_{L^2}+\|m\|_{C^2}\|u_\xi\|_{L^2}\|e^{\alpha\xi}v\|_{L^2}+\|m\|_{C^1}\|e^{\alpha\xi}v_\xi\|_{L^2})|\bar q|.\end{aligned}$$

Next we consider variations in u. We have

$$m(\xi,q,u(\xi)+\bar u(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi)=\int_0^1 m_u(\xi,q,u(\xi)+t\bar u(\xi))\,dt\,\bar u(\xi)v(\xi).$$

Therefore

$$\|m(\xi,q,u+\bar u)v-m(\xi,q,u)v\|_{L^2}\le\|m\|_{C^1}\|\bar u\|_{L^2}\|v\|_{L^2}$$

and

$$\|e^{\alpha\xi}(m(\xi,q,u+\bar u)v-m(\xi,q,u)v)\|_{L^2}\le\|m\|_{C^1}\|\bar u\|_{L^2}\|e^{\alpha\xi}v\|_{L^2}.$$

Also,

$$\begin{aligned}&(m(\xi,q,u(\xi)+\bar u(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi))_\xi=\\ &\int_0^1 m_{u\xi}(\xi,q,u(\xi)+t\bar u(\xi))\,dt\,\bar u(\xi)v(\xi)+\int_0^1 m_{uu}(\xi,q,u(\xi)+t\bar u(\xi))(u_\xi+t\bar u_\xi)\,dt\,\bar u(\xi)v(\xi)\\ &\qquad+\int_0^1 m_u(\xi,q,u(\xi)+t\bar u(\xi))\,dt\,\bar u_\xi v(\xi)+\int_0^1 m_u(\xi,q,u(\xi)+t\bar u(\xi))\,dt\,\bar u(\xi)v_\xi.\end{aligned}$$

Therefore

$$\begin{aligned}&\|(m(\xi,q,u(\xi)+\bar u(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi))_\xi\|_{L^2}\\ &\qquad\le\|m\|_{C^2}\|\bar u\|_{L^2}\|v\|_{L^2}+\|m\|_{C^2}\|u_\xi\|_{L^2}C\|\bar u\|_{H^1}\|v\|_{H^1}\\ &\qquad+\frac12\|m\|_{C^2}\|\bar u_\xi\|_{L^2}C\|\bar u\|_{H^1}\|v\|_{H^1}+\|m\|_{C^1}\|\bar u_\xi\|_{L^2}\|v\|_{L^2}+\|m\|_{C^1}\|\bar u\|_{L^2}\|v_\xi\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}\|e^{\alpha\xi}&(m(\xi,q,u(\xi)+\bar{u}(\xi))v(\xi)-m(\xi,q,u(\xi))v(\xi))_\xi\|_{L^2}\\ &\le \|m\|_{C^2}\|\bar{u}\|_{L^2}\|e^{\alpha\xi}v\|_{L^2}+\|m\|_{C^2}\|u_\xi\|_{L^2}C\|\bar{u}\|_{H^1}\|e^{\alpha\xi}v\|_{H^1}\\ &+\frac{1}{2}\|m\|_{C^2}\|\bar{u}_\xi\|_{L^2}C\|\bar{u}\|_{H^1}\|e^{\alpha\xi}v\|_{H^1}+\|m\|_{C^1}\|\bar{u}_\xi\|_{L^2}\|e^{\alpha\xi}v\|_{L^2}\\ &\qquad+\|m\|_{C^1}\|\bar{u}\|_{L^2}\|e^{\alpha\xi}v_\xi\|_{L^2}.\end{aligned}$$

Finally, we consider variations in v. We have

$$m(\xi,q,u(\xi))(v(\xi)+\bar{v}(\xi))-m(\xi,q,u(\xi))v(\xi)=m(\xi,q,u(\xi))\bar{v}(\xi).$$

Estimates are left to the reader.

Using the separate Lipschitz estimates for variations in q, u, and v, one can easily show that the mappings are Lipschitz on the given sets.

To prove the estimates when $m(\xi,0,u)=0$, we note that this assumption implies that $\|m\|_{C^0}\le C|q|$ and $\|m\|_{C^1}\le C|q|$ on the given sets, then use (5.33)–(5.36).

5.7. **Proof of Proposition 5.13.** Since $V(t)$ is a mild solution of (5.20) in $\mathcal{E}^2$, it satisfies the integral equation

$$V(t)=e^{t\mathcal{A}}V^0+\int_0^t e^{(t-s)\mathcal{A}}\Big(G(V,q(s))+\kappa(V,q(s))H'(\xi-q(s))\Big)\,ds. \tag{5.37}$$

Since $V^0\in\mathcal{E}^2$ by assumption, and $G(V,q(s))+\kappa(V,q(s))H'(\xi-q(s))$ is in $\mathcal{E}^2$, Proposition 5.2 implies that $e^{t\mathcal{A}}V^0=e^{t\mathcal{A}_\alpha}V^0$ and

$$\begin{aligned}e^{(t-s)\mathcal{A}}&\Big(G(V,q(s))+\kappa(V,q(s))H'(\xi-q(s))\Big)\\ &=e^{(t-s)\mathcal{A}_\alpha}\Big(G(V,q(s))+\kappa(V,q(s))H'(\xi-q(s))\Big).\end{aligned}$$

Therefore (5.37) holds with $\mathcal{A}$ replaced by $\mathcal{A}_\alpha$. In addition, $V^0\in\mathrm{R}(\mathcal{P}^{\mathrm{s}}_\alpha)$, and we recall from Section 5.3 that $G(V,q(s))+\kappa(V,q(s))H'(\xi-q(s))$ is in $\mathrm{R}(\mathcal{P}^{\mathrm{s}}_\alpha)$. Therefore (5.37) holds with $\mathcal{A}$ replaced by $\mathcal{A}_\alpha\mathcal{P}^s_\alpha$.

By Theorem 5.3 and Proposition 5.11 (1) and (4),

$$\begin{aligned}\|V(t)\|_\alpha&\le Ke^{-\nu t}\|V^0\|_\alpha\\ &+\int_0^t Ke^{-\nu(t-s)}C(\|V(s)\|_0+|q(s)|)(1+\|H'(\cdot-q(s))\|_\alpha)\|V(s)\|_\alpha\,ds.\end{aligned}$$

Using the *a priori* bound (5.24), one finds a constant C_1 so that

$$\|V(t)\|_\alpha\le Ke^{-\nu t}\|V^0\|_\alpha+C_1\delta\int_0^t e^{-\nu(t-s)}\|V(s)\|_\alpha\,ds. \tag{5.38}$$

Choosing $\delta_2<\frac{\nu}{2C_1}$ and using Gronwall's inequality for the function $u(t)=e^{\nu t}\|V(t)\|_\alpha$ (see, e.g., [21, Section 1.2.1]), we arrive at the first estimate in (5.25).

From Proposition 5.11 (3), the *a priori* bound (5.24) and the first estimate in (5.25), we have

$$\begin{aligned}|\dot q(t)| = |\kappa(V(t), q(t))| &\le C_2(|q(t)| + \|V(t)\|_0)\|V(t)\|_\alpha \\ &\le C_2 K\delta e^{-\nu t/2}\|V^0\|_\alpha = Ce^{-\nu t/2}\|V^0\|_\alpha, \quad (5.39)\end{aligned}$$

where $C = C_2 K\delta$. Using (5.39) and

$$q(t) = q(s) + \int_0^t \dot q(\tau)\, d\tau, \quad 0 \le t < T_{\max}(\delta, \rho), \tag{5.40}$$

we obtain the second estimate in (5.25):

$$|q(t) - q^0| \le \int_0^t |\dot q(\tau)|\, d\tau \le C\|V^0\|_\alpha \int_0^t e^{-\nu\tau/2}\, d\tau \le \frac{2C}{\nu}\|V^0\|_\alpha. \tag{5.41}$$

Finally, if $T_{\max}(\delta, \rho) = \infty$ then (5.39) implies that in (5.40), $\lim_{t\to\infty} q(t) = q^*$ exists. From (5.40) and (5.39) we have

$$|q^* - q(t)| \le \int_t^\infty |\dot q(\tau)|\, d\tau \le \frac{2C}{\nu} e^{-\nu t/2}\|V^0\|_\alpha.$$

5.8. **Proof of Proposition 5.14.** Using $A = L + BR(\xi)$, (5.10), (5.17), and (5.16), we rewrite (5.20) as:

$$\partial_t V = LV + BR(\xi - q)V + N(V, q) + \kappa(V, q)H'(\xi - q). \tag{5.42}$$

Let

$$\begin{aligned}
g_1(\xi, q) &= h_2(\xi - q)\rho'(h_1(\xi - q)), && (5.43)\\
g_2(\xi, q) &= e^{-\beta} - \rho(h_1(\xi - q)), && (5.44)\\
g_3(\xi, v_1, q) &= h_2(\xi - q)\rho_2(h_1(\xi - q), v_1)v_1, && (5.45)\\
f_1(\xi, v_1, q) &= \rho_1(h_1(\xi - q), v_1)v_1, && (5.46)\\
a(\xi, v_1, q) &= -\beta e^{-\beta} - \beta f_1(\xi, v_1, q), && (5.47)\\
G_1(\xi, V, q) &= g_1(\xi, q)v_1 + \rho(h_1(\xi - q))v_2 + f_1(\xi, v_1, q)v_2 && (5.48)\\
&\quad + g_3(\xi, v_1, q)v_1 + \kappa(V, q)h_1'(\xi - q), && (5.49)\\
G_2(\xi, V, q) &= -\beta g_1(\xi, q)v_1 + \beta g_2(\xi, q)v_2 - \beta g_3(\xi, v_1, q)v_1 + \kappa(V, q)h_2'(\xi - q). && (5.50)
\end{aligned}$$

We then rewrite (5.42) as

$$\partial_t v_1 = (\partial_{\xi\xi} + c\partial_\xi)v_1 + G_1(\xi, V, q), \tag{5.51}$$
$$\partial_t v_2 = (c\partial_\xi + a(\xi, v_1, q))v_2 + G_2(\xi, V, q). \tag{5.52}$$

Let δ_2 and C be given by Proposition 5.13. Let $\delta \in (0, \delta_2)$, let $0 < \rho < \delta$, let $(V^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}) \times \mathbb{R}$ satisfy (5.23). We consider the following nonautonomous linear system related to (5.51)–(5.52):

$$\partial_t \hat v_1 = (\partial_{\xi\xi} + c\partial_\xi)\hat v_1 + G_1(\xi, V(t), q(t)), \tag{5.53}$$
$$\partial_t \hat v_2 = (c\partial_\xi + a(\xi, q(t), v_1(t)))\hat v_2 + G_2(\xi, V(t), q(t)), \tag{5.54}$$

where $(V, q)(t) = (V, q)(t, V^0, q^0)$. Since $(V, q)(t)$ is a fixed solution of (5.20)–(5.21) in $\mathcal{E}^2 \times \mathbb{R}$, we can regard (5.53)–(5.54) as a nonautonomous linear system on $\mathcal{E}^2$. The solution with the value V^0 at $t = 0$ is of course $V(t, V^0, q^0)$.

We claim:

- If we regard (5.53)–(5.54) as a nonautonomous linear equation on $\mathcal{E}_0^2$ with the initial condition $\hat{V}(0) = V^0$, the solution, which we denote $\hat{V}(t, V^0)$, is again $V(t, V^0, q^0)$.

To see this, we first note that the nonhomogeneous linear system

$$\partial_t \hat{v}_1 = (\partial_{\xi\xi} + c\partial_\xi)\hat{v}_1 + h_1(t, \xi), \tag{5.55}$$

$$\partial_t \hat{v}_2 = (c\partial_\xi - (\beta e^{-\beta} + b_2(t, \xi)))\hat{v}_2 + h_2(t, \xi), \tag{5.56}$$

may be solved in $\mathcal{E}_0^2$ on $0 \le t \le T$ provided (h_1, b_2, h_2) is a continuous function from $[0, T]$ to $\mathcal{E}_0^3$ and the initial condition $\hat{V}^0 \in \mathcal{E}_0^2$. We have:

The solution of (5.55)–(5.56) in $\mathcal{E}_0^2$ on $0 \le t \le T$ depends continuously on $(\hat{V}(0), h_1, b_2, h_2)$. (5.57)

Let V^{0k} be a sequence of C^2 functions that converges to V^0 in $\mathcal{E}$ as $k \to \infty$. The solution of (5.20)–(5.21) in $\mathcal{E}^2 \times \mathbb{R}$ on $0 \le t \le T$ with initial condition (V^{0k}, q^0) is $(V, q)(t, V^{0k}, q^0)$. Then $(V, q)(t, V^{0k}, q^0) \to (V, q)(t, V^0, q^0)$ in $\mathcal{E}^2$, so

$$(V, q)(t, V^{0k}, q^0) \to (V, q)(t, V^0, q^0) \text{ in } \mathcal{E}_0^2. \tag{5.58}$$

Now consider (5.55)–(5.56) on $\mathcal{E}_0^2$ with

$$h_1(t, \xi) = G_1(\xi, (V, q)(t, V^{0k}, q^0)), \tag{5.59}$$

$$b_2(t, \xi) = \beta f_1(\xi, (V, q)(t, V^{0k}, q^0)), \tag{5.60}$$

$$h_2(t, \xi) = G_2(\xi, (V, q)(t, V^{0k}, q^0)), \tag{5.61}$$

$$\hat{V}(0) = V^{0k}. \tag{5.62}$$

The solution $\hat{V}(t, V^{0k})$ is of course the classical solution $V(t, V^{0k}, q^0)$.

From (5.58),

$$(G_1, f_1, G_2)(\xi, (V, q)(t, V^{0k}, q^0)) \to (G_1, f_1, G_2)(\xi, (V, q)(t, V^0, q^0)) \text{ in } \mathcal{E}_0^2. \tag{5.63}$$

To see this, use Corollary 5.7 on each summand in the definitions of these functions.

By (5.63) and (5.57), with the formulas (5.59)–(5.62), $V(t, V^{0k}, q^0) = \hat{V}(t, V^{0k}) \to \hat{V}(t, V^0)$ in $\mathcal{E}_0^2$. From (5.58) we conclude that $\hat{V}(t, V^0) = V(t, V^0, q^0)$, which completes the proof of the claim.

We easily see:

$$\begin{aligned} &g_i(\xi, q)e^{-\alpha\xi} \in \mathcal{E}_0, \quad i = 1, 2, 3, \quad \|g_3(\xi, q)e^{-\alpha\xi}\| \le C\|v_1\|_0, \\ &\qquad\qquad f_1(\xi, v_1(\xi), q) \in \mathcal{E}_0, \quad \|f_1(\xi, v_1(\xi), q)\|_0 \le \|\rho_1(h_1(\xi - q))\|_{C^0}\|v_1\|_0. \end{aligned} \tag{5.64}$$

(The fact that $f_1(\xi, v_1(\xi), q) \in \mathcal{E}_0$ follows from Corollary 5.7.) Let $W(\xi) = e^{\alpha\xi}V(\xi)$. Using (5.64), Proposition 5.11 (5), the *a priori* bound (5.24), and the exponential decay of

$\|W(t)\|_0 = \|V(t)\|_\alpha$ in (5.25), we have the estimates:

$$\begin{aligned}\|G_1(\xi, V, q)(t)\|_0 &\le C(\|g_1 e^{-\alpha\xi}\|_0 \|W(t)\|_0 + \|\rho(h_1(\xi - q))\|_{C^0}\|v_2(t)\|_0 + \|f_1\|_0\|v_2(t)\|_0 \\ &\quad + \|g_3 e^{-\alpha\xi}\|_0\|W(t)\|_0 + e^{-\alpha q(t)}(\|V(t)\|_0 + |q(t)|)\|W(t)\|_0) \\ &\le C(\|W(t)\|_0 + (1+\delta)\|v_2(t)\|_0 + e^{\alpha\delta}\delta\|W(t)\|_0) \le C(\|v_2(t)\|_0 + e^{-\nu t/2}\|W^0\|_0); \quad (5.65)\end{aligned}$$

$$\begin{aligned}\|G_2(\xi, V, q)(t)\|_0 &\le C(\|g_1 e^{-\alpha\xi}\|_0\|W(t)\|_0 + \|g_2 e^{-\alpha\xi}\|_0\|W(t)\|_0 \\ &\quad + \|v_1(t)\|_0\|W(t)\|_0 + e^{-\alpha q(t)}(\|V(t)\|_0 + |q(t)|)\|W(t)\|_0) \\ &\le C(1 + \delta + e^{\alpha\delta}\delta)\|W(t)\|_0 \le C e^{-\nu t/2}\|W^0\|_0. \quad (5.66)\end{aligned}$$

We shall first use the fact that (5.54), regarded as a nonautonomous linear equation on $\mathcal{E}_0$, has the solution $\hat{v}_2(t) = v_2(t)$ to show (5.28). The solution of the linear equation

$$\partial_t \hat{v}_2 = (c\partial_\xi + a(t,\xi))\hat{v}_2 \quad (5.67)$$

on $\mathcal{E}_0$ or $\mathcal{E}$ is

$$\hat{v}_2(t,\xi) = \Lambda(t,s)v_2(s,\xi) = \exp\left(\int_s^t a(\tau, \xi + c(t-\tau))\,d\tau\right)\hat{v}_2(s, \xi + c(t-s));$$

the propagator $\Lambda(t,s)$ of (5.67) is defined by this equation. Therefore the mild solution of (5.54), which because of the claim we denote $v_2(t)$, satisfies the equation

$$\begin{aligned}v_2(t) = \Lambda(t,0)v_2(0) + \int_0^t \Lambda(t,\tau)G_2(V(\tau), q(\tau))\,d\tau \\ = \exp\left(\int_0^t a(\tau, \xi + c(t-\tau))\,d\tau\right) v_2(0, \xi + ct) \\ + \int_0^t \exp\left(\int_\tau^t a(s, \xi + c(t-s))\,ds\right) G_2(V(\tau, \xi + c(t-\tau)), q(\tau))\,d\tau. \quad (5.68)\end{aligned}$$

From the *a priori* bound (5.24) and (5.64), there is a constant C_1 independent of δ such that for all $\xi \in \mathbb{R}$ we have:

$$|f_1(\xi, v_1(\xi), q)| \le \|f_1(\cdot, v_1(\cdot), q)\|_{C^0} \le \|f_1(\cdot, v_1(\cdot), q)\|_0 \le C_1\|v_1\|_0 \le C_1\delta. \quad (5.69)$$

Using this constant C_1, we fix $\delta_3 \in (0, \delta_2)$ so small that

$$\delta_3 < \frac{1}{2C_1}e^{-\beta}. \quad (5.70)$$

According to (5.47), for $\delta \in (0, \delta_3)$ we then arrive at the estimate

$$a(t,\xi) \le -\beta e^{-\beta} + \beta\|f_1(\cdot, v_1(t,\cdot), q(t))\|_{C^0} \le -\frac{\beta}{2}e^{-\beta}. \quad (5.71)$$

Using (5.71) and (5.66), we obtain the following estimate in (5.68):

$$\|v_2(t)\|_0 \le e^{-t(\beta/2)e^{-\beta}}\|v_2(0)\|_0 + C\int_0^t e^{-(t-\tau)(\beta/2)e^{-\beta}} e^{-\nu\tau/2}\,d\tau\,\|W^0\|_0. \quad (5.72)$$

Since $\nu < \beta e^{-\beta}$ by (5.9), the required inequality (5.28) follows.

Finally, we use the fact that (5.53), regarded as a nonautonomous linear equation on $\mathcal{E}_0$, has the solution $\hat{v}_1(t) = v_1(t)$ to show (5.27). Let $\hat{L} = \partial_{\xi\xi} + c\partial_\xi$. Then $e^{t\hat{L}}$ can be written explicitly using the heat kernel; from this expression we see that the norm of the operator $e^{t\hat{L}}$ on $\mathcal{E}_0$ is uniformly bounded for $t \geq 0$. The mild solution of (5.53), which because of the claim we denote $v_1(t,\xi)$, satisfies the integral equation

$$v_1(t,\xi) = e^{t\hat{L}} v_1(0,\xi) + \int_0^t e^{(t-\tau)\hat{L}} G_1(\xi, V(\tau,\xi), q(\tau))\, d\tau.$$

Applying (5.65), the exponential decay of W as in (5.25), and the inequality (5.28) just proved, we infer (5.27):

$$\|v_1(t)\|_0 \leq C\|v_1(0)\|_0 + C\int_0^t \Big(\|v_2(\tau)\|_0 + e^{-\nu\tau/2}\|W^0\|_0\Big)\, d\tau$$
$$\leq C(\|V^0\| + |q^0|)(1 + \int_0^t e^{-\nu\tau/2}\, d\tau).$$

6. Extensions

We have studied (1.1)–(1.3) with one of the boundary conditions being $u_1(\infty) = 0$. Thus we have taken $u_1 = 0$ to be both the ignition temperature (the value above which $\rho > 0$) and the background temperature $u_1(\infty)$. Let us consider various alternatives to these choices.

(1) It is generally considered that the true ignition temperature is absolute 0. Thus we should use (1.1)–(1.3) with $u_1 = 0$ representing absolute 0, and the right boundary condition should be $u_1(\infty) = u_1^* > 0$. This choice presents the "cold boundary difficulty" [6]: there is no traveling wave solution (because the boundary is not cold enough). The assumptions we have used represent the minimal change that allows a traveling wave.

(2) A more drastic change in the model would be to take the ignition temperature to be higher than the background temperature. Thus if $u_1 = 0$ represents ignition temperature, one could use (1.1)–(1.3) together with the right boundary condition $u_1(\infty) = u_1^* < 0$. The traveling wave equation gives rise to a phase portrait qualitatively like that in Figure 2.1. However, the desired traveling wave connects the hyperbolic equilibrium on the positive u_1-axis to one of the semihyperbolic equilibria in $u_1 < 0$. Such a solution of the traveling wave equation automatically approaches both end states exponentially. Our results hold for this case.

(3) Another possible change to the model is to allow a linear convection term in the first equation, so that (1.1) becomes

$$\partial_t u_1 = \partial_{xx} u_1 + a\partial_x u_1 + \omega(u_1, u_2).$$

Thus heat, in addition to diffusing, is convected, for example by a flow of gas over the solid that is not otherwise modeled. This situation is important in oil recovery [11]. The symmetry $x \mapsto -x$ is broken, so one should consider traveling waves with both positive and negative velocities. Results similar to ours should hold with minor adjustments.

(4) In all these situations the unit reaction rate $\rho(u_1)$ can be replaced by any function in $C^2(\mathbb{R})$ that equals 0 for $u_1 \leq 0$ and is positive for $u_1 > 0$. (For existence of the front, see [24].) The theorems still hold; however, we don't know how computations like that described in Appendix A would turn out.

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Appendix A. Numerical computation of the Evans function

Figure A.1 (a) shows part of the Evans function $D(\lambda)$ for $\beta = 1$. We have allowed λ to vary on the curve Γ in the complex plane given by: (1) $\operatorname{Re}\lambda = -0.05$, $0 \le \operatorname{Im}\lambda \le 1000$, (2) $-0.05 \le \operatorname{Re}\lambda \le 1000$, $\operatorname{Im}\lambda = 1000$, (3) $\operatorname{Re}\lambda = 1000$, $0 \le \operatorname{Im}\lambda \le 1000$. As λ traverses this curve in the clockwise direction, $D(\lambda)$ traces out the curve shown in the clockwise direction. Since Γ has two corners, the curve shown has two corners. Both occur in the small boxed region; a blow-up of this region is shown in Figure A.1 (b).

Let $\bar\Gamma = \{\bar\lambda : \lambda \in \Gamma\}$. Then $\Gamma \cup \bar\Gamma$ is a curve in the complex plane that surrounds the origin once. $D(\Gamma \cup \bar\Gamma)$ is the curve shown together with its reflection across the imaginary axis. Since $D(\Gamma \cup \bar\Gamma)$ clearly winds once around 0, the analytic function $D(\lambda)$ has just one zero, of order one, in the interior of $\Gamma \cup \bar\Gamma$. This is of course $\lambda = 0$. Thus we have numerical evidence that the only zero of $D(\lambda)$ with $\operatorname{Re}\lambda \ge 0$ is $\lambda = 0$.

Let us describe how the computation was done. To calculate $D(\lambda)$ exactly, given $\beta > 0$, one first needs a pair $(\sigma, (u_1, u_2)(\xi))$, $\sigma \in \mathbb{R}$, $-\infty < \xi < \infty$, $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$, such that:

(E1) $(u_1, u_2)(\xi)$ is a solution of (2.12)–(2.13)
(E2) $(u_1, u_2)(-\infty) = (\frac{1}{\beta}, 0)$.
(E3) $(u_1, u_2))(\infty) = (0, 1)$.

Next, in the expressions for $\mu(\lambda)$, $E(\xi)$, F, $\hat{\mathbf{z}}(\lambda)$, and $\hat{\boldsymbol\psi}(\lambda)$ in Section 3, we replace c by σ, and think of all these expressions as functions of the pair (σ, λ), although we suppress σ in the notation. Then, given the pair $(\sigma, (u_1, u_2)(\xi))$ just found and $\lambda \in \mathbb{C}$, one needs $\mathbf{z}(\xi)$, $0 \le \xi < \infty$, $\mathbf{z} \in \mathbb{C}^3$, and $\boldsymbol\psi(\xi)$, $-\infty < \xi \le 0$, $\boldsymbol\psi \in (\mathbb{C}^3)^T$, such that:

(E4) $\mathbf{z}(\xi)$ is a solution of $\mathbf{z}' = (E(\xi) + \lambda F)\mathbf{z}$.
(E5) $\lim_{\xi\to\infty} e^{-\mu(\lambda)\xi}\mathbf{z}(\xi) = \hat{\mathbf{z}}(\lambda)$.

(E6) $\boldsymbol{\psi}(\xi)$ is a solution of $\boldsymbol{\psi}' = -\boldsymbol{\psi}(E(\xi) + \bar{\lambda}F)$.
(E7) $\lim_{\xi\to-\infty} e^{\mu(\bar{\lambda})\xi}\boldsymbol{\psi}(\xi) = \hat{\boldsymbol{\psi}}(\bar{\lambda})$.

The value of the Evans function D at λ is then $\bar{\boldsymbol{\psi}}(0)\mathbf{z}(0)$.

We approximate this calculation by one that uses a finite interval $T_- \le \xi \le T_+$, with $T_- < 0 < T_+$. Given $\beta > 0$, we first need a pair $(\sigma, (u_1, u_2)(\xi))$, $\sigma \in \mathbb{R}$, $T_- \le \xi \le T_+$, $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}$, such that:

(A1) $(u_1, u_2)(\xi)$ is a solution of (2.12)–(2.13).
(A2) $\hat{\boldsymbol{\psi}}(0)((u_1, u_2)(T_-) - (\frac{1}{\beta}, 0)) = 0$.
(A3) $u_2(T_+) = 1$.

Condition (2) says that $(u_1, u_2)(T_-)$ is in the first-order approximation to the unstable manifold of (2.12)–(2.13) at $(\frac{1}{\beta}, 0)$, and condition (3) says that $(u_1, u_2)(T_+)$ is in the first-order approximation to the stable manifold of (2.12)–(2.13) at $(0, 1)$.

Next, given the pair $(\sigma, (u_1, u_2)(\xi))$ just found and $\lambda \in \mathbb{C}$, we need $\mathbf{z}(\xi)$, $0 \le \xi \le T_+$, $\mathbf{z} \in \mathbb{C}^3$, and $\boldsymbol{\psi}(\xi)$, $T_- \le \xi \le 0$, $\boldsymbol{\psi} \in (\mathbb{C}^3)^T$, such that:

(A4) $\mathbf{z}(\xi)$ is a solution of $\mathbf{z}' = (E(\xi) + \lambda F)\mathbf{z}$.
(A5) $\mathbf{z}(T_+)$ is a multiple of $\hat{\mathbf{z}}(\lambda)$.
(A6) $\boldsymbol{\psi}(\xi)$ is a solution of $\boldsymbol{\psi}' = -\boldsymbol{\psi}(E(\xi) + \bar{\lambda}F)$.
(A7) $\boldsymbol{\psi}(T_-)$ is a multiple of $\hat{\boldsymbol{\psi}}(\bar{\lambda})$

Since solutions of (A1)–(A3) are only unique up to a phase shift, and solutions of (A4)–(A5) and (A6)–(A7) are only unique up to multiplication by a complex constant, we also need three more conditions:

(A8) $(u_1, u_2)(\xi)$ satisfies a phase condition.
(A9) $\mathbf{z}(\xi)$ satisfies a boundary condition.
(A10) $\boldsymbol{\psi}(\xi)$ satisfies a boundary condition.

Taking into account the parameter σ, there is a three-dimensional space of solutions of (A1); the three conditions (A2), (A3), (A8) pick out one of these solutions, including the value of σ. For each (σ, λ), the space of solutions of (A4) has three complex dimensions; conditions (A5) and (A9) constitute three complex conditions, which pick out a unique solution. A similar argument applies to (A6). Once $\mathbf{z}(\xi)$ and $\boldsymbol{\psi}(\xi)$ are found, the value of the approximate Evans function at λ is again $\bar{\boldsymbol{\psi}}(0)\mathbf{z}(0)$.

The phase condition used in (A8) is an integral one suited to the computation of heteroclinic solutions [12]. In (A9) and (A10), we used $(z_1^2 + z_2^2 + z_3^2)(0) - 1 = 0$ and $(\psi_1^2 + \psi_2^2 + \psi_3^2)(0) - 1 = 0$. These conditions are not guaranteed to produce moderate values of $D(\lambda)$, but did so in practice. We note that in order that $D(\lambda)$ be analytic, (A9) and (A10) must be analytic in λ, which they are. The change in boundary conditions results in multiplying the function $D(\lambda)$ from Section 3 by an analytic function that is nonzero on the domain in which we are interested; this change does not alter the winding number about 0.

The AUTO computation has, in addition to the parameters (β, σ, λ), the parameters T_- and T_+. To start the computation we used a graphical ODE solver to find a value of σ for which there is solution $(u_1, u_2)(\xi)$ that, when restricted to a short enough interval $T_- \le \xi \le T_+$, appears to be approximately heteroclinic. Using this solution we then compute approximations to $\mathbf{z}(\xi)$ and $\boldsymbol{\psi}(\xi)$ for $\lambda = 0$ by setting $\mathbf{z}(\xi)$ equal to a multiple of $(u_1'(\xi), u_2'(\xi), 0)$ and using Proposition 3.3. AUTO then uses this starting point to compute a pair $(\sigma, (u_1, u_2)(\xi))$ that "exactly" solves (2.12)–(2.13) on $T_- \leq \xi \leq T_+$, and "exact" solutions $\mathbf{z}(\xi)$ and $\boldsymbol{\psi}(\xi)$ for $\lambda = 0$. One can then use AUTO's continuation routines to increase $|T_-|$ and T_+ and to vary λ. The computation shown used $T_- = -103.5$ and $T_+ = 295$.

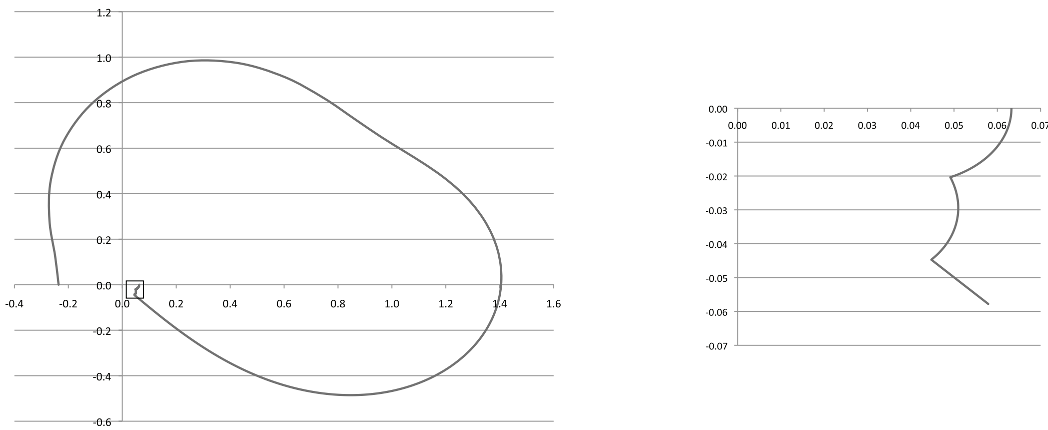

FIGURE A.1. Evans function.

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