

Stability of fronts and transient behavior in KPP systems

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Abstract

We consider a system of two reaction diffusion equations with the KPP type nonlinearity which describes propagation of pressure driven flames. It is known that the system admits a family of traveling wave solutions parameterized by their velocity. In this paper we show that these traveling fronts are stable under the assumption that perturbations belong to an appropriate weighted L^2 space. We also discuss an interesting meta-stable pattern the system exhibits in certain cases.

Key words: KPP systems, traveling fronts, stability, meta-stable regimes

AMS Subject Classifications: 35K57, 35B35, 35B40

1 Introduction

In this paper we consider the following system of reaction diffusion equations,

$$\begin{aligned} u_t &= u_{zz} + (1-v)(hu + (1-h)v), \\ v_t &= \varepsilon v_{zz} + (1-v)(hu + (1-h)v), \quad x \in \mathbb{R} \quad t > 0 \end{aligned} \quad (1.1)$$

where $h \in [0, 1]$, $\varepsilon \in (0, 1]$ are parameters of the system.

The model (1.1) describes propagation of the pressure driven flames in porous media under assumption of linear reaction kinetics. In this case the variable v and u are certain combinations of normalized temperature and pressure. The model was originally derived in [2] and details can be found in [8, 7].

There is also a different perspective that makes the system (1.1) interesting. When $h = 0$ the second equation of the system degenerates to the classical KPP equation [11, 4]. When $h = 1$, the system (1.1) takes the form of the system describing quadratic autocatalysis [1]. Thus, we see the system (1.1) as a system that provides a natural link between two well-known and well-studied equations.

In this paper we continue our work on the analysis of dynamical features of the system (1.1). In our recent paper [7] we studied traveling front solutions for the system (1.1) invading unstable equilibrium $(u, v) = (0, 0)$, that is special solutions of the form $(u, v)(t, z) = (U, V)(z - ct)$, where c is an a priori unknown speed of the traveling front. Substituting the traveling front ansatz into (1.1) we obtain the following nonlinear eigenvalue problem,

$$\begin{aligned} U'' + cU' + (1-V)(hU + (1-h)V) &= 0, \\ \varepsilon V'' + cV' + (1-V)(hU + (1-h)V) &= 0, \end{aligned} \quad (1.2)$$

$$(1.3)$$

where the derivative is with respect to $x = z - ct$, together with the boundary-like condition,

$$(U, V) \rightarrow (1, 1) \quad \text{as } x \rightarrow -\infty, \quad (U, V) \rightarrow (0, 0) \quad \text{as } x \rightarrow \infty. \quad (1.4)$$

In [7] it was shown that existence of solution for the problem (1.2)-(1.4) is fully determined by the existence of positive, exponentially decaying to the equilibrium, solutions of the linearization of the system (1.2) about

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the unstable equilibrium $(U, V) = (0, 0)$. Specifically, linearizing (1.2) around $(U, V) = (0, 0)$ and substituting $(U, V) \sim \exp(-\lambda x)$ one obtains the solvability condition,

$$(\lambda^2 - c\lambda + h)(\varepsilon\lambda^2 - c\lambda + (1 - h)) = h(1 - h). \quad (1.5)$$

Equation (1.5) has two solutions that provide a relation between the velocity of traveling wave c and the rate of exponential decay λ . However, only one of the solutions of (1.5) corresponds to non-negative traveling waves. This solution reads,

$$c(\lambda) = \frac{1}{2} \left[\frac{1}{\lambda} + (1 + \varepsilon)\lambda + \sqrt{\left(\frac{1}{\lambda} + (1 - \varepsilon)\lambda\right)^2 - 4(1 - \varepsilon)(1 - h)} \right]. \quad (1.6)$$

It is not hard to see that there exists a unique $0 < c^* < \infty$ such that the equation (1.6) has no solution when $c < c^*$, one solution λ^* when $c = c^*$, and two distinct solutions λ_+ and λ_- ($\lambda_+ < \lambda_-$) when $c > c^*$. We call this critical value c^* of the front's velocity the critical velocity. The following theorem has been proved in [7].

Theorem 1.1 *For any $c \geq c^*$, the system (1.2) has a unique, up to translation, positive traveling wave solution (front) $U(x), V(x)$ asymptotically connecting the equilibrium $(U, V) = (1, 1)$ as $x \rightarrow -\infty$ to the equilibrium $(U, V) = (0, 0)$ as $x \rightarrow \infty$. The solution converges to both of equilibria exponentially fast and the rate of convergence to $(U, V) = (0, 0)$ is $\lambda = \lambda_+$ where λ_+ is the minimal solution of (1.6). The system (1.2) has no positive traveling wave solutions for $c < c^*$.*

The question of the existence of the fronts is resolved in Theorem 1.1. The next immediate question is the stability of the fronts. In [12, 13] the authors investigated a similar system, but with a small h , in other words, they considered a system that is a singular perturbation of the system (1.1) with $h = 0$. It was shown in [12, 13] that a reduction principle can be used to prove that the stability properties of the scalar KPP equation dominate the properties of the fronts in the perturbed system. For the system (1.1), we will demonstrate that the stability properties are inherited from the scalar KPP equation even for h that is not so small: each traveling wave in the co-moving coordinate frame is stable provided that the initial perturbation decays faster than traveling wave for x large. In the next section we will show that this is indeed the case if $h \in (0, 3/4]$ and $\varepsilon \in (0, 1]$. More precisely, for a front we will consider a space of perturbations that decay at $+\infty$ faster than the front and show that in the co-moving frame the front is stable with respect to the weighted L^2 -norm. Moreover, we will show that this result translates into convective behavior in a sense that, for any fixed $L \in \mathbb{R}$, within the interval $[L, \infty)$ initial perturbation will decay pointwise. In the case when $h > 3/4$ we prove a similar stability result for fronts that move at sufficiently fast speeds. The case of $h = 0$ will be discussed separately in Sect. 3.

System (1.1), unlike the system considered in [12, 13], also allows the diffusion coefficients to be quite different ($\varepsilon \in (0, 1]$). This additional multi-scale structure also serves as a complication. Instead of exploiting smallness of parameters, i.e. their closeness to singular limits, we try to obtain results that are valid for more general parameter regimes. This is done by employing energy type estimates conceptually similar to these used by Focant and Gallay in [5] for analysis of stability of fronts for reaction diffusion systems with more general reaction term, but diffusion coefficients which are close to each other.

We also use the technique that works for both the case of supercritical fronts ($c > c^*$) and the critical front ($c = c^*$). Raugel and Kirchgässner prove that the admissible perturbations to the fronts with non-critical velocities decay exponentially in time [12], whereas perturbations to the critical front decay algebraically [13]. These two cases are quite different (the reasons are discussed below) and, in general, require different techniques. Our unified approach yields the information about the exponential decay of perturbations in to the fronts with $c > c^*$, but provides weaker information about the stability of the critical front: the perturbations decay, but the rate is unspecified.

Exponentially weighted L^∞ spaces were used to prove the orbital stability of supercritical fronts by Sattinger [14, 15]. For our technique, and as in [12, 13], exponentially weighted Hilbert spaces are more suitable. The necessity of using weights is determined by the presence of the unstable essential spectrum of the linearization of the system about the given front.

To stabilize the front, at least on the linear level, one uses an exponential weight, in our case with a positive rate. This works well for the supercritical fronts: the essential spectrum is shifted to the open left-hand side of the complex plane. On the other hand, the nonlinearity is not well defined in the exponentially weighted spaces, therefore, the nonlinear stability of the non-critical fronts can not be deduced from the linear stability using Henry's theory [9], and a different approaches are needed.

The situation is even more complicated for the critical front. The essential spectrum cannot be separated from the imaginary axis, therefore even on linear level, the completely stable spectrum can not be achieved. As it is mentioned in [12, 13], the Evans function approach that have been successfully used for some parabolic equations (see for example, [6, 10]) is not applicable for KPP-type equations since the necessary for the construction of the Evans function condition of "consistent splitting" does not hold.

In addition to stabilizing (or almost stabilizing in the critical case) of the essential spectrum, the weights, that one has to use here, remove an eigenvalue at 0 since the derivative of the front does not belong to that weighted space. The family of the fronts that we are considering is parameterized by two parameters: the velocity of the fronts and the translation for a front with each fixed velocity. It seems that using that particular exponential weight allows us to concentrate on a front with a fixed velocity - no shifts in the velocity are allowed, and fixes the translation invariance of the system - the stability is not orbital, i.e. it is of an individual front not within the family.

2 Stability of traveling fronts

In this section we study stability of the traveling fronts for the system (1.1). For brevity we will denote

$$w = hu + (1 - h)v, \quad W = hU + (1 - h)V. \quad (2.1)$$

For further analysis we consider the problem in the system of coordinates that moves with the velocity of the corresponding front, that is $(t, z) \rightarrow (t, x) = (t, z - ct)$. In the new system of coordinates the system (1.1) reads:

$$\begin{aligned} u_t &= u_{xx} + cu_x + (1 - v)w, \\ v_t &= \varepsilon v_{xx} + cv_x + (1 - v)w. \end{aligned} \quad (2.2)$$

We now consider the problem (2.2) with the initial data $u_0(x), v_0(x)$ satisfying the following conditions:

$$0 \leq u_0(x) \leq K, \quad 0 \leq v_0(x) \leq 1, \quad (2.3)$$

where $K > 0$ is some constant. Moreover, we assume that initial data is perturbation of the traveling front in the following sense

$$u_0(x) = U(x) + \tilde{u}_0(x), \quad v_0(x) = V(x) + \tilde{v}_0(x), \quad (2.4)$$

where (U, V) is a traveling wave solution corresponding to $c(\lambda)$ and

$$\begin{aligned} \int_{-\infty}^{\infty} (\tilde{u}_0(x)e^{sx})^2 dx &< \infty, \quad \int_{-\infty}^{\infty} (\tilde{v}_0(x)e^{sx})^2 dx < \infty, \\ \int_{-\infty}^{\infty} \left(\frac{d\tilde{u}_0(x)}{dx} e^{sx} \right)^2 dx &< \infty, \quad \int_{-\infty}^{\infty} \left(\frac{d\tilde{v}_0(x)}{dx} e^{sx} \right)^2 dx < \infty, \end{aligned} \quad (2.5)$$

where it is assumed that either $s = \lambda$ or $s = \lambda + \delta$ with $\delta > 0$ sufficiently small. In other words, we assume that the initial perturbation to the front decays slightly faster at $x \rightarrow \infty$ than the front itself.

We now seek the solution of the problem (2.2) in the following form

$$u(t, x) = U(x) + \tilde{u}(t, x), \quad v(t, x) = V(x) + \tilde{v}(t, x). \quad (2.6)$$

Substituting (2.6) into (2.2) and taking into account (1.2) we have the following system describing dynamics of perturbation (\tilde{u}, \tilde{v}) :

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + c\tilde{u}_x + h(1 - V(x) - \tilde{v}(t, x))\tilde{u} + [(1 - h)(1 - V(x) - \tilde{v}(t, x)) - W]\tilde{v}, \\ \tilde{v}_t &= \varepsilon\tilde{v}_{xx} + c\tilde{v}_x + h(1 - V(x) - \tilde{v}(t, x))\tilde{u} + [(1 - h)(1 - V(x) - \tilde{v}(t, x)) - W]\tilde{v}. \end{aligned} \quad (2.7)$$

For brevity, and technical convinience (see (2.15) below), we will use the notation $v(t, x) = V(x) + \tilde{v}(t, x)$ when working with (2.7).

Now we can formulate the main result of this section:

Theorem 2.1 Consider problem (2.2) with $\varepsilon \in (0, 1]$ and $h \in (0, 3/4]$. Assume initial data satisfy (2.3), (2.4) and condition (2.5) with $s = \lambda$. Then for any $c \geq c^*$ and $L \in \mathbb{R}$ one has,

$$\sup_{x \in [L, \infty)} |U(x) - u(t, x)| \rightarrow 0, \quad \sup_{x \in [L, \infty)} |V(x) - v(t, x)| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (2.8)$$

Moreover, if condition (2.5) is imposed with $s = \lambda + \delta$ with $\delta > 0$ and $c > c^*$ then there exist two constants M and ω such that,

$$\sup_{x \in [L, \infty)} |U(x) - u(t, x)| < Me^{-\omega t}, \quad \sup_{x \in [L, \infty)} |V(x) - v(t, x)| < Me^{-\omega t}, \quad \text{for all } t > 0. \quad (2.9)$$

For the sake of convenience we first make a following change of variables $(\tilde{u}, \tilde{v})(t, x) = (\phi, \psi)(t, x)e^{-sx}$. In new variables the system (2.7) reads:

$$\begin{aligned} \phi_t &= \phi_{xx} + (c - 2s)\phi_x + [s^2 - cs + h(1 - v)]\phi + [(1 - h)(1 - v) - W]\psi, \\ \psi_t &= \varepsilon\psi_{xx} + (c - 2\varepsilon s)\psi_x + h(1 - v)\phi + [\varepsilon s^2 - cs + (1 - h)(1 - v) - W]\psi. \end{aligned} \quad (2.10)$$

The proof of Theorem 2.1 uses energy type estimates and we will need the following non-negative functionals evaluated on solutions of (2.10),

$$\begin{aligned} E_0(t) &= h\|\phi\|^2 + (1 - h)\|\psi\|^2, \quad E_0^x = h\|\phi_x\|^2 + \varepsilon(1 - h)\|\psi_x\|^2, \\ E_1(t) &= \langle W\psi, \psi \rangle, \quad E_1^x(t) = \langle W\psi_x, \psi_x \rangle, \\ E_2(t) &= \langle W\phi, \phi \rangle, \quad E_1^x(t) = \langle W\phi_x, \phi_x \rangle. \end{aligned} \quad (2.11)$$

Here $\|\psi\|(t) = \left[\int_{-\infty}^{\infty} (\psi)^2(t, \cdot) \right]^{1/2}$ and $\langle \psi, \phi \rangle(t) = \int_{-\infty}^{\infty} \psi(t, \cdot) \phi(t, \cdot)$ are the standard norm and inner products in $L^2(\mathbb{R})$. We also note that, thanks to (2.5), all functionals in (2.11) are well defined at time $t = 0$ provided s satisfies the assumption of Theorem 2.1.

Conceptually our construction is similar to the one used in [5].

Lemma 2.1 Let the assumption of Theorem 2.1 be satisfied, then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0(t) &= h(s^2 - cs + h)\|\phi\|^2 + (1 - h)(\varepsilon s^2 - cs + 1 - h)\|\psi\|^2 + h\langle [2(1 - h) - W]\phi, \psi \rangle \\ &\quad - E_0^x(t) - (1 - h)E_1(t) - \langle v(h\phi + (1 - h)\psi), (h\phi + (1 - h)\psi) \rangle \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0(t) &\leq h(s^2 - cs + h)\|\phi\|^2 + (1 - h)(\varepsilon s^2 - cs + 1 - h)\|\psi\|^2 + 2h(1 - h)\langle |\phi|, |\psi| \rangle \\ &\quad - E_0^x(t) - (1 - h)E_1(t) - \langle v(h\phi + (1 - h)\psi), (h\phi + (1 - h)\psi) \rangle. \end{aligned} \quad (2.13)$$

Proof. Multiplying the first and the second equations of (2.10) by ϕ and ψ respectively and integrating by parts we have,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi\|^2 &= -\|\phi_x\|^2 + (s^2 - cs + h)\|\phi\|^2 - h\langle v\phi, \phi \rangle + \langle [(1 - h)(1 - v) - W]\psi, \phi \rangle, \\ \frac{1}{2} \frac{d}{dt} \|\psi\|^2 &= -\varepsilon\|\psi_x\|^2 + (\varepsilon s^2 - cs + 1 - h)\|\psi\|^2 + \langle h(1 - v)\phi, \psi \rangle - \langle [(1 - h)v + W]\psi, \psi \rangle. \end{aligned} \quad (2.14)$$

Multiplying the first and the second equations in (2.14) by h and $1 - h$ respectively and adding the results, we obtain (2.12)

It was shown in [7] that for the component $v(t, x)$ of the solution of the problem (2.2) satisfying (2.3)

$$0 \leq v(t, x) \leq 1 \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0 \quad (2.15)$$

and the traveling wave solution W satisfies

$$0 < W(x) < 1 \quad \text{for all } x \in \mathbb{R}. \quad (2.16)$$

These bounds imply that the last term in (2.12) is non-positive. Moreover, one can see that

$$h|2(1 - h) - W(x)| \leq 2h(1 - h) \quad \text{for each } x \in \mathbb{R}, \quad \text{provided that } h \in [0, 3/4].$$

Incorporating these observations in (2.12) we obtain (2.13). ■

Lemma 2.2 *Let the assumption of Theorem 2.1 be satisfied, then*

$$\varepsilon\lambda^2 - c\lambda + 1 - h < 0, \quad \lambda^2 - c\lambda + h < 0 \quad (2.17)$$

and the quadratic form

$$\mathbf{A}(s) = \begin{pmatrix} h(s^2 - cs + h) & h(1-h) \\ h(1-h) & (1-h)(\varepsilon s^2 - cs + 1 - h) \end{pmatrix} \quad (2.18)$$

is negatively defined for $s = \lambda + \delta$ with $\delta > 0$ sufficiently small and $c > c^*$.

Proof. First observe that $\text{Trace } \mathbf{A}(\lambda) < 0$. Indeed, let

$$p_1 = \varepsilon\lambda^2 - c\lambda + 1 - h, \quad p_2 = \lambda^2 - c\lambda + h, \quad (2.19)$$

then

$$\text{Trace } \mathbf{A}(\lambda) = (1-h)p_1 + hp_2. \quad (2.20)$$

By (1.5) $p_1 p_2 = (1-h)h > 0$. On the other hand,

$$p_1 + p_2 = (1+\varepsilon)\lambda^2 + 1 - 2c\lambda. \quad (2.21)$$

Substituting definition of $c(\lambda)$ given by (1.6) into (2.21), we have

$$p_1 + p_2 = -\lambda \sqrt{\left(\frac{1}{\lambda} + (1-\varepsilon)\lambda\right)^2 - 4(1-\varepsilon)(1-h_\varepsilon)} < 0. \quad (2.22)$$

Therefore $p_1, p_2 < 0$ and thus

$$\text{Trace } \mathbf{A}(\lambda) < 0. \quad (2.23)$$

By continuity we thus have $\text{Trace } \mathbf{A}(\lambda + \delta) < 0$ for a sufficiently small δ .

Next, by straightforward computations, we obtain,

$$\begin{aligned} \text{Det } \mathbf{A}(\lambda + \delta) &= h(1-h)[(\lambda^2 - c\lambda + h)(\varepsilon\lambda^2 - c\lambda + (1-h)) - h(1-h)] + \\ &\delta[(2\lambda - c)(\varepsilon\lambda^2 - c\lambda + (1-h)) + (2\varepsilon\lambda - c)(\lambda^2 - c\lambda + h)] + O(\delta^2). \end{aligned} \quad (2.24)$$

Observe that $\text{Det } \mathbf{A}(\lambda) = 0$ by (1.5). Differentiating (1.5) with respect to λ we have

$$(2\lambda - c)p_2 + (2\varepsilon\lambda - c)p_1 = \lambda \frac{dc}{d\lambda} (p_1 + p_2). \quad (2.25)$$

It was shown in [7] that for supercritical waves ($\lambda < \lambda^*$) one has $dc/d\lambda < 0$. Therefore,

$$(2\lambda - c)p_2 + (2\varepsilon\lambda - c)p_1 = k^2 \quad (2.26)$$

for some $k^2 > 0$. So that

$$\text{Det } \mathbf{A}(\lambda + \delta) = k^2\delta + O(\delta^2) \quad (2.27)$$

and, thus, $\text{Det } \mathbf{A}(\lambda + \delta) > 0$ for $\delta > 0$ sufficiently small. ■

Lemma 2.3 *Let the assumptions of Theorem 2.1 be satisfied, then $E_1(t), E_1^x(t) \rightarrow 0$ as $t \rightarrow 0$*

Proof. First, we set $s = \lambda$ in (2.12), (2.13) and obtain estimates on evolution of the functional $E_0(t)$. By straightforward computations and taking into account (1.5), we have

$$\begin{aligned} h(\lambda^2 - c\lambda + h)\|\phi\|^2 + (1-h)(\varepsilon\lambda^2 - c\lambda + 1 - h)\|\psi\|^2 + 2h(1-h)\langle\psi, \phi\rangle &= -\mu^2\langle\phi - \nu^2\psi, \phi - \nu^2\psi\rangle, \\ h(\lambda^2 - c\lambda + h)\|\phi\|^2 + (1-h)(\varepsilon\lambda^2 - c\lambda + 1 - h)\|\psi\|^2 + 2h(1-h)\langle|\psi|, |\phi|\rangle &= -\mu^2\langle|\phi| - \nu^2|\psi|, |\phi| - \nu^2|\psi|\rangle, \end{aligned}$$

where

$$\mu^2 = -h(\lambda^2 - c\lambda + h) > 0, \quad \nu^4 = \frac{(1-h)(\varepsilon\lambda^2 - c\lambda + 1-h)}{h(\lambda^2 - c\lambda + h)} > 0, \quad (2.28)$$

based on Lemma 2.2. Hence by (2.12), (2.13) and (2.28) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0(t) &= -E_0^x(t) - (1-h)E_1(t) - h\langle W\phi, \psi \rangle \\ &\quad - \langle v(h\phi + (1-h)\psi), (h\phi + (1-h)\psi) \rangle - \mu^2 \langle \phi - \nu^2\psi, \phi - \nu^2\psi \rangle \end{aligned} \quad (2.29)$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_0(t) &\leq -E_0^x(t) - (1-h)E_1(t) \\ &\quad - \langle v(h\phi + (1-h)\psi), (h\phi + (1-h)\psi) \rangle - \mu^2 \langle |\phi| - \nu^2|\psi|, |\phi| - \nu^2|\psi| \rangle. \end{aligned} \quad (2.30)$$

Next, we obtain equation for evolution of $E_1(t)$. Multiplying second equation of (2.10) by $W\psi$ and integrating by parts we obtain

$$\frac{1}{2} \frac{d}{dt} E_1(t) = -\varepsilon \langle W\psi_x, \psi_x \rangle + \langle \rho W\psi, \psi \rangle + h \langle W\phi, \psi \rangle - h \langle vW\phi, \psi \rangle, \quad (2.31)$$

where

$$\rho(x) = \left[\frac{\varepsilon}{2} \frac{W_{xx}}{W} - \frac{(c-2\varepsilon\lambda)}{2} \frac{W_x}{W} + \varepsilon\lambda^2 - c\lambda + (1-h)(1-v) - W \right]. \quad (2.32)$$

It was shown in [7] that W is a monotone C^2 function connecting equilibrium points $(0,0)$ and $(1,1)$ as $x \rightarrow \pm\infty$ with bounded derivatives. Standard regularity results imply that $|W_{xx}/W|$ and $|W_x/W|$ are uniformly bounded and $C_1 = \sup_{x \in \mathbb{R}} |\rho(x)|$ is finite. Here and below C_i are positive constants. Hence,

$$\frac{1}{2} \frac{d}{dt} E_1(t) \leq -\varepsilon E_1^x(t) + C_1 E_1(t) + h \langle W\phi, \psi \rangle - h \langle vW\phi, \psi \rangle. \quad (2.33)$$

We differentiate the second equation of (2.10) with respect to x ,

$$\psi_{xt} = \varepsilon \psi_{xxx} + (c-2\varepsilon s)\psi_{xx} + h(1-v)\phi_x + [\varepsilon s^2 - cs + (1-h)(1-v) - W] \psi_x - hv_x\phi - [(1-h)v_x + W_x] \psi,$$

multiply the result by $W\psi_x$,

$$\begin{aligned} \psi_{xt} W\psi_x &= \varepsilon \psi_{xxx} W\psi_x + (c-2\varepsilon s)\psi_{xx} W\psi_x + h(1-v)\phi_x W\psi_x \\ &\quad + [\varepsilon s^2 - cs + (1-h)(1-v) - W] \psi_x W\psi_x - hv_x\phi W\psi_x - [(1-h)v_x + W_x] \psi W\psi_x, \end{aligned}$$

and integrate with respect to x . We then integrate some of the terms by parts and obtain the equation for the evolution of $E_1^x(t)$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_1^x(t) &= -\varepsilon \langle W\psi_{xx}, \psi_{xx} \rangle + \langle vW\phi, \psi_{xx} \rangle + \langle W^2\psi, \psi_{xx} \rangle + \langle [\rho + (1-h)v + W] W\psi_x, \psi_x \rangle \\ &\quad + h \langle W\phi_x, \psi_x \rangle + h \langle vW_x\phi, \psi_x \rangle + (1-h) \langle vW_x\psi, \psi_x \rangle + \langle WW_x\psi, \psi_x \rangle \end{aligned} \quad (2.34)$$

Using Young's inequality and the fact that $0 \leq v, W \leq 1$ and $|W_x| < 1 + h/2c$ (see [7]) we have

$$\begin{aligned} -\varepsilon \langle W\psi_{xx}, \psi_{xx} \rangle + \langle vW\phi, \psi_{xx} \rangle + \langle W^2\psi, \psi_{xx} \rangle &\leq \frac{1}{2\varepsilon} \langle v\psi, \psi \rangle + \frac{1}{2\varepsilon} \langle W\psi, \psi \rangle, \\ \langle [\rho + (1-h)v + W] W\psi_x, \psi_x \rangle + h \langle W\phi_x, \psi_x \rangle &\leq (C_1 + 2 - \frac{h}{2}) \|\psi_x\|^2 + \frac{h}{2} \|\phi_x\|^2, \\ h \langle vW_x\phi, \psi_x \rangle + (1-h) \langle vW_x\psi, \psi_x \rangle + \langle WW_x\psi, \psi_x \rangle &\leq \frac{1+h}{4c} (2 \langle v\phi, \phi \rangle + (1-h) \langle v\psi, \psi \rangle + \langle W\psi, \psi \rangle). \end{aligned} \quad (2.35)$$

These estimates imply that there exists a constant $\tilde{C}_2 > 0$ such that

$$\frac{1}{2} \frac{d}{dt} E_1^x(t) \leq \tilde{C}_2 (E_0^x(t) + E_1(t) + \langle v\phi, \phi \rangle + \langle v\psi, \psi \rangle).$$

It is also possible to choose a constant $C_2 > \tilde{C}_2 > 0$ such that

$$\frac{1}{2} \frac{d}{dt} E_1^x(t) \leq C_2 (E_0^x(t) + E_1(t) + h^2 \langle v\phi, \phi \rangle + \langle v((1-h) + W/2)^2 \psi, \psi \rangle). \quad (2.36)$$

Next we add (2.29) and (2.33) together and obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (E_0(t) + E_1(t)) &\leq C_3 E_1(t) - \mu^2 \langle \phi - \nu^2 \psi, \phi - \nu^2 \psi \rangle \\ &\quad - \langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi) \rangle, \end{aligned} \quad (2.37)$$

with

$$C_3 = C_1 - 1 + h + \sup_{t>0, x \in \mathbb{R}} v(t, x) [(1-h)W + W^2/4] = C_1 + \frac{1}{4}.$$

We multiply (2.36) by σ/C_2 and add the result to (2.37) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(E_0(t) + E_1(t) + \frac{\sigma}{C_2} E_1^x(t) \right) &\leq \sigma E_0^x(t) + (C_3 + \sigma) E_1(t) + \sigma (h^2 \langle v\phi, \phi \rangle + \langle v[(1-h) + W/2]^2 \psi, \psi \rangle) \\ &\quad - \langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi) \rangle - \mu^2 \langle \phi - \nu^2 \psi, \phi - \nu^2 \psi \rangle \end{aligned}$$

We then rewrite the last three terms on the right as follows,

$$\begin{aligned} &\sigma (h^2 \langle v\phi, \phi \rangle + \langle v[(1-h) + W/2]^2 \psi, \psi \rangle) - \langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi) \rangle \\ &\quad - \mu^2 \langle \phi - \nu^2 \psi, \phi - \nu^2 \psi \rangle - 2\sigma h \langle v[1-h + W/2]\phi, \psi \rangle - \mu^2 (\|\phi\|^2 + \nu^4 \|\psi\|^2 - 2\nu^2 \langle \psi, \phi \rangle) \\ &= -(1-\sigma) \langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi) \rangle - \mu^2 \|\phi\|^2 - \mu^2 \nu^4 \|\psi\|^2 \\ &\quad + 2\mu^2 \nu^2 \langle \phi, \psi \rangle - 2\sigma h \langle v[1-h + W/2]\phi, \psi \rangle \end{aligned}$$

We then choose σ so small that the following bound holds,

$$2\mu^2 \nu^2 \langle \phi, \psi \rangle - 2\sigma h \langle v[1-h + W/2]\phi, \psi \rangle \leq 2\mu^2 \nu^2 \langle |\phi|, |\psi| \rangle,$$

and arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(E_0(t) + E_1(t) + \frac{\sigma}{C_2} E_1^x(t) \right) &\leq \sigma E_0^x(t) + (C_3 + \sigma) E_1(t) - \mu^2 \langle |\phi| - \nu^2 |\psi|, |\phi| - \nu^2 |\psi| \rangle \\ &\quad - (1-\sigma) \langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi) \rangle \end{aligned} \quad (2.38)$$

From (2.30),

$$\frac{1}{2} \frac{d}{dt} E_0(t) \leq -E_0^x(t) - (1-h) E_1(t). \quad (2.39)$$

We denote

$$C_4 = \frac{C_3 + \sigma}{1-h} + 1.$$

Multiplying (2.39) by $C_4 - 1$ and adding the result to (2.37) and (2.38), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (C_4 E_0(t) + E_1(t)) &\leq 0, \\ \frac{1}{2} \frac{d}{dt} \left(C_4 E_0(t) + E_1(t) + \frac{\sigma}{C_2} E_1^x(t) \right) &\leq 0. \end{aligned} \quad (2.40)$$

We observe that nonnegative functionals $E_0(t)$ and $C_4 E_0(t) + E_1(t)$ and $C_4 E_0(t) + E_1(t) + \frac{\sigma}{C_2} E_1^x(t)$ are non-increasing by (2.39) and (2.40). Hence, $E_0(t)$, $C_3 E_0(t) + E_1(t)$, $C_4 E_0(t) + E_1(t) + \frac{\sigma}{C_2} E_1^x(t)$ converge as $t \rightarrow \infty$. That implies that $E_1(t)$ and $E_1^x(t)$ also converge as $t \rightarrow \infty$.

We integrate (2.39) to obtain

$$\int_0^\infty E_0^x(t) dt + (1-h) \int_0^\infty E_1(t) dt \leq \frac{1}{2} (E_0(0) - E_0(\infty)) < \infty. \quad (2.41)$$

Since $E_0^x(t) \geq \varepsilon(1-h)E_1^x(t) + hE_2^x(t)$ for all t , we have

$$\int_0^\infty E_1(t) dt \leq \frac{1}{2(1-h)} (E_0(0) - E_0(\infty)) < \infty, \quad (2.42)$$

$$\begin{aligned} \int_0^\infty E_1^x(t) dt &\leq \frac{1}{2\varepsilon(1-h)} (E_0(0) - E_0(\infty)) < \infty, \\ \int_0^\infty E_2^x(t) dt &\leq \frac{1}{2h} (E_0(0) - E_0(\infty)) < \infty. \end{aligned} \quad (2.43)$$

First two inequalities in (2.42) guarantee integrability of $E_1(t)$ and $E_1^x(t)$ and thus, since $E_1(t)$ and $E_1^x(t)$ converge as $t \rightarrow \infty$, they must converge to zero. The last inequality in (2.42) will be used in the next lemma which establishes the same result for $E_2(t)$ and $E_2^x(t)$. ■

Lemma 2.4 *Under the assumptions of Theorem 2.1, $E_2(t), E_2^x(t) \rightarrow 0$ as $t \rightarrow \infty$*

Proof. We multiply the first equation of (2.10) by $W\phi$ and integrate some of the terms by parts to obtain

$$\frac{1}{2} \frac{d}{dt} E_2(t) = -\langle W\phi_x, \phi_x \rangle + \langle \gamma W\phi, \phi \rangle + \langle W[(1-h)(1-v) - W]\psi, \phi \rangle, \quad (2.44)$$

where

$$\gamma(x) = \left[\frac{1}{2} \frac{W_{xx}}{W} - \frac{(c-2\lambda)}{2} \frac{W_x}{W} + \lambda^2 - c\lambda + h(1-v) \right]. \quad (2.45)$$

Since $|W_{xx}/W|$ and $|W_x/W|$ are uniformly bounded we have $C_5 = \sup_{x \in \mathbb{R}} |\gamma(x)| < \infty$. From (2.44) we have

$$\frac{1}{2} \frac{d}{dt} E_2(t) \leq C_5 E_2(t) + \langle W|\psi|, |\phi| \rangle \quad (2.46)$$

The following two inequalities are easy to check,

$$\begin{aligned} \phi^2 &= (\phi - \nu^2\psi + \nu^2\psi)^2 = (\phi - \nu^2\psi)^2 + \nu^4\psi^2 + 2\nu^2(\phi - \nu^2\psi)\psi \leq (\nu^4 + \nu^2)\psi^2 + (1 + \nu^2)(\phi - \nu^2\psi)^2, \\ |\phi||\psi| &= |\psi||\phi - \nu^2\psi + \nu^2\psi| \leq \nu^2\psi^2 + |\psi||\phi - \nu^2\psi| \leq (\nu^2 + 1/2)\psi^2 + \frac{1}{2}(\phi - \nu^2\psi)^2. \end{aligned} \quad (2.47)$$

Therefore

$$\begin{aligned} E_2(t) &\leq (\nu^4 + \nu^2)E_1(t) + (1 + \nu^2)\langle W(\phi - \nu^2\psi), (\phi - \nu^2\psi) \rangle, \\ \langle W|\phi|, |\psi| \rangle &\leq (\nu^2 + 1/2)E_1(t) + \frac{1}{2}\langle W(\phi - \nu^2\psi), (\phi - \nu^2\psi) \rangle. \end{aligned} \quad (2.48)$$

From (2.46) and (2.48) we obtain

$$\frac{1}{2} \frac{d}{dt} E_2(t) \leq -E_2(t) + C_6 E_1(t) + C_7 \langle W(\phi - \nu^2\psi), (\phi - \nu^2\psi) \rangle, \quad (2.49)$$

where

$$C_6 = (C_5 + 1)(\nu^2 + \nu^4) + \nu^2 + \frac{1}{2}, \quad C_7 = (C_5 + 1)(1 + \nu^2) + \frac{1}{2}.$$

We multiply (2.37) by C_7/μ^2 , (2.39) by $C_8 = (C_2 + C_2 C_6/\mu^2)/(1-h)$ and add the results to (2.49). Taking into account that $\langle W(\phi - \nu^2\psi), (\phi - \nu^2\psi) \rangle \leq \langle (\phi - \nu^2\psi), (\phi - \nu^2\psi) \rangle$ for all $t \geq 0$, we then obtain

$$\frac{1}{2} \frac{d}{dt} (E_2 + \frac{C_7}{\mu^2} E_1(t) + \left(C_8 + \frac{C_7}{\mu^2} \right) E_0(t)) \leq -E_3(t). \quad (2.50)$$

From (2.50) the functional $E_2 + \frac{C_7}{\mu^2} E_1(t) + \left(C_8 + \frac{C_7}{\mu^2} \right) E_0(t)$ is not increasing and thus converge as $t \rightarrow \infty$. Since functionals E_0 and E_1 converge as $t \rightarrow \infty$ we conclude that E_2 also converges as $t \rightarrow \infty$.

Integrating (2.50) we have

$$\int_0^\infty E_2(t) dt \leq E_2(0) - E_2(\infty) + \frac{C_7}{\mu^2} (E_1(0) - E_1(\infty)) + \left(C_8 + \frac{C_7}{\mu^2} \right) (E_0(0) - E_0(\infty)) < \infty. \quad (2.51)$$

Therefore, $E_2(t)$ converges to zero as $t \rightarrow \infty$.

Next, differentiating first equation of (2.10) with respect to x multiplying by $W\phi_x$ and integrating by parts we have

$$\begin{aligned} \frac{d}{dt} E_2^x(t) = & -\langle W\phi_{xx}, \phi_{xx} \rangle + h\langle vW\phi, \phi_{xx} \rangle + (1-h)\langle vW\psi, \phi_{xx} \rangle + \langle W^2\psi, \phi_{xx} \rangle + \langle (\gamma + hv)W\phi_x, \phi_x \rangle \\ & + (1-h)\langle W\phi_x, \psi_x \rangle + h\langle vW_x\phi, \phi_x \rangle + (1-h)\langle vW_x\psi, \phi_x \rangle + \langle WW_x\psi, \phi_x \rangle. \end{aligned} \quad (2.52)$$

Applying Young's inequality to the right hand side of (2.52) similar to (2.35) we have

$$\frac{1}{2} \frac{d}{dt} E_2^x(t) \leq C_9 \left(E_0^x(t) + E_1(t) + h^2 \langle v\phi, \phi \rangle + ((1-h) + W/2)^2 \langle v\psi, \psi \rangle \right), \quad (2.53)$$

where $C_9 > 0$ is sufficiently large.

Multiplying (2.53) by σ/C_9 with a sufficiently small $\sigma > 0$ and adding the result to (2.37) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(E_0(t) + E_1(t) + \frac{\sigma}{C_9} E_2^x(t) \right) \leq & \sigma E_0^x(t) + (C_3 + \sigma) E_1(t) - \mu^2 \langle |\phi| - \nu^2 |\psi|, |\phi| - \nu^2 |\psi| \rangle \\ & - (1 - \sigma) \langle v(h\phi + [(1-h) + W/2]\psi), (h\phi + [(1-h) + W/2]\psi) \rangle. \end{aligned} \quad (2.54)$$

Multiplying (2.39) by $(C_3 + \sigma)/1 - h$ and adding the result to (2.54), we get

$$\frac{1}{2} \frac{d}{dt} \left(C_4 E_0(t) + E_1(t) + \frac{\sigma}{C_9} E_2^x(t) \right) \leq 0, \quad (2.55)$$

which implies that the non-negative functional $C_4 E_0(t) + E_1(t) + \frac{\sigma}{C_9} E_2^x(t)$ is non increasing. This, together with the fact that non increasing functionals $E_0(t)$ and $E_1(t)$ converge as $t \rightarrow \infty$, immediately yields convergence of $E_2^x(t)$ as $t \rightarrow \infty$.

In Lemma 2.3 we already established that $\int_0^\infty E_2^x(t) dt < \infty$ (see equation (2.42), therefore $E_2^x(t) \rightarrow 0$ as $t \rightarrow \infty$. ■

Lemma 2.5 *Let f be such that $\langle Wf, f \rangle(t) + \langle Wf_x, f_x \rangle(t)$ is well defined and set $g(x) = f(x)e^{-\lambda x}$. Then for any given $L \in \mathbb{R}$ there exists a constant $C_{10} > 0$ independent of t and L such that*

$$\sup_{x \in [L, \infty)} |g(t, x)| \leq C_{10} e^{-\lambda L/2} (\langle Wf, f \rangle(t) + \langle Wf_x, f_x \rangle(t))^{1/2}. \quad (2.56)$$

Proof. First, we observe that

$$|W(x)f^2(x)| = \left| \int_x^\infty [W(z)f^2(t, z)]_z dz \right| \leq \int_x^\infty |W_z| f^2(t, z) dz + 2 \int_x^\infty |Wf(t, z)f(t, z)_z| dz. \quad (2.57)$$

Since $|W_z/W| < C_{11}$ for some finite constant $C_{11} > 0$, we have

$$|W(x)f^2(x)| \leq C_{11} \int_x^\infty Wf^2(t, z) dz + 2 \int_x^\infty |Wf(t, z)f(t, z)_z| dz \leq (C_{11} + 1) \langle Wf, f \rangle + \langle Wf_x, f_x \rangle. \quad (2.58)$$

Thus,

$$|g(x)| \leq \left((C_{11} + 1) \frac{e^{-2\lambda x}}{W(x)} (\langle Wf, f \rangle + \langle Wf_x, f_x \rangle) \right)^{1/2}. \quad (2.59)$$

W is a monotone decreasing function approaching unity as $x \rightarrow -\infty$ and zero as $x \rightarrow \infty$. Moreover $W \sim O(e^{\lambda x})$ for large x . Therefore, given $L \in \mathbb{R}$

$$\sup_{x \in [L, \infty)} \frac{e^{-2\lambda x}}{W(x)} < C_{12} e^{-\lambda L}, \quad (2.60)$$

where C_{12} is independent of L . Combining (2.59) and (2.60) we have (2.56) ■

Lemma 2.6 *Let the assumption of Theorem 2.1 be satisfied with $s = \lambda + \delta$ and $c > c^*$, then there exist two constants $N > 0$ and $\alpha > 0$ such that*

$$E_0(t) < N e^{-\alpha t}, \quad E_0^x < N e^{-\alpha t}. \quad (2.61)$$

Proof. It was shown in Lemma 2.2 that the quadratic form

$$h(s^2 - cs + h)\|\phi\|^2 + (1-h)(\varepsilon s^2 - cs + 1-h)\|\psi\|^2 + 2h(1-h)\|\phi\|\|\psi\| = \mathbf{J}^T(t) \cdot \mathbf{A}(s) \cdot \mathbf{J}(t), \quad (2.62)$$

where $\mathbf{J}(t) = (\|\phi\|, \|\psi\|)(t)$ and

$$\mathbf{A}(s) = \begin{pmatrix} h(s^2 - cs + h) & h(1-h) \\ h(1-h) & (1-h)(\varepsilon s^2 - cs + 1-h) \end{pmatrix} \quad (2.63)$$

is negatively defined for $s = \lambda + \delta$ and a small enough $\delta > 0$. Therefore, there exist $\alpha > 0$ such that

$$\mathbf{J}^T(t) \cdot \mathbf{A}(\lambda + \delta) \cdot \mathbf{J}(t) < -\alpha E_0(t). \quad (2.64)$$

Thus, from (2.13) we have,

$$\frac{1}{2} \frac{dE_0(t)}{dt} \leq -\alpha E_0(t) - E_0^x(t). \quad (2.65)$$

We differentiate the system (2.10) with respect to x , multiply the first and the second equation by ϕ_x and ψ_x respectively, and integrate some of the terms by parts, to obtain

$$\begin{aligned} \frac{1}{2} \frac{d\|\phi_x\|^2}{dt} &= -\|\phi_{xx}\|^2 + h\langle v\phi, \phi_{xx} \rangle + \langle [(1-h)v + W]\psi, \phi_{xx} \rangle + (s^2 - cs + h)\|\phi_x\|^2 + (1-h)\langle \psi_x, \phi_x \rangle, \\ \frac{1}{2} \frac{d\|\psi_x\|^2}{dt} &= -\varepsilon\|\psi_{xx}\|^2 + h\langle v\phi, \psi_{xx} \rangle + \langle [(1-h)v + W]\psi, \psi_{xx} \rangle + h\langle \phi_x, \psi_x \rangle + (\varepsilon s^2 - cs + 1-h)\|\psi_x\|^2. \end{aligned} \quad (2.66)$$

Applying Young's inequality to the right hand side of (2.66)

$$\begin{aligned} \frac{1}{2} \frac{d\|\phi_x\|^2}{dt} &\leq C_{13}(E_0(t) + E_0^x(t)) \\ \frac{1}{2} \frac{d\|\psi_x\|^2}{dt} &\leq C_{14}(E_0(t) + E_0^x(t)), \end{aligned} \quad (2.67)$$

for sufficiently large C_{13} and C_{14} . Multiplying the first and the second equations of (2.67) by h and $\varepsilon(1-h)$ respectively and adding the results together, we have

$$\frac{1}{2} \frac{dE_0^x(t)}{dt} \leq C_{15}(E_0(t) + E_0^x(t)), \quad (2.68)$$

with $C_{15} = hC_{13} + \varepsilon(1-h)C_{14}$. Multiplying (2.68) by $\alpha/2C_{15}$ and adding the result to (2.65) we have

$$\frac{1}{2} \frac{d}{dt} \left(E_0(t) + \frac{\alpha}{2C_{15}} E_0^x(t) \right) \leq -\frac{\alpha}{2} (E_0(t) + E_0^x(t)). \quad (2.69)$$

If we take C_{15} sufficiently large so that $\alpha/2C_{15} \leq 1$, then

$$\frac{1}{2} \frac{d}{dt} \left(E_0(t) + \frac{\alpha}{2C_{15}} E_0^x(t) \right) \leq -\frac{\alpha}{2} \left(E_0(t) + \frac{\alpha}{2C_{15}} E_0^x(t) \right). \quad (2.70)$$

Integrating this equation we get

$$\left(E_0(t) + \frac{\alpha}{2C_{15}} E_0^x(t) \right) \leq \left(E_0(0) + \frac{\alpha}{2C_{15}} E_0^x(0) \right) e^{-\alpha t/2}, \quad (2.71)$$

which immediately implies (2.61). ■

Proof. (Theorem 2.1) By setting $f = (\phi, \psi)$ and $g = (\tilde{u}, \tilde{v})$ in Lemma 2.5 we have

$$\begin{aligned} \sup_{x \in [L, \infty)} |\tilde{v}(t, x)| &\leq C_{16} e^{-\lambda L/2} (E_1(t) + E_1^x(t)), \\ \sup_{x \in [L, \infty)} |\tilde{u}(t, x)| &\leq C_{17} e^{-\lambda L/2} (E_2(t) + E_2^x(t)). \end{aligned} \quad (2.72)$$

Let $s = \lambda$. Then, according to Lemmas 2.3 and 2.4, $E_1(t)$, $E_1^x(t)$, $E_2(t)$, $E_2^x(t) \rightarrow 0$ as $t \rightarrow \infty$ which proves (2.8).

We set $s = \lambda + \delta$ and observe that $E_1(t) + E_2(t) + E_1^x(t) + E_2^x(t) < C_{18}(E_0(t) + E_0^x(t))$ to obtain from (2.72)

$$\begin{aligned} \sup_{x \in [L, \infty)} |\tilde{v}(t, x)| &\leq C_{19} e^{-\lambda L/2} (E_0(t) + E_0^x(t)), \\ \sup_{x \in [L, \infty)} |\tilde{u}(t, x)| &\leq C_{20} e^{-\lambda L/2} (E_0(t) + E_0^x(t)). \end{aligned} \quad (2.73)$$

By Lemma 2.6 $E_0(t)$ and $E_0^x(t)$ converge to zero exponentially fast. This proves (2.9). ■

Remark 2.1 For $h \in (3/4, 1]$ we will show that sufficiently fast supercritical fronts are stable under perturbations that decay at $x \rightarrow \infty$ exponentially at faster rates, although the obtained bounds on the rates of convergence are not optimal.

Indeed, by adding the first and the second equations in (2.10) we have

$$\frac{1}{2} \frac{d}{dt} \tilde{E}_0(t) \leq (s^2 - cs + h) \|\phi\|^2 + (\varepsilon s^2 - cs + 1 - h) \|\psi\|^2 + \|\phi\| \|\psi\| - \tilde{E}_0^x(t), \quad (2.74)$$

where

$$\tilde{E}_0(t) = \|\phi\|^2 + \|\psi\|^2 \geq 0, \quad \tilde{E}_0^x(t) = \|\phi_x\|^2 + \varepsilon \|\psi_x\|^2 \geq 0. \quad (2.75)$$

The first term in the right hand side of (2.74) is associated with the quadratic form $\mathbf{J}^T(t) \cdot \tilde{\mathbf{A}}(s) \cdot \mathbf{J}(t)$ where $\mathbf{J}(t) = (\|\phi\|, \|\psi\|)(t)$ and

$$\tilde{\mathbf{A}}(s) = \begin{pmatrix} (s^2 - cs + h) & 1/2 \\ 1/2 & (\varepsilon s^2 - cs + 1 - h) \end{pmatrix}. \quad (2.76)$$

Arguments similar to ones in the proof of Lemma 2.2 can be used to show that the quadratic form $\mathbf{J}^T(t) \cdot \tilde{\mathbf{A}}(\lambda + \delta) \cdot \mathbf{J}(t)$ is negatively definite provided $c > c^\dagger$ and perturbations satisfy (2.5) with $s > \lambda$, where

$$c^\dagger = \min_{s \in (0, \infty)} c(s) = \frac{1}{2} \left[\frac{1}{\lambda} + (1 + \varepsilon)\lambda + \sqrt{\left(\frac{1}{\lambda} + (1 - \varepsilon)\lambda \right)^2 + 1 - 4(1 + h - \varepsilon)(1 - h)} \right], \quad (2.77)$$

and λ is the smallest solution of

$$c(s) = \frac{1}{2} \left[\frac{1}{s} + (1 + \varepsilon)s + \sqrt{\left(\frac{1}{s} + (1 - \varepsilon)s \right)^2 + 1 - 4(1 + h - \varepsilon)(1 - h)} \right]. \quad (2.78)$$

In this case,

$$\mathbf{J}^T(t) \cdot \tilde{\mathbf{A}}(s) \cdot \mathbf{J}(t) \leq -\beta \tilde{E}_0(t), \quad (2.79)$$

for some $\beta > 0$, and thus

$$\frac{1}{2} \frac{d}{dt} \tilde{E}_0(t) \leq -\beta \tilde{E}_0(t) - \tilde{E}_0^x(t). \quad (2.80)$$

Computations similar to these of Lemma 2.6 also give

$$\frac{1}{2} \frac{d}{dt} \tilde{E}_0^x(t) \leq C_{21} (\tilde{E}_0(t) + \tilde{E}_0^x(t)). \quad (2.81)$$

Combining (2.80) and (2.81) as in Lemma 2.6, we obtain $\tilde{E}_0(t), \tilde{E}_0^x(t) \rightarrow 0$ as $t \rightarrow \infty$ and arguments identical to ones in the proof of Theorem 2.1 gives (2.9).

3 Transient behavior in the limit of weak coupling.

In this section we will discuss an interesting dynamics that the system (1.1) exhibits when $\varepsilon < 1/2$ and h is sufficiently small. Let us note that when $h = 0$ the systems (1.1) decouples. The second equation of (1.1) becomes classical KPP equation and the first equation of the system (1.1) is driven by the second one. This situation is rather degenerate, however is worth some additional discussion. When $h = 0$ the system (1.2) that describes traveling fronts takes the form,

$$\begin{aligned} U_0'' + cU_0' + (1 - V_0)V_0 &= 0, \\ \varepsilon V_0'' + cV_0' + (1 - V_0)V_0 &= 0, \end{aligned} \quad (3.1)$$

with the boundary conditions

$$(U_0, V_0) \rightarrow (1, 1) \quad \text{as } x \rightarrow -\infty, \quad (U_0, V_0) \rightarrow (0, 0) \quad \text{as } x \rightarrow \infty. \quad (3.2)$$

Linearizing the system (3.1) around unstable equilibrium $(U_0, V_0) = (0, 0)$ and substituting $(U_0, V_0) = (M, N) \exp(-\lambda x)$ we obtain

$$\begin{aligned} (\lambda^2 - c\lambda)M + N &= 0, \\ (\varepsilon\lambda^2 - c\lambda + 1)N &= 0. \end{aligned} \quad (3.3)$$

In order to have nontrivial solution of the second equation we must have

$$c = \varepsilon\lambda + \frac{1}{\lambda}. \quad (3.4)$$

In order to have nontrivial positive solution of the system one needs, in addition to (3.4), to satisfy

$$c \geq \lambda. \quad (3.5)$$

If $\varepsilon \geq 1/2$ the condition (3.5) is satisfied if (3.4) holds. However, for smaller ε the condition (3.5) is restrictive. Elementary algebraic calculations can be used to conclude that the system (3.3) has nontrivial positive solution for

$$c \geq c^* = \begin{cases} 2\sqrt{\varepsilon} & \text{if } \varepsilon \geq \frac{1}{2}, \\ \frac{1}{\sqrt{1-\varepsilon}} & \text{if } \varepsilon < \frac{1}{2}. \end{cases} \quad (3.6)$$

It is not hard to verify that the equilibrium point $(U_0, V_0) = (0, 0)$ is hyperbolic and thus all solutions of the system (3.1), (3.2), if they exist, approach this equilibrium at an exponential rate. Therefore, there exist no positive traveling wave solutions for the system (3.1)- (3.2) for $c < c^*$. Moreover, following steps of Theorem 2.1 of [7], one can show that for any $c \geq c^*$, the system (3.1) has a unique, up to translation, positive monotone traveling wave solution (front) $U_0(x), V_0(x)$ asymptotically connecting the equilibrium $(U_0, V_0) = (1, 1)$ as $x \rightarrow -\infty$ to the equilibrium $(U_0, V_0) = (0, 0)$ as $\xi \rightarrow \infty$. The solution converges to both of equilibria exponentially fast.

As noted above when $h = 0$ the system (1.1) decouples and the second equation of the system (3.1) can be considered independently, that is

$$\varepsilon \hat{V}_0'' + c\hat{V}_0' + (1 - \hat{V}_0)\hat{V}_0 = 0, \quad (3.7)$$

with the boundary-like conditions

$$\hat{V}_0 \rightarrow 1 \quad \text{as } \xi \rightarrow -\infty, \quad \hat{V}_0 \rightarrow 0 \quad \text{as } \xi \rightarrow \infty. \quad (3.8)$$

It is well known [11] that the problem (3.7), (3.8) has a family of positive traveling wave solutions parameterized by the speed of propagation c . The smallest, critical, speed of propagation is $c^{**} = 2\sqrt{\varepsilon}$. It is the minimal value of c for which the expression (3.4), that comes from the formal linearization of (3.7) at $V_0 = 0$, has real solution. All fronts converge to the equilibria exponentially fast. The rate of convergence of the critical front to the unstable equilibrium is $\lambda^{**} = 1/\sqrt{\varepsilon}$, whereas the rate of convergence to zero for super-critical fronts is defined as the smallest solution of (3.4). Therefore, when $\varepsilon < 1/2$, along with the family of traveling wave solutions for the system (3.1) with the speed $c \in [c^*, \infty) = [\frac{1}{\sqrt{1-\varepsilon}}, \infty)$, there exists an interval of velocities $[c^{**}, c^*) = [2\sqrt{\varepsilon}, \frac{1}{\sqrt{1-\varepsilon}})$ for which there exist traveling wave solutions of the second equation of the system considered separately from the full system as it is. We call such solutions semi-fronts to emphasize this "semi-existence" result.

Thus, in relation with existence of semifronts and fronts, for the system (3.1) with $\varepsilon < 1/2$ the following is true:

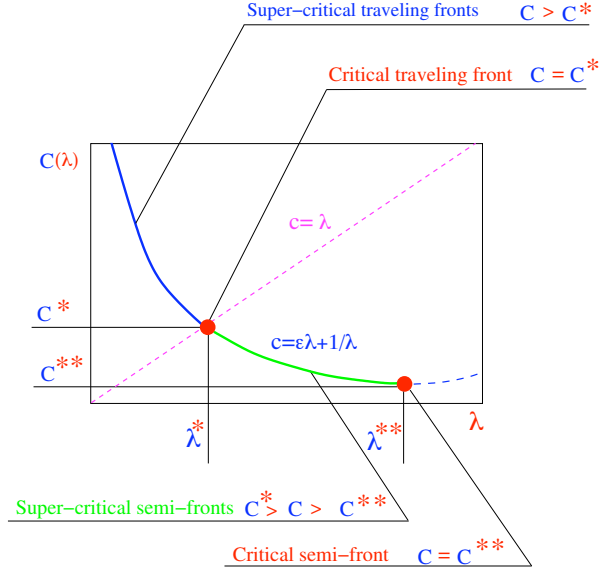


Figure 1: Propagation velocity of traveling fronts and traveling semi-fronts vs. their rate of decay

- (i) There exists a critical semi-front that propagates with the speed $c^{**} = 2\sqrt{\varepsilon}$, with the rate of decay to the equilibrium at $V_0 = 0$ given by $\lambda^{**} = 1/\sqrt{\varepsilon}$
- (ii) There exist super-critical semi-fronts propagating with speeds $c \in (c^{**}, c^*)$
- (iii) There exists a critical front $c^* = 1/\sqrt{1-\varepsilon}$ with the rate of decay to the equilibrium at $(U_0, V_0) = (0, 0)$ given by $\lambda^* = 1/\sqrt{1-\varepsilon}$
- (iv) There exist super-critical fronts propagating with speeds $c \in (c^*, \infty)$.

The situation described above is illustrated on Figure 3.

Note that when ε is sufficiently small the critical semi-front is very slow and decays to zero very fast, whereas the critical front propagates with the velocity of order unity which accidentally coincides with its rate of exponential decay. Thus, there is a transparent separation of scales in the problem. It is clear that for any $h > 0$ semi fronts fail to persist, however we expect that, when h is small, propagation of disturbances in the problem (1.1) with initial conditions which resemble a semi-front will have a scaling of the velocity of that semi-front for a time interval $[0, \tau(h))$ such that $\tau(h) \rightarrow \infty$ as $h \rightarrow 0$. In order to validate this conjecture, we performed a number of numerical

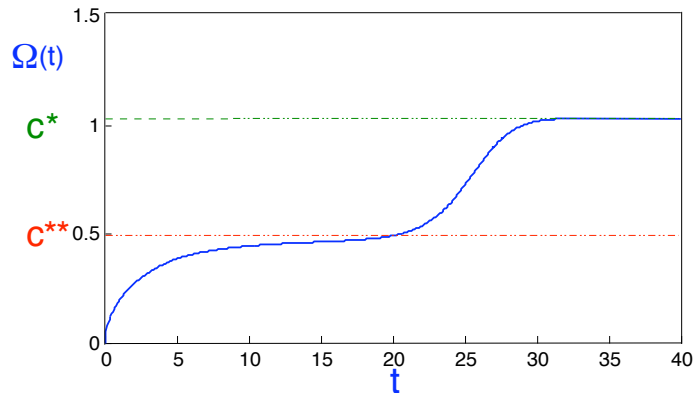


Figure 2: Bulk burning rate Ω evaluated on the solution of the problem (1.1) with initial conditions (3.9). Here $\varepsilon = 0.06$ and $h = 10^{-5}$, c^* and c^{**} are the velocities of the critical front and the semi-front respectively.

experiments with the system (1.1) and initial data

$$u_0(x) = 0, \quad v_0(x) = \chi_{(-\infty, 0]}(x), \quad (3.9)$$

where $\chi_{(-\infty, 0]}(x)$ is characteristic function of $(-\infty, 0]$. In order to measure the characteristic velocity of propagation in the system (1.1), we use the bulk burning rate [3] that is the average of the reaction rate,

$$\Omega(t) = \int (1 - v(t, \cdot))w(t, \cdot). \quad (3.10)$$

This quantity is fairly standard tool for measuring velocity of propagation of disturbances in reactive systems.

Numerical simulations of (1.1) and (3.9) showed that, as expected, the bulk burning rate $\Omega(t)$ initially approaches the velocity of critical semi-front c^{**} and after some time that scales as $\tau(h) \sim \log(1/h)$ jumps to the velocity of the critical front c^* . The typical dynamics of the bulk burning rate is shown on Figure 3. This behavior and scaling of $\tau(h)$ are observed to be very robust and rather insensitive to the initial data v_0 provided it is squeezed between the characteristic function of the interval and the critical semi-front.

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References

- [1] Billingham J., Needham D.J., The Development of Travelling Waves in Quadratic and Cubic Autocatalysis with Unequal Diffusion Rates. I. Permanent Form Travelling Waves, *Philosophical Transactions: Physical Sciences and Engineering*, **334** (1633) (1991) 1-24.
- [2] Brailovsky I., Goldshtein V., Shreiber I., and Sivashinsky G., On combustion waves driven by diffusion of pressure. *Combust. Sci. and Tech.*, **124** (1997) 145-165.
- [3] Constantin P., Kiselev A., Oberman A., and Ryzhik L., Bulk Burning Rate in Passive-Reactive Diffusion, *Arch. Rational Mech. Anal.* **154** (2000) 53-91.
- [4] Fisher R. A., The advance of advantageous genes, *Ann. Eugenics* **7** (1937), 355-369.
- [5] Focant S., Gallay Th., Existence and stability of propagating fronts for an autocatalytic reaction-diffusion system, *Physica D* **120** (1998) 346-368.
- [6] Gardner R., Jones C. K. R. T., Kapitula T., Stability of travelling waves for nonconvex scalar viscous conservation laws, *Comm. Pure Appl. Math.* **XLVI** (1993), 505-526.
- [7] Ghazaryan A., Gordon P., KPP type flame fronts in porous media, *Nonlinearity* **21** (5) (2008) 973-992.
- [8] Gordon P., Recent mathematical results on combustion in hydraulically resistant porous media, *Mathematical Modelling of Natural Phenomena (MMNP)* **2** (2) (2007) 56-76.
- [9] Henry D., Geometric theory of semilinear parabolic equations, Springer-Verlag, Berlin/New-York, 1980.
- [10] Kapitula T., On the stability of travelling waves in weighted L^∞ spaces, *S. Differen. Eq.* **112** (1994), 179-215.
- [11] Kolmogorov A.N., Petrovsky I.G., and Piskunov N.S., A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem, *Bull. Moscow Univ. Math. Mech.* **1** (6) (1937), 1-26.
- [12] Raugel G., Kirchgässner K., Stability of fronts for a KPP-system: the noncritical case, in *Dynamics of nonlinear waves in dissipative systems: reduction, bifurcation and stability*. Pitman Research Notes in Mathematics Series **352** (1996) 147-208.
- [13] Raugel G., Kirchgässner K., Stability of fronts for a KPP-system, II: the critical case, *J. Differ. Eq.* **146** (1998) 399-456.
- [14] Sattinger D. H., On the stability of waves of nonlinear parabolic systems, *Adv. Math.* **22** (1976), 312-355.
- [15] Sattinger D. H., Weighted norms for the stability of travelling waves, *J. Differ. Eq.* **25** (1977), 130-144.