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Traveling Waves in Porous Media Combustion: Uniqueness of Waves for Small Thermal Diffusivity

Anna Ghazaryan,[1,3] Peter Gordon,[2] and Christopher K. R. T Jones[1]

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We study traveling wave solutions arising in Sivashinsky's model of subsonic detonation which describes combustion processes in inert porous media. Subsonic (shockless) detonation waves tend to assume the form of a reaction front propagating with a well defined speed. It is known that traveling waves exist for any value of thermal diffusivity [5]. Moreover, it has been shown that, when the thermal diffusivity is neglected, the traveling wave is unique. The question of whether the wave is unique in the presence of thermal diffusivity has remained open. For the subsonic regime, the underlying physics might suggest that the effect of small thermal diffusivity is insignificant. We analytically prove the uniqueness of the wave in the presence of non-zero diffusivity through applying geometric singular perturbation theory.

KEY WORDS: Geometric singular perturbation theory; traveling waves; subsonic detonation; porous media combustion.

1. INTRODUCTION

Gaseous detonation is one of the classical topics of combustion theory. In the past decade, there has been significant progress in the study of the key features of this phenomenon. However, it is still far from completely understood. Recently, Sivashinsky proposed a model of subsonic detonation that describes propagation of the combustion fronts in highly resistible media [7]. The assumption of high resistance of the media provides a

Dedicated to Mr. Brunovsky in honor of his 70th birthday.

[1] Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA. E-mail: ghazarya@email.unc.edu

[2] Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102, USA.

[3] To whom correspondence should be addressed.

natural simplification of the system of governing equations while still preserving many of the qualitative features of the original system. In particular, the model is capable of describing the transition from deflagration to detonation which remains one of the major challenges in the combustion theory. The model reads [1]:

$$\begin{aligned} T_t - (1-\gamma^{-1})P_t &= \varepsilon T_{xx} + Y\Omega(T), \\ P_t - T_t &= P_{xx}, \\ Y_t &= \varepsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y \Omega(T). \end{aligned} \tag{1}$$

Here P, T and Y are the appropriately scaled pressure, temperature and concentration of the deficient reactant; $\gamma > 1$ is the specific heat ratio, ε is a ratio of molecular and pressure diffusivities, Le is a Lewis number and $Y\Omega(T)$ is the reaction rate. The first and third equations of the system (1) represent the partially linearized equations for the conservation of energy and deficient reactant, while the second one is a linearized continuity equation taking into account the equations of state and momentum (Darcy's law).

One of the most distinctive features of premixed combustion is its ability to form a reaction wave that assumes the shape of a sharp front propagating subsonically or supersonically at a well defined speed. Therefore, traveling wave solutions of the system (1) are indispensable for understanding the underlying physical process, especially ones which approach their rest states exponentially fast. In this paper we are interested in the traveling wave solutions of Eq. (1), that is solutions of the form $T(x,t) = T(\xi)$, $P(x,t) = P(\xi)$, $Y(x,t) = Y(\xi)$, where $\xi = x - ct$ and c is the *a priori* unknown front speed. Substituting this ansatz into Eq. (1) we obtain a reduced system of ordinary differential equations

$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= \varepsilon T'' + Y\Omega(T), \\ P'' &= c(T' - P'), \\ cY' + \varepsilon Y'' &= \gamma Y \Omega(T) \end{aligned} \tag{2}$$

with the front-like boundary conditions:

$$\begin{aligned} P(-\infty) = 1, \quad T(-\infty) = 1, \quad Y(-\infty) = 0, \\ T(+\infty) = 0, \quad P(+\infty) = 0, \quad Y(+\infty) = 1. \end{aligned} \tag{3}$$

We have set Le $=1$ in Eq. (2) for simplicity. Note that, unlike the situation in some other thermo-diffusive systems, a Lewis number not equal to one would not change the results of this paper. We assume that the function

$\Omega(T)$ is of the Arrhenius type with an ignition cut-off, that is, $\Omega(T)$ vanishes on an interval $[0, \Theta]$ and is positive for $T > \Theta$:

$$\Omega(T) = 0 \quad \text{for} \quad 0 \le T < \Theta < 1. \tag{4}$$

Moreover, $\Omega(T)$ is an increasing, Lipschitz continuous function, except for a possible discontinuity at the ignition temperature $T = \Theta$. In what follows, we fix the translational invariance of the system by assuming that fronts under consideration reach the ignition temperature Θ at $\xi = 0$.

Most relevant for applications is the case when ε is small (for realistic materials ε varies in the range $\varepsilon \sim 10^{-2} - 10^{-5}$). This observation makes ε a natural small parameter of the problem. For ε small one may formally distinguish two separate regimes of propagation: deflagration associated with small (order of $\sqrt{\varepsilon}$) thermal diffusivity and detonation associated with order one diffusion of pressure [7]. Setting $\varepsilon = 0$, that is ignoring the thermal diffusivity, is very attractive for studying the subsonic detonation regime. For the leading order asymptotics, the system of governing equations (2) reduces to the following one [7]

$$\begin{aligned} -cT' + c(1 - \gamma^{-1})P' &= Y\Omega(T), \\ P'' &= c(T' - P'), \\ cY' &= \gamma Y \Omega(T). \end{aligned} \tag{5}$$

From the physical viewpoint, the last system is believed to reflect the correct phenomenon [7]. From the point of view of mathematics, this is not clear at all because of the singular nature of the perturbation. Moreover, in some limiting situations, the details of propagation of the detonation waves are very sensitive to the value of the thermal diffusivity [8].

Solutions of the problem (5) are well understood. In particular, it is known that there is a unique value of the speed $c = c_0$ for which an appropriate solution exists [4]. The properties of this solution have been investigated in [4], where the structure of the equations in (5) has been exploited to show that for any ξ

$$\begin{aligned} &T(\xi) > 0, \ T'(\xi) < 0, \\ &P(\xi) > 0, \ P'(\xi) < 0, \\ &Y(\xi) > 0, \ Y'(\xi) \ge 0. \end{aligned} \tag{6}$$

Moreover, as was shown in Gordon and Ryzhik [5], solutions of the system (2) are strictly monotone and converge to that of Eq. (5) as $\varepsilon \to 0$.

The uniqueness of the wave solutions of the system (2) has not however yet been established. The possibility that the presence of non-zero thermal diffusivity can lead to a significant change in the qualitative behavior of the solutions has also not been ruled out. Therefore, it is of interest to see if uniqueness of the wave solution of Eq. (5) is robust under perturbation Eq. (2) with non-zero ε. We show here that the wave solution of the system (2) is, indeed, unique for small $\varepsilon > 0$. Our main result can be stated as follows:

Theorem 1. *There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, there is a unique value of c, depending on ε, for which system* (2) *has an orbit satisfying* (3). *Moreover the orbit is unique, and hence so is the traveling wave up to translation.*

The system of ordinary differential equations (2), governing the traveling wave solution, is a singular perturbation of Eq. (5) as the higher order derivatives are being added in Eq. (5). Thus it is natural to attempt to study the problem using the techniques of geometric singular perturbation theory. Singular perturbations can be studied using Fenichel's invariant manifold theory [3]. This approach allows us to show that the traveling wave of Eq. (5) perturbs to a unique wave of Eq. (2) for $\varepsilon > 0$ small enough.

The strategy of the proof of our main result can be described as follows. We first construct a smooth manifold M_0 on which the traveling wave of the unperturbed system (5) lives. This manifold is normally hyperbolic and therefore, by Fenichel [2] for small enough $\varepsilon > 0$, perturbs to a unique invariant manifold M_ε of the system (2). It is worth mentioning that, because of the discontinuity of the nonlinear terms, generally speaking, M_0 can perturb to a discontinuous manifold. That significantly complicates further analysis. To overcome this difficulty, we use the linear structure of the system for low temperatures.

As a next step, we show that in a neighborhood of M_ε no traveling wave can exist off M_ε. Here the continuity of M_ε is important. Restricting the flow to M_ε brings us to a lower dimensional problem. On the other hand, the smooth dependence of M_ε on $\varepsilon (0 \leq \varepsilon \ll 1)$ allows us to extrapolate the information about the existence of a unique front on M_0 to M_ε.

More precisely, we extend the phase space of the system describing the flow on M_0 by adding a direction corresponding to the velocity c. Using properties obtained in Gordon et al. [4] we show that the front is represented as a transversal intersection of two invariant manifolds suspended in $M_0 \times \{c \text{ near } c_0\}$. The intersection occurs at a unique value $c = c_0$. Upon switching on a sufficiently small $\varepsilon > 0$ the transversal intersection perturbs with a nearby c_ε replacing c_0, thus proving the existence of a, unique up to translation, front solution of Eq. (2).

The construction of the front using methods of geometric singular perturbation theory also implies that the ε-dependent family of fronts supported by Eq. (2) converge to the front of Eq. (5) as $\varepsilon \to 0$, thus providing a proof alternative to the one presented in Gordon and Ryzhik [5].

2. CONSTRUCTION OF THE ORBIT IN $\varepsilon = 0$ CASE

It has been shown in Gordon and Ryzhik [5] that there are no traveling wave solutions to the system (2) with $c = 0$. Therefore, in what follows, $c > 0$. We introduce a new variable

$$Q = \frac{1}{c}\int_{\xi}^{+\infty} Y\Omega(T)\,\mathrm{d}x.$$

We integrate each of the equations of (2) using the boundary conditions (3) at $+\infty$ to obtain an equivalent system:

$$\begin{aligned} Q' &= -c^{-1}Y\Omega(T), \\ P' &= c(T-P), \\ \varepsilon T' &= c(1-\gamma^{-1})P - cT + cQ, \\ \varepsilon Y' &= c(1-Y) - \gamma c Q. \end{aligned} \tag{7}$$

We call Eq. (7) the slow system as opposed to the fast system obtained from Eq. (7) in the fast scaling $\eta = \frac{1}{\varepsilon}\xi$

$$\begin{aligned} \dot{Q} &= -\varepsilon c^{-1}Y\Omega(T), \\ \dot{P} &= \varepsilon c(T-P), \\ \dot{T} &= c(1-\gamma^{-1})P - cT + cQ, \\ \dot{Y} &= c(1-Y) - \gamma c Q. \end{aligned} \tag{8}$$

The two systems (7) and (8) are equivalent when $\varepsilon > 0$. The problem amounts to finding values of c for which a certain heteroclinic orbit, in either of these systems, exists.

Satisfying the boundary conditions Eq. (3) determines the critical points at the ends of the desired heteroclinic orbit. We seek an orbit (Q, P, T, Y) such that

$$(Q, P, T, Y) \to (\gamma^{-1}, 1, 1, 0) \ \text{ at } \ -\infty, \tag{9}$$

and

$$(Q, P, T, Y) \to (0, 0, 0, 1) \ \text{ at } \ +\infty. \tag{10}$$

When $\varepsilon = 0$, a reduced system can be derived formally as follows: In Eq. (7), the last two equations reduce to algebraic relations when $\varepsilon = 0$:

$$\begin{aligned} (1-\gamma^{-1})P - T + Q &= 0, \\ 1 - Y - \gamma Q &= 0, \end{aligned} \tag{11}$$

which, when inserted into the equations for Q and P, give the limiting slow equations:

$$\begin{aligned} Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P), \\ P' &= -c\gamma^{-1}P + cQ. \end{aligned} \tag{12}$$

From the results found in Gordon and Ryzhik [5] we know that Eq. (12) possesses a heteroclinic orbit which connects the equilibrium at $(Q, P) = (\gamma^{-1}, 1)$ as $\xi \to -\infty$ to $(Q, P) = (0, 0)$ as $\xi \to +\infty$. Moreover, the speed $c = c_0 > 0$ is unique at which such a heteroclinic orbit exists. The linearization of Eq. (12) at $(\gamma^{-1}, 1)$ has eigenvalues $-c\gamma^{-1} < 0$ and $\gamma c^{-1}\Omega(1) > 0$, thus, the equilibrium $(\gamma^{-1}, 1)$ is a saddle and so has a one-dimensional unstable manifold (denoted W^u). At $(0, 0)$ the eigenvalues are 0 and $-c\gamma^{-1}$. So, there is a one-dimensional stable manifold W^s and a one-dimensional center manifold W^c. The center manifold W^c is due to the presence in some neighborhood of $(0, 0)$ of the curve of equilibria $P = \gamma Q$ which contains $(0, 0)$. Since the equation is actually linear in a neighborhood of $(0, 0)$, due to the ignition cut-off, the flow is easily analyzed. The heteroclinic solution is formed as an intersection of $W^u(\gamma^{-1}, 1)$ with $W^s(0, 0)$. The intersection will be formed at some fixed value of c, say $c = c_0$.

Appending an equation for the parameter c to Eq. (12), we obtain:

$$\begin{aligned} Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P), \\ P' &= -c\gamma^{-1}P + cQ, \\ c' &= 0, \end{aligned} \tag{13}$$

which has as critical points the following two subsets of the three-dimensional $Q \times P \times c$ phase space of Eq. (13): a one-dimensional curve $Q = \gamma^{-1}$, $P = 1$ and a subset P_Θ of a two-dimensional plane $P = \gamma Q$ defined by the condition $P < \Theta$. For any fixed c we form a two-dimensional center-unstable manifold for $(\gamma^{-1}, 1, c)$, denoted W^{cu}. The situation at the equilibrium $(0, 0, c)$, which belongs to P_Θ, is more complicated. The center manifold of $(0, 0, c)$ includes P_Θ. We are not interested in the full center-stable manifold. Rather, we form a stable manifold for the submanifold of the center manifold of $(0, 0, c)$ which does not include any equilibria from P_Θ other than $Q = 0$, $P = 0$. With an abuse of notation, we call this two-dimensional manifold W^{cs}. It is important to notice that, as

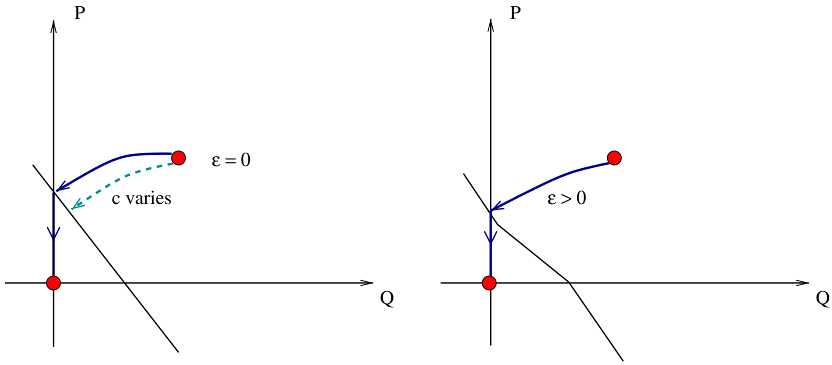

Figure 1. The invariant manifolds intersect transversally as c varies near c_0.

it follows from the expressions for the relevant eigenvalues, both W^{cu} and W^{cs} depend on c. The following lemma is an extension of Gordon and Ryzhik [5, Theorem 2]:

Lemma 1. $W^{cu}(\gamma^{-1}, 1, c_0)$ *transversely intersects* $W^{cs}(0, 0, c_0)$.

The lemma is proved using a standard calculation of Melnikov's type (see, for example, [6]).

Proof. We consider the intersections of W^{cu} and W^{cs} with a plane in the phase space $Q \times P \times c$

$$Q = \Theta - (1 - \gamma^{-1})P, \tag{14}$$

where Θ is the ignition temperature (see Fig. 1). Since this plane is independent of c and, as we mentioned above, W^{cu} and W^{cs} are c-dependent, the intersections of the plane (14) with W^{cu} and W^{cs} are at most one-dimensional. We also mention that, since a solution of Eq. (5) exists [4], each of these intersections is not empty. We denote them $P = h^u(c)$ and $P = h^s(c)$, respectively, and think of them as curves in the plane (14).

The proof of Lemma 1 amounts to checking the transversality condition [6, Section 1.4]:

$$\left(\frac{\partial h^u}{\partial c} - \frac{\partial h^s}{\partial c}\right)\bigg|_{c=c_0} \neq 0. \tag{15}$$

We fix the translational invariance of the system (13) in the following way. The properties Eq. (6) of the solutions of Eq. (5) imply that the temperature T is a monotone function of ξ. The ignition value is reached at some finite ξ, which, without loss of generality, we can assume to be $\xi = 0$.

We notice that for temperatures below ignition $Q=0$, the intersection of the plane (14) with $P=h^s(c)$ is actually a point $Q=0$, $P=(1-\gamma^{-1})^{-1}\Theta$, which does not move when c is varied. Therefore

$$\left.\frac{\partial h^s}{\partial c}\right|_{c=c_0}=0. \tag{16}$$

This also can be easily seen from the explicit expression for the solution of Eq. (12) for $\xi\geq 0$

$$Q=0,\quad P=\frac{\Theta}{1-\gamma^{-1}}e^{-\frac{c}{\gamma}\xi}.$$

Indeed, the derivative of P with respect to c at $\xi=0$ is zero.

Next we concentrate on the temperatures above ignition. We start by identifying vectors tangent to W^{cu} and W^{cs}. One of them is

$$H_1=\left(\frac{\partial Q}{\partial c},\frac{\partial P}{\partial c},1\right)=(\delta Q^-,\delta P^-,1).$$

Another one is the vector field of Eq. (13)

$$H_2=(f_1(Q,P,c),f_2(Q,P,c),0), \tag{17}$$

where

$$\begin{aligned} f_1(Q,P,c) &= c^{-1}(\gamma Q-1)\Omega(Q+(1-\gamma^{-1})P),\\ f_2(Q,P,c) &= -c\gamma^{-1}P+cQ. \end{aligned}$$

To find the sign of $\frac{\partial h^u}{\partial c}$ at $c=c_0$ we look at the following vector product on the interval $\xi\leq 0$

$$\begin{vmatrix} i & j & k\\ f_1 & f_2 & 0\\ \delta Q^- & \delta P^- & 1\end{vmatrix}=f_2 i-f_1 j+(f_1\delta P^- - f_2\delta Q^-)k. \tag{18}$$

Note that on the plane (14)

$$H_1=\left(-(1-\gamma^{-1})\frac{\partial h^u}{\partial c},\frac{\partial h^u}{\partial c},1\right). \tag{19}$$

Therefore, at $\xi=0$, Eq. (18) reads

$$\begin{vmatrix} i & j & k\\ f_1 & f_2 & 0\\ -(1-\gamma^{-1})\dfrac{\partial h^u}{\partial c} & \dfrac{\partial h^u}{\partial c} & 1\end{vmatrix}=f_2 i-f_1 j+(f_1+(1-\gamma^{-1})f_2)\frac{\partial h^u}{\partial c}k,$$

thus imposing a condition

$$f_1 \delta P^- - f_2 \delta Q^- = (f_1 + (1-\gamma^{-1}) f_2) \frac{\partial h^u}{\partial c}. \tag{20}$$

Differentiating the quantity

$$w = f_1 \delta P^- - f_2 \delta Q^-, \tag{21}$$

with respect to ξ we obtain

$$\begin{aligned} w' = & \left(\frac{\partial f_1}{\partial Q} f_1 + \frac{\partial f_1}{\partial P} f_2\right) \delta P^- + f_1 \left(\frac{\partial f_2}{\partial Q} \delta Q^- + \frac{\partial f_2}{\partial P} \delta P^- + \frac{\partial f_2}{\partial c}\right) \\ & - \left(\frac{\partial f_2}{\partial Q} f_1 + \frac{\partial f_2}{\partial P} f_2\right) \delta Q^- - f_2 \left(\frac{\partial f_1}{\partial Q} \delta Q^- + \frac{\partial f_1}{\partial P} \delta P^- + \frac{\partial f_1}{\partial c}\right) \\ = & \left(\frac{\partial f_1}{\partial Q} + \frac{\partial f_2}{\partial P}\right) w + f_1 \frac{\partial f_2}{\partial c} - f_2 \frac{\partial f_1}{\partial c} \end{aligned} \tag{22}$$

Then w satisfies a differential equation

$$w' = \left(-\frac{c}{\gamma} + \frac{\gamma}{c}\Omega(T) - \frac{1}{c}(1-\gamma Q)\Omega'(T)\right) w + \frac{2}{c}\left(Q - \frac{1}{\gamma} P\right)(\gamma Q - 1)\Omega(T), \tag{23}$$

where $T = T(P, Q) = Q + (1-\gamma^{-1})P$. We know from Gordon et al. [4] that if a solution asymptotically connecting $(\gamma^{-1}, 1)$ and $(0, 0)$ exists, then for the temperatures above the ignition temperature

$$T < P, \quad \text{and} \quad Y > 0.$$

These inequalities imply

$$Q - \gamma^{-1} P = T - P < 0, \quad \gamma Q - 1 = -Y < 0$$

and therefore

$$F = \frac{2}{c}(Q - \frac{1}{\gamma} P)(\gamma Q - 1)\Omega(T) > 0, \tag{24}$$

as well as

$$\begin{aligned} f_1 &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P) < 0, \\ f_2 &= -c\gamma^{-1} P + cQ < 0. \end{aligned} \tag{25}$$

Functions Q and P at $-\infty$ decay to $(\gamma^{-1}, 1)$ at the rate $e^{\frac{\gamma}{c}\Omega(1)\xi}$. Therefore $w\,e^{\left(\frac{c}{\gamma}-\frac{\gamma}{c}\Omega(1)\right)\xi}$ approaches 0 at $-\infty$ and we can write the solution to Eq. (23) as

$$w(\xi) = e^{a_-(\xi)} \int_{-\infty}^{\xi} e^{-a_-(z)} F(z)\,\mathrm{d}z, \tag{26}$$

where

$$a_-(\xi) = \left[-\frac{c}{\gamma} + \frac{\gamma}{c}\Omega(1)\right]\xi - \int_{-\infty}^{\xi} \left(\frac{1}{c} Y\Omega'(T(z)) - \frac{\gamma}{c}[\Omega(T(z)) - \Omega(1)]\right) \mathrm{d}z.$$

From Eq. (26), we obtain that

$$w(0) > 0.$$

On the other hand, Eq. (25) implies that

$$f_1 + (1-\gamma^{-1}) f_2 < 0$$

for any $\xi \le 0$. Using the last two inequalities in Eq. (20) we conclude that

$$\left.\frac{\partial h^u}{\partial c}\right|_{c=c_0} < 0, \tag{27}$$

which combined with (16) completes the proof of the transversality condition (15). □

3. CONSTRUCTION OF THE ORBIT FOR A SMALL ε

Geometric singular perturbation theory, see Jones [6], gives a clear prescription as to how to construct a heteroclinic orbit (traveling wave) for the system (2) in the case when ε is small but non-zero. The idea is to construct a manifold on which a perturbation of Eq. (12) governs the flow. This manifold will be invariant under the full Eqs. (8) or, equivalently, Eq. (7). Since the intersection that creates the heteroclinic orbit in Eq. (13) is transverse, it will perturb to Eq. (7).

The manifold in question is constructed as a perturbation of the critical manifold for the limiting ($\varepsilon = 0$) system. Indeed, in the fast scaling Eq. (8) with $\varepsilon = 0$ reads

$$\begin{aligned} \dot{Q} &= 0, \\ \dot{P} &= 0, \\ \dot{T} &= c(1-\gamma^{-1})P - cT + cQ, \\ \dot{Y} &= c(1-Y) - c\gamma Q. \end{aligned} \tag{28}$$

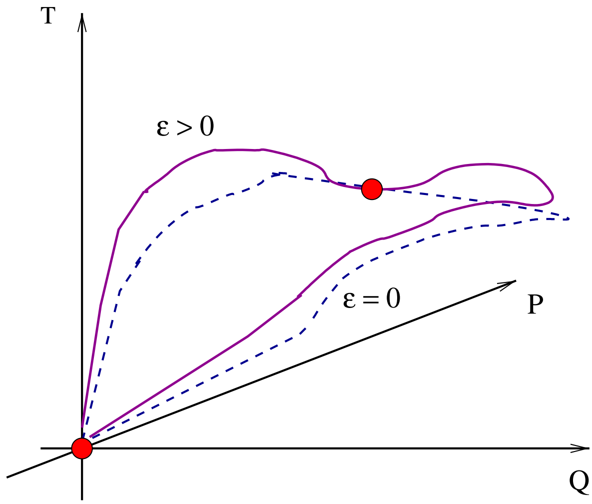

Figure 2. Continuous case: The critical manifold M_0, at least over compact sets, perturbs to an invariant manifold M_ε.

There is then a large set of critical points given by the condition:

$$\begin{aligned} T &= (1-\gamma^{-1})P + Q, \\ Y &= 1-\gamma Q. \end{aligned} \tag{29}$$

which we call M_0. The manifold M_0 is a manifold of critical points for the system (28). This system linearized about each critical point on M_0 has two double eigenvalues: 0 and $-c$. The double zero eigenvalue corresponds to the dimension of M_0. The other eigenvalues being negative ($c > 0$) implies that M_0 is attracting. Each point on M_0 has a two-dimensional stable manifold and a two-dimensional center manifold, which is M_0 itself. Therefore M_0 is an invariant and normally hyperbolic manifold for Eq. (28).

At this point we additionally assume that $\Omega(T)$ is smooth and $\Omega(T) = 0$ for $T < 0$. Under these conditions invariant manifold theory by Fenichel is applicable. More precisely, by Fenichel's First Theorem, see Refs. [3] or [6], the critical manifold M_0, at least over compact sets, perturbs to an invariant manifold for Eq. (8) with $\varepsilon > 0$ but small (see Fig. 2). We call this manifold M_ε. The distance between M_0 and M_ε is of order ε. If ε is small enough, M_ε is normally hyperbolic and attracting on the fast scale $\eta = \xi/\varepsilon$.

On M_ε the flow on the slow scale ξ is determined by equations that are an $O(\varepsilon)$ perturbation of Eq. (12). Indeed, by Fenichel's First Theorem, M_ε is given by

$$\begin{aligned} T &= \Gamma_1(Q,P,\varepsilon) = (1-\gamma^{-1})P + Q + O(\varepsilon), \\ Y &= \Gamma_2(Q,P,\varepsilon) = 1-\gamma Q + O(\varepsilon). \end{aligned} \tag{30}$$

The equations for the flow on M_ε are then given by the following system

$$\begin{aligned} Q' &= -c^{-1}(1-\gamma Q + O(\varepsilon))\Omega((1-\gamma^{-1})P + Q + O(\varepsilon)), \\ P' &= -c\gamma^{-1}P + cQ + O(\varepsilon). \end{aligned} \tag{31}$$

According to Fenichel [3] any invariant set for Eq. (8) which is sufficiently close to M_0 is located on M_ε. Therefore the equilibria $(Q,P,T,Y) = (\gamma^{-1},1,1,0)$ and $(Q,P,T,Y) = (0,0,0,1)$ belong to M_ε.

The heteroclinic orbit of interest is now constructed as a value of c at which the unstable manifold W^u of $(Q,P) = (\gamma^{-1},1)$ intersects transversely the stable manifold W^s of $(Q,P) = (0,0)$ as the speed parameter c varies. This will be done on M_ε as a perturbation of the same construction on M_0. The following lemma guaranties the validity of the reduction of the problem to M_ε.

Lemma 2. *For sufficiently small* $\varepsilon > 0$, *any heteroclinic orbit connecting* (9) *to* (10) *must lie in* M_ε.

Proof. In deriving the critical manifold M_0 for the case $\varepsilon = 0$, we obtained that M_0 is an attracting set. The slow manifold M_ε exists for sufficiently small ε and is also attracting. Therefore the unstable manifold of the equilibrium $(Q,P,T,Y) = (\gamma^{-1},1,1,0)$ must lie in M_ε for ε sufficiently small. □

Next let us show that the structure of the flow at $\xi \to \pm\infty$ for small ε is similar to the one for $\varepsilon = 0$.

The flow at $\xi \to -\infty$, i.e., at the critical point $(\gamma^{-1},1)$ in M_0, is a saddle with a one-dimensional unstable manifold W^u and a one-dimensional stable manifold W^s. This structure perturbs to the case $\varepsilon > 0$. We thus want to follow $W^u(\gamma^{-1},1,c,\varepsilon)$ as c is varied in M_ε.

The situation is more complicated at the boundary $\xi \to \infty$. This boundary corresponds to the critical point $(Q,P) = (0,0)$ in both cases $\varepsilon = 0$ and $\varepsilon > 0$, on M_0 and M_ε, respectively.

First we consider the case $\varepsilon = 0$. From Eq. (29) it is obvious that there is a neighborhood of $(Q,P) = (0,0)$ in M_0 where $T \le \Theta$. In that neighborhood, or, in other words, for temperatures below the ignition, the equation for the flow is very simple as it is given by a linear vector field:

$$\begin{aligned} Q' &= 0, \\ P' &= -c\gamma^{-1}P + cQ. \end{aligned} \tag{32}$$

The eigenvalues of the linearization of Eq. (32) about $(Q, P) = (0, 0)$ are 0 and $-c\gamma^{-1}$. As pointed out above, it is the stable manifold W^s of $(Q, P) = (0, 0)$, that we want to hit with $W^u(\gamma^{-1}, 1)$. The flow of Eq. (32) can be easily analyzed and it is not hard to see that the only way to decay to $(0, 0)$ is by being on $W^s(0, 0, c)$.

The behavior near $(0, 0)$ when $\varepsilon > 0$ is essentially identical to the behavior described above. Indeed, the flow on M_ε near $(0, 0)$ is given by

$$\begin{aligned} Q' &= 0, \\ P' &= -c\gamma^{-1}P + cQ + O(\varepsilon). \end{aligned} \tag{33}$$

Here $O(\varepsilon) = \varepsilon f(\varepsilon, Q, P)$ where f is a continuous function uniformly bounded for small $\varepsilon \geq 0$. Therefore the flow near $(0, 0)$ is structurally the same as for the $\varepsilon = 0$ case: the eigenvalues of the linearization of Eq. (33) about $(Q, P) = (0, 0)$ are 0 and $-c\gamma^{-1} + O(\varepsilon)$. For small enough ε, the second eigenvalue is negative and therefore there is a one-dimensional stable manifold $W^s(0, 0, c, \varepsilon)$.

If we again extend Eq. (32) by allowing c to vary, then the relevant invariant manifolds can be defined exactly as in the $\varepsilon = 0$ case. We denote them $W^{cs}(0, 0, c, \varepsilon)$ and $W^{cu}(\gamma^{-1}, 1, c, \varepsilon)$.

It suffices then to prove that for every $0 < \varepsilon \ll 1$ there exists c_ε such that

$$W^{cu}(\gamma^{-1}, 1, c_\varepsilon, \varepsilon) \bigcap W^{cs}(0, 0, c_\varepsilon, \varepsilon) \neq \varnothing. \tag{34}$$

Lemma 1 shows that Eq. (34) is true when $\varepsilon = 0$. The transversality condition (15) proved for the case $\varepsilon = 0$ also allows us to claim that for the perturbed problem the intersection persists [6].

We consider the plane

$$Q = \Theta - (1 - \gamma^{-1})P + O(\varepsilon), \tag{35}$$

where $O(\varepsilon)$ is as in the first equation of Eq. (30). Within the manifold M_ε, the intersections of $W^{cu}(\gamma^{-1}, 1, c, \varepsilon)$ and $W^{cs}(0, 0, c, \varepsilon)$ with this plane are given by $P = h^u(c, \varepsilon)$ and $P = h^s(c, \varepsilon)$, respectively. An intersection point $(c_\varepsilon, P_\varepsilon)$ of h^u and h^s is found by solving

$$\begin{aligned} P &= h^u(c, \varepsilon), \\ P &= h^s(c, \varepsilon) \end{aligned}$$

for P and c as functions of ε (see Fig. 1). This system has a unique solution if the determinant

$$\det \begin{vmatrix} -1 & \frac{\partial h^u}{\partial c} \\ -1 & \frac{\partial h^s}{\partial c} \end{vmatrix} = \frac{\partial h^u}{\partial c} - \frac{\partial h^s}{\partial c}$$

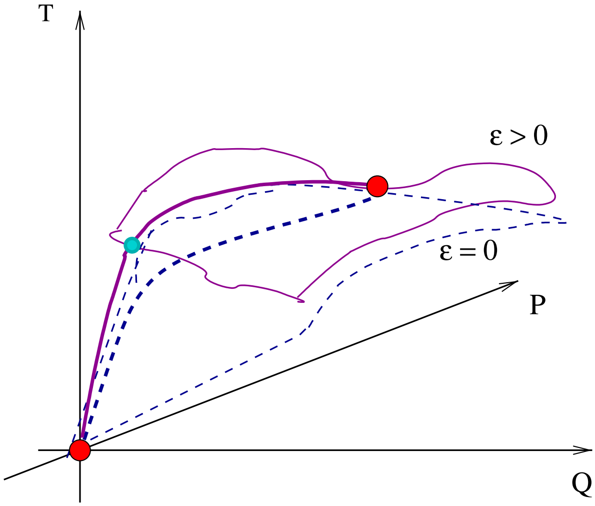

Figure 3. The case of the discontinuous Ω: The solution can leave M_ε^- through the boundary only and must go through the ignition point $T_{ign} = \Theta$, $Q_{ign} = 0$, $P_{ign} = \Theta(1-\gamma^{-1})^{-1} + O(\varepsilon)$.

is nonzero at $c = c_0$ and $\varepsilon = 0$. This condition coincides with Eq. (15). By the Implicit Function Theorem, Eq. (34) then holds when $\varepsilon > 0$ also but with a nearby c_ε replacing c_0. The unique solution restricted to M_ε is constructed.

Remark 1. Generally speaking, the manifold M_ε is not unique. In fact, it depends on the specific way the flow is modified in order to restrict the flow to a compact manifold. Nevertheless, any such manifold contains the point $(Q, P) = (1/\gamma, 1)$ and the one-dimensional unstable manifold of that point. In other words, various modifications of the flow in the neighborhood of the boundary of the compact subset of M_0 yield various M_ε, but all of these manifolds coincide along the subset $\bigcup W^u(1/\gamma, 1)$, where the union is taken with respect to c near c_0. The uniqueness of the solution then follows from the transversality argument above.

If we allow the reaction function Ω to be discontinuous at the ignition temperature Θ, then the situation is different. The c-dependent critical manifold M_ε, now possibly discontinuous along $T = \Theta$, consists of two sets: M_ε^-, which corresponds to the temperatures above the ignition, and M_ε^+ for lower temperatures.

Let us denote $M_0^- = M_0 \cap \{T \geq \Theta\}$ which is a compact manifold with a boundary. We assume that the flow (7) is modified in a standard way to make Fenichel's invariant manifold theory applicable, i.e., under the modified flow the manifold is overflowing invariant and compact. Then, for small $\varepsilon > 0$, there exists an overflowing invariant manifold M_ε^- which is a perturbation of M_0 of order ε.

Since M_ε^- is attracting, instead of Lemma 2 we have the following statement: the solutions converging towards the burnt state (9) at $\xi = -\infty$ do so along the unstable manifold of the burnt state and thus belong to M_ε^- (see Fig. 3 for the illustration). On the other hand the solutions converging to the unburnt state are solutions of a linear system. We recall that solutions of the full system (7) exist and are monotone functions of the spacial variable ξ ([5]). At the unique point on an orbit where $T = \Theta$, the continuity of the solution implies $Q = 0$. It is easy to see that from the Implicit Function Theorem the manifold M_ε^- intersects with $T = \Theta$ and $Q = 0$ at exactly one point which we will call the ignition point. When $\varepsilon = 0$ the manifold M_0 does not depend on c and the coordinates of the ignition point are $T_{ign} = \Theta$, $Q_{ign} = 0$, $Y_{ign} = 1$, $P_{ign} = \Theta(1-\gamma^{-1})^{-1}$. When $\varepsilon > 0$ small enough, by the Implicit Function Theorem the coordinates of the ignition point on M_ε^- are $T_{ign} = \Theta$, $Q_{ign} = 0$, $Y_{ign} = 1 - O(\varepsilon)$, $P_{ign} = \Theta(1-\gamma^{-1})^{-1} + O(\varepsilon)$, where terms of order ε possibly depend on c. When $\varepsilon = 0$, from Eq. (27) we know that the one-dimensional slow unstable manifold W^u of $(Q, P) = (1, 1/\gamma, 1)$ which lies on M_0 reaches the ignition point if and only if $c = c_0$. Therefore, for nonzero ε, using the Implicit Function Theorem again we obtain that the unstable manifold W^u of $(Q, P) = (1/\gamma, 1)$ which lies on M_ε^- reaches the ignition point at a unique value of c which is ε-close to c_0. Now the full orbit can be constructed using the linearity of Eq. (7) for temperatures below ignition.

In order to obtain M_ε^- as a perturbation of M_0^- it is necessary to modify the flow near the boundary of M_0^- to transform the manifold to an overflowing invariant compact manifold. The perturbed manifold M_ε^- depends on the modification. Nevertheless, the argument in Remark 1 still applies and the uniqueness of the orbit follows.

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